Memorandum M-513

THE CRACK GROWTH RATE OF A CRACK IN A W.T.D.C.B.-SPECIMEN IN BOTH A CONSTANT AND LINEARLY VARYING EXTERNAL LOAD CONDITION

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INTRODUCTION

Because of the increased interest in fracture toughness properties, the width-tapered-double-cantilever-beam (W.T.D.C.B.) test is actually an intensively used test method for obtaining energy release rate \( G_{IC} \) data of both adhesives and composites. The shape of the specimen is chosen in such a way that the energy release rate is only a function of the applied load, which is mostly constant during crack growth. In some cases however (severe plasticity of the resin system or adhesive, plasticising effect of the environment) there can be a significant increase in load during crack growth.

In this paper, equations for the crack growth rate are derived for both a constant and linearly varying applied load. The theoretical results are compared with experimental data.

THEORY

1: load is constant during crack growth

in general: \( v = C.P \) \hspace{1cm} (1)

where \( v \): displacement
\( C \): compliance of the specimen
\( P \): applied load

\[
\frac{dv}{dt} = P \cdot \frac{dC}{dt}
\]

\[
= P \cdot \frac{dC}{da} \cdot \frac{da}{dt}
\]

\[
= P \cdot 2B_t \cdot \frac{1}{2B_t} \cdot \frac{dC}{da} \cdot \frac{da}{dt}
\]

- for a W.T.D.C.B.-specimen:

\[
B_t = \frac{1}{3} a
\]

\[
\frac{1}{2B_t} \cdot \frac{dC}{da} = \text{constant} = Q
\]

- for a displacement controlled test

\[
\frac{dv}{dt} = \text{constant}
\]

\[
\frac{da}{dt} = \frac{3}{2PQa} \cdot \frac{dv}{dt}
\]

\[
2PQ ada = 3 \cdot \frac{dv}{dt} \cdot dt
\]

\[
P a^2 = 3 \cdot \frac{dv}{dt} \cdot t + C_1
\]
at $t=0$ ; $a=a_0$

$C_1 = \frac{PQa_0}{2}$

\[ a = \left( \frac{3}{PQ} \cdot \frac{\text{d}v}{\text{d}t} . t + a_0^2 \right)^{1/2} \]  

(3)

**Remark**: for a height tapered specimen, the derivation is largely simplified:

\[ \frac{\text{d}v}{\text{d}t} = P \cdot \frac{\text{d}C}{\text{d}a} \cdot \frac{\text{d}a}{\text{d}t} \]

in this case $P$, $\frac{\text{d}v}{\text{d}t}$, $\frac{\text{d}C}{\text{d}a}$ are constant which gives:

\[ a = \frac{1}{PQ} \cdot \frac{\text{d}v}{\text{d}t} . t + a_0 \]

(4)

where $Q = \frac{\text{d}C}{\text{d}a}$

**2: load is linearly varying during crack growth**

\[ \frac{\text{d}P}{\text{d}t} = C_1 \hspace{1cm} (C_1 \text{ is a constant}) \]

\[ P = C_1 t + C_2 \]

\[ = C_1 t + P_0 \]

(5)

where $P_0$: load level at $t=0$

for a variable load one gets for equation 1:

\[ v = C.P \]

\[ \frac{\text{d}v}{\text{d}t} = P \frac{\text{d}C}{\text{d}t} + C \frac{\text{d}P}{\text{d}t} \]

\[ = 2PB_t \cdot \frac{1}{2B_t} \cdot \frac{\text{d}C}{\text{d}a} \cdot \frac{\text{d}a}{\text{d}t} + C \frac{\text{d}P}{\text{d}t} \]

(6)

in this case: $\frac{\text{d}v}{\text{d}t} = \text{constant} = C_3$

$\frac{\text{d}P}{\text{d}t} = \text{constant} = C_1$
\[
\frac{1}{2B_t} \cdot \frac{dc}{da} = \text{constant} = Q
\]

\[B_t = \frac{1}{3} a\]

\[C_3 = \frac{2}{3} aPQ \frac{da}{dt} + C_i C\]  \hspace{1cm} (7)

\[C_3 = \frac{2}{3} (C_i t + P_o) aQ \frac{da}{dt} + C_i C\]

Before this differential equation can be solved, one has to find an expression for the compliance \(C\).

For a W.T.D.C.B.-specimen, the compliance can be approximated by:

\[\frac{dc}{da} = \frac{8}{EB_t} \cdot \left(\frac{3a^2}{h^3} + \frac{1}{h}\right)\]  \hspace{1cm} (8)

with \(B_t = \frac{1}{3} a\) one obtains:

\[\frac{dc}{da} = \frac{24}{Ea} \left(\frac{3a^2}{h^3} + \frac{1}{h}\right)\]

integration results in:

\[C = \frac{12}{Eh} \left(\frac{3a^2}{h^2} + 2 \ln(a)\right) + C_0\]  \hspace{1cm} (9)

combining eq. 7 and 9:

\[C_3 = \frac{2}{3} \left(C_i t + P_o\right) aQ \frac{da}{dt} + C_i \left(\frac{12}{Eh} \left(\frac{3a^2}{h^2} + 2 \ln(a)\right) + C_0\right)\]  \hspace{1cm} (10)

equation 10 can be rewritten as:

\[C_9 = (C_4 + C_5) a \frac{da}{dt} + C_6 a^2 + C_7 \ln(a)\]  \hspace{1cm} (11)

where:

\[\frac{2}{3} C_1 Q = C_4\]

\[\frac{24}{Eh} = C_1 = C_7\]

\[\frac{2}{3} P_o Q = C_5\]

\[C_i C_0 = C_8\]

\[\frac{36C_1}{Eh} = C_6\]

\[C_3 - C_8 = C_9\]
\[
\frac{da}{dt} = \frac{C_9 - C_6a^2 - C_7 \ln(a)}{a(C_4 + C_5)}
\]

or:
\[
\frac{a}{C_9 - C_6a^2 - C_7 \ln(a)} \, da = \frac{1}{C_4 t + C_5} \, dt
\]

the left and right part of equation 12 can be solved independently; the right part results in:
\[
F'(t) = \frac{1}{C_4 t + C_5} \, dt
\]
\[
F(t) = \frac{1}{C_4} \ln(C_4 t + C_5)
\]

(13)

Because the left part of eq. 12 cannot be directly integrated, 2 possible approximations will be discussed. In fig. 1, the two approximations are compared with the original function for realistic values of \(a\), \(C_9\), \(C_6\) and \(C_7\).

1: \(F'(a) = \frac{a}{C_9}\)
\[
F(a) = \frac{a^2}{2C_9}
\]

combined with eq. 13 one gets:
\[
\frac{a^2}{2C_9} = \frac{1}{C_4} \ln(C_4 t + C_5) + C_{10}
\]

at \(t=0\): \(a=a_0\)
\[
C_{10} = \frac{a_0^2}{2C_9} - \frac{1}{C_4} \ln C_5
\]

general solution:
\[
a^2 = \frac{2C_9}{C_4} \ln(C_4 t + C_5) + a_0^2 - 2 \frac{C_9 \ln C_5}{C_4}
\]
\[
a = \left( \frac{2C_9}{C_4} \ln(C_4 t + C_5) + a_0^2 - 2 \frac{C_9 \ln C_5}{C_4} \right)^{1/2}
\]

(14)

2: \(F'(a) = \frac{a}{C_9 - C_6a^2}\)
\[ F(a) = -\frac{1}{2C_6} \ln (C_9 - C_6a^2) \]

**General Solution:**

\[-\frac{1}{2C_6} \ln(C_9 - C_6a^2) = \frac{1}{C_4} \ln(C_4t + C_5) + C_{11} \]

At \( t=0, \ a=0 \)

\[ C_{11} = -\frac{1}{2C_6} (\ln(C_9 - C_6a_0^2) + \frac{2C_6}{C_4} \ln C_5) \]  \hspace{1cm} (15)

\[ \ln(C_9 - C_6a^2) = -2C_6 \left( \frac{1}{C_4} \ln C_4t + C_5 \right) - 2C_6a_{11} \]

\[ a = \left( \frac{C_9}{C_6} - \frac{1}{C_6} e^{-2C_6 \left( \frac{1}{C_4} \ln(C_4t + C_5) + C_{11} \right)} \right)^{1/2} \]  \hspace{1cm} (16)

**Comparison with Experimental Data**

In the fig. 3 and 4, the theoretical crack growth rates are compared with experimental data. The tests are performed on a W.T.D.C.B.-specimen as shown in fig 2. The specimen consists of:

- 2 thick adherend beams: 7075 T6
- 1 adhesive layer: AF 163-2 (3-M company)

The specimen dimensions are chosen such that plastic deformation of the adherend beam does not occur. During the test, the initial constants (\( F_0, a_0, C_0 \)) have been measured.

The first specimen (fig 3) was tested in laboratory air and showed a constant load during crack growth.

**Experimental Details:**

- \( b_t = \frac{1}{3} a \)
- \( \frac{dv}{dt} = 5 \text{ (mm/h)} \)
- \( a_o = 182 \text{ (mm)} \)
- \( Q = 1.57 \times 10^{-3} \text{ (1/Nm)} \)
- \( P = 1375 \text{ (N)} \)

As can be seen in fig. 3, the measured and calculated (eq. 3) crack growth rate fits quite well. The theoretical result shows a small overestimating of the crack growth rate.
The experimental data shown in fig. 4 are obtained from a specimen tested in distilled water environment at 50 °C. During crack growth the load continuously increased, most probably because of increased plasticity (because of the shape of the specimen, the area available for plastic deformation increases with increasing crack length).

**Experimental details:**

\[
\frac{dp}{dt} = 1.316 \quad (\text{N/min})
\]

\[
\frac{dv}{dt} = 5 \quad (\text{mm/h})
\]

\[Q = 1.57 \times 10^{-3} \quad (1/\text{Nm})\]

\[C_o = 1.165 \times 10^{-2} \quad (\text{mm/N})\]

\[a_o = 132 \quad (\text{mm})\]

\[P_o = 1116 \quad (\text{N})\]

The experimental result is compared with 3 theoretical models:

1: an average load is calculated and the constant load equation (eq. 3) is used to calculate the crack growth rate

2: the first approximation of the crack length is used \((F(a) = a/C_o)\) with variable load

3: the second approximation of the crack length is used \((F(a) = a/(C_o - C_o a^2))\) with variable load

In fig. 4 it is clear that the 3rd model is by far the most accurate, except in the starting phase where the crack growth rate is overestimated. The accuracy of the three models is clearly shown in table 1, where the crack growth rate of the three models is compared with the experimental result (in the measuring range between 45 and 180 minutes, the curves are for this purpose linearised).

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<tr>
<th></th>
<th>Crack growth rate (mm/min)</th>
<th>Correlation coefficient</th>
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<tbody>
<tr>
<td>Measurement</td>
<td>0.219</td>
<td>0.995</td>
</tr>
<tr>
<td>Model 1</td>
<td>0.371</td>
<td>0.998</td>
</tr>
<tr>
<td>Model 2</td>
<td>0.326</td>
<td>0.997</td>
</tr>
<tr>
<td>Model 3</td>
<td>0.241</td>
<td>0.996</td>
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*Table 1*
CONCLUSION

The crack growth rate of a crack in a W.T.D.C.B.- specimen can be well predicted in both a constant and linearly varying load condition. In the latter case, a constant load approach does not result in accurate data.

NOMENCLATURE

\[ a: \text{crack length} \]
\[ a_0: \text{initial crack length} \]
\[ B_t: \text{width of the specimen (variable; function of crack length)} \]
\[ C: \text{compliance of specimen} \]
\[ C_0: \text{initial compliance} \]
\[ C_{1-11}: \text{constants} \]
\[ E: \text{Young's modulus} \]
\[ h: \text{height of the specimen (constant)} \]
\[ P: \text{load} \]
\[ P_0: \text{initial load level (for model 2 and 3)} \]
\[ Q: \text{specimen constant} \]
\[ t: \text{time} \]
\[ v: \text{displacement of loading point} \]
Fig 1: Two possible approximations for \( F(a) = \frac{a}{c_9 - c_6 a^2 - c_7 \ln a} \)

\[ c_9 = 7 \times 10^{-2} \]
\[ c_6 = 7 \times 10^{-7} \]
\[ c_7 = 4 \times 10^{-5} \]

Fig 2: Specimen configuration
Fig 3: Comparison of experimental and theoretical data, obtained with a constant load model

\[ a = \left( \frac{3}{PQ} \cdot \frac{dv}{dt} \cdot t + a_o^2 \right)^{1/2} \]

Fig 4: Comparison of experimental and theoretical data, obtained with a constant load model (model 1) and two different varying load models (model 2 and 3)