Half-integer Shapiro steps in single-plaquette Josephson-junction arrays in a magnetic field

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We derive an equation for a single-plaquette array of Josephson junctions for an arbitrary number of flux quanta per unit cell f . We show that for $f = \frac{1}{2}$ this equation is equivalent to one derived for a 2×2 Josephson-junction array in the $f = \frac{1}{2}$ ground state. We show that in the presence of an rf drive with frequency ν , the system exhibits, for all nonzero values of f , integer and half-integer Shapiro steps at $\langle V \rangle = nh\nu/4e$, where $n = 1, 2, 3, \dots$. In addition, very small subharmonic Shapiro steps at $\langle V \rangle = nh\nu/2em$, where $m = 1, 2, 3, \dots$ and not equal to n , are observed for all f not equal to zero. These particular steps, however, are found to be consistently smaller than the integer and half-integer steps concurrently present in the array I - V characteristics. Contrary to recent suggestions, we show that single-plaquette array behavior is *not* entirely consistent with that of large arrays of Josephson junctions and hence *not* responsible for the display of fractional giant Shapiro steps in these large arrays. Such behavior may, however, be responsible for the recent observation of half-integer steps in high- T_c grain-boundary junctions.

Two-dimensional (2D) arrays of Josephson junctions exhibit a remarkably rich response to alternating fields. This response includes the displays of integer giant Shapiro steps at voltages

$$V_n = n \frac{Nh\nu}{2e}, \quad n = 1, 2, 3, \dots, \quad (1a)$$

where N is the number of junctions in the direction of the external current flow and ν is the applied rf frequency; and fractional giant Shapiro steps at voltages

$$V_n = n \frac{Nh\nu}{q2e}, \quad n = 1, 2, 3, \dots \text{ and } q = 2, 3, 4, \dots \quad (1b)$$

Here q corresponds to a transverse magnetic field $f = p/q$, with f defined as the number of flux quanta per unit cell and p an integer. Both integer giant and fractional giant Shapiro steps have been thoroughly investigated¹⁻⁴ and are attributed to the driven motion of a $q \times q$ superlattice of field-induced vortices commensurate with the underlying array lattice.²⁻⁴ An additional response of 2D arrays in the display of subharmonic steps at voltages⁵

$$V_n = \frac{n}{m} \frac{Nh\nu}{q2e}, \quad m = 1, 2, 3, \dots \neq n. \quad (1c)$$

Note that in Eq. (1c) it is not possible to resolve between, for example, a half-integer subharmonic step corresponding to $n = 1$, $m = 2$ and the fractional giant Shapiro step [Eq. (1b)] corresponding to $n = 1$ and $q = 2$ when $f = \frac{1}{2}$.

Initially, only dynamic simulations of rf-biased arrays showed subharmonic steps corresponding to those expressed by Eq. (1c).³ These steps, however, were eventually shown to be the result of employing free boundary conditions which consequently distorted the ground state of the $q \times q$ superlattice.^{4,6} Further investigation of rf-biased arrays subsequently led to the experimental observation of half-integer ($Nh\nu/4e$) subharmonic steps,⁷ in

zero field and in fields f not equal to $\frac{1}{2}$.⁵ This experimental observation initially suggested that the difference between subharmonic and fractional giant Shapiro steps may be quite subtle. However, subsequent experiments have shown that self-field effects, present for both zero and nonzero fields, are the primary cause of subharmonic steps in real arrays.^{5,8,9} Additional studies have determined that disorder,^{10,11} inductive effects,^{5,12} and non-sinusoidal current-phase relationships¹³ can all lead to subharmonic steps in both real¹⁴ and simulated arrays.

Recently, there have been studies performed on single-plaquette arrays¹⁵⁻¹⁷ which show that these arrays exhibit half-integer and subharmonic steps. As a consequence of these studies, there have been both direct and indirect suggestions^{16,17} that the fractional giant Shapiro steps observed in large arrays can be attributed to the behavior of a single-plaquette array. This appears to be a valid suggestion since a single-plaquette array is the unit cell of a large array *and* shows equally rich behavior, including integer and subharmonic steps. In this paper, we use our study of single-plaquette arrays to suggest otherwise, that (a) the behavior of a single-plaquette array does *not* contribute to the observed response in large experimental or simulated arrays and (b) the origin of subharmonic steps in a single-plaquette array is quite different from that of a large array.

In Fig. 1, we show our definition of the gauge-invariant phase differences for a single-plaquette array in which we have assumed all the junctions to have identical critical current. From fluxoid quantization and net current flow across the cell, we obtain

$$\alpha + \gamma + 2\beta = 2\pi f \pmod{2\pi}, \quad (2a)$$

$$\frac{d\gamma}{d\tau} - \frac{d\alpha}{d\tau} + \sin\gamma - \sin\alpha = I_{\text{tot}}, \quad (2b)$$

where $I_{\text{tot}} = I_{\text{dc}} + I_{\text{ac}} \sin(\Omega\tau)$ is the total applied dc and rf

FIG. 1. Gauge-invariant phase differences in a single-plaquette Josephson-junction array. Each bond represents a junction connecting two superconducting islands.

current normalized to the junction critical current i_c , $\Omega = \hbar\nu/2e i_c r_n$ is the normalized drive frequency with r_n being the junction normal-state resistance, and $\tau = (2e i_c r_n / \hbar)t$ is the normalized time. We assume that $\beta = \beta'$ because of the symmetry of the system and the fact that only supercurrent flows through these junctions, resulting in a zero voltage drop across them. By applying Kirchoff's current conservation law to the two nodes on the right-hand side of the plaquette, we can write the following two equations:

$$\frac{d\alpha}{d\tau} - \frac{d\beta}{d\tau} + \sin\alpha - \sin\beta = \frac{I_{\text{tot}}}{2}, \quad (2c)$$

$$\frac{d(2x)}{d\tau} \left[1 + \frac{4[1 - \cos(2x)]}{[3 + \cos(2x)]^2} \right] + \frac{2[2 \cos(\pi f) \sin x + \sin(2x)]}{[6 + 2 \cos(2x) + 8 \cos(\pi f) \cos x]^{1/2}} = I_{\text{tot}}. \quad (5)$$

This equation is like that of a single junction with phase difference $2x$, instead of x . It also resembles the one derived for a 2×2 cell when $f = \frac{1}{2}$.¹⁸ That particular equation was found to produce integer Shapiro steps at voltages corresponding to Eq. (1a), where $N = 1$, across the whole 2×2 cell or half-integer ones at voltages corresponding to Eq. (1b), with $q = 2$, across a *single cell* in the array. In contrast to the 2×2 case where steps are observed at voltages $V_n = \hbar\nu/2qe$ for $f = p/q$, Shapiro steps in the single plaquette occur *mainly* at the half-harmonic and its multiples $V = (n/2)\hbar\nu/2e$ where $n = 1, 2, 3, \dots$ for all values of f not equal to zero. This can be seen from Eq. (5) which contains only terms in $\sin(2x)$ and $\sin(x)$ for all values of the flux quanta per plaquette. Since the supercurrent terms are not completely harmonic in Eq. (5), there will also be subharmonic steps for every value of $f = (p/q)$ not equal to zero. As

$$\frac{d\beta}{d\tau} - \frac{d\gamma}{d\tau} + \sin\beta - \sin\gamma = \frac{I_{\text{tot}}}{2}, \quad (2d)$$

Subtracting (2d) from (2c), we obtain

$$\frac{d\gamma}{d\tau} - \frac{d\alpha}{d\tau} - 2 \frac{d\beta}{d\tau} + \sin\gamma - \sin\alpha - 2 \sin\beta = 0. \quad (3)$$

Defining $x = (\gamma - \alpha)/2$ and $y = (\gamma + \alpha)/2$, Eqs. (2) and (3) become

$$2 \frac{dx}{d\tau} + 2 \cos y \sin x = I_{\text{tot}}, \quad (4a)$$

$$2 \frac{dy}{d\tau} - \sin(\pi f - y) + \sin y \cos x = 0. \quad (4b)$$

For $f = \frac{1}{2}$, we find that Eqs. (4a) and (4b) are identical to those obtained by Rzchowski, Sohn, and Tinkham¹⁸ for a 2×2 array in the $f = \frac{1}{2}$ ground state. The equality of these two sets of equations, for this particular value of f , is coincidental and arises because the current in the single-plaquette array, by symmetry, divides equally between two current branches. The only difference between the two differently sized arrays is that the voltage is $d(x + y)/d\tau$ or $d(x - y)/d\tau$ across the single-plaquette array and $d(2x)/d\tau$ across the 2×2 cell. This difference is unimportant when discussing the I - V characteristics of two arrays since $\langle d\alpha/d\tau \rangle = \langle d\gamma/d\tau \rangle$. It follows then that the I - V characteristics, normalizing voltages and currents to the number of junctions, of the 2×2 cell are identical to those of the single-plaquette array for $f = \frac{1}{2}$. Furthermore, all aspects, including the power and frequency dependence of step widths, found for 2×2 array at $f = \frac{1}{2}$,¹⁸ will also be true for the single plaquette when $f = \frac{1}{2}$.¹⁹

We can extend some of the conclusion of Rzchowski, Sohn, and Tinkham¹⁸ to the single-cell case at arbitrary f by considering Eq. (4) in the low-drive frequency and amplitude limit for which $dy/d\tau$ is small.²⁰ Solving for $\cos(y)$ in Eq. (4b) to first order in $dy/d\tau$ and substituting the result into Eq. (4a), we obtain

discussed later, these steps are always quite small compared to the half-integer subharmonic and integer steps.²¹

Figure 2 shows a typical I - V curve we obtain by numerically integrating Eq. (4) with $f = \frac{1}{3}$. Clearly, integer giant and half-integer giant Shapiro steps, with extremely small subharmonic ones, are present. For all $f = p/q$ tried, we found that the half-integer steps are the dominant subharmonic steps [of the $n/(mq)$ type].²² More importantly, the half-integer subharmonic steps are significantly larger than the q th fractional steps corresponding to the $f = p/q$ tried. (This is unlike what is found in large arrays where the q th fractional steps dominate any self-field-induced steps.) The origin of these subharmonic steps can be seen quite clearly by looking at the current-phase relationship in Eq. (5) as a function of f . Figure 3 shows that relationship for $f = 0, \frac{1}{2}, \frac{1}{3}$, and $\frac{1}{5}$. As f decreases from $\frac{1}{2}$ to 0, the current-phase relation-

FIG. 2. Current-voltage characteristic of a single plaquette in the presence of a magnetic field corresponding to $f = \frac{1}{3}$. The normalized ac frequency used is $\Omega = 1.0$ and the normalized drive current is $I_{ac} = 1.0$. Inset: Magnification of region near origin. Integer and half-integer giant Shapiro steps are clearly present. Subharmonic steps are also present but are extremely small.

ship smoothly transforms from a sinusoid with period π , to a nonsinusoid for intermediate values, to again a sinusoid with period 2π . Correspondingly, we observe in our simulations a smooth transition from half-integer steps with extremely small subharmonic ones, to a mixture of both with very small subharmonic steps, to solely integer steps for the smallest values of f . We thus conclude that the subharmonic steps in single plaquettes are due to the *nonsinusoidal current-phase relationship* in Eq. (5) with half-integer steps and its harmonics being the main response of the single-plaquette array.²³

In contrast to the fractional giant Shapiro steps exhibited by large arrays,²⁻⁴ we have found that only half-integer rather than q th fractional steps are predominant for such f values as $f = \frac{1}{3}$ and $\frac{1}{5}$ in a single-plaquette array. In addition, these half-integer steps do *not* have the same frequency or power dependence that $f = \frac{1}{2}$ fractional Shapiro steps do.¹⁸ These differences, i.e., the dominance of half-integer steps above all other types of steps for all nonzero f and the differing frequency and power dependence, clearly distinguishes between the two cases and is consistent with experiments of large arrays where self-field effects have been canceled using a normal-metal ground plane.^{5,7} Only q th fractional and not subharmonic steps are observed for $f = p/q$ in these experiments of large arrays. This is contrary to Kim and Lee's results which are claimed to be consistent with the ones, experimental or numerical, obtained with large arrays.¹⁶ It also rules out any subsequent suggestion that single-plaquette behavior may be the origin of fractional giant Shapiro steps in large arrays. If this suggestion were correct, then the half-harmonic response would be dominant in experiments of large arrays for all values of f not equal to zero. We emphasize that q th fractional and not subharmonic steps are observed for $f = p/q$ in large arrays where self-fields have been excluded; in contrast, subharmonic steps are readily observed in single-plaquette arrays where

FIG. 3. Current-phase relationship of Eq. (5) for different values of f . The nonsinusoidal relationship is responsible for the display of half-integer steps and their harmonics by single-plaquette arrays.

self-field effects have also been excluded. We therefore conclude that origin of subharmonic steps in single-plaquette arrays is different than that of large arrays.

The presence of half-integer steps in the single-plaquette array in no way invalidates, as Kim and Lee¹⁶ had suggested, the moving-vortex superlattice model¹⁶ often used to explain the occurrence of fractional giant Shapiro steps in large arrays. First, this model is only a *qualitative* description of the dynamics in arrays. Second, the behavior of a single plaquette (i.e., the predominance of half-integer steps for all f not equal to zero) may be thought of in an analogous qualitative way: A single vortex is present in the plaquette, and it alternates between rotating in the clockwise or counterclockwise direction during an entire Josephson cycle. The half-integer steps are the result of synchronizing the period of the rf drive to a full cycle in which the vortex oscillates between these two states. Our simulations confirm this qualitative picture. We emphasize that this picture of an alternating vortex in a single-plaquette array is true for all f , in contrast with large arrays where it is only true for $f = \frac{1}{2}$ and not arbitrary $f = p/q$. In other words, while it is true that the vortex motion in an single-plaquette array will lead to half-integer steps, it does not lead to the other fractional steps experimentally observed.

The concept of a vortex oscillating, in synchrony with an applied rf, between two states is very similar to that presented by Vanneste *et al.*²⁴ to explain the origin of half-integer states in a dc superconducting quantum interference device (SQUID). They attribute these steps to a periodic switching between adjacent fluxoid states of the SQUID synchronized to the applied rf. The main difference between the case of the SQUID and the single-plaquette array is that in the case of the former, half-integer steps only appear if significant inductance is present and if the field is near a half-integral of flux quanta in the SQUID. In contrast with this, our model explains the origin of half-integer steps without the need to

take inductance into account and predicts that the steps should be present for other values of f . Recently, Early, Clark, and Char²⁵ have reported the observation of half-integer steps in rf-biased high- T_c grain-boundary junctions for various f . Their observation of only half-integer steps has led them to suggest that multiple junctions in parallel in the grain boundaries might be the origin of these steps. Given the size of their junctions, the geometrical inductance might be large enough to induce the half-integer steps. However, it is important to note that the half-integer steps Early, Clark, and Char observed in the high- T_c grain-boundary junctions appear for different values of f , not just $f = \frac{1}{2}$.²⁵ This is in contrast to the results of Vanneste *et al.*'s simulations²⁴ where half-integer steps only occur near $f = \frac{1}{2}$. Thus, it might be possible that our analysis could explain the results of Early, Clark, and Char, as it does not need to invoke inductance. The two models, i.e., ours and Vanneste *et al.*'s,²⁴ can only be differentiated by a detailed comparison of the power and frequency dependence of the half-integer steps to the experimental data.

In conclusion, we have shown that the main response

to an rf drive of a single-plaquette array is a half-integer subharmonic response for all nonzero values of the external magnetic flux per plaquette f . Subharmonic response of other orders are also present due to the nonsinusoidal current-phase relationship; however, these particular steps are always quite small compared to the integer and half-integer steps for all values of f not equal to zero. In fact, their small stepwidths make it extremely difficult to observe them in real experiments. This behavior is in contrast to the fractional steps exhibited by large arrays. Such steps are easily observable and display a definite power and frequency dependence. We thus clearly distinguish the difference between subharmonic and fractional steps in single-plaquette arrays and rule out single-plaquette behavior as an explanation for fractional giant Shapiro steps in large arrays.

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⁷To date, we are aware of only half-integer subharmonic steps having been experimentally observed. This is in contrast to the Devil's staircase series of subharmonic steps found in simulations.

⁸Current-induced vortices, due to self-fields, lead to subharmonic steps in large arrays. See S. P. Benz, Ph.D. thesis, Harvard University, 1990.

⁹L. L. Sohn and M. Rzchowski (unpublished). A normal-metal ground plane was placed on top of an 1000×300 array and current was injected through it such that the current-induced fields in the array were canceled. All previous displays of subharmonic steps in the array disappeared, thus reinforcing the concept of self-field effects inducing subharmonic steps. See also Refs. 5 and 9.

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¹⁷The analysis of Hebboul and Garland (see Ref. 11) suggests that the appearance of subharmonic steps in large Josephson-junction arrays are governed by the dynamic behavior of a single-unit cell in the array.

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¹⁹This raises an interesting point that the dynamics of a large array of underdamped junctions would be the same in a single plaquette for $f = \frac{1}{2}$. See for instance M. Octavio, C. B. Whan, U. Geigenmüller, and C. J. Lobb, *Phys. Rev. B* **47**, 1141 (1993).

²⁰Although $dy/d\tau$ being small cannot be justified in general, we find that this approximation is consistent with numerical simulations.

²¹While we have not explored the full parameter space to show that there are no significant subharmonic steps of higher order, the approximation we have performed is again consistent with numerical simulations.

²²The exception being of course that at $f = \frac{1}{2}$, the half-integer steps displayed by the single-plaquette array are to be considered fractional steps. To avoid confusion, all mention of half-integer subharmonic steps thereafter in this paper should be considered as such for all nonzero f and f not equal to $\frac{1}{2}$.

²³This is in contrast to the conclusion drawn by Hebboul and Garland (see Ref. 11) which stated that a *nonuniform critical current distribution* and *cell inductances* are responsible for subharmonic steps, particularly at $f = 0$, appearing in single-plaquette arrays.

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