A SIMPLE METHOD FOR THE CALCULATION OF STANDARD-GRAPHS TO BE USED IN GEO-ELECTRICAL PROSPECTING

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN
DOCTOR IN DE TECHNISCHE WETENSCHAP
AAN DE TECHNISCHE HOGESCHOOL TE
DELFT OP GEZAG VAN DE RECTOR MAG-
NIFICUS PROF. IR. H. J. DE WIJS, HOOG-
LERAAR IN DE AFDELING DER MIJNBOUW-
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UITGEVERIJ WALTMAN — DELFT

STELLINGEN

1.

De door FLATHE opgegeven tijdsduren voor het berekenen van standaard-krommen volgens de door hem aangegeven methode zijn geflatteerd.

FLATHE, H. 1955. A practical method of calculating geoelectrical model graphs for horizontally stratified media. *Geophysical Prospecting*, Vol III, No 3, pp. 268-294.

2.

Gezien de overeenkomst die er bestaat tussen de wet van OHM en die van DARCY is de in dit proefschrift ontwikkelde methode voor het bepalen van elektrische bodemweerstand ϱ overdraagbaar op een denkbeeldige bepaling van de doorlaatfactoren $k(\approx 1/\varrho)$ van horizontale, homogene, isotrope lagen in een met grondwater verzadigd bodemprofiel, waarin geen verticale toevoer van water door neerslag plaatsvindt noch onttrekking door verdamping, en dat wordt onderworpen aan een stationaire pompproef, waarbij de put van zeer kleine afmetingen is en de afpompings verwaarloosbaar klein is ten opzichte van de met water verzadigde dikte van de bovenste laag.

3.

De hydrologie staat nog slechts aan het begin van haar ontwikkeling.

4.

De doorsnee civiel-ingenieur is nog te weinig bekend met de toepassingsmogelijkheden van het geofysisch onderzoek in zijn vakgebied.

5.

Een herhaling van de thans in uitvoering zijnde geo-elektrische verkenning van de ondergrond van het lage deel van Nederland zal pas zinvol zijn over enkele tientallen jaren.

6.

Het nemen van ongestoorde grondmonsters behoort tot de onmogelijkheden.

7.

Een eventuele geofysische methode ter bepaling van de drukhoogteverdeling van het grondwater door metingen aan de aardoppervlakte zou van groot nut zijn in geo-hydrologische onderzoeken.

8.

De aanwezigheid van een dunne, in hydraulisch opzicht, slecht doorlatende laag in een overigens homogene bodem wordt op indirecte wijze door elektrische weerstandsmetingen aangetoond indien de interpretatie van die metingen duidt op de aanwezigheid van zout grondwater boven zoet grondwater. De geïnterpreteerde diepte van het grensvlak tussen het zoute en het zoete grondwater komt dan overeen met die van de slecht doorlatende laag.

9.

In de door MAZURE ontwikkelde methode voor het berekenen van de afvoer van een kanaalvormige boezem door een spuisluis aan getijwater – waarbij de invloeden van de wateronttrekking, de traagheid en de stromingsweerstand afzonderlijk in rekening worden gebracht – wordt het door de stromingsweerstand bepaalde gedeelte van de daling van de waterstand aan de binnenzijde van de sluis berekend ten opzichte van de waterstand aan het einde van de boezem. Het zou juister zijn deze daling te berekenen ten opzichte van de gemiddelde boezemstand en wel omdat bij het in rekening brengen van de daling door wateronttrekking ook de gemiddelde boezemstand wordt berekend.

MAZURE, J. P. 1933. Berekening van een sluisgang voor een kanaalboezem. De Ingenieur, 10 nov. 1933. Afd. B.

10.

Voor de berekening van de afvoer per sluisgang door een sluis aan getijwater, waarbij de getijkromme de veel voorkomende enkelvoudige golving vertoont en waarbij gedurende de gehele sluisgang uitsluitend gestuwde afvoer optreedt kan de navolgende formule worden gebezigd:

$$Q = CA\sqrt{TO}$$

waarin:

Q = totale afvoer per sluisgang in m^3

$$C = 0,96\mu b \sqrt{\frac{2g\alpha}{1.15}} \text{ in } m^{3/2}/\text{sec}$$

μ = contractiecoëfficiënt

b = sluisbreedte in m

g = versnelling van de zwaartekracht in m/sec^2

α = verhouding:

$$\frac{\text{energiehoogte bovenstrooms} - \text{waterstand in de sluis}}{\text{energiehoogte bovenstrooms} - \text{waterstand benedenstrooms}}$$

($\alpha > 1$)

A = hoogte in m boven de sluisdrempel van het snijpunt van de getijkromme met de verticaal die de spuitijd halveert

T = tijdsduur van de sluisgang in sec

O = oppervlak, ingesloten tussen de getijkromme en het verloop van de binnenwaterstand in m^2

11.

De door SOROKIN, zonder bronvermelding, beschreven hulppuntsmethode voor de interpretatie van weerstandskrommen bij drielagen profielen is dezelfde als die welke in 1943 door EBERT werd gepubliceerd in „Beiträge zur Angewandte Geophysik”.

SOROKIN, L. W. 1950. Moskou. Duitse vertaling, 1953, Berlin: „Lehrbuch der geophysikalischen Methoden zur Erkundung von Erdölvorkommen”.

DIT PROEFSCHRIFT IS GOEDGEKEURD DOOR DE PROMOTOR
PROF. O. KOEFOED

Aan mijn vrouw

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PREFACE

The earth puts at the disposal of mankind a variety of natural resources. The exploitation of these resources depends both on human needs and on technical and economic possibilities. The growth of the world-population and the rising standard of living lead to increasing demands on these resources. On the other hand, the progress in technical achievement facilitates an increasing intensity of exploitation.

Any exploitation should be preceded by exploration; first the occurrence and the extent of the resources must be determined. The discovery of these hidden treasures of the earth is one of the most fascinating tasks.

The author of the present thesis, being an officer of the Service for Water Resources Development of the Netherlands, deems it a privilege to be in charge of a survey of the ground-water resources in the lower part of this country. With this aim he applies the geo-electrical method of geophysical prospecting. The evaluation of the geo-electrical measurements is an interesting problem in itself. This problem involves the interpretation of surface measurements in terms of electrical resistivity of the soil and interface-depths. The theoretical aspects of it have long been known, but the practical application has, until now, been rather time-consuming.

In his practice the author felt the need of simpler interpretation techniques. This feeling gave rise to a hobby. By trial and error and by using several artifices a new and simple calculation-method was born. The application of this method saves time, or a greater number of calculations can be performed within the time available. In the latter case the application of this method contributes to a greater certainty in the interpretation of exploration data.

It is my sincere hope that this may lead to an even more efficient exploitation of the resources and finally contribute to human welfare.

INTRODUCTION

The author, being in charge of the geo-electrical survey of the lower part of the Netherlands, had to interpret resistivity soundings in geo-hydrologic terms [4, 5, 6, 7 and 9].¹⁾ In a deltaic area, as in this country, the subsoil consists to great depths of almost horizontal layers of different lithological properties – gravel, sand, clay, peat – which are saturated with ground water of different salinity. So it is obvious that the first step in the interpretation procedure should be a mathematical interpretation of the apparent resistivities measured at the surface into profiles consisting of a limited number n of horizontal, homogeneous and isotropic layers of different thicknesses t_m and different electric resistivities ρ_m (fig. 1.1). The next step is to correlate the interface depths

Fig. 1.1 Resistivity profile consisting of n horizontally stratified isotropic media

and the true formation resistivities with geo-hydrologic properties, which can be done after comparing a number of resistivity profiles with borelogs, including ground water sampling. The final interpretation is brought about only after a combination of both steps.

¹⁾ A figure in [] refers to the corresponding number of the References.

This thesis concerns the first stage of the interpretation procedure: the determination of interface depths and true formation resistivities from a measured apparent resistivity curve. An apparent resistivity curve represents the apparent resistivities measured between two potential electrodes, a distance a apart, as a function of the distance L between two current electrodes at the earth surface, or of a fraction of L . With regard to the setting of the potential electrodes two configurations are common, namely that according to WENNER (fig. 1.2), where $a/L = 1/3$, and that according to SCHLUMBERGER (fig. 1.3), where $a/L \rightarrow 0$. In addition to these, several other electrode configurations have also been put into practice.

Fig. 1.2 Electrode configuration according to WENNER

Fig. 1.3 Electrode configuration according to SCHLUMBERGER

Since the nineteen thirties, many authors have contributed to the solution of the problem of interpretation, either empirically or theoretically. A review of these methods has been given by MOONEY [20].

Some authors endeavoured to arrive at a direct method of interpretation. A first attempt was made by SLICHTER [27] in 1933 for a continuous vertical resistivity variation. In 1940 PEKERIS [24] developed a method applicable to a number of horizontal layers with the only restriction that each layer is thicker than the one above it. The most recent of these efforts is that by VOZOFF [32] in 1958. After assuming the number of layers and the resistivity contrasts, the assumed profile can be adjusted step by step by fitting the relevant SLICHTER kernel to that integrated from the field data. This procedure requires the use of a computer. Unavoidably there are limits to the solubility of this method.

Usually the mathematical interpretation is performed in an indirect way. The indirectness must be understood as follows. The measured resistivity curve is compared with a collection of graphs which has been calculated for assumed resistivity profiles. The calculated graph of best fit or an interpolation between some of these graphs indicates the interpretation to be assigned to the measured curve.

At present the following collections are available:

1. 5 two-layer graphs and 7 three-layer graphs for WENNER arrangement published by HUMMEL in 1929 [13];
2. 20 two-layer graphs for WENNER arrangement first calculated and published by TAGG [30, 31] in 1930 and 1934 and again by ROMAN [25] in 1934, who introduced the bilogarithmic scale for plotting, which has been usual since that time;
3. 41 three-layer graphs and 8 four-layer graphs for WENNER arrangement published by WATSON [34] in 1934;
4. 480 three-layer graphs for SCHLUMBERGER arrangement published by La Compagnie Générale de Géophysique [2] in 1955. These graphs have been calculated for 12 different values of the ratio ρ_2/ρ_1 of the resistivities of the second layer to the first one, at 10 different values of the ratio d_2/d_1 of the depth of the second interface to that of the first one and 4 values of the ratio ρ_3/ρ_1 viz: 0, 1, ∞ and $(\rho_2/\rho_1)^2$;
5. 2700 three- and four-layer graphs for WENNER arrangement published by MOONEY and WETZEL [21] in 1956. This collection includes also 90 three-layer graphs, which were published by WETZEL and McMURRY in 1937 [33] and the set of two-layer graphs published by TAGG and by ROMAN;
6. 31 three-layer graphs for WENNER arrangement calculated by KOEFOED [16] in 1955;
7. 72 five-layer graphs for SCHLUMBERGER arrangement published by FLATHE [12] in 1963.

Furthermore SOROKIN [28] mentions the existence of a collection of 720 graphs for SCHLUMBERGER arrangement. As appears from that statement the set includes all graphs published by La Compagnie Générale de Géophysique and moreover those where $\rho_3/\rho_1 = (\rho_2/\rho_1)^{1/2}$ and $\rho_3/\rho_1 = (\rho_2/\rho_1)^{3/2}$.

It is a well known fact that some geophysical prospecting companies and public services have calculated collections of graphs for their own use.

With a resistivity profile consisting of n layers, where the thicknesses and the resistivities of the various layers are expressed in the thickness t_1 and the resistivity ρ_1 of the first layer, there are $2n-3$ degrees of freedom and consequently ∞^{2n-3} possibilities. This implies that in many practical cases the measured curve cannot be matched sufficiently accurately with any of the finite number of calculated graphs and even interpolation may be troublesome. Consequently in such cases there is no accurate interpretation possible. In those cases where the available collection of standard-graphs fails, the geophysicist should be able to calculate the required curves himself. These calculations however are rather time-consuming and therefore they are usually omitted. Of course this leads to a greater portion of guesswork in the inter-

pretation. The use of a computer is efficient only for the calculation of great numbers of curves.

It is true that some empirical methods have been developed to construct graphs for three layers with the help of two-layer graphs as envelope curves: one for the case where the second layer reaches to infinite depth and one where the first two layers are replaced by a substitutional layer. HUMMEL [13] and SOROKIN [28] gave formulae and CAGNIARD [3] presented some diagrams, which were generalized by KOEFOED [17], for the thickness and resistivity of the substitutional layer. This procedure can easily be extended to more than three layers. A serious objection of these empirical methods is that the construction of that reach of the graph which is between the two two-layer graphs is uncertain. The objection is the stronger the greater the number of layers. So this rapid method is only a rough one.

The conclusion is justified that there is need of a simple and rapid calculation method, which can be applied by the geophysicist in the field without using a computer. Some approximation is acceptable, provided that an idea of the error can be gained. The derivation of such a method is the subject of this thesis.

The author is well aware that even when employing the most simple calculation method some ambiguity in interpretation will be unavoidable because of the principle of equivalence and the principle of suppression as described by MAILLET [19].

REVIEW OF EXISTING METHODS FOR THE CALCULATION OF STANDARD-GRAPHS IN GEO-ELECTRICAL PROSPECTING

2.1 Types of methods

Standard-graphs for any electrode arrangement can be calculated from the distribution of the surface potential V , generated by one pole placed at the surface of a horizontally stratified medium and emitting a current of strength I ; V being a function of I and the distance r to that pole. In the case of an electrode arrangement according to SCHLUMBERGER the apparent resistivities ρ_a follow simply from:

$$\rho_a = - \frac{2\pi}{I} r^2 \frac{dV}{dr} \dots \dots \dots (2.1)$$

where r equals $L/2$ (fig. 1.3) and in case of an arrangement according to WENNER, from:

$$\rho_a = \frac{2\pi r}{I} \cdot \Delta V \dots \dots \dots (2.2)$$

where r equals a (fig. 1.2).

The difficulty however is in the calculation of that potential distribution. In that respect the existing calculation methods can be subdivided into two main groups, viz:

1. those where the stratified semi-infinite medium with one pole of unit strength at the surface, is replaced by a homogeneous medium of infinite extent, with twice the original pole and moreover an infinite series of image poles producing the same potential distribution in a horizontal plane through the original pole;
2. those starting from the mathematical expression for the potential distribution. This mathematical expression is in the form of a definite integral, first derived by STEFANESCO [29], [26].

As a result of his study STEFANESCO arrived at the following expression for the potential V in a point P at the surface at a distance r from the pole O , emitting a current of strength I (fig. 1.1):

$$V = \frac{\rho_1 I}{2\pi} \left\{ \frac{1}{r} + 2 \int_0^\infty \Theta_1(\lambda) J_0(\lambda r) d\lambda \right\}$$

where ϱ_1 = resistivity of the upper layer

J_0 = Bessel function of the first kind and zero order

Θ_1 = the so-called kernel function, determined from the boundary conditions at the interfaces between the various layers. The index 1 refers to the first layer, as V is the potential distribution at the earth surface. As in the present problem no other index-figure occurs it will be subsequently omitted in order to make way for another index n referring to the number of layers. So with an n -layer resistivity profile Θ_n is a function of the reflection-coefficients

$$k_m = \frac{\varrho_{m+1} - \varrho_m}{\varrho_{m+1} + \varrho_m}$$

at the interfaces and of the depths d_m of the interfaces ($m < n$)

λ = integration variable

Methods belonging to both groups will be discussed in the paragraphs 2.2 and 2.3 respectively. Finally in paragraph 2.4 the relationship between both types of methods will be shown.

2.2 Methods with image poles

HUMMEL [13] was the first to develop a method with image poles for two- and three-layer cases. In this the semi-infinite layered medium with one pole of unit strength at the earth surface was replaced by a fictive infinite medium, the resistivity ϱ of which equals that of the upper layer, with twice the original pole and an infinite series of image poles.

Fig. 2.1 gives a picture of that infinite series of image poles for the case of a two-layer resistivity profile. Any image pole below the earth surface has an equal counterpart above it, in order to assure the condition that there is no passage of current through the original earth surface. In the calculation of the potential distribution at the earth surface any of these pairs of image poles can be replaced by one image pole located at the same depth below surface and of the twofold strength.

For three-layer cases HUMMEL introduced an infinite number of infinite series of image poles, again symmetrical with respect to the earth surface. As far as these poles are situated below the earth surface each of these semi-series consists of poles at equal intervals of $2t_1$. The poles of the first semi-series range from $2t_1$ to infinity, those of the second semi-series from $2t_2 + 2t_1$ to infinity, those of the third semi-series from $4t_2 + 2t_1$ to infinity and so on, so that the poles of the q^{th} semi-series range from $2(q-1)t_2 + 2t_1$ to infinity.

The strengths of the various image poles are functions of the reflection-

Fig. 2.1 System of image poles for a two-layer case according to HUMMEL

Fig. 2.2 System of layers of unit thickness t , with the corresponding depths of the image poles, as introduced by EHRENBURG and WATSON

coefficients k_1 and k_2 . For the semi-series starting from $2t_1$, these strengths are $2k_1^j$, where j is the serial-number of an image pole within that series; for the semi-series starting from $2t_2 + 2t_1$, they are $2(1 - k_1^2)k_2^j k_1^{j-1}$, where again j is counted from 1, at the depth $2t_2 + 2t_1$ to infinity, etc. As result HUMMEL gave an expression for the potential distribution at the earth surface due to all poles

above as well as below the earth surface and derived from it another expression for the apparent resistivities for electrode configurations in WENNER arrangement.

A similar, but simpler method was presented by EHRENBURG and WATSON [10] in 1932 and further refined by WATSON [32] in 1934. This method is applicable to any number of layers the thicknesses of which are equal to or a whole multiple of some unit thickness t . After introducing interfaces at any depth interval equal to one unit t (fig. 2.2) a general solution could be given. Compared with HUMMEL's system, the image poles of the infinite number of series coincide, and so only one infinite system of image poles at depths and heights $2ht$ below and above the original earth surface is left, h running from 1 to infinity. The strengths $2S_{2ht}$ of these poles have been given by WATSON and are expressed in the reflection-coefficients k at all interfaces:

$$\begin{aligned} S_{2t} &= k_1 \\ S_{4t} &= (1 - k_1^2)k_2 + k_1S_{2t} \\ S_{6t} &= (1 - k_1^2)(1 - k_2^2)k_3 + (k_1 - k_1k_2)S_{4t} + k_2S_{2t} \\ S_{8t} &= (1 - k_1^2)(1 - k_2^2)(1 - k_3^2)k_4 + (k_1 - k_1k_2 - k_2k_3)S_{6t} \\ &\quad + (k_2 - k_1k_3 + k_1k_2k_3)S_{4t} + k_3S_{2t} \\ S_{10t} &= (1 - k_1^2)(1 - k_2^2)(1 - k_3^2)(1 - k_4^2)k_5 + (k_1 - k_1k_2 - k_2k_3 - k_3k_4)S_{8t} \\ &\quad + (k_2 - k_1k_3 - k_2k_4 + k_1k_2k_3 + k_1k_3k_4 - k_1k_2k_3k_4)S_{6t} \\ &\quad + (k_3 - k_1k_4 + k_1k_2k_4 + k_2k_3k_4)S_{4t} + k_4S_{2t} \text{ etc.} \end{aligned}$$

In actual cases a great number of these k -values are equal to zero. So in fact the strengths of the image poles are functions of the actual interface depths d_m as well as the actual reflection-coefficients $k_m \neq 0$. Thus for any particular depth configuration, simplified formulae can easily be derived from the general expressions of WATSON. The formulae for the deeper poles occur as a recurrence formula e.g. if $d_1 = t$ and $d_2 = 4t$, then:

$$\begin{aligned} S_{2t} &= +k_1 \\ S_{4t} &= +k_1S_{2t} \\ S_{6t} &= +k_1S_{4t} \\ S_{8t} &= +k_1S_{6t} - k_1^2k_2 + k_2 \\ S_{10t} &= +k_1S_{8t} - k_1k_2S_{4t} + k_2S_{2t} \\ S_{12t} &= +k_1S_{10t} - k_1k_2S_{6t} + k_2S_{4t} \text{ etc.} \end{aligned}$$

The accuracy of the calculation depends on the depths to which the equidistant image poles are taken into account.

In 1955 KOEFOED [16] gave a refinement of EHRENBURG and WATSON's method for three-layer cases, especially for those where $\varrho_3 = \varrho_1$. In the first place he arrived at a more rapid convergency by calculating ϱ_a directly, instead of calculating the potential distribution first. Moreover, a further sim-

plification was proposed. After a number of image poles all deeper ones were replaced by one image pole of such a strength that the aggregate strength of all poles remained unchanged. Finally KØEFOED introduced a series expansion for the effect of the deeper poles.

2.3 Methods based on the integral of Stefanesco

The solution of the integral of STEFANESCO is troublesome. The difficulty is in the kernel function Θ_n . Any attempt to solve this problem must start with a decomposition of the kernel function Θ_n or the Bessel function J_0 or both into such fractions that integration becomes possible. The decomposition should be either exact or a series expansion such that the first few terms give a sufficiently accurate approximation. This decomposition has been carried out in different ways by several authors. They are MUSKAT (1933), FLATHE (1955), La Compagnie Générale de Géophysique (1955), MOONEY and WETZEL (1956) and ONODERA (1963). A summary of each of these solutions follows below.

MUSKAT [22] was the first one who solved this problem by means of series expansions of the Bessel function. Different procedures are given for small and large distances. As a result MUSKAT arrived at the potential distribution about an electrode on the surface of the earth. This method is not very popular.

FLATHE [11] developed a method to be used by the geophysicist in the field. His method is restricted to those profiles where any layer-thickness equals a whole multiple of some unit thickness. In itself this is no limitation as the unit thickness can be taken very small. The solution is exact for any resistivity profile where an even number of layer units overlies a perfectly conducting or a perfectly insulating substratum. In case the substratum is neither perfectly conducting nor perfectly insulating, an approximate resistivity curve can be obtained by adding either a conducting or an insulating substratum at some depth below the deepest actual interface. The deeper the additional interface the better the approximation.

The decomposition of the kernel function Θ_n into fractions according to FLATHE, is such that any fraction leads to a so-called "prime element" $D(\kappa_i, r)$. A prime element is the resistivity graph for a resistivity profile as shown in fig. 2.3.

Fig. 2.3 Elementary resistivity profile of FLATHE's "prime element"

The only parameter in this system of prime elements is the reflection-coefficient

$$\kappa = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$$

A set of these prime elements is shown in a diagram in FLATHE's paper. The resistivity curve to be calculated is a linear function of a finite number of these prime elements:

$$\frac{\rho_a}{\rho_1} = \sum \alpha_i D(\kappa_i, r)$$

The coefficients α_i and κ_i can be calculated according to FLATHE. This requires the determination of the roots of a polynomial, the degree of which is equal to one half of the even number of layer units. For great numbers of layer units this is a time-consuming task. So for a rapid calculation one has to be content with a limited number of layer units.

In the same year as FLATHE presented his method described above, La Compagnie Générale de Géophysique published a set of graphs [2], which were calculated in the years 1933–1936. The method of calculation of the standard graphs for profiles with perfectly conducting substratum or with perfectly insulating substratum as described in that publication is very similar to FLATHE's method. By decomposition of the kernel function the apparent resistivities to be calculated can be built up as a linear function of the apparent resistivities belonging to the following cases:

- one profile consisting of a layer of unit thickness on a perfectly insulating substratum (fig. 2.4a);
- one profile consisting of a layer of unit thickness on a perfectly conducting substratum (fig. 2.4b);
- a finite number of profiles consisting of two layers of unit thickness underlain by a perfectly conducting substratum (fig. 2.4c), every profile having a different ratio $k_s = \frac{\rho_s - \rho_1}{\rho_s + \rho_1}$ determined by the ratio of the resistivities ρ_s of the second layer to that of the first.

$$k_1 = +1 \quad \frac{\rho_1 = 1 \quad t_1 = 1}{\rho_2 = \infty} \quad d_1 = 1$$

Fig. 2.4a

$$k_1 = -1 \quad \frac{\rho_1 = 1 \quad t_1 = 1}{\rho_2 = 0} \quad d_1 = 1$$

Fig. 2.4b

$$k_1 = k_s \quad \frac{\rho_1 = 1 \quad t_1 = 1}{\rho_2 = \rho_s \quad t_2 = 1} \quad d_1 = 1$$

$$k_2 = -1 \quad \frac{\rho_3 = 0}{\rho_2 = 0} \quad d_2 = 2$$

Fig. 2.4c

Fig. 2.4a, b and c Elementary resistivity profiles used in the calculation method of La C.G.G.

In fact this is also a linear combination of prime elements as introduced by FLATHE. Therefore a set of such curves (abaques) has also been included in that publication.

The four-layer standard graphs published by MOONEY and WETZEL were calculated by means of a computer. Thus their method was suited to the use of a computer, and consequently it is not a rapid method for the use of the geophysicist in the field. To this end, MOONEY and WETZEL [21] subdivided the complete integration interval of the integral of STEFANESCO, ranging from 0 to ∞ into 24 different sub-intervals and replaced the kernel function by polynomials of the second degree in the first 23 of these intervals and by a remainder term in the last interval. Of course this is an approximation, which introduces an error. This error however is acceptable as long as the resistivity contrasts are not too great and the layers not too thin. If a greater accuracy is desired, the number of intervals could be increased.

The most recent solution for the decomposition of the kernel function is that by ONODERA [23], who expands the kernel function according to a system of LEGENDRE polynomials of least squares approximations. In this way STEFANESCO's integral can be replaced by a number of integrals of the form:

$$\int_0^{\infty} A_\nu e^{-2\nu\lambda} \mathcal{J}_0(\lambda r) d\lambda$$

As this is the LIPSCHITZ integral these integrals equal $\frac{A_\nu}{\sqrt{r^2 + 4\nu^2}}$, which lends itself to numerical calculations.

2.4 Correspondence between both groups of calculation methods

The kernel function in STEFANESCO's integral is a fraction. Both the numerator and the denominator consist of a sum of various powers of $e^{-2\lambda}$; the powers being governed by the depths of the interfaces. By simple division this kernel function can be expanded as an infinite series of terms in the form of $e^{-D\lambda}$.

So the integral $\int_0^{\infty} \Theta_n(\lambda) \mathcal{J}_0(\lambda r) d\lambda$ can be replaced by an infinite series of integrals of the form:

$$\int_0^{\infty} K e^{-D\lambda} \mathcal{J}_0(\lambda r) d\lambda$$

This integral equals $\frac{K}{\sqrt{r^2 + D^2}}$. The expression $\frac{K}{\sqrt{r^2 + D^2}}$ occurs in the methods

based on image poles. Physically it represents K times the reciprocal value of the distance of a pole at depth D to a point at the surface at a distance r from the original pole. The values of K depend on the reflection-coefficients k and turn out to be equal to the strengths of the image poles.

DERIVATION OF A SIMPLE CALCULATION METHOD

3.1 Exact theory

The derivation of the new method for the calculation of standard-graphs for a resistivity profile consisting of n horizontally stratified isotropic media of different thicknesses t_m and different electrical resistivities ϱ_m , as shown in fig. 1.1, originates with the potential distribution of a pole at the surface as given by STEFANESCO [29]:

$$V = \frac{\varrho_1 I}{2\pi} \left\{ \frac{1}{r} + 2 \int_0^\infty \Theta_n(\lambda) J_0(\lambda r) d\lambda \right\} \quad \dots \quad (3.1)$$

and described in paragraph 2.1.

The kernel function Θ_n can be written as follows:

$$\Theta_n(\lambda) = \frac{P_n(u)}{Q_n(u)} = \frac{P_n(u)}{H_n(u) - P_n(u)} \quad \dots \quad (3.2)$$

where $u = e^{-2\lambda}$.

$$\text{For two layers:} \quad P_2(u) = k_1 u^{d_1} \quad \dots \quad (3.3)$$

$$\text{and } H_2(u) = 1 \quad \dots \quad (3.4)$$

$$\text{thus } Q_2(u) = 1 - k_1 u^{d_1} \quad \dots \quad (3.5)$$

For any greater number of layers n the functions $P_n(u)$, $H_n(u)$ and $Q_n(u)$ can be calculated from the corresponding functions for $n-1$ layers: $P_{n-1}(u)$, $H_{n-1}(u)$ and $Q_{n-1}(u)$. When creating an additional interface at a depth d_{n-1} with a reflection-coefficient k_{n-1} somewhere in the original substratum of an $(n-1)$ -layer case, the number of layers becomes n . In this case FLATHE's "law of formation" applies, resulting in the following recurrence formulae:

$$P_n(u) = P_{n-1}(u) + H_{n-1}(u^{-1}) k_{n-1} u^{d_{n-1}} \quad \dots \quad (3.6)$$

$$H_n(u) = H_{n-1}(u) + P_{n-1}(u^{-1}) k_{n-1} u^{d_{n-1}} \quad \dots \quad (3.7)$$

$$Q_n(u) = Q_{n-1}(u) - Q_{n-1}(u^{-1}) k_{n-1} u^{d_{n-1}} \quad \dots \quad (3.8)$$

Table 3.I comprises the complete expressions for $P_6(u)$, $H_6(u)$ and $Q_6(u)$, as obtained by repeated application of the formulae (3.6), (3.7) and (3.8). The complete expressions $P_n(u)$, $H_n(u)$ and $Q_n(u)$ with $n < 6$ follow from those of table 3.I by omitting all terms comprising coefficients k_m with $m \geq n$.

Table 3.I Complete expressions for $P_6(u)$, $H_6(u)$ and $Q_6(u)$

$$\begin{aligned}
 P_6(u) = & k_1 u^{d_1} + k_2 u^{d_2} + k_3 u^{d_3} + k_4 u^{d_4} + k_5 u^{d_5} \\
 & + k_1 k_2 k_3 u^{d_3-d_2+d_1} + k_1 k_2 k_4 u^{d_4-d_2+d_1} + k_1 k_2 k_5 u^{d_5-d_2+d_1} \\
 & + k_1 k_3 k_4 u^{d_4-d_3+d_1} + k_1 k_3 k_5 u^{d_5-d_3+d_1} \\
 & + k_1 k_4 k_5 u^{d_5-d_4+d_1} \\
 & + k_2 k_3 k_4 u^{d_4-d_3+d_2} + k_2 k_3 k_5 u^{d_5-d_3+d_2} \\
 & + k_2 k_4 k_5 u^{d_5-d_4+d_2} \\
 & + k_3 k_4 k_5 u^{d_5-d_4+d_3} \\
 & + k_1 k_2 k_3 k_4 k_5 u^{d_5-d_4+d_3-d_2+d_1}
 \end{aligned}$$

$$\begin{aligned}
 H_6(u) = & 1 + k_1 k_2 u^{d_2-d_1} + k_1 k_3 u^{d_3-d_1} + k_1 k_4 u^{d_4-d_1} + k_1 k_5 u^{d_5-d_1} \\
 & + k_2 k_3 u^{d_3-d_2} + k_2 k_4 u^{d_4-d_2} + k_2 k_5 u^{d_5-d_2} \\
 & + k_3 k_4 u^{d_4-d_3} + k_3 k_5 u^{d_5-d_3} \\
 & + k_4 k_5 u^{d_5-d_4} \\
 & + k_1 k_2 k_3 k_4 u^{d_4-d_3+d_2-d_1} + k_1 k_2 k_3 k_5 u^{d_5-d_3+d_2-d_1} \\
 & + k_1 k_2 k_4 k_5 u^{d_5-d_4+d_2-d_1} + k_1 k_3 k_4 k_5 u^{d_5-d_4+d_3-d_1} \\
 & + k_2 k_3 k_4 k_5 u^{d_5-d_4+d_3-d_2}
 \end{aligned}$$

$$\begin{aligned}
 Q_6(u) = & 1 - k_1 u^{d_1} - k_2 u^{d_2} - k_3 u^{d_3} - k_4 u^{d_4} - k_5 u^{d_5} \\
 & + k_1 k_2 u^{d_2-d_1} + k_1 k_3 u^{d_3-d_1} + k_1 k_4 u^{d_4-d_1} + k_1 k_5 u^{d_5-d_1} \\
 & + k_2 k_3 u^{d_3-d_2} + k_2 k_4 u^{d_4-d_2} + k_2 k_5 u^{d_5-d_2} \\
 & + k_3 k_4 u^{d_4-d_3} + k_3 k_5 u^{d_5-d_3} \\
 & + k_4 k_5 u^{d_5-d_4} \\
 & - k_1 k_2 k_3 u^{d_3-d_2+d_1} - k_1 k_2 k_4 u^{d_4-d_2+d_1} - k_1 k_2 k_5 u^{d_5-d_2+d_1} \\
 & - k_1 k_3 k_4 u^{d_4-d_3+d_1} - k_1 k_3 k_5 u^{d_5-d_3+d_1} \\
 & - k_1 k_4 k_5 u^{d_5-d_4+d_1} \\
 & - k_2 k_3 k_4 u^{d_4-d_3+d_2} - k_2 k_3 k_5 u^{d_5-d_3+d_2} \\
 & - k_2 k_4 k_5 u^{d_5-d_4+d_2} \\
 & - k_3 k_4 k_5 u^{d_5-d_4+d_3} \\
 & + k_1 k_2 k_3 k_4 u^{d_4-d_3+d_2-d_1} + k_1 k_2 k_3 k_5 u^{d_5-d_3+d_2-d_1} \\
 & + k_1 k_2 k_4 k_5 u^{d_5-d_4+d_2-d_1} + k_1 k_3 k_4 k_5 u^{d_5-d_4+d_3-d_1} \\
 & + k_2 k_3 k_4 k_5 u^{d_5-d_4+d_3-d_2} \\
 & - k_1 k_2 k_3 k_4 k_5 u^{d_5-d_4+d_3-d_2+d_1}
 \end{aligned}$$

3.2 Series expansion

The kernel function Θ_n can, under certain conditions, be replaced by a convergent infinite series of terms. This is based on the series expansion of $y = \frac{1}{1-x}$ into:

$$y = \lim_{p \rightarrow \infty} 1 + x + x^2 + \dots + x^p \quad (3.9)$$

This series converges only if $|x| < 1$, and convergency is the stronger, the nearer x approaches 0. The new and simple calculation method to be described in this chapter is based on the principle that for practical purposes it will be satisfactory to take only the first few terms of this series expansion.

Applying the series expansion to the kernel function Θ_n yields:

$$\Theta_n(\lambda) = \frac{P_n(u)}{Q_n(u)} = P_n(u) \frac{1}{1 - \{1 - Q_n(u)\}} =$$

$$= P_n(u) \times \lim_{p \rightarrow \infty} [1 + \{1 - Q_n(u)\} + \{1 - Q_n(u)\}^2 + \dots + \{1 - Q_n(u)\}^p] \quad \dots (3.10)$$

In p^{th} degree of approximation:

$$\Theta_n(\lambda) \approx P_n(u) [1 + \{1 - Q_n(u)\} + \{1 - Q_n(u)\}^2 + \dots + \{1 - Q_n(u)\}^{p-1}] \quad \dots (3.11)$$

It should be noticed here that this approximation is only adequate when the series converges, so in the case:

$$|1 - Q_n(u)| < 1 \quad \dots \dots \dots (3.12)$$

or where:

$$0 < Q_n(u) < +2 \quad \dots \dots \dots (3.13)$$

This limiting factor will be investigated in chapter 4.

If the above condition is fulfilled, the expression for V according to STEFANESCO can be replaced by:

$$V \approx \frac{e_1 I}{2\pi} \left[\frac{1}{r} + 2 \int_0^\infty P_n(u) \mathcal{J}_0(\lambda r) d\lambda \right.$$

terms to be added in:

1st

2nd

3rd

p^{th}

}

$$+ 2 \int_0^\infty P_n(u) \{1 - Q_n(u)\} \mathcal{J}_0(\lambda r) d\lambda$$

$$+ 2 \int_0^\infty P_n(u) \{1 - Q_n(u)\}^2 \mathcal{J}_0(\lambda r) d\lambda$$

$$\dots \dots \dots$$

$$+ 2 \int_0^\infty P_n(u) \{1 - Q_n(u)\}^{p-1} \mathcal{J}_0(\lambda r) d\lambda \left. \vphantom{\int_0^\infty} \right] \quad \dots \dots \dots (3.14) \quad \text{degree of approximation}$$

$P_n(u)$ and $\{1 - Q_n(u)\}$ are both polynomials in $u = e^{-2\lambda}$. From the recurrence formulae for $P_n(u)$ and $Q_n(u)$ given by FLATHE and worked out in table 3.I, it appears that the coefficients of the terms in u are products of the reflection-coefficients k_m only. Also the exponents of the powers of u are linear functions of the depths d_m of the various interfaces only. Thus any product $P_n(u) \{1 - Q_n(u)\}^{p-1}$ is built up of a finite number of terms in the form of:

$$K e^{-D\lambda} \quad \dots \dots \dots (3.15)$$

K being functions of the reflection-coefficients k_m only and D being linear functions of the depths d_m of the interfaces only. So the form within braces in

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the exact expression (3.1) for V comprises an infinite number of terms:

$$\int_0^\infty K e^{-D\lambda} J_0(\lambda r) d\lambda = \frac{K}{\sqrt{D^2 + r^2}} \dots \dots \dots (3.16)$$

Thus

$$V = \frac{\varrho_1 I}{2\pi} \left\{ \frac{1}{r} + \text{an infinite number of terms } \frac{K}{\sqrt{D^2 + r^2}} \right\} \dots \dots (3.17)$$

In any successive degree of approximation p another group of terms $\frac{K}{\sqrt{D^2 + r^2}}$ is added to the previous groups. The number of terms within these groups is greater with higher degree of approximation p , for any fixed number of layers n .

The coefficients K and the depths D , occurring in the terms $\frac{K}{\sqrt{D^2 + r^2}}$ to be added to the form within braces in expression (3.17) when proceeding from the $(p-1)^{\text{th}}$ degree of approximation to the p^{th} degree of approximation, can be evaluated by multiplication of $P_n(u)$, borrowed from table 3.I, with $\{1 - Q_n(u)\}^{p-1}$; $Q_n(u)$ also from table 3.I. This operation will be postponed to a later stage.

First a study will be made of how these terms appear in the expressions for ϱ_a when applying electrode configurations according to SCHLUMBERGER and to WENNER.

3.3 Apparent resistivities

In SCHLUMBERGER arrangement:

$$\varrho_a = \frac{-2\pi}{I} r^2 \frac{dV}{dr} \dots \dots \dots (3.18)$$

The potential gradient $\frac{dV}{dr}$ follows from expression (3.17):

$$\frac{dV}{dr} = \frac{\varrho_1 I}{2\pi} \left\{ \frac{-1}{r^2} - \text{an infinite number of terms } \frac{Kr}{(D^2 + r^2)^{3/2}} \right\} \dots \dots \dots (3.19)$$

Substituting expression (3.19) in (3.18) yields:

$$\varrho_a = \varrho_1 \left\{ 1 + \text{an infinite number of terms } K \left(\frac{r}{\sqrt{D^2 + r^2}} \right)^3 \right\} \dots (3.20)$$

or
$$\varrho_a = \varrho_1 \left\{ 1 + \text{an infinite number of terms } K \left(\frac{1}{\sqrt{\left(\frac{D}{r}\right)^2 + 1}} \right)^3 \right\} (3.21)$$

In WENNER arrangement:

$$\varrho_a = \frac{2\pi r \Delta V}{I} \quad \dots \dots \dots (3.22)$$

The potential difference $\Delta V = 2\{V(r) - V(2r)\}$ equals:

$$\Delta V = \frac{\varrho_1 I}{\pi} \left\{ \left(\frac{1}{r} - \frac{1}{2r} \right) + \text{an infinite number of terms } K \left(\frac{1}{\sqrt{D^2 + r^2}} - \frac{1}{\sqrt{D^2 + (2r)^2}} \right) \right\} \quad (3.23)$$

Thus $\varrho_a = 2r\varrho_1 \left\{ \left(\frac{1}{r} - \frac{1}{2r} \right) + \text{an infinite number of terms} \right.$

$$\left. K \left(\frac{1}{\sqrt{D^2 + r^2}} - \frac{1}{\sqrt{D^2 + (2r)^2}} \right) \right\} \quad \dots \dots \dots (3.24)$$

hence:

$$\varrho_a = \varrho_1 \left\{ 1 + \text{an infinite number of terms } K \left(\frac{2}{\sqrt{\left(\frac{D}{r}\right)^2 + 1}} - \frac{2}{\sqrt{\left(\frac{D}{r}\right)^2 + 4}} \right) \right\} \quad (3.25)$$

The expression (3.25) appears to be similar to those given by HUMMEL [13] and by EHRENBURG and WATSON [10 and 34] for electrode configurations in WENNER arrangement. In paragraph 3.7 it will be shown that the coefficients K are the contributions to the strengths of the image poles as introduced by HUMMEL and by EHRENBURG and WATSON and that the depths D are equal to the depths of the poles below the earth surface. Therefore the coefficients K and the depths D will also be assigned to pole strengths and pole depths respectively. It must be noticed here that the same values K and D occur in the expressions (3.25) and (3.21), for electrode arrangements according to both WENNER and SCHLUMBERGER.

3.4 Composition of standard-graphs

In this stage an analysis of the values of the pole strengths K and pole depths D is required first. Such an analysis has been performed for 3-, 4-, 5- and 6-layer cases in the tables 3.II, 3.III, 3.IV and 3.V. The 3-layer case, having the smallest number of parameters d_m and k_m (instead of $\frac{\varrho_m}{\varrho_{m-1}}$), has been elaborated to a higher degree of approximation than the more complicated 4-, 5- and 6-layer cases. As such, table 3.II referring to three-layer cases can best serve for explanation and further fundamental considerations.

The first column "number of line" has been added for ease when performing

numerical calculations in the calculation tables to be described later in this chapter. The column "depth of the image poles" contains all depths D , expressed in the interface depths d_1 and d_2 , occurring when proceeding to the 7th degree of approximation. The next columns show the contributions K to the strengths of the image poles, to be added in the successive degrees of approximation. The following example may illustrate how these values K and D have been obtained.

According to the derivation in paragraph 3.2 the terms to be added in 2nd degree of approximation ($p = 2$) in this 3-layer case ($n = 3$) originate from:

$$2 \int_0^{\infty} P_3(u) \{1 - Q_3(u)\}^{2-1} J_0(\lambda r) d\lambda \quad (3.26)$$

in the expression (3.14) for V .

Substituting

$$P_3(u) = k_1 e^{-2\lambda d_1} + k_2 e^{-2\lambda d_2} \quad (3.27)$$

and

$$Q_3(u) = 1 - k_1 e^{-2\lambda d_1} - k_2 e^{-2\lambda d_2} + k_1 k_2 e^{-2\lambda(d_2 - d_1)} \quad (3.28)$$

as borrowed from table 3.I, the term (3.26) becomes:

$$\begin{aligned} & 2 \int_0^{\infty} (k_1 e^{-2\lambda d_1} + k_2 e^{-2\lambda d_2}) (k_1 e^{-2\lambda d_1} + k_2 e^{-2\lambda d_2} - k_1 k_2 e^{-2\lambda(d_2 - d_1)}) J_0(\lambda r) d\lambda = \\ & 2 \int_0^{\infty} (k_1^2 e^{-4\lambda d_1} + 2k_1 k_2 e^{-2\lambda(d_2 + d_1)} - k_1^2 k_2 e^{-2\lambda d_2} + k_2^2 e^{-4\lambda d_2} - k_1 k_2^2 e^{-2\lambda(2d_2 - d_1)}) J_0(\lambda r) d\lambda \end{aligned} \quad (3.29)$$

This expression points to:

- a pole at depth $D = 4d_1$ with strength $K = +2k_1^2$ (line 2 in table 3.II)
- a pole at depth $D = 2d_2 + 2d_1$ with strength $K = +4k_1 k_2$ (line 10 in table 3.II)
- a pole at depth $D = 2d_2$ with strength $K = -2k_1^2 k_2$ (line 9 in table 3.II)
- a pole at depth $D = 4d_2$ with strength $K = +2k_2^2$ (line 18 in table 3.II)
- a pole at depth $D = 4d_2 - 2d_1$ with strength $K = -2k_1 k_2^2$ (line 17 in table 3.II)

The same procedure has been followed for all other degrees of approximation in this table as well as in the tables 3.III, 3.IV and 3.V for 4,5 and 6 layers respectively.

Table 3.II shows great regularity. So it is easy to detect series of coefficients and exponents. From equation (3.28) it follows that the contribution K to the strength of a pole at depth D in the column to be added in p^{th} degree of approximation equals:

- $+k_1$ times the contribution to the pole strength at depth $D-2d_1$, added in $(p-1)^{\text{th}}$ degree of approximation,
- $+k_2$ times the contribution to the pole strength at depth $D-2d_2$, added in $(p-1)^{\text{th}}$ degree of approximation,
- $-k_1k_2$ times the contribution to the pole strength at depth $D-2d_2+2d_1$, added in $(p-1)^{\text{th}}$ degree of approximation.

Frequently one or two of the contributions to the pole strengths at the depths: $D-2d_1$; $D-2d_2$ and $D-2d_2+2d_1$ added in the $(p-1)^{\text{th}}$ degree of approximation are equal to zero. This fact greatly simplifies the calculation of the K -values. Applying this rule to table 3.II all contributions to the strength of any pole occurring in the table have been evaluated.

In the column "Completion" all contributions to those poles, in the successive degrees of approximation over the 7th degree, have been compiled. Therefore this procedure has been performed up to the 14th degree of approximation. This does not mean that the table is complete up to that degree of approximation. All pole depths to be introduced over the 7th degree of approximation have been omitted.

Finally in table 3.II a column "Complete strengths of the image poles" has been inserted. In the left half of this column, the sum of all contributions to the various poles is written in a more elegant form. In the right half of the column the complete strength of the poles is expressed in terms of that of three shallower poles, as obtained by a consistent application of the above-mentioned regularity.

3.5 Calculation tables

In order to facilitate the numerical calculation of standard-graphs three types of calculation tables have been developed.

Owing to the fact that, in the new method, the pole strengths are merely determined by the reflection-coefficients k_m , it is possible to use calculation table no. 1, as represented in table 3.VI, for the calculation of the numerical values of the strengths of the image poles at any depth configuration of a given sequence of n layers with given resistivities ρ_m . The columns of this table should be filled in on the basis of one of the tables 3.II, 3.III, 3.IV or 3.V, depending on the number of layers considered.

Similarly it is possible to use calculation table no. 2, as shown in table 3.VII, for the numerical values of the depths of the image poles with given interface depths d_m , regardless of the resistivities of the layers. The columns of this table should also be filled in on the basis of one of the tables 3.II, 3.III, 3.IV or 3.V.

The separation into strength- and depth-tables saves much work when

Table 3.VIII Calculation table no. 3

Resistivity profile		Degree of approximation	Calculation table no. 3 Apparent resistivities ϱ_a																																	
			Distances r		1		1.4		2		2.8		4		5.6		7.5		10		14		20		28		40		56		75		100			
					Depths of the image poles	Strengths	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
$\varrho_1 =$	$d_1 =$																																			
$\varrho_2 =$	$d_2 =$																																			
$\varrho_3 =$	$d_3 =$																																			
$\varrho_4 =$	$d_4 =$																																			
$\varrho_5 =$	$d_5 =$																																			
$\varrho_6 =$																																				

Table 3.IX Calculation table no. 3 applied to the resistivity profile of fig. 3.2 in second degree of approximation

Resistivity profile		Degree of approximation	Calculation table no. 3 Apparent resistivities ϱ_a																																			
			Distances r		1		1.4		2		2.8		4		5.6		7.5		10		14		20		28		40		56		75		100					
					Depths of the image poles	Strengths	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−		
			+	−			+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−		
$\varrho_1 = 1$	$d_1 = 1$ $d_2 = 3$ $d_3 = -$ $d_4 = -$ $d_5 = -$ $d_6 = -$	2	2	+1.000	1.000	−	1.000	−	1.000	−	1.000	−	1.000	−	1.000	−	1.000	−	1.000	−	1.000	−	1.000	−	1.000	−	1.000	−	1.000	−	1.000	−	1.000	−				
$\varrho_2 = 3$					4	+0.500	0.089	−	0.188	−	0.355	−	1.000	0.455	1.000	0.280	1.000	0.162	1.000	0.097	1.000	0.057	1.000	0.030	1.000	0.015	1.000	0.007	1.000	0.004	1.000	0.002	1.000	0.001	1.000	0.001		
$\varrho_3 = 1/3$							0.007	−	0.017	−	0.044	−	0.095	−	0.179	−	0.500	0.230	0.500	0.156	0.500	0.100	0.500	0.056	0.500	0.030	0.500	0.015	0.500	0.007	0.500	0.004	0.500	0.002	0.500	0.001		
$\varrho_4 = -$							6	−1.200	−	0.005	−	0.014	−	0.036	−	0.088	−	0.205	−	0.380	−	0.570	0.440	1.200	0.272	1.200	0.148	1.200	0.079	1.200	0.039	1.200	0.020	1.200	0.010	1.200	0.006	1.200
$\varrho_5 = -$							8	−1.600	−	0.003	−	0.008	−	0.022	−	0.056	−	0.144	−	0.300	−	0.512	−	0.760	0.560	1.600	0.325	1.600	0.180	1.600	0.092	1.600	0.048	1.600	0.027	1.600	0.015	1.600
$\varrho_6 = -$							10	−0.640	−	0.001	−	0.002	−	0.005	−	0.012	−	0.032	−	0.074	−	0.138	−	0.230	0.298	0.640	0.186	0.640	0.107	0.640	0.056	0.640	0.030	0.640	0.017	0.640	0.009	0.640
							12	+1.280	0.001	−	0.002	−	0.006	−	0.015	−	0.041	−	0.098	−	0.196	−	0.345	−	0.570	−	1.280	0.470	1.280	0.285	1.280	0.154	1.280	0.082	1.280	0.047	1.280	0.027
			Σ	−0.660	1.097	0.009	1.207	0.024	1.405	0.063	2.110	0.611	2.220	0.661	2.598	1.146	2.696	1.473	3.285	2.347	4.200	3.526	4.439	3.955	4.146	3.747	3.967	3.605	3.878	3.528	3.834	3.490	3.810	3.469				
					1.088		1.183		1.342		1.499		1.559		1.452		1.223		0.938		0.674		0.484		0.399		0.362		0.350		0.344		0.341					

Table 3.VI Calculation table no. 1

Calculation table no. 1 Numerical values of the strengths of the image poles										
Resistivities		k	k^2	k^3	k^4	k^5	k^6	k^7		
$\varrho_1=$		$k_1=$								
$\varrho_2=$		$k_2=$								
$\varrho_3=$		$k_3=$								
$\varrho_4=$		$k_4=$								
$\varrho_5=$		$k_5=$								
$\varrho_6=$										
Number of line	Contributions to the strengths of the image poles in the indicated degrees of approximation							Summation of the contributions in		
	7th									
	6th									
	5th									
	4th									
	3rd									
	2nd									
	1st							degree of approximation		
1										
2										
3										
...										
80										

Table 3.VII Calculation table no. 2

Calculation table no. 2 Numerical values of the depths of the image poles											
Number of line	$d_1 =$										
	$d_2 =$										
	$d_3 =$										
	$d_4 =$										
	$d_5 =$										
1											
2											
3											
...											
80											

calculating sets of standard-graphs, keeping either the depth configuration or the resistivity distribution unchanged.

Having prepared the calculation tables no. 1 and no. 2, the apparent resistivities ϱ_a at selected distances r can be calculated in calculation table no. 3, shown in table 3.VIII. Therefore first the strengths of the poles as determined on the horizontal lines in calculation table no. 1 are summed for any individual pole depth according to calculation table no. 2. Then the pole depths and their total strengths $\bar{K}$ are transferred to the relevant columns in calculation table no. 3.

3.6 Unit-graphs and complement unit-graphs

One task is still left, namely the multiplication by the so-called distance factors:

$$f_s\left(\frac{r}{D}\right) = \left(\frac{1}{\sqrt{\left(\frac{D}{r}\right)^2 + 1}}\right)^3 \cdot \dots \cdot \dots \cdot \dots \quad (3.30)$$

in SCHLUMBERGER arrangement and

$$f_w\left(\frac{r}{D}\right) = \left(\frac{2}{\sqrt{\left(\frac{D}{r}\right)^2 + 1}} - \frac{2}{\sqrt{\left(\frac{D}{r}\right)^2 + 4}}\right) \cdot \dots \cdot \dots \cdot \dots \quad (3.31)$$

in WENNER arrangement.

This is a time-consuming task. This trouble however can be overcome by the use of an auxiliary graph.

Both factors are functions of $\frac{D}{r}$ and consequently also of $\frac{r}{D}$. At $\frac{r}{D} = 0$, $f_s\left(\frac{r}{D}\right) = 0$ and also $f_w\left(\frac{r}{D}\right) = 0$; and for $\frac{r}{D} \rightarrow \infty$, $f_s\left(\frac{r}{D}\right) \rightarrow 1$ and also $f_w\left(\frac{r}{D}\right) \rightarrow 1$. With logarithmic scale for the variable $\frac{r}{D}$ and linear ordinate scale both functions plot S-shaped.

The calculations to be made are of the following type:

$$\frac{\varrho_a}{\varrho_1} = 1 + \Sigma \bar{K} f \cdot \dots \cdot \dots \cdot \dots \cdot \dots \quad (3.32)$$

where f is a symbol for either $f_s(r/D)$ or $f_w(r/D)$. The numerous multiplications $\bar{K}f$ can be reduced to graphical additions when using logarithmic scales for $\bar{K}$ and for f . This can be realized by plotting $\log f$ versus $\log r/D$ on tracing paper (fig. 3.1) and then shifting this "unit-graph" f over a sheet of bilogarithmic

Fig. 3.1 Unit-graphs and complement unit-graphs for SCHLUMBERGER and for WENNER arrangement, with their asymptotes

paper, bearing the distances r on the horizontal axis, into such a position that the unit of the vertical scale on the transparent sheet matches with the desired value of $\bar{K}$ on the underlying sheet of bilogarithmic paper and that at the same time the vertical line for $r/D = 1$ on the transparent sheet matches with $r = D$ in the r -scale of the underlying bilogarithmic paper. The products $\bar{K}f$ for any desired distance r can now be read from the underlying bilogarithmic paper.

The numerical examples to be described later have been calculated with the help of a unit-graph on a sheet of tracing paper of $1\text{ m} \times 1\text{ m}$ with a subdivision of 25 centimetres per factor 10, in combination with bilogarithmic paper of the same dimensions.

Notwithstanding the great advantages of the logarithmic scale for the unit-graphs there is one drawback, namely the impossibility of accurate readings at f -values near 1. On the other hand high accuracy is required, as very often small apparent resistivities must be obtained as differences of relatively great values of $\bar{K}f$. In order to overcome this difficulty a dodge has been applied:

$$\bar{K}f = \bar{K}\{1 - (1 - f)\} = \bar{K} - \bar{K}(1 - f) \dots \dots \dots (3.33)$$

For values of f near 1, $1 - f$ is small and thus $\bar{K}(1 - f)$ can be read accurately, and $\bar{K}$ is known precisely. Therefore from $f = 1/2$, readings should be made on the "complement unit-graph", $1 - f$. The complement unit-graphs for both electrode arrangements are also drawn in fig. 3.1 for $f \geq 1/2$.

It should be noticed here that similar unit-graphs and complement unit-graphs can be drawn up for any other electrode arrangement. The system of poles applies regardless of the electrode arrangement.

3.7 Comparison with the methods of Hummel and of Ehrenburg and Watson

The newly derived system of poles continued into an infinite degree of approximation yields the same pole depths and pole strengths as the twofold infinite system of HUMMEL. This appears clearly by a comparison of table 3.II with HUMMEL's expression for ϱ_a/ϱ_1 for three-layer cases, as briefly described in paragraph 2.2.

Making a comparison between EHRENBURG and WATSON's system of poles (see paragraph 2.2) and the system derived in this chapter it turns out that in the new system the pole depths D are functions of the interface depths d_m only, and so the pole strengths are built up of functions $\bar{K}$ of the reflection-coefficients k_m only. In EHRENBURG and WATSON's system there is a continuous series of image poles at depth intervals equal to $2t$, where t is equal to the greatest common divisor of the various layer thicknesses t_m (see fig. 1.1 and fig. 2.2). As

shown in paragraph 2.2 the strengths of these poles are functions of the interface depths d_m as well as the reflection-coefficients k_m .

Depending on the magnitude of the various depths d_m , several pole depths D in the new system will be equal. Addition of the complete pole strengths of these poles at equal depth yields the same total strength as obtained in EHRENBURG and WATSON's system for that particular resistivity profile. For the three-layer case this appears once more from the last column of table 3.II, which shows the origin of the recurrence formulae for the pole strengths as described in paragraph 2.2.

This leads to a further comparison between the calculation of a standard-graph for an n -layer case with the help of either HUMMEL's or EHRENBURG and WATSON's system of poles and a calculation of the same graph using the new system. Therefore it will be assumed that the new system is applied to a degree of approximation p such that the deepest pole, which according to the tables 3.II, 3.III, 3.IV and 3.V occurs at the depth $2pd_{n-1}$, is at the same depth as that to which the series of poles according to HUMMEL or to EHRENBURG and WATSON have been taken into account. Then, in general, using the new method not all pole depths occurring in these methods are taken into account. Moreover, in general, the pole strengths are not complete for the following reasons. First, a number of contributions to the pole strength on each line of the relevant table 3.II, 3.III, 3.IV or 3.V, has been omitted depending on the degree of approximation p and the pole depth D . Secondly, not every line of that table or its downward extension bearing a pole not deeper than $2pd_{n-1}$ is included in the p^{th} degree of approximation of the new method.

Regardless of the accuracy, which will be investigated in chapter 4, it can be stated now that when taking the deepest pole at the same depth in the new method, fewer poles occur and the strength of these is less complicated than in the two existing methods.

3.8 Example

By way of illustration, apparent resistivity curves have been calculated for the resistivity profile of fig. 3.2 both according to the new method and according to EHRENBURG and WATSON.

The calculations according to the new method have been performed in

Fig. 3.2 Resistivity profile

Fig. 3.3 Approximate resistivity curves for the resistivity profile of fig. 3.2. Calculations according to the new method

first, second and third degree of approximation. The approximate resistivity curves are represented in fig. 3.3. Table 3.IX shows the numerical elaboration of calculation table no. 3 for the curve in second degree of approximation. The pole depths and strengths introduced in this table have been borrowed from the relevant calculation tables no. 1 and no. 2.

Fig. 3.4 shows the approximate resistivity curves based on calculations according to EHRENBURG and WATSON. The deepest poles involved in the

Fig. 3.4 Approximate resistivity curves for the resistivity profile of fig. 3.2. Calculations according to EHRENBURG and WATSON

Fig. 3.5 Approximate resistivity curves for the resistivity profile of fig. 3.2. Calculations according to EHRENBURG and WATSON as improved by KOEFOED

calculations for the curves indicated by $p = 1, 2$ and 3 are $6, 12$ and 18 respectively.

Finally fig. 3.5 represents the corresponding curves for calculations according to EHRENBURG and WATSON, as improved by KOEFOED. For each of these curves the strength of the deepest pole has been made such that the aggregate strength of all poles is correct.

ON THE APPLICABILITY AND THE ACCURACY OF THE NEW CALCULATION METHOD

4.1 Applicability

In paragraph 3.2 it has already been mentioned that the proposed calculation method converges only if $0 < Q_n(u) < +2$. So this is the criterion for the applicability. The function $Q_n(u)$ depends both on the resistivities ϱ_m of the various layers and the depths d_m of the interfaces. So in fact the best way to check the applicability of the new method to any particular resistivity profile, is to investigate the function $Q_n(u)$ (see table 3.I). This function is a polynomial of $u = e^{-2\lambda}$. The numerical values of each term of this polynomial at various values of λ can simply be evaluated with the help of semilogarithmic paper. On such paper these exponential functions of λ , on the linear axis, plot as straight lines, the slopes of which are indicated by the powers of u . Having evaluated the polynomial for various values of λ , it is possible to make a graphical representation of $Q_n(u)$ as a function of λ and so one could check if the condition $0 < Q_n(u) < +2$ is satisfied or not. By way of example this check has been performed in fig. 4.2 for the resistivity profile shown in fig. 4.1.

Fig. 4.1 Resistivity profile

Fig. 4.2 $Q_4(u)$ at the resistivity profile of fig. 4.1

The above test of the applicability is complete and can be performed very rapidly. However an even more rapid but incomplete test can be made if the depth configuration is not involved in the test. This means that the test should

cover even the most unfavourable depth configuration. Thus the incompleteness must be understood as follows. If, for a given sequence of resistivities ϱ_m , the simplified test to be worked out below indicates that $0 < Q_n(u) < +2$ even for the most unfavourable depth configuration, then no further analysis is required for the resistivity profile under consideration. If, on the other hand, the simplified test shows that there is a possibility that for some value of λ $Q_n(u) > 2$ then a complete test is necessary to check whether, depending on the depth configuration, $Q_n(u) > +2$ for some value of λ or $Q_n(u) < +2$ for any value of λ . $Q_n(u) < 0$ is impossible, as will be shown later.

To arrive at the incomplete test the depths d_m in the expressions for $Q_n(u)$ are replaced by the relevant sums of the layer thicknesses $t_1 + t_2 + \dots + t_m$ (see fig. 1.1). By repeated application of the recurrence formula (3.8) in the form of:

$$Q_n(u) = Q_{n-1}(u) - Q_{n-1}(u^{-1})k_{n-1}u^{t_1+t_2+\dots+t_{n-1}} \dots \dots \dots (4.1)$$

the expressions appear in the following form:

$$Q_1(u) = 1 \dots \dots \dots (4.2)$$

$$Q_2(u) = 1 - k_1 u^{t_1} \dots \dots \dots (4.3)$$

$$Q_3(u) = 1 - k_1 u^{t_1} - k_2 u^{t_2}(u^{t_1} - k_1) \dots \dots \dots (4.4)$$

$$Q_4(u) = (1 + k_2 k_3 u^{t_3})(1 - k_1 u^{t_1}) - (k_2 u^{t_2} + k_3 u^{t_2+t_3})(u^{t_1} - k_1) \dots \dots (4.5)$$

$$Q_5(u) = (1 + k_2 k_3 u^{t_3} + k_2 k_4 u^{t_3+t_4} + k_3 k_4 u^{t_4})(1 - k_1 u^{t_1}) - (k_2 u^{t_2} + k_3 u^{t_2+t_3} + k_4 u^{t_2+t_3+t_4} + k_2 k_3 k_4 u^{t_2+t_3+t_4})(u^{t_1} - k_1) \dots \dots (4.6)$$

$$Q_6(u) = (1 + k_2 k_3 u^{t_3} + k_2 k_4 u^{t_3+t_4} + k_2 k_5 u^{t_3+t_4+t_5} + k_3 k_4 u^{t_4} + k_3 k_5 u^{t_4+t_5} + k_4 k_5 u^{t_5} + k_2 k_3 k_4 k_5 u^{t_3+t_4+t_5})(1 - k_1 u^{t_1}) - (k_2 u^{t_2} + k_3 u^{t_2+t_3} + k_4 u^{t_2+t_3+t_4} + k_5 u^{t_2+t_3+t_4+t_5} + k_2 k_3 k_4 u^{t_2+t_3+t_4} + k_2 k_3 k_5 u^{t_2+t_3+t_4+t_5} + k_2 k_4 k_5 u^{t_2+t_3+t_5} + k_3 k_4 k_5 u^{t_2+t_4+t_5})(u^{t_1} - k_1) \dots \dots \dots (4.7)$$

The extreme values of $Q_n(u)$ occur if, depending on the magnitudes of the reflection-coefficients k_m , some of the layer thicknesses t_m take the extreme value $t_m = 0$, so that $u^{t_m} = 1$, and the others are infinite so that $u^{t_m} = 0$ for them. In the first instance the magnitude of a reflection-coefficient is governed by its sign. For example, for a two-layer case with a negative reflection-coefficient k_1 :

$$Q_2(u)_{\max} = 1 - k_1 \text{ for } t_1 = 0$$

$$Q_2(u)_{\min} = 1 \quad \text{for } t_1 = \infty$$

If in this case a third layer is added below with a positive reflection-coefficient k_2 :

Table 4.I Incomplete test of $Q_n(u)$ – Potential extreme values

k_1	$Q_2(u)$		k_2	$Q_3(u)$		k_3	$Q_4(u)$	
	max.	min.		max.	min.		max.	min.
+	1	$1-k_1$	+	$1+k_1k_2$	$(1-k_1)(1-k_2)$	+	$1+k_1k_2+k_2k_3+k_1k_3$	$(1-k_1)(1-k_2)(1-k_3)$
						-	$\frac{1+k_1k_2}{(1-k_1)(1-k_2)(1-k_3)}$	$(1-k_1)(1-k_2)$
			-	$\frac{1}{(1-k_1)(1-k_2)}$	$1-k_1$	+	$\frac{1}{1+k_1k_2+k_2k_3+k_1k_3} \cdot \frac{1}{(1-k_1)(1-k_2)}$	$\frac{(1-k_1)(1+k_2k_3)}{(1-k_1)(1-k_2)(1-k_3)}$
						-	$\frac{1+k_2k_3}{(1-k_1)(1-k_2)(1-k_3)}$	$\frac{1-k_1}{1+k_1k_2+k_2k_3+k_1k_3}$
-	$1-k_1$	1	+	$1-k_1$	$\frac{1+k_1k_2}{(1-k_1)(1-k_2)}$	+	$(1-k_1)(1+k_2k_3)$	$\frac{1+k_1k_2}{1+k_1k_2+k_2k_3+k_1k_3} \cdot \frac{1}{(1-k_1)(1-k_2)(1-k_3)}$
						-	$\frac{1-k_1}{(1+k_1k_2+k_2k_3+k_1k_3)} \cdot \frac{1}{(1-k_1)(1-k_2)(1-k_3)}$	$\frac{1+k_2k_3}{1+k_1k_2} \cdot \frac{1}{(1-k_1)(1-k_2)}$
			-	$(1-k_1)(1-k_2)$	1	+	$(1-k_1)(1-k_2)$	$\frac{1+k_2k_3}{1+k_1k_2+k_2k_3+k_1k_3} \cdot \frac{1}{(1-k_1)(1-k_2)(1-k_3)}$
						-	$(1-k_1)(1-k_2)(1-k_3)$	1

Table 4.II Extreme values of α , as functions of $Q_n(u)$ and p , with their limits

p	$Q_n(u)$	$\alpha_{\max}$	$\alpha_{\min}$
even	$+1 \geq Q_n(u)_{\max} > Q_n(u)_{\min} > 0$	$+1 \geq 1 - \{1 - Q_n(u)_{\max}\}^p > 0$	$+1 > 1 - \{1 - Q_n(u)_{\min}\}^p > 0$
	$+2 > Q_n(u)_{\max} > +1 > Q_n(u)_{\min} > 0$	$+1$	$+1 > 1 - \{1 - Q_n(u)_{\min}\}^p > 0$
	with $1 - Q_n(u)_{\min} > Q_n(u)_{\max} - 1$	$+1$	$+1 > 1 - \{1 - Q_n(u)_{\max}\}^p > 0$
	with $1 - Q_n(u)_{\min} < Q_n(u)_{\max} - 1$	$+1 \geq 1 - \{1 - Q_n(u)_{\min}\}^p > 0$	$+1 > 1 - \{1 - Q_n(u)_{\max}\}^p > 0$
odd	$+1 \geq Q_n(u)_{\max} > Q_n(u)_{\min} > 0$	$+1 \geq 1 - \{1 - Q_n(u)_{\max}\}^p > 0$	$+1 > 1 - \{1 - Q_n(u)_{\min}\}^p > 0$
	$+2 > Q_n(u)_{\max} > +1 > Q_n(u)_{\min} > 0$	$+2 > 1 - \{1 - Q_n(u)_{\max}\}^p > +1$	$+1 > 1 - \{1 - Q_n(u)_{\min}\}^p > 0$
	$+2 > Q_n(u)_{\max} > Q_n(u)_{\min} \geq +1$	$+2 > 1 - \{1 - Q_n(u)_{\max}\}^p > +1$	$+2 > 1 - \{1 - Q_n(u)_{\min}\}^p \geq +1$

$$Q_3(u)_{\max} = 1 - k_1 \text{ for } t_1 = 0 \text{ and } t_2 = \infty$$

$$Q_3(u)_{\min} \text{ is either } 1 + k_1 k_2 \text{ for } t_1 = \infty \text{ and } t_2 = 0$$

$$\text{or } (1 - k_1)(1 - k_2) \text{ for } t_1 = 0 \text{ and } t_2 = 0,$$

so the smallest of these two potential minima should be taken.

This procedure can be continued to any number of layers with any sequence of positive and negative reflection-coefficients k_m . The results of such analysis for any possible sequence of positive and negative reflection-coefficients up to 4 layers are set out in table 4.I as the "Incomplete test of $Q_n(u)$ ". In any partition of the table the potential maxima or minima are given. The maximum equals the greatest of the recorded potential maximum values and the minimum equals the least of the recorded potential minima. It can easily be proved that the potential minima can never be negative.

Application of this incomplete test to the resistivity profile of fig. 4.1, with $k_1 = -0.6$; $k_2 = -0.6$ and $k_3 = +0.5$ yields:

$$Q_4(u)_{\max} = (1 - k_1)(1 - k_2) = 2.56$$

and $Q_4(u)_{\min}$ is either: $1 + k_2 k_3 = 0.7$

$$\text{or: } 1 + k_1 k_2 + k_2 k_3 + k_1 k_3 = 0.76$$

$$\text{or: } (1 - k_1)(1 - k_2)(1 - k_3) = 1.28,$$

so that $Q_4(u)_{\min} = 0.7$.

As $Q_4(u)_{\max} > 2$ the incomplete test for this resistivity profile leaves the possibility of non-applicability of the method and should therefore be followed by the complete test (fig. 4.2).

4.2 Accuracy

A study of the accuracy of the proposed calculation method should arrive at the accuracy of the approximate value of $\frac{\rho_a}{\rho_1}$, to be denoted below as $\left(\frac{\rho_a}{\rho_1}\right)^*$.

Owing to the nature of the method it will be simpler to analyse the accuracy of the approximate value of $\frac{\rho_a}{\rho_1} - 1$, to be denoted as $\left(\frac{\rho_a}{\rho_1} - 1\right)^*$. Anticipating a study of the relative accuracy of $\left(\frac{\rho_a}{\rho_1} - 1\right)^*$, which is

$$\beta = \frac{\left(\frac{\rho_a}{\rho_1} - 1\right)^*}{\frac{\rho_a}{\rho_1} - 1} \dots \dots \dots (4.8)$$

Fig. 4.3 Relation between $(\varrho_a/\varrho_1)^*$ and ϱ_a/ϱ_1 at various values of $\beta = \frac{(\varrho_a/\varrho_1 - 1)^*}{\varrho_a/\varrho_1 - 1}$

fig. 4.3 shows the relation between $\left(\frac{\varrho_a}{\varrho_1}\right)^*$ and $\frac{\varrho_a}{\varrho_1}$ at various values of β .

The study of the relative accuracy of $\left(\frac{\varrho_a}{\varrho_1} - 1\right)^*$ starts with a study of the error introduced in

$$\frac{1}{Q_n(u)} \text{ or } \frac{1}{1 - \{1 - Q_n(u)\}},$$

when approximating it by the series expansion:

$$1 + \{1 - Q_n(u)\} + \{1 - Q_n(u)\}^2 + \dots + \{1 - Q_n(u)\}^{p-1}$$

in p^{th} degree of approximation. This error is:

$$\{1 - Q_n(u)\}^p \cdot \frac{1}{Q_n(u)},$$

which implies that the approximation equals:

$$[1 - \{1 - Q_n(u)\}^p] \cdot \frac{1}{Q_n(u)}.$$

So the relative accuracy α of the approximation of $\frac{1}{Q_n(u)}$ is:

$$\alpha = 1 - \{1 - Q_n(u)\}^p \dots \dots \dots (4.9)$$

The limits for α in p^{th} degree of approximation follow from the extreme values of $Q_n(u)$; these extremes being derived from either a complete or an incomplete test. The maximum and minimum values of α as well as the limits of these extremes are listed in table 4.II for different ranges of $Q_n(u)$ in odd and even degrees of approximation. The extremes for α can easily be visualized by means of the auxiliary fig. 4.4.

Fig. 4.4 Extreme values of α as functions of $Q_n(u)$ and p

4.3 Further analysis

The relative accuracy of the approximation for $\frac{1}{Q_n(u)}$ being established, the next step is to evaluate the accuracy of $(\varrho_a/\varrho_1 - 1)^*$.

As in SCHLUMBERGER arrangement:

$$\varrho_a = -\frac{2\pi}{I} r^2 \frac{dV}{dr} \dots \dots \dots (4.10)$$

and in WENNER arrangement:

$$\varrho_a = \frac{4\pi r \{V(r) - V(2r)\}}{I} \dots \dots \dots (4.11)$$

whereas:

$$V = \frac{\varrho_1 I}{2\pi} \left\{ \frac{1}{r} + 2 \int_0^\infty P_n(u) \cdot \frac{1}{Q_n(u)} \cdot \mathcal{J}_0(\lambda r) \cdot d\lambda \right\} \dots \dots \dots (4.12)$$

it appears that in SCHLUMBERGER arrangement:

$$\frac{\varrho_a}{\varrho_1} - 1 = 2r^2 \int_0^\infty P_n(u) \cdot \frac{1}{Q_n(u)} \cdot \mathcal{J}_1(\lambda r) \cdot \lambda \cdot d\lambda \dots \dots \dots (4.13)$$

and in WENNER arrangement:

$$\frac{\varrho_a}{\varrho_1} - 1 = 4r \int_0^\infty P_n(u) \cdot \frac{1}{Q_n(u)} \cdot \{\mathcal{J}_0(\lambda r) - \mathcal{J}_0(2\lambda r)\} \cdot d\lambda \dots \dots \dots (4.14)$$

For a clear insight into the expressions for $\frac{\varrho_a}{\varrho_1} - 1$ the various components of the integrant have been plotted in fig. 4.5 for SCHLUMBERGER arrangement with the example given in fig. 4.1. Fig. 4.5a shows $\lambda \cdot P_4(u)$, fig. 4.5b shows $\frac{1}{Q_4(u)}$; both are determined by the resistivity profile under consideration. Fig. 4.5c represents the function $\mathcal{J}_1(\lambda r)$ for $r = 1, 2, 4$ and 8 ; the ordinates being determined by the current-electrode spacing r . For values of $\lambda r \rightarrow \infty$, $\mathcal{J}_1(\lambda r)$ approximates: $\sqrt{\frac{2}{\pi \lambda r}} \sin\left(\lambda r - \frac{\pi}{4}\right)$. Owing to the oscillating character of the functions $\mathcal{J}_1(\lambda r)$, (and the same applies to $\mathcal{J}_0(\lambda r)$), the integrant, being the product of the functions of the figs. 4.5a, b and c displays an infinite series of oscillations as shown in fig. 4.5d. This integrant has also been plotted for $r = 1, 2, 4$ and 8 . From this example it will be clear that the greatest error

Fig. 4.5 Analysis of the components of $\int_0^\infty P_n(u) \cdot \frac{1}{Q_n(u)} \cdot J_1(\lambda r) \cdot \lambda \cdot d\lambda$

a. $\lambda \cdot P_4(u)$; b. $\frac{1}{Q_4(u)}$; c. $J_1(\lambda r)$ for $r = 1, 2, 4$ and 8 ; d. $\frac{\lambda \cdot P_4(u)}{Q_4(u)} \cdot J_1(\lambda r)$

in $\frac{\varrho_a}{\varrho_1} - 1$ occurs if the greatest ordinates of the integrant (fig. 4.5d) coincide

with the smallest relative accuracy in the approximation of $\frac{1}{Q_n(u)}$, which in turn occurs at those values of $Q_n(u)$, which differ most from 1. In order to obtain an impression of the values of r for which the worst approximations may be expected, a joint scale has been added below fig. 4.5c, indicating the location of the top of the first positive oscillation of $J_1(\lambda r)$ at various values of r .

The above example has been elaborated in SCHLUMBERGER arrangement. Of course, for WENNER arrangement the same conclusions could be drawn from similar figures.

4.4 Another approximation

The extremes of $\left(\frac{\varrho_a}{\varrho_1} - 1\right)^*$, in SCHLUMBERGER as well as in WENNER arrangement, occur if:

1. all positive oscillations of the integrant are combined with $\alpha_{\max}$, the maximum value of α , and all negative oscillations with $\alpha_{\min}$, the minimum value of α ;
2. all positive oscillations are combined with $\alpha_{\min}$ and all negative ones with $\alpha_{\max}$.

So the upper limit for $\left(\frac{\varrho_a}{\varrho_1} - 1\right)^*$ is:

$$\begin{aligned} \left(\frac{\varrho_a}{\varrho_1} - 1\right)_{\max}^* &= \alpha_{\max} \cdot g(r) \int_0^{\infty} (\text{positive ordinates of the integrant}) d\lambda \\ &+ \alpha_{\min} \cdot g(r) \int_0^{\infty} (\text{negative ordinates of the integrant}) d\lambda \quad \quad (4.15) \end{aligned}$$

where $g(r) = 2r^2$ in SCHLUMBERGER arrangement and $g(r) = 4r$ in WENNER arrangement.

From equation (4.15) it follows that:

$$\left(\frac{\varrho_a}{\varrho_1} - 1\right)_{\max}^* > \alpha_{\min} \cdot g(r) \int_0^{\infty} (\text{integrant}) d\lambda = \alpha_{\min} \left(\frac{\varrho_a}{\varrho_1} - 1\right) \quad . . \quad (4.16)$$

and also:

$$\left(\frac{\varrho_a}{\varrho_1} - 1\right)_{\max}^* > \alpha_{\max} \cdot g(r) \int_0^{\infty} (\text{integrant}) d\lambda = \alpha_{\max} \left(\frac{\varrho_a}{\varrho_1} - 1\right) \quad . . \quad (4.17)$$

and thus:

$$\frac{1}{\alpha_{\min}} \left(\frac{\varrho a}{\varrho_1} - 1 \right)_{\max}^* > \frac{\varrho a}{\varrho_1} - 1 \quad \dots \dots \dots (4.18)$$

and $\frac{1}{\alpha_{\max}} \left(\frac{\varrho a}{\varrho_1} - 1 \right)_{\max}^* > \frac{\varrho a}{\varrho_1} - 1 \quad \dots \dots \dots (4.19)$

Similarly $\left(\frac{\varrho a}{\varrho_1} - 1 \right)_{\min}^*$ is:

$$\begin{aligned} \left(\frac{\varrho a}{\varrho_1} - 1 \right)_{\min}^* &= \alpha_{\min} \cdot g(r) \int_0^{\infty} (\text{positive ordinates of the integrant}) \, d\lambda \\ &\quad + \alpha_{\max} \cdot g(r) \int_0^{\infty} (\text{negative ordinates of the integrant}) \, d\lambda \dots \dots (4.20) \end{aligned}$$

From equation (4.20) it follows that:

$$\left(\frac{\varrho a}{\varrho_1} - 1 \right)_{\min}^* < \alpha_{\max} \cdot g(r) \int_0^{\infty} (\text{integrant}) \, d\lambda = \alpha_{\max} \left(\frac{\varrho a}{\varrho_1} - 1 \right) \quad \dots \dots \dots (4.21)$$

and also:

$$\left(\frac{\varrho a}{\varrho_1} - 1 \right)_{\min}^* < \alpha_{\min} \cdot g(r) \int_0^{\infty} (\text{integrant}) \, d\lambda = \alpha_{\min} \left(\frac{\varrho a}{\varrho_1} - 1 \right) \quad \dots \dots \dots (4.22)$$

and thus:

$$\frac{\varrho a}{\varrho_1} - 1 > \frac{1}{\alpha_{\max}} \left(\frac{\varrho a}{\varrho_1} - 1 \right)_{\min}^* \quad \dots \dots \dots (4.23)$$

and $\frac{\varrho a}{\varrho_1} - 1 > \frac{1}{\alpha_{\min}} \left(\frac{\varrho a}{\varrho_1} - 1 \right)_{\min}^* \quad \dots \dots \dots (4.24)$

It depends on the signs of $\left(\frac{\varrho a}{\varrho_1} - 1 \right)_{\max}^*$ and $\left(\frac{\varrho a}{\varrho_1} - 1 \right)_{\min}^*$,

whether $\frac{1}{\alpha_{\min}} \left(\frac{\varrho a}{\varrho_1} - 1 \right)_{\max}^* > \frac{1}{\alpha_{\max}} \left(\frac{\varrho a}{\varrho_1} - 1 \right)_{\max}^*$ or just the reverse, and whether

$\frac{1}{\alpha_{\max}} \left(\frac{\varrho a}{\varrho_1} - 1 \right)_{\min}^* > \frac{1}{\alpha_{\min}} \left(\frac{\varrho a}{\varrho_1} - 1 \right)_{\min}^*$ or the reverse.

It is of no practical value to evaluate these limits for $\frac{\varrho a}{\varrho_1} - 1$. For practical purposes the following reasoning can be applied. As for convergent cases:

$$+ 2 > \alpha_{\max} > \alpha_{\min} > 0 \quad \dots \dots \dots (4.25)$$

and consequently

$$+\infty > \frac{1}{\alpha_{\min}} > \frac{1}{\alpha_{\max}} > +1/2 \dots \dots \dots (4.26)$$

and
$$\left(\frac{\rho_a}{\rho_1} - 1\right)_{\max}^* > \left(\frac{\rho_a}{\rho_1} - 1\right)^* > \left(\frac{\rho_a}{\rho_1} - 1\right)_{\min}^* \dots \dots \dots (4.27)$$

it follows that:

$$\frac{1}{\alpha_{\min}} \left(\frac{\rho_a}{\rho_1} - 1\right)_{\max}^* > \frac{1}{\alpha_{\min}} \left(\frac{\rho_a}{\rho_1} - 1\right)^* > \frac{1}{\alpha_{\min}} \left(\frac{\rho_a}{\rho_1} - 1\right)_{\min}^* \dots \dots (4.28)$$

and

$$\frac{1}{\alpha_{\max}} \left(\frac{\rho_a}{\rho_1} - 1\right)_{\max}^* > \frac{1}{\alpha_{\max}} \left(\frac{\rho_a}{\rho_1} - 1\right)^* > \frac{1}{\alpha_{\max}} \left(\frac{\rho_a}{\rho_1} - 1\right)_{\min}^* \dots \dots (4.29)$$

These last two lines teach us that $\frac{1}{\alpha_{\min}} \left(\frac{\rho_a}{\rho_1} - 1\right)^*$ and $\frac{1}{\alpha_{\max}} \left(\frac{\rho_a}{\rho_1} - 1\right)^*$ are within the same limits as have been derived for $\frac{\rho_a}{\rho_1} - 1$, in the equations (4.18),

(4.19), (4.23) and (4.24). This does not necessarily imply that $\frac{\rho_a}{\rho_1} - 1$ should be within $\frac{1}{\alpha_{\min}} \left(\frac{\rho_a}{\rho_1} - 1\right)^*$ and $\frac{1}{\alpha_{\max}} \left(\frac{\rho_a}{\rho_1} - 1\right)^*$. However $\frac{1}{\alpha_{\min}}$ and $\frac{1}{\alpha_{\max}}$ both converge to 1 and $\left(\frac{\rho_a}{\rho_1} - 1\right)^*$ converges to $\frac{\rho_a}{\rho_1} - 1$, so that the approximate

apparent resistivity curves represented by $1 + \frac{1}{\alpha_{\min}} \left(\frac{\rho_a}{\rho_1} - 1\right)^*$ and by $1 + \frac{1}{\alpha_{\max}} \left(\frac{\rho_a}{\rho_1} - 1\right)^*$ converge to $1 + \left(\frac{\rho_a}{\rho_1} - 1\right) = \frac{\rho_a}{\rho_1}$, as also does $1 + \left(\frac{\rho_a}{\rho_1} - 1\right)^*$.

If, for some subsequent degrees of approximation, the approximate apparent resistivity curves according to $1 + \left(\frac{\rho_a}{\rho_1} - 1\right)^*$, $1 + \frac{1}{\alpha_{\min}} \left(\frac{\rho_a}{\rho_1} - 1\right)^*$ and $1 + \frac{1}{\alpha_{\max}} \left(\frac{\rho_a}{\rho_1} - 1\right)^*$ are plotted it will soon be clear within which range the apparent resistivity curve ρ_a/ρ_1 is.

For each of the three different sets of approximations it holds that the mutual intersections of the curves of the same convergent set are at or near the correct ordinate $\frac{\rho_a}{\rho_1}$.

In many cases either $\frac{1}{\alpha_{\min}}$ or $\frac{1}{\alpha_{\max}}$ equals 1. So $\frac{1}{\alpha_{\max}} = 1$ in all even degrees of approximation. If $Q_n(u) < 1$, then $\frac{1}{\alpha_{\max}} = 1$ in any degree of approximation, which implies that then actually only two different sets of approximate curves exist.

These simple rules enable the apparent resistivity curve to be sketched sufficiently accurately if it has been calculated to some degrees of approximation. In the practical application of the method however, the resistivity curve is calculated to one degree of approximation only, but mostly it is obvious where the exact curve is located with respect to the approximate curve and its two allied ones.

4.5 Example

The following example illustrates the application of the above derived rules.

Fig. 4.6 Resistivity profile

For the resistivity profile of fig. 4.6 calculations have been performed in 1st, 2nd, 4th and 5th degree of approximation. The incomplete test yielded:

$$Q_{\max} = 1 \text{ and } Q_{\min} = 0.500$$

$$\text{So } \alpha_{\max} = 1 \text{ and } \alpha_{\min} = 1 - (0.500)^p.$$

Therefore in this case actually there are only two sets of approximations, the first set as from the calculations: $1 + \left(\frac{Q_a}{Q_1} - 1\right)^*$, the second set being $1 + \frac{1}{\alpha_{\min}} \left(\frac{Q_a}{Q_1} - 1\right)^*$. These two sets of approximations have been plotted in fig. 4.7 and fig. 4.8. The curve $1 + \left(\frac{Q_a}{Q_1} - 1\right)^*$ with $p = 5$ differs very little from that with $p = 4$. Therefore it has been omitted in fig. 4.7. The curves of the first set intersect practically all in A , those of the second set practically in B .

Left from A and right from B the apparent resistivity curve $\frac{Q_a}{Q_1}$ is enclosed between the two sets of approximate curves. Between A and B , $\frac{Q_a}{Q_1}$ is slightly below the curves of both sets.

Fig. 4.7 Approximate resistivity curves $1 + (\rho_a/\rho_1 - 1)^*$ for the resistivity profile of fig. 4.6

Fig. 4.8 Approximate resistivity curves $1 + \frac{1}{\alpha_{\min}} (\rho_a/\rho_1 - 1)^*$ for the resistivity profile of fig. 4.6

4.6 Internal asymptotes

Another approach to the accuracy of the proposed calculation method is one where the error of the asymptotes and the internal asymptotes is investigated. An internal asymptote of an n -layer curve is defined as the asymptote of the m -layer curve, which comes into being if from the n -layer case all layers below the m^{th} one are replaced by one with the same specific resistivity as the m^{th} layer. The tables 3.II, 3.III, 3.IV and 3.V showing the composition of standard-graphs for 3, 4, 5 and 6 layers show that any n -layer case includes completely and unchanged the $(n-1)$ -layer case. The n^{th} layer only brings about an additional number of lines related with k_{n-1} and d_{n-1} . This means that the internal asymptotes of an n -layer curve converge as the asymptotes of the relevant m -layer curves. The asymptotic value of an m -layer curve in p^{th} degree of approximation is equal to the sum of all coefficients K occurring in the relevant table in p^{th} degree of approximation. From the derivation of the calculation method in chapter 3 it follows that the coefficients K in the column for p^{th} degree of approximation in a table for an m -layer case originate from the coefficients in $P_m(u)$ and in $\{1 - Q_m(u)\}$. If the sums of the coefficients in $P_m(u)$ and in $\{1 - Q_m(u)\}$ are denoted as P_m and $1 - Q_m$, the asymptotic value of an m -layer curve in p^{th} degree of approximation is found to be:

$$\frac{Q_a}{Q_1} = 1 + P_m \{ 1 + (1 - Q_m) + (1 - Q_m)^2 + \dots + (1 - Q_m)^{p-1} \} \dots \quad (4.30)$$

and the relative error in $\frac{Q_a}{Q_1} - 1$ is again $(1 - Q_m)^p$. From the expressions for $Q_m(u)$ (see paragraph 4.2) it appears that:

$$Q_m = (1 - k_1)(1 - k_2) \dots (1 - k_{m-1}) \dots \quad (4.31)$$

This shows afterwards that a great number of the potential extreme values of $Q_n(u)$ as indicated in table 4.I are just the products Q_m governing the convergencies of the asymptote and the internal asymptotes of the resistivity curve.

STUDY OF THE SAVING OF WORK WHEN APPLYING THE NEW CALCULATION METHOD

5.1 Standard of comparison

In paragraph 3.7 it has already been pointed out that the proposed calculation method shows a great deal of similarity both with HUMMEL's method and with EHRENBURG and WATSON's method. In this chapter a study on the saving of work is envisaged. Therefore both the amount of work to be carried out in the application of the new method and that according to EHRENBURG and WATSON's method will be compared with that according to HUMMEL's method. For a fair comparison each of these approximate methods should be applied to such a degree of approximation that the results obtained by them have comparable accuracy.

To arrive at a standard of comparable accuracy: first the pole depths, occurring on the lines of the tables 3.II, 3.III, 3.IV and 3.V have been expressed as functions of the various layer thicknesses t_m instead of the interface depths d_m (see table 5.I). From these tables together the following conclusions can be drawn:

1. The new calculation method applied to an n -layer profile in p^{th} degree of approximation displays poles at depths ranging from $2t_1$ up to $2pd_{n-1}$ inclusive.
2. Any following degree of approximation $p+1$ entails an additional set of poles within the range from $2t_1 + 2pt_s$ up to $2(p+1)d_{n-1}$ inclusive, where t_s is the smallest layer thickness.
3. Addition of an $(n+1)^{\text{th}}$ layer – by inserting a new n^{th} layer of thickness t_n at the top of the substratum which then becomes the $(n+1)^{\text{th}}$ layer – gives rise to another set of poles the depths of which range from $2(t_1 + t_n)$ up to $2pd_n$ inclusive, in p^{th} degree of approximation.
4. HUMMEL's calculation method for three-layer resistivity profiles displays an infinite number of infinite series of poles. Each of these series consists of poles at equal intervals of $2t_1$, ranging from $2(q-1)t_2 + 2t_1$ to infinity, as described in paragraph 2.2. The number of series as well as the deepest pole of each of these series to be taken into account is governed by the desired accuracy.
5. The calculation method of EHRENBURG and WATSON shows one continuous series of poles from the depth $2t_1$, at intervals equal to $2t$, where t is equal

to the greatest common divisor of the various layer-thicknesses t_m (see fig. 1.1). Again the deepest pole to be taken into account is governed by the desired accuracy.

The nature of convergency, when applying the new method, in comparison with the convergencies with the methods by HUMMEL and by EHRENBURG and WATSON, which are similar in this respect, will now be described.

The new method yields in any degree of approximation a curve, the shape of which is similar to the resistivity curve to be calculated. In successive degrees of approximation the various ordinates converge to the correct values. On the other hand any approximation according to EHRENBURG and WATSON's method is a curve which for the smaller values of $r = L/2$ is very accurate, whereas the tail is not of the correct form and often rather far from the correct curve. This is illustrated by fig. 3.3 and fig. 3.4, representing approximate curves for the same resistivity profile, calculated according to the new method and to EHRENBURG and WATSON's method.

Of course, when adjusting the strength of the deepest pole in EHRENBURG and WATSON's method according to KOEFOED the approximation of the tail becomes much better, as shown in fig. 3.5.

In order to fulfill the concept of comparable accuracy the maximum pole depths for calculations according to HUMMEL and to EHRENBURG and WATSON will be chosen at $2pd_{n-1}$, when comparing with a calculation according to the new method for n layers in p^{th} degree of approximation. This choice is made only after a study has been made of the convergency of the asymptotic value of ϱ_3/ϱ_1 for the three-layer cases, indicated by points in the diagram of fig. 5.1. The convergency of this asymptote according to the new method is

Fig. 5.1 Diagram showing the resistivity configurations, for which the convergency of the asymptote has been investigated

$$\begin{array}{c}
 \text{-----} \\
 k_1 = +\frac{1}{2} \frac{\varrho_1 = 1}{\varrho_2 = 3} d_1 = 1 \\
 k_2 = -\frac{1}{2} \frac{\varrho_3 = 1}{\varrho_2 = 3} d_2
 \end{array}$$

Fig. 5.2 Resistivity configuration R

Fig. 5.3 Convergence of the asymptotes of some resistivity curves for the resistivity configuration of fig. 5.2

- a. according to EHRENBURG and WATSON, with $d_2 = 1\frac{1}{2}, 2, 4$ and 8;
- b. according to the new method, in heavy dots

independent of the depth configuration, whereas in the application of the two existing calculation methods several depths of the second interface had to be considered. This study was based on the principle of equal depths of the deepest poles in both types of methods at any degree of approximation. An example is shown in fig. 5.3 for the resistivity configuration R indicated in fig. 5.1 and shown in fig. 5.2. The new method is not applicable to the points within the hatched area of fig. 5.1. Apart from those cases, the convergence according to the new method was generally far better than with the two other systems. Allowing for the better convergence of the two existing calculation methods at smaller electrode distances, the chosen standard of comparable accuracy appears to be reasonable.

5.2 Subdivision of the work

The standard of comparison being established, a quantitative comparison is made between both methods with respect to:

1. the number of lines in the relevant table (3.II, 3.III, 3.IV or 3.V): l
2. the number of image poles: i ($l \geq i$)
3. the number of contributions to the strengths of the image poles: c ($c \geq l \geq i$)
4. the work-index w , defined as $w = c + i$

The work-index is a measure for the amount of work to be carried out when calculating a resistivity curve. This work comprises first the calculation of the strengths and depths of the image poles and secondly the calculation of the various ordinates ϱ_a/ϱ_1 .

In the new method the calculation of the strengths and depths of the image poles is done with the help of the calculation tables no. 1 and no. 2 (tables 3.VI and 3.VII). Any contribution to the strength of an image pole involves one multiplication. The number of multiplications is considered as a criterion for the amount of work involved in this part of the calculation. Thus the time required for this first part of the calculation is roughly proportional to the number c of contributions to the strengths of the image poles.

The same holds in HUMMEL's method for a greater number i of poles and a greater number c of contributions to their strengths, as described in paragraph 3.7.

EHRENBURG and WATSON's method, as improved by WATSON makes use of a recurrence formula for the pole strengths. As described in paragraph 2.2, for each particular resistivity profile the total pole strengths can be calculated as linear functions of a few shallower poles. With three-layer profiles only three shallower poles have to be taken into account, involving only three multiplications. So for three-layer profiles the number of contributions c to the strengths of the image poles is three for each image pole. For the shallowest poles even simpler expressions exist and therefore c is less.

The calculation of the various ordinates ϱ_a/ϱ_1 is conceived as being performed for each one of the three methods considered making use of calculation table no. 3 (table 3.VIII). The contributions to those ordinates due to a number i of image poles must be read with the help of the auxiliary graph of fig. 3.1 and added. This work is proportional to the number of poles i .

Practical experience taught that the time required to calculate the pole strengths and pole depths divided by the time required to calculate ϱ_a/ϱ_1 from these poles, is equal to c divided by i . Thus $w = c + i$ is proportional to the time required to calculate a resistivity curve. This provides a standard for the contemplated study on the saving of work.

5.3 Total amount of work

For a two-layer case in p^{th} degree of approximation: $l = p$; $i = p$; $c = p$; $w = 2p$, with the new method as well as with HUMMEL's system of poles and

with that according to EHRENBURG and WATSON. This can be read from table 3.II.

For three-layer cases this study has been elaborated for p is 1 up to 7 inclusive, at various depths of the second interface d_2 expressed in d_1 as unit. The result is recorded in table 5.II. For any of the selected depth configurations the table shows in five columns successively the numbers l , i , c and w for calculations according to:

1. HUMMEL
2. EHRENBURG and WATSON
3. the new method

and further the ratios of the numbers for l , i , c and w for calculations according to:

4. EHRENBURG and WATSON to those by HUMMEL, expressed as a percentage
5. the new method to those by HUMMEL, expressed as a percentage.

For calculations according to EHRENBURG and WATSON with a correction of the aggregate pole strength as proposed by KOEFOED the numbers l and i remain unchanged. Only c and consequently w could be increased by 1. This is to account for one additional contribution to the strength of the deepest pole.

From table 5.II it may be concluded that both EHRENBURG and WATSON's method as well as the new method save a lot of work compared with HUMMEL's

Table 5.III Study of the amount of work

H = HUMMEL		EW = EHRENBURG and WATSON			Nm = New method	
Depth configuration		$d_1 = 0$ $d_2 = 1$ $d_3 = 2$	also $d_1 = 1$ $d_2 = 1$ $d_3 = 2$	$d_1 = 1$ and $d_2 = 2$ $d_3 = 2$		
Degree of approximation according to the new method	Item	H	EW	Nm	EW/H %	Nm/H %
1st	l	∞	∞	4	100	0
	i	2	2	2	100	100
	c	∞	∞	4		0
	w	∞	∞	6		0
2nd	l	∞	∞	18	100	0
	i	4	4	4	100	100
	c	∞	∞	25		0
	w	∞	∞	29		0
3rd	l	∞	∞	48	100	0
	i	6	6	6	100	100
	c	∞	∞	92		0
	w	∞	∞	98		0

Tabel 5.II Study of the amount of work

H = HUMMEL; EW = EHRENBURG and WATSON; Nm = New method

Depth configuration		$d_1 = 1$ $d_2 = 1$					$d_1 = 1$ (4) $d_2 = 1,25$ (5)					$d_1 = 1$ (2) $d_2 = 1,5$ (3)				
Degree of approximation according to the new method	Item	H	EW	Nm	$\frac{EW}{H}$ %	$\frac{Nm}{H}$ %	H	EW	Nm	$\frac{EW}{H}$ %	$\frac{Nm}{H}$ %	H	EW	Nm	$\frac{EW}{H}$ %	$\frac{Nm}{H}$ %
1st	<i>l</i>	∞	∞	2	100	0	2	2	2	100	100	2	2	2	100	100
	<i>i</i>	1	1	1	100	100	2	2	2	100	100	2	2	2	100	100
	<i>c</i>	∞	∞	2	50	0	3	2	2	67	67	3	2	2	67	67
	<i>w</i>	∞	∞	3	50	0	5	4	4	80	80	5	4	4	80	80
2nd	<i>l</i>	∞	∞	6	100	0	10	10	6	100	60	9	9	6	100	60
	<i>i</i>	2	2	2	100	100	7	7	6	100	86	5	5	5	100	100
	<i>c</i>	∞	∞	7	80	0	19	12	7	63	37	16	10	7	63	44
	<i>w</i>	∞	∞	9	80	0	26	19	13	73	50	21	15	12	71	50
3rd	<i>l</i>	∞	∞	12	100	0	24	24	12	100	50	20	20	12	100	60
	<i>i</i>	3	3	3	100	100	12	12	12	100	100	8	8	8	100	100
	<i>c</i>	∞	∞	16	78	0	54	27	16	50	30	43	19	16	44	30
	<i>w</i>	∞	∞	19	78	0	66	39	28	59	42	51	27	24	53	44
4th	<i>l</i>	∞	∞	20	100	0	45	45	20	100	44	36	36	20	100	50
	<i>i</i>	4	4	4	100	100	17	17	17	100	100	11	11	11	100	100
	<i>c</i>	∞	∞	30	71	0	115	42	30	36	26	89	28	30	31	30
	<i>w</i>	∞	∞	34	71	0	132	59	47	45	36	100	39	41	39	44
5th	<i>l</i>	∞	∞	30	100	0	72	72	30	100	42	56	56	30	100	50
	<i>i</i>	5	5	5	100	100	22	22	22	100	100	14	14	14	100	100
	<i>c</i>	∞	∞	50	65	0	208	57	50	27	24	158	37	50	23	30
	<i>w</i>	∞	∞	55	65	0	230	79	72	34	31	172	51	64	30	30
6th	<i>l</i>	∞	∞	42	100	0	105	105	42	100	40	81	81	42	100	50
	<i>i</i>	6	6	6	100	100	27	27	27	100	100	17	17	17	100	100
	<i>c</i>	∞	∞	77	59	0	339	72	77	21	23	255	46	77	18	30
	<i>w</i>	∞	∞	83	59	0	366	99	104	27	28	272	63	94	23	30
7th	<i>l</i>	∞	∞	56	100	0	140	140	56	100	40	110	110	56	100	50
	<i>i</i>	7	7	7	100	100	32	32	32	100	100	20	20	20	100	100
	<i>c</i>	∞	∞	112	54	0	504	87	112	17	22	384	55	112	14	20
	<i>w</i>	∞	∞	119	54	0	536	119	144	22	27	404	75	132	19	30

$d_1 = 1$ $d_2 = 2$					$d_1 = 1$ $d_2 = 4$					$d_1 = 1$ $d_2 = 8$					$d_1 = 0$ $d_2 = 1$				
H	EW	Nm	$\frac{EW}{H}$ %	$\frac{Nm}{H}$ %	H	EW	Nm	$\frac{EW}{H}$ %	$\frac{Nm}{H}$ %	H	EW	Nm	$\frac{EW}{H}$ %	$\frac{Nm}{H}$ %	H	EW	Nm	$\frac{EW}{H}$ %	$\frac{Nm}{H}$ %
3	3	2	100	67	5	5	2	100	40	9	9	2	100	22	∞	∞	2	100	0
2	2	2	100	100	4	4	2	100	50	8	8	2	100	25	2	2	2	100	100
4	3	2	75	50	6	5	2	83	33	10	9	2	90	20	∞	∞	2	133	0
6	5	4	83	67	10	9	4	90	40	18	17	4	94	22	∞	∞	4	133	0
10	10	6	100	60	15	15	6	100	40	27	27	6	100	22	∞	∞	6	100	0
4	4	4	100	100	8	8	6	100	75	16	16	6	100	38	3	3	3	100	100
17	7	7	41	41	23	17	7	74	30	39	33	7	85	18	∞	∞	7	117	0
21	11	11	52	52	31	25	13	81	42	55	49	13	89	24	∞	∞	10	117	0
21	21	12	100	57	30	30	12	100	40	54	54	12	100	22	∞	∞	12	100	0
6	6	6	100	100	12	12	12	100	100	24	24	12	100	50	4	4	4	100	100
43	11	16	26	37	56	29	16	52	29	96	57	16	59	17	∞	∞	16	100	0
49	17	22	35	45	68	41	28	60	41	120	81	28	68	23	∞	∞	20	100	0
36	36	20	100	56	51	51	20	100	39	90	90	20	100	22	∞	∞	20	100	0
8	8	8	100	100	16	16	16	100	100	32	32	20	100	63	5	5	5	100	100
86	15	30	17	35	112	41	30	37	27	190	81	30	43	16	∞	∞	30	87	0
94	23	38	24	40	128	57	46	45	36	222	113	50	51	23	∞	∞	35	87	0
55	55	30	100	55	77	77	30	100	39	135	135	30	100	22	∞	∞	30	100	0
10	10	10	100	100	20	20	20	100	100	40	40	30	100	75	6	6	6	100	100
50	19	50	13	33	195	53	50	27	26	330	105	50	32	15	∞	∞	50	76	0
60	29	60	18	38	215	73	70	34	33	370	145	80	39	22	∞	∞	56	76	0
78	78	42	100	54	108	108	42	100	39	189	189	42	100	22	∞	∞	42	100	0
12	12	12	100	100	24	24	24	100	100	48	48	42	100	88	7	7	7	100	100
39	23	77	10	32	310	65	77	21	25	525	129	77	25	15	∞	∞	77	68	0
51	35	89	14	35	334	89	101	27	30	573	177	119	31	21	∞	∞	84	68	0
05	105	56	100	53	145	145	56	100	39	252	252	56	100	22	∞	∞	56	100	0
14	14	14	100	100	28	28	28	100	100	56	56	56	100	100	8	8	8	100	100
57	27	112	8	31	464	77	112	17	24	784	153	112	20	14	∞	∞	112	61	0
71	41	126	11	34	492	105	140	21	28	840	209	168	25	20	∞	∞	120	61	0

method. In many cases, especially in the lower degrees of approximation and with ratios t_2/t_1 differing more from one, the new method saves even more work than EHRENBURG and WATSON's method.

Table 5.III shows the results of such a study for some selected four-layer cases in first, second and third degree of approximation. The depth configurations indicated at the head of this table have two interfaces in common and furthermore they are chosen in such a way that either t_1 , t_2 or t_3 equals zero. This application of layers of thickness zero anticipates chapter 6.

A figure ∞ for l , c and w in the columns "H" and "EW" of the tables 5.II and 5.III indicates that these methods are not applicable to the considered depth configuration. This occurs if $d_m = d_{m-1}$ or $t_m = 0$.

5.4 Special advantage of the new method

Actually, in many cases, still greater reductions of the amount of work are obtained when applying the new method. For instance, if a resistivity curve is to be calculated for some depth configuration of a resistivity sequence for which calculation table no. 1 (table 3.VI) has already been prepared on a previous occasion, then the amount of work w , according to the new method is limited to $w = \varepsilon c + i$, where ε is a small fraction of 1. This is because all multiplications have been done already. The various contributions must only be summed up in a different way. This advantage is due to the segregation of pole strengths and pole depths in the new method. It saves a lot of work when calculating sets of standard-graphs.

METHODS OF ENLARGING THE APPLICABILITY OF THE METHOD AND OF SPEEDING UP THE CALCULATIONS

6.1 Principle

Contrary to EHRENBURG and WATSON's method, the new method does not preclude the calculation of resistivity curves for profiles comprising one or even several layers of thickness $t = 0$. At first sight this fact does not seem to be of practical importance as such profiles never occur in practice. However an n -layer resistivity profile comprising n' layers ($n' < n$) of thickness $t = 0$ can be understood as a more general description of an $(n - n')$ -layer case, the equivalent reflection-coefficients of which are determined by those of the original profile as follows:

$$k_m = \frac{\frac{\rho_{m+1} - 1}{\rho_{m+1} + \rho_m}}{\frac{\rho_{m+1}}{\rho_m} + 1} \dots \dots \dots (6.1)$$

So
$$\frac{\rho_{m+1}}{\rho_m} - 1 = k_m \frac{\rho_{m+1}}{\rho_m} + k_m$$

$$(1 - k_m) \frac{\rho_{m+1}}{\rho_m} = 1 + k_m$$

and thus

$$\frac{\rho_{m+1}}{\rho_m} = \frac{1 + k_m}{1 - k_m} \dots \dots \dots (6.2)$$

Consequently

$$\frac{\rho_{m+2}}{\rho_{m+1}} = \frac{1 + k_{m+1}}{1 - k_{m+1}} \dots \dots \dots (6.3)$$

From equation (6.2) and equation (6.3) it follows that:

$$\frac{\rho_{m+2}}{\rho_m} = \frac{(1 + k_m)(1 + k_{m+1})}{(1 - k_m)(1 - k_{m+1})} \dots \dots \dots (6.4)$$

If $t_{m+1} = 0$, the equivalent reflection-coefficient of k_m and k_{m+1} equals:

$$k_m' = \frac{\frac{(1+k_m)(1+k_{m+1})}{(1-k_m)(1-k_{m+1})} - 1}{\frac{(1+k_m)(1+k_{m+1})}{(1-k_m)(1-k_{m+1})} + 1} = \frac{k_m + k_{m+1}}{1 + k_m k_{m+1}} \dots \dots \dots (6.5)$$

This relationship is represented in the diagram of fig. 6.1, showing lines of equal equivalent reflection-coefficient k_m' .

Conversely it is possible to convert a given n -layer resistivity profile into an infinite number of $(n+n'')$ -layer profiles, which actually represent the same

Fig. 6.1 Equivalent reflection-coefficient $k_m' = \frac{k_m + k_{m+1}}{1 + k_m k_{m+1}}$

resistivity profile. This provides a useful means of speeding up the calculation of resistivity curves. To explain this, it has been assumed that the approximate resistivity curves are calculated both for a given n -layer profile and for an equivalent $(n+n'')$ -layer profile. It is also assumed that the curves are calculated to such a degree of approximation that the same amount of work, in

terms of the work-index w , is required in both cases (see chapter 5). A far better accuracy can then be obtained with a properly selected $(n+n'')$ -layer profile than with the original n -layer profile.

6.2 Possibilities

The insertion of layers of thickness $t = 0$, can be realized in three different ways viz:

1. Substitution of an original reflection-coefficient k_m by an equivalent combination of coefficients k_{ma} , k_{mb} , according to fig. 6.1 or even by more than two substitute reflection-coefficients.

For the general two-layer case of fig. 6.2 this principle has been worked out with two substitute reflection-coefficients in fig. 6.2a.

2. Addition of one or several layers of thickness $t = 0$ on top of the upper layer.

Again starting from fig. 6.2 this principle is illustrated in fig. 6.2b for one additional layer and in fig. 6.2c for two of them. Any reflection-coefficient k_0 at these fictive interfaces satisfies the original resistivity profile. So there is an unlimited freedom in the selection of the additional reflection-coefficients.

3. Insertion of a layer of thickness $t = 0$ within one of the original layers.

Obviously this is only justified if the additional reflection-coefficients are selected such that the same resistivity occurs below the extra interface as above it. Thus for two additional reflection-coefficients these coefficients should be $+k_d$ and $-k_d$. For the two-layer case of fig. 6.2, this principle is shown in fig. 6.2d and fig. 6.2e.

Fig. 6.2 Resistivity profile

Fig. 6.2a

Fig. 6.2b

Fig. 6.2c

Fig. 6.2d

Fig. 6.2e

Fig. 6.2f

Fig. 6.2a–6.2f Equivalent resistivity profiles of fig. 6.2 $0 \leq \gamma \leq 1$

Beside these three different methods any combination of them satisfies as well. So, for the same two-layer case, fig. 6.2f represents a combination of the methods 1 and 2.

For the general original three-layer case of fig. 6.3 the following possibilities appear:

- a. with one additional layer:
 - according to method 1: fig. 6.3a and fig. 6.3b
 - according to method 2: fig. 6.3c
- b. with two additional layers:
 - according to method 1: fig. 6.3d
 - according to method 2: fig. 6.3e
 - according to method 3: fig. 6.3h, fig. 6.3i and fig. 6.3j
 - combining the methods 1 and 2: fig. 6.3f and fig. 6.3g.

Fig. 6.3
Resistivity profile

Fig. 6.3a

Fig. 6.3b

Fig. 6.3c

Fig. 6.3d

Fig. 6.3e

Fig. 6.3f

Fig. 6.3g

Fig. 6.3h

Fig. 6.3i

Fig. 6.3j

Fig. 6.3a–6.3j Equivalent resistivity profiles of fig. 6.3 $0 \leq \gamma \leq 1$

In order to arrive at directions for this selection a study must precede of the rate of change of $Q_n(u)$ by insertion of layers of thickness $t = 0$. For the original profile of fig. 6.2: $Q_2(u) = 1 - k_1 u^{t_1}$. $Q_3(u)$ and $Q_4(u)$ for the equivalent profiles, according to the equations (4.4) and (4.5) in a somewhat adapted form, are:

$$\begin{aligned}
 \text{at fig. 6.2a } Q_3(u) &= 1 - k_{1a} u^{t_1} - k_{1b} u^{t_1} + k_{1a} k_{1b} u^0 = 1 + k_{1a} k_{1b} - (k_{1a} + k_{1b}) u^{t_1} \\
 &= (1 + k_{1a} k_{1b}) (1 - k_1 u^{t_1}) = (1 + k_{1a} k_{1b}) Q_2(u) \\
 \text{at fig. 6.2b } Q_3(u) &= 1 - k_0 u^0 - k_1 u^{t_1} + k_0 k_1 u^{t_1} = (1 - k_0) (1 - k_1 u^{t_1}) = (1 - k_0) Q_2(u) \\
 \text{at fig. 6.2c } Q_4(u) &= 1 - k_{0a} u^0 - k_{0b} u^0 - k_1 u^{t_1} + k_{0a} k_{0b} u^0 + k_{0a} k_1 u^{t_1} + k_{0b} k_1 u^{t_1} \\
 &\quad - k_{0a} k_{0b} k_1 u^{t_1} \\
 &= 1 - k_{0a} - k_{0b} + k_{0a} k_{0b} - k_1 u^{t_1} (1 - k_{0a} - k_{0b} + k_{0a} k_{0b}) \\
 &= (1 - k_{0a}) (1 - k_{0b}) (1 - k_1 u^{t_1}) = (1 - k_{0a}) (1 - k_{0b}) Q_2(u) \\
 \text{at fig. 6.2d } Q_4(u) &= 1 - k_a u^{\gamma t_1} + k_a u^{\gamma t_1} - k_1 u^{t_1} - k_a^2 u^0 + k_a k_1 u^{(1-\gamma)t_1} - k_a k_1 u^{(1-\gamma)t_1} \\
 &\quad + k_a^2 k_1 u^{t_1} \\
 &= 1 - k_a^2 - k_1 u^{t_1} (1 - k_a^2) = (1 - k_a^2) (1 - k_1 u^{t_1}) = (1 - k_a^2) Q_2(u) \\
 \text{at fig. 6.2e } Q_4(u) &= 1 - k_1 u^{t_1} - k_a u^{t_1+t_a} + k_a u^{t_1+t_a} + k_1 k_a u^{t_a} - k_1 k_a u^{t_a} - k_a^2 u^0 \\
 &\quad + k_1 k_a^2 u^{t_1} \\
 &= 1 - k_a^2 - k_1 u^{t_1} (1 - k_a^2) = (1 - k_a^2) (1 - k_1 u^{t_1}) = (1 - k_a^2) Q_2(u) \\
 \text{at fig. 6.2f } Q_4(u) &= 1 - k_0 u^0 - k_{1a} u^{t_1} - k_{1b} u^{t_1} + k_0 k_{1a} u^{t_1} + k_0 k_{1b} u^{t_1} + k_{1a} k_{1b} u^0 \\
 &\quad - k_0 k_{1a} k_{1b} u^0 \\
 &= 1 - k_0 + k_{1a} k_{1b} - k_0 k_{1a} k_{1b} - u^{t_1} (k_{1a} + k_{1b} - k_0 k_{1a} - k_0 k_{1b}) \\
 &= (1 - k_0) (1 + k_{1a} k_{1b}) - u^{t_1} (k_{1a} + k_{1b}) (1 - k_0) \\
 &= (1 - k_0) (1 + k_{1a} k_{1b}) (1 - k_1 u^{t_1}) = (1 - k_0) (1 + k_{1a} k_{1b}) Q_2(u)
 \end{aligned}$$

Similarly for the equivalent profiles of the original three-layer profile of fig. 6.3 with $Q_3(u) = 1 - k_1 u^{t_1} - k_2 u^{t_2} + k_1 k_2 u^{t_2}$, the functions $Q_4(u)$ and $Q_5(u)$, according to the adapted equations (4.5) and (4.6) are:

$$\begin{aligned}
 \text{at fig. 6.3a } Q_4(u) &= (1 + k_{1a} k_{1b}) Q_3(u) \\
 \text{at fig. 6.3b } Q_4(u) &= (1 + k_{2a} k_{2b}) Q_3(u) \\
 \text{at fig. 6.3c } Q_4(u) &= (1 - k_0) Q_3(u) \\
 \text{at fig. 6.3d } Q_5(u) &= (1 + k_{1a} k_{1b}) (1 + k_{2a} k_{2b}) Q_3(u) \\
 \text{at fig. 6.3e } Q_5(u) &= (1 - k_{0a}) (1 - k_{0b}) Q_3(u) \\
 \text{at fig. 6.3h } Q_5(u) &= (1 - k_a^2) Q_3(u) \\
 \text{at fig. 6.3i } Q_5(u) &= (1 - k_a^2) Q_3(u) \\
 \text{at fig. 6.3j } Q_5(u) &= (1 - k_a^2) Q_3(u) \\
 \text{at fig. 6.3f } Q_5(u) &= (1 - k_0) (1 + k_{1a} k_{1b}) Q_3(u) \\
 \text{at fig. 6.3g } Q_5(u) &= (1 - k_0) (1 + k_{2a} k_{2b}) Q_3(u)
 \end{aligned}$$

For any greater number of original layers similar relationships can be found.

The above simple rules can be summarized as follows:

1. application of method 1 yields:

$$Q_{n+1}(u) = (1 + k_{ma}k_{mb})Q_n(u) \dots \dots \dots (6.6)$$

2. application of method 2 yields:

$$Q_{n+1}(u) = (1 - k_0)Q_n(u) \dots \dots \dots (6.7)$$

3. application of method 3 yields:

$$Q_{n+2}(u) = (1 - k_d^2)Q_n(u) \dots \dots \dots (6.8)$$

These recurrence formulae provide an efficient means of making $Q(u)$ as close to 1 as possible.

6.3 Application

Considering the original function $Q_n(u)$ it is worthy of note that $Q_n(u)$ tends asymptotically to 1 when λ approaches infinity. From the analysis in paragraph 4.3 and from fig. 4.5 it was clear that, owing to this fact the values of $\varrho_a/\varrho_1 - 1$ at smaller values of $r = L/2$ have only little error. Multiplying $Q_n(u)$ with a factor as indicated by the equations (6.6), (6.7) or (6.8) makes this no longer valid. However as $\varrho_a/\varrho_1 - 1$ at the smaller values of $r = L/2$ is only small, a relatively great error β can be allowed without substantial error in the approximate resistivity curve. Fig. 4.3 shows this clearly. An exception should be made for the case of an additional layer of thickness $t = 0$ on top of the actual upper layer. This case will be described later.

In many cases it is desirable to have the asymptote of the curve at its correct value from the first degree of approximation. Then the value of $Q(u)$ at $\lambda = 0$ (i.e. Q , see paragraph 4.6) should be equal to 1, which can easily be realized with the methods 1 or 3. It is true that the same result can be obtained when applying method 2, but this method has the draw-back that the values of $\varrho_a/\varrho_0 - 1$ in general are not small at the small values of $r = L/2$.

If before calculating it can be foreseen that a resistivity curve will display apparent resistivities which are far from ϱ_1 for a long reach of the curve then it is recommended to add one or more resistivity contrasts on top of the actual upper layer (method 2), so that ϱ_0 is about the same as the apparent resistivity ϱ_a in that reach. Hereby $\varrho_a/\varrho_0 - 1$ is made small for that reach of the curve. So a relatively great error in $\varrho_a/\varrho_0 - 1$ still gives only small error in ϱ_a/ϱ_0 . In the case of at least two additional layers the conditions of ideal convergency of the asymptote can be fulfilled simultaneously. In case of only one layer with ϱ_0 on top, the condition of ideal convergency of the asymptote can still be fulfilled by applying the methods 1 or 3 too.

Application of method 2 as recommended for the above case brings about a bad convergency of the ordinates of the resistivity curve at the smaller values

$r = L/2$. However these ordinates can either be estimated or be calculated without an additional layer at the top.

6.4 Determination of additional reflection-coefficients

The results to be obtained with regard to $Q(u)$ by either of the three methods are the same. For ideal convergency of the asymptote the procedures are as follows:

In method 1: k_{ma} and k_{mb} should satisfy both the equations

$$1 + k_{ma}k_{mb} = \frac{1}{Q_n} \quad \dots \dots \dots (6.9)$$

and $\frac{k_{ma} + k_{mb}}{1 + k_{ma}k_{mb}} = k_m \quad \dots \dots \dots (6.10)$

This system of equations is equivalent to the system consisting of:

$$(1 - k_{ma})(1 - k_{mb}) = \frac{1 - k_m}{Q_n} \quad \dots \dots \dots (6.11)$$

Fig. 6.4 Diagram of: $(1 - k_{ma})(1 - k_{mb}) = 1 - k_m/Q_n$

Fig. 6.5 Graphical determination of two substitute reflection-coefficients k_{ma} and k_{mb}

and
$$\frac{k_{ma} + k_{mb}}{1 + k_{ma}k_{mb}} = k_m \dots \dots \dots (6.10)$$

The diagram of fig. 6.4 shows for the case of two substitute resistivity-contrasts with reflection-coefficients k_{ma} and k_{mb} a set of hyperbolae, satisfying equation (6.11) for values of $\frac{1 - k_m}{Q_n}$ ranging from 0 to 2.

The determination of the substitute reflection-coefficients k_{ma} and k_{mb} is visualized in fig. 6.5 as the intersection of the locus of all combinations k_{ma}, k_{mb} which are equivalent to k_m – according to equation (6.10) (or (6.5)) and represented in fig. 6.1 – with the hyperbola according to equation (6.11). In many cases there are two intersections and consequently two solutions for the combination of k_{ma} and k_{mb} . For practical application one should first estimate k_{ma} and k_{mb} from these two diagrams and proceed by iteration using the equations (6.11) and (6.10). If there are no intersections, as occurs for the greater values of k_m and for the greater values of $\frac{1 - k_m}{Q_n}$, one should take:

$$k_{ma} = k_{mb} = \frac{1 - \sqrt{1 - k_m^2}}{k_m} \dots \dots \dots (6.12)$$

thus making $(1 - k_{ma})(1 - k_{mb})$ as close to $\frac{1 - k_m}{Q_n}$ as possible, or take more than two substitute resistivity contrasts.

In method 2: k_0 follows from

$$1 - k_0 = \frac{1}{Q_n} \dots \dots \dots (6.13)$$

If there is another requirement for Q_0 , expressed by

$$k_0 = \frac{Q_1 - Q_0}{Q_1 + Q_0} \dots \dots \dots (6.14)$$

then:

$$(1 - k_{0a})(1 - k_{0b}) = \frac{1}{Q_n} \dots \dots \dots (6.15)$$

and

$$\frac{k_{0a} + k_{0b}}{1 + k_{0a}k_{0b}} = k_0 \dots \dots \dots (6.16)$$

In this case the determination of k_{0a} and k_{0b} runs as in method 1.

In method 3: k_d follows directly from

$$(1 - k_d^2) = \frac{1}{Q_n} \dots \dots \dots (6.17)$$

This equation can also be satisfied by imaginary values of k_d . It can be proved that the imaginary pole strengths, occurring with imaginary k -values, cancel out.

Compared with the two other methods, method 3 has the disadvantage that two additional layers are needed instead of one to fulfill one requirement.

It should be mentioned here that, when applying method 3 or a combination of some of the above described methods to one resistivity profile, the products of some terms $(1 + k_{ma}k_{mb})$, $(1 - k_0)$ and $(1 - k_d^2)$ may never exceed the value of 2. This is because for the original function $Q_n(u)$: $\lim_{\lambda \rightarrow \infty} Q_n(u) = 1$ whereas the condition $Q(u) < 2$ should be met to assure convergency.

6.5 Examples

The above directions can best be illustrated by the following simple, but most instructive examples. The resistivity profile given in fig. 6.6 is successively replaced by the equivalent profiles in fig. 6.6a, fig. 6.6b and fig. 6.6c. The

$$k_1 = -1 \frac{\varrho_1 = 1}{\varrho_2 = 0} d_1 = 1$$

Fig. 6.6 Resistivity profile

$$k_{1a} = +1/2 \frac{\varrho_1 = 1}{\varrho_{1a} = 3} d_{1a} = 1$$

$$k_{1b} = -1 \frac{\varrho_1 = 1}{\varrho_2 = 0} d_1 = 1$$

Fig. 6.6a

$$k_{1a} = -1 \frac{\varrho_1 = 1}{\varrho_{1a} = 0} d_{1a} = 1$$

$$k_{1b} = +1/2 \frac{\varrho_1 = 1}{\varrho_2 = 0} d_1 = 1$$

Fig. 6.6b

$$k_0 = +1/2 \frac{\varrho_0 = 1/3}{\varrho_1 = 1} d_0 = 0$$

$$k_1 = -1 \frac{\varrho_1 = 1}{\varrho_2 = 0} d_1 = 1$$

Fig. 6.6c

Fig. 6.6a–6.6c Equivalent resistivity profiles of fig. 6.6

Fig. 6.7 $Q(u)$ at the resistivity profiles of the figs. 6.6 and 6.6a, b and c

Fig. 6.8 Approximate resistivity curves for the resistivity profile of fig. 6.6

Fig. 6.8ab Approximate resistivity curves for the resistivity profiles of fig. 6.6a and fig. 6.6b

corresponding $Q_2(u)$ and $Q_3(u)$ curves are drawn in fig. 6.7. The approximate resistivity curves for each of these profiles in some subsequent degrees of approximation are shown in the figs. 6.8, 6.8ab and 6.8c respectively.

A second example, somewhat more complicated is that of the resistivity profile given in fig. 6.9. This profile could be replaced by the equivalent profile of fig. 6.9a.

$$Q_4(u) = 1 + 0.6u^1 - 0.3u^2 + 0.18u^3 + 0.6u^4 - 0.3u^5 - 0.5u^6 \dots \quad (6.18)$$

and $Q_4 = 1.28$. See fig. 6.10.

Fig. 6.8c Approximate resistivity curves for the resistivity profile of fig. 6.6c

Fig. 6.9 Resistivity profile

Fig. 6.9a Equivalent resistivity profile of fig. 6.9

Fig. 6.10 $Q(u)$ at the resistivity profiles of fig. 6.9 and fig. 6.9a

Fig. 6.11 Approximate resistivity curves for the resistivity profile of fig. 6.9

Fig. 6.11a Approximate resistivity curves for the resistivity profile of fig. 6.9a

With $k_0 = +0.219$: $Q_5 = (1 - 0.219) \times Q_4 = 1$ and $\varrho_0 = 0.641$, making $\varrho_a/\varrho_0 - 1$ small.

The function

$$Q_5(u) = 0.781 + 0.468u^1 - 0.234u^2 + 0.141u^3 + 0.468u^4 - 0.234u^5 - 0.390u^6 \quad (6.19)$$

is also plotted in fig. 6.10.

Fig. 6.11 and fig. 6.11a represent the approximate resistivity curves in various degrees of approximation for the resistivity profiles of fig. 6.9 and fig. 6.9a respectively.

The work-indices are:

for the profile of fig.:		6.9	6.9a
in 1 st	degree of approximation	8	15
in 2 nd	degree of approximation	33	74
in 3 rd	degree of approximation	83	—

SUMMARY

The quantitative interpretation of geo-electrical depth-soundings, measured over a horizontally stratified subsoil, is in the first instance a mathematical procedure. This first stage of interpretation yields a resistivity profile consisting of a limited number of horizontal layers of different thickness and different resistivity. The next stage is to correlate the resistivity profile with the available geologic or geohydrologic information as obtained from boreholes.

This thesis concerns the first stage of interpretation. Although some authors (e.g. SLICHTER, PEKERIS, VOZOFF) developed methods to derive the resistivity profile directly from the apparent resistivity curve the current interpretation technique is still an indirect one. This means that a measured resistivity curve is matched with a graph out of a collection of so-called standard-graphs. Standard-graphs are calculated for assumed resistivity profiles. The resistivity profile belonging to the graph of best fit is then ascribed to the subsoil at the location where the apparent resistivity curve has been measured. Collections of standard-graphs have been published by La Compagnie Générale de Géophysique, MOONEY and WETZEL and by FLATHE, whereas several institutions and firms have their own unpublished collections. As the number of possible resistivity profiles is unlimited, any collection of standard-graphs is incomplete. Consequently the need is often felt to enlarge the available number of graphs by calculation. The calculation starts with the potential distribution at the surface due to one pole at the surface of a horizontally stratified medium. This calculation is rather time-consuming.

The existing calculation methods can be subdivided into two groups viz:

1. those making use of an infinite number of image poles at various depths below the surface of a homogeneous earth, producing the same surface potential distribution. Such methods have been developed by HUMMEL (1929) and by EHRENBURG and WATSON (1932, 1934);
2. those starting from the mathematical expression for the potential distribution as given by STEFANESCO (1930). Such methods have been developed by MUSKAT, FLATHE, La Compagnie Générale de Géophysique, MOONEY and WETZEL and by ONODERA.

The present thesis describes a new and simple calculation method, which starting from STEFANESCO's integral arrives at a system of image poles similar to but much simpler than those by HUMMEL and EHRENBURG and WATSON.

The strengths and depths of the image poles for 3-, 4-, 5- and 6-layer resistivity profiles are laid down in tables. In every next degree of approximation an additional group of image poles occurs and new contributions are added to the strengths of the image poles introduced in lower degrees of approximation. When performing this procedure to an infinite degree of approximation the complete system of image poles according to HUMMEL would have been obtained. So, when comparing with HUMMEL's method and taking the deepest poles in both methods at the same depth, the new method displays fewer image poles, the strengths of which are given by much simpler expressions. Compared with EHRENBURG and WATSON's method, the striking difference is the separation of the influences of the interface depths and of the resistivity contrasts on the depths and strengths of the image poles. The pole depths are determined by the interface depths only and the pole strengths by the reflection-coefficients only. This implies that for the calculation of a set of graphs where the parameter is either an interface depth or the magnitude of a resistivity contrast, only part of the calculations has to be repeated, whereas EHRENBURG and WATSON's method requires a complete calculation for every subsequent graph.

In order to facilitate the performance of the calculations, calculation tables have been designed, one for the pole depths and another for the pole strengths. Both tables correspond to the above-mentioned tables for 3-, 4-, 5- and 6-layer resistivity profiles.

The calculation of apparent resistivities requires the multiplication of the calculated pole strengths by the so-called distance factors, which are a function of the depth of the relevant pole and the electrode distance. These functions depend on the electrode configuration, e.g. according to WENNER or to SCHLUMBERGER. As multiplications reduce to summations on logarithmic scale, unit-graphs and complement unit-graphs for the distance factors should be constructed on bilogarithmic scale. This has been done for the two above-mentioned electrode configurations. The contributions to the apparent resistivities due to the various image poles can be read from the unit-graph (or complement unit-graph) on tracing paper over a sheet of bilogarithmic paper. The horizontal position of the unit-graph accounts for the pole depth, the vertical position for the pole strength. The various contributions can be written down and summed up in calculation table no. 3, which has been designed for this purpose.

In the derivation of the new method one condition has been made, namely the denominator $Q_n(u)$ of the kernel function in STEFANESCO's integral should be less than 2. This condition limits the application of the new method somewhat. Also the accuracy of the new approximation method is the better, the closer $Q_n(u)$ is to 1. So by testing the function $Q_n(u)$, the applicability of the

new method can be checked and one gets an impression of the accuracy. It is even possible to trace the less accurate sections of the approximate resistivity graph. For a rapid test of the applicability an incomplete test has been developed. If it yields $Q_n(u) < 2$ the applicability is assured. If it yields $Q_n(u) > 2$ it should be followed by a complete test of $Q_n(u)$.

An impression of the accuracy can also be obtained by checking the convergency of two or more successive approximations. Another system of successive approximations is that where every approximate value $(\varrho_a/\varrho_1 - 1)$ is multiplied by the reciprocal value of either the maximum or the minimum value of $\alpha = 1 - \{1 - Q_n(u)\}^p$, where p is the degree of approximation. The convergency of this system can also be checked, or one could check the degree of agreement of the graphs in corresponding degrees of approximation according to both systems. These checks have been illustrated by means of numerical examples.

It should also be mentioned that the nature of approximation is more appropriate with the new method than with those by HUMMEL or by EHRENBURG and WATSON. Graphs according to the new method display the proper shape from the first degree of approximation, whereas when using one of the two existing methods the accuracy decreases with increasing electrode distance in any degree of approximation.

One chapter has been dedicated to a study on the saving of work, when applying the new calculation method. The total amount of work to be carried out for the calculation of a resistivity graph is thought to be proportional to the number of multiplications in the determination of the pole strengths plus the number of image poles, each of them involving another position of the unit-graph. This criterion being established, the so-called work-index has been introduced. The work-index is the sum of the number of multiplications and the number of image poles. In order to be able to compare the amount of work involved in the various methods a standard of comparable accuracy has been established first. It is assumed that the calculations according to the various methods have comparable accuracy if the deepest poles taken into consideration with those methods have the same depth. With this standard of comparison the work-index has been determined for several depth configurations and several degrees of approximation for HUMMEL's method, EHRENBURG and WATSON's method and the new method. It turned out that both EHRENBURG and WATSON's method and the new method lead to considerable reductions of the work-index compared with HUMMEL's method. The reductions, expressed as a percentage, increase with increasing degree of approximation. In many cases the reductions obtained with the new method were much greater than those by EHRENBURG and WATSON, regardless of the above-mentioned saving of work when calculating sets of graphs.

The above described limitation of the applicability of the new method can be overcome and also the accuracy can be improved by an artifice. By inserting one or even several layers of zero thickness, either on top of the profile or at one of the existing interfaces or at an arbitrary depth within a layer the actual resistivity profile remains unaffected, but all ordinates of the function $Q_n(u)$ are reduced in the same ratio. The ratio depends on the resistivity of the inserted layer only. This suggests a method whereby the resistivity of the inserted layer is selected, such that $Q_n(u)$ is as close to 1 as possible so that the accuracy of the resistivity graph to be calculated is optimal. Sometimes it is desirable to have the asymptote of the resistivity graph at its correct value from the first degree of approximation. This can also be realized by another proper selection of the inserted resistivity. Several examples illustrate these techniques. The insertion of an additional layer leads on the one hand to higher work-indices in corresponding degrees of approximation, on the other hand to a greater accuracy, which makes a lower degree of approximation and consequently a lower work-index capable of giving satisfactory results.

SUMMARY IN DUTCH

De kwantitatieve interpretatie van weerstandskrommen, welke werden verkregen door geo-elektrische metingen in een gebied met horizontaal gelaagde ondergrond, is in eerste instantie een wiskundige procedure. Als resultaat van deze eerste fase van de interpretatie wordt een weerstandsprofiel gevonden bestaande uit een beperkt aantal horizontale lagen van verschillende dikte en verschillende specifieke elektrische weerstand. In de daarop volgende fase wordt het aldus verkregen weerstandsprofiel gecorreleerd met de uit boringen bekende geologische of geo-hydrologische gegevens.

Dit proefschrift heeft betrekking op de eerste fase van de interpretatie. Hoewel verschillende schrijvers (SLICHTER, PEKERIS, VOZOFF) methoden ontwikkeld hebben om het weerstandsprofiel direct uit de weerstandskrommen af te leiden is de gangbare interpretatietechniek nog steeds indirect. Dit wil zeggen, dat de gemeten weerstandskromme tot dekking wordt gebracht met een kromme uit een collectie van zogenaamde standaardkrommen. Standaardkrommen zijn weerstandskrommen die werden berekend voor aangenomen weerstandsprofielen. Het weerstandsprofiel dat behoort bij de passende standaardkromme wordt dan toegeschreven aan de ondergrond ter plaatse van de meting. Er zijn verzamelingen van standaardkrommen gepubliceerd door La Compagnie Générale de Géophysique, MOONEY en WETZEL en door FLATHE. Bovendien beschikken diverse diensten en firma's over eigen, niet gepubliceerde, verzamelingen. Daar het aantal mogelijke weerstandsprofielen onbeperkt is, is geen verzameling van standaardkrommen ooit volledig. Bijgevolg bestaat er dikwijls behoefte aan vergroting van het aantal standaardkrommen door berekening. De berekening gaat uit van de potentiaalverdeling aan de oppervlakte als gevolg van één stroombron aan de oppervlakte van een horizontaal gelaagd medium. De berekening hiervan is zeer tijdrovend.

De bestaande berekeningsmethoden kunnen worden onderverdeeld in twee groepen, te weten:

1. die, waarbij gerekend wordt met een oneindig aantal spiegelpolen op verschillende diepten onder de oppervlakte van een homogene ondergrond, zodanig dat dezelfde potentiaalverdeling aan de oppervlakte ontstaat. Dergelijke methoden zijn ontwikkeld door HUMMEL (1929) en door EHRENBURG en WATSON (1932, 1934);
2. die, welke uitgaan van de wiskundige uitdrukking voor de potentiaalverdeling, zoals die werd afgeleid door STEFANESCO (1930). Dergelijke me-

thoden werden ontwikkeld door MUSKAT, FLATHE, La Compagnie Générale de Géophysique, MOONEY en WETZEL en ONODERA.

In dit proefschrift wordt een nieuwe en eenvoudige berekeningswijze beschreven die, uitgaande van de integraal van STEFANESCO uitkomt op een stelsel van spiegelpolen soortgelijk aan maar veel eenvoudiger dan die van HUMMEL en van EHRENBURG en WATSON. De sterkten en diepten van de spiegelpolen voor weerstandsprofielen van 3-, 4-, 5- en 6-lagen zijn in tabellen vastgelegd. In elke volgende graad van benadering komt er een nieuwe groep spiegelpolen bij en worden er nieuwe bijdragen toegevoegd aan de sterkten van de spiegelpolen die reeds in lagere graden van benadering werden ingevoerd. Indien dit stelsel tot in oneindig grote graad van benadering zou worden uitgewerkt zou het volledige stelsel van spiegelpolen volgens HUMMEL worden verkregen. Dit houdt in dat in de nieuwe methode in vergelijking tot de methode HUMMEL, mits zover doorgevoerd dat de diepste polen volgens beide methoden even diep gelegen zijn, minder spiegelpolen voorkomen, waarvan bovendien de sterkten nog door veel eenvoudiger uitdrukkingen worden gegeven. Vergeleken met de methode van EHRENBURG en WATSON is het opvallende verschil de scheiding van de invloeden van de diepten van de grensvlakken en van de weerstandscontrasten op de diepten en sterkten van de spiegelpolen. De diepten van de spiegelpolen worden uitsluitend door de diepten van de grensvlakken bepaald, de poolsterkten uitsluitend door de contrastfactoren. Dit impliceert dat bij de berekening van een bundel weerstandskrommen, waarvan de parameter hetzij de diepte van een grensvlak is, dan wel de grootte van een weerstandscontrast, slechts een gedeelte van de berekeningen behoeft te worden herhaald, terwijl volgens de methode van EHRENBURG en WATSON voor elke kromme een volledige berekening nodig is.

Teneinde de uitvoering van de berekeningen te vergemakkelijken zijn berekeningstabellen ontworpen, één voor de pooldiepten en één voor de poolsterkten. Beide tabellen sluiten aan bij voornoemde tabellen voor 3-, 4-, 5- en 6-lagen weerstandsprofielen.

Voor de berekening van de schijnbare weerstanden moeten de berekende poolsterkten worden vermenigvuldigd met de zogenaamde afstandsfactoren. Deze worden bepaald door de diepte van de desbetreffende pool en de stroomelektrodenafstand en hangen tevens af van de elektrodenopstelling, zoals bijvoorbeeld volgens WENNER of SCHLUMBERGER. Daar vermenigvuldigingen overgaan in sommaties bij toepassing van een logaritmische schaal moeten voor de afstandsfactoren eenheidskrommen en complementaire eenheidskrommen worden samengesteld op dubbellogaritmische schaal. Dit is gebeurd voor de twee genoemde elektrodenopstellingen. De bijdragen tot de schijnbare weerstanden door de verschillende spiegelpolen kunnen worden afgelezen op

een vel dubbellogaritmisch papier, waaroverheen een op transparant papier getekende eenheidskromme c.q. complementaire eenheidskromme is gelegd. Door de plaats van de eenheidskromme in horizontale richting wordt de pooldiepte in rekening gebracht; door de plaats in verticale zin de poolsterkte. De verschillende bijdragen kunnen worden genoteerd en gesommeerd in berekeningstabel no. 3, die speciaal daarvoor is ontworpen.

In de afleiding van de nieuwe methode is een voorwaarde gesteld, namelijk dat de noemer $Q_n(u)$ van de kernfunctie in de integraal van STEFANESCO kleiner dan 2 zou zijn. Deze voorwaarde beperkt de toepassing van de nieuwe methode enigszins. Ook is de nauwkeurigheid van de nieuwe benaderingsmethode des te beter, naarmate $Q_n(u)$ minder van 1 verschilt. Derhalve kan de toepasbaarheid van de nieuwe methode worden gecontroleerd door de functie $Q_n(u)$ te onderzoeken, waarbij men tevens een indruk krijgt van de nauwkeurigheid. Zelfs is het mogelijk de minst nauwkeurige trajecten van de benaderde weerstandskromme aan te geven. Voor een snel onderzoek naar de toepasbaarheid is een onvolledige toets ontwikkeld. Als de onvolledige toets aangeeft dat $Q_n(u) < 2$ is de toepasbaarheid verzekerd. Als deze toets aangeeft dat $Q_n(u) > 2$ dient de onvolledige toets gevolgd te worden door een volledig onderzoek van $Q_n(u)$.

Een indruk van de nauwkeurigheid kan ook worden verkregen door de convergentie na te gaan van twee of meer opeenvolgende benaderingen. Een ander stelsel van opeenvolgende benaderingen is dat waarbij elke benaderde waarde van $(\rho_a/\rho_1 - 1)$ vermenigvuldigd wordt met het omgekeerde van hetzij de maximum dan wel de minimum waarde van $\alpha = 1 - \{1 - Q_n(u)\}^p$, waarin p de graad van benadering is. De convergentie van dit stelsel kan eveneens worden onderzocht, of men zou de mate van overeenstemming kunnen nagaan van de krommen in een overeenkomstige graad van benadering volgens beide stelsels. Deze controles zijn geïllustreerd met numerieke voorbeelden.

Er dient nog vermeld te worden, dat de aard van de benadering geschikter is bij de nieuwe methode dan bij die van HUMMEL of van EHRENBURG en WATSON. Krommen volgens de nieuwe methode vertonen reeds vanaf de eerste graad van benadering de juiste vorm, terwijl bij de beide gevestigde berekeningsmethoden de nauwkeurigheid afneemt met toenemende elektrodenafstand in elke graad van benadering.

In dit proefschrift is ook een hoofdstuk gewijd aan een onderzoek naar de werkbesparing, die bereikt wordt met de nieuwe berekeningsmethode. De totale hoeveelheid werk, die verricht moet worden voor de berekening van een weerstandskromme is evenredig gesteld aan het aantal vermenigvuldigingen in de bepaling van de poolsterkten vermeerderd met het aantal spiegelpolen, die immers ieder een nieuwe positie van de eenheidskromme met zich meebrengen. Na vaststelling van dit criterium is het zogenaamde werkcijfer ingevoerd. Het

werkcijfer is de som van het aantal vermenigvuldigingen en het aantal spiegelpolen. Teneinde een vergelijking te kunnen trekken tussen de hoeveelheden werk die de verschillende methoden met zich meebrengen werd eerst een norm van vergelijkbare nauwkeurigheid ingevoerd. Daartoe werd verondersteld, dat berekeningen volgens de diverse methoden vergelijkbare nauwkeurigheid hebben indien de diepste polen die daarbij worden beschouwd even diep gelegen zijn. Met deze norm werd het werkcijfer bepaald voor diverse laagdikte-verhoudingen in verschillende graden van benadering voor de methode van HUMMEL, die van EHRENBURG en WATSON en de nieuwe methode. Daaruit bleek dat zowel de methode van EHRENBURG en WATSON als de nieuwe methode tot belangrijke reducties van het werkcijfer leidden ten opzichte van de methode HUMMEL. De reducties, in procenten uitgedrukt, worden groter bij toenemende graad van benadering. In vele gevallen waren de reducties die met de nieuwe methode bereikt werden aanzienlijk groter dan die volgens EHRENBURG en WATSON, afgezien nog van de reeds vermelde werkbeparing bij het berekenen van krommenbundels.

De eerder genoemde beperking in de toepasbaarheid van de nieuwe methode kan worden overwonnen en ook de nauwkeurigheid kan worden verbeterd door een kunstgreep. Door tussenvoeging van één of zelfs meerdere lagen met een dikte nul, hetzij aan de oppervlakte of op een van de bestaande grensvlakken dan wel op willekeurige diepte in een laag, blijft het eigenlijke weerstandsprofiel onveranderd, maar worden alle ordinaten van de functie $Q_n(u)$ in dezelfde verhouding vergroot of verkleind. Deze verhouding hangt uitsluitend van de weerstand van de tussengevoegde laag af. Hieruit volgt, dat het mogelijk is de weerstand van de tussengevoegde laag zodanig te kiezen dat $Q_n(u)$ zo weinig mogelijk van 1 verschilt, opdat de nauwkeurigheid van de te berekenen weerstandskromme optimaal is. Soms is het gewenst de asymptoot van de weerstandskromme reeds vanaf de eerste graad van benadering op zijn juiste plaats te hebben. Dit kan eveneens gerealiseerd worden door een andere geschikte keuze van de tussengevoegde weerstand. Deze technieken worden door verschillende voorbeelden geïllustreerd. De invoeging van een extra laag leidt enerzijds tot een hoger werkcijfer in overeenkomstige graden van benadering, anderzijds tot een grotere nauwkeurigheid, die omgekeerd weer maakt dat met een lagere graad van benadering en bijgevolg een lager werkcijfer reeds bevredigende resultaten worden bereikt.

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LIST OF SYMBOLS

symbol	denotation	dimension
a	potential electrode spacing; in WENNER arrangement electrode spacing	L
c	number of contributions to the strengths of the image poles	—
$d_1, d_2, \dots, d_m, \dots, d_{n-1}$	depth of the 1 st , 2 nd , $\dots$, m^{th} , $\dots$, $(n-1)^{\text{th}}$ interface	L
d_n	depth of an interface added in the substratum	L
d_0	additional interface depth equal to zero	L
d_{1a}	inserted interface depth equal to d_1	L
e	base of Napierian logarithms	—
f	general symbol for either $f_S(r/D)$ or $f_W(r/D)$	—
$f_S(r/D)$	"unit-graph" of distance factors in SCHLUMBERGER arrangement	—
$f_W(r/D)$	"unit-graph" of distance factors in WENNER arrangement	—
$g(r)$	$2r^2$ with SCHLUMBERGER arrangement or $4r$ with WENNER arrangement	L^2 L
h	serial number of an image pole in EHRENBURG and WATSON's system of poles	—
i	number of image poles	—
j	serial-number of a pole in a series of poles according to HUMMEL	—
$k_1, k_2, \dots, k_m, k_{m+1}, \dots, k_{n-1}$	reflection-coefficient at the 1 st , 2 nd , $\dots$, m^{th} , $(m+1)^{\text{th}}$, $\dots$, $(n-1)^{\text{th}}$ interface	—
k_m'	equivalent reflection-coefficient of k_m and k_{m+1}	—
$k_{1a}, k_{1b}; k_{2a}, k_{2b}; \dots; k_{ma}, k_{mb}$	equivalent combination of reflection-coefficients of $k_1, k_2, \dots, k_m$	—
k_0, k_{0a}, k_{0b}	additional reflection-coefficient at interfaces added on top of a resistivity profile	—
$+k_d, -k_d$	additional reflection-coefficients at two coinciding interfaces, inserted in a resistivity profile	—
k_s	reflection-coefficient at the first interface in the elementary three-layer profile according to La C.G.G.	—
l	number of lines in a table	—
m	number of a layer	—
n	total number of layers	—
n'	fraction of n , whole number	—
n''	number of additional layers of zero thickness	—
p	degree of approximation	—
q	serial number of a series of poles according to HUMMEL	—
r	distance from a point source at the surface	L
t	unit layer thickness	L
$t_1, t_2, \dots, t_m, t_{m+1}, \dots, t_{n-1}$	thickness of the 1 st , 2 nd , $\dots$, m^{th} , $(m+1)^{\text{th}}$, $\dots$, $(n-1)^{\text{th}}$ layer	L
t_a	arbitrary thickness	L
t_n	thickness of a layer added at the top of the substratum	L
t_s	smallest layer thickness	L
u	$e^{-2\lambda}$	—
w	work-index	—
x	variable	—
y	function of x	—
A_v	coefficient, used by ONODERA	—
D	general symbol for the depths of the image poles	L
$D(x_i, r)$	"prime element" according to FLATHE	—
$H_2(u), \dots, H_{n-1}(u), H_n(u)$	function occurring in the denominator of the kernel function Θ_1 with $2, \dots, n-1, n$ layers	—

symbol	denotation	dimension
I	strength of electric current	I
J_0	Bessel function of the first kind and zero order	—
J_1	Bessel function of the first kind and the first order	—
K	general symbol for the strengths of the image poles	—
$\bar{K}$	general symbol for the sum of the strengths of the image poles at equal depth	—
L	current electrode spacing	L
$P_2(u), \dots, P_m(u), \dots, P_{n-1}(u), P_n(u)$	numerator of the kernel function Θ_1 with $2, \dots, m, \dots, n-1, n$ layers	—
P_m	value of $P_m(u)$ at $u = 1$ or $\lambda = 0$	—
$Q_2(u), Q_3(u), \dots, Q_m(u), \dots, Q_n(u), Q_{n+1}(u), Q_{n+2}(u)$	denominator of the kernel function Θ_1 with $2, 3, \dots, m, \dots, n, n+1, n+2$ layers	—
$Q_2, Q_3, \dots, Q_m, \dots, Q_n$	value of $Q_2(u), Q_3(u), \dots, Q_m(u), \dots, Q_n(u)$ at $u = 1$ or $\lambda = 0$	—
$Q_{\max}, Q_{\min}$	maximum, minimum value of $Q_n(u)$	—
$S_{2t}, S_{4t}, \dots, S_{2ht}$	strength of the image pole at depth $2t, 4t, \dots, 2ht$ in EHRENBURG and WATSON's system of poles	—
V	surface potential due to one point source at the earth surface	V
ΔV	potential difference between two potential electrodes in WENNER arrangement	V
α	relative accuracy of the approximation of $1/Q_n(u)$ (Eq. (4.9))	—
$\alpha_{\max}, \alpha_{\min}$	maximum, minimum value of α	—
α_i	coefficient to be applied to FLATHE's prime element $D(\alpha_i, r)$	—
β	relative accuracy of $(\varrho_a/\varrho_1 - 1)^*$. (Eq. (4.8))	—
γ	coefficient ($0 \leq \gamma \leq 1$)	—
ε	fraction of 1 ($0 \leq \varepsilon \leq 1$)	—
κ_i	reflection-coefficient at the first interface in the elementary resistivity profile of FLATHE's prime element	—
λ	integration variable	L^{-1}
ν	whole number, used by ONODERA	L
π	3,1416	—
$\varrho_1, \varrho_2, \dots, \varrho_m, \varrho_{m+1}, \dots, \varrho_n$	resistivity of the 1 st , 2 nd , $\dots$, m^{th} , $(m+1)^{\text{th}}$, $\dots$, n^{th} layer	RL
ϱ_a	apparent resistivity	RL
ϱ_s	resistivity of the second layer in the elementary three layer profile according to La C.G.G.	RL
$\varrho_{1a}, \varrho_{2a}$	resistivity of an additional layer of zero thickness inserted at the base of the original 1 st , 2 nd layer	RL
ϱ_0	resistivity of one or the upper additional layer of zero thickness added on top of a resistivity profile	RL
ϱ_{0a}	resistivity of a second additional layer of zero thickness, added on top of a resistivity profile	RL
ϱ_d	resistivity of an additional layer of zero thickness added somewhere in a resistivity profile	RL
$(\varrho_a/\varrho_1)^*$	approximation of ϱ_a/ϱ_1	—
$(\varrho_a/\varrho_1 - 1)^*$	approximation of $\varrho_a/\varrho_1 - 1$	—
$(\varrho_a/\varrho_1 - 1)_{\max}^*, (\varrho_a/\varrho_1 - 1)_{\min}^*$	maximum, minimum value of $(\varrho_a/\varrho_1 - 1)^*$	—
Σ	summation	—
Θ_1	kernel function, according to STEFANESCO. As only the index 1 occurs it has been subsequently omitted in order to make room for another index referring to the number of layers n , so that	—
Θ_n	kernel function according to STEFANESCO for an n -layer resistivity profile	—

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ANNEX A.

```
'begin' write('e-73/18-11-63/harkema/v.d. hoeven/resistivitycurves');
```

```
'begin' integer i,j,s,n; 'real' ro3,ro2,k1,k2;
'array' a[1:12],uit[1:12,0:7],d[1:7];
'procedure' vasko; 'code';
```

```
read(n,a); d[1]:=1; d[2]:=2; d[3]:=3;
d[4]:=5; d[5]:=1/2; d[6]:=1/3; d[7]:=1/5;
'for' j:=1 'step' 1 'until' 12 'do'
'begin' uit[j,0]:=a[j]; a[j]:=2/a[j] 'end';
'for' i:=1 'step' 1 'until' n 'do'
```

```
'begin' 'real' som; 'integer' t; 'array' q[-7:400];
read(ro2,ro3); q[0]:=1;
'for' j:=-7 'step' 1 'until' -1 'do' q[j]:=0;
k1=(ro2-1)/(ro2+1); k2=(ro3-ro2)/(ro3+ro2);
'for' t:=1 'step' 1 'until' 7 'do'
```

```
'begin' 'integer' x,s,v; 'boolean' b0; b0:= 'true';
'if' d[t] 'less' 0.9 'then' 'go to' hoeve1;
b0:= 'false'; x:=d[t]; j:=0; 'go to' hoevez;
```

```
hoeve1: x:=1/d[t]; j:=x-1;
'for' v:=1 'step' 1 'until' j 'do'
q[v]:=0;
```

```
hoevez: j:=j+1;
'if' 'not' b0 'and' abs(j-x) 'less'
0.5 'then'
'begin' q[j]:=q[j-1]*k1;
j:=j+1
'end';
q[j]:=k2*q[j-1-x]+k1*('if' b0 'then'
q[j-x]-k2*q[j-1] 'else' q[j-1]-k2*q[j-x]);
'if' abs(q[j]) 'greater' 0.00025 'or'
j 'less' 30 'then' 'go to'
hoevez;
```

```
'for' s:=1 'step' 1 'until' 12 'do'
'begin' 'real' ad,y; som:=0;
ad:=a[s]*('if' b0 'then' d[t] 'else' 1);
'for' v:=0 'step' 1 'until' j 'do'
'begin' y:=(v*ad) 'power' 2;
som:=som+q[v]*(4/sqrt(1+y)-4/sqrt(4+y));
'end';
```

```
uit[s,t]:=som-1
'end' einde s-lus;
'end' einde d-lus, results are put out using the procedure
vasko (Delft);
'end'
'end'
'end'
```