Water Induced Crack Initiation in an Aquifer.
An Elastic Analytical Approach.

Master Thesis of D.P.J. Oostveen

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Preface

This report is the final result of the graduation project “Water Induced Crack Initiation in an Aquifer, An Elastic Analytical Approach.” (Department of Water Management, Faculty of Civil Engineering and Geosciences, Delft University of Technology). This graduation project is part of the NOVEM-BSE project “Increasing the maximum well discharge by enlarging the allowable injection pressures for energy storage in the subsurface”.

I wish to thank the graduation committee and members of the NOVEM-BSE project for their support, advise and feedback. They proved to be a valuable source of information. I also wish to thank NOVEM-BSE without whom this project wasn’t possible in the first place.
Summary

In the drinking water industry as well as in the heat and cold storage industry water is injected into the soil. When a too high infiltration pressure is applied, cracks occur, resulting in collapse of the well by leakage. In order to determine this maximum infiltration pressure the Olsthoorn Rule is used. In the field however, the impression has risen that the Olsthoorn Rule is too strict.

The Olsthoorn Rule assumes that cracks will occur in the aquifer when the minimum principal stress is reduced to zero. At that moment the water is assumed to form a wedge and causes cracks in the soil. This point is reached if the injection pressure is equal to the horizontal effective stress. However, this appears to be somewhat unlikely because no pressure head exists transversal to the flow direction. Also the way in which the horizontal effective stress is determined is open to improvement. Because the horizontal effective stress is significantly reduced during the construction of the well and the development of stresses after placing well is unknown, the minimal known value of horizontal stress of sand (the active soil pressure) is used.

The literature does not give a solution for this particular case. For similar cases, however in poorly permeable soil, huge differences in given solutions and in measurements occur. This has been described to the strong dependency of these cases to the horizontal stress. On basis of this literature study the complete problem was considered too broad for this master thesis and one aspect, the crack initiation in the aquifer was singled out. The other aspects, crack initiation and stress development (induced by swelling) in the clay ball and confining clay layer and crack propagation through the clay ball were left out of this study.

The crack initiation in the aquifer has been investigated using the cavity expansion theory. The aquifer is proposed to be a homogeneous elastic axial symmetric disc bound to either plane strain or plane stress conditions. The infiltration point is located at its centre. The disc is loaded by a seepage force induced by the radial infiltration. This seepage force will cause a radial deformation of the disc. An annular crack will initiate when the effective radial stress at the inner boundary reduces to zero. Until this moment no displacement can occur at the inner boundary. Therefore, the crack initiation criterion for annular cracks is the infiltration pressure at which the radial effective stress at the inner boundary reduces to zero under condition of no displacement at the inner boundary. Since sand is not completely elastic also plasticity might occur. This study does not deal with this phenomenon but if the radial effective stress is reduced to zero, plastic behaviour in non-cohesive soils also causes the tangential effective stress to reduce to zero. This makes the soil around the well also vulnerable to radial cracks.

If the outer boundary goes to infinity the crack initiation criterion reduces to the Olsthoorn Rule. This unfortunately gives no reduction of the Olsthoorn Rule.

The minimal horizontal stress however can be determined more precisely by measuring the active soil pressure with a CPT-test. This could lead to a significant increase of the estimated horizontal effective stress. It has however to be kept in mind that the crack initiation in the confining clay layer might become prevalent. This crack initiation can also be determined with the Olsthoorn Rule. However in clay the active soil pressure cannot be measured and has to be estimated at its lower end. This still results in a possible increase of the infiltration pressure up to 34%.
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# Notations

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<th>Description</th>
<th>Unit</th>
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<tr>
<td>$A$</td>
<td>Cross-sectional area</td>
<td>[m$^2$]</td>
</tr>
<tr>
<td>$a$</td>
<td>Relation between the inner and outer boundary</td>
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</tr>
<tr>
<td>$C$</td>
<td>Integration Constants</td>
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<td>Young’s modulus</td>
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<td>$F$</td>
<td>Body force</td>
<td>[kN m$^{-1}$]</td>
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<td>$f$</td>
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<td>$N^*_s$</td>
<td>Bearing capacity factor</td>
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Introduction

Both in the drinking water industry and in the cold and heat storage industry infiltration wells are used to inject water into the soil. When a too high infiltration pressure is applied the well will collapse. As a collapse criterion the Olsthoorn Rule is used. In the field however the impression has risen that this rule is too strict. Therefore a NOVEM-BSE project “Increasing the maximum well discharge by enlarging the allowable injection pressures for energy storage in the subsurface” has initiated to investigate whether the Olsthoorn Rule could be loosened. This master thesis is made within the framework of this project.

In the context of this graduation project an orientating literature study has been made which is incorporated in this report. The literature revealed some deficiencies of knowledge. The precise process of crack initiation in a non-clogged aquifer was unknown. The exact developments of tensions in the confining clay layer and above all in the clayball were also unknown. Because in the course of this literature study the subject appeared to be to broad this master thesis is limited to the limiting factor, the behaviour of the aquifer. Therefore in this report the first deficiency is solved and a theory for the crack initiation in an elastic aquifer has been derived based. In this solution the aquifer is modelled as an infinite completely elastic layer in both plane strain and plane stress conditions. The plastic behaviour is not taken into account.

In order to make this report more readable the literature study and the more theoretical part have been put in the appendices.
In chapter 1 the problem is analysed resulting into the research question and goals. The literature study is summarised in chapter 2 and the full study is given in appendices A to D. In chapter 3 the process of solving the research question is dealt with and a possible relaxation of the Olsthoorn Rule is presented (For a more detailed discussion see also appendix E). Finally in chapter 4 and 5 the conclusions and recommendations are given.
1 Problem analysis

1.1 Problem description

Infiltration wells (see Figure 1) are used to inject water into the ground. The main application areas are drinking water storage and treatment, cold and heat storage for energy conservation purposes and recharging wells around underground construction sites. The scope of this study will be narrowed to infiltration into a confined aquifer (sealed with a clay ball where the well has perforated the confining clay layer) and thus excluding most of the recharging wells around underground construction sites. If the injection pressure is too high the well will collapse because of crack-induced leakage. On the surface this collapse is observed when water exfiltrates alongside the shaft or somewhere in the field around the infiltration well. To determine this point of collapse the Olsthoorn Rule is used. In the field the impression has however risen that the Olsthoorn Rule is too strict and could even be loosened by a factor 2. Since the design rule is rather conservative, collapse does not occur frequently and is mostly caused by a failing safety system. Therefore almost no measurement data of the collapse conditions are known.

The Olsthoorn Rule is often the limiting factor in well design. Loosening it could therefore mean a possible increase of the capacity of most individual wells and thereby the increase of the feasibility of planned infiltration wells. For the heat-cold storage this will lead to an increase of durable energy.

Figure 1 Cross-section of an infiltration well
1.2 **Main research question**

The main goal of this project is to investigate the possibility of improving the Olsthoorn Rule for infiltration well design. The aim of the research can best be formulated in the form of a main research question:

*Can the Olsthoorn Rule for design of infiltration wells be loosened and if necessary, which design rule should replace it?*

1.3 **Sub questions**

In order to derive an answer to this question a literature study has been made (See chapter 2 and appendices A to D). In this literature study an answer to the following questions has been sought:

1. What is an injection well, how is it constructed and what is its operating regime? (See appendix A, page 25)
2. What is the background and validity of the Olsthoorn Rule? (See appendix B, page 35)
3. What, if any, other rules for this particular problem can be found in the literature? (See appendix C, page 38)
4. Which mechanisms influence the behaviour of soil and groundwater, and how can they be mathematically described? (See appendix D, page 42)

This literature study revealed that the mechanisms on which the Olsthoorn Rule was based were not valid for a permeable layer. It also showed that there were huge uncertainties concerning the exact stresses in the sand layer. The swelling behaviour of the clay ball and the influences it has on stresses in the clay layer are also uncertain. Because of the size of the deficiencies this master thesis will only deal with the sand layer. Therefore, only an answer to the following questions has been sought:

a. At what infiltration-pressure do cracks initiate in a non-clogged homogeneous elastic aquifer? (See chapter 3.1)
b. Can a better estimation for the horizontal stress in the sand layer be made? (See chapter 3.2)
c. Can the design rule on basis of the previous answers be loosened? (See chapter 3.3)

1.4 **Research goal**

The goal of this research can best be described as:

*Replace the Olsthoorn Rule by a more accurate design rule by taking into account the behaviour of a non-clogged sand layer and a more accurate estimation of the horizontal stresses in this sand layer.*
1.5 Assumptions

In order to give answers to the research questions these following assumptions have been made:

1. The layers are all homogeneous and elastic.
2. The layers continue into infinity.
3. The thickness of a layer does not vary.
4. The well contains no construction errors and the clay-ball in the confining layer is well applied and seals perfectly.
5. The Olsthoorn Rule can be used to determine the crack-birth in the confining clay layer.

Ad 1-3: In the field the layers are obviously not homogeneous and infinite, but these inhomogeneities cannot be taken into account in an analytical solution. Besides measuring these inhomogeneities in the field is impractical if not virtually impossible.

Ad 4: It is well known that problems can occur when the clay-ball is not applied properly. However if the clay ball is applied with care no problems should occur and it can safely be assumed that the clay ball to seals perfectly.

Ad 5: As has been stated above, the mechanism described by the Olsthoorn Rule is still applicable for the clay layer. Since however the horizontal stress in the clay layer is unknown, also the conservative estimation of the horizontal stress made by Olsthoorn is considered to be valid. This assumption is valid although rather conservative since the estimation by Olsthoorn is very conservative.
2 Summary of the literature study

Infiltration wells are used to inject water into the ground. This is done either for drinking water storage and treatment, cold and heat storage or recharging purposes. During the drilling process the soil around the well is destressed and the refilling of the well will only restress the soil to a certain extent. Since the exact amount of restressing is unknown (and moreover because for most applications no special measurements are taken to increase the restressing) we can only assume the minimum elastic horizontal stress possible:

\[ \sigma_h^' = K_{activ} \sigma_v^' \]  

(2.1)

The clay ball that seals the pierced confining clay layer will swell, but since the extent is unknown this cannot be taken into account and should be examined. This is however considered out of scope for this study but it should be studied in another study. (See also appendix A, page 25)

A high infiltration pressure in infiltration wells can cause the well to collapse. To determine the maximum infiltration pressure the Olsthoorn Rule is used. In the field the impression has risen that this rule is too strict and that the actual infiltration capacity of a well is much higher and could even exceed a factor 2.

The Olsthoorn Rule generally stated that the maximum infiltration pressure might not be higher than the minimum horizontal stress:

\[ \Delta p \leq \sigma_h^' = K_{activ} \sigma_v^' \]  

(2.2)

Since \( K \) is unknown the minimum possible value for \( K_{activ} \) is assumed (0.22 for sand and 0.3 for clay).

The Olsthoorn Rule is originally developed to determine the maximum infiltration pressure that can be used to remediate a clogged well [Olsthoorn 1982]. A later addition has been made to include non-clogged wells [NN 1985]. This addition however assumes that when the water pressure elevation is higher than the least principal stress the water will form a wedge and push the grains aside causing a fissure. In an aquifer water can flow freely. Therefore transversal to the flow direction no gradient of the pressure head exists and therefore no force pushing the soil aside. This rule is therefore not applicable as a crack birth criterion in an aquifer. In an aquitard however it is applicable. (See also appendix B, page 35.)

Literature gives several theories for the development of water-induced fractures. But all these theories are in poorly permeable soils. Therefore these fracturing models are not applicable for the specific case. Within these models strong variations appear. Bjerrum [1972] attributes these variations to variations in horizontal effective stress. (See also appendix C, page 38.) In general cracks are considered to occur when one of the principal stresses is reduced to zero.

An absolute sealing is the moment at which uplift occurs, when the water pressure is higher than the downward pressure of the soil above the aquifer:

\[ \Delta p \leq \sigma_v^' \]  

(2.3)
3 Fracture initiation in the aquifer

3.1 Basic theory

In order to determine the moment at which a crack appears in the aquifer an elastic cavity expansion calculation has been made. The complete derivation and Maple output can be found in appendix E (page 63). The basic idea and the result of this calculation are given below.

For this calculation the aquifer has been modelled as a perfectly elastic homogeneous disk (see Figure 2) that is either bound to plane strain or plane stress conditions. With plane strain the strain in the vertical is taken constant and with plane stress the stress in the vertical is taken constant. The actual field situation is somewhere in-between, the deeper one gets the more it converges to the plane strain case. Since the infiltration wells are not too close to the surface the actual field situation will be closer to plane strain than to plane stress. For completeness however also the plane stress situation has also been elaborated.

Combining the axial symmetric equilibrium equation with the stress-strain relation and the groundwater flow formula of Thiem results into a set of expressions for the radial and tangential stresses. As derived in appendix E this results for the plane strain case in:

\[
\sigma_r'(r) = -\frac{1}{2} \frac{C_1}{r^2} - \frac{1}{2} \frac{C_0 \ln(r)}{\nu - 1} + C_2 \\
\sigma_\theta'(r) = \frac{1}{2} \frac{C_1}{r^2} - \frac{1}{2} \frac{(\nu + \ln(r))C_0}{\nu - 1} + C_2
\]

(3.1)

(3.2)
And for plane stress case we get:

$$\sigma_r (r) = -\frac{1}{2} \frac{C_1}{r^2} + \left( \nu + \frac{1}{2} \right) C_0 \ln (r) + C_2$$  (3.3)\\

$$\sigma_\theta (r) = \frac{1}{2} \frac{C_1}{r^2} + \left( \nu - \frac{1}{2} \right) C_0 + \left( \nu + \frac{1}{2} \right) C_0 \ln (r) + C_2$$  (3.4)

Where $C_1$ and $C_2$ are integration constants dependent on the boundary conditions and $C_0$ follows from the Formula of Thiem for water pressure change induced by a well:

$$p (r) = - \frac{Q}{2\pi kD} \ln \left( \frac{r}{r_{\max}} \right) = -C_0 \ln \left( \frac{r}{r_{\max}} \right)$$  (3.5)

In order to know the precise form of the stress development the boundary conditions are needed. At the outer boundary either the total displacement or the stress increment is taken zero. So at a large distance of the well the influence of this well disappears. The crack in the form of a cavity between the well casing and the surrounding soil will initiate when the inner boundary starts to move outward. So as long as there is no displacement at the inner boundary there is no crack. Therefore at the inner boundary the displacement has been taken zero.

These boundary conditions result in expressions for $C_1$ and $C_2$:

For plane strain with no displacement at $r_{\max}$:

$$C_1 = -\frac{r_{\max}^2 r_{\min}^2 (2\nu - 1) \ln \left( \frac{r_{\min}}{r_{\max}} \right) C_0}{(1 - \nu) r_{\max}^2 - (1 - \nu) r_{\min}^2}$$  (3.6)\\

$$C_2 = -\frac{1}{2} C_0 \frac{(1 - \nu - \ln (r_{\max})) r_{\max}^2 - (1 - \nu - \ln (r_{\min})) r_{\min}^2}{(1 - \nu) r_{\max}^2 - (1 - \nu) r_{\min}^2}$$  (3.7)

For plane strain with no stress increment at $r_{\max}$:

$$C_1 = \frac{\left( 2\nu^2 - 3\nu + 1 + (2\nu - 1) \ln \left( \frac{r_{\min}}{r_{\max}} \right) \right) r_{\min}^2 r_{\max}^2 C_0}{(2\nu^2 - 3\nu + 1) r_{\min}^2 + (1 - \nu) r_{\max}^2}$$  (3.8)\\

$$C_2 = \frac{1}{2} C_0 \frac{(2\nu^2 - 3\nu + 1 - (1 - 2\nu) \ln (r_{\min})) r_{\min}^2 + \ln (r_{\max}) r_{\max}^2}{(2\nu^2 - 3\nu + 1) r_{\min}^2 + (1 - \nu) r_{\max}^2}$$  (3.9)
For plane stress with no displacement at \( r_{\text{max}} \):

\[
C_1 = C_0 \left( \frac{(2\nu - 1)\ln \left( \frac{r_{\text{min}}}{r_{\text{max}}} \right)}{r_{\text{max}}^2 - r_{\text{min}}^2} \right) (3.10)
\]

\[
C_2 = \frac{1}{2} C_0 \left( \frac{2\nu - 1 \left( r_{\text{max}}^2 - r_{\text{min}}^2 \right) + (2\nu^2 + \nu - 1) \left( \ln \left( r_{\text{min}} \right) r_{\text{min}}^2 - \ln \left( r_{\text{max}} \right) r_{\text{max}}^2 \right) + 2\nu \ln \left( \frac{r_{\text{min}}}{r_{\text{max}}} \right) r_{\text{min}}^2}{(1 - \nu) \left( r_{\text{min}}^2 - r_{\text{max}}^2 \right)} \right)
\] (3.11)

For plane stress with no stress increment at \( r_{\text{max}} \):

\[
C_1 = \left( \frac{(2\nu^2 + \nu - 1)\ln \left( \frac{r_{\text{min}}}{r_{\text{max}}} \right) - 2\nu + 1}{(1 + \nu) r_{\text{max}}^2 + (1 - \nu) r_{\text{min}}^2} \right) C_0
\] (3.12)

\[
C_2 = \frac{(2\nu - 1) r_{\text{min}}^2 + (1 + 3\nu + 2\nu^2) \ln \left( r_{\text{max}} \right) r_{\text{max}}^2 + (1 + \nu - 2\nu^2) \ln \left( r_{\text{min}} \right) r_{\text{min}}^2 - 2\nu \ln \left( \frac{r_{\text{min}}}{r_{\text{max}}} \right) r_{\text{min}}^2}{(1 + \nu) r_{\text{max}}^2 + (1 - \nu) r_{\text{max}}^2}
\] (3.13)

For all four calculated cases we can now calculate and plot the development of the stresses (see Figure 4).

![Figure 4](image-url)

**Figure 4** Development of radial (R) and tangential (T) stresses with \( r_{\text{min}} = 0.1, r_{\text{max}} = 1000, \sigma_0 = -100 \) and \( \nu = 0.3 \) for: A: plane strain with no displacement at \( r_{\text{max}} \), B: plane strain with no stress increment at \( r_{\text{max}} \), C: plane stress with no displacement at \( r_{\text{max}} \) and D: plane stress with no stress increment at \( r_{\text{max}} \). (See also appendix E, Figure 48, page 67 (A); Figure 54, page 72 (B); Figure 57, page 74 (C) and Figure 62, page 78 (D) for larger graphs)
We can see that the water infiltration causes radial stresses to decrease and tangential stresses to increase around the well. We can also see that when the radial stress increment is taken zero at \( r_{\text{max}} \) the radial stress is zero at \( r_{\text{max}} \) and that when the displacement at \( r_{\text{max}} \) is taken zero the radial stress will increase at \( r_{\text{max}} \) because the soil is kept in its place where it tends to move outward. This appears to be most strongly the case in the plane stress case.

We compare the case for permeable layer (with plane strain with no stress increment at \( r_{\text{max}} \) with the cavity expansion case for an impermeable layer loaded at the inner boundary with a certain water pressure (see Figure 5, see also Appendix D, page 42). We see that the shapes are somewhat similar but the radial and tangential stresses are swapped.

For the clogged case only changes in the stress-state occur when the infiltration pressure exceeds the horizontal stress \( (\sigma_{h}) \). At this moment an annular gap appears and when the water pressure increases the gap will grow and the tangential stress decreases until it reaches zero and cracks in radial direction occur. In the non-clogged case the stresses will react immediately if the infiltration pressure is increased and the radial stress will decrease until it reaches zero. At that moment a annular gap will occur. At that moment the soil around the well has already become plastic and in the cohesionless case also the tangential stress has become zero, making the soil also vulnerable for cracks in radial direction.

Figure 5. Development of radial stresses (A; R), tangential stresses (B; T) and both stresses (C) in a clogged well (left) and an unclogged well (right). See also appendix D (Figure 33, Figure 34 and Figure 35 on page 55 and further) and E (Figure 52, Figure 53 and Figure 54 on page 71 and further) for more detailed graphs.
The loading caused by the infiltration causes a tensile force. Soil, just like concrete, cannot endure tensile forces and will crack when tensile forces appear. Because before loading there exists a certain horizontal pressure in the ground, there will be, just like in the tension zone of a pre-stressed concrete beam, a decrease of the pressure until it is reduced to zero. At this moment the soil will move away from the well casing and an annular crack will appear. So as long as the radial stress at the inner boundary does not exceed the initial horizontal pressure in the no crack exists:

\[-\Delta \sigma_r \leq K \sigma_v\]  \hfill (3.14)

In order to determine this point we now have to calculate at what infiltration pressure the radial stress at the inner boundary is reduced to zero. If the outer boundary goes to infinity all four approaches give the same ‘crack-birth’ criterion for cracks in the aquifer:

\[\Delta p_{\text{well}} = K \sigma_v\]  \hfill (3.15)

This ‘crack-birth’ criterion is equal to the Olsthoorn Rule but based on a different and more appropriate collapse mechanism. It is highly dependent on the initial horizontal stress in the aquifer. This horizontal stress is however unknown and difficult to measure\(^1\). Especially because due to drilling the soil is deloaded. Before the well is refilled the radial pressure at the edge is reduced to the drilling fluid pressure. The refilling process causes the soil to destress a little, but since no special measures are taken, this will probably only increase the radial stress to a point not much higher than the minimal horizontal stress. This minimal horizontal stress is about 0.22 to 0.3 times the vertical stress for sand and 0.3 to 0.5 times the vertical stress for clays. If special measures were taken and the soil is densified (e.g. by vibrating the well casing during and after refilling the shaft), the horizontal stress is expected to return to the original horizontal stress before drilling.

---

\(^1\) With the LCPM the horizontal pressure can be measured. This method is however not standard and expensive. It could however be used for research purposes in order to measure the behavior off stresses around the well.
3.2 The minimum horizontal stress

This minimum horizontal stress is dependent on the internal friction angle ($\phi$):

$$K_{\text{activ}} = \frac{1 + \sin(\phi)}{1 - \sin(\phi)} \quad (3.16)$$

The internal friction angle can in non-cohesive soils, like sand, be measured with a CPT-test. The bearing capacity factor ($N^*_q$) is related to the vertical stress and the measured cone resistance ($q_c$):

$$N^*_q = \frac{q_c}{\sigma'_{v}} \quad (3.17)$$

For the relation between the bearing capacity factor and the internal friction angle several empirical relations exists (see Figure 6) they can be found in most geotechnical handbooks. (e.g. those by van Tol [1996] and the CROW [2004])

![Figure 6 Relation between the bearing capacity factor $N^*_q$ and the internal friction angle at deep penetration [CROW 2004]](image)

With these relations we can estimate the minimum horizontal stress in the aquifer and the initial estimated value for $K_{\text{activ}}$ in the aquifer (of 0.22) can be measured with a CPT-test. This
can, dependent on the actual situation, result in a significant increase of the value for $K_{active}$ and therefore also of the maximum infiltration pressure.

The $K_{active}$ of the confining clay layer, however, cannot be measured since clay is a cohesive material. Therefore the $K_{active}$ value of clay has to be estimated at about 0.3. If the $K_{active}$ of the sand-layer exceeds this value, the collapse mechanism of the confining clay layer might become predominant.

### 3.3 Design rule

If the internal friction angle is not measured with a CPT-test the value for $K_{active}$ has to be estimated at its lower end, resulting in the formula that is currently used:

$$\Delta p_{well} = 0.22 \sigma_v'$$

(3.18)

When the internal friction angle is measured with a CPT-test, this can lead to a significant increase of the maximum infiltration pressure. However if the measured value of $K_{active}$ is higher than 0.3, the collapse of the confining clay layer might become prevalent.

$$\Delta p_{well} = K_{active} \sigma_v' \quad (with \ K_{active} \leq 0.3)$$

$$\Delta p_{well} = 0.3 \sigma_v' \quad (with \ K_{active} \geq 0.3)$$

(3.19)

This means a increase of the infiltration pressure is possible up to 36%.
3.4 Critical Remarks

The crack in the aquifer may not lead to well collapse, since it is not certain if the crack will propagate through the confining clay layer. The mechanism of crack propagation is left out of this study. In order to determine the moment of collapse this has to be studied. The moment of crack penetration will be somewhere between the moment of crack initiation in the aquifer and the moment at which a crack will initiate into the confining layer on its own. For the moment of crack initiation in the confining layer the Olsthoorn Rule applies. Since the minimal horizontal stress in a clay layer is higher than that of a sand layer ($K$ is about 0.3) this will mean that there is a maximum increase of the capacity of about 36%. If the clay ball swells this can possibly even be more but in order to quantify this further research is needed.

The performed calculation is completely based on the assumption that the soil behaves elastic. So plasticity is not taken into account. Plasticity occurs if the radial stress decreases to a point where the relation between this stress and the vertical stress reaches the Mohr-Coulomb criterion (see appendix D, page 42). From this point on the soil behaves plastic, and the relation between the radial stress and the vertical stress remains given by the Mohr-Coulomb criterion. This means that either the vertical stress will be pulled down by the decrease of the radial stress, or the vertical stress pushes the radial stress up (or a situation in-between both extremes). In any case radial stress does not decrease more than in the elastic model, and the plastic behaviour can only have a positive (if any) influence on the crack-birth criterion.

Another criterion of collapse that is often used in combination with the elastic cavity expansion theory is the point at which the relation radial and tangential reaches the Mohr-Coulomb criterion and the soil becomes theoretically plastic. However, since normally the vertical stress is much higher the relation between the horizontal stresses and the vertical stress will reach the Mohr-Coulomb criterion sooner. This causes the soil to be already plastic before this criterion is reached. There is also no possible mechanism for crack development given, it is only assumed. Just like the point of collapse.

For the determination of the crack-birth criterion the limit of $r_{\text{max}}$ to infinity has been taken. One could also choose a certain distance of the well for $r_{\text{max}}$, where the influence of the well is considered to be neglectable. If this means that some point has to be chosen arbitrary, a certain sound basis would lack. However this point can sometimes be based on natural boundaries or neighbouring wells in a well field, which gives it the required sound basis.

In this calculation the annular crack is expected to initiate at the transition zone between the casing and the filter gravel. The crack could however also initiate between the filter gravel and the aquifer. For $r_{\text{max}}$ goes to infinity the crack initiation criterion also applies for this second case since $r_{\text{min}}$ drops out of the formula.
4 Conclusions

- The Olsthoorn Rule is not suitable for the calculation of the maximum capacity of infiltration wells since it is based on a collapse mechanism that cannot occur in a permeable medium.

- The literature does not give a solution for this particular case. For similar cases, however in poorly permeable soil, huge differences in given solutions and in measurements occur. This has been described to the huge dependency of these cases to the horizontal stress. This horizontal stress however is difficult to determine.

- The moment of crack-birth of an annular gap in the aquifer can be determined by using the cavity expansion theory to determine the moment at which tension appears around the well. Crack-birth occurs if the infiltration pressure is equal to the horizontal soil pressure: \( \Delta p_{well} = \sigma_h' = K \sigma_v' \). The horizontal soil pressure however is difficult to determine and therefore the minimum possible horizontal pressure is taken (\( K_{activ} \sigma_v' \)). This formula is equal to the Olsthoorn Rule if for \( K_{activ} \) the minimum value for sand is taken (0.22).

- The \( K_{activ} \) value however can in non-cohesive materials, like sand, be estimated with the CPT-test. This can in most cases result in an increase of the maximum infiltration pressure, since at the moment a very conservative value (of 0.22) is used. However one has to keep in mind that another collapse mechanism might become prevalent. In this case a crack might develop in the confining clay layer.

- It is not certain if and when the crack in the aquifer will propagate through the confining layer. When the moment of crack-propagation is later or does not occur at all the Olsthoorn Rule can be loosened. Others have already studied the propagation of cracks, but since it has been considered out of the scope of this graduation project it has to be sorted out in a later study.

- If the aquifer-crack does not occur or propagate, another crack will initiate in the confining layer if the infiltration overpressure is equal to the horizontal soil pressure in the confining layer. Clay is a cohesive material and therefore the \( K_{activ} \) value of this layer cannot be determined with a CPT-test and remains unknown. Therefore the minimum value for \( K_{activ} \) in clays is used (\( K_{activ} = 0.3 \)). So if the measured \( K_{activ} \) in the sand layer exceeds 0.3 the collapse of the clay layer becomes prevalent and the infiltration pressure might not exceed \( 0.3 \sigma_v' \). This still means a possible increase of the infiltration pressure up to 36%.

- As a design rule we can now use (where \( K_{activ} \) is taken 0.22 if it is not measured):

\[
\begin{align*}
\Delta p_{well} &= K_{activ} \sigma_v' \quad (\text{with } K_{activ} \leq 0.3) \\
\Delta p_{well} &= 0.3 \sigma_v' \quad (\text{with } K_{activ} \geq 0.3)
\end{align*}
\]  

(4.1)
The horizontal pressure in the confining clay layer is strongly dependent on the swelling capacity of the clay ball. Also the way the clay ball is applied can influence the swelling capacity. Since these exact influences are unknown the exact positive effect of the swelling on the horizontal pressure is unknown. A higher horizontal pressure in the clay layer can lead to a further loosening of the Olsthoorn Rule and therefore further research into the exact behavior of the clay ball is needed.
5 Recommendations

- It is advisable to determine the value of $K_{act}$ with a CPT-test. In most cases this value will be higher than the value that is normally used in the Olsthoorn rule.

- In this study the moment of crack-birth in the aquifer is examined. It is assumed that this crack will immediately propagate through the confining clay layer and cause leakage. It is however uncertain if this is the case. Therefore the precise moment of crack-penetration through the confining layer has to be sorted out. There are some scientific studies on this subject, which could be a good lead.

- If the crack does not penetrate directly the precise stress development of the clay ball needs to be examined since then the horizontal stress in the confining layer determines the maximum infiltration capacity. In order to assess the development of clay balls both experimental and model studies can be used.

- The tension in the soil especially in and around the clay ball can be measured with an LCPM. This device can measure the horizontal pressure in the soil. It is however an expensive and not standard measurement device.

- If the LCPM proves to be too expensive one could try to measure the tension development in the soil by attaching several strain gauges to the well pipe and measuring its deformation due to the stress development of the clay ball. From these measured deformation the stress increment on the well pipe can be calculated.

- With a computer model the process both the destressing caused by drilling and the relaxation caused by swelling of the clay ball can be calculated.
Literature

Baumann, K., Niehhuens, B., Tholen, M., Treskatis, C. [2003]

“Hydraulic fracturing in field permeability testing”, *Geotechnique*, 22, pp. 319-332.

Boekelman, Ir. R.H. et al. [2001]
Geohydrology I, lecture notes CT4420, Delft University of Technology.

Boekelman, R.H., Maas, C., Bolier, G., Kop, J., Gehrels, J.C. [2003]
Geohydrological survey, lecture notes CT5330, Geohydrology II, Delft University of Technology.

CROW [2004]
Handboek Zandboek, CROW - Ede.

“Similarity solutions for drained and undrained cavity expansions in soils”, *Geotechnique*, 44(1), pp. 21-34.

De Josselin de Jong, ir G. and Geertsma, ir. J. [1953]

Toepassing van ruimte expansie theorie bij horizontaal boren, Grondmechanica Delft, SE-701088.

Detay, M. [1997]

IF-technology [2004]

Knorr, W. [2000]

Lehr, J., Hurlburt, S., Gallagher, B. and Voytek, J. [1988]

Havinga, H.R., Petschl, R.O., and Hergarden, H. [1995]
Berekening van de stabiliteit van een diep boorgat in zand, rapport project stabiliteit diepe schachten, Grondmechanica Delft, Rapport nr.: CO-356670/5.
Kuijvenhoven, C., Roo, K de, Schutte, M., Spies, J., Velthorst, A., Wessels, A. [2003]
Bentoniet zwelt het of zwelt het niet, inventariserend onderzoek naar het gedrag van bentoniet als deze in contact komt met de verontreiniging per, concept, Saxion Hogeschool IJselland.

Maas, C. [2003]
Syllabus pompproeven, Delft University of Technology.


Murdoch, L.C. [2002]

Murdoch, L.C. [1993]

Nieheus, B., Baumann, K., Tholen, M., and Treskatis, C. [2003]
“Qualitätskriterien für Abdichtungssuspensionen im Brunnenbau”, *bbbr Fachmagazin für Brunnen- und Leitungsba*, 04/03, pp. 24-32.

NN [1985]

Olsthoorn, Ir. T.N. [1982]
Verstopping van Persputten, Werkgroep Persputten, Rijswijk, KIWA.

Rubbert, T. [2003]

Ruygrok, P., Van den Berg, P. [1997]
Scheurvorming in grondconstructies, een voor GD nieuwe activiteit?, concept, GeoDelft.

Suthersan S.S. [1999]

Terratech [s.a.]
Mikolit sealing pellets, Optimum sealing, Easy and efficient application, leaflet of the manufacturer Terratech, Sittard, The Netherlands.

Terzaghi, K., Peck, R.B. [1967]

Timoshenko, S.P., and Goodier, J.N. [1970]
Trimpin, M. [2003]
“Umläufigkeiten im Ringraum”, bbr Fachmagazin für Brunnen- und Leitungsbaus, 05/03, pp. 24-31.

Vaziri, H.H., and Byrne, P.M. [1990]

Van Tol, Prof.ir. A.F. [1996]
Funderingstechnieken, lecture notes CTWA303, Delft University of Technology.

Verruijt, Prof. Dr. Ir. A. [s.a.]
Grondmechanica, vijfde druk, Delft University Press.

Wicklein, A. and Steussloff, S. et al. [2002]

Yu, Hai-Sui [2000]
Appendix A: Infiltration wells

A.1 Use

Infiltration wells are used to inject water into the ground. There are three main applications for infiltration wells. They are used in the drinking-water industry both for water storage (see Figure 7) and water treatment (e.g. in the Dutch dunes).

The second use is in the energy industry for cold and heat storage (Figure 8).

One application is that in the summer, when there is an excess of heat, water is pumped up and heated with either direct solar energy or by using it as cooling water for the air-conditioning system (normally 20º to 25ºC) and subsequently infiltrated into the ground. When there is a shortage of heat this water is pumped up, and used to preheat for instance the central heating system. In this way precious energy can be saved. Another application involves the infiltration of cold water (normally about 5ºC) in the winter and the exfiltration for cooling systems in the summer.

Also higher temperatures up to 90ºC are applied. The advantage of these high temperatures is that the water can be directly used for heating. But since water of a higher temperature is more reactive a higher water-quality standard is needed. This requires the use of a water treatment plant. Warm water is lighter and therefore due to buoyancy it tends to rise. When the water is again extracted also cold water is pumped up. This effect increases with the increase of temperature. Another not directly technical disadvantage of high-temperature infiltration is that it gives rise to more discussion at the licensing stage. Obviously licensing authorities want to be sure that the high storage temperatures do not have a negative effect on the environment. These factors increase the costs of high-temperature infiltration.
A third application for infiltration wells is at recharging groundwater near extraction wells around underground construction sites. Since the recharge is used to restore the natural water level, the infiltration pressure will not be as large as with injection wells. These wells are also shallower than the other types of infiltration wells. Therefore these wells are not taken into account.

These different applications have different regimes. Where the infiltration wells for water treatment only infiltrate the water, the wells for heat storage also have to exfiltrate the water since the heated water that has been infiltrated also needs to be exfiltrated when the heat is needed. But not only the flow directions can differ with the seasons also the discharge may vary, for heat storage the well capacity is somewhere between 50 and 150 m$^3$/h but depends strongly on the availability (or lack) of heat.

In this study we will mainly be looking at the heat storage application but some of the results could also be used for other applications

**A.2 Layout**

Infiltration wells consist of several components. A cross-section of an infiltration well for heat storage with common dimension ranges is given in Figure 9.

The pump is located in a pump chamber just below the surface. A well casing with a diameter of 160 to 315 mm is connected to the pump. At the end of this well casing the filter casing is situated. This filter casing has slits to allow water to be either infiltrated or exfiltrated. The height of the filter casing depends on the needed discharge and the thickness of the aquifer and varies normally between 10 an 40 meters. Around the filter casing filter gravel is applied. This rather course material enhances the permeability and flow velocity and decreases well clogging. Directly above the filter, a clayball with a minimum height of about 3 meter is situated. This clayball seals the penetrated impermeable layer. The other impermeable layers are also sealed with a clayball, and the remainder is filled with refill gravel.

![Figure 9 Cross-section infiltration well](image-url)
A.3 Construction

Drilling methods

There are several methods to drill a well. In principle there are two main categories. One which displaces the soil while the other removes the soil. The first method has the disadvantage that it is only suited for small boreholes. It also densifies the soil and inserting the filter might prove to be problematic. Therefore this method is unsuited for drilling wells. A stress analysis for this kind of drilling is given by Murdoch [Murdoch 2003].

The second method removes the soil. This method can be divided into several sub categories:

Pulse drilling

This method consists of hoisting a heavy tube (c.a. 1.5 m. in length) with a non-return valve and dropping it on the terrain that has to be pierced. When drilling in solid formations no feed pipes are needed. When drilling in soft unconsolidated soil feed pipes are normally necessary. Every now and then the drilling bit has to be brought to the surface in order to clean the shaft. This causes the drilling speed to decrease with depth from 10 to 20 m/day for the first 50 meters to about 1 to 10 m/day for the deeper part. Pulse drilling can be used for the construction of infiltration wells but since it is slow and ill adapted for soft or unstable terrain and casing are needed it is not often used.

Wash drilling

The formation is loosened by the eroding force of a hydraulic jet, inserted into the borehole with an injection nozzle. The soil material is transported to the surface by the water stream. Sometimes feed pipes are used for great depths. This is a quick method that is used for shallow wells for monitoring purposes and dewatering of building pits.

Figure 10 Pulse drilling [Boekelman et al. 2003]
Rotary drilling

Rotary drilling is the most common method. A drill bit is attached to the end of a string of screwed drill pipes. This assembly is driven either mechanically or hydraulically with a rotary movement into the ground. For the upper 5 to 15 meters feed pipes might be needed, beneath the borehole is supported by the drilling fluid. As drilling fluid either clear water, bentonite mud or mud with a synthetic biodegradable polymer is used. There are basically two methods to insert the drilling fluid. At the first method, straight flush, the drill fluid is injected through the hollow drill pipe, and resurfaces alongside the shaft. With the second method, reverse circulation, the whole process is reversed, the drilling fluid is injected alongside the shaft, and resurfaces through the drill pipe.

Straight-flush

This method is quick with a drilling speed of 200 m/day. Since a certain flow rate is needed to uplift the removed soil the borehole diameter is limited to about 10 to 50 cm. Because the debris is transported alongside the borehole wall this can become clogged, decreasing the capacity of the future well, even if only water is used as a drilling fluid.

Reverse circulation

This method is with its 20 to 50 m/day slower than the straight-flush method, but since the upward flow rate is not influenced by the borehole diameter, bigger diameters can be established. Also since the debris is transported through the drill pipes the borehole wall wonot get clogged (if water is used as a drill fluid). Therefore this method is, although it is rather expensive, mostly used for construction of infiltration (and exfiltration) wells.

Other drilling methods

Also other drilling methods such as auger borings [Terzaghi and Peck, 1967], the Benoto process or down-the-hole (DTH) hammer drilling [Detay 1997] might be used, but are not common practice.
The clayball

After placing both the filter and well casing the remaining hole is filled. Around the filter-casing filter gravel is applied. The pierced impermeable layers are sealed with a clayball in order to avoid leakage from one aquifer into another. This clayball consists of either bentonite based granulate or grouts (cement-clay mixture). The remainder is filled with refill gravel. (see Figure 9). The fill material is normally loosely dumped, so no high horizontal stress will develop in the borehole. Therefore the horizontal stress in the filter gravel is not expected to be much higher than the active earth pressure. The horizontal stress in the clayball depends on the swelling properties of the used material. The clayball directly above the filter has normally a minimum height of about 3 m. If the clay layer is large, a clayball of 3 m. height is applied both at the bottom and at the top of the layer. In between refill gravel is applied.

Bentonite based granulate

In the Netherlands bentonite based granulates are most commonly used for sealing in- and exfiltration wells. The precise composition and application of the clayball varies. For the clayball direct above the filter a higher quality material (e.g. micoliet300 or better) is used and for the other seals a lesser material is used (e.g. micoliet100). The swell pressure of bentonite is in the order of 60 kPa in laboratory conditions. Because bentonite starts swelling immediately when it gets in contact with water, it is difficult to apply at greater depths (> 200 m.). In that case grout is used. Another disadvantage of bentonite is that it tends to crack in combination with halogen compounds (tri and per).

There has not been much research done to the behaviour of bentonite barriers. A recent experiment by students of the Saxion College in cooperation with Tauw [Kuijvenhoven et al. 2003] (see Table 1) indicates that the most commonly applied granulate micolit300 does not swell. This could be one of the weak points of the infiltration well. In order to verify this result more research will be needed, especially in connection to the effect on the permeability and affixation. (Further investigation is not within the scope of this graduation project.) When micolit300 proves problematic other granulates containing more natrium-bentonite should be used.

<table>
<thead>
<tr>
<th>Material</th>
<th>Swell capacity</th>
<th>Time influence on swell</th>
<th>Influence of PER</th>
<th>Influence of salt</th>
</tr>
</thead>
<tbody>
<tr>
<td>QSE 700%</td>
<td>+ +</td>
<td>None</td>
<td>None</td>
<td>Negative</td>
</tr>
<tr>
<td>Mikolit300</td>
<td>-</td>
<td>Negative</td>
<td>Negative</td>
<td>Negative</td>
</tr>
<tr>
<td>Cebogel Premium</td>
<td>+ +</td>
<td>None</td>
<td>Negative</td>
<td>None</td>
</tr>
<tr>
<td>QSM 200%</td>
<td>+</td>
<td>None</td>
<td>None</td>
<td>Negative</td>
</tr>
</tbody>
</table>

Table 1 Representation of the results of the Saxion-College-research [Kuijvenhoven et al. 2003]

Another question that remains is the in situ swelling stress induced by the swelling of the granulate. For the swelling process water is needed. Therefore if not enough water can be attracted the maximum swelling capacity can not be reached. This might be a problem in the centre of the clay ball since the permeability decreases when the clay-ball swells.
Commonly used bentonite based granulates are (see also Figure 14):

*Micolit100:* Is not exactly a bentonite but a tertiary clay from Limburg (Montford) with about 20%-30% smectite. The swell capacities are minimal but according to the manufactures the granule swells sufficiently to transform into a well sealing clay layer, similar to “in-situ” clay layers (see also Figure 13).

*Micolit300:* The same as micoliet100 but enriched with 30% natrium-bentonite. The swell capacity is greater than that of micoliet100 and brings absolute sealing according to the manufacturer (see Figure 13), but the Saxion-HBO research indicates that it does not swell but further research is needed to prove these results especially since the research has been made for educational purposes and not by a certified research institute (see Table 1).

*MicolitB:* Consists for 100% of natrium-bentonite and does swell strongly. The pellets have been improved to delay swelling in order to enhance application.

*QS Bentoniet:* Also consists for 100% of natrium-bentonite and is available in granular form or in pallets. The latter start to swell later and can easier be applied at greater depth.

*Wyoming bentonite:* Is the best bentonite available when swelling power is concerned but is also rather expensive.
Grouts
More research has been done into the behaviour of grouts. Recently laboratory and field experiments commissioned by the DVGW where conducted in Germany. [Knorr 2000, Rubbert 2003, Niehheus et al. 2003, Trimpin 2003, Baumann et al. 2003.] In the Netherlands grout are however not commonly used for infiltration wells, the bentonite based granulates are prevalent if not omnipotent.

This recent German research was triggered by new improved measurement techniques, which indicated that many clayballs where actually leaky. This indicated that the design parameters based on standard laboratory parameters were inaccurate. These laboratory experiments only measured the permeability of the grout and not of the whole system. The experiments indicated that the material permeability (c.a. $10^{-9}$ to $10^{-11}$ m/s) is correct for the grout itself, but at the borders the permeability is less (c.a. $10^{-7}$ to $10^{-8}$ m/s, see Figure 16). This is caused by the decreased affixion with especially the smooth well casing and also the surrounding clay layer (see Figure 15). When mud is used as a drilling fluid due to the lesser ascend rate at the border zones the drilling fluid is not properly displaced causing a mixed-zone.

![Figure 15 Potential weak points looked at in the German research](Baumann et al. 2003)

![Figure 16 System permeability measured in time with both a low and high CMC concentrations](Baumann et al. 2003)
Tracer experiments confirm the existence of these weak points. Other results of these tracer experiments are [Nieheus et al. 2003]:

- The material itself is waterproof.
- Important weak points are inhomogeneities and cracks in the material.
- Suspensions of low viscosity are more sensitive for the formation of inhomogeneities.
- Solid, relative brittle materials appear to have a greater tendency for water-induced cracks then the more plastic materials.
- The most important weak point is the border area between the clayball and the well casing.
- In the border area with the surrounding loose rock there appears, at least for these particular experiments, to be no preferable path. But these experiments did not look at cases where the clayball is surrounded by a clay layer, as is the common case in the Netherlands.
- The quality of the clayball decreases when there are interruptions during the construction.
- In all the cases there was a mixed zone.

All the wells had a subsidence of several decimetres within several hours after the construction was completed. There are three principle causes for this subsidence:

- The afterward displacement of drilling mud or air from interspaces within the clayball and the filling of holes in the borehole wall.
- Infiltration of grout into the pores of the surrounding sand.
- The loss of water from the clayball (consolidation).

Other results of the German research are:

- The higher the casing resistance the lower the permeability.
- The system-permeability appears to increase with the increase of the hydraulic head (see Figure 16).
- A high CMC-concentration in the drill fluid has a bad influence on the clayball quality (see also Figure 16)
- A high percentage of water in the grout causes a decrease in the strength of the clayball and increases the permeability.
- The mixed-zone intensifies when grouts with a low viscosity are used.

\[^2\] Also the settlement of the filter can be considered as a cause of the subsidence. This is however not mentioned in the German research.
A.4 Stress analysis around a borehole

Due to the drilling process the stresses around the borehole decrease. These stresses around a borehole can be approached by cavity expansion methods (see appendix D, page 42). This can be done either with an elastic soil model (see Figure 18) or an elastic-plastic soil model (Mohr-Coulomb, 2D see Figure 19 and 3D see Figure 20).

For the elastic model a straightforward calculation can be made:

\[
\sigma_r = \left(\sigma_{r \text{ at } r_{\min}} - \sigma_0\right) \frac{r_{\min}^2}{r^2} + \sigma_0
\]

\[
\sigma_\theta = -\left(\sigma_{r \text{ at } r_{\min}} - \sigma_0\right) \frac{r_{\min}^2}{r^2} + \sigma_0
\]  

(0.1)

Where \( r_{\min} \) can be calculated from the initial diameter of the borehole (\( r_{\min, \text{ before loading}} \)):

\[
r_{\min} = \frac{1}{1 - \frac{\sigma_{r \text{ at } r_{\min}} - \sigma_0}{2G}} r_{\min, \text{ before loading}}
\]

(0.2)

Figure 18 Elastic Cavity Expansion
For the elasto-plastic calculation this is not so straightforward (see also appendix D.2). Most authors use for the Mohr-Coulomb criterion the radial and tangential stresses, where actually the radial and vertical stresses are, especially for the more shallow cases, normative. Some however do take the vertical stresses into account, but assume the soil to be homogeneous over depth, which is of course in most cases not realistic. In order to obtain a more exact solution (taking into account the layered character of the soil) an analytical computer calculation with e.g. Plaxis can be useful.

However the precise plastic behaviour is in this case not very important. When the borehole is refilled the soil will restress and the plastic zone will disappear, thus making the plastic calculation for a global determination of the horizontal stress after refilling unnecessary. The amount of restressing is however unknown and therefore the minimum elastic horizontal stress has to be assumed:

\[ \sigma_h = K_{\text{activ}} \sigma_v \]  

(0.3)

Where \( K_{\text{activ}} \) for sand is somewhere between 0.22 and 0.33 and for clay somewhere between 0.3 and 0.5. This assumption seems realistic since during the refilling nothing is done to increase the horizontal stress.

The influence of the swelling of the clay ball can be significant for the horizontal stress in the clay layer, but since the exact behaviour is unknown the minimum horizontal stress has to be assumed.
Appendix B: The Olsthoorn Rule

Nowadays the maximum injection pressure at geotechnical collapse of infiltration wells is calculated with the Olsthoorn Rule. This rule was originally intended for rehabilitation of clogged wells. Later the rule was adapted to include the non-clogged situation. However field experience indicates that the rule is rather strict and in some cases the maximum infiltration pressure can even be doubled.

B.1 Derivation of the Olsthoorn Rule

The Olsthoorn Rule was originally developed for clogged wells in the KIWA publication “Verstopping van Persputten” by Ir. T.N. Olsthoorn [Olsthoorn 1982]. This derivation is resumed below.

The horizontal stress caused by the injection well may, according to the Olsthoorn Rule, not exceed the current (natural) horizontal effective stress.

\[
\sigma_{wh} \leq \sigma_{bh}
\]  

When this horizontal stress is larger, vertical cracks may occur due to hydraulic fracturing because these cracks cannot be closed and collapse may occur. Furthermore, it is assumed that the overpressure in the well is directly imposed on the surrounding soil.

\[
\sigma_{wh} = \rho_w g \Delta \varphi
\]

According to the Coulomb-Navier relation, the horizontal effective stress can be between a minimum value (active earth pressure) and a maximum value (passive earth pressure). These minimum and maximum values depend on the vertical effective stress. The actual horizontal effective stress is somewhere between those two values. Because this value is not exactly known, the worst case is taken, the minimum value:

\[
\sigma_{bh} \geq \sigma_{ev} \tan^2 \left( \frac{45^\circ - \phi}{2} \right) - 2c \tan \left( \frac{45^\circ - \phi}{2} \right)
\]

At some depth the cohesion, c, is relatively small in relation to the effective stress and can be considered negligible.

\[
\sigma_{bh} \geq \sigma_{ev} \tan^2 \left( \frac{45^\circ - \phi}{2} \right)
\]

---

3 Keuringsinstituut voor Waterleidingartikelen KIWA N.V.
The load of the topping layers and the water pressure determines the vertical effective stress:

\[
\sigma_v = \int_0^h \rho(z) g \delta z 
\]

\[
\sigma_{kv} = \sigma_v - p 
\]

The Olsthoorn Rule considers the topping layers to be homogeneous, and the hydraulic head to be at ground level. This results into:

\[
\sigma_{kv} = \sigma_v - p = (\rho - \rho_w) gh
\]

Combined with equation (0.7) this results into an expression for the maximum allowable overpressure in the well.

\[
\Delta \varphi \leq \frac{\rho - \rho_w}{\rho_w} h \tan^2 \left( \frac{45^\circ - \phi}{2} \right) = \lambda h
\]

In the worst-case scenario ($\rho = 2000 \text{ kg/m}^3; \rho_w = 1000 \text{ kg/m}^3; \phi = 40^\circ$, for sand) this equation can be written as:

\[
\Delta \varphi \leq 0.22h \quad (\lambda = 0.22)
\]

With $\Delta \varphi$ is the overpressure in the well and $h$ either the top of the filter casing or the top of the aquifer (whichever comes first). Note that with formula (0.11), depending on the soil characteristics, also other values for the relation between $\Delta \varphi$ and $h$ can be determined. This rule can also be rewritten into:

\[
\Delta p_{\text{max}} = K_{\text{actv}} \sigma_v
\]

This rule has later been adapted to the non-clogging situation [NN 1985]. It assumed that water acts as a wedge, which opens the fissure as it goes along. In loose formations (i.e. clay, peat, sand and gravel) this phenomena is assumed to occur when the water pressure exceeds the soil pressure. This means that fissuring will occur when the least principal effective stress drops to zero. In other words when the water pressure exceeds the minimum effective stress. This minimum horizontal effective stress is estimated in the same way as for the clogged case, resulting in the same crack birth criterion:

\[
\Delta p_{\text{max}} = K_{\text{actv}} \sigma_v
\]
B.2 Critical analysis of the Olsthoorn Rule

The first theory for the clogged well is suited for the clogged situation and therefore this approach is not valid for an infiltration well. The second theory is based on the assumption that water will form a wedge when the minimum stress is equal to the water pressure. In an aquifer water can flow freely. Therefore transversal to the flow direction no gradient of the pressure head exists and therefore no force pushes the soil aside. Therefore this assumption is not valid in an aquifer. The wedge theory is however valid for the clay ball. The water can form a wedge either between the casing and the clay ball, or between the clay ball and the surrounding confining clay layer. If the water pressure is higher than the horizontal effective stress it will squeeze itself upward alongside the well pipe or the shaft (see Figure 24).

Also the choice of the minimum horizontal pressure is conservative and if it is possible to determine either the horizontal stress or at least the $K_{\text{act}}$ more accurate this will in most cases definitely lead to a higher maximum infiltration pressure.

Figure 23 Wedge: the water pressure pushes the soil aside in the direction of the minimum stress (black arrow) causing a crack that propagates transversal to it (blue arrow).

Figure 24 Collapse mechanisms of the clayball
Appendix C: Fracturing

Unfortunately no specific research of the collapse-mechanisms of an infiltration-well has been done. There are however several articles concerning fracturing in low permeable layers. In these layers the equilibrium between the water pressure gradient and the flow pressure is not established immediately and therefore the water pressure works in a complete other way. Where in the normal permeable layer the flow pressure is the only force induced by the water flow, in the low permeable layers the water pressure is mostly considered as a force on a certain surface, eg transition zone from the filter to the surrounding soil. Since the aquifer is however permeable these fracturing models are not completely applicant for the specific case of fracturing in an aquifer. An overview is given below.

C.1 Global

One of the first fields that relates to the well collapse problem is fracturing. Generally there are three types of cracks (see Figure 25) [Ruygrok 1997]

1. Tensile track.
2. Sliding mode
3. Tearing mode

The first kind, tensile fracture is most common in soil since it can only cope with pressure and not with tension. So the occurrence of zero stress is a criterion for crack formation in soils.

Cracks can occur in coherent soils (clay, cemented granulate, sand under capillary cohesion). For clay we can distinguish between:

1. Brittle, and eventually relative slow deformation processes. This happens in clay with a water content below the plastic limit, so rather dry clay, where the groundwater level is mostly 2 to 3 meters lower. This is not the case with the infiltration wells.
2. Plastic, and relative (very) high deformation speed.

C.2 Hydraulic fracturing

A description of the hydraulic fracturing process can be found in handbooks for oil and gas extraction and also in newer handbooks for soil remediation such as [Suthersan 1999].

Hydraulic fracturing is the fracturing of rock or soils due to high water pressure. In controlled form it is used for yield improvement of low permeable layers. Primary this technique was used in the mining industry for oil recovery, but nowadays it is also used in shallow fine-grained formations for in situ remediation purposes.
Hydraulic fracturing involves injecting a fluid, usually water, at modest rates and high pressures into the soil matrix to be fractured. A high-pressure water nozzle (cutting jet) is used to cut a disc-shaped notch. This notch becomes the starting point for the fracture and by manipulating the direction of this notch, the fracture can be aimed in its preferred horizontal direction. (see Figure 26).

The crack initiation is forced by a localised high pressure induced by the nozzle. Unintended hydraulic fracturing is not caused by a water nozzle but by a uniform increase of the water pressure in the filter. The hydraulic fraction is also mostly closely monitored, and the injection pressure is interactively adjusted. Therefore handbooks do not often give a relationship for the crack initiation.

Also air or gas might be used instead of water. This process is called pneumatic fracturing.

### C.3 Crack propagation

Murdoch and Slack [2002, 1993] did research into the forms of hydraulic fractures in shallow fine-grained formations. These hydraulic fractures were created for remediation purposes, and during the process filled with sand to avoid closure. The research was not aimed at fracture initiation and therefore this research is of less value. Important results from this research are:

- The shallow fractures tend to be gently dipping upward but some are steeply dipping and vent in the vicinity of the borehole.
- Most fractures develop a preferred direction of propagation. This preferred direction is commonly related with the state of stress and the layering. Tending toward the path of minimum resistance.
- Generally, the dips range between 0.8 and 22° with an average of 4.4 and the 95% confidence interval ranging from 3.0 to 8.2°.
- Normally the dips increase with depth (even more than 70°), but also deep gently dipping fractures were found.

Especially the relation between the least principal stress and the crack propagation direction is important.

Suthersan [1999] indicates that fractures in highly liquid and plastic soils might not propagate as well as in more brittle materials.
C.4 Cavity expansion and hydraulic fracturing

Bjerrum [1972] has researched the relation between cavity expansion and hydraulic fracturing in fine-grained clay layers. This research was triggered by erroneous permeabilities in field experiments caused by a too high infiltration pressure (at the Dead Sea, Israel and in Oslo). Field experiments indicate that locally there is a relationship between the infiltration pressure and the vertical effective stress, however this relationship can vary from site to site:

\[
\frac{\Delta p}{\sigma_{v,0}'} = \lambda
\]  

(0.15)

With the Olsthoorn Rule this \(\lambda\) is about 0.22. Field studies at the Dead Sea, Israel, give a \(\lambda\) of about 0.4 to 0.5 but also lower values of 0.2 where repeatedly observed. Bjerrum attributes these low values to a lower vertical effective stress than the overburden weight of fill, which was being carried by shear stresses along the walls of the trench and ‘arching’ was taking place. (It might also be caused by the natural cracks in the Death Sea area caused by tectonic activity). Other experiments in Fornebu, Oslo, give for \(\lambda\) a value of about 0.8. These results already indicate that \(\lambda\) varies greatly between different locations. Bjerrum already indicates that the maximum infiltration pressure does not directly depend on the vertical effective stress but on the horizontal effective stress. The variations of \(\lambda\) can then be attributed to variations in \(K\).

Other used values for \(\lambda\) found by Bjerrum [1972] in the literature are:

Terzaghi and Peck uplift:

\[
\frac{\Delta p}{\sigma_{v,0}'} > 1
\]  

(0.16)

Ahrens, “one pound per foot of depth”:

\[
\frac{\Delta p}{\sigma_{v,0}'} = 1.65
\]  

(0.17)

Petroleum engineers: “0.65 lb/sq” for fully saturated regions where the least principal effective stress is horizontal and equal to one-third of the effective overburden pressure:

\[
\frac{\Delta p}{\sigma_{v,0}'} = \frac{1}{3}
\]  

(0.18)

Where the vertical effective stress is least, this analysis indicates:

\[
\frac{\Delta p}{\sigma_{v,0}'} = 1
\]  

(0.19)
Bjerrum gives the following enumeration of most important factors that influence the maximum allowable infiltration pressure at permeability tests:

a. The initial state of stress and water pressure in the ground.
b. The changes in these stresses arising from installing the piezometer and any other disturbance of the soil. This can be calculated with the cavity expansion theory for borehole expansion. In the case of the infiltration well this will be the effect of well construction on the stresses, which can be calculated with the cavity expansion theory for contraction (see chapter appendix A and D).
c. The magnitude of the pressure head adopted and its variation with time.
d. The deformation characteristics and the degree of homogeneity of the soil around the piezometer.
e. The geometry of the piezometer tip.

Due to the hydraulic excess, the soil skeleton will experience an outward drag force. This will be associated near the piezometer with both radial and circumferential extension. As long as the applied head is low no fracture will occur. At a certain value of the applied head, the circumferential stress will be reduced to zero and a fracture will occur. This can be calculated with the cavity expansion theory. Bjerrum derives the following equation assuming that around the piezometer a condition of plane radial strain holds:

$$\frac{\Delta p}{\sigma_{h0}} = \left(1 - \nu\right) \left[K + \frac{\sigma'_t}{\sigma_{h0}}\right]$$  \hspace{1cm} (0.20)

A second collapse mechanism occurs when the soil has moved away from the piezometer tip:

$$\frac{\Delta p}{\sigma_{h0}} = \left(1 - \nu\right) \left[2K + \frac{\sigma'_t}{\sigma_{h0}}\right]$$  \hspace{1cm} (0.21)

Where $\sigma'_t$ is the maximum tensile effective stress that the soil skeleton can sustain. The implications for different soil types are listed below in Table 2.

<table>
<thead>
<tr>
<th>Method of installation</th>
<th>Soil type</th>
<th>Ratio of horizontal/vertical principal effective stress, $K$ [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.3</td>
</tr>
<tr>
<td>Ideal installation</td>
<td>All soils</td>
<td>0.4</td>
</tr>
<tr>
<td>(no disturbance)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Driven</td>
<td>High compressibility</td>
<td>0.4-0.5</td>
</tr>
<tr>
<td></td>
<td>Medium compressibility</td>
<td>0.5-0.7</td>
</tr>
<tr>
<td></td>
<td>Low compressibility</td>
<td>0.7-1.0</td>
</tr>
</tbody>
</table>

Table 2 Values of $\lambda$ for different soil types [Bjerrum et al. 1972]

---

4 Note that the influence of driving the piezometer into the soil has been eliminated from these formulas.
Appendix D: Cavity Expansion Theory

In order to describe the behavior of soil around a well the cavity expansion theory is used. The cavity expansion theory describes the elastic and plastic behavior between an inner \( (r_{\text{min}}) \) and an outer \( (r_{\text{max}}) \) boundary of either an infinitely thick (plane strain) or an infinitely thin (plane stress) circular layer of material (see Figure 2).

In this appendix first the elastic cavity expansion theory will be elaborated for both loading (e.g. in a clogged well) and unloading cases (e.g. during construction of the well) where \( r_{\text{max}} \) goes to infinity. In order to arrive at the ruling differential equation the differential equation for rectangular coordinates is derived and translated into polar coordinates. This polar differential equation forms the basis of the cavity expansion theory and by adding boundary conditions the solutions for different loading cases can be calculated.

Thereafter the plastic extension of the cavity expansion theory is dealt with in a more global way since it is at the moment of less importance for this study.

Please not concerning the stresses that tension is taken positive and that all stresses below are stress increments, unless differently is noted. The total stress can be calculated as follows:

\[
\sigma_{\text{total}}(r) = \sigma_0(r) + \sigma(r)
\]  

**D.1 Elastic Cavity Expansion**

**D.1.1 Rectangular Coordinates**

The elastic behaviour in rectangular coordinates is pretty standard can be found in most handbooks such as Timoshenko and Goodier [1970], Verruijt [2001].

Hooke’s Law for rectangular coordinates is written as:

\[
\begin{align*}
\varepsilon_{xx} &= \frac{1}{E} \left( \sigma'_{xx} - \nu \left( \sigma'_{yy} + \sigma'_{zz} \right) \right) \\
\varepsilon'_{yy} &= \frac{1}{E} \left( \sigma'_{yy} - \nu \left( \sigma'_{xx} + \sigma'_{zz} \right) \right) \\
\varepsilon'_{zz} &= \frac{1}{E} \left( \sigma'_{zz} - \nu \left( \sigma'_{xx} + \sigma'_{yy} \right) \right) \\
\gamma_{xy} &= \frac{1}{G} \sigma_{xy} \\
\gamma_{yz} &= \frac{1}{G} \sigma_{yz} \\
\gamma_{zx} &= \frac{1}{G} \sigma_{zx}
\end{align*}
\]  

(0.23)

With:

\[
G = \frac{E}{2(1+\nu)}
\]  

(0.24)
For the plane strain case ($\varepsilon_{zz}, \varepsilon_{yz}, \varepsilon_{xz} = 0$) formula (0.23) can be rewritten into:

$$\sigma'_{zz} = \nu \left( \sigma'_{xx} + \sigma'_{yy} \right) \quad (0.25)$$

and:

$$\varepsilon_{xx} = \frac{1}{E} \left( (1-\nu^2) \sigma'_{xx} - \nu (1+\nu) \sigma'_{yy} \right)$$

$$\varepsilon_{yy} = \frac{1}{E} \left( (1-\nu^2) \sigma'_{yy} - \nu (1+\nu) \sigma'_{zz} \right)$$

$$\gamma_{xy} = \frac{1}{G} \sigma_{xy} = \frac{2(1+\nu)}{E} \sigma_{xy} \quad (0.26)$$

Figure 28 Strains [Verruijt 2001]

For the plane stress case ($\sigma_{zz}, \sigma_{yz}, \sigma_{xz} = 0$) we can rewrite (0.23) into:

$$\varepsilon_{xx} = \frac{1}{E} \left( \sigma'_{xx} - \nu \sigma'_{yy} \right)$$

$$\varepsilon_{yy} = \frac{1}{E} \left( \sigma'_{yy} - \nu \sigma'_{zz} \right)$$

$$\gamma_{xy} = \frac{1}{G} \sigma_{xy} = \frac{2(1+\nu)}{E} \sigma_{xy} \quad (0.27)$$

The equilibrium of a small rectangular block can be written as (see also Figure 29):

$$\frac{\partial \sigma'_{xx}}{\partial x} + \frac{\partial \sigma'_{xy}}{\partial y} + F_x = 0$$

$$\frac{\partial \sigma'_{yy}}{\partial y} + \frac{\partial \sigma'_{xy}}{\partial x} + F_y = 0 \quad (0.28)$$

Figure 29 Stresses
Formula (0.28) can now be rewritten into:

\[
\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = - \frac{\partial^2 \sigma'_{xy}}{\partial x^2} - \frac{\partial F_y}{\partial x} \tag{0.29}
\]

\[
\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = - \frac{\partial^2 \sigma'_{xy}}{\partial y^2} - \frac{\partial F_y}{\partial y} \tag{0.30}
\]

Combining formula (0.29) and (0.30) gives:

\[
\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = - \frac{1}{2} \left( \frac{\partial^2 \sigma'_{xx}}{\partial x^2} + \frac{\partial F_y}{\partial x} + \frac{\partial^2 \sigma'_{yy}}{\partial y^2} + \frac{\partial F_y}{\partial y} \right) \tag{0.31}
\]

For the two-dimensional case we consider three strain components:

\[
\varepsilon_x = \frac{\partial u_x}{\partial x} \tag{0.32}
\]

\[
\varepsilon_y = \frac{\partial u_y}{\partial y} \tag{0.33}
\]

\[
\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \tag{0.34}
\]

We can now differentiate equation (0.34) to x and y:

\[
\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u_y}{\partial x^3} + \frac{\partial^3 u_y}{\partial x^2 \partial y} \tag{0.35}
\]

\[
\frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^3 u_x}{\partial x \partial y^2} \tag{0.36}
\]

\[
\frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 u_y}{\partial x^2 \partial y} \tag{0.37}
\]

These formulas (0.35) (0.36) (0.37) can be combined into a differential relation (condition of compatibility):

\[
\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{0.38}
\]

Substituting (0.26) and (0.31) into (0.38) we finally get a differential equation for the plane strain case:

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) = -\frac{1}{1-V} \left( \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} \right) \tag{0.39}
\]

In order to obtain the stress state, this differential equation has to be solved in combination with the equilibrium equation (0.28).
Assuming that the body force have a potential we can write:

\[ F_x = -\frac{\partial V}{\partial x} \quad F_y = -\frac{\partial V}{\partial y} \]  
(0.40)

And thus rewriting equation (0.28) into:

\[ \frac{\partial}{\partial x}(\sigma_x - V) + \frac{\partial \tau_{xy}}{\partial y} = 0 \]
\[ \frac{\partial}{\partial y}(\sigma_y - V) + \frac{\partial \tau_{xy}}{\partial x} = 0 \]  
(0.41)

These equations can be satisfied by using this stress function:

\[ \sigma_x - V = \frac{\partial^2 \phi}{\partial y^2} \quad \sigma_y - V = \frac{\partial^2 \phi}{\partial x^2} \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \]  
(0.42)

Resulting into one differential equation:

\[ \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y} + \frac{\partial^4 \phi}{\partial y^2} = \frac{1-2\nu}{1-\nu} \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \]  
(0.43)

We can deal with the plane stress case in the same way by substituting (0.27) and (0.31) into (0.38) we finally get a differential equation:

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) = -(1+\nu) \left( \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} \right) \]  
(0.44)

With the stress function (0.42) we get:

\[ \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y} + \frac{\partial^4 \phi}{\partial y^2} = -(1-\nu) \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \]  
(0.45)

In order to obtain the stress state for a particular case this equation (in combination with the boundary conditions) has to be solved.
D.1.2 Polar Coordinates

In order to solve a perfectly elastic boundary problem in polar coordinates we have to translate the differential equation from Cartesian to polar coordinates. This has already been done by Timoshenko and Goodier [1970] and we will mainly follow their approach.

Hooke’s Law for polar coordinates can be written as:

\[
\begin{align*}
\varepsilon_r &= \frac{1}{E} \left( \sigma_r - \nu (\sigma_\theta + \sigma_z) \right) \\
\varepsilon_\theta &= \frac{1}{E} \left( \sigma_\theta - \nu (\sigma_r + \sigma_z) \right) \\
\varepsilon_z &= \frac{1}{E} \left( \sigma_z - \nu (\sigma_\theta + \sigma_r) \right) \\
\gamma_{r\theta} &= \frac{1}{G} \tau_{r\theta} \\
\gamma_{r\theta} &= \frac{1}{G} \tau_{r\theta} \\
\gamma_{\theta z} &= \frac{1}{G} \tau_{\theta z}
\end{align*}
\] (0.46)

For plane strain this can be rewritten into:

\[
\begin{align*}
\varepsilon_r &= \frac{1}{E} \left( (1-\nu^2) \sigma_r - \nu (1+\nu) \sigma_\theta \right) \\
\varepsilon_\theta &= \frac{1}{E} \left( (1-\nu^2) \sigma_\theta - \nu (1+\nu) \sigma_r \right) \\
\gamma_{r\theta} &= \frac{1}{G} \tau_{r\theta} = \frac{2(1+\nu)}{E} \tau_{r\theta}
\end{align*}
\] (0.47)

For plane stress this can be rewritten into:

\[
\begin{align*}
\varepsilon_r &= \frac{1}{E} \left( \sigma_r - \nu \sigma_\theta \right) \\
\varepsilon_\theta &= \frac{1}{E} \left( \sigma_\theta - \nu \sigma_r \right) \\
\gamma_{r\theta} &= \frac{1}{G} \tau_{r\theta} = \frac{2(1+\nu)}{E} \tau_{r\theta}
\end{align*}
\] (0.48)

The equilibrium now is:

\[
\begin{align*}
\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \sigma_r - \sigma_\theta + R &= 0 \\
\frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2 \tau_{r\theta}}{r} + S &= 0
\end{align*}
\] (0.50)

In order to derive the differential equation conversion rules between the polar and rectangular case are needed. For the reader’s convenience these rules are elaborated below.

For stresses these rules are:

\[
\begin{align*}
\sigma_r &= \sigma_x \cos^2 \theta + \sigma_s \sin^2 \theta + 2 \tau_{x\theta} \sin \theta \cos \theta \\
\sigma_\theta &= \sigma_x \sin^2 \theta + \sigma_s \cos^2 \theta - 2 \tau_{x\theta} \sin \theta \cos \theta \\
\tau_{r\theta} &= \left( \sigma_y - \sigma_s \right) \sin \theta \cos \theta + \tau_{x\theta} \left( \cos^2 \theta - \sin^2 \theta \right)
\end{align*}
\] (0.51)
The relation between the stresses can also be written as:

\[ \sigma_x + \sigma_y = \sigma_r + \sigma_\theta \]  

(0.53)

For the coordinates we can write:

\[ r^2 = x^2 + y^2 \quad \theta = \arctan \frac{y}{x} \]  

(0.54)

Which can be rewritten into:

\[ \frac{\partial (r^2)}{\partial x} = \frac{\partial (x^2)}{\partial x} + \frac{\partial (y^2)}{\partial x} = 2r \frac{\partial r}{\partial x} = 2x + \frac{\partial (y^2)}{\partial x} \]  

(0.55)

This results into:

\[ \frac{\partial r}{\partial x} = \frac{x}{r} = \cos \theta \]  

(0.56)

In the same way we can derive:

\[ \frac{\partial r}{\partial y} = \frac{y}{r} = \sin \theta \]  

(0.57)

This results into the following conversion rules:

\[ \frac{\partial \theta}{\partial x} = \frac{\partial \left( \arctan \left( \frac{y}{x} \right) \right)}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{y}{r^2} = -\frac{\sin \theta}{r} \]  

(0.58)

\[ \frac{\partial \theta}{\partial y} = \frac{\partial \left( \arctan \left( \frac{y}{x} \right) \right)}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}} = \frac{1}{\sqrt{x^2 + y^2}} = \frac{x}{r} = \cos \theta \]  

(0.59)

\[ \frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial x} = \cos \theta \frac{\partial f}{\partial r} - \sin \theta \frac{\partial f}{\partial \theta} \]  

(0.60)

\[ \frac{\partial f}{\partial y} = \sin \theta \frac{\partial f}{\partial r} + \cos \theta \frac{\partial f}{\partial \theta} \]  

(0.61)

\[ \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial^2 f}{\partial x^2} \right) = \cos^2 \theta \frac{\partial^2 f}{\partial r^2} + r \sin^2 \theta \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} - 2 \sin \theta \cos \theta \frac{\partial f}{\partial r} \frac{1}{r} \frac{\partial^2 f}{\partial \theta \partial r} \]  

(0.62)
\[ \frac{\partial^2 f}{\partial y^2} = \sin^2 \theta \frac{\partial^2 f}{r^2 \partial \theta^2} + \cos^2 \theta \left( \frac{1}{r \partial r} + \frac{1}{r^2 \partial \theta^2} \right) + 2 \sin \theta \cos \theta \frac{\partial}{\partial r} \left( \frac{1}{r \partial \theta} \right) \]  
\[ - \frac{\partial^2 f}{\partial x \partial y} = \sin \theta \cos \theta \left( \frac{1}{r \partial r} + \frac{1}{r^2 \partial \theta^2} \right) - \cos^2 \theta - \sin^2 \theta \frac{\partial}{\partial r} \left( \frac{1}{r \partial \theta} \right) \]  
(0.63)  
(0.64)

They can be rewritten into:

\[ \left( \frac{\partial^2 }{\partial x^2} + \frac{\partial^2 }{\partial y^2} \right) f = \left( \frac{\partial^2 }{r^2 \partial \theta^2} + \frac{1}{r \partial r} + \frac{1}{r^2 \partial \theta^2} \right) f \]  
(0.65)

Substituting (0.65) and (0.53) into (0.39) we get for zero body force:

\[ \left( \frac{\partial^2 }{\partial r^2} + \frac{1}{r \partial r} + \frac{1}{r^2 \partial \theta^2} \right) (\sigma_r + \sigma_\theta) = 0 \]  
(0.66)

And with:

\[ \text{div} \left( \begin{array}{c} F_x \\ F_y \end{array} \right) = \frac{1}{r} \text{div} \left( \begin{array}{c} F_r \\ F_\theta \end{array} \right) \]  
(0.67)

\[ \nabla f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r \partial r} + \frac{1}{r^2 \partial \theta^2} \]  
(0.68)

We get the following differential equation for the plane strain case:

\[ \left( \frac{\partial^2 }{\partial r^2} + \frac{1}{r \partial r} + \frac{1}{r^2 \partial \theta^2} \right) (\sigma_r + \sigma_\theta) = - \frac{1}{1-\nu} \left( \frac{1}{r \partial r} \left( r F_r \right) + \frac{1}{r \partial \theta} \frac{\partial F_\theta}{\partial \theta} \right) \]  
(0.69)

And for the plane stress case:

\[ \left( \frac{\partial^2 }{\partial r^2} + \frac{1}{r \partial r} + \frac{1}{r^2 \partial \theta^2} \right) (\sigma_r + \sigma_\theta) = -(1+\nu) \left( \frac{1}{r \partial r} \left( r F_r \right) + \frac{1}{r \partial \theta} \frac{\partial F_\theta}{\partial \theta} \right) \]  
(0.70)

These equations can be solved in combination with the equilibrium equation.
D.1.3 Polar Coordinates without body force

Solutions of this set of differential equation derived in the previous paragraph can be found in Timoshenko and Goodier [1970], Yu, Has-Sui [2000], Adel and Luger [1987].

When we take for \( f \) the stress-function \( \phi(x, y) \) we can rewrite the equations (0.42) – but then without body forces \( (V = 0) \) - with respectively equation (0.62), (0.63) and (0.64) into a new set of expressions for \( \sigma_r, \sigma_\theta, \) and \( \tau_{r\theta} \). These new expressions can be substituted into equations (0.51) resulting into expressions for the stress-function in polar coordinates:

\[
\sigma_r = \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \\
\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} \\
\tau_{r\theta} = \frac{1}{r^2} \frac{\partial \phi}{\partial \theta} - \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right)
\]

These expressions can be substituted into either (0.69) or (0.70) – with body forces taken zero – resulting into the following differential equation:

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) = 0
\]  

In order to obtain the stress state for a particular case without body forces this equation (in combination with the boundary conditions) has to be solved.

Since completely axis-symmetric, no changes in the tangential direction and therefore:

\[
\frac{\partial}{\partial \theta} = 0
\]

Resulting into:

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right) = 0
\]

This differential equation has a general solution:

\[
\phi = C_1 \log r + C_2 r^2 \ln r + C_3 r^2 + C_4
\]
Which can with equations (0.71) and (0.72) be rewritten into:

\[
\sigma_r = \frac{C_1}{r^2} + C_2 \left( 1 + 2 \ln r \right) + 2C_3 \tag{0.78}
\]

\[
\sigma_\theta = \frac{C_1}{r^2} + C_2 \left( 3 + 2 \ln r \right) + 2C_3 \tag{0.79}
\]

\[
\tau_{r\theta} = 0 \tag{0.80}
\]

The strains in radial coordinates are defined as [Timoshenko and Goodier 1970]:

\[
\varepsilon_r = \frac{\partial u}{\partial r} \tag{0.81}
\]

\[
\varepsilon_\theta = \frac{u}{r} + \frac{\partial v}{\partial \theta} \tag{0.82}
\]

\[
\gamma_{r\theta} = \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \tag{0.83}
\]

Equation (0.81) combined with equation (0.48) gives for plane strain case:

\[
\varepsilon_r = \frac{\partial u}{\partial r} = \frac{1}{E} \left( (1 + \nu) \frac{C_1}{r^2} + \left( 1 - 3\nu - 4\nu^2 \right) C_2 + \left( 2 - 2\nu - 4\nu^2 \right) C_2 \ln r + \left( 1 - \nu - 2\nu^2 \right) 2C_3 \right) \tag{0.84}
\]

We can differentiate this equation resulting into an expression for the radial displacement \( u \):

\[
u = \frac{1}{E} \left( -(1 + \nu) \frac{C_1}{r} + 2 \left( 1 - \nu - 2\nu^2 \right) C_2 r \ln r - (1 + \nu) C_2 r + 2 \left( 1 - \nu - 2\nu^2 \right) C_3 r \right) + f(\theta) \tag{0.85}
\]

Where \( f(\theta) \) is a function of \( \theta \) only.

Combining equation (0.85), (0.82) and (0.48) gives the following expression:

\[
\frac{\partial v}{\partial \theta} = r \varepsilon_r - u = \frac{(4 - 4\nu^2) C_2 r}{E} - f(\theta) \tag{0.86}
\]

We can again differentiate this equation to arrive at an expression for the tangential displacement \( v \):

\[
\frac{d v}{d \theta} = \frac{(4 - 4\nu^2) C_2 r \theta}{E} - \int f(\theta) d\theta + f_i(r) \tag{0.87}
\]

Where \( f_i(r) \) is a function of \( r \) only.

In case of a closed ring the tangential displacement \( v \) at \( \theta = n \) and \( \theta = n + 2\pi \) has to be equal. This can only be the case if \( C_2 \) is zero. In a completely rotation symmetrical case also \( f(\theta) \) has to be zero.
In the same way we can make this calculation for the plane stress case. By combining equation (0.81) and (0.49) we get:

\[ \varepsilon_r = \frac{\partial u}{\partial r} = \frac{1}{E} \left( (1 + \nu) \frac{C_1}{r^2} + (1 - 3\nu) C_2 + 2(1 - \nu) C_3 \ln r + 2(1 - \nu) C_3 \right) \]  

(0.88)

By differentiation we get (where \( f(\theta) \) is a function of \( \theta \) only):

\[ u = \frac{1}{E} \left( - (1 + \nu) \frac{C_1}{r} + 2(1 - \nu) C_2 r \ln r - (1 + \nu) C_2 r + 2(1 - \nu) C_3 r \right) + f(\theta) \]  

(0.89)

When we combine expressions (0.89), (0.82) and (0.49) we get:

\[ \frac{\partial v}{\partial \theta} = r \varepsilon_r - u = \frac{4C_2 r}{E} - f(\theta) \]  

(0.90)

By differentiation we get (where \( f_1(r) \) is a function of \( r \) only.):

\[ v = \frac{4C_2 r \theta}{E} - \int f(\theta) d\theta + f_1(r) \]  

(0.91)

In case of a closed ring the tangential displacement \( v \) at \( \theta = n \) and \( \theta = n + 2\pi \) has to be equal. This can only be the case if \( C_2 \) is zero. In a completely rotation symmetrical case also \( f(\theta) \) has to be zero.

This results in these expressions for the radial and tangential stress-increments:

\[ \sigma_r = \frac{C_1}{r^2} + 2C_3 \]  

(0.92)

\[ \sigma_\theta = - \frac{C_1}{r^2} + 2C_3 \]

We can now consider a well with no stress increment at infinity and a stress increment caused by a change of stress from \( \sigma_0 \) to \( \sigma_{r \text{op} r \text{fin}} \) at the inner boundary.

Substituting the boundary condition at infinity gives:

\[ \text{at } r = \infty, \sigma_r = 0 \rightarrow C_3 = 0 \]  

(0.93)

And the condition at the inner boundary gives:

\[ \text{at } r = r_{\text{min}}, \sigma_r = \sigma_{r \text{op} r \text{fin}} - \sigma_0 = \frac{C_1}{r_{\text{min}}} \rightarrow C_3 = (\sigma_{r \text{op} r \text{fin}} - \sigma_0) r_{\text{min}}^{-2} \]  

(0.94)
Resulting into:

\[
\sigma_r = \left( \sigma_{\text{rop max}} - \sigma_0 \right) \frac{r_{\text{min}}^2}{r^2} \\
\sigma_\theta = - \left( \sigma_{\text{rop max}} - \sigma_0 \right) \frac{r_{\text{min}}^2}{r^2}
\]  

(0.95)

Taking into account an initial stress of \(\sigma_0\) before loading this results into:

\[
\sigma_{r,\text{total}} = \left( \sigma_{\text{rop max}} - \sigma_0 \right) \frac{r_{\text{min}}^2}{r^2} + \sigma_0 \\
\sigma_{\theta,\text{total}} = - \left( \sigma_{\text{rop max}} - \sigma_0 \right) \frac{r_{\text{min}}^2}{r^2} + \sigma_0
\]  

(0.96)

**Displacement**

Base on equation (0.82) we can now give an expression for the displacement in radial direction:

\[
u = r \varepsilon_\theta
\]  

(0.97)

In plane strain this leads to:

\[u = \frac{1 + \nu}{E} \left( \sigma_{\text{rop max}} - \sigma_0 \right) \frac{r_{\text{min}}^2}{r}
\]  

(0.98)

And at \(r = r_{\text{min}}\) we get:

\[u = \frac{1 + \nu}{E} \left( \sigma_{\text{rop max}} - \sigma_0 \right) r_{\text{min}} = \frac{1}{2G} \left( \sigma_{\text{rop max}} - \sigma_0 \right) r_{\text{min}}
\]  

(0.99)

And with \(r_{\text{min}} = r_{\text{min},0} + u\) we get the following expression for the diameter of the borehole:

\[r_{\text{min}} = \frac{1}{1 - \frac{\sigma_{\text{rop max}} - \sigma_0}{2G} r_{\text{min},0}}
\]  

(0.100)

In the plane stress case we can write for the displacement:

\[u = \frac{2(1 + \nu)}{E} \left( \sigma_{\text{rop max}} - \sigma_0 \right) \frac{r_{\text{min}}^2}{r}
\]  

(0.101)
and for the diameter of the hole:

\[
 r_{\text{min}} = \frac{1}{1 - \frac{\sigma_{\text{op ran}} - \sigma_0}{G}} r_{\text{min},0} \tag{0.102}
\]

With these formulas we can now describe the elastic behaviour for several situations. In soil mechanics the soil mass is in most cases modelled as a infinitely thick layer, and therefore the plane strain approach is most often used. This is also the case in the next two elaborated cases:

**Stresses during the construction of the well.**

During the construction of a well the radial stress at \( r_{\text{min}} \) is equal to the support pressure of the drilling fluid. Therefore the \( \sigma_{\text{op ran}} \) is equal to the support pressure \( p_{\text{well}} \). This leads to these formulas (see also Figure 32):

\[
\begin{align*}
\sigma_{r,\text{total}} &= (p_{\text{well}} - \sigma_0) \frac{r_{\text{min}}^2}{r^2} + \sigma_0 \\
\sigma_{\theta,\text{total}} &= -(p_{\text{well}} - \sigma_0) \frac{r_{\text{min}}^2}{r^2} + \sigma_0
\end{align*}
\tag{0.103}
\]

where \( r_{\text{min}} \) can depends on the original diameter of the well \( r_{\text{min},0} \):

\[
r_{\text{min}} = \frac{1}{1 - \frac{p_{\text{well}} - \sigma_0}{2G}} r_{\text{min},0} \tag{0.104}
\]

**Stresses during injection in a completely clogged well**

When water is injected into a completely clogged well, water pressure will build up on the clogged surface. This water pressure will partially or completely take over the horizontal radial pressure between the well and the clogged soil. Only when the water pressure exceeds this horizontal pressure it will act as an extra load on the clogged surface since before this point the decrease of the pressure between the well and the clogged soil is equal to the increase of the water pressure. From this point on the \( \sigma_{\text{op ran}} \) is equal to the support pressure \( p_{\text{well}} \). This leads to these formulas (see also Figure 35):

\[
\begin{align*}
\sigma_{r,\text{total}} &= \begin{cases} 
(p_{\text{well}} - \sigma_0) \frac{r_{\text{min}}^2}{r^2} + \sigma_0 & \text{for } p_{\text{well}} \geq \sigma_0 \\
- (p_{\text{well}} - \sigma_0) \frac{r_{\text{min}}^2}{r^2} + \sigma_0 & \text{for } 0 \leq p_{\text{well}} \leq \sigma_0
\end{cases} \\
\sigma_{\theta,\text{total}} &= \begin{cases} 
\sigma_0 & \text{for } p_{\text{well}} \geq \sigma_0 \\
0 & \text{for } 0 \leq p_{\text{well}} \leq \sigma_0
\end{cases}
\end{align*}
\tag{0.105}
\]

with:

\[
r_{\text{min}} = \frac{1}{1 - \frac{p_{\text{well}} - \sigma_0}{2G}} r_{\text{min},0} \tag{0.106}
\]
Figure 30 Radial stresses in a perfectly elastic soil during construction of a well (with $\sigma_{\text{min}} = 100$ kPa)

Figure 31 Tangential stresses in a perfectly elastic soil during construction of a well (with $\sigma_{\text{min}} = 100$ kPa)
Figure 32 Stresses in a perfectly elastic soil during construction of a well (with $\sigma_{\text{min}} = 100$ kPa)

Figure 33 Radial stresses in a perfectly elastic soil during infiltration in a completely clogged well (with $\sigma_{\text{min}} = 100$ kPa)
Figure 34 Tangential stresses in a perfectly elastic soil during infiltration in a completely clogged well (with $\sigma_{\text{min}} = 100$ kPa)

Figure 35 Stresses in a perfectly elastic soil during infiltration in a completely clogged well (with $\sigma_{\text{min}} = 100$ kPa)
D.2 Plastic extensions to the cavity expansion theory

Plasticity occurs when the relation between the minimum and maximum principal stress exceeds a certain value\(^5\). From this point on the soil behaves plastic. Soil becomes plastic when the stresses reach the yield criterion:

\[ \alpha \sigma_k - \sigma_i = Y \quad \text{with: } \sigma_i \leq \sigma_j \leq \sigma_k \]  

(0.107)

With for the Mohr-Coulomb criterion\(^6\):

\[ \alpha = \frac{1 + \sin \phi}{1 - \sin \phi} \quad Y = 2C \frac{\cos \phi}{1 - \sin \phi} \]  

(0.108)

In the plastic zone the elastic stress-strain relationship does not hold anymore. Instead the principal stress remains on the yield criterion. Therefore everywhere in the plastic zone the yield criterion (0.107) has to hold.

D.2.1 Only taking into account stresses in the r-\(\theta\)-plane

In literature the plastic cavity expansion theory is mostly only elaborated in two dimensions (\(r\) and \(\theta\)). The vertical stresses are considered irrelevant and not of influence on the yield criterion. This is a good approach in cases where the stresses in the \(r-\theta\)-plane are normative but when this is not the case this approach is doubtful because the plastic zone is larger.

This two-dimensional approach can be found in Den Adel and Luger [1987] and Yu [2000]. We will discuss them in a more global way in order to give insight in the solution procedure. For a more extensive and detailed elaboration we refer to Yu [2000].

The principal stresses coincide with \(\sigma_r\) and \(\sigma_\theta\). The development of these stresses in the plastic zone has therefore to satisfy the yield criterion:

\[ \alpha \sigma_\theta - \sigma_r = Y \quad \text{for } \sigma_r \leq \sigma_\theta \quad \text{(expansion)} \]  

\[ \alpha \sigma_r - \sigma_\theta = Y \quad \text{for } \sigma_\theta \leq \sigma_r \quad \text{(contraction)} \]  

(0.109)

Also the equilibrium has to hold, which can be derived from equation (0.50):

\[ (\sigma_\theta - \sigma_r) = r \frac{\partial \sigma_r}{\partial r} \]  

(0.110)

For the expansion case equation (0.109) and (0.110) result into a differential equation:

\[ \mathcal{D}_r \]  

---

\(^5\) All stresses in this section are total stresses.

\(^6\) Also other criterions could be used, but in soil mechanics the Mohr-Coulomb criterion is most common.
The general solution for this equation is:

\[ \sigma_r = \frac{Y}{\alpha-1} + C_1 r^{\frac{1-\alpha}{\alpha}} \]  

(0.112)

Also \( \sigma_\theta \) can now be calculated with equation (0.109):

\[ \sigma_\theta = \frac{Y}{\alpha-1} + C_1 r^{\frac{1-\alpha}{\alpha}} \]  

(0.113)

We now have the stresses in the plastic zone. For the elastic zone the general form of the stresses has been found in the previous paragraph (D.1):

\[ \sigma_r = \frac{C_2}{r^2} + \sigma_0 \]  

(0.114)

\[ \sigma_\theta = -\frac{C_2}{r^2} + \sigma_0 \]  

(0.115)

The continuity at the elastic-plastic interface at \( r = r_{\text{plastic}} \) can be used to determine the integration constants \( C_1 \) and \( C_2 \):

\[ C_1 = \left( -Y + (\alpha - 1) \sigma_0 \right) \frac{2\alpha}{\alpha^2 - 1} r_{\text{plastic}}^{\frac{\alpha - 1}{\alpha}} \]  

(0.115)

\[ C_2 = \frac{Y + (1 - \alpha) \sigma_0}{1 + \alpha} r_{\text{plastic}}^2 \]  

(0.116)

Since at the inner boundary (\( r_{\text{min}} \)) the radial stress is known \( (\sigma_{\text{rmin}}) \) we can derive from (0.112)

\[ \frac{r_{\text{plastic}}}{r_{\text{min}}} = \left( \frac{\alpha + 1}{2\alpha} \left( Y - (\alpha - 1) \sigma_{\text{rmin}} \right) \right)^{\frac{\alpha}{\alpha - 1}} \]  

(0.117)

However both \( r_{\text{min}} \) and \( r_{\text{plastic}} \) remain unknown. In order to derive both we have to take plastic deformation into account. The easiest way is by assuming no volume change in the plastic zone [Luger 1976].

1. Assume no displacement at \( r_{\text{min}} \) \( (r_{\text{min}} = r_{\text{min,old}}) \)
2. Calculate value for \( r_{\text{plastic,new}} \) with equation (0.117)
3. Calculate location of \( r_{\text{plastic}} \) before deformation:
\[ r_{\text{plastic before}} = \frac{C_3}{C_3 + 1} r_{\text{plastic after}} \]  

(0.118)

with:

\[ C_3 = E \left( (1 + \nu) \frac{C_2}{c_{\text{after}}} + (1 - \nu - \nu^2) \sigma_0 \right) \]  

(0.119)

4. Calculate displacement of \( r_{\text{min}} \) by assuming no volume change:

\[ A_{\text{before}} = A_{\text{after}} = \pi r_{\text{plastic before}}^2 - \pi r_{\text{min before}}^2 = \pi r_{\text{plastic after}}^2 - \pi r_{\text{min after}}^2 \]  

(0.120)

resulting into:

\[ r_{\text{min after}} = \sqrt{r_{\text{plastic after}}^2 - r_{\text{plastic before}}^2 + r_{\text{min before}}^2} \]  

(0.121)

5. Iterate step 2 to 4 until satisfaction.

Others like Yu [2000] make use of more complicated flow rules for plastic deformation. Further elaborations are considered out of the scope of this thesis.

---

**Figure 36** 2D Elastic-Plastic Cavity Expansion [Vaziri and Byrne 1990]
D.2.2 Also taking into account vertical stresses

De Josseling de Jong [1953] has elaborated elastic and plastic stress development around a borehole in 3D. He however did not take displacements into account. A short description is given below, it is however not considered necessary to give a description of the calculations.

He makes a distinction between two plastic zones:
The first ‘faillible’ zone, where both $\sigma_1$ and $\sigma_{III}$ are the biggest principal stress and $\sigma_{II}$ is the smallest.
And the second ‘faillible’ zone, where either $\sigma_1$ or $\sigma_{III}$ is the biggest principal stress and $\sigma_{II}$ is the smallest.

We see that when the problem is approaches in 3D also the shear stresses are taking playing a part. Therefore the principal directions do not coincide with $z$, $r$ and $\theta$ (see Figure 37). When the soil becomes plastic it can’t always bear the soil above, it has to be ‘detoured’ to the soil that can cope with some extra vertical pressure. The vertical pressure is horizontally redistributed by these shear stresses. This mechanism results into a ‘dome’ around the weak point.

Figure 37 Schematisation in 3D [De Josseling de Jong 1953]

Figure 38 Stresses for the case where the radial and tangential stress (in combination with the [schuifspanning]) primary govern the plasticity criterion [De Josseling de Jong 1953]

Figure 39 Stresses for the case where the radial and vertical stress (in combination with the [schuifspanning] primary govern the plasticity criterion [De Josseling de Jong 1953]
To obtain a more realistic picture of the stresses around a borehole, also a calculation with a 3D program (e.g. Plaxis) can be made.

In order to analyse the general form of the stresses around a borehole a Plaxis study by GeoDelft [Havinga et al. 1995] has been used. This study investigates the effect of a large borehole (diameter: 5 m. and depth: 50 m.) by comparing it to the effect of a small borehole (diameter: 5 m. and depth: 50 m.). We see that for the small borehole the plastic zone is much smaller (see Figure 40) than for the large borehole (see Figure 42). We see that dependent on the size of the borehole at smaller boreholes only the vertical and the radial stresses are on the yield criterion, and when the borehole is larger there are two plastic zones, the inner one with all stress on the yield criterion and the other with only the vertical and radial stresses on the yield criterion.

Figure 40 Plastic zone around a small borehole (1 m. diameter) [Havinga et al. 1995]

Figure 41 Stresses around a small borehole (1 m. diameter) [Havinga et al. 1995]
Figure 42 Plastic zone around a large borehole (5 m. diameter) [Havinga et al. 1995]

Figure 43 Stresses around a large borehole (5 m. diameter) [Havinga et al. 1995]
Appendix E: Crack initiation in the aquifer.

In order to determine the moment at which a crack appears in the aquifer, an elastic cavity expansion calculation has been made. For this derivation the program Maple 6 has been used. The source code of this derivation can be found at the end of this section.

![Figure 44 Schematisation of the confined Aquifer](image)

The aquifer is modelled as a homogeneous confined layer of uniform height that extends into infinity. The initial vertical stress is determined by the weight of the confining soil and the initial water pressure (before loading). The initial horizontal stress is considered homogeneous and linearly dependent on the initial vertical stress:

\[ \sigma_{r,\text{init}} = K \sigma_{v,\text{init}} \]  

(0.122)

All the pressures and stresses used below will be increments (unless differently stated), and should be added to this initial stress.

The aquifer is loaded by an increasing water pressure caused by the infiltration well. The water pressure \( p \) can be found with the formula of Thiem:

\[ p(r) = -\frac{Q \gamma_w}{2\pi kD} \ln \left( \frac{r}{r_{\text{max}}} \right) = -C_0 \ln \left( \frac{r}{r_{\text{max}}} \right) \]  

(0.123)

The equilibrium equation (see also Figure 45) for this particular case is\(^7\):

\[ \frac{\partial \sigma'_r}{\partial r} + \frac{\partial p}{\partial r} + \frac{\sigma'_r - \sigma'_\theta}{r} = 0 \]  

(0.124)

\(^7\) Since the flow pressure is caused by the water pressure it would mean counting double if it as also taken into account.
When we combine formula (0.124) with (0.123) we get for the equilibrium of a wedge:

$$\frac{\partial \sigma'_r}{\partial r} - \frac{C_0}{r} + \frac{\sigma'_r - \sigma'_\theta}{r} = 0$$  \hspace{1cm} (0.125)

For the relation between strain and displacement we can write:

$$\varepsilon_r = \frac{\partial u(r)}{\partial r}$$ \hspace{1cm} (0.126)

$$\varepsilon_\theta = \frac{u(r)}{r}$$ \hspace{1cm} (0.127)

The linear elastic stress-strain relationship is described by Hooke’s Law and can in a plane be described in two ways. Either the strain in the vertical is taken constant (plane strain) or the stress in the vertical is taken constant (plane stress). The actual field situation is somewhere in-between. Therefore both cases will be elaborated below.

### E.1 Plane Strain

Hooke’s Law for the plane strain situation gives:

$$\varepsilon_r = \frac{1}{E} \left( (1-\nu^2) \sigma'_r - \nu (1+\nu) \sigma'_\theta \right) = \frac{\partial u(r)}{\partial r}$$ \hspace{1cm} (0.128)

$$\varepsilon_\theta = \frac{1}{E} \left( (1-\nu^2) \sigma'_\theta - \nu (1+\nu) \sigma'_r \right) = \frac{u(r)}{r}$$ \hspace{1cm} (0.129)

Equation (0.129) gives an expression for $u(r)$. This can be combined with equation (0.128) to give an expression with only $\sigma_r$ and $\sigma_\theta$:

$$(1+\nu) (\sigma_r - \sigma_\theta) + (\nu + \nu^2) r \frac{\partial \sigma_r}{\partial r} - (1-\nu^2) r \frac{\partial \sigma_\theta}{\partial r} = 0$$ \hspace{1cm} (0.130)
Since equation (0.125) gives an expression for $\sigma_\theta$, we can combine equation (0.125) with (0.130) to obtain a differential equation with only $\sigma_r$:

\[
\left(v^2-1\right)\left(r^2\frac{\partial^2 \sigma_r (r)}{\partial r^2} + 3r\frac{\partial \sigma_r (r)}{\partial r}\right) + (1+v)C_0 = 0
\]  
(0.131)

The general solution of this differential equation is:

\[
\sigma_r (r) = -\frac{C_1}{r^\frac{3}{2}} - \frac{1}{2} \frac{C_0}{v-1} \ln (r) + C_2
\]  
(0.132)

Where $C_1$ and $C_2$ are integration constants and $C_0$ is a constant dependent on the hydraulic properties of the aquifer and the induced hydraulic head in the well. Combining this equation (0.132) with equation (0.125) gives an expression for the stress in the tangential direction:

\[
\sigma_\theta (r) = \frac{1}{2} \frac{C_1}{r^2} - \frac{1}{2} \left(\frac{2v-1+\ln (r)}{v-1}\right) C_0 + C_2
\]  
(0.133)

In order to get an expression for the displacement we combine equation (0.132), (0.133) and (0.128) with equation (0.127):

\[
u (r) = \frac{r\left((v^3 - \frac{1}{3} v^2 - v + \frac{1}{3}) C_0 + (v^2 + \frac{1}{3} v^2 - \frac{1}{3}) \ln (r) C_0 + \left(\frac{1}{2} v^2 - \frac{1}{2}\right) r^2 C_1 + \left(-2v^3 + v^2 + 2v - 1\right) C_2\right)}{E (v-1)}
\]  
(0.134)

At a large distance from the well the influence of the well is minimal. At this point the displacement of the well is zero. This is the first boundary condition:

\[
u (r_{\text{max}}) = 0
\]  
(0.135)

The assumption reads that as long as there is no displacement at the well surface there is no crack between the soil and the well. So as long as there is no cavity between the well and the surrounding soil this condition holds:

\[
u (r_{\text{min}}) = 0
\]  
(0.136)

With these conditions the integration constants can be calculated:

\[
C_1 = -\frac{r_{\text{max}}^2 r_{\text{min}}^2 \left(2v-1\right) \ln \left(\frac{r_{\text{min}}}{r_{\text{max}}}\right) C_0}{(1-v) r_{\text{max}}^2 - (1-v) r_{\text{min}}^2}
\]  
(0.137)

\[
C_2 = -\frac{1}{2} C_0 \frac{1-v \ln \left(r_{\text{max}}\right)}{(1-v) r_{\text{max}}^2 - (1-v) r_{\text{min}}^2}
\]  
(0.138)
For any set of given values for \( r_{\text{max}} \), \( r_{\text{min}} \) and \( \nu \) we can now compute the development of the stresses (see also Figure 46, Figure 47 and Figure 48). Unfortunately these integration constants do not converge if \( r_{\text{max}} \) goes to infinity (see also Figure 49).

**Figure 46** Development of radial stresses in a finite plane (plane strain case with no displacement at \( r_{\text{max}} \) and \( r_{\text{max}} = 1000 \text{ m}; r_{\text{min}} = 0.1 \text{ m}; \nu = 0.3 \))

**Figure 47** Development of tangential stresses in a finite plane (plane strain case with no displacement at \( r_{\text{max}} \) and \( r_{\text{max}} = 1000 \text{ m}; r_{\text{min}} = 0.1 \text{ m}; \nu = 0.3 \))
Figure 48 Development of stresses in a finite plane (plane strain case with no displacement at $r_{\text{max}}$ and $r_{\text{max}}$ = 1000 m; $r_{\text{min}}$ = 0.1 m; $\nu$ = 0.3)

Figure 49 Behaviour of the integration constants (for $r_{\text{min}}$ = 1).
A cavity between the well casing and the surrounding soil (at $r_{\text{min}}$) will only appear if the radial stress at the well casing ($r_{\text{min}}$) is zero. So at the point of ‘cavity birth’ this equation holds:

$$\sigma'_{r} (r_{\text{min}}) = -\frac{1}{2} \frac{C_{1}}{r_{\text{min}}^{2}} - \frac{1}{2} \frac{C_{0} \ln(r_{\text{min}})}{\nu - 1} + C_{2} = -K \sigma'_{v}$$

(0.139)

Since $C_{1}$ and $C_{2}$ are known and only depend on $r_{\text{min}}$ and $r_{\text{max}}$ we can now express the value of $C_{0}$ at collapse as a function of $r_{\text{max}}$, $r_{\text{min}}$ and the initial horizontal stress:

$$C_{0} = 2K \sigma'_{v} \frac{r_{\text{max}}^{2} - r_{\text{min}}^{2}}{r_{\text{max}}^{2} - r_{\text{min}}^{2} + 2 \ln(r_{\text{min}}/r_{\text{max}}) r_{\text{max}}^{2}}$$

(0.140)

We can now give an expression for the water pressure by combining (0.123) with (0.140):

$$p(r) = 2K \sigma'_{v} \frac{\left(r_{\text{max}}^{2} - r_{\text{min}}^{2}\right)}{r_{\text{max}}^{2} - r_{\text{min}}^{2} + 2 \ln(r_{\text{min}}/r_{\text{max}}) r_{\text{max}}^{2}} \ln \left(\frac{r}{r_{\text{max}}}\right)$$

(0.141)

The maximum infiltration pressure is equal to the pressure at collapse at $r_{\text{min}}$. By multiplying this value with $1/\gamma_{w}$ we get the maximum infiltration over pressure in meters as a function of $r_{\text{max}}$ and $r_{\text{min}}$:

$$\varphi_{p_{\text{max}}} = \frac{1}{\gamma_{w}} 2K \sigma'_{v} \frac{\left(r_{\text{max}}^{2} - r_{\text{min}}^{2}\right) \ln \left(\frac{r_{\text{min}}}{r_{\text{max}}}\right)}{2 \ln \left(\frac{r_{\text{min}}}{r_{\text{max}}}\right) r_{\text{max}}^{2} - r_{\text{min}}^{2} + r_{\text{max}}^{2}}$$

(0.142)
When we let $r_{\text{max}}$ go to infinity this formula reduces to:

$$
\varphi_w = \lim_{r_{\text{max}} \to \infty} \frac{1}{\gamma_w} 2K\sigma_v \frac{\left( r_{\text{max}}^2 - r_{\text{min}}^2 \right) \ln \left( \frac{r_{\text{min}}}{r_{\text{max}}} \right)}{2 \ln \left( \frac{r_{\text{min}}}{r_{\text{max}}} \right) r_{\text{max}}^2 - r_{\text{min}}^2 + r_{\text{max}}^2} = \frac{1}{\gamma_w} K \sigma_v \quad (0.143)
$$

Which is equal to the old Olsthoorn Rule, but based on the more accurate annular gap initiation mechanism.

The influence of the boundaries can be expressed by dividing equation (0.142) by the solution of (0.143). This results into a multiplication factor that has to be multiplied with equation (0.143). This can be rewritten into (see also Figure 51):

$$
f = \frac{2 \left( 1 - a^2 \right) \ln \left( a \right)}{2 \ln \left( a \right) - a^2 + 1} \quad (0.144)
$$

With:

$$
a = \frac{r_{\text{min}}}{r_{\text{max}}} \quad (0.145)
$$

We see that the multiplication factor reduces to unity in the limit when $\frac{r_{\text{min}}}{r_{\text{max}}} \downarrow 0$ (when $r_{\text{max}}$ goes to infinity).

![Figure 51 Multiplication factor $f$ as a function of $a=\frac{r_{\text{min}}}{r_{\text{max}}}$](image-51)
We can also, instead of assuming no displacement, assume no radial stress increment at $r_{\text{max}}$ ($\sigma_r (r_{\text{max}}) = 0$). This leads in the same way as above to an expression for $C_1$, $C_2$ and $C_0$:

$$C_1 = \frac{2\nu^2 - 3\nu + 1 + (2\nu - 1)\ln \left(\frac{r_{\text{min}}}{r_{\text{max}}}\right)\left(r_{\text{min}}^2 r_{\text{max}}^2 C_0\right)}{\left(2\nu^2 - 3\nu + 1\right)\left(r_{\text{min}}^2 + (1 - \nu) r_{\text{max}}^2\right)}$$  \hspace{1cm} (0.146)

$$C_2 = \frac{1}{2} C_0 \frac{2\nu^2 - 3\nu + 1 - (1 - 2\nu)\ln \left(\frac{r_{\text{min}}}{r_{\text{max}}}\right)\left(r_{\text{min}}^2 + \ln \left(\frac{r_{\text{max}}}{r_{\text{min}}}\right)\right) r_{\text{max}}^2}{\left(2\nu^2 - 3\nu + 1\right)\left(r_{\text{min}}^2 + (1 - \nu) r_{\text{max}}^2\right)}$$  \hspace{1cm} (0.147)

$$C_0 = 2K\sigma_v \frac{r_{\text{max}}^2 - (2\nu - 1) r_{\text{min}}^2}{\left(2\nu - 1\right)\left(r_{\text{max}}^2 - r_{\text{min}}^2\right) + 2\ln \left(\frac{r_{\text{min}}}{r_{\text{max}}}\right) r_{\text{max}}^2}$$  \hspace{1cm} (0.148)

In the same way as above we can now give an expression for the water pressure:

$$p(r) = -2K\sigma_v^v \frac{r_{\text{max}}^2 - (2\nu - 1) r_{\text{min}}^2}{\left(2\nu - 1\right)\left(r_{\text{max}}^2 - r_{\text{min}}^2\right) + 2\ln \left(\frac{r_{\text{min}}}{r_{\text{max}}}\right) r_{\text{max}}^2} \ln \left(\frac{r}{r_{\text{max}}}\right)$$  \hspace{1cm} (0.149)

And when we let $r_{\text{max}}$ go to infinity this formula reduces to again to the Olsthoorn Rule:

$$\varphi = \lim_{r_{\text{min}} \to \infty} \frac{1}{2K\sigma_v^v} \left(\frac{r_{\text{max}}^2 - (2\nu - 1) r_{\text{min}}^2}{\left(2\nu - 1\right)\left(r_{\text{max}}^2 - r_{\text{min}}^2\right) + 2\ln \left(\frac{r_{\text{min}}}{r_{\text{max}}}\right) r_{\text{max}}^2}\right) = \frac{1}{2K\sigma_v^v}$$  \hspace{1cm} (0.150)
Figure 52 Development of radial stresses in a finite plane (plane strain case with no stress increment at \(r_{\text{max}}\) and \(r_{\text{max}} = 1000 \text{ m}; r_{\text{min}} = 0.1 \text{ m}; \nu = 0.3\))

Figure 53 Development of tangential stresses in a finite plane (plane strain case with no stress increment at \(r_{\text{max}}\) and \(r_{\text{max}} = 1000 \text{ m}; r_{\text{min}} = 0.1 \text{ m}; \nu = 0.3\))
**E.2 Plane Stress**

The plane stress situation can be calculated in the same way as the plane strain case, but now with another stress-strain relation:

\[
\varepsilon_r = \frac{1}{E} \left( \sigma'_r - \nu \left( \sigma'_\theta - p \right) \right)
\]

\[
\varepsilon_\theta = \frac{1}{E} \left( \sigma'_\theta - \nu \left( \sigma'_r - p \right) \right)
\]

\[
\varepsilon_z = \frac{1}{E} \left( -p - \nu \left( \sigma'_r + \sigma'_\theta \right) \right)
\]

Where the vertical stress has been considered constant:

\[
\sigma_z = p + \sigma'_z = 0
\]

Equation (0.125), (0.126), (0.127) and (0.151) combined give this differential equation:

\[
r^2 \left( \frac{\partial^2 \sigma_r (r)}{\partial r^2} \right) + 3r \left( \frac{\partial \sigma_r (r)}{\partial r} \right) + (1 - 2\nu) C = 0
\]
The global solution of this differential equation is:

$$\sigma_r (r) = -\frac{1}{2} \frac{C_1}{r^2} + (\nu + \frac{1}{2}) C_0 \ln (r) + C_2$$  \hspace{1cm} (0.154)$$

In the same way as above we now can derive the following expressions:

$$\sigma_\theta (r) = \frac{1}{2} \frac{C_1}{r^2} + (\nu - \frac{1}{2}) C_0 \ln (r) + C_2$$  \hspace{1cm} (0.155)$$

$$u (r) = \frac{1}{E} \left( (\nu - \frac{1}{2}) C_0 r + \left(-\nu^2 + \frac{1}{2} \nu + \frac{1}{2}\right) C_0 r \ln (r) - \nu C_0 r \ln \left(\frac{r}{r_{\text{max}}}\right) + \frac{1}{2} (\nu + 1) C_1 \frac{1}{r} + (1 - \nu) C_2 r \right)$$  \hspace{1cm} (0.156)$$

Considering the displacement at \( r_{\text{max}} \) and \( r_{\text{min}} \) to be zero we can in the same way as in the plane strain case derive:

$$C_1 = C_0 \frac{(2\nu - 1) \ln \left(\frac{r_{\text{min}}}{r_{\text{max}}}\right)}{r_{\text{max}}^2 - r_{\text{min}}^2}$$  \hspace{1cm} (0.157)$$

$$C_2 = \frac{1}{2} C_0 \frac{(2\nu - 1) \left(r_{\text{max}}^2 - r_{\text{min}}^2\right) + (2\nu^2 + \nu - 1) \left(\ln (r_{\text{min}}) r_{\text{min}}^2 - \ln (r_{\text{max}}) r_{\text{max}}^2\right) + 2\nu \ln \left(\frac{r_{\text{min}}}{r_{\text{max}}}\right)}{(1 - \nu) \left(r_{\text{min}}^2 - r_{\text{max}}^2\right)}$$  \hspace{1cm} (0.158)$$

![Development of radial stresses around an infiltration well](image)

**Figure 55** Development of radial stresses in a finite plane (plane stress case with no displacement at \( r_{\text{max}} \) and \( r_{\text{min}} = 1000 \text{ m}; \ r_{\text{min}} = 0.1 \text{ m}; \ \nu = 0.3 \)**
Figure 56 Development of tangential stresses in a finite plane (plane stress case with no displacement at \( r_{\text{max}} \) and \( r_{\text{max}} = 1000 \text{ m}; \ r_{\text{min}} = 0.1 \text{ m}; \ \nu = 0.3 \))

Figure 57 Development of stresses in a finite plane (plane stress case with no displacement at \( r_{\text{max}} \) and \( r_{\text{max}} = 1000 \text{ m}; \ r_{\text{min}} = 0.1 \text{ m}; \ \nu = 0.3 \))
We can now derive an expression for $C_0$:

$$C_0 = 2K\sigma, \quad \frac{(1-\nu) r_{\min}^2 - (1-\nu) r_{\max}^2}{(2\nu - 1)(r_{\max}^2 - r_{\min}^2) + (2\nu - 2) \ln \left( \frac{r_{\min}}{r_{\max}} \right) r_{\max}^2 + 2\nu \ln \left( \frac{r_{\min}}{r_{\max}} \right) r_{\min}^2}$$

(0.159)

Again we can give an expression for the water pressure:

$$p(r) = -2K\sigma, \quad \frac{(1-\nu) r_{\min}^2 - r_{\max}^2}{(2\nu - 1)(r_{\max}^2 - r_{\min}^2) + (2\nu - 2) \ln \left( \frac{r_{\min}}{r_{\max}} \right) r_{\max}^2 + 2\nu \ln \left( \frac{r_{\min}}{r_{\max}} \right) r_{\min}^2} \ln \left( \frac{r}{r_{\max}} \right)$$

(0.160)

And when we let $r_{\max}$ go to infinity this formula reduces to the Olsthoorn Rule:

$$\varphi = \lim_{r_{\max} \to \infty} \left( \frac{1}{2K\sigma} \frac{(1-\nu) r_{\min}^2 - r_{\max}^2} {2\nu - 1}(r_{\max}^2 - r_{\min}^2) + (2\nu - 2) \ln \left( \frac{r_{\min}}{r_{\max}} \right) r_{\max}^2 + 2\nu \ln \left( \frac{r_{\min}}{r_{\max}} \right) r_{\min}^2 \right) = \frac{1}{\gamma_w} K\sigma_v$$

(0.161)

Again we can derive a multiplication factor to indicate the influence of the boundary (see also Figure 59):

$$\varphi = \frac{(1-\nu)(a^2 - 1) \ln(a)}{(1-2\nu)(a^2 - 1) + (2\nu - 1) \ln(a) + 2\nu \ln(a) a^2}$$

(0.162)

With:

$$a = \frac{r_{\min}}{r_{\max}}$$

(0.163)
We can also, instead of assuming no displacement, assume no radial stress increment at $r_{\text{max}}$ ($\sigma_r(r_{\text{max}}) = 0$). This leads in the same way as above to an expression for $C_1$, $C_2$ and $C_0$:

$$C_1 = \frac{(2\nu^2 + \nu - 1)\ln \left(\frac{r_{\text{min}}}{r_{\text{max}}}\right) - 2\nu + 1}{(1 + \nu) r_{\text{max}}^2 + (1 - \nu) r_{\text{min}}^2} C_0$$  \hspace{1cm} (0.164)

$$C_2 = \frac{(2\nu - 1) r_{\text{min}}^2 + (1 + 3\nu + 2\nu^2) \ln \left(\frac{r_{\text{max}}}{r_{\text{min}}}\right) r_{\text{max}}^2 + (1 + \nu - 2\nu^2) \ln \left(\frac{r_{\text{min}}}{r_{\text{max}}}\right) r_{\text{min}}^2}{(1 + \nu) r_{\text{max}}^2 + (1 - \nu) r_{\text{min}}^2} C_0$$  \hspace{1cm} (0.165)

$$C_0 = -2K\sigma_r \frac{(1 + \nu) r_{\text{max}}^2 + (1 - \nu) r_{\text{max}}^2}{(1 - 2\nu) \left(r_{\text{max}}^2 - r_{\text{min}}^2\right) - (2 + 2\nu) \ln \left(\frac{r_{\text{min}}}{r_{\text{max}}}\right) r_{\text{max}}^2 - 2\nu \ln \left(\frac{r_{\text{min}}}{r_{\text{max}}}\right) r_{\text{min}}^2}$$  \hspace{1cm} (0.166)

For the water pressure we can now write:

$$p(r) = 2K\sigma_r \frac{(1 + \nu) r_{\text{max}}^2 + (1 - \nu) r_{\text{max}}^2}{(1 - 2\nu) \left(r_{\text{max}}^2 - r_{\text{min}}^2\right) - (2 + 2\nu) \ln \left(\frac{r_{\text{min}}}{r_{\text{max}}}\right) r_{\text{max}}^2 - 2\nu \ln \left(\frac{r_{\text{min}}}{r_{\text{max}}}\right) r_{\text{min}}^2} \ln \left(\frac{r}{r_{\text{max}}}\right)$$  \hspace{1cm} (0.167)
And when we let \( r_{\text{max}} \) go to infinity this formula like in the previous cases reduces to the Olsthoorn Rule:

\[
\varphi = \lim_{r_{\text{max}} \to \infty} \left[ \frac{1}{\gamma_w} \frac{2K\sigma_v}{(2\nu-1)(r_{\text{max}}^2 - r_{\text{min}}^2) + (2 + 2\nu) \ln \left( \frac{r_{\text{min}}}{r_{\text{max}}} \right) r_{\text{max}}^2 + 2\nu \ln \left( \frac{r_{\text{min}}}{r_{\text{max}}} \right) r_{\text{min}}^2} \right] = \frac{1}{\gamma_w} K\sigma_v
\]

\[(0.168)\]

Figure 60 Development of radial stresses in a finite plane (plane stress case with no stress increment at \( r_{\text{max}} \) and \( r_{\text{max}} = 1000 \text{ m}; \ r_{\text{min}} = 0.1 \text{ m}; \ \nu = 0.3)\]
Figure 61 Development of tangential stresses in a finite plane (plane stress case with no stress increment at $r_{\text{max}}$ and $r_{\text{max}} = 1000$ m; $r_{\text{min}} = 0.1$ m; $\nu = 0.3$)

Figure 62 Development of stresses in a finite plane (plane stress case with no stress increment at $r_{\text{max}}$ and $r_{\text{max}} = 1000$ m; $r_{\text{min}} = 0.1$ m; $\nu = 0.3$)
E.3 Maple listing plane strain

> restart;
> with(DEtools):
> with(PDEtools):

Flow:

> p(r) := -C*ln(r/rmax);

\[ p(r) = -C \ln \left( \frac{r}{r_{\text{max}}} \right) \]

Equilibrium:

> impulsbalans1 := diff(sigma[R](r),r)+diff(p(r),r)+(sigma[R](r) - sigma[theta](r))/r = 0;

\[ \text{impulsbalans1} := \left( \frac{\partial}{\partial r} \sigma_R(r) \right) - \frac{C}{r} + \frac{\sigma_R(r) - \sigma_\theta(r)}{r} = 0 \]

Strains and displacement:

> eq1 := epsilon[R](r) = diff(u(r),r);

\[ e_R(r) = \frac{\partial}{\partial r} u(r) \]

> eq2 := epsilon[theta](r) = u(r)/r;

\[ e_\theta(r) = \frac{u(r)}{r} \]

Hooke’s Law:

> hookel1 := epsilon[R](r) = 1/E*((1-nu^2)*sigma[R](r) - nu*(1+nu)*sigma[theta](r));

\[ \text{hookel1} := e_R(r) = \frac{(1 - \nu^2) \sigma_R(r) - \nu (1 + \nu) \sigma_\theta(r)}{E} \]

> hookel2 := epsilon[theta](r) = 1/E*((1-nu^2)*sigma[theta](r) - nu*(1+nu)*sigma[R](r));

\[ \text{hookel2} := e_\theta(r) = \frac{(1 - \nu^2) \sigma_\theta(r) - \nu (1 + \nu) \sigma_R(r)}{E} \]

Combine Hooke’s Law with eq1 and eq2

> eq3 := subs(hookel1,eq1);

\[ eq3 := \frac{(1 - \nu^2) \sigma_R(r) - \nu (1 + \nu) \sigma_\theta(r)}{E} = \frac{\partial}{\partial r} u(r) \]

> eq4 := subs(hookel2,eq2);

\[ eq4 := \frac{(1 - \nu^2) \sigma_\theta(r) - \nu (1 + \nu) \sigma_R(r)}{E} = \frac{u(r)}{r} \]

Derive \( u(r) \) from eq4

> eq5 := u(r) = solve(eq4,u(r));
At a certain point, \( r_{\text{max}} \), far from the well the displacement in zero.

Combine eq6 with eq7 leaves an expression with only stresses in radial direction

\[
eq 5 := u(r) = - \frac{(-\sigma_0(r) + \sigma_0(r) v^2 + v \sigma_r(r) + v^2 \sigma_r^2(r)) r}{E}
\]

Substitute eq5 in eq3

\[
eq 6 := (\text{eval}(\text{subs}(\text{eq5}, \text{eq3}))) = (1 - v^2) \sigma_r(r) - v (1 + v) \sigma_0(r)
\]

Derive expression for \( \sigma_\theta(r) \) from the impulsbalans1

\[
eq 7 := \sigma_\theta(r) = \text{solve impulsbalans1, sigma[theta]}(r); \]

Combine eq6 with eq7 leaves an expression with only stresses in radial direction

\[
eq 8 := \text{simplify} \left( \text{subs}(\text{eq7}, \text{eq6}) \right); \]

This expression can be solved

\[
eq 9 := \text{dsolve} \left( \text{eq8, sigma[R]}(r) \right); \]

And for the displacement we can now write

\[
eq 11 := u(r) = \text{solve} \left( \text{subs}(\text{eq10, subs( eq9, eq4) }), u(r) \right); \]

At a certain point, \( r_{\text{max}} \), far from the well the displacement in zero.
\[
eq 12 := \text{simplify}(E \times (\text{subs}(u(r_{\text{max}})=0, \text{subs}(r=r_{\text{max}}, \text{eq11}))));
\]
\[
eq 12 := 0 = \frac{1}{2} (C \text{r}_{\text{max}}^2 - _C1 - 2 C \text{r}_{\text{max}}^2 v - C \ln(\text{r}_{\text{max}}) \text{r}_{\text{max}}^2 - 2 _C2 \text{r}_{\text{max}}^2
+ 4 _C2 \text{r}_{\text{max}}^2 v - v^2 C \text{r}_{\text{max}}^2 + v^2 _C1 + 2 v^3 C \text{r}_{\text{max}}^2 + 2 v^2 C \ln(\text{r}_{\text{max}}) \text{r}_{\text{max}}^2
+ 2 v^2 _C2 \text{r}_{\text{max}}^2 - 4 v^3 _C2 \text{r}_{\text{max}}^2 + v C \ln(\text{r}_{\text{max}}) \text{r}_{\text{max}}^2))/(\text{r}_{\text{max}} (-1 + v))
\]

As long as there is horizontal stress in the r direction at the well there will be no displacement and the following equation must be valid:
\[
eq 13 := \text{simplify}(E \times (\text{subs}(u(r_{\text{min}})=0, \text{subs}(r=r_{\text{min}}, \text{eq11}))));
\]
\[
eq 13 := 0 = \frac{1}{2} (C \text{r}_{\text{min}}^2 - _C1 - 2 C \text{r}_{\text{min}}^2 v - C \ln(\text{r}_{\text{min}}) \text{r}_{\text{min}}^2 - 2 _C2 \text{r}_{\text{min}}^2
+ 4 _C2 \text{r}_{\text{min}}^2 v - v^2 C \text{r}_{\text{min}}^2 + v^2 _C1 + 2 v^3 C \text{r}_{\text{min}}^2 + 2 v^2 C \ln(\text{r}_{\text{min}}) \text{r}_{\text{min}}^2
+ 2 v^2 _C2 \text{r}_{\text{min}}^2 - 4 v^3 _C2 \text{r}_{\text{min}}^2 + v C \ln(\text{r}_{\text{min}}) \text{r}_{\text{min}}^2))/(\text{r}_{\text{min}} (-1 + v))
\]

Combination of these equations (eq12 and eq13) give an expression for _C1 and _C2
\[
sol := \text{solve}({eq12,eq13},{_C1,_C2});
\]
\[
sol :=\left\{-_C2 = \frac{1}{2} C (-\ln(\text{r}_{\text{max}}) \text{r}_{\text{max}}^2 - \text{r}_{\text{min}}^2 + \text{r}_{\text{min}}^2 v - \text{r}_{\text{max}}^2 v + \ln(\text{r}_{\text{min}}) \text{r}_{\text{min}}^2 + \text{r}_{\text{min}}^2)
+ (-1 + 2 v) \ln(\text{r}_{\text{min}}) \text{r}_{\text{min}}^2 C \text{r}_{\text{max}}^2 - \text{r}_{\text{min}}^2 v + \text{r}_{\text{min}}^2 v + \text{r}_{\text{max}}^2 - \text{r}_{\text{min}}^2\right\}
\]

\[
>C1(\text{r}_{\text{max}}) := \text{subs}(\text{nu}=0.3, \text{subs}(\text{r}_{\text{min}}=1, \text{subs}(C=1, \text{eval}(_C1, \text{sol}))));
\]
\[
C1(\text{r}_{\text{max}}) := -0.4 \ln\left(\frac{1}{\text{r}_{\text{max}}}\right) \text{r}_{\text{max}}^2 - 0.7
\]

\[
>C2(\text{r}_{\text{max}}) := \text{subs}(\text{nu}=0.3, \text{subs}(\text{r}_{\text{min}}=1, \text{subs}(C=1, \text{eval}(_C2, \text{sol}))));
\]
\[
C2(\text{r}_{\text{max}}) := \frac{1}{2} -\ln(\text{r}_{\text{max}}) \text{r}_{\text{max}}^2 - 0.7 + 0.7 \text{r}_{\text{max}}^2 + \ln(1)
\]

\[
>\text{plot}({C1(\text{r}_{\text{max}}),C2(\text{r}_{\text{max}})}, \text{r}_{\text{max}}=1..\text{infinity}, \text{labels}=[\text{r}_{\text{max}}, C], \text{labeldirections}=[\text{HORIZONTAL}, \text{VERTICAL}]);
\]
At the well the change of $\sigma_R$ is:

\[ eq14 := \sigma_R = \frac{1}{2} \left( -C_1 \right) \ln(\min) - \frac{1}{2} \left( C_1 \right) \ln(\max) + C_2 \]

The original horizontal stress equals:

\[ eq15 := \sigma_R = K \sigma_v \]

We can now calculate for what value of $C$ and $p$ (in meters overpressure), $\sigma_R$ reaches this value

\[ eq16 := C = solve(subs(sol, subs(eq15, eq14)), C) \]

\[ eq17 := solve(subs(sol, subs(eq15, eq14)), C) * 1/10 * \ln(min/max) \]

And when $r_{max}$ goes to infinity we get:

\[ answ := phi = limit(eq17, r_{max}=\infty) \]

\[ eq18 := \sigma_R(min) = 0, subs(r=r_{max}, eq9) \]
eq18 := \( 0 = \frac{1}{2} \frac{-C_1}{(-1 + v) \text{rmax}^2} - \frac{1}{2} \frac{C_1 v}{(-1 + v) \text{rmax}^2} - \frac{1}{2} \frac{C \ln(\text{rmax})}{-1 + v} + C_2 \)

\[
\begin{align*}
\text{sol2} &:= \text{solve}\{\text{eq18}, \text{eq13}\}, \{C_1, C_2\}; \\
\text{sol2} &:= \left\{ C_2 = -\frac{1}{2} C (-2 v^2 \text{rmin}^2 + 3 \text{rmin}^2 v - 2 v \ln(\text{rmin}) \text{rmin}^2 - \text{rmin}^2 \\
&\quad + \ln(\text{rmin}) \text{rmin}^2 + \ln(\text{rmax}) \text{rmax}^2) \right\} \left\{ 2 v^2 \text{rmin}^2 - 3 \text{rmin}^2 v + \text{rmin}^2 + \text{rmax}^2 - \text{rmax}^2 v, \\
&\quad C_1 = \frac{2 v^2 - 3 v + 1 - \ln(\frac{\text{rmin}}{\text{rmax}}) + 2 \ln(\frac{\text{rmin}}{\text{rmax}}) v}{2 v^2 \text{rmin}^2 - 3 \text{rmin}^2 v + \text{rmin}^2 + \text{rmax}^2 - \text{rmax}^2 v} \right\}
\end{align*}
\]

eq19 := \( C = \text{solve}\{\text{subs(sol2,subs(\text{eq15},\text{eq14})},C); \\
eq 2 K \sigma_v (2 v^2 \text{rmin}^2 - 3 \text{rmin}^2 v + \text{rmin}^2 + \text{rmax}^2 - \text{rmax}^2 v) / \left( 2 \text{rmax}^2 v^2 \\
- 2 v^2 \text{rmin}^2 - 3 \text{rmax}^2 v + 3 \text{rmin}^2 v + 2 \ln(\frac{\text{rmin}}{\text{rmax}}) \text{rmax}^2 v + \ln(\text{rmax}) \text{rmax}^2 \\
- \ln(\text{rmin}) \text{rmax}^2 - \ln(\frac{\text{rmin}}{\text{rmax}}) \text{rmax}^2 + \text{rmax}^2 - \text{rmin}^2 \right) \}
\]

\[
\begin{align*}
\text{eq20} &:= \text{solve}\{\text{subs(sol2,subs(\text{eq15},\text{eq14})},C) * 1/10 * \ln(\text{rmin}/\text{rmax}); \\
\text{eq20} &:= -\frac{1}{5} K \sigma_v (2 v^2 \text{rmin}^2 - 3 \text{rmin}^2 v + \text{rmin}^2 + \text{rmax}^2 - \text{rmax}^2 v) \ln(\frac{\text{rmin}}{\text{rmax}}) / \left( 2 \text{rmax}^2 v^2 - 2 v^2 \text{rmin}^2 - 3 \text{rmax}^2 v + 3 \text{rmin}^2 v + 2 \ln(\frac{\text{rmin}}{\text{rmax}}) \text{rmax}^2 v \\
+ \ln(\text{rmax}) \text{rmax}^2 - \ln(\text{rmin}) \text{rmax}^2 - \ln(\frac{\text{rmin}}{\text{rmax}}) \text{rmax}^2 + \text{rmax}^2 - \text{rmin}^2 \right) \\
\text{answ2} &:= \text{phi = limit(eq20, \text{rmax}=\infty);} \\
\text{answ2} &:= \phi = \frac{1}{10} K \sigma_v
\end{align*}
\]

### E.4 Maple listing plane stress

\[
\begin{align*}
\text{p(r)} &:= -C * \ln(\frac{r}{\text{rmax}}); \\
\text{p(r)} &:= -C \ln(\frac{r}{\text{rmax}})
\end{align*}
\]

Equilibrium
\[
\begin{align*}
\text{impulsbalans1} &:= \text{diff(\sigma[R](r),r)} + \text{diff(p(r),r)} + (\sigma[R](r) - \sigma[\theta](r))/r = 0;
\end{align*}
\]
impulsbalans1 := \left( \frac{\partial}{\partial r} \sigma_r(r) \right) - \frac{C}{r} + \frac{\sigma_r(r) - \sigma_\theta(r)}{r} = 0

Strains and displacement
> eq1 := \text{epsilon}[R](r) = \text{diff}(u(r), r);
> eq1 := \varepsilon_R(r) = \frac{\partial}{\partial r} u(r)

> eq2 := \text{epsilon}[\theta](r) = \frac{u(r)}{r};
> eq2 := \varepsilon_\theta(r) = \frac{u(r)}{r}

Hooke’s Law
> hookel := \text{epsilon}[R](r) = \frac{1}{E} \left( \sigma_r(r) - \nu \left( \sigma_\theta(r) - p(r) \right) \right);
> hookel := \varepsilon_R(r) = \frac{\sigma_r(r) - \nu \left( \sigma_\theta(r) + C \ln \left( \frac{r}{r_{max}} \right) \right)}{E}

> hooke2 := \text{epsilon}[\theta](r) = \frac{1}{E} \left( \sigma_\theta(r) - \nu \left( \sigma_r(r) - p(r) \right) \right);
> hooke2 := \varepsilon_\theta(r) = \frac{\sigma_\theta(r) - \nu \left( \sigma_r(r) + C \ln \left( \frac{r}{r_{max}} \right) \right)}{E}

Combine Hooke’s Law with eq1 and eq2
> eq3 := \text{subs}(hookel, eq1);
> eq3 := \frac{\sigma_r(r) - \nu \left( \varepsilon_\theta(r) + C \ln \left( \frac{r}{r_{max}} \right) \right)}{E} = \frac{\partial}{\partial r} u(r)

> eq4 := \text{subs}(hooke2, eq2);
> eq4 := \frac{\sigma_\theta(r) - \nu \left( \varepsilon_R(r) + C \ln \left( \frac{r}{r_{max}} \right) \right)}{E} = \frac{u(r)}{r}

Derive \( u(r) \) from eq4
> eq5 := u(r) = \text{solve}(eq4, u(r));
> eq5 := u(r) = \left( \frac{\sigma_\theta(r) - \nu \sigma_r(r) - \nu C \ln \left( \frac{r}{r_{max}} \right)}{E} \right) r

Substitute eq5 in eq3
> eq6 := \text{eval} (\text{subs}(eq5, eq3));
> eq6 := \sigma_r(r) - \nu \left( \varepsilon_\theta(r) - \frac{\partial}{\partial r} \sigma_r(r) - \nu \frac{C}{r} \right) r + \frac{\sigma_\theta(r) - \nu \sigma_r(r) - \nu C \ln \left( \frac{r}{r_{max}} \right)}{E}

Derive expression for \( \sigma_\theta(r) \) from the impulsbalans1
eq7 := \sigma_\theta(r) = \text{solve(impulsbalans1, \sigma_\theta(r))};

Combine eq6 with eq7 leaves an expression with only stresses in radial direction

eq8 := simplify(subs(eq7,eq6));

eq 0 = \frac{\partial}{\partial r} \sigma_r(r) + \sqrt{r} + \sqrt{C} - \sqrt{\sigma_r(r)} + \sqrt{C \ln \left( \frac{r}{r_{\text{max}}} \right)} \right) \frac{1}{E}

This expression can be solved

eq9 := dsolve(eq8, \sigma_R(r));

eq 0 = \frac{\partial}{\partial r} \sigma_r(r) = \frac{\text{C1}}{r^2} + \ln(r) \sqrt{r} + \frac{1}{2} \ln(r) C + _C2

And for the displacement we can now write

eq11 := u(r) = solve(subs(eq10, subs(eq9, eq4)), u(r));

eq 0 = -\frac{1}{2} \left( -2 C \sqrt{r} + C \sqrt{r}^2 - _C1 - \ln(r) \sqrt{r} C \sqrt{r}^2 - 2 \ _C2 \sqrt{r}^2 \\
- v \ _C1 + 2 \ln(r) \sqrt{v} C \sqrt{r}^2 + 2 \ _C2 \sqrt{r}^2 + 2 \sqrt{v} \ C \ln \left( \frac{r}{r_{\text{max}}} \right) \sqrt{r}^2 \right) (E \ r)

At a certain point, r[\text{max}], far from the well the displacement in zero.

eq12 := simplify(E * (subs(u(rmax)=0, subs(r=rmax, eq11))));

eq 0 = -\frac{1}{2} \left( -2 C \text{rmax}^2 \sqrt{v} + C \text{rmax}^2 - _C1 - \ln(\text{rmax}) \sqrt{v} C \text{rmax}^2 \\
- \ln(\text{rmax}) C \text{rmax}^2 - 2 \ _C2 \text{rmax}^2 - v \ _C1 + 2 \ln(\text{rmax}) \sqrt{v} C \text{rmax}^2 \\
+ 2 \sqrt{v} \ _C2 \text{rmax}^2) / \text{rmax}

As long a there is horizontal stress in the r direction at the well there will be no displacement
and the following equation must be valid:

eq13 := simplify(E * (subs(u(rmin)=0, subs(r=rmin, eq11))));

eq 0 = -\frac{1}{2} \left( -2 C \text{rmin}^2 \sqrt{v} + C \text{rmin}^2 - _C1 - \ln(\text{rmin}) \sqrt{v} C \text{rmin}^2 \\
- \ln(\text{rmin}) C \text{rmin}^2 - 2 \ _C2 \text{rmin}^2 - v \ _C1 + 2 \ln(\text{rmin}) \sqrt{v} C \text{rmin}^2 \\
+ 2 \sqrt{v} \ _C2 \text{rmin}^2 + 2 \sqrt{v} C \ln \left( \frac{\text{rmin}}{r_{\text{max}}} \right) \text{rmin}^2 \right) / \text{rmin}

combination of these equations (eq12 and eq13) give an expression for _C1 and _C2

sol := solve({eq12, eq13}, {_C1, _C2});
sol := \[
-C2 = -\frac{1}{2} C (2 v rmin^2 - rmin^2 - \ln(rmin) v rmin^2 + \ln(rmin) rmin^2
- \ln(rmax) v rmax^2 - 2 \ln(rmin) v^2 rmin^2 - 2 v rmax^2 + 2 rmin^2 v \ln(rmax)
- \ln(rmax) rmax^2 + 2 \ln(rmax) v^2 rmax^2 + rmax^2) / (rmin^2 - v rmin^2 - rmax^2 + v rmax^2),
_C1 = -\frac{rmax^2 C rmin^2 (2 v - 1) \ln\left(\frac{rmax}{rmin}\right)}{-rmin^2 + rmax^2}
\]

> C1(rmax) := subs(nu=.3,subs(rmin=1,subs(C=1,eval(_C1,sol))));

> C2(rmax) := subs(nu=.3,subs(rmin=1,subs(C=1,eval(_C2,sol))));

> plot({C1(rmax),C2(rmax)},rmax=1..infinity,labels=[r_max,C],labelsdirections=[HORIZONTAL,VERTICAL]);

> plot({C1(rmax),C2(rmax)},rmax=1..10000,labels=[r_max,C],labelsdirections=[HORIZONTAL,VERTICAL]);

At the well the change of sigma[R] is:
The original horizontal stress equals:
\[ \sigma_R = K \sigma_v \]

We can now calculate for what value of C and p (in meters overpressure), \( \sigma_R \) reaches this value
\[ C = \frac{1}{5} K \sigma_v \left( r_{\text{min}}^2 - \nu r_{\text{min}}^2 - r_{\text{max}}^2 + \nu r_{\text{max}}^2 \right) \ln \left( \frac{r_{\text{min}}}{r_{\text{max}}} \right) + \frac{2}{5} \frac{r_{\text{max}}^2 \ln(r_{\text{min}})}{r_{\text{max}}} + \frac{2}{5} \frac{r_{\text{max}}^2 \ln(r_{\text{max}})}{r_{\text{max}}} + 2 \nu r_{\text{max}}^2 + \frac{2}{5} r_{\text{max}}^2 + 3 \frac{r_{\text{max}}^2 \ln(\frac{r_{\text{max}}}{r_{\text{min}}})}{r_{\text{max}}} \ln(\frac{r_{\text{max}}}{r_{\text{min}}}) + 3 \frac{r_{\text{max}}^2 \ln(\frac{r_{\text{max}}}{r_{\text{min}}})}{r_{\text{max}}} \ln(\frac{r_{\text{max}}}{r_{\text{min}}}) + 2 \nu r_{\text{min}}^2 + 2 \nu r_{\text{min}}^2 \ln(\frac{r_{\text{max}}}{r_{\text{min}}}) \]

And when \( r_{\text{max}} \) goes to infinity we get:
\[ \phi = \frac{1}{10} K \sigma_v \]
\[
\text{sol2} := \{ \_C1 = C \, r_{\text{max}}^2 \, r_{\text{min}}^2 \left( \ln(r_{\text{max}}) \, v - 2 \ln(r_{\text{max}}) \, v^2 + 2 \ln(r_{\text{max}}) \right) / (r_{\text{max}}^2 + r_{\text{min}}^2 + v \, r_{\text{max}}^2 - v \, r_{\text{min}}^2) \}, \_C2 = -\frac{1}{2} C \left( 2 \, v \, r_{\text{min}}^2 - r_{\text{min}}^2 \right)
\]

\[
> \text{eq19} := C = \text{solve(subs(sol2,subs(eq15,eq14)),C)};
\]

\[
\text{eq19} := C = -2 \, K \, \sigma_v \left( r_{\text{max}}^2 + r_{\text{min}}^2 + v \, r_{\text{max}}^2 - v \, r_{\text{min}}^2 \right) / \left( -r_{\text{min}}^2 + r_{\text{max}}^2 \right)
\]

\[
+ 2 \ln(r_{\text{max}}) \, r_{\text{max}}^2 - 2 \, r_{\text{max}}^2 \ln(r_{\text{min}}) + 2 \, v \, r_{\text{min}}^2 - 4 \, r_{\text{max}}^2 \ln(r_{\text{min}}) \, v
\]

\[
+ 4 \ln(r_{\text{max}}) \, v \, r_{\text{max}}^2 + 2 \, r_{\text{max}}^2 \, v \ln\left( \frac{r_{\text{min}}}{r_{\text{max}}} \right) - 2 \, v \ln\left( \frac{r_{\text{min}}}{r_{\text{max}}} \right) r_{\text{min}}^2 - 2 \, v \, r_{\text{max}}^2 \right) \}
\]

\[
> \text{eq20} := \text{solve(subs(sol2,subs(eq15,eq14)),C)*1/10*ln(r_{\text{min}}/r_{\text{max}})};
\]

\[
\text{eq20} := -\frac{1}{5} K \, \sigma_v \left( r_{\text{max}}^2 + r_{\text{min}}^2 + v \, r_{\text{max}}^2 - v \, r_{\text{min}}^2 \right) \ln\left( \frac{r_{\text{min}}}{r_{\text{max}}} \right) / \left( -r_{\text{min}}^2 + r_{\text{max}}^2 \right)
\]

\[
+ 2 \ln(r_{\text{max}}) \, r_{\text{max}}^2 - 2 \, r_{\text{max}}^2 \ln(r_{\text{min}}) + 2 \, v \, r_{\text{min}}^2 - 4 \, r_{\text{max}}^2 \ln(r_{\text{min}}) \, v
\]

\[
+ 4 \ln(r_{\text{max}}) \, v \, r_{\text{max}}^2 + 2 \, r_{\text{max}}^2 \, v \ln\left( \frac{r_{\text{min}}}{r_{\text{max}}} \right) - 2 \, v \ln\left( \frac{r_{\text{min}}}{r_{\text{max}}} \right) r_{\text{min}}^2 - 2 \, v \, r_{\text{max}}^2 \right) \}
\]

\[
> \text{answ2} := \phi = \text{limit(eq20,r_{\text{max}}=\infty)};
\]

\[
\text{answ2} := \phi = \frac{1}{10} K \, \sigma_v
\]