The influence of breaking waves on the vertical velocity distribution in the surf zone

Master Thesis
September 1994

TU Delft
Delft University of Technology
delft hydraulics
The influence of breaking waves on the vertical velocity distribution in the surf zone

D.J.R. Walstra
Abstract

In this report we describe the development and calibration of a model which predicts the vertical distribution of the horizontal velocities induced by waves, the so-called undertow. Location of interest is the nearshore region where the flow is induced by both breaking and non-breaking waves.

The model consists out of three modules. The first module computes the properties of the incoming (breaking) waves, a second module converts the resulting data from the first module so that it can be used as input for the third module which is a 2DV model. The 2DV model is a 2DV version of DELFT HYDRAULICS' TRISULA which is a 3D hydrostatic tidal model.

The waves influence the flow in three ways. A first effect is a shear stress at the water surface which originates from the breaking of waves. Also is shown in this thesis that the waves induce a mass flux which has as a result that there is a net flow directed off-shore. The third effect is the influence of the orbital motion on the viscosity distribution. All these effects are included in the model described in this thesis.

The model has been tested against measured data in the Delta flume to see how accurate the model predicts the undertow.
Contents

Abstract
List of tables
List of figures
List of symbols

1 Introduction .................................................. 1

1.1 General .................................................. 1
1.2 Aim ..................................................... 1
1.3 Methodology ............................................. 1

2 The shallow water equations .............................. 3

2.1 Introduction ............................................. 3
2.2 Properties of the fluid .................................. 3
2.3 Short derivation of the Navier-Stokes equation 11
2.3.1 The expression for the stress tensor ............... 4
2.3.2 The deviatoric stress tensor ....................... 5
2.3.3 Time averaging ..................................... 6
2.3.4 Reynolds-stresses .................................. 8

2.4 Shallow water equations ................................. 9

2.4.1 Hydrostatic flow model including net effects of waves ... 10

2.5 The sigma-transformation ................................. 12

2.5.1 Introduction ........................................ 12
2.5.2 The sigma-coordinate ................................ 12
2.5.3 The $x^*$, $y^*$ and $t^*$ coordinates .................. 12
2.5.4 First order partial derivatives ...................... 13
2.5.5 Transformed velocity components ................... 14
2.5.6 The equations of motion ............................. 14
2.5.7 Depth averaged continuity equation ................. 17
## Contents (continued)

3 Shear stress distribution in dissipative water waves ........................................ 20

3.1 Introduction ........................................ 20
3.2 The radiation stresses ........................................ 20
3.3 Dissipation in the wave boundary layer ........................................ 22
  3.3.1 Exchange of energy ........................................ 22
  3.3.2 The water motion in dissipative waves ........................................ 24
  3.3.3 The shear stress distribution ........................................ 25
  3.3.4 Inserting terms in shallow water equations ........................................ 26
3.4 Spilling breakers and broken waves ........................................ 27
  3.4.1 The exchange of energy ........................................ 27
  3.4.2 Vertical shear stress distribution ........................................ 30

4 Wave model ........................................ 33

4.1 Introduction ........................................ 33
4.2 A dissipation model for random waves ........................................ 33
  4.2.1 Introduction ........................................ 33
  4.2.2 Wave height distribution ........................................ 34
  4.2.3 Determination of Breaker Height ........................................ 35
  4.2.4 Mean energy dissipation in a breaking wave field ........................................ 36
  4.2.5 Energy flux ........................................ 36

4.3 Surface roller effect ........................................ 37
  4.3.1 Introduction ........................................ 37
  4.3.2 Properties of the roller, Svendsen (1984a) ........................................ 37
  4.3.3 Kinetic energy in roller, Okayasu (1989) extended ........................................ 38
  4.3.4 Potential energy of roller ........................................ 40
  4.3.5 Roller equation ........................................ 40
  4.3.6 Roelvink (1993) ........................................ 41
  4.3.7 Okayasu extended ........................................ 42

4.4 Mass flux in the surf zone ........................................ 42
  4.4.1 Introduction ........................................ 42
  4.4.2 Mass flux due to wave motion ........................................ 43
  4.4.3 Mass flux due to the roller ........................................ 44
## Contents (continued)

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.5</td>
<td>The shear stress</td>
<td>45</td>
</tr>
<tr>
<td>4.5.1</td>
<td>Introduction</td>
<td>45</td>
</tr>
<tr>
<td>4.5.2</td>
<td>Applying results of wave model</td>
<td>45</td>
</tr>
<tr>
<td>5</td>
<td><strong>TRISULA as a 2DV-model</strong></td>
<td>48</td>
</tr>
<tr>
<td>5.1</td>
<td>Introduction</td>
<td>48</td>
</tr>
<tr>
<td>5.2</td>
<td>Flow model</td>
<td>48</td>
</tr>
<tr>
<td>5.3</td>
<td>The turbulence model</td>
<td>49</td>
</tr>
<tr>
<td>5.3.1</td>
<td>The k- and e-equations</td>
<td>49</td>
</tr>
<tr>
<td>5.3.2</td>
<td>The empirical constants</td>
<td>50</td>
</tr>
<tr>
<td>5.3.3</td>
<td>Incorporation of wave induced turbulence</td>
<td>50</td>
</tr>
<tr>
<td>5.4</td>
<td>Boundary conditions</td>
<td>52</td>
</tr>
<tr>
<td>5.4.1</td>
<td>Introduction</td>
<td>52</td>
</tr>
<tr>
<td>5.4.2</td>
<td>Wall boundaries</td>
<td>52</td>
</tr>
<tr>
<td>5.4.3</td>
<td>Free surface boundary</td>
<td>56</td>
</tr>
<tr>
<td>5.5</td>
<td>Summarizing wave effects</td>
<td>57</td>
</tr>
<tr>
<td>5.6</td>
<td>The staggered grid</td>
<td>58</td>
</tr>
<tr>
<td>6</td>
<td><strong>Numerical model</strong></td>
<td>58</td>
</tr>
<tr>
<td>6.1</td>
<td>Introduction</td>
<td>58</td>
</tr>
<tr>
<td>6.2</td>
<td>Physical experiments</td>
<td>58</td>
</tr>
<tr>
<td>6.2.1</td>
<td>Introduction</td>
<td>58</td>
</tr>
<tr>
<td>6.2.2</td>
<td>Beach geometries</td>
<td>58</td>
</tr>
<tr>
<td>6.2.3</td>
<td>Wave conditions</td>
<td>58</td>
</tr>
<tr>
<td>6.2.4</td>
<td>Measurement set-up</td>
<td>58</td>
</tr>
<tr>
<td>6.2.5</td>
<td>Measured parameters</td>
<td>59</td>
</tr>
<tr>
<td>6.3</td>
<td>Numerical model</td>
<td>59</td>
</tr>
<tr>
<td>6.3.1</td>
<td>Introduction</td>
<td>59</td>
</tr>
<tr>
<td>6.3.2</td>
<td>Bottom boundary</td>
<td>60</td>
</tr>
<tr>
<td>6.3.3</td>
<td>Water level, water depth and wave height</td>
<td>60</td>
</tr>
<tr>
<td>6.3.4</td>
<td>Vertical boundaries</td>
<td>60</td>
</tr>
<tr>
<td>6.3.5</td>
<td>Input parameters</td>
<td>60</td>
</tr>
</tbody>
</table>
Contents (continued)

7 Comparison with experiments ........................................... 62

7.1 Introduction ......................................................... 62

7.2 Wave models ....................................................... 62
   7.2.1 Introduction .................................................. 62
   7.2.2 Battjes and Janssen (1978) ................................. 62
   7.2.3 Roller models ............................................... 62
   7.2.4 Turbulence model ........................................... 67
   7.2.5 Mass flux .................................................... 70

7.3 Flow model ........................................................ 71
   7.3.1 Introduction .................................................. 71
   7.3.2 Number of layers ............................................ 71
   7.3.3 Accuracy criterion ......................................... 71
   7.3.4 Varying horizontal mesh-size ............................. 71
   7.3.5 Comparison with test series II ............................ 72

8 Evaluation and recommendations ...................................... 73

8.1 Introduction ......................................................... 73

8.2 Wave model ........................................................ 73

8.3 Turbulence model .................................................. 74

8.4 Mass flux induced by (breaking) waves ........................... 74

8.5 Recommendations .................................................. 75

9 Acknowledgements ..................................................... 76

References

Figures

Appendix A: Derivation of the continuity equation, and the inertial terms of the
            momentum equation

Appendix B: Shear stress distribution in dissipative waves

Appendix C: Discretisation
List of tables

5.1 Empirical constants in $k$-$\varepsilon$ model
6.1 Set-up of various sub-tests
6.2 Measured parameters
7.1 Total performance of various models
7.2 Total performance for test-series II
# List of figures

2.1 Coordinate system with x,y-plane in undisturbed water level
2.2 Sigma transformation, boundary fitted in the vertical plane
2.3 Wave induced mass flux between trough level and instantaneous water level

3.1 Vertical distribution of radiation shear stress from pressure, $S_{xx}^1$ and momentum, $S_{xx}^2$
3.2 Control volume (Eulerian description) on which the momentum equation is applied
3.3 Definition sketch and forces acting on a surface roller
3.4 Forces acting on a vertical column of the roller
3.5 Flow below solid body transported with velocity c
3.6 Control volume for the momentum equation applied on the roller

4.1 Forces induced by roller
4.2 Velocity distribution in roller
4.3 Potential energy of roller

5.1 Vertical distribution of wave induced turbulence
5.2 The staggered grid for the 2DV-model

6.1 Schematized geometry of LIP experiments in the Delta flume

7.1 LIP 11D Delta flume experiments (Test IA, IB and IC)
7.2 LIP 11D Delta flume experiments (Test IIA, IIB and IIC)
7.3 Wave height distributions
7.4 Results of wave model (Test IA)
7.5 Results of wave model (Test IB)
7.6 Results of wave model (Test IC)
7.7 Horizontal velocities (0.1 [m] above bottom)
7.8 Horizontal velocities (0.2 [m] above bottom)
7.9 Horizontal velocity results (Test IA)
7.10 Horizontal velocity results (Test IB)
7.11 Horizontal velocity results (Test IC)
7.12 Relative and absolute deviation per horizontal plane
7.13 Averaged absolute deviation per vertical
7.14 Averaged relative deviation per vertical
7.15 Comparison of measured and computed set-up
7.16 Vector plot of the velocities (Test IA)
7.17 Vector plot of the velocities (Test IB)
7.18 Vector plot of the velocities (Test IC)
7.19 Results of wave model (varying slope of wave front)
7.20 Horizontal velocity results (comparison of different values for $\beta$)
7.21 Turbulence properties (Test IC)
7.22 Turbulence properties at bottom boundary (Test IC)
7.23 Horizontal velocity results (test of turbulent energy dissipation in free surface boundary, Test IB)
7.24 Turbulence properties (Test IB)
List of figures (continued)

7.25  Horizontal velocity results (test of the potential energy in the roller, Test IB)
7.26  Horizontal velocity results (test of the correction factor for the wave energy, Test IB)
7.27  Horizontal velocity results (comparison of results obtained with a different number of layers, Test IB)
7.28  Horizontal velocity results (different accuracy criterium, Test IB)
7.29  Horizontal velocity results (varying horizontal mesh-size)
7.30  Horizontal velocity results (Test IIA)
7.31  Horizontal velocity results (Test IIB)
7.32  Horizontal velocity results (Test IIC)
7.33  Horizontal velocity results (Test IIC)
# List of symbols

**Roman letters**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>cross sectional area of roller</td>
</tr>
<tr>
<td>$a_n$</td>
<td>fourier constant</td>
</tr>
<tr>
<td>$B$</td>
<td>horizontal momentum</td>
</tr>
<tr>
<td>$b_n$</td>
<td>fourier constant</td>
</tr>
<tr>
<td>$C_d$</td>
<td>empirical constant</td>
</tr>
<tr>
<td>$C_{\mu}$</td>
<td>empirical constant</td>
</tr>
<tr>
<td>$C_{ei}$</td>
<td>empirical constant</td>
</tr>
<tr>
<td>$C_{e2}$</td>
<td>empirical constant</td>
</tr>
<tr>
<td>$C_f$</td>
<td>drag coefficient</td>
</tr>
<tr>
<td>$c$</td>
<td>wave celerity</td>
</tr>
<tr>
<td>$c_g$</td>
<td>group celerity</td>
</tr>
<tr>
<td>$c_s$</td>
<td>empirical constant</td>
</tr>
<tr>
<td>$D$</td>
<td>dissipation</td>
</tr>
<tr>
<td>$D_r$</td>
<td>dissipation in roller</td>
</tr>
<tr>
<td>$D_w$</td>
<td>dissipation in wave</td>
</tr>
<tr>
<td>$d_{ij}$</td>
<td>deviatoric stress tensor</td>
</tr>
<tr>
<td>$dt$</td>
<td>time step</td>
</tr>
<tr>
<td>$dx$</td>
<td>mesh size</td>
</tr>
<tr>
<td>$E$</td>
<td>wave energy</td>
</tr>
<tr>
<td>$E_1$</td>
<td>energy of rotational component of roller</td>
</tr>
<tr>
<td>$E_2$</td>
<td>energy of parallel component of roller</td>
</tr>
<tr>
<td>$E_3$</td>
<td>total energy of the roller</td>
</tr>
<tr>
<td>$E_{br}$</td>
<td>energy density of the broken waves</td>
</tr>
<tr>
<td>$E_p$</td>
<td>potential energy of wave</td>
</tr>
<tr>
<td>$E_{r,k}$</td>
<td>kinetic energy of roller</td>
</tr>
<tr>
<td>$E_{\mu}$</td>
<td>potential energy of roller</td>
</tr>
<tr>
<td>$E_{r}$</td>
<td>energy density of breaking waves</td>
</tr>
<tr>
<td>$F$</td>
<td>shear force</td>
</tr>
<tr>
<td>$F_x$</td>
<td>energy flux</td>
</tr>
<tr>
<td>$F_{\tau}$</td>
<td>shear force induced by roller</td>
</tr>
<tr>
<td>$F(H)$</td>
<td>probability function of the wave heights</td>
</tr>
<tr>
<td>$f$</td>
<td>wave frequency</td>
</tr>
<tr>
<td>$f_c$</td>
<td>coriolis coefficient</td>
</tr>
<tr>
<td>$f_p$</td>
<td>peak wave frequency</td>
</tr>
<tr>
<td>$g$</td>
<td>earth’s gravitation</td>
</tr>
<tr>
<td>$H$</td>
<td>wave height</td>
</tr>
<tr>
<td>$H_0$</td>
<td>wave height at $x=0$</td>
</tr>
<tr>
<td>$H_b$</td>
<td>trough to crest wave height of a breaking wave</td>
</tr>
<tr>
<td>$H_m$</td>
<td>maximum wave height</td>
</tr>
<tr>
<td>$H_{m0}$</td>
<td>spectral first order wave height</td>
</tr>
<tr>
<td>$H$</td>
<td>modal wave height</td>
</tr>
<tr>
<td>$H$</td>
<td>random wave height</td>
</tr>
<tr>
<td>$H_{rms0}$</td>
<td>root mean square wave height in deep water</td>
</tr>
<tr>
<td>$H_{rms}$</td>
<td>root mean square wave height</td>
</tr>
</tbody>
</table>
List of symbols (continued)

\( h \) total water depth
\( i \) unit base vector
\( j \) unit base vector
\( k \) unit base vector
\( k \) wave number
\( k' \) turbulent kinetic energy
\( k' \) fluctuating turbulent kinetic energy
\( k \) averaged turbulent kinetic energy
\( k' \) amplitude of fluctuating turbulent kinetic energy
\( k_w \) kinetic turbulent energy due to waves
\( L \) wave length
\( L_{\text{p0}} \) wave length in deep water
\( L_{R} \) length of roller
\( l_t \) turbulent length scale
\( m_r \) mass of roller per unit width
\( M \) mass flux per unit width
\( M_{\text{liw}} \) mass flux according to linear wave theory
\( M_{\gamma} \) mass flux from roller
\( M_{r,k} \) mass flux originating from kinetic energy of roller
\( M_{r,p} \) mass flux originating from potential energy of roller
\( M_{\text{sfm}} \) mass flux according to stream function method
\( M_t \) total mass flux
\( M_w \) mass flux due to wave motion
\( N \) number of experiments
\( n \) number of layers
\( P \) horizontal pressure force
\( P_{\text{e}} \) wave induced turbulent energy dissipation
\( P_{\text{t}} \) wave induced turbulent energy production
\( p_s \) normal pressure
\( p^* \) additional pressure
\( p_s \) surface pressure
\( Q_b \) percentage of breaking waves
\( Q_{b}' \) empirically modified \( Q_b \)
\( q_{ij} \) Reynolds stresses
\( r \) radial coordinate
\( r_c \) bottom roughness
\( S \) setup
\( S_{\text{xx1}} \) radiation stress
\( S_{\text{xx2}} \) radiation stress originating from the pressure
\( S_{\text{xx}} \) radiation stress originating from the orbital motion
\( s_0 \) deep water steepness
\( s_{ij} \) normal Reynolds stresses
\( T \) wave period
\( T_p \) peak wave period
\( t \) time
List of symbols (continued)

\( t_0 \)  reference time
\( t' \)  time in \( \sigma \)-space
\( \text{tr}(a) \)  trace of matrix \( a \)
\( u_0 \)  horizontal orbital velocity
\( u' \)  friction velocity
\( u_{\text{orb}} \)  orbital motion just outside wave boundary layer
\( \bar{u}_{\text{orb}} \)  velocity amplitude
\( u''_{w} \)  wave friction velocity
\( u \)  velocity
\( \langle u \rangle \)  averaged velocity in x-direction
\( u \)  orbital motion x-direction
\( u' \)  turbulent velocity x-direction
\( u_{1} \)  rotational component of velocity in roller
\( u_{2} \)  parallel component of velocity in roller
\( u_{a} \)  depth averaged velocity
\( v \)  velocity in y-direction
\( \langle v \rangle \)  averaged velocity in y-direction
\( v' \)  turbulent velocity y-direction
\( W \)  time averaged rate of work
\( W_{r} \)  work done by shear stress
\( w \)  velocity in z-direction
\( \langle w \rangle \)  averaged velocity in z-direction
\( w \)  orbital motion z-direction
\( w' \)  turbulent velocity z-direction
\( w \)  additional vertical velocity
\( x \)  horizontal coordinate
\( x' \)  x-coordinate in \( \sigma \)-space
\( y \)  horizontal coordinate
\( y' \)  y-coordinate in \( \sigma \)-space
\( z \)  vertical coordinate
\( z' \)  dimensionless wall distance
\( z' \)  alternative z-coordinate
\( z_{0} \)  roughness height at bottom
\( z_{b} \)  bottom profile
\( z_{t} \)  trough level
\( z_{w} \)  roughness height at free surface
List of symbols (continued)

Greek letters

\( \alpha \) constant in expression for the wave dissipation
\( \beta \) steepness of the wave front
\( \gamma \) ratio between maximum wave height and mean water depth
\( \Delta \) \( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \)
\( \delta_{ij} \) kronecker delta, if \( i=j \) then \( \delta_{ij}=1 \) else \( \delta_{ij}=0 \)
\( e \) energy dissipation
\( e \) dimensionless parameter
\( \xi \) instantaneous water surface
\( \xi^* \) additional water surface elevation
\( \eta \) orthogonal curvilinear coordinate
\( \theta \) direction of wave propagation with respect to positive \( x \)-axis
\( \kappa \) Von Karman constant (0.4)
\( \mu \) viscosity of the fluid
\( \mu_t \) horizontal turbulent kinematic viscosity
\( \nu_t \) turbulent viscosity
\( \xi \) orthogonal curvilinear coordinate
\( \pi \) constant
\( \rho \) density of the water per unit of volume
\( \Sigma \) summation
\( \sigma \) transformed vertical coordinate
\( \sigma_{ij} \) stress tensor
\( \tau \) shear stress
\( \tau_{xx} \) shear stress at water surface in \( x \)-direction
\( \tau_{xy} \) shear stress at water surface in \( y \)-direction
\( \tau_r \) shear stress of roller
\( \psi \) ratio between the square of the wave height and the area of the roller
\( \omega \) wave angular frequency
\( \omega \) transformed vertical velocity
\( \nabla \) \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \)
1 Introduction

1.1 General

In many coastal engineering problems sediment transport plays an important role. Frequently there is a shortage of material at some location while at other places an overabundance of material can be equally troublesome. An important goal of coastal engineering research is to predict the sediment transport rates in a coastal region.

In a coastal region, however, the problem is very complex because both current and waves are of importance for the sediment transport. The current mainly influences the transport velocity, while the waves mainly influence the amount of material to be transported.

For the prediction of coastal changes computer models are available in which the influence of waves is incorporated. Until now these models follow a depth averaged approach (2DH). In a coastal zone however the water movement is characterized by its three dimensional (3D) character of turbulence generated by breaking waves, undertow caused by the gradient in the radiation stress, etcetera. For the prediction of coastal changes in areas where 3D related phenomena dominate it is obvious a 2DH approach can not give accurate results.

DELFT HYDRAULICS has developed TRISULA, a 3D hydrostatic tidal model making use of orthogonal curvilinear coordinates in the horizontal plane and an unsteady non-orthogonal $\sigma$-transformation of the vertical. For the incorporation of wave effects, additional relations have to be found in order to make a good approximation of the flow in the near coastal region.

1.2 Aim

The aim of this study is to develop a model which describes the undertow induced by both breaking and non-breaking waves. The model includes:

- a wave model (Battjes and Janssen, 1978) which is extended with a roller equation (Roelvink, 1993) to take the effects of breaking waves into account;
- a new theory concerning the wave induced mass flux;
- the generation of wave induced turbulence which is incorporated in a two-equation turbulence model.

1.3 Methodology

The basis of this study is a wave model in which the influence of breaking waves is included. The result of the wave model is then used to prescribe boundary conditions for TRISULA. Since we only take waves travelling perpendicular to the coast into account a 2DV version of TRISULA is used. This means that the vertical and only one horizontal direction are considered. The model was tested by comparison of computed and results measured in a flume.
First the shallow water equations are derived from the full Navier-Stokes equations in Chapter 2. It is also shown how the flow can be separated into different types of flow. Generally two parts can be distinguished in the total velocity field: a steady and a fluctuating part. The fluctuating part can be divided in two types of flow: the turbulent flow and the orbital motion. Each of these types of flow has its own physical properties (which are mainly correlated to the adjacent time-scale) and require different approaches.

In Chapter 3 expressions are derived for the wave related terms. They are evaluated for the two following cases: waves with dissipation in the boundary layer and waves with energy dissipation due to spilling breakers or broken waves. A wave model in which the influence of breaking waves is taken into account is described in Chapter 4. Next, the 2DV-model and the incorporated two-equation turbulence model are described in Chapter 5. In Chapter 6 the construction of the numerical model based on flume experiments is highlighted. In Chapter 7 the computed tests are compared with the measured results. Chapter 8 is devoted to a discussion of the results and recommendations for further development of the model.
2 The shallow water equations

2.1 Introduction

TRISULA solves the shallow water equations which consist of the momentum equations, the continuity equation and the transport equation. In the horizontal momentum equations the Coriolis force, density gradients, the turbulent viscosity, the shear stresses exerted by the turbulent flow on the bottom and the wind stress are included. Since we will implement shear stresses at the water surface to account for dissipation by waves (see Chapter 3) it is useful to give a derivation of the shallow water equations as they are used in TRISULA.

First, general properties of the fluid are given which form the basis for all derivations in this report. Next, a short derivation of the Navier-Stokes equations is given followed by a section in which the shallow water equations are treated. These partial differential equations are based on the Navier-Stokes equations and the continuity equation in an Eulerian coordinate system. Since TRISULA uses a scaled coordinate, the so-called $\sigma$-coordinate, a $\sigma$-transformation on the shallow water equations will be performed in the final section.

2.2 Properties of the fluid

(1) Continuum. The physical and mechanical properties of the fluid should not approach infinity or suffer a jump at any isolated points.

(2) Viscosity. For laminar flows, the fluid can be described as a Newtonian fluid, in which molecular viscosity plays an important role. For turbulent flows, it is actually a non-Newtonian fluid featured by its turbulent viscosity.

(3) Incompressibility. The density of a fluid element does not change during its motion. The effect of volumetric variation on the flow can be neglected.

(4) Homogeneity. The fluid, as a medium in mass transportation and heat conduction, is well mixed, or the spatial distribution of its density has no influence on the flow. Flow and transportation can be calculated separately, by taking the resulting water level and flow velocity as inputs of transport equations. Homogeneity together with incompressibility means that the density $\rho$ is constant.

(5) Isotropy. Parameters of material properties, such as the viscosity coefficient $\mu$, do not vary with direction.

2.3 Short derivation of the Navier-Stokes equation

We will define a Cartesian coordinate system with the coordinates $x$ and $y$ in the horizontal plane of the undisturbed water surface. The $z$-axis is chosen perpendicular to this plane positive upwards (see Fig. 2.1).

The derivation of the Navier-Stokes equation that is presented below is based on Batchelor (1967).

First we will give the continuity equation and equation of motion for an incompressible fluid.
The equation of continuity is:

\[ \nabla \cdot \mathbf{u} = 0 \]  \hspace{1cm} (2.1)

The equation of motion:

\[ \rho \frac{D\mathbf{V}}{Dt} = \nabla \cdot \mathbf{\sigma} + \rho \mathbf{F}_i \]  \hspace{1cm} (2.2)

where \( \nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \) (i,j,k are unit basis vectors), \( \mathbf{\sigma} \) denotes a symmetric order-2 Cartesian tensor, called the stress tensor, \( \rho \) is the density of the water and \( u_i \) are the velocities in \( x, y \) and \( z \)-direction.

The equations above equations represent conservation relations for mass and momentum, respectively. In Eq. 2.2 volume and surface forces lead to an effective generation of momentum per unit volume at a rate given by the right-hand side. Surface forces contribute to the acceleration of the fluid only if the stress tensor varies with position in the fluid.

The volume force acting on a fluid in many cases is due simply to the earth’s gravitational field, for which \( F = g \). The stress tensor presents more of a problem, since it is a manifestation of internal reaction in the fluid.

![Coordinate system with x, y-plane in undisturbed water level](image)

Figure 2.1 Coordinate system with x, y-plane in undisturbed water level

### 2.3.1 The expression for the stress tensor

In a fluid at rest only normal stresses are exerted, the normal stress is independent of the direction of the normal to the surface element across which it acts. The stress tensor has the form:

\[ \mathbf{\sigma}_i = -p \delta_{ij} \]  \hspace{1cm} (2.3)

where the parameter \( p \) is the static fluid pressure and may be a function of the position in the fluid and \( \delta_{ij} \) is the Kronecker delta. There is no reason to expect these results to be valid for a fluid in motion, and it is clear from observation that they are not: the tangential stresses then are non-zero, in general, and the normal components of the stress acting across a surface element depend on the direction of the normal to the element. The simple notion of a pressure acting equally in all directions is lost in most cases of fluid in motion.
It is useful nevertheless to have available a scalar quantity characterizing a moving fluid which is analogous to the static fluid pressure in the sense that it is a measure of the local intensity of the squeezing of the fluid. We define the pressure at a point in a moving fluid to be the mean normal stress with sign reversed, and denoted by \( p \):

\[
p = -\frac{1}{3} \sigma_{ii} = \frac{1}{3} \sigma_{ij} \delta_{ij}
\]

(2.4)

It is convenient now to regard the stress tensor \( \sigma_{ij} \) as the sum of an isotropic part \(-p \delta_{ij}\), having the same form as the stress tensor in a fluid at rest and a remaining non-isotropic part, \( d_{ij} \), contributing the tangential stresses:

\[
\sigma_{ij} = -p \delta_{ij} + d_{ij}
\]

(2.5)

The non-isotropic part \( d_{ij} \) may be termed the deviatoric stress tensor, and has the distinctive property of being due entirely to the existence of motion in the fluid.

Since the stress at any point in the fluid is an expression of the mutual reactions of adjacent parts of fluid near that point, it is natural to consider the connection between the stress and the local properties of the fluid. In the case of a fluid in relative motion, the connection between the stress and the local properties of the fluid is more complicated compared to a fluid at rest: the stress tensor contains a non-isotropic part as well as isotropic part.

2.3.2 The deviatoric stress tensor

In this section we will establish a relation between the deviatoric stress tensor \( d_{ij} \) and the local properties of the fluid.

The part of the flux of momentum across a material surface element which results from frictional interaction of the matter in relative motion on the two sides of the element and which is represented by the deviatoric stress is assumed to depend only on the instantaneous distribution of fluid velocity in the neighbourhood of the element. The hypothesis can be expressed as:

\[
d_{ij} = \mu \frac{\partial u_i}{\partial x_j}
\]

(2.6)

where \( \mu \) is called the viscosity of the fluid expressing the proportionality between the rate of shear and the tangential force per unit area.

Introducing Eqs. 2.5 and 2.6 into the momentum equation gives the Navier-Stokes equation of motion in vector form:

\[
\rho \frac{DV}{Dt} = -\nabla p + \rho F_i + \mu \Delta u_i
\]

(2.7)

where

\[
\Delta = \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]
The Navier-Stokes equations in a rectangular coordinate system for incompressible flow are listed below:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

(2.8)

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = F_{Bx} - \frac{1}{\rho} \frac{\partial p}{\partial x} + v \Delta u
\]

(2.9)

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = F_{By} - \frac{1}{\rho} \frac{\partial p}{\partial y} + v \Delta v
\]

(2.10)

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = F_{Bz} - \frac{1}{\rho} \frac{\partial p}{\partial z} + v \Delta w
\]

(2.11)

From now on we will refer to the Navier-Stokes equations as the complete system, including the equation of continuity and the equations of motion. In Appendix A another more direct approach in the derivation of the inertial terms and the continuity equation is given which has a more explicit physical meaning.

The velocities in the Navier-Stokes equations above include averaged flow, turbulent flow and orbital motion. Each of these velocities have their own physical properties which implies that the Navier-Stokes equations including all three types of flow can not be solved. Since they vary on time scales which differ from each other, they can be distinguished by performing a time averaging on the velocities. This will be elucidated in the following section.

2.3.3 Time averaging

As stated in the previous section three types of flow can be distinguished: a mean flow, a wave and a turbulent related velocity:

\[
u = \langle u \rangle + \bar{u} + u'\]

(2.12)

From a mathematical point of view, the velocity field can be assumed to be a random function defined on a sample space. One can imagine for instance that we record the longitudinal flow velocity at a given location in a flume: if the experiment is repeated \( N \) times under the same conditions, one obtains \( N \) realizations of the velocity evolution, each of them corresponding to the same point in the sample space.

In a statistical description of the flow, we consider an ensemble average, i.e. a statistical average performed on an infinite number of independent realizations. The ensemble average operator will be noted as \( \langle . \rangle \): for example in the above experiment, let \( u^i(x,t) \) be any component of the velocity at location \( x \) and time \( t \) measured during the experiment \( i \). The ensemble average of the product of a component at a location \( x_n \) and at time \( t_0 \) will be given by:

\[
\langle u(x_n, t_0) \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} u^i(x_n, t_0)
\]

(2.13)
When the flow is stationary the ergodic hypothesis allows one to calculate an ensemble average as a temporal average (for more information the reader is referred to Lesieur, 1987):

\[
\lim_{t \to \infty} \frac{1}{t_0^{-1/2}} \int_{t_0}^{t_0+t} u(t_0,x; t) dt = \frac{1}{t_0^{-1/2}} \int_{t_0}^{t_0+t} u(t,x) dt = \bar{u}(t_0,x) \tag{2.14}
\]

which is independent of \( t_0 \). However, not very often such long series are available and moreover the flow will change during time. Still there are circumstances possible in which a time average will deviate negligible from a ensemble average when a time scale is chosen which is: short compared to the time scale over which the main flow changes and long enough to effectively average turbulent and orbital movement.

We assume that orbital motion has a time scale which is large enough to let the turbulent fluctuations vanish (which in fact is not true since the turbulent properties vary at a much larger time scale than wave orbital motion).

If the turbulence properties are assumed to vary at a time scale of the order of the wave period, the present elaboration makes little sense, unless more is known about this short-term variation of turbulence. Besides, a rather strong interaction between the wave orbital motion and the net current has to be expected in that case. The present knowledge on this aspect of wave current interaction is insufficient to provide a solid basis for incorporation in TRISULA.

This forces us to stick to the present state-of-the-art, in which the assumption of wave invariant turbulence plays a key role. More or less consistently, the wave orbital motion is usually described using theories that take no account of net current or turbulence. This assumption allows us to perform a time averaging on the flow over a wave period \( T \):

\[
\frac{1}{T} \int_{0}^{t} u dt = \langle u \rangle, \quad \frac{1}{T} \int_{0}^{t} \mu dt = 0 \quad \text{and} \quad \frac{1}{T} \int_{0}^{t} u' dt = 0 \tag{2.15}
\]

Applying Eq. 2.15 on the equations of motion and continuity gives the time averaged Navier-Stokes equations known as the Reynolds equation:

\[
\begin{align*}
\langle \nabla \mu \rangle &= \nabla \langle u \rangle = 0 \\
\frac{\partial \langle u \rangle}{\partial t} + \nabla \langle u \rangle \cdot \nabla \langle u \rangle + \frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \langle u \rangle \cdot \nabla \langle u \rangle = \frac{1}{\mu} \nabla^2 \langle u \rangle = \langle F \rangle
\end{align*} \tag{2.16}
\]

As can be seen in Eq. 2.16 two terms (viz. the third and fourth term on the left-hand side of Eq. 2.18) appear which originate from time averaging of the non-linear term:

\[
\begin{align*}
\frac{1}{T} \int_{0}^{t} (u \mu) dt &= \frac{1}{T} \int_{0}^{t} \left( \langle u \rangle + \tilde{u}_i + u_i' \right) \langle u \rangle + \tilde{u}_i + u_i' \right) dt \\
&= \frac{1}{T} \int_{0}^{t} \left( \langle u \rangle \langle u \rangle + \langle u \rangle \tilde{u}_i + u_i' \langle u \rangle + \tilde{u}_i \langle u \rangle + \tilde{u}_i u_i' + u_i' \langle u \rangle + u_i' \langle u \rangle \right) dt \tag{2.17}
\end{align*}
\]

\[
\begin{align*}
&= \langle u \rangle \langle u \rangle + \frac{1}{T} \int_{0}^{t} \langle \tilde{u}_i \rangle \langle u \rangle + \frac{1}{T} \int_{0}^{t} \langle u_i' \rangle \langle u \rangle = \langle u \rangle \langle u \rangle + \langle \tilde{u}_i \rangle + \langle u_i' \rangle
\end{align*}
\]
In Eq. 2.17 the time averaging of the non-linear term is elucidated. In the intermediate expression of Eq. 2.17 it can be seen that the assumption concerning the time scales of orbital motion and turbulent flow prevents the existence of additional terms related to both orbital and turbulent motion (the sixth and seventh term in the intermediate expression of Eq. 2.17). The term related to the turbulent velocities is commonly referred to as the Reynolds-stress:

\[ q_{ij} = \rho \langle u_i' u_j' \rangle \]  \hspace{1cm} (2.18)

Physically, these Reynolds stresses can be understood as fictive stresses which allow one to consider the mean motion as a real flow motion. In the following section these Reynolds-stresses are approximated with the aid of an eddy-viscosity assumption.

Another term which appears is: \( <\bar{a}_{ij}> \) and is caused by the wave orbital motion. This term will be extensively elaborated in Chapter 3, here we just mention that they introduce wave induced stresses.

### 2.3.4 Reynolds-stresses

The Reynolds-stresses can be divided into normal stresses \( q_{11}, q_{22} \) and \( q_{33} \) and the shear stresses \( q_{ij} (\text{with } i \neq j) = s_{ij} \). Except in a thin viscous layer near the bottom the viscous shear stress components, \( \nu \Delta <u_*> \), can be neglected compared to the turbulent shear stress components, \( \nabla s_{ij} \), when developed turbulent flow is considered. The gradients of the normal turbulent stresses, however, can be neglected compared to the gradient of the normal pressure \( \rho \).

When neglecting the viscous shear stress the expression for the shear stress tensor becomes:

\[ \tau^{ij} = -s_{ij} \hspace{1cm} (i \neq j) \]  \hspace{1cm} (2.19)

Hence:

\[ \frac{1}{\rho} (\nabla \cdot \tau) = -\frac{1}{\rho} (\nabla \cdot s) \]  \hspace{1cm} (2.20)

For the turbulent component (the Reynolds-stresses) Boussinesq derived an empirical relationship between the Reynolds-stresses (both normal and shear) and the time-averaged velocity:

\[ \frac{\tau_{xx}}{\rho} = \mu_t \left( \frac{\partial u}{\partial x} \right), \hspace{0.5cm} \frac{\tau_{xy}}{\rho} = \mu_t \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \hspace{0.5cm} \frac{\tau_{yx}}{\rho} = \mu_t \left( \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \right) \]  \hspace{1cm} (2.21)

where

\[ \mu_t = \text{the horizontal turbulent kinematic viscosity also known as the eddy-viscosity. } \]
Under the assumption that \( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \), the Reynolds-stress is written as:

\[
\frac{1}{\rho_h} \left[ \frac{\partial (\tau_{xx})}{\partial x} + \frac{\partial (\tau_{yy})}{\partial y} + \frac{\partial (\tau_{zz})}{\partial z} \right] = \left[ \frac{\partial}{\partial x} \left( \nu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) + \frac{\partial}{\partial y} \left( \nu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) + \frac{\partial}{\partial z} \left( \nu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \right] = \nabla \nu \nabla \langle u \rangle \tag{2.22}
\]

Since we made the approach of wave invariant turbulence the turbulent viscosity, \( \nu_t \), is wave invariant. The same derivation can be made for the \( y \)- and \( z \)-direction.

In Eq. 2.22 a basic difference between the molecular and the turbulent viscosity can be distinguished. If the result of Eq. 2.22 is compared with the last terms on the right-hand side of Eqs. 2.9, 2.10 and 2.11 it can be seen that the turbulent viscosity is included in the spatial derivatives indicating that the turbulent viscosity is a property of the flow while the molecular viscosity is a property of the fluid independent of the flow.

### 2.4 Shallow water equations

Using the results of the time averaging procedure and the Boussinesq approximation gives the full time-dependent 3D Navier-Stokes equations in which the three types of flow are separated. The first terms on the left-hand side of Eqs. 2.24 to 2.26 represent the mean flow, the turbulent flow is represented by the last term on the left-hand side of Eqs. 2.24 to 2.26 and the wave effects are represented by the terms on the right-hand side of these equations. The second term on the left-hand side of Eqs. 2.24 and 2.25 are the coriolis forces. Since our area of interest is relatively small they are neglected from hereon. Notice that in the continuity equation only the mean flow is present.

\[
\frac{\partial \langle u \rangle}{\partial x} + \frac{\partial \langle v \rangle}{\partial y} + \frac{\partial \langle w \rangle}{\partial z} = 0 \tag{2.23}
\]

\[
\frac{D \langle u \rangle}{Dt} - f_c \langle v \rangle = \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x} - \nabla \nu \nabla \langle u \rangle = -\frac{\partial \langle u u \rangle}{\partial x} - \frac{\partial \langle u v \rangle}{\partial y} - \frac{\partial \langle u w \rangle}{\partial z} \tag{2.24}
\]

\[
\frac{D \langle v \rangle}{Dt} + f_c \langle u \rangle = \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial y} - \nabla \nu \nabla \langle v \rangle = -\frac{\partial \langle u v \rangle}{\partial x} - \frac{\partial \langle v v \rangle}{\partial y} - \frac{\partial \langle v w \rangle}{\partial z} \tag{2.25}
\]

\[
\frac{D \langle w \rangle}{Dt} + g + \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial z} - \nabla \nu \nabla \langle w \rangle = -\frac{\partial \langle u w \rangle}{\partial x} - \frac{\partial \langle v w \rangle}{\partial y} - \frac{\partial \langle w w \rangle}{\partial z} \tag{2.26}
\]

For the inertial terms the Lagrangian notation is used, which gives information of a particle in coordinate system travelling with the same speed:

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \tag{2.27}
\]

If \( \xi(x,y,t) \) is the instantaneous water surface elevation and viscous effects are disregarded, the free surface boundary conditions can be written as:

\[\text{Delft Hydraulics}\]
(1) Kinematic boundary condition

\[ w = \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} \]  

(2.28)

(2) Dynamic boundary condition

\[ \tau_{nx} = 0, \quad \tau_{ny} = 0, \quad \rho = p_s \]  

(2.29)

in which \( p_s \) denotes a prescribed surface pressure.

At the bottom of the water body (no-slip):

\[ u = v = w = 0 \]  

(2.30)

### 2.4.1 Hydrostatic flow model including net effects of waves

We now take a look at the momentum equation in \( z \)-direction. For reasons of transparency we write the equation in its original form. Since we made the assumption of wave invariant turbulence the Reynolds-stresses can be neglected here:

\[ \frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial w\zeta}{\partial z} + g = -\frac{1}{\rho} \frac{\partial p}{\partial z} \]  

(2.31)

Eq. 2.31 will be depth integrated from an arbitrary point \( z \) to the water surface \( \zeta \):

\[ \int_0^\zeta \frac{\partial w}{\partial t} dz + \int_x^\zeta \frac{\partial uw}{\partial x} dz + \int_y^\zeta \frac{\partial vw}{\partial y} dz + \int_z^\zeta \frac{\partial w\zeta}{\partial z} dz + \int_g^\zeta g dz = -\frac{1}{\rho} \int_0^\zeta \frac{\partial p}{\partial z} dz \]  

(2.32)

Applying the Leibnitz rule yields for Eq. 2.32:

\[ \frac{\partial}{\partial t} \int_z^\zeta w dz - w|_{z-\zeta} \frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} \int_z^\zeta uw dz - uw|_{z-\zeta} \frac{\partial \zeta}{\partial x} \]  

\[ + \frac{\partial}{\partial y} \int_z^\zeta vw dz - vw|_{z-\zeta} \frac{\partial \zeta}{\partial y} \]  

\[ + \int_z^\zeta w\zeta - z + g(\zeta - z) = \frac{1}{\rho}(p - p_s) \]  

(2.33)

Inserting the kinematic boundary condition (Eq. 2.28) and averaging the equation over one wave period \( T \) transforms Eq. 2.33 to:

\[ \frac{1}{T \zeta} \int_0^\zeta w dz dt + \frac{1}{T_x} \int_0^\zeta \left( \langle u \rangle \langle w \rangle + \langle w \rangle \zeta + \langle u \zeta \rangle + \zeta \zeta \right) dz dt \]  

\[ + \frac{1}{T_y} \int_0^\zeta \left( \langle v \rangle \langle w \rangle + \langle w \rangle \zeta + \langle v \zeta \rangle + \zeta \zeta \right) dz dt \]  

\[ - \frac{1}{T_0} \int_0^\zeta \left( \langle w \rangle \langle w \rangle + 2 \langle w \rangle \zeta + \zeta \zeta \right) dz dt \]  

\[ + \int_0^\zeta g(\zeta - z) = \frac{1}{T} \int_0^\zeta p dt \]  

(2.34)
The terms in Eq. 2.34 are treated term by term. \( p_z \) is omitted since it does not introduce additional gradients. The variables are now subdivided into an averaged, \( < . > \), and wave related part, see also Section 2.3.3 where the separation of the flow is treated.

A basic assumption underlying the TRISULA-concept is, that the pressure distribution without wave effects is hydrostatic. This means that in the vertical momentum equation all terms related to \( < w > \) are disregarded. So Eq. 2.34 now becomes after time averaging:

\[
\frac{1}{T_0} \frac{\partial}{\partial z} \int \tilde{\omega} dz dt + \frac{1}{T_0} \frac{\partial}{\partial x} \int \tilde{u} \tilde{\omega} dz dt + \frac{1}{T_0} \frac{\partial}{\partial y} \int \tilde{v} \tilde{\omega} dz dt
- \frac{1}{T_0} \int \tilde{\omega} \mid_{z_1} dz dt + g(<\zeta> - z) = \frac{1}{T \rho_0} \int \rho dt
\]  
(2.35)

Also the integration boundaries are divided into two parts: from \( z \) to trough level \( z_t \), and from \( z_t \) to \( \zeta \). This is shown in Eq. 2.36 for the first term of Eq. 2.35:

\[
\frac{1}{T_0} \frac{\partial}{\partial z} \int \tilde{\omega} dz dt = \frac{1}{T_0} \frac{\partial}{\partial z} \int \tilde{\omega} dz dt + \frac{1}{T_0} \frac{\partial}{\partial z_t} \int \tilde{\omega} dz dt
\]  
(2.36)

The first term in Eq. 2.36 on the right-hand side is zero by definition and the second term is negligible because it only is present from trough level to the instantaneous water surface and is therefore of a smaller order. The second and third term on the left-hand side of Eq. 2.35 are approximately zero since the horizontal and vertical orbital motion are out of phase. The vertical pressure distribution is now found to be:

\[-<\tilde{\omega}^2> \mid_{z_t} + g(<\zeta> - z) = \frac{1}{\rho} <\rho> - <\rho> - \rho_o = -\rho <\tilde{\omega}^2> \mid_{z_t} \]  
(2.37)

Eqs. 2.23, 2.24, 2.25 and 2.37 are the shallow water equations in which the net wave effects and the assumption of hydrostatic pressure are included. The shallow water equations as they are used in TRISULA do not take the net wave effects into account. By omitting the wave related terms, denoted by a tilde, the shallow water equations as they are incorporated in TRISULA are obtained.

The terms representing the wave effects, on the right-hand side of Eqs. 2.24, 2.25 and 2.26, originate from the time averaging of the inertial terms which are non-linear. They can be seen as the contribution of the orbital motion to the inertial terms. Since they originate from a time averaging procedure (Eq. 2.17), the first order derivative with respect to time disappears. Integrating the wave related terms over the water depth would result in the well-known radiation stresses.
2.5 The sigma-transformation

2.5.1 Introduction

TRISULA solves the flow equations for an incompressible fluid, under the shallow water assumption. In the horizontal directions, orthogonal curvilinear (ξ, η) coordinates are used, whereas in the vertical direction a scaled coordinate, the so-called σ-coordinate, introduced by Phillips (1957), is used. σ = -1 at the sea bed and σ = 0 at the surface. The coordinate system is boundary fitted both in the horizontal and the vertical plane (see Fig. 2.2).

![Sigma transformation, boundary fitted in the vertical plane](image)

A three-dimensional model consists of a number of layers. In a σ-coordinate system, the layer interfaces are chosen following planes of constant σ, so the number of layers is constant over the vertical computational field. Applying the σ-coordinate system has the advantage that the layers are of varying size since the z-coordinate is normalized. For each layer, a set of coupled conservation equations is solved.

2.5.2 The sigma-coordinate

The σ-coordinate can be defined as:

$$\sigma = \frac{z - \zeta(x, y, t)}{h(x, y, t)}$$  \hspace{1cm} (2.38)

Definition in Eq. 2.38 yields for the free surface $z = \zeta$: $\sigma = 0$ and for the bottom $z = -d$: $\sigma = -1$. Since $h$, $d$ and $\zeta$ do not depend on $\sigma$ the following relations hold:

$$\frac{\partial h}{\partial \sigma} = 0, \quad \frac{\partial d}{\partial \sigma} = 0, \quad \frac{\partial \zeta}{\partial \sigma} = 0$$  \hspace{1cm} (2.39)

2.5.3 The $x^*$, $y^*$ and $t^*$ coordinates

For reasons of transparency, we will use the Cartesian $x$ and $y$ as horizontal coordinates. The transformed time is equal to the time in the original Cartesian system. Introduction of the new coordinates $x^*$, $y^*$ and $t^*$ seems unnecessary, but we will show in the following section that the partial derivative of a scalar quantity with respect to e.g. $x$ ($y, z$ and $t$
constant) is not equal to the partial derivative with respect to \( x^* \) (\( y^* \), \( \sigma \) and \( t^* \) constant). Summarizing the transformation of time and space-coordinates we get:

\[
\begin{align*}
t^* &= t \\
x^* &= x \\
y^* &= y \\
\sigma &= \frac{z - \xi}{h} = \sigma(x, y, z, t)
\end{align*}
\]  
(2.40)

2.5.4 First order partial derivatives

Applying the chain rule to express the first order derivatives of the Cartesian system into the \( \sigma \)-system yields:

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial t^*} + \frac{\partial}{\partial x^*} \frac{\partial x^*}{\partial t} + \frac{\partial}{\partial y^*} \frac{\partial y^*}{\partial t} + \frac{\partial}{\partial \sigma} \frac{\partial \sigma}{\partial t}
\]
(2.41)

\[
= \frac{\partial}{\partial t^*} + \frac{\partial}{\partial \sigma} \frac{\partial \sigma}{\partial t}
\]

For all four (subscripts denote the variables that are held constant) the transformation yields:

\[
\left( \frac{\partial}{\partial t} \right)_{x,y,t} = \left( \frac{\partial}{\partial t^*} \right)_{x^*,y^*,\sigma} + \left( \frac{\partial}{\partial x^*} \right)_{x,y,t} \left( \frac{\partial}{\partial \sigma} \right)_{x^*,y^*,t^*}
\]

\[
\left( \frac{\partial}{\partial x} \right)_{y,z,t} = \left( \frac{\partial}{\partial x^*} \right)_{y^*,z^*,t^*} + \left( \frac{\partial}{\partial x} \right)_{x,y,t} \left( \frac{\partial}{\partial \sigma} \right)_{x^*,y^*,t^*}
\]

\[
\left( \frac{\partial}{\partial y} \right)_{z,x,t} = \left( \frac{\partial}{\partial y^*} \right)_{z^*,x^*,t^*} + \left( \frac{\partial}{\partial y} \right)_{y,z,t} \left( \frac{\partial}{\partial \sigma} \right)_{x^*,y^*,z^*,t^*}
\]

\[
\left( \frac{\partial}{\partial \sigma} \right)_{x,y,t} = \left( \frac{1}{h} \frac{\partial}{\partial \sigma} \right)_{x^*,y^*,t^*}
\]
(2.42)

We will omit the subscripts from hereon. Combining Eqs. 2.40 and 2.42, the following properties for \( h \), \( d \) and \( \xi \) hold:

\[
\frac{\partial \zeta}{\partial x^*} = \frac{\partial \zeta}{\partial x} \frac{\partial x}{\partial x^*} \quad \frac{\partial h}{\partial \zeta} = \frac{\partial h}{\partial x} \frac{\partial x}{\partial x^*} \quad \frac{\partial \xi}{\partial \sigma} = \frac{\partial \zeta}{\partial \sigma} \frac{\partial x}{\partial x^*}
\]
(2.43)

\[
\frac{\partial \zeta}{\partial \zeta^*} = \frac{\partial \zeta}{\partial y} \frac{\partial y}{\partial \zeta^*} \quad \frac{\partial h}{\partial \zeta^*} = \frac{\partial h}{\partial y} \frac{\partial y}{\partial \zeta^*} \quad \frac{\partial \xi}{\partial \sigma} = \frac{\partial \zeta}{\partial \sigma} \frac{\partial y}{\partial \zeta^*}
\]

When \( c \) stands for \( x \), \( y \) and \( t \) the following relation can be found:

\[
\frac{\partial \sigma}{\partial c} = \frac{\partial}{\partial \sigma} \left( \frac{z - \zeta}{h} \right) = -\frac{1}{h^2} \left( h \frac{\partial \zeta}{\partial c} + (z - \zeta) \frac{\partial h}{\partial c} \right) = -\frac{1}{h} \left( \frac{\partial \zeta}{\partial c} + \sigma \frac{\partial h}{\partial c} \right)
\]
(2.44)
2.5.5 Transformed velocity components

In the horizontal plane the velocity components remain the same. The components \( u \) and \( v \) are transformed into \( u^* \) and \( v^* \) respectively, having equal magnitudes at the same locations.

\[
u^*(x^*, y^*, \sigma, t^*) = u(x, y, z, t) \quad v^*(x^*, y^*, \sigma, t^*) = v(x, y, z, t)
\]  
(2.45)

Analogous to \( w = \frac{Dz}{Dt} \) we can define the vertical velocities in TRISULA, \( \omega \), as multiplying the 'original' transformed vertical velocity \( \frac{D\sigma}{Dt^*} \) with the total water depth \( h \):

\[
\omega = h \frac{D\sigma}{Dt^*} = H \frac{D}{Dt} \left( \frac{z - \zeta}{h} \right) = \frac{1}{h} \left( h \frac{Dz}{Dt} - \frac{DH}{Dt} - \frac{Dh}{Dt} + \zeta \frac{Dh}{Dt} \right)
\]

\[
= w - \frac{D\zeta}{Dt} - \frac{1}{h} \frac{Dh}{Dt} (z - \zeta) = w - \left( \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} \right) - \sigma \left( \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} \right)
\]

or:

\[
w = \omega + \left( \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} \right) + \sigma \left( \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} \right)
\]

(2.47)

2.5.6 The equations of motion

In this Section we will derive the transformation of Eqs. 2.23, 2.24, 2.25, 2.26 and 2.37, for this purpose we will use Eqs. 2.40, 2.42 and 2.47.

The equation of continuity:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

(2.48)

substituting Eq. 2.42 transforms Eq. 2.48 to:

\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial \sigma} \frac{\partial \sigma}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial \sigma} \frac{\partial \sigma}{\partial y} + \frac{1}{h} \frac{\partial w}{\partial \sigma} = 0
\]

(2.49)

Multiplying by \( h \) and substituting Eq. 2.47 yields:

\[
h \frac{\partial u}{\partial x} + h \frac{\partial u}{\partial \sigma} \frac{\partial \sigma}{\partial x} + h \frac{\partial v}{\partial y} + h \frac{\partial v}{\partial \sigma} \frac{\partial \sigma}{\partial y} + \frac{\partial \omega}{\partial \sigma}
\]

\[
+ \frac{\partial}{\partial \sigma} \left( \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} \right) + \frac{\partial}{\partial \sigma} \left( \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} \right) = 0
\]

(2.50)

With:

\[
\frac{\partial \sigma}{\partial x} = -\frac{1}{h} \left( \frac{\partial \zeta}{\partial x} + \sigma \frac{\partial h}{\partial x} \right) \text{ and } \frac{\partial \sigma}{\partial y} = -\frac{1}{h} \left( \frac{\partial \zeta}{\partial y} + \sigma \frac{\partial h}{\partial y} \right)
\]

(2.51)
Eq. 2.50 becomes:

\[
\begin{align*}
\frac{h}{\partial x} \frac{\partial u}{\partial x} - \frac{\partial u}{\partial \sigma} \left( \frac{\partial \zeta}{\partial x} + \sigma \frac{\partial h}{\partial x} \right) + \frac{\partial u}{\partial \sigma} \left( \frac{\partial \zeta}{\partial x} + \sigma \frac{\partial h}{\partial x} \right) + h \frac{\partial v}{\partial y} \\
- \frac{\partial v}{\partial \sigma} \left( \frac{\partial \zeta}{\partial y} + \frac{\partial h}{\partial y} \right) + \frac{\partial v}{\partial \sigma} \left( \frac{\partial \zeta}{\partial y} + \frac{\partial h}{\partial y} \right) + \frac{\partial \omega}{\partial \sigma} + \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} &= 0
\end{align*}
\]  
(2.52)

If it is assumed that the bottom is constant in time this reduces to:

\[
\frac{\partial \zeta}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} + \frac{\partial \omega}{\partial \sigma} = 0
\]  
(2.53)

Eq. 2.53 is the transformed equation of continuity which is used in TRISULA.

The same transformation will now be made for the momentum equation in \(x\)-direction:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \text{REST}
\]  
(2.54)

where \(\text{REST} = \text{Reynolds-stresses}\).

This transforms to:

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial \sigma}{\partial \sigma} \frac{\partial u}{\partial \sigma} + u \left( \frac{\partial u}{\partial \sigma} + \frac{\partial \sigma}{\partial \sigma} \frac{\partial u}{\partial \sigma} \right) + \nu \left( \frac{\partial u}{\partial \sigma} + \frac{\partial \sigma}{\partial \sigma} \frac{\partial u}{\partial \sigma} \right) \\
+ \left( \omega + u \frac{\partial \zeta}{\partial \sigma} + v \frac{\partial \zeta}{\partial \sigma} + \frac{\partial h}{\partial \sigma} + u \frac{\partial h}{\partial \sigma} + v \frac{\partial h}{\partial \sigma} \right) \left( \frac{1}{h} \frac{\partial u}{\partial \sigma} \right) \\
= -\frac{1}{\rho} \left( \frac{\partial p}{\partial x} + \frac{\partial \sigma}{\partial \sigma} \frac{\partial p}{\partial \sigma} \right) + \text{REST}
\end{align*}
\]  
(2.55)

Inserting Eq. 2.42 for \(x, y\) and \(t\) gives:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial \sigma} = -\frac{1}{\rho} \left( \frac{\partial p}{\partial x} + \frac{\partial \sigma}{\partial \sigma} \frac{\partial p}{\partial \sigma} \right) + \text{REST}
\]  
(2.56)

According to Eq. 2.37 the pressure gradient can be written as:

\[
\frac{\partial <p>}{\partial z} = -p \left( g - \frac{\partial <\dot{\omega}>}{\partial z} \right)
\]  
(2.57)

using Eq. 2.42 transforms Eq. 2.57 to:

\[
\frac{\partial <p>}{\partial \sigma} = -p \left( gh + \frac{\partial <\dot{\omega}>}{\partial \sigma} \right)
\]  
(2.58)

The same routine can be applied on the Reynolds-stresses, but in the horizontal Reynolds-stresses we maintain the second partial derivative in the Cartesian system. This can be explained by the fact that the horizontal and vertical length scale are of a different order. In fact the horizontal Reynolds-stresses can be neglected if we consider the horizontal scales (see e.g. Van Kester et al., 1989).
The transformed Reynolds-stresses:

\[
\begin{align*}
\frac{\partial \tau_{ux}}{\partial x} &= \frac{\partial}{\partial x} \left( 2 \nu \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial \alpha} \frac{\partial \alpha}{\partial x} \right) \right) \\
\frac{\partial \tau_{uy}}{\partial y} &= \frac{\partial}{\partial y} \left( \nu \left( \frac{\partial u}{\partial y} + \frac{\partial u}{\partial \alpha} \frac{\partial \alpha}{\partial y} + \frac{\partial v}{\partial \alpha} \frac{\partial \alpha}{\partial y} \right) \right) \\
\frac{\partial \tau_{uz}}{\partial \alpha} &= \frac{1}{\mu^2 \frac{\partial u}{\partial \alpha}} \\
\end{align*}
\]

(2.59)

The same derivation can be made for the momentum equation in y-direction.

For the transformation of the wave related terms we have to keep in mind that the time scale on which the \( \alpha \)-planes vary is much larger then the time scale of orbital motion. The transformed wave related terms:

\[
\frac{\partial \langle \ddot{u} \ddot{w} \rangle}{\partial x} = \frac{1}{h} \left( \frac{\partial \epsilon}{\partial x} + \frac{\partial \epsilon}{\partial \alpha} \frac{\partial \alpha}{\partial x} \right) + \frac{\partial \langle \ddot{u} \ddot{v} \rangle}{\partial y} = \frac{1}{h} \left( \frac{\partial \epsilon}{\partial y} + \frac{\partial \epsilon}{\partial \alpha} \frac{\partial \alpha}{\partial y} \right) + \frac{\partial \langle \ddot{u} \ddot{w} \rangle}{\partial \alpha} \frac{1}{h} \frac{\partial \epsilon}{\partial \alpha} \]

(2.60)

Since the horizontal orbital motion is assumed to be constant over the water depth the second and fourth in Eq. 2.60 are zero.

The resulting equations are summarized below:

Equation of continuity:

\[
\frac{\partial \zeta}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} + \frac{\partial \omega}{\partial \alpha} = 0
\]

(2.61)

Momentum equation in x-direction:

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial \alpha} &= \frac{1}{\rho} \left( \frac{\partial p}{\partial x} + \frac{\partial \rho}{\partial \alpha} \right) \\
&- \frac{\partial}{\partial x} \left( \nu \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial \alpha} \frac{\partial \alpha}{\partial x} \right) \right) - \frac{\partial}{\partial y} \left( \nu \left( \frac{\partial u}{\partial y} + \frac{\partial u}{\partial \alpha} \frac{\partial \alpha}{\partial y} + \frac{\partial v}{\partial \alpha} \frac{\partial \alpha}{\partial y} \right) \right) \\
&- \frac{1}{\mu^2 \frac{\partial u}{\partial \alpha}} \frac{\partial \langle \ddot{u} \ddot{w} \rangle}{\partial x} - \frac{\partial \langle \ddot{u} \ddot{v} \rangle}{\partial y} \frac{1}{h} \frac{\partial \epsilon}{\partial \alpha} \\
\end{align*}
\]

(2.62)

Momentum equation in y-direction:

\[
\begin{align*}
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial \alpha} &= \frac{1}{\rho} \left( \frac{\partial p}{\partial y} + \frac{\partial \rho}{\partial \alpha} \right) \\
&- \frac{\partial}{\partial y} \left( \nu \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial \alpha} \frac{\partial \alpha}{\partial y} \right) \right) - \frac{\partial}{\partial x} \left( \nu \left( \frac{\partial v}{\partial y} + \frac{\partial v}{\partial \alpha} \frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \alpha} \frac{\partial \alpha}{\partial x} \right) \right) \\
&- \frac{1}{\mu^2 \frac{\partial v}{\partial \alpha}} \frac{\partial \langle \ddot{u} \ddot{w} \rangle}{\partial y} - \frac{\partial \langle \ddot{u} \ddot{v} \rangle}{\partial x} \frac{1}{h} \frac{\partial \epsilon}{\partial \alpha} \\
\end{align*}
\]

(2.63)
Hydrostatic pressure relation:

\[
\frac{\partial p}{\partial \sigma} = -\rho \left( gh + \frac{\partial \langle \omega \rangle}{\partial \sigma} \right)
\]  

(2.64)

We now have four equations and five unknown variables: \( \xi, u, v, \omega \) and \( p \). The continuity equation gives, when \( u, v \) and \( \xi \) are known, the vertical velocity \( \omega \). For the determination of the water level, \( \xi \), we need an additional depth averaged continuity equation.

2.5.7 Depth averaged continuity equation

The depth averaged continuity equation can be written as:

\[
\int_{-d}^{d} \frac{\partial u}{\partial x} \, dz + \int_{-d}^{d} \frac{\partial v}{\partial y} \, dz + \int_{-d}^{d} \frac{\partial \omega}{\partial z} \, dz = 0
\]  

(2.65)

Changing the order of integration and differentiation in Eq. 2.65 gives:

\[
\frac{\partial}{\partial x} \left[ \int_{-d}^{d} u \, dz \right] - u \frac{\partial \xi}{\partial x} \bigg|_{z=t} + \frac{\partial}{\partial y} \left[ \int_{-d}^{d} v \, dz \right] - v \frac{\partial \xi}{\partial y} \bigg|_{z=t} + [\omega]_{z=d}^{z=-d} = 0
\]  

(2.66)

Inserting the boundary conditions Eqs. 2.28 and 2.30 results in:

\[
\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial x} \left[ \int_{-d}^{d} u \, dz \right] + \frac{\partial}{\partial y} \left[ \int_{-d}^{d} v \, dz \right] = 0
\]  

(2.67)

We will have to take the averaged and fluctuating parts under consideration (as will become clear later on). As stated before we will assume that the turbulent related fluctuations are negligible:

\[
u = \langle \nu \rangle + \tilde{\nu}
\]  

(2.68)

A time averaging is performed on Eq. 2.67:

\[
\frac{\partial \langle \nu \rangle}{\partial t} + \frac{T}{1-T} \int \frac{\partial}{\partial x} \left[ \int_{-d}^{d} u \, dz \, dt \right] + \frac{T}{1-T} \int \frac{\partial}{\partial y} \left[ \int_{-d}^{d} v \, dz \, dt \right] = 0
\]  

(2.69)

The second term now is treated separately:

\[
\frac{T}{1-T} \int \frac{\partial}{\partial x} \left[ \int_{-d}^{d} u \, dz \, dt \right] = \frac{\partial}{\partial x} \left( \frac{T}{1-T} \int \int_{-d}^{d} u \, dz \, dt \right)
\]  

\[
= \frac{1}{T} \frac{\partial}{\partial x} \left( \int \int_{z=t}^{z=-t} \langle \nu \rangle \, dz \, dt + \int \int_{z=t}^{z=-t} \tilde{\nu} \, dz \, dt \right)
\]  

(2.70)

Where the vertical integration of the fluctuating part is divided into two parts: from the bottom, \(-d\), to trough level, \(z_t\), and from trough level to instantaneous water level, \(\xi(x,y,t)\) (see also Fig. 2.3). The second term in Eq. 2.70 is zero since by definition \(\langle \tilde{\nu} \rangle = 0\).
For the third term in Eq. 2.69 the same derivation can be made. If \( \langle u \rangle \) is assumed to be independent of \( z \) between trough and water level. The upper boundary of the vertical integration can be replaced by \( \langle t \rangle \). The continuity equation now becomes:

\[
\frac{\partial \langle \zeta \rangle}{\partial t} + \frac{\partial}{\partial x} \int_{-d}^{z_t} \left< u \right> dz + \frac{\partial}{\partial y} \int_{-d}^{z_t} \left< v \right> dz + \frac{\partial}{\partial x} \left( \frac{1}{T} \int_{z_t}^{z} \tilde{w} dz dt \right) + \frac{\partial}{\partial y} \left( \frac{1}{T} \int_{z_t}^{z} \tilde{v} dz dt \right) = 0 \tag{2.71}
\]

Figure 2.3 Wave induced mass flux between trough level and instantaneous water level

The last two terms in Eq. 2.71 represent gradients of the wave induced mass flux between trough level and water level. If we assume the orbital velocity to be constant over the top layer we can make the following derivation:

\[
M_x = \frac{1}{T} \int_{z_t}^{\langle \zeta \rangle} \rho \tilde{u} dz dt = \frac{1}{T} \int_{z_t}^{\langle \zeta \rangle} \rho \tilde{u} |_{z_t}^{\langle \zeta \rangle} (\zeta - z_t) dt \tag{2.72}
\]

For the orbital motion and water elevation we can write according to linear wave theory:

\[ \tilde{u} |_{z_t}^{\langle \zeta \rangle} = \frac{H_c}{2d} \cos(kx - \omega t) \tag{2.73} \]

\[ \zeta = \frac{H_c}{2} \cos(kx - \omega t) \tag{2.74} \]

Inserting these equations into Eq. 2.72 we get:

\[ M_x = \frac{\rho}{T} \int_{z_t}^{\langle \zeta \rangle} \frac{H_c^2}{4d} \cos^2(kx - \omega t) - \frac{H_c}{2d} \cos(kx - \omega t) dt \]

\[ = \frac{\rho}{T} \int_{z_t}^{\langle \zeta \rangle} \frac{H_c^2}{8d} dt = \rho \frac{H_c^2}{8d} = \frac{2c_1}{g d^{16}} \rho g H^2 - \frac{2E_p}{c} \tag{2.75} \]

In which we assume that the wave velocity \( c \) can be approximated by \( \sqrt{gd} \).
If we consider oblique waves we can write for the mass flux:

\[ M_x = \frac{E}{c} \cos \theta \]  \hspace{1cm} (2.76)

\[ M_y = \frac{E}{c} \sin \theta \]  \hspace{1cm} (2.77)

where \( E \) is the energy density, \( c \) the phase celerity and \( \theta \) is the direction of the wave propagation with respect to the positive \( x \)-axis.

The depth averaged continuity equation including the wave induced mass drift now can be written as:

\[ \frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} \left( \zeta u \right) + \frac{\partial}{\partial y} \left( \zeta v \right) + \frac{1}{\rho} \frac{\partial M_x}{\partial x} + \frac{1}{\rho} \frac{\partial M_y}{\partial y} = 0 \]  \hspace{1cm} (2.78)

With Eqs. 2.61, 2.62, 2.63, 2.64 and 2.78 a closed system is obtained which can be solved. In Eq. 2.78 the averaging symbols have been omitted since the water level variations and velocities which TRISULA computes are the averaged ones.
3 Shear stress distribution in dissipative water waves

3.1 Introduction

In the previous chapter we derived the shallow water equations and showed how additional terms originate from the Navier-Stokes equations. We have distinguished three types of water flows: the averaged flow (e.g. tide, wind driven), wave induced flow (e.g. long shore current, undertow) and turbulent flow. The averaged flow and turbulent flow are incorporated in TRISULA.

In this chapter we will derive relations for the wave induced vertical shear stress distribution. It is then possible to compute the vertical distribution of the time averaged flow by solving the shallow water equations with the wave induced stress as the driving force.

The circulation pattern in the surf zone is of interest, because it is of importance for the on–offshore sediment transport, and thus the development of the coastal profile. It is characteristic for the two-dimensional circulation that the flow directed offshore close to the bed is forming the so called undertow.

The theory outlined in this chapter relies on an imbalance in the vertical distribution of the radiation stress which induces a circulation current.

3.2 The radiation stresses

The radiation stress $S_{xx}$ consists of two parts, one originating from the pressure $S_{xx}^{1}$ and another originating from the momentum of the orbital motion $S_{xx}^{2}$ (see e.g. Longuet-Higgins and Stewart, 1964).

![Diagram of vertical distribution of radiation shear stress from pressure and momentum](image)

**Figure 3.1** Vertical distribution of radiation shear stress from pressure, $S_{xx}^{1}$ and momentum, $S_{xx}^{2}$.
In linear shallow water waves, the contribution to $S_{xx}^1$ is located around MWL as sketched in Fig. 3.1 and contributes with one third of the total to the radiation stress. It originates from the pressure as will be shown later on. The contribution $S_{xx}^2$ is evenly distributed over the water depth, this stress originates from the gradient of the horizontal orbital movement.

After wave-breaking, the wave height decreases in the shoreward direction, creating a gradient in the radiation stress. This will give a slope of the mean water level, defined as the set-up. However, because of the uneven vertical distribution of $S_{xx}$ the gradient in the radiation stress becomes a function of the distance from the bed $z$.

To calculate in detail the consequence of this, a control volume as sketched in Fig. 3.2 is considered. The control volume is fixed in time and space.

The shape of the radiation stress gradient is similar to the vertical distribution of $S_{xx}$, shown in Fig. 3.1. Because of the set-up, a pressure gradient is introduced, which is uniformly distributed over the depth. In order to obtain a time-averaged force balance on the control volume a resulting shear stress $\tau$ has to be introduced on the bottom of the control volume.

![Control volume](image)

Figure 3.2 Control volume (Eulerian description) on which the momentum equation is applied

The exchange of momentum through the bottom of the control volume of Fig. 3.2 is neglected in studies made so far. If this momentum flux is neglected no force balance on the control volume is possible. We will show that taking into account the momentum flux through the bottom of the control volume leads to a shear stress distribution which does not have this inconsistency.

The term $\overline{uw}$ will be non-zero if there is a phase shift between horizontal and vertical orbital motion. We will show that in the case of non-uniform water waves a phase shift is present between horizontal and vertical orbital motion.

We will omit the tilde in this chapter. All velocities are orbital velocities unless mentioned otherwise. The time-averaging symbols are replaced by an overline for reasons of transparency.

This contribution is evaluated for the two following cases.

(a) Waves with energy dissipation in the boundary layer only (laminar or turbulent).
(b) Waves with energy dissipation due to spilling breakers or broken waves.
It turns out that it is of vital importance to include this $\vec{u}w$ term in order to obtain the correct vertical shear stress distribution. As mentioned in the introduction we will restrict ourselves to considering linear shallow-water wave theory, i.e. hydrostatic pressure distribution. The theory outlined in this chapter is based on Deigaard and Fredsøe (1989). In Appendix B the theory is elaborated more extensively.

### 3.3 Dissipation in the wave boundary layer

We consider progressive, linear shallow-water waves. The sea bed is horizontal and the only deformation of the waves is assumed to be caused by the energy dissipation in the wave boundary layer. The wave condition is steady, and the only variation in wave height is in the direction of propagation. The derivations made are valid for a turbulent as well as a laminar wave boundary layer.

If the energy dissipation as a first approximation is neglected, the surface elevation $\zeta$ and the horizontal orbital velocity $u_0$ are given by:

\[
\zeta = \frac{H}{2} \cos(kx - \omega t) \tag{3.1}
\]

\[
u_0 = \frac{Hc}{2d} \cos(kx - \omega t) \tag{3.2}
\]

in which:

- $H$ = wave height
- $d$ = mean water depth
- $t$ = time
- $x$ = horizontal coordinate
- $\omega$ = wave angular frequency
- $k$ = wave number

The variable $c$ is the wave celerity which for shallow water waves is equal to $\sqrt{gd}$, $g$ being the acceleration of gravity.

The vertical orbital velocity is:

\[
w = \frac{z + d \partial \zeta}{d \partial t} \tag{3.3}
\]

where $z$ is the distance from the undisturbed water level.

#### 3.3.1 Exchange of energy

In Appendix B it is shown that energy is extracted from the outer potential flow and is transported to the wave boundary layer where it is converted to turbulence and heat. The time-averaged rate of energy dissipation per unit of the bed, $D$ is:

\[
D = \rho u \nu_u |u_u| \tag{3.4}
\]
Longuet-Higgins (1953) showed that spatial and temporal gradients in the velocity deficit of the wave boundary layer create a vertical velocity, which is small compared to the orbital velocities. This additional vertical velocity attains the value \( w_{\infty} \) outside the wave boundary layer, while it decreases through the boundary layer to be zero at the bed.

For \( w_{\infty} \) the following expression is found:

\[
    w_{\infty} = -\frac{\tau_b}{c\rho} = -\frac{u_*|u_*|}{c} \tag{3.5}
\]

in which:

\[
    u_* = \text{the shear stress velocity } \sqrt{\tau_b/\rho} \\
    \tau_b = \text{the bottom shear stress}
\]

The wave boundary layer acts as a small periodic disturbance which travels along the bed with the phase velocity \( c \) of the wave motion. Such a travelling disturbance will generate a surface wave with increasing wave height in the direction of the propagation.

In the following calculations we take the vertical velocity generated by the bottom perturbation to be constant to \( w_{\infty} \) over the entire depth outside the boundary layer, so the horizontal velocities become equal to zero. This choice can be justified by the following reasons:

Apart from the non-uniform growing wave any small wave can be superposed on the system without interacting with the wave motion at the order under consideration. The wave with \( w = w_{\infty} \) is, however, the wave with the minimum energy loss in the wave boundary layer, because no horizontal velocities are generated. Hence, this wave will dominate any other dissipative wave motion after a distance.

\( w_{\infty} \) is associated with an additional water surface elevation of:

\[
    \zeta^* = \int w_{\infty} dt = -\int_{-\infty}^{1} u_*|u_*|dt \tag{3.6}
\]

\( \zeta^* \) gives rise to an additional horizontal pressure gradient \( \partial p^*/\partial x \), which carries out a resulting work on the horizontal orbital velocity of the wave motion. The time-averaged rate of work per unit bed area is given by:

\[
    W = -\rho u_*u_*|u_*| \tag{3.7}
\]

which is of exact the same magnitude as the dissipation in the wave boundary layer.

The extraction of energy is evenly distributed over the depth. As the dissipation takes place in the bottom boundary layer, energy must be transferred from the outerflow to the boundary layer. The downward energy transport is performed by the work done by the hydrostatic pressure on the vertical flow velocity. This time-averaged rate of work is calculated as:

\[
    W = -\rho g \zeta^* u_*|u_*|/c = -\rho u_*u_*|u_*|z \tag{3.8}
\]
So \( W \) is at the bed equal to the work found in Eq. 3.7, this implies that the energy extracted from the wave motion is transferred to the wave boundary layer.

### 3.3.2 The water motion in dissipative waves

First the non-uniform wave field with dissipation in the wave boundary layer is analyzed. The dissipation is assumed to be weak, and the wave field is calculated by perturbation analysis. The effect of the pressure gradient from the displacement in the wave boundary layer is thus first calculated from the unperturbed wave condition (Eqs. 3.1, 3.2 and 3.3) and is then inserted into the flow equation in order to obtain a solution for the dissipative waves. In Appendix B the complete derivation is shown.

This results in the following expressions for the for the perturbed orbital motion:

\[
\zeta = \frac{H}{2} \cos(kx - \omega t) \tag{3.9}
\]

\[
u = \frac{Hc}{2d} \cos(kx - \omega t - \frac{1}{2} \epsilon) \tag{3.10}
\]

where the dimensionless parameter \( \epsilon \) is inverse proportional to \( H \) and assumed to be a small quantity:

\[
\epsilon = \frac{8 \rho g d^2}{\omega c^2 H^2} \left| u_1 \right| \left| u_0 \right| < 1 \tag{3.11}
\]

The wave height is given by:

\[
H = H_0 \exp\left(-\frac{1}{2} k \epsilon x\right) \tag{3.12}
\]

\( H_0 \) is the wave height at \( x=0 \). In the derivation of Eqs. 3.9, 3.10 and 3.12 all terms of order \( \epsilon^2 \) and higher order are neglected.

The vertical flow velocity is found from the continuity equation:

\[
w = \zeta \frac{d}{dt} \frac{\partial \zeta}{\partial t} + w^* \tag{3.13}
\]

The perturbed solution has two important features:

(a) The wave amplitude is damped in the direction of propagation; it is easily shown that the decay gives a gradient in the energy flux corresponding to the energy-dissipation (Eq. 3.4).

(b) A phase shift has been introduced between the surface elevation and the horizontal orbital velocity. This means that the horizontal and vertical velocity are not completely out of phase, which is important for determining the shear stress distribution.
3.3.3 The shear stress distribution

The shear stress is determined by Eulerian considerations, i.e. by considering the momentum balance for a control volume fixed in space. The shear stress as a function of $z$ is found from the horizontal projection of the momentum equation for the control volume shown in Fig. 3.2.

The momentum equation reads:

$$\frac{dB}{dt} = P + F$$  \hspace{1cm} (3.14)

where the left-hand side represents the change in the horizontal momentum including the momentum flux through the boundaries of the control volume, $P$ is the horizontal pressure force and $F$ is the shear force acting on the horizontal bottom of the control body. All quantities are averaged over a wave period.

For the change in horizontal momentum the following expression can be found:

$$\frac{dB}{dt} = -\rho \left[ 2u \frac{\partial u}{\partial x} dz + \bar{uw} dx \right]$$  \hspace{1cm} (3.15)

The first term on the right-hand side of the equation above represents the momentum contribution of the radiation stress gradient and is evenly distributed over the water depth (see Fig. 3.1). The second term represents the momentum flux through the bottom of the control volume.

The pressure term is found from the hydrostatic forces and yields:

$$\bar{P} = -\rho g \zeta \frac{\partial \zeta}{\partial x} dx + \rho g \zeta S dx$$  \hspace{1cm} (3.16)

The shear force at the bottom of the control body is given by:

$$\bar{F} = -\tau dx$$  \hspace{1cm} (3.17)

where $\tau$ is the shear stress due to viscous effects or turbulent Reynolds stress.

Substituting Eqs. 3.15, 3.16 and 3.17 into Eq. 3.14 and dividing by $dx$ and $\rho$ gives the reduced momentum equation:

$$\frac{\tau}{\rho} = 2u \frac{\partial u}{\partial x} z - g \zeta \frac{\partial \zeta}{\partial x} + Sg \zeta + \bar{uw}$$  \hspace{1cm} (3.18)

Using the results of the perturbation analysis shown in the previous section (Eqs. 3.9, 3.10, 3.12 and 3.13) the following expression for the reduced momentum equation can be found:
\[
\frac{\tau}{\rho} = \frac{1}{4} g \frac{H_0}{d} \frac{dH}{dx} z - \frac{1}{8} H_0 g \frac{dH}{dx} + S g z + \frac{1}{8} H_0 g \frac{dH}{dx} \left( 1 - \frac{z}{d} \right) \\
= \frac{g}{d} \left( \frac{1}{4} H_0 \frac{dH}{dx} + S d \right)
\]

(3.19)

In the intermediate expression of Eq. 3.19 the first two terms represent the contributions to the shear stress from the momentum and pressure of the radiation stress gradient. The last term represents the momentum flux through the bottom of the control volume. It is seen that without this term \( \tau \) will be different from zero at the water surface. The radiation stress originating from the pressure which is located around \( MWL \) is balanced by the \( \bar{uw} \)-term at \( MWL \).

The wave set-up is normally estimated as the mean water surface slope, required to balance the radiation stress gradient. This is because this set-up gives zero mean bed stress, and is therefore assumed to create only a very weak mean flow. From Eq. 3.19 it is seen that the shear stress in fact becomes zero over the entire water depth (outside the wave boundary layer) for a wave set-up of:

\[
S = \frac{1}{8} \frac{H}{d} \frac{dH}{dx}
\]

(3.20)

which is only one third of the equilibrium set-up commonly attributed to waves with dissipation in the wave boundary layer. It is thus possible to obtain a perfect balance between the radiation stress gradient and a wave set-up without any shear stresses, except in the wave boundary layer.

### 3.3.4 Inserting terms in shallow water equations

The shear stress distribution was derived by considering the momentum balance for a control volume fixed in space. In the previous chapter we saw that different types of flow could be distinguished: mean flow, turbulent flow and orbital motion. These different types of flow all resulted in additional terms in the shallow water equations. In this section we show that when the expressions derived in this chapter are inserted into the shallow water equations the results are identical.

The momentum equation for the wave related terms yields:

\[
\frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{uw}}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x}
\]

(3.21)

When we take a look at the horizontal momentum equation for the wave related terms (Eq. 3.21) we can see that the difference between that equation and the expression of vertical shear stress distribution (Eq. 3.19) is a spatial derivative in the vertical. The forces present at the water surface disappear when this derivation is made. Since the radiation stress at \( MWL \) is balanced by the \( \bar{uw} \)-term no resulting forces remain, so it is allowed to skip both terms.
For the first term in Eq. 3.21 the following expression can be derived:

\[ \frac{\partial u^2}{\partial x} = \frac{Hg}{4d} \frac{dH}{dx} \]  
(3.22)

The second term in Eq. 3.21 becomes:

\[ \frac{\partial u w}{\partial z} = -\frac{Hg}{8d} \frac{dH}{dx} \]  
(3.23)

Inserting Eqs. 3.22 and 3.23 into Eq. 3.21 yields for the pressure gradient:

\[ -\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{g}{8d} \frac{\partial \zeta}{\partial x} = \frac{Hg}{8d} \frac{dH}{dx} \rightarrow S = \frac{\partial \zeta}{\partial x} = -\frac{1}{8} \frac{H}{d} \frac{dH}{dx} \]  
(3.24)

So the set-up as it was found in Eq. 3.19 results.

The momentum transfer due to the \( \bar{uw} \)-term appears to be important for all non-uniform or unsteady wave conditions. In the case of shoaling waves without energy loss there is also a phase shift between the surface elevation and the horizontal orbital velocities. This phase gives a \( \bar{uw} \)-term, which is necessary to include in the analysis. Otherwise the momentum equation will not give the correct solution, which is that shoaling does not introduce any shear stresses, being described by potential flow theory. This is shown by M. Piepers (1994) in his thesis. The relevance of this is that because of the fact shoaling does not gives rise to shear stresses it is allowed to assume a horizontal bottom for our derivation of the shear stress distribution. So the bottom slope itself does not introduce any shear stresses. The shear stresses are only caused by the dissipation in the boundary layer.

### 3.4 Spilling breakers and broken waves

#### 3.4.1 The exchange of energy

The present section treats spilling breakers and the inner part of the surf zone. The plunging breakers have been transformed to bore-like waves, denoted broken waves.

In spilling and broken waves the dissipation of energy mainly occurs in the region in and near the surface roller, and in this region energy loss in the wave boundary layer is neglected since it is of a smaller order. We will use a description of the spilling breaker and broken waves where the surface rollers are represented by a mass of water which is moving forward with the wave with the celerity \( c \).

In the following a simple dynamic model is introduced in order to describe the extraction of energy from the wave motion to the region of energy dissipation near the surface. The considerations are based on a mass of water -the surface roller- which is stationary relative to the wave crest.

The wave is sketched in Fig. 3.3. The surface elevation without the roller is called \( \zeta \), while the local thickness of the roller is called \( \zeta' \).
The water in a surface roller is moving with the wave front without any large vertical accelerations. It is therefore assumed that the pressure in the roller is hydrostatic.

Assuming hydrostatic pressure in the roller, it is seen that a pressure \( p^* \) and a shear stress \( \tau_s \) are introduced at the interface between the roller and the water below, which take part in the wave motion. The forces acting on a column of water are shown in Fig. 3.4.

The vertical and horizontal force balance give:

\[
 p^* = \zeta^* \rho g \quad (3.25)
\]

and for the shear stress when neglecting terms containing \( (dx)^2 \) and dividing by \( dx \) gives:

\[
 7\tau_s = -\rho g \zeta^* \left( \frac{\partial \zeta^*}{\partial x} + \frac{\partial \zeta}{\partial x} \right) \quad (3.26)
\]

Analogous to the additional surface elevation caused by the wave boundary layer, \( \zeta^* \) creates an additional pressure gradient in the fluid, taking part in the orbital motion below the roller:

\[
 \frac{\partial p^*}{\partial x} = \rho g \frac{\partial \zeta^*}{\partial x} \quad (3.27)
\]

This pressure gradient performs a work on the fluid, acting on the orbital motion. Besides the pressure also the shear stress \( \tau_s \) performs a work on the water outside the roller. If we—as an idealization—consider the roller as a solid body a sketched in Fig. 3.5, a boundary layer will be formed below this roller.

The roller is moving with the velocity \( c \) which also must be the flow velocity at the top of the boundary layer.

Therefore the work done on the fluid outside the roller by the shear stress \( \tau_s \) is given by:

\[
 W_c = -\tau_c c \quad (3.28)
\]
The work given by Eq. 3.28 is not applied to extract or add energy directly from or to the wave motion, but simply represents energy dissipated in the boundary layer between the roller and the surrounding fluid. By application of the same argument as behind Eq. 3.4, it turns out that the dissipation $D$ is given by:

$$D = c \tau_z$$  \hspace{1cm} (3.29)

So energy is extracted from the wave motion by the additional pressure gradient in the water with ordinary orbital motion, caused by the surface roller above the wave.

This extraction of energy is evenly distributed over the depth. The dissipation of energy takes place in the intense shear zone of the roller (the dissipation in the bottom boundary layer is neglected), so energy must be transferred up to the roller from the wave motion by the work done by the hydrostatic pressure against the vertical flow velocities.

In Appendix B it is shown that the work done by the additional pressure gradient is equal to the energy transfer up to the boundary layer between the roller and the water body below.

We now have obtained a model in which energy is dissipated in the boundary layer between the roller and the surrounding fluid and the energy is transported to the boundary layer from the surrounding fluid by the work done by the hydrostatic pressure against the vertical flow velocities.
3.4.2 Vertical shear stress distribution

In order to calculate the shear stress distribution, it is as in the case of energy dissipation in the wave boundary layer necessary to know the $uw$-term originating from the organized orbital motion. The principle is again to determine the phase between $u$ and $\zeta$:

The surface elevation of the wave motion is assumed to be periodic in time and is described by a shape function:

$$\zeta = \frac{H(x)}{2} f(kx - \omega t)$$  \hspace{1cm} (3.30)

where $H$ is the local wave height. $f(kx - \omega t)$ can be described by a Fourier series:

$$f(kx - \omega t) = \sum_{n=1}^{\infty} \left[ a_n \cos(n(kx - \omega t)) + b_n \sin(n(kx - \omega t)) \right]$$  \hspace{1cm} (3.31)

The vertical shear stress distribution is just as in the case of dissipation in the boundary layer determined by the momentum equation Eq. 3.14, applied on the control volume shown in Fig. 3.6.

![MWL](image)

Figure 3.6 Control volume for the momentum equation applied on the roller

The energy loss in the wave implies a gradient in the wave height, which locally can be approximated as (first order):

$$H(x) = H_0 + \frac{dH}{dx} x$$  \hspace{1cm} (3.32)

where $H_0$ is the wave height at $x=0$ and $\frac{dH}{dx}$ the wave gradient at $x=0$. The gradient in wave height is assumed to be weak, which is reasonable in the inner zone. Now the horizontal orbital velocity is determined from the continuity equation and Eqs. 3.30 and 3.32 to be (see Appendix B for a detailed derivation):
\[ u = \frac{H_0 \omega}{2d} \sum_{n=1}^{\infty} a_n \left( \cos[n(kx - \omega t)] - \frac{1}{H_0} \frac{dH}{dx} \frac{1}{kn} \sin[n(kx - \omega t)] \right) \]
\[ + \frac{H_0 \omega}{2d} \sum_{n=1}^{\infty} b_n \left( \sin[n(kx - \omega t)] + \frac{1}{H_0} \frac{dH}{dx} \frac{1}{kn} \cos[n(kx - \omega t)] \right) \]  

(3.33)

The vertical velocity is given by:

\[ \omega = \frac{z + d}{d} \frac{\partial \zeta}{\partial t} = \frac{z + d}{d} \frac{H_0}{2} \sum_{n=1}^{\infty} \left[ a_n n \omega \sin[n(kx - \omega t)] - b_n n \omega \cos[n(kx - \omega t)] \right] \]  

(3.34)

\[ u \text{ (Eq. 3.33)} \text{ and } w \text{ (Eq. 3.34)} \] are known so the term \( \bar{uw} \) can be found:

\[ \bar{uw} = -\frac{1}{8} g H_0 \frac{dH}{dx} \frac{z + d}{d} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \]  

(3.35)

Again assuming hydrostatic pressure, the pressure term in Eq. 3.14 is still given by Eq. 3.16, which can be rewritten to:

\[ \bar{P} = -\rho g \zeta \frac{\partial \zeta}{\partial x} + \rho gzSdx - \left( \frac{1}{2} \rho g \left( \frac{\partial \zeta}{\partial x} \right)^2 - \rho g \frac{\partial \zeta}{\partial x} - \frac{1}{2} \rho g \left( \frac{\partial \zeta}{\partial x} \right)^2 \right) \]  

(3.36)

Neglecting terms containing \((dx)^2\) gives Eq. 3.16 again.

The average change in momentum is given by Eq. 3.15. An additional contribution arises from the roller, so Eq. 3.15 now becomes:

\[ \frac{dB}{dt} = -\rho \left[ 2\bar{u} \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} (e^2 \zeta^*) dx + \bar{uw} dx \right] \]  

(3.37)

Inserting these equations into the momentum equation gives after some algebra the expression for the vertical shear stress distribution (by application of Eqs. 3.14, 3.17, 3.36 and 3.37):

\[ \tau = -\frac{1}{4} \rho g H_0 \frac{dH}{dx} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \left( 1 - \frac{1}{2} \frac{z}{d} \right) - \rho \frac{\partial}{\partial x} (e^2 \zeta^*) + \rho gzS \]  

(3.38)

The first term on the right hand side has the character of dissipation. The mean energy flux of the wave motion can be defined as:

\[ F_e = \rho g \int_0^T \zeta u dt = \rho g d \bar{u} = \frac{\rho g H_0^2}{8} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \]  

(3.39)

For \( \bar{\zeta^*} \) we have:

\[ \bar{\zeta^*} = \frac{1}{T} \int_0^T \zeta^* dt = \frac{1}{T c_0} \int \zeta^* dx \]  

(3.40)

where \( L_n \) is the horizontal length of the roller (Fig. 3.3).
Therefore, defining $A$ as the area of the roller in vertical projection, we get:

$$\overline{\zeta^*} = \frac{A}{L}; \quad A = \int_0^{L_z} \zeta^* \, dx \quad (3.41)$$

$A$ was measured by Duncan (1981) for breaking waves behind a hydrofoil and found by Svendsen (1984a) to be approximated by:

$$A \approx 0.9H^2 \quad (3.42)$$

In Chapter 4 we will derive wave theory in which the area of the roller is one of the results. We therefore leave the area of the roller unspecified.

The shear stress can now be written as:

$$\tau = -\frac{1}{c} \frac{dF_s}{dx} \left(1 - \frac{z}{2d}\right) + \rho g S_z - \rho \frac{d}{dx} \left(\frac{Ac}{T}\right) \quad (3.43)$$

where $A$ is the cross sectional area of the surface roller.

At the water surface ($z=0$) Eq. 3.43 becomes:

$$\tau = -\frac{1}{c} \frac{dF_s}{dx} - \rho \frac{d}{dx} \left(\frac{Ac}{T}\right) \quad (3.44)$$

For a wave set-up balancing the radiation stress gradient (no mean bed shear stress) the shear stress distribution is given by:

$$\tau = \frac{z+\nu}{d} \left(\frac{1}{c} \frac{dF_s}{dx} + \frac{\rho}{T} \frac{d}{dx} \left(\frac{Ac}{T}\right)\right) \quad (3.45)$$

An important feature in Eq. 3.45 is the fact that the shear stress for breaking waves is consisting out of two parts: a wave part and a correction to take the roller into account. In the following chapter a wave theory is derived which gives us the input parameters to compute the shear stress at the water surface. Since we will use a 2DV-version of TRISULA only a shear stress at the water surface has to be defined. TRISULA itself will compute the shear stress distribution in side the water body.
4 Wave model

4.1 Introduction

In the previous chapter we have derived relations which can be used to compute the shear stress at the surface due to breaking waves and the shear stress distribution in dissipative water waves. In this chapter a wave model is described which assumes that energy dissipated in waves is only due to the breaking of waves. In the surfzone the wave height decay due to energy dissipated in the wave boundary layer will be neglected.

The wave model is based on the dissipation model for random waves by Battjes and Janssen (1978). This theory is elaborated in the following section. Next, an additional equation is discussed which introduces a lag between the maximum gradient in short wave energy and the maximum gradient of the return flow in breaking waves. This is followed by a section in which the mass flux due to breaking and non-breaking waves is discussed. The final section shows how the expressions for the shear stress found in the previous chapter can be modified so that the results of the wave model can be used as input for the flow model.

4.2 A dissipation model for random waves

4.2.1 Introduction

For given incident wave parameters and beach profile, the variation of mean wave energy density with distance to the shoreline can in principle be calculated from the wave energy balance:

\[ \frac{\partial F_x}{\partial x} + D_w = 0 \]  

(4.1)

in which \( F_x \) is the \( x \)-component of the time mean energy flux per unit of length, \( x \) is a horizontal coordinate perpendicular to the still-water line and \( D_w \) is the time-mean dissipated energy per unit area.

The equation above can be integrated numerically since \( F_x \) and \( D_w \) depend on known parameters.

As indicated in the previous chapter we can distinguish two areas in the near shore region: the surf zone where the energy decay is primarily due to breaking waves and the area outside the surf zone where the decay of wave energy is mainly due dissipation in a thin wave boundary layer. From here on we will neglect the dissipation outside the surf zone and focus on the dissipation inside the surf zone due to breaking waves.

In the following section we will discuss the parameters of which \( F_x \) and \( D_w \) depend. First the wave height distribution will be elaborated followed by a section on the breaker height. Then it is possible to derive an expression for the mean energy dissipation in a random breaking wave field. Finally an expression for the energy flux \( F_x \) will be derived.
4.2.2 Wave height distribution

The basic idea behind the theory outlined here is that as waves propagate shore wards there height is limited as the water depth decreases. So the decreasing water depth limits the larger wave heights in the distribution. It is now possible to derive a relation in which the mean square value from a distribution of wave heights is limited by the local water depth.

Battjes and Janssen (1978) wrote the assumptions stated above in terms of a probability distribution of the wave heights, $F(H)$. The shape of the lower, non-broken wave heights is assumed to be the same as in the absence of breaking waves. This distribution is the Rayleigh-type:

$$ F(H) = Pr(H \leq H) = 1 - e^{-\frac{1}{2} \frac{H^2}{\bar{H}^2}} \text{ for } 0 \leq H < H_m$$

$$ = 1 \text{ for } H_m \leq H $$

(4.2)

In which $H$ is a random variable, $\bar{H}$ is some modal wave height which represents the random wave field and is left unspecified for the moment and $H_m$ the maximum wave height.

The root mean square of the wave height distribution can be defined as:

$$ H_{rms} = \left( \int_0^\infty H^2 dF(H) \right)^{\frac{1}{2}} $$

(4.3)

We now define the $Q_b$ which is the probability that a wave of certain height will break at a certain point as:

$$ Q_b = Pr(H = H_m) $$

(4.4)

Using the expression for $Q_b$ we can write for $H_{rms}$:

$$ H_{rms}^2 = \lim_{H' \to H_m} H'^2 \int_0^{H'} dF(H) + Q_b H_m^2 $$

(4.5)

With:

$$ Pr(H=H_m) = Q_b $$

(4.6)

and:

$$ Pr(H>H_m) = 0 $$

(4.7)

we can write for $F(H)$:

$$ \lim_{H \to H_m} F(H) = 1 - Q_b $$

(4.8)
which yields for $Q_b$:

\[ Q_b = e^{\left(-\frac{1}{2} \frac{H_m^2}{H^2}\right)} \]  

(4.9)

By using Eqs. 4.2 and 4.9 the first term on the right-hand side of Eq. 4.5 becomes:

\[
\lim_{H^* \to H_m} \int_{H_m}^{H^*} H^2 dF(H) = \lim_{H^* \to H_m} \left[ -H^2 e^{\left(-\frac{1}{2} \frac{H_m^2}{H^2}\right)} \right]_0^{H^*} + \lim_{H^* \to H_m} \int_{H_m}^{H^*} 2He^{\left(-\frac{1}{2} \frac{H_m^2}{H^2}\right)} dH
\]

\[
= -H_m^2 Q_b - \lim_{H^* \to H_m} \left[ 2H^2 e^{\left(-\frac{1}{2} \frac{H_m^2}{H^2}\right)} \right]_0^{H^*}
\]

\[
= -H_m^2 Q_b - 2H^2 (Q_b - 1)
\]

(4.10)

Inserting the result into Eq. 4.5 yields for the expression for $H_{rms}$:

\[ H_{rms}^2 = 2(1 - Q_b)H^2 \]  

(4.11)

It is now possible to eliminate the unspecified $H$ by inserting Eq. 4.9 into Eq. 4.11:

\[
\frac{1 - Q_b}{\ln Q_b} = \left( \frac{H_{rms}}{H_m} \right)^2
\]

(4.12)

From this equation $Q_b$ can be solved as a function of $H_{rms}/H_m$. The local value for $H_{rms}$ is found by solving the differential equation (Eq. 4.1) numerically. We will show in Section 4.2.4 that the Dissipation $D_m$ is proportional to $Q_b$.

### 4.2.3 Determination of Breaker Height

In this section we will specify the breaker height $H_m$. In the model of Battjes and Janssen $H_m$ was based on Miche's criterion for the maximum wave height which was adapted through the inclusion of a parameter $\gamma$ to account for influences of bottom slope and incident wave steepness:

\[ H_m = \frac{0.88}{k_p} \tanh \left( \frac{\gamma k_p h}{0.88} \right) \]  

(4.13)

in which $k_p$ is the wave number based on the linear wave theory dispersion relation for gravity waves with frequency $f_p$.

Battjes and Stive (1985) performed a study on the dissipation model as it was described above and came up with a new empirical relation for $\gamma$:

\[ \gamma = 0.5 + 0.4 \tanh(33s_0) \]  

(4.14)
in which \( s_0 \) is the deep-water steepness defined as:

\[
s_0 = \frac{H_{rms}}{L_p} \tag{4.15}
\]

in which the subscript indicates that the wave height and wave length are determined in deep water conditions.

### 4.2.4 Mean energy dissipation in a breaking wave field

Following LeMéhauté (1962), and others the energy dissipation in a breaking wave is modelled after that in a bore connecting two regions of uniform flow. This results in the following order-of-magnitude relation for the rate of energy dissipation per unit of horizontal area, \( D \):

\[
D = \frac{1}{4} \rho \nu H_b^2 \tag{4.16}
\]

in which \( f \) is the wave frequency and \( H_b \) the trough to crest height of the breaking wave. If we want to use a similar expression for the case of random waves it is necessary to compute the expected value of \( D \), written as \( \bar{D} \), taking into account the randomness of the wave field.

The frequency at the peak of the energy spectrum is used for the frequency in Eq. 4.16, whereas the mean square of the random variable \( H_b \) is equated to the square of the nominal depth limited height of periodic waves, \( H_{rms} \), in water of local mean water depth, \( h \).

Substituting the approximations in Eq. 4.16 yields:

\[
\bar{D} = \frac{1}{4} \alpha Q_\nu f \rho \nu H_m^2 \tag{4.17}
\]

in which \( \alpha \) is a constant expected to be of order 1.

### 4.2.5 Energy flux

Finally we have to establish an expression for the onshore energy flux \( F_x \) per unit width in order to integrate the energy balance Eq. 4.1. The energy flux is estimated as:

\[
\bar{F}_x = E c_g \tag{4.18}
\]

in which the energy \( E \) is given by:

\[
E = \frac{1}{8} \rho \nu H_{rms}^2 \tag{4.19}
\]

and \( c_g \) is the group velocity according to linear theory for \( f = f_p \).

We now have obtained a closed model enabling the application of the energy balance to the surf zone.
4.3 Surface roller effect

4.3.1 Introduction

Svendsen (1978, 1984a) pointed at the importance of surface rollers. He described the existence of a transition zone between the unbroken wave shape and the turbulent bore (here denoted as roller). We will assume that the energy dissipation mainly takes place in the roller, in this way a lag is introduced between the dissipation in the wave and the dissipation in the roller because of the fact that the roller has to build up before energy can be dissipated. The properties of the roller as they were found by Svendsen (1984a) are shown in the following section. Okayasu (1989) extended the theory outlined by Svendsen (1984a) and found a new expression for the kinetic energy in the roller. We modified the theory of Okayasu which is elaborated in Section 4.3.3. In Section 4.3.4 the potential energy of the roller is derived.

Roelvink (1993) constructed an additional differential equation which is solved simultaneously with the wave energy balance according to Battjes and Janssen (1978). He used the properties derived by Svendsen (1984a). This additional equation is highlighted in Section 4.3.5.

In Sections 4.3.6 and 4.3.7 it is shown how the properties of the roller according to Svendsen (1984a) and the extended theory of Okayasu can be implemented in the additional equation (which is referred to as the roller equation from here on).

4.3.2 Properties of the roller, Svendsen (1984a)

In this section some properties of the roller are derived, these derivations are based on Svendsen (1984a). We assume that the density of the roller is $\rho$ and that it travels with the wave celerity $c$. The cross-sectional area of the roller is $A$.

The kinetic energy of the roller, $E_{r,\text{Kin}}$, averaged over a wave length can be written as:

$$E_{r,\text{Kin}} = \frac{1}{2} \frac{m_c^2}{L} = \rho A \frac{c^2}{2L} = \rho A \frac{c}{2T} \quad (4.20)$$

For the derivation of the shear stress induced by the roller we can make use of the fact that the position of the roller does not change relative to the underlying wave as the wave propagate shorewards. This implies that the shear stress induced by the roller should balance the downward force exerted on the roller due to gravity. This is schematically depicted in Fig. 4.1.
We can write for the shear force, \( F_r \):

\[
F_r = \sin \beta F_g = \sin \beta \rho g A
\]

(4.21)

The slope of the wave front is quite small so we can replace \( \sin \beta \) by \( \beta \). Furthermore, we assume that the shear stress is present over the total wave (time averaging over a wave period) which implies that \( F_r \) has to be divided by the wave length \( L \). This gives:

\[
\bar{\tau}_r = \beta \rho g \frac{A}{L}
\]

(4.22)

The dissipation of the roller, \( D_r \), can now be written as:

\[
D_r = \bar{\tau}_r c = \beta \rho g \frac{A}{T}
\]

(4.23)

For the mass flux of the roller, \( M_{x,r} \), we can write (again time averaged over a wave period):

\[
M_{x,r} = \frac{m_r c}{L} = \rho \frac{A}{T}
\]

(4.24)

### 4.3.3 Kinetic energy in roller, Okayasu (1989) extended

Okayasu (1989) extended the theory by assuming a velocity distribution as shown in Fig. 4.2. We will adapt his approach concerning the kinetic energy of the roller. The main difference between his and our derivation is that we leave the area of the roller unspecified. We assume that the shape of the roller is a circle (implying that the area of the roller is proportional to the square of the wave height).

Although the water in the front part of the roller decelerates and moves downward and the back side of the roller accelerates and moves upwards, we assume the mean horizontal velocity to be equal to the wave celerity. The diameter of the roller is assumed to be proportional to the wave height \( H (A = \psi^* H) \).
The energy, $E_1$, of the rotational component per unit width is defined as:

$$E_1 = \int_0^{\psi H_{rms}} \frac{1}{2} \rho u_1^2 2\pi r dr$$

(4.25)

in which $r$ is the radial coordinate.

With:

$$u_1 = \frac{2c}{H_{rms}} r$$

(4.26)

Eq. 4.25 can be written as:

$$E_1 = \frac{1}{16} \pi \rho c^2 H_{rms}^2 \psi^4$$

(4.27)

For the parallel component, $E_2$, the same derivation can be made which results in:

$$E_2 = \frac{1}{8} \pi \rho c^2 H_{rms}^2 \psi^2$$

(4.28)

The total kinetic energy of the roller is now found as:

$$E_3 = E_1 + E_2 = \frac{1}{16} \pi \rho c^2 H_{rms}^2 \psi^2 (2 + \psi^2)$$

(4.29)

The total kinetic energy should be proportional to $c^2 H_{rms}^2$ which is in agreement with Svendsen (1984a). The kinetic energy per unit width and length can now be expressed as:

$$E_{rk} = \frac{E_3}{L} = \frac{\pi \rho c^2 H_{rms}^2 \psi^2 (2 + \psi^2)}{16L}$$

(4.30)
4.3.4 Potential energy of roller

For the potential energy of the roller, $E_{r,\rho}$, we also have to derive an expression, in Fig. 4.3 the schematisation for the potential energy is shown.

![Diagram of roller with labels: 1/4H_{rms}, MWL, H_{rms}, centre of gravity.]

Figure 4.3 Potential energy of roller

According to this schematisation the potential energy can be written as:

$$E_{r,\rho} = \frac{\rho g A}{L} \frac{1}{4} H_{\text{rms}} = \frac{\rho g \psi^2 H_{\text{rms}}^3}{16L} \quad (4.31)$$

With $k = \frac{2\pi}{L}$ the total energy of the roller yields:

$$E_{r} = \frac{\rho H_{\text{rms}}^2 \psi^2 k}{32} (c^2 (2 + \psi^2) + g H_{\text{rms}}) \quad (4.32)$$

4.3.5 Roller equation

As mentioned in the introduction we assume that the lag effect is due to the fact that the roller has to build up before any energy can be dissipated in the roller. This is taken into account by considering a total energy balance in which the total energy is decomposed into two parts: a wave part which is assumed to be twice the potential energy according to linear wave theory and a roller part of which its properties were exemplified in the previous sections. The total energy balance then reads:

$$\frac{\partial E_{c}}{\partial x} + \frac{\partial E_{c}}{\partial x} = -D_r \quad (4.33)$$

In this equation it is assumed that all the dissipation is located in the roller. The total energy balance can be split up into two parts:

$$\frac{\partial E_{c}}{\partial x} = -D_w \quad (4.34)$$
which is the equation described in Section 4.2.1. The dissipation in the wave part is a sink in this equation and brought into the equation for the roller as a source since all the dissipation is assumed to take place in the roller. This equation is written as:

\[
\frac{\partial E_{C}}{\partial x} = D_{w} - D_{r}
\]  \hspace{1cm} (4.35)

With this equation a lag is introduced between the dissipation in the wave and the dissipation in the roller. Roelvink (1993) used the properties of Svendsen (cf. Section 4.3.2) to obtain a closed system of equations. This is shown in the following section. It is also possible to use the expression for the kinetic energy found in Section 4.3.3 to construct a closed system, this shown in Section 4.3.7.

4.3.6 Roelvink (1993)

As stated above the properties of Svendsen (1984a) are used to construct a closed system of equations. With the properties presented in Section 4.3.2 it is possible to express the dissipation in the roller, \(D_{r}\), in terms of the kinetic energy of the roller:

\[
D_{r} = 2\beta g \frac{E_{r,k}}{c}
\]  \hspace{1cm} (4.36)

In the roller equation the kinetic energy flux of the roller is computed. In the wave energy balance the wave height is predicted in which the potential energy of the roller is included implicitly. The wave energy equation is calibrated on measurements made out- and inside the surfzone. This implies that roller effects are included in the measurements and by calibrating the wave energy equation on these measurements the roller effects are included as far as the (total) potential energy is concerned.

Eq. 4.35 now becomes:

\[
\frac{\partial E_{C}}{\partial x} = D_{w} - 2\beta g \frac{E_{r,k}}{c}
\]  \hspace{1cm} (4.37)

where \(\beta\) is the steepness of the wave front and found by Roelvink (1993) to be 0.10.

Now a closed set of equations is obtained in which no area of the roller has to be specified. Eq. 4.37 is calibrated by Roelvink (1993) using the measured set-up in flume tests. He showed that when a lag is introduced between the dissipation in the wave and the dissipation in the roller the results of computations compared with measurements of the set-up in a flume improved considerably.

In the next section we will use the extended theory of Okayasu to obtain a closed system of equations.
4.3.7 Okayasu extended

In this section we will use the expressions derived in Section 4.3.3. For the Dissipation in the roller we will use the expressions given by Svendsen (Section 4.3.2).

Inserting Eq. 4.30 into Eq. 4.22 gives the dissipation in the roller, \( D_r \), in terms of the kinetic energy of the roller according to Okayasu extended:

\[
D_r = \beta g \frac{E_{r,k}}{c} \left( \frac{4}{2 + \psi^2} \right)
\]  

(4.38)

Inserting Eqs. 4.30 and 4.38 into Eq. 4.35 gives a differential equation in which the only unknown variable is \( \psi \):

\[
\frac{\partial E_{r,k}}{\partial x} = D_w - \beta g \frac{E_{r,k}}{c} \left( \frac{4}{2 + \psi^2} \right)
\]

\[
\frac{\partial}{\partial x} \left( \frac{\rho c^2 H_{rms}^2 \psi^2 k}{32} \frac{1}{2 + \psi^2} \right) = D_w - \beta \rho g c H_{rms}^2 k \psi^2 \frac{4}{8}
\]

(4.39)

This equation is solved in \( \psi \) whereas the equation according to Roelvink is solved in \( E_{r,k} \). In Eq. 4.36 can be seen that the ratio between the dissipation and the kinetic energy in the roller is constant. In Okayasu extended however the ratio between the dissipation and the kinetic energy in the roller is not constant but proportional to \( \frac{4}{2 + \psi^2} \) (compare Eqs. 4.36 and 4.38). As a result of this the dissipation in the roller in Okayasu extended has a varying sensitivity for changes of the dissipation in the wave whereas the dissipation in the roller in the equation according to Roelvink has a constant sensitivity for the dissipation in the wave.

As mentioned we use the expression for the dissipation, \( D_r \), found by Svendsen (1984a) for both methods. \( D_r \) is in fact derived under the assumption that the roller travels with a velocity \( c \) over the underlying water body. The assumption by Okayasu concerning the velocity distribution has as a result that in the transition zone between the roller and the underlying water no velocity difference is present. So no dissipation is present in that case. At the front- and rear-side of the roller, however, there is an exchange of water between the roller and the underlying water body which is assumed to create the shear stress from which the dissipation according to Svendsen (1984a) originates.

4.4 Mass flux in the surf zone

4.4.1 Introduction

In Section 2.5.7 we showed that if we include the orbital motion in the continuity equation an additional term appeared which expresses the wave induced mass flux. Outside the surf zone the mass flux is caused only by wave motion. In the surf zone however also surface rollers are present (for a definition sketch is referred to Chapter 3 Fig. 3.3) which contribute to the total mass flux. Hereby the total mass flux in the surf zone is given by:
\[ M_r = M_w + M_r \]  
(4.40)

In which \( M_w \) is the mass flux due to wave motion and \( M_r \) is the mass flux due to the surface roller. In the next section the mass flux due to wave motion is treated followed by a section dealing with the mass flux due to the roller.

### 4.4.2 Mass flux due to wave motion

Dean (1965) showed that the wave particle velocity calculated by the stream method is very accurate for nearly breaking waves. One of the disadvantages of this method is that the computations are time consuming and the fact that theories including higher order terms are only valid for a certain area; up till there is now no higher order wave theory available which is valid throughout the whole surfzone. Okayasu (1989) related the mass flux according to the linear wave theory with the mass flux according to the stream function method based on experiments in a flume. We will use this relation which makes it possible to correct the mass flux according to the linear wave theory. Okayasu found that the mass flux computed with linear wave theory is overestimating the mass flux with 20\%, which yields:

\[ M_w = M_{\text{str}} = 0.8M_{\text{str}} \]  
(4.41)

First we will derive the mass flux according to the linear wave theory and then perform a correction according to Eq. 4.41.

The mass flux in Section 2.5.7 was derived as:

\[ M_x = \frac{1}{T} \int_0^T \rho \bar{u} \mid_{z_0} (\zeta - z) \, dt \]  
(4.42)

which with:

\[ \zeta = \frac{H_c}{c} \cos(kx - \omega t) \]  
(4.43)

\[ \bar{u} \mid_{z_0} = \frac{H_c}{2d} \cos(kx - \omega t) \]  
(4.44)

transforms to:

\[ M_x = \frac{1}{T} \int_0^T \left( \frac{H_c}{4d} \cos^2(kx - \omega t) - \frac{H_c}{2d} \cos(kx - \omega t) \right) dx \]  
(4.45)

\[ = \frac{\rho}{T} \int_0^T \frac{H_c^2}{8d} dt = \rho \frac{H_c^2}{8d} = \frac{2c}{16g} \rho g H^2 \]

The potential energy is defined as:

\[ E_p = \frac{1}{T} \int_0^T \rho g (\zeta - <\zeta>) \, dt = \frac{1}{2} \rho g (\zeta - <\zeta>)^2 \]  
(4.46)
If we assume a periodic sinusoidal wave with an amplitude of $0.5*H$ this can be written as:

$$E_p = \frac{1}{16} \rho g H^2$$  \hspace{1cm} (4.47)

The mass flux according linear wave theory now becomes:

$$M_w = \frac{2E_p}{c}$$  \hspace{1cm} (4.48)

Applying Eq. 4.41 yields for the mass flux due to wave motion:

$$M_w = \frac{1.6E_p}{c} = \frac{0.8E}{c}$$  \hspace{1cm} (4.49)

### 4.4.3 Mass flux due to the roller

In this section the mass flux due to the roller is derived. Similar to the previous section we have to distinguish the new theory based on Okayasu and the theory according to Roelvink which is based on Svendsen (1984a). The difference again is the fact that when using the extended theory an additional factor is appearing in the mass flux. First we will give the general theory concerning the mass flux followed by an elaboration according to Roelvink and Okayasu extended, respectively.

First the mass flux due to the wave motion has to be corrected since otherwise the total mass flux is overpredicted. The mass flux due to wave motion is originating from the potential energy. In the surf zone breaking waves are present which do not contribute to this mass flux. The mass flux due to waves is originating from the potential energy of the waves as derived in the previous section while the mass flux due to rollers can be described with the kinetic energy of the roller. In the potential wave energy also the potential energy of the roller is included as was explained in Section 4.3.6. This implies that the potential energy of the waves should be corrected to take the roller into account. This is possible by subtracting the potential energy of the roller (Eq. 4.31) from the potential energy of the wave motion (Eq. 4.47). The total mass flux in the surf zone then becomes:

$$M_t = M^{c.w} + M_r = \frac{0.8E_p}{c} - \frac{E_{r,k}}{c} + M_r$$  \hspace{1cm} (4.50)

We will use the expression as it was found by Svendsen (Eq. 4.24) and express the mass flux in terms of kinetic energy. If we just as in Chapter 3 consider the roller to travel with wave velocity shorewards, the mass flux of the roller, $M_r$, is given by:

$$M_r = \frac{\rho A_c}{L} = \rho \frac{A}{T}$$  \hspace{1cm} (4.51)

By inserting Eq. 4.20 into Eq. 4.51 it is possible to express the mass flux in terms of the kinetic roller energy, $E_{r,k}$:

$$M_r = \frac{2E_{r,k}}{c}$$  \hspace{1cm} (4.52)
It is also possible to use the kinetic energy according to Okayasu extended (Eq. 4.30) which yields for the mass flux:

\[ M_r = \rho \frac{A}{T} = \frac{E_{r,t}}{c^2 + \psi^2} \tag{4.53} \]

The total mass flux due to breaking waves and wave motion can be described in terms of the potential energy according to linear wave theory and the kinetic energy of the roller. The difference between the Svendsen and Okayasu extended is the term: \[ \frac{4}{2 + \psi^2} \]. The parameter \( \psi \) is small in deep water and increases towards the shore since the roller grows and the wave height decreases. In this way the mass flux due to the roller near the shore is reduced.

4.5 The shear stress

4.5.1 Introduction

With the wave model as it was described above it is now possible to compute the shear stress at the surface. For this purpose we use the results that were derived in Chapter 3. In this section we will transform Eq. 3.44 in order to use the results of the wave computation to compute the shear stress at the water surface to take the breaking waves into account. We will elucidate the transformations and show how the equations are modified if we also use the dissipation in the roller instead of the dissipation in the waves.

4.5.2 Applying results of wave model

For clearness’s sake we depict the equation below:

\[ \tau = -\frac{1}{4} \rho g H \frac{dH}{dx} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{\rho}{T} \frac{dAc}{dx} \tag{4.54} \]

This is the shear stress at the water surface and \( a_n^2 \) and \( b_n^2 \) are fourier coefficients.

In Chapter 3 the energy flux was found to be:

\[ F_x = \frac{1}{8} \rho g H_0^2 c \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \tag{4.55} \]

The shear stress at the water surface can now be written as:

\[ \tau = -\frac{1}{c} \frac{dF_x}{dx} = \frac{\rho}{T} \frac{dAc}{dx} \tag{4.56} \]

This result of Chapter 3 allows us to express the shear stress in terms of dissipation rather than in terms of energy density gradients (which as stated above makes it possible to use the results of the wave model).
Since we will use root mean square wave height from the energy flux equation which represents itself a random wave height distribution it is assumed that the term \( \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \) is of order one. This is a similar approach followed by Stive and De Vriend (1994) who introduced the relation:

\[
\frac{dE}{dx} = \frac{1}{c} \frac{dF_x}{dx_x} = \frac{D_w}{c} \tag{4.57}
\]

By using Eqs. 4.56 and 4.57 we can now express the shear stress at the water surface as:

\[
\tau = -\frac{dE}{dx} = \frac{\rho}{T} \frac{dAc}{dx} = \frac{D_w}{c} \tag{4.58}
\]

By this a simple expression is found expressing the shear stresses in terms of the dissipation in the roller (making it possible to use the \( D \), computed according to Eq. 4.35).

Stive and De Vriend (1994), however, follow a somewhat different approach in determining the dissipation in the roller. Instead of using the additional differential equation, they derived an expression for the second term in Eq. 4.56:

\[
\frac{\rho}{T} \frac{dAc}{dx} = 1.15k \frac{d(E^*h)}{dx} \tag{4.59}
\]

in which \( E^* \) is the energy density of those waves which are breaking, i.e. which are effectively contributing to the breaking related dissipation. In the case of random waves they suggest for \( E^* \):

\[
E^* = Q_b^* E_{br} \tag{4.60}
\]

in which \( E_{br} \) is the energy density of the broken waves and \( Q_b^* \) is the breaking wave fraction based on the Battjes and Janssen model which is empirically modified because of the fact that the original \( Q_b \) is significantly underestimating the breaking wave fraction:

\[
Q_b' = \begin{cases} 
7Q_b & \text{for } Q_b \leq 0.1 \\
1 - 548(0.3 - Q_b)^{4.67} & \text{for } 0.1 < Q_b < 0.3 \\
1.0 & \text{for } Q_b \geq 0.3
\end{cases} \tag{4.61}
\]

An advantage of this method is that there no need for an additional differential equation which is the case for Roelvink (1993) and Okayasu extended. It is possible to express the properties of the roller with local parameters which will reduce the computational costs. For this method it is essential however to use the actual breaking wave fraction. It should be questioned if the empirically modified \( Q_b \) gives an accurate approximation of the actual breaking wave fraction (it is seen in Chapter 7 that this is not the case). Some effort has been made in determining the actual wave breaking fraction. Up till now no reliable expression is found for this parameter (see e.g. Beyer, 1994).
We now have described three methods to take the roller effect into account:

- Roelvink (1993)
- Okayasu extended
- Stive and De Vriend (1994)

In Chapter 7 these three methods will be compared.
5 TRISULA as a 2DV-model

5.1 Introduction

In the three previous chapters we described the shallow water equations, what kind of stresses are exerted on the water body if waves are present and a wave model, respectively. In this chapter a flow model is described in which the results of the previous three chapters are introduced. In Section 5.2 the program which solves the shallow water equations is briefly discussed followed by Section 5.3 where an elaboration about the incorporation of wave induced turbulence in the turbulence model is given, furthermore the turbulence model itself is highlighted. In Section 5.4 the boundary conditions for the turbulence model are treated. Next, a summary is given in Section 5.5 of the wave related terms which are incorporated in the 2DV-model. In the final section the staggered grid which is used by the 2DV-program is highlighted.

5.2 Flow model

In Chapter 2 we derived the shallow water equations for the three dimensional case, the flow model which we are using is a so-called 2DV flow model. This model also solves the shallow water equations but only in the vertical and one horizontal direction. We have done this because it reduces the computation time and since we are only considering incident waves perpendicular to the coast a 3D model is not necessary. The equations which were derived in Chapter 2 remain valid only the equation with respect to the y-direction is omitted, the same can be said for all gradients in y-direction. Similar to TRISULA the flow model uses the scaled vertical σ-coordinate (see Section 2.5). For clearness' sake the equations which are solved in the 2DV flow model are depicted below:

the continuity equation:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial \omega}{\partial \sigma} = 0$$

(5.1)

with:

$$\sigma = \frac{z - \zeta}{h}$$

(5.2)

the equation of motion yields:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \omega \frac{\partial u}{\partial \sigma} - \frac{\partial}{\partial z} \left( v \frac{\partial u}{\partial z} \right) + g \frac{\partial \zeta}{\partial x} = 0$$

(5.3)

The equations presented above solve the velocities in each layer. To compute the new water level the same equations are solved but now in the depth averaged form:
\[
\frac{\partial \zeta}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{1}{\rho} \frac{\partial M_e}{\partial x} = 0
\]  \hspace{1cm} (5.4)

\[
\frac{\partial u_z}{\partial t} + \frac{1}{\rho} \int_{-\epsilon}^{\epsilon} \left( u \frac{\partial u}{\partial x} + \frac{\omega}{h} \frac{\partial u}{\partial \sigma} \right) dz + v \frac{\partial u}{\partial z} \bigg|_{-\epsilon}^{\epsilon} - v \frac{\partial u}{\partial z} \bigg|_{-\epsilon}^{\epsilon} + g \frac{\partial \zeta}{\partial x} = 0 \hspace{1cm} (5.5)
\]

For more information about how the equations are solved numerically the reader is referred to Piepers (1994).

### 5.3 The turbulence model

#### 5.3.1 The \( k \)- and \( \epsilon \)-equations

In the 2DV-model a so-called \( k \)-\( \epsilon \) model is incorporated. This two-equation model eliminates the need for specifying the turbulence length scale as a function of position throughout the flow. Modelled forms of the \( k \)- and \( \epsilon \)-equations are considered in the following. The two equations are presented below.

**Kinetic energy equation:**

\[
\frac{Dk}{Dt} = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left( \frac{\mu_s}{\sigma_k} \frac{\partial k}{\partial x_j} \right) + \frac{\mu_s}{\rho} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \epsilon \hspace{1cm} (5.6)
\]

where \( k = \) turbulent energy per unit of mass, \( \sigma_k = \) Prandtl-Schmidt number and \( \epsilon = \) energy dissipation per unit of mass.

This is the high Reynolds number form of the transport equation for \( k \) and is not applicable to the viscous sublayer near walls. The term on the left-hand side represents the rate of change and convective transport of kinetic energy. The first term on the right-hand side represents the diffusive transport, the second term the production and the third term represents the dissipation of kinetic energy.

**Dissipation equation:**

\[
\frac{De}{Dt} = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left( \frac{\mu_s}{\sigma_k} \frac{\partial \epsilon}{\partial x_j} \right) + \frac{C_{\epsilon \mu} \mu_s}{\rho} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{C_{\epsilon \mu} \epsilon^2}{k} \hspace{1cm} (5.7)
\]

where \( \sigma_k, C_{\epsilon \mu}, C_{\epsilon 2} \) are constants which have been determined in various experiments. The same can be said of the Prandtl-Schmidt number \( \sigma_k \) which appeared in the previous equation. In the following equations also some constants appear, they will be elaborated in the next section. The meaning of the various terms in the dissipation equation is similar to those in the kinetic energy only in this case with respect to energy dissipation.
The turbulence length scale $l_t$ is computed according to:

$$l_t = C_d \frac{k^{3/2}}{\epsilon}$$  \hspace{1cm} (5.8)

where $C_d$ is a constant.

Now the turbulent viscosity can be computed according to the Kolmogorov-Prandtl expression:

$$\nu_t = \frac{C_n}{C_d} \sqrt{k l_t}$$  \hspace{1cm} (5.9)

### 5.3.2 The empirical constants

The values of the various constants are based on extensive examination of free turbulent flows. The $k-\epsilon$ model in the form described above has been applied successfully to many two-dimensional wall boundary layers, recirculating flows and also to three-dimensional wall boundary layers (for more information the reader is referred to Rodi, 1984).

<table>
<thead>
<tr>
<th>$C_n$</th>
<th>$C_d$</th>
<th>$C_{\nu}$</th>
<th>$\sigma_k$</th>
<th>$\sigma_{\epsilon}$</th>
<th>$C_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.09</td>
<td>1.44</td>
<td>1.92</td>
<td>1.0</td>
<td>1.3</td>
<td>0.1925</td>
</tr>
</tbody>
</table>

Table 5.1 empirical constants

### 5.3.3 Incorporation of wave induced turbulence

In the model described above the turbulence generated by breaking waves is not accounted for. Experimental studies however showed that turbulence generated by breaking waves is an important factor for various processes, such as wave transformation, sediment transport and nearshore current generation, within the surf zone.

In order to include the wave induced turbulence an additional production term should be introduced into the kinetic energy equation. We have done this under the assumption that the energy decay in the wave model (the dissipation) is transferred to turbulent energy (see e.g. Roelvink and Stive, 1989).

$$Production = \frac{\partial E c_{\epsilon}}{\partial x}$$  \hspace{1cm} (5.10)

where $E$ is the organized energy density and $c_{\epsilon}$ is the wave group velocity.

The vertical distribution of energy production is taken as a straight line with its maximum value at the water surface and its minimum at a half wave height below the water surface. Deigaard et al. (1986) assume a parabolic distribution over the vertical vanishing at the water surface and one wave height beneath the surface based on measurements by Rouse et al.
(1958). Our model however has its water surface at MWL which means that the dissipation is maximal at MWL. In Fig. 5.1 both distributions are shown.

\[ P_t(<\zeta>) = \frac{4D_w}{H_{rms}} \]  
(5.11)

at a half wave height below the water surface:

\[ P_t(<\zeta>-0.5*H_{rms}) = 0 \]  
(5.12)

This gives the following expression for the wave induced turbulence production:

\[ P_t(z') = P_t(z'=0)\left(1 - \frac{2z'}{H_{rms}}\right) \]  
(5.13)

where \(z'\) is a coordinate with its origin in the water surface and positive downwards.

Inserting \(P_t\) into Eq. 5.6 gives:

\[ \frac{Dk}{Dt} = \frac{1}{\rho} \frac{\partial}{\partial x_i} \left( \mu_l \frac{\partial k}{\partial x_j} \right) + \frac{\mu_l}{\rho} \left( \frac{\partial U_i}{\partial x_j} \frac{\partial U_j}{\partial x_i} - \frac{\partial U_i}{\partial x_i} \frac{\partial U_j}{\partial x_j} \right) + \frac{P_t(z')}{\rho} - \epsilon \]  
(5.14)

A similar expression has to be introduced into the \(\epsilon\)-equation to take the increase of turbulent energy dissipation due to waves into account. We will adopt the expression found by Mocke et al. (1993). These authors however derived an expression for a depth averaged turbulence model, we assume that the source term accounting for the increase of turbulent energy dissipation has the same vertical distribution as the turbulent energy production:

\[ P_t(z') = c_t \left( \frac{P_t(z')^2}{\frac{1}{2}H_{rms}} \right) \frac{1}{3} \]  
(5.15)

where \(c_t\) is a constant which is of order \(I\).
Inserting the above expression into the dissipation equation (Eq. 5.7) yields:

\[
\frac{D\epsilon}{Dt} = \frac{1}{\rho} \frac{\partial}{\partial x} \left( \frac{\nu_t \partial \epsilon}{\sigma_\epsilon \partial x} \right) + \frac{C_\epsilon \nu_t \epsilon}{k} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j} + \frac{P(\epsilon)}{\rho} - C'_\epsilon \frac{\epsilon^2}{k} \tag{5.16}
\]

With Eqs. 5.14 and 5.16 a \(k-\epsilon\) model is obtained in which, with the right boundary conditions, the wave generated turbulence is included in a physically correct way. In the following section the boundary conditions which have to be prescribed at the surface and at the bottom are treated.

### 5.4 Boundary conditions

#### 5.4.1 Introduction

At both the surface and the bottom there are physical boundaries which have to be incorporated in our \(k-\epsilon\) model. The surface is defined as the mean water level (MWL). The bottom is assumed to be impermeable and not moving. Similar to the \(k-\epsilon\) model itself we will have to take the wave effects into account both at the water surface and at the bottom boundary.

#### 5.4.2 Wall boundaries

Near the wall the turbulent shear stresses are dominating and are in approximation independent of the distance to the wall. The Reynolds stresses in this area can then be written as:

\[
s_{xx} = -\rho |u_*| u_* \tag{5.17}
\]

According to the mixing-length hypothesis of Prandtl we can also write for the Reynolds stresses:

\[
s_{xx} = -\rho l_y \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} \tag{5.18}
\]

So we now can write for the resultant friction velocity:

\[
u_* = l_y \frac{\partial u}{\partial x} \tag{5.19}
\]

In the area near the wall the turbulent length scale, \(l_y\), will be approximately proportional to the distance of the wall yielding:

\[
l_y \sim \kappa z \tag{5.20}
\]

where \(\kappa\) is the von Karman constant.
Inserting Eq. 5.20 into Eq. 5.19 and integrating the subsequent equation gives:

$$u_* = \frac{\langle u \rangle \kappa}{\ln \frac{z}{z_0}}$$  \hspace{1cm} (5.21)

where \( z \) is the distance from the wall, \( z_0 \) is a roughness parameter which is in fact an integration constant and \( \kappa \) is the von Karman constant.

Now integration through the viscous-sublayer is not necessary since an expression is obtained which connects the wall conditions to the dependent variables just outside the viscous sublayer.

This law should be applied in the region where the turbulent viscosity is dominating and where the turbulent length scale is proportional to the distance of the wall. Quantifying this, a dimensionless wall distance can be defined: \( z^+ = \frac{z u_*}{v} \) whose value should be in the range \( 30 < z^+ < 100 \) (for more information is referred to Rodi, 1984).

In this region convection and diffusion of turbulent energy are negligible so that local equilibrium prevails, this implies that in the boundary region there is a balance between the production and dissipation of turbulent energy. Using this together with the fact that the shear stress is approximately equal to the wall shear stress gives:

$$k = \frac{1}{\sqrt{C_u}} u_*^2$$  \hspace{1cm} (5.22)

For \( \epsilon \) the following boundary condition results:

$$\epsilon = C_d \frac{k^{1.5}}{k z_0}$$  \hspace{1cm} (5.23)

Apart from the influence of the flow on the bottom shear stress it should be expected that also the presence of waves has a certain influence on the bottom shear stress and it is found from experiments that this is the case (see e.g. Hattori and Aono, 1985).

Several authors have proposed solutions in which the roughness height, \( z_0 \), is increased to take the influence of the orbital motion into account (see e.g. van Rijn, 1988 and Sleath, 1991). But with these methods however large errors can be observed in the near-surface region; furthermore they are based on the assumption of a logarithmic velocity profile. This assumption however does not hold in the surfzone: the net flow is small and the influence of waves is predominant which leads to velocity profiles with opposite velocities at the upper and lower regions of the water body (see Fig. 7.10).

To incorporate the influence of waves we assume that the influence of the flow and the orbital motion can be superimposed without interacting with each other.

To take the generation of turbulence at the bottom boundary due to orbital motion into account we assume that there is also some dissipation of wave energy in this layer. This
seems in contradiction with earlier assumptions where was stated that the wave height decay is mainly due to the breaking of waves and that the dissipation in the wave boundary layer could be neglected. This assumption still stands but the influence of the waves on the turbulence in the boundary layer can not be neglected. This implies that there is in fact a wave height gradient due to dissipation in the boundary layer. The influence of the dissipation in the boundary layer can be neglected as far as the wave height computations is concerned but its influence on the turbulence production near the bottom has to be taken into account.

To overcome these problems a new type of boundary condition has to be developed at the bottom to account for the orbital motion. This is shown below.

To incorporate the wave influence on the turbulence in the boundary layer we can just as in Chapter 2 subdivide the turbulent kinetic energy, $k$, and the turbulent dissipation, $\varepsilon$, into an averaged and a fluctuating part:

$$ k = \overline{k} + k' \quad \text{and} \quad \varepsilon = \overline{\varepsilon} + \varepsilon' \quad (5.24) $$

For the turbulent viscosity, $\nu_t$, we can write:

$$ \nu_t = C_p \frac{k^2}{\varepsilon} \quad (5.25) $$

Inserting the expressions for $k$ and $\varepsilon$ into Eq. 5.23 gives for $\nu_t$ using $\frac{\varepsilon'}{\varepsilon} < 1$:

$$ \overline{\nu_t} = C_p \frac{\overline{(k + k')^2}}{\overline{\varepsilon} + \varepsilon'} = C_p \frac{\overline{k^2 + k'^2}}{\overline{\varepsilon} + \varepsilon'} \frac{\overline{1 + \frac{k'}{k}}} {\left(1 + \frac{\varepsilon'}{\varepsilon}\right)} = C_p \frac{k^2}{\varepsilon} \left(1 - \frac{\varepsilon'}{\varepsilon}\right) \quad (5.26) $$

From the derivation above it can be seen that in the bottom boundary condition only the expression for $k$ has to be extended to take the influence of waves into account while the expression for $\varepsilon$ does not have to be modified.

The wave boundary layer is in general turbulent so we can write for the bottom shear stress:

$$ \tau_b = \frac{1}{2} C_f \rho |\hat{u}_{orb}| \hat{u}_{orb} \quad (5.27) $$

where $\tau_b$ is the shear stress exerted by the water on the bottom, $\hat{u}_{orb}$ is the amplitude of the orbital motion just outside the wave boundary layer according to linear wave theory, $C_f$ is a drag coefficient which according to measurements can vary with the Reynolds number and the roughness of the bottom. We can write for $C_f$:
\[ C_f = \exp \left[ -s \cdot \frac{\bar{u}_{arb}}{\bar{u}_w} \right] \]  

(5.28)

where \( r_c \) is a bottom roughness defined as: \( r_c = 33\varepsilon_0 \).

For \( \bar{u}_{arb} \) we can write:

\[ \bar{u}_{arb} = \frac{\omega H_{rms}}{2 \sinh(kh)} \]  

(5.29)

Similar to \( u \), we can define a wave friction velocity \( u_* \) by using Eq. 5.27:

\[ \hat{u}_{w*} = \frac{1}{2} \sqrt{C_f \bar{u}_{arb}} \]  

(5.30)

where the hat indicates that the expression is an amplitude.

We now can write for \( k' \):

\[ k' = \frac{\hat{u}_{w*}^2}{\sqrt{C_\mu}} = \frac{C_f \hat{u}_{arb}^2}{4\sqrt{C_\mu}} \]  

(5.31)

As derived in Eq. 5.26 we are interested in the averaged result of the quadratic term of the fluctuation. This term, \( k_w \), can be written as:

\[ k_w = \frac{1}{2} \sqrt{k_{1/2}^2} = \frac{1}{2} \sqrt{k_{2/2}} \]

\[ = \frac{1}{8} \sqrt{C_\mu} \hat{u}_{arb}^2 \]

\[ = \frac{1}{8} \sqrt{C_\mu} \hat{u}_{arb}^2 \]  

(5.32)

The boundary condition for \( k \) in which wave effects are included now becomes:

\[ k = \sqrt{k_f^2 + k_w^2} = \sqrt{\frac{1}{C_\mu} \left( \frac{u_{arb}^4}{64} + \frac{C_f^2 \hat{u}_{arb}^4}{64} \right)} \]  

(5.33)

The boundary condition for \( \nu \), now yields:

\[ \nu = C_\varepsilon \frac{k^2}{k_{1/2}^2} = \frac{C_\mu k_f^2 + k_w^2}{C_d k_{1/2}^2} \]  

(5.34)

This boundary condition however will give rise to numerical instabilities in the computations. This is because of the fact that in \( \varepsilon \) the velocity just above the bottom boundary is included to the third power which has as a result that the viscosity at the boundary will show large fluctuations.

For this reason we developed a new boundary condition which is similar to the expression in Eq. 5.9. We assume similar to e.g. van Rijn (1989) that the bottom roughness increases
because of the influence of waves and that we can represent this apparent bottom roughness increase by an increasing \( z_0 \) (denoted as \( z_a \)). We suggest the following expression for \( z_a \):

\[
z_a = \left( \frac{k_w}{k_f} + 1 \right) z_0 \quad \text{with} \quad \frac{k_w}{k_f} \leq 10
\]  
(5.35)

The first term between brackets can have the maximum value of 10 which is the same limitation van Rijn (1989) proposes.

The implemented boundary condition for \( v \), now becomes:

\[
v_t = \frac{C_u (k_f^2 + k_w^2 \times z_a)}{C_d}
\]  
(5.36)

### 5.4.3 Free surface boundary

When a shear layer is created by forces near the surface the boundary conditions described above for wall boundaries seem appropriate. This is certainly only an approximation, but there is little experimental information on surface effects on turbulence. In the case of wind induced-shear stresses it seems plausible to say that the presence of a free surface reduces the length scale of turbulence. In our case however waves (breaking and non-breaking) are present at the surface which implies that the length scale of turbulence is not reduced by the free surface to the extent as in the case of absence of waves.

Again using the hypothesis that there is a balance between the production and dissipation of turbulent energy it is possible to say that at the free surface the dissipation balances the production of turbulence induced by the breaking of waves (see Eq. 5.10). This gives us the boundary condition at the free surface:

\[
\epsilon(\zeta) = \frac{P(\zeta)}{\rho}
\]  
(5.37)

for the turbulent energy, \( k \), is found:

\[
k(\zeta) = \left( \frac{\epsilon(\zeta) \times z_w}{C_d} \right)^{2/3}
\]  
(5.38)

where \( z_w \) is the roughness height at the free surface. Because of the presence of waves the roughness height is assumed to be: \( 0.5 \times H_{rms} \).

The viscosity at the water surface can now be determined with Eq. 5.9

It is noted that we in fact use the same boundary conditions for both types of boundaries. Rodi (1984) also uses the same boundary conditions but in his case they are related to the shear stress exerted on the free surface, which is common practise in the case of wind induced shear stress. In the presence of waves this boundary condition did not give satisfactory results.
5.5 Summarizing wave effects

For reasons of transparency a summary of how the wave effects are incorporated in TRISULA is given below.

A first wave effect appeared in the depth averaged continuity equation if the wave related terms were included. This term is the so-called wave induced mass flux of which its gradient is included in the continuity equation. Since we performed a time averaging over one wave period $T$ we only take averaged effects of waves into account. The wave term appearing in the continuity equation can be interpreted as the varying mass of the waves. As the wave height decreases when a wave propagates shore wards also the amount of mass in the wave decreases, this mass is transferred from the wave to the underlying water body. The mass flux is influenced both by breaking and non-breaking waves.

A second effect is the shear stress exerted at the water surface as waves break. This shear stress is related to the dissipation in the roller. It is assumed that a roller is travelling over the underlying water body with the wave celerity $c$.

The third term is the effect of the wave dissipation on the turbulence. As a first approximation the turbulent energy production due to waves is proportional to the dissipation in the wave at the water surface. Also the influence of waves on the viscosity near the bottom has to be included.

5.6 The staggered grid

The shallow water equations are solved on a staggered grid. The grid which is used in the 2DV-model is shown in Fig. 5.2. A staggered grid has the advantage that it is possible to locate the various variables at different positions in the grid. In this way averaging procedures, which introduce inaccuracies into the computation, can be avoided. Because averaging procedures are avoided, the number of grid points which are included in the discretisation of the shallow water equations is limited which is leading to a sparse system.

In Appendix C a detailed overview is given of the position of various parameters in the staggered grid.

Figure 5.2 The staggered grid for the 2DV-model
6 Numerical model

6.1 Introduction

In this chapter we describe how the numerical model is constructed based on the physical experiments. In Section 6.2 the physical experiments are elaborated; the beach geometries, incident wave conditions, measurement set-up and measured parameters are highlighted. Next the construction of the numerical model is treated in Section 6.3.

6.2 Physical experiments

6.2.1 Introduction

Within the framework of the European Large Installation Plan (LIP) a programme of detailed measurements of hydrodynamic and sediment transport in the surf zone has been carried out in DELFT HYDRAULICS’ Delta flume. The objective of the study was the generation of high quality and high resolution data on hydrodynamics and sediment transport dynamics on a natural 2DV beach under equilibrium, erosive and accretive conditions.

6.2.2 Beach geometries

The profile is restricted by the Delta flume geometry. The flume has a maximum length of approximately 250 m and depth in front of the wave generator of 9 m. Depending on the maximum wave height, run-up and dune height the working water depth was between 4 and 5 m. Using a water depth of 4.1 m the length of the constructed bottom profile is approximately 183 m. The bottom slope was to be changed gradually from 1/30 for the deep water region \( h > 3.7 \) m to 1/20 for the near shore region \( h < 1.6 \) m. For schematized geometry is referred to Fig. 6.1.

6.2.3 Wave conditions

Narrow banded, random waves were chosen such that the wave steepness at the peak frequency in combination with the water level were expected to result in a stable, erosive and accretive beach. During each of these sub-tests the water level and wave conditions were kept constant. The duration of each sub-test was determined by the number of possible wave hours per day, but was chosen to be at least long enough to obtain accurate sediment transport estimates from profile measurements.

Each sub-test used the resulting profile from the previous test as the initial geometry. In Table 6.1 the incident wave height, wave period and sub-test duration for the different sub-tests are presented.

6.2.4 Measurement set-up

Each sub-test was subdivided into a number of wave hours. During each wave hour, a time series of exactly one hour of waves was generated. Starting when the wave paddle was started until after the last waves had died away, measurements were taken by instruments attached to the flume wall, by instruments attached to a moveable carriage and by three
instruments attached to moveable wave gauges. All moveable instruments remained in a fixed position during each wave hour.

<table>
<thead>
<tr>
<th>Test code</th>
<th>Initial geometry</th>
<th>$H_{w0}$ [m]</th>
<th>$T_r$ [s]</th>
<th>Water level [m]</th>
<th>Duration [h]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A</td>
<td>Dean-type</td>
<td>0.9</td>
<td>5</td>
<td>4.1</td>
<td>12</td>
</tr>
<tr>
<td>1B</td>
<td>result of 1A</td>
<td>1.4</td>
<td>5</td>
<td>4.1</td>
<td>18</td>
</tr>
<tr>
<td>1C</td>
<td>result of 1B</td>
<td>0.6</td>
<td>8</td>
<td>4.1</td>
<td>13</td>
</tr>
<tr>
<td>2A</td>
<td>Dean-type with dune</td>
<td>0.9</td>
<td>5</td>
<td>4.1</td>
<td>12</td>
</tr>
<tr>
<td>2B</td>
<td>result of 2A</td>
<td>1.4</td>
<td>5</td>
<td>4.1</td>
<td>12</td>
</tr>
<tr>
<td>2E</td>
<td>result of 2B</td>
<td>1.4</td>
<td>5</td>
<td>4.6</td>
<td>18</td>
</tr>
<tr>
<td>2C</td>
<td>result of 2E</td>
<td>0.6</td>
<td>8</td>
<td>4.1</td>
<td>21</td>
</tr>
</tbody>
</table>

Table 6.1 Set-up of various sub-tests

6.2.5 Measured parameters

Below we will give an overview of the parameters which were measured and are interesting for us to compare with the results of the model or are used as boundary conditions.

<table>
<thead>
<tr>
<th>Boundary related parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>bottom profile</td>
<td>$z_0$ [m]</td>
</tr>
<tr>
<td>mean water level (set-up)</td>
<td>$\xi$ [m]</td>
</tr>
</tbody>
</table>

Wave related parameters

| spectral first order wave height | $H_{w0}$ [m] |
| spectral peak wave period       | $T_r$ [s] |

Velocity related parameters

| mean velocity in horizontal direction | $<u>$ [m/s] |
| mean velocity in vertical direction  | $<w>$ [m/s] |

Table 6.2 Measured parameters

6.3 Numerical model

6.3.1 Introduction

In this section the numerical model is described. We will construct a model which has the same size as the physical model. First the bottom which has to be prescribed is highlighted followed by an elaboration about the incident wave height, the water level and water depth. Next, the vertical boundaries in the flow model are treated and finally the most important input parameters are treated.
6.3.2 Bottom boundary

The bottom in the physical experiment is changing continuously under wave action. In our model we can not take this changing bottom into account. Our main goal is to model the undertow. Therefore it would be possible to make a computation with the bottom belonging to the time at which the horizontal velocity was measured. In this way we need to make 8 computations per test. This would be to time consuming taking the limited time under consideration. It was therefore decided to use the bottom belonging to the time half way the test of interest.

![Diagram](image)

Figure 6.1 Schematized geometry of LIP experiments in the Delta flume

6.3.3 Water level, water depth and wave height

Each computation started with a horizontal water level and a water depth as indicated in Table 6.1. The prescribed wave height in the wave model was modified so that the measured and computed wave height agreed in deep water.

6.3.4 Vertical boundaries

Both vertical boundaries are closed: the shoreward boundary is closed and cut off at a water depth of approximately 0.1 m since no drying and flooding procedures are included in the flow model.

At the seaward boundary relatively large vertical velocities appear which are due to the mass flux. Since we have constructed a closed boundary no mass can be transported through this boundary which implies that at the boundary the wave induced mass flux is zero, while just inside the model waves are present inducing mass flux. This does not have an influence on the area of interest, but it is only a local effect.

6.3.5 Input parameters

In the computations which were made various input parameters have to be prescribed. In this section we will give the values for these parameters. If they are modified in a computation it will be mentioned explicitly.
Time step

Since we are interested in a steady state situation, the time step is chosen as large as possible. In this way the computation time is reduced. It is mentioned here that the results of the computations do not depend on the time step which was chosen. In all cases a time step of 0.3 s was chosen.

Accuracy criterium

The accuracy criterium determines to what extent the computation should converge. In the flow model this is determined by the water elevation:

$$
\epsilon_{\text{max}} = \frac{\zeta(t+dt) - \zeta(t)}{dt}
$$

(6.1)

Its value is set to $1.0 \cdot 10^{-5}$.

Roughness height

The roughness height is an integration constant which determines at which height above the bed the velocity profile has its origin. It especially has influence on the velocities near the bottom. Based on comparison of computed and measured velocities it has been given the value of 0.0002 m.

Number of layers

The number of layers should be reduced to a minimum without having a negative influence on the results of a computation. This parameter is also tested but for the first computations the number of layers is 20.

Mesh-size

The mesh-size should be chosen so that all the characteristics of the bottom are reproduced correctly. In the computations the mesh-size is 1 m which gives approximately 180 points in the horizontal direction. With 20 layers the number of computational cells is approximately 3600.
7 Comparison with experiments

7.1 Introduction

We have implemented the various equations which were derived in the previous chapters. In this chapter we will compare the results of the computations with those measured in the physical experiment described in Chapter 6. Two types of tests can be distinguished in this chapter. First the different methods to take the wave and roller effects into account are compared. The wave and roller effects are incorporated by: a shear stress at the surface, wave induced turbulence and a wave and a roller induced mass flux. Various expressions for these effects are tested. In the second part of this chapter the 2DV-model is tested with respect to varying mesh-size, varying the number of layers etcetera. The figures in this chapter are enclosed at the end of the chapter.

In Figs. 7.1 and 7.2 the bottom profiles and the mean water level (MWL) are plotted. As mentioned earlier the bottom profiles measured half way the physical experiment of interest are used for the computation.

7.2 Wave models

7.2.1 Introduction

First the wave height predictions computed with the wave energy balance according to Battjes and Janssen are compared with the measured wave height distributions. This is followed by a section in which the various roller models are compared. The three models which we will compare are the roller model according to Roelvink (1993), Okayasu extended (referred to as present) and Stive and De Vriend (1994).

7.2.2 Battjes and Janssen (1978)

The results of the wave height computations are shown in Fig. 7.3 together with the measured values. As can be seen the prediction of the wave height is good although deviations can be observed near the bars around $x = 140 m$. The results of the wave energy balance are used in the roller models which are tested in the following section.

7.2.3 Roller models

First the implemented expressions according to Stive and De Vriend (1994), Roelvink (1993) and Okayasu extended are discussed briefly, followed by a discussion of the results. Below only a summary is given, for the complete derivation the reader is referred to Chapter 4.

Stive and De Vriend (1994)

As described in Section 4.5.2 Stive and De Vriend found a term to take the effect of the roller into account. The implemented expressions are summarized below.
The expression found for the shear stress at the water surface can be written as:

\[ \tau = \frac{D_w}{c} - 1.15 \frac{d(E^* kh)}{dx} \]  

(7.1)

for \( E^* \) we can write:

\[ E^* = Q_b E \]  

(7.2)

\( Q_b \) is the breaking wave fraction based on the Battjes Janssen model (Eq. 4.12) which is empirically modified. The expression for \( Q_b \) is given in Eq. 4.59. \( E \) is the wave energy according to linear wave theory.

For the kinetic energy in the roller we use the expression derived by Svendsen which was found to be:

\[ E_{r,k} = \rho \frac{Ac}{2T} = 0.9 \rho \frac{Q_b H_{rms}^2}{2T} \]  

(7.3)

The mass flux can be written as:

\[ M_x = \frac{0.8E}{c} - \frac{E_{r,p}}{c} + \frac{2E_{r,k}}{c} = \frac{\rho g H_{rms}^2}{10c} + \frac{0.9 \rho g Q_b H_{rms}^2}{T} \]  

(7.4)

where we can write for \( E_{r,p} \):

\[ E_{r,p} = \frac{\rho g A}{4L} H_{rms} \]  

(7.5)

Roelvink (1993)

Roelvink constructed an additional differential equation (roller equation) to take the effect of the roller into account:

\[ \frac{\partial E_c}{\partial x} = D_w - D_r \]  

(7.6)

Inserting the properties of the roller found by Svendsen (Section 4.3.2) transforms the equation above to:

\[ \frac{\partial E_{r,k}}{\partial x} = D_w - 2 \rho g \frac{E_{r,k}}{c} \]  

(7.7)

where \( D_w \) is the dissipation in the wave and:

\[ E_{r,k} = \rho \frac{Ac}{2T} \]  

(7.8)
The shear stress at the water surface can be written as:

\[ \tau = \frac{D_f}{c} \]  

(7.9)

For the mass flux we use:

\[ M_x = \frac{0.8E}{c} \frac{E_{r,p}}{c} + \frac{2E_{r,k}}{c} \]  

(7.10)

\( E_{r,p} \) can be written according to Eq. 7.5

**Present**

The roller equation is the same as the one used by Roelvink. The expression for the kinetic energy of the roller is different:

\[ E_{r,k} = \frac{\pi \rho c^2 H_{rms}^2 \psi^2 (2 + \psi^2)}{16L} \]  

(7.11)

Inserting Eq. 7.11 into the roller equation (Eq. 7.7) gives:

\[ \frac{\partial}{\partial x} \left( \frac{\rho c^2 H_{rms}^2 \psi^2 k}{32} (2 + \psi^2) \right) = D_w - \frac{\beta \rho g c H_{rms}^2 \psi^2}{8} \]  

(7.12)

The shear stress at the water surface can be computed with Eq. 7.9

The mass flux can be computed according to Eq. 7.10 in which the kinetic energy of the roller according to Eq. 7.11 is inserted. For the potential energy we can write:

\[ E_{r,p} = \frac{\rho \pi g \psi^2 H_{rms}^2}{16L} \]  

(7.13)

**Comparison of different roller models**

In Fig. 7.4 the dissipation in the roller, the kinetic energy of the roller and the wave induced mass flux are depicted respectively. The dissipation in the roller according to Stive and De Vriend is rather rocky while the \( D_w \), according to Roelvink and the present method changes gradually. The fact that a roller has to build up before energy can be dissipated by the roller is clear with the two latter methods; the dissipation in the roller is increasing and decreasing slower as the dissipation in the wave. The difference between Roelvink and the present method is the changing sensitivity. In test IA they have a similar sensitivity in the deeper region while in the near shore region the sensitivity of the present method decreases. This can be explained by the fact that the roller according to the present method has more kinetic energy although the mass is approximately equal to the mass in the roller according to Roelvink (see Fig. 7.4). This additional kinetic energy is originating from the internal spinning motion (see also Fig. 4.2 for the internal velocity distribution). This can also be observed as one is at the beach and looks at the incoming breaking waves travelling towards the coast; in the deeper regions the rollers increase relatively faster than they decrease near shore.
The kinetic energy of the roller shows the same development as $D_w$. The wave induced mass flux for the three methods does not show much difference.

The reason for the rocky development of $D_t$ according to Stive and De Vriend is not fully understood. But one of the main reasons is the modified $Q$, which is not accurate enough. It is therefore essential that the expression for the predicted $Q$ is improved. This however will need a better wave model since $Q$ is based on the $Q$ according to Battjes and Janssen which is in fact a breaking wave fraction prediction within an equation which is empirically calibrated. This implies that the $Q$ according to Battjes and Janssen is a parameter which only can be used in the wave energy balance and can not be used to predict the actual breaking wave fraction.

Because of the reasons mentioned above we will not test the method according to Stive and De Vriend for the remaining physical experiments.

In Figs. 7.5 and 7.6 the results for test IB and IC are shown. The development for the various parameters is similar as it was seen for test IA.

In Figs. 7.7 and 7.8 the horizontal velocities at 0.1 m and 0.2 m above the bottom are shown respectively. The results of Stive and De Vriend show large errors. The same rocky development as was observed for the dissipation in the roller is present. In general it can be said that both the present method and Roelvink (1993) overestimate the velocities in the deeper regions ($x < 120 \text{ m}$). In the near shore region ($x > 140 \text{ m}$) the computational results show a rapid change in velocity ($x = 152 \text{ m}$ for test IA, $x = 158 \text{ m}$ for test IB and test IC) which is not present in the measured velocities. This implies that the dissipation in the roller drops and increases again in these areas. If we look at the Figs. 7.4 to 7.6 it can be seen that this is indeed the case. This can only be improved if the dissipation in the wave does not have those rapid changes in that area. This reveals the fundamental problem that the dissipation in the wave in the wave energy balance is in fact a fictive dissipation. To solve this a new improved wave model should be developed in which the actual dissipation in the wave is determined. This however is beyond the scope of this thesis.

In Figs. 7.9, 7.10 and 7.11 the vertical distribution of the horizontal velocities is shown at various locations of the horizontal. The results of Stive and de Vriend for test IA are not shown because in Figs. 7.7 and 7.8 already it is shown that large errors can be observed.

The two other methods give identical results in the deep water region. At the near shore region, however, differences can be observed. For test IA both methods give reasonable results. In test IB a bar is present at $x = 140 \text{ m}$. In Fig. 7.10 can be seen that the present method is performing better in that region. Generally it can be said that for both methods reproduction of the vertical distribution of the horizontal velocities is reasonable but that the present method gives better results, especially in areas where a bar is present.

In order to give an objective judgement on both methods we have calculated the relative and absolute deviations for all the tests. In Fig. 7.12 the relative and absolute deviations for each horizontal plane are shown. It is obvious from Fig. 7.12 that the present method has the best results, only for test IA the performance of both methods is comparable. This is because in that test no bar is present in the bottom profile. In Figs. 7.13 and 7.14 the averaged relative and absolute deviation per vertical are shown respectively. In these figures it can be seen
that Roelvink (1993) gives better results in the shallow area \((x = 160 \text{ m} \text{ for test IA and } x = 170 \text{ m} \text{ for test IB and test IC})\). But generally it can be said that the present method shows better results.

The total averaged relative and absolute deviation together with their standard deviations are presented in the table below.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>rel. dev.</td>
<td>rel. st. dev.</td>
<td>abs. dev.</td>
</tr>
<tr>
<td>IA</td>
<td>0.37 [-]</td>
<td>0.41 [-]</td>
<td>0.024 [m/s]</td>
</tr>
<tr>
<td></td>
<td>1.1 [-]</td>
<td>1.07 [-]</td>
<td>0.075 [m/s]</td>
</tr>
<tr>
<td></td>
<td>rel. dev.</td>
<td>rel. st. dev.</td>
<td>abs. dev.</td>
</tr>
<tr>
<td></td>
<td>0.29 [-]</td>
<td>0.27 [-]</td>
<td>0.035 [m/s]</td>
</tr>
<tr>
<td></td>
<td>0.84 [-]</td>
<td>0.86 [-]</td>
<td>0.028 [m/s]</td>
</tr>
<tr>
<td></td>
<td>0.98 [-]</td>
<td>0.93 [-]</td>
<td>0.037 [m/s]</td>
</tr>
</tbody>
</table>

Table 7.1 Total performance of various models

In Fig. 7.15 the computed and measured set-up are plotted. In all tests it can be seen that the model overestimates the set-up. One of the reasons for this is the fact that the set-down is not predicted. The deviation is however mainly caused by the fact that in the physical experiments water is stored in the profile which was constructed of sand. This will effectively reduce the water elevation in the near shore area. For a better comparison between the measured and computed set-up it is suggested to use physical tests performed in flumes with solid bottom profiles.

To get an impression of how the total flow field looks three vector plots of the velocity are included of test IA, IB and IC respectively, according to the present method (Figs. 7.16, 7.17 and 7.18). To compensate the distorted scale the vertical velocities are scaled so that the ratio between the vertical and horizontal velocities is similar to the ratio between the \(x\)- and \(z\)-axis.
Varying the slope of the wave front

An empirical constant which appears in the roller equation is the slope of the wave front ($\beta$). As stated earlier it was found by Roelvink to be 0.10. We have made three computations in which the $\beta$ was set to 0.10, 0.07 and 0.05 respectively for Test IB. In Fig. 7.19 can be seen that the dissipation in the roller reacts slower to the dissipation in the wave as $\beta$ is decreased. Also can be seen that the amount of kinetic energy in the roller is sensitive for this parameter. The mass flux increases also when $\beta$ is decreased but not to the extent as the kinetic energy of the roller. In Fig. 7.20 the computations with $\beta$ set to 0.10 and 0.07 are compared. It can be seen that the computation in which $\beta$ is set to 0.10 performs better.

7.2.4 Turbulence model

Wall boundaries

In Section 5.4.3 we derived a new type of wall boundary condition for the $k$-$\epsilon$ model which includes the generation of wave induced turbulence. In this section we will test this new type of boundary by comparing it to the classic formulation. It is mentioned that the present method is used in the roller equation for the remaining tests. To see the influence of the new bottom boundary condition we will use test IC because of the fact that in the deeper region almost no dissipation is present at the surface implying that the turbulence generated near the bottom is relative important for the velocity distribution. We will compare both methods at two locations: at $x = 65.0$ m and $x = 138.0$ m. First both types of boundaries are shown below.

The implemented new type of wall boundary:

$$\nu_t = \frac{C_d}{C_d''} \sqrt{k \kappa z_o}$$

(7.14)

in which:

$$k = \sqrt{k_f^2 + k_w^2} = \sqrt{\frac{1}{C_u} \left( \frac{u_w^4}{64} + \frac{C_f^2}{12} \bar{u}_{orb} \right)}$$

(7.15)

and:

$$z_o = \left( \frac{k_w}{k_f} + 1 \right) z_0 \text{ with } k_w \leq 10$$

(7.16)

For $\epsilon$ we use:

$$\epsilon = C_d' \frac{k_f^{1.5}}{k \kappa z_0}$$

(7.17)
In the classical boundary condition we do not take the influence of the orbital motion on the turbulence into account. We can write for \( v_r \):

\[
v_r = \frac{C_u}{C_d} \sqrt{k_f} z_0 \tag{7.18}
\]

in which:

\[
k_f = \frac{1}{\sqrt{C_r}} u_r^2 \tag{7.19}
\]

For \( \varepsilon \) we use the boundary condition given in Eq. 7.17.

As stated above we will compare the results for both types of boundary conditions at two locations. The first location is at \( x = 65.0 \, m \) where there is very little dissipation at the surface. The second location at \( x = 138.0 \, m \) is positioned at the bar. At this location there will be more dissipation at the surface and also the velocities just above the bed will be larger then at the first location.

In Fig. 7.21 the viscosity, the kinetic turbulence energy, the turbulent energy dissipation and the velocity distribution are shown respectively for both locations. At \( x = 65.0 \, m \) the influence of the new boundary condition can clearly be seen. The wave induced turbulence at the bottom is dominating the viscosity distribution over the total water depth. The same can be said for the kinetic turbulent energy. Since we use the same boundary condition for the turbulent energy dissipation it could be expected that there is little difference between both types.

Finally the velocity distribution at both locations is shown. Especially at \( x = 65 \, m \) the influence of the prescribed boundary is significant. With the new boundary conditions the velocity distribution shows a considerable improvement. At \( x = 138 \, m \) the difference between both types of boundary conditions is decreased. This is because of the fact that in this region the influence of the waves on the viscosity distribution is relatively smaller; the viscosity induced by the flow is dominant at this location.

Still large deviations of the horizontal velocities can be observed in test IC (see also Fig. 7.11). One of the reasons for these deviations is the fact that streaming is not included in the present model. This implies that in the case of waves which lose wave energy mainly due to dissipation in the wave boundary layer the velocity distribution can not be represented correctly in the present model. This should be subject of further research. Also the new type of boundary condition proposed in this thesis should be investigated further. Especially the fact that there is no turbulent energy dissipation at the boundary due to orbital motion is dubious.

In Fig. 7.22 the development of the viscosity, kinetic turbulent energy and turbulent energy dissipation at the bottom are depicted respectively. For the viscosity and the kinetic turbulent energy it can be seen that the influence of the waves is dominant at the bottom. For the turbulent energy dissipation it can be seen that when using the classical method it is increasing faster at the near shore region then when the new boundary type is used. This is caused by the fact that there are higher velocities just above the bed when using the classical method in stead of the new method.
Free surface boundary

The free surface boundary in which the wave induced turbulence is included was based on Roelvink and Stive (1989) and Deigaard et al. (1986) as far as the kinetic turbulent energy is concerned (see Sections 5.3.3 and 5.4.3). The influence of the waves on the turbulent energy dissipation was based on Mocke et al. (1993). The latter is tested in this subsection since it is somewhat controversial. The former is not tested since it is generally accepted as a boundary condition in the $k$-$\varepsilon$ model.

The assumed distribution of the turbulent kinetic energy and the turbulent energy dissipation is similar as it is shown in Fig. 5.1. The vertical distribution of the turbulent kinetic energy is written as:

$$P_{t}(z') = P_{t}(z'=0) \left(1 - \frac{2z'}{H_{ms}}\right) \quad (7.20)$$

where $z'$ is a coordinate with its origin in the water surface and positive downwards.

For the turbulent energy dissipation we derived in Section 5.3.3:

$$P_{e}(z') = C_{r} \left(\frac{P_{t}^{2}(z')}{\frac{1}{2} H_{ms}}\right)^{\frac{1}{3}} \quad (7.21)$$

where $C_{r}$ is an empirical constant of order 1.

Two computations are made: one including the term in Eq. 7.23 and one without this term. The computations are made for test IB.

In Fig. 7.23 the vertical velocity distribution is plotted. The velocities of the computation in which the turbulent energy dissipation is excluded show a less curved distribution than the one in which the turbulent energy dissipation is included. Also it can be seen that the former agrees better with measured velocities in the deeper regions while the latter agrees better in the near shore region. In order to decide which computation performs better we have to realize that the developed model will be used to predict sediment transport in the coastal region. This implies that the prediction of the velocities near the bed ($z = 0.1 \, m$ and $z = 0.2 \, m$) are more important than the velocities higher above the bed ($z \geq 0.4 \, m$).

In Fig. 7.24 the viscosity, the kinetic energy, the turbulent energy dissipation and the velocities are shown respectively for $x = 65.0 \, m$ and $x = 138.0 \, m$. As can be seen in the first figure the turbulent energy dissipation influences the viscosity distribution over the total water depth. Also can be seen that the viscosity distribution for the computation in which the turbulent energy dissipation is neglected has its maximum just below the water surface. This seems to be in contradiction with measurements (see e.g. Mocke et al. 1993). Taking the reasons stated above under consideration we can say that the computation in which the turbulent energy dissipation is included performs better than the one in which it was neglected.
7.2.5 Mass flux

In Section 7.2.3 the implemented expressions for the mass flux were shown. In this section two alternative expressions for the mass flux are tested. The first test concerns the potential energy, the second test concerns the correction factor for the wave energy.

In our model we assumed that the mass flux induced by the waves is originating from the potential energy of the waves and the mass flux of the roller is originating from the kinetic energy of the roller. Therefore the potential energy of the waves should be corrected with the potential energy of the roller. We will compare the method in which the potential energy is subtracted from the wave energy with a test in which the correction is not included.

The mass flux in which the correction with the potential energy is included is written as:

\[ M_x = \frac{0.8E}{c} - \frac{E_{sp}}{c} + \frac{2E_{rk}}{c} \]
\[ = \frac{\rho gh_{rms}^2}{10c} - \frac{\rho \pi g \psi^2 H_{rms}^3}{16cL} + \frac{\rho \pi c H_{rms}^2 \psi^2(2 + \psi^2)}{16L} \]  (7.22)

The expression above is compared with a computation in which the correction is excluded:

\[ M_x = \frac{0.8E}{c} + \frac{2E_{rk}}{c} \]
\[ = \frac{\rho gh_{rms}^2}{10c} + \frac{\rho \pi c H_{rms}^2 \psi^2(2 + \psi^2)}{16L} \]  (7.23)

Both expressions are compared for test IB. The results of both computations are shown in Fig. 7.25. The influence of the correction is small in general, but especially at the near shore area \((x \geq 160 \text{ m})\) an improvement can be observed.

A second test is the correction factor for the potential energy of the waves. Okayasu (1989) found that the potential wave energy according to linear wave theory is over-estimating the potential energy in the surf zone with approximately 20%. We will make two computations for test IB in which the correction is included and excluded, respectively.

mass flux with the corrected potential wave energy:

\[ M_x = \frac{0.8E}{c} - \frac{E_{sp}}{c} + \frac{2E_{rk}}{c} \]
\[ = \frac{\rho gh_{rms}^2}{10c} - \frac{\rho \pi g \psi^2 H_{rms}^3}{16cL} + \frac{\rho \pi c H_{rms}^2 \psi^2(2 + \psi^2)}{16L} \]  (7.24)

mass flux without the correction:

\[ M_x = \frac{E}{c} - \frac{E_{sp}}{c} + \frac{2E_{rk}}{c} \]
\[ = \frac{\rho gh_{rms}^2}{8c} - \frac{\rho \pi g \psi^2 H_{rms}^3}{16cL} + \frac{\rho \pi c H_{rms}^2 \psi^2(2 + \psi^2)}{16L} \]  (7.25)
In Fig. 7.26 the result of both computations is shown. The computation with the correction gives better results then the computation without the correction factor. The only difference between both computations is a shift of the velocities. The distribution itself does not change, the same can be said for the previous test.

From both tests the conclusion can be drawn that the correction factor for the potential wave energy and the correction of the wave induced with the potential energy of the roller improve the overall results.

7.3 Flow model

7.3.1 Introduction

In this section we will test some parameters of the 2DV-model. First we will make a computation with 5 in stead of 20 layers to see how the model performs. Furthermore the accuracy criterium is tested.

7.3.2 Number of layers

In computations made so far the number of layers was 20. To see how the model performed we made a computation in which only 5 layers were present. The result is shown in Fig. 7.27. It can be seen that the model performs remarkably well, at the deeper region some deviations can be observed but at the near shore region there is almost no difference.

The number of layers also has to depend on the wave heights present in the model. Since we assume linear distribution at the surface area concerning the turbulent energy and turbulent energy dissipation (see Fig. 5.1) it is important that this area (a half $H_{rms}$ below the water surface) is covered by at least one layer. This can be written formally as:

$$dz_{\text{max}} \leq \frac{1}{2} H_{rms}$$  (7.26)

7.3.3 Accuracy criterium

Until now the accuracy criterium ($\epsilon_{\text{max}}$) was set to $1e-5$. To see if this criterium was strict enough we will make a computation in which $\epsilon_{\text{max}}$ is set to $1e-7$. The result of both computations is shown in Fig. 7.28. As can be seen there is almost no difference between both computations. This means that $\epsilon_{\text{max}} = 1e-5$ is strict enough for the computations.

7.3.4 Varying horizontal mesh-size

A computation has been made in which the horizontal mesh-size was increased to $2 \, m$. The computation was made for Test IC. In Fig. 7.29 the results are compared with the computation in which the mesh-size was $1 \, m$. The results show good resemblance but the computation in which the mesh-size was set to $1 \, m$ performs better. The mesh-size is limited by the fact that the bottom geometry should be represented well. In Test IC a bar of approximately $7 \, m$ is present. With a mesh-size of $2 \, m$ it is covered by 3 points. If the mesh-
size is increased more the bar will not be represented well which has a negative effect on the results of the computation.

7.3.5 Comparison with test series II

In Figs 7.30 to 7.33 the vertical distribution of the horizontal velocities are shown respectively. Comparing the results for the four tests it can be seen that the model gives the best results for the experiments in which the dissipation of wave energy is mainly due to breaking (test IIB and IIE). For the experiments in which the dissipation in the wave boundary layer can not be neglected (Test IIC and to some extent Test IIA) the computational results deviate from the measured results in the deeper regions. In the near shore region the resemblance is reasonable. This can also be seen in the total deviations which are in Table 7.2:

<table>
<thead>
<tr>
<th></th>
<th>Test IIA</th>
<th>Test IIB</th>
<th>Test IIC</th>
<th>Test IIE</th>
</tr>
</thead>
<tbody>
<tr>
<td>rel. dev.</td>
<td>0.58</td>
<td>0.34</td>
<td>1.00</td>
<td>0.44</td>
</tr>
<tr>
<td>rel. st. dev.</td>
<td>0.63</td>
<td>0.34</td>
<td>1.39</td>
<td>0.37</td>
</tr>
<tr>
<td>abs. dev.</td>
<td>0.031 [m/s]</td>
<td>0.036 [m/s]</td>
<td>0.034 [m/s]</td>
<td>0.051 [m/s]</td>
</tr>
<tr>
<td>abs. st. dev.</td>
<td>0.022 [m/s]</td>
<td>0.027 [m/s]</td>
<td>0.025 [m/s]</td>
<td>0.035 [m/s]</td>
</tr>
</tbody>
</table>

Table 7.2 Total performance for test-series II

From this the conclusion can be drawn that the effect of streaming should be included in the model in order to give reliable predictions in the case of waves in which dissipation in the wave boundary can not be neglected.
8 Evaluation and recommendations

8.1 Introduction

In this thesis we have described the development and calibration of a model which predicts
the vertical distribution of the horizontal velocities, the so-called undertow. In the model
three modules can be distinguished. The wave module in which the wave height distribution
is computed and in which also the roller effects are taken into account by an additional roller
equation. The results of the wave module are converted in a conversion module so that the
data of the wave module can be used as input for the 2DV model, which is the third module.
This 2DV model is in fact a 2DV version of TRISULA.

A new model was designed to take the roller effects into account which was tested against
the roller equation constructed by Roelvink (1993). For the incorporation of wave induced
turbulence in the two equation turbulence (k-ε) model also some new boundary conditions
have been developed. In the k-ε model the empirical values are set to the most commonly
used values which are valid for a wide range of flow-types. Another effect of (breaking)
waves is the mass flux which induces an off-shore current. An alternative expression is
suggested for this mass flux.

In Section 8.2 the various ways to take the roller effects into account are discussed. Next,
the influence of the orbital motion on the viscosity distribution is treated. The wave induced
mass flux and the influence of the roller on this flux is evaluated in Section 8.3.

8.2 Wave model

For the prediction of the wave height distribution the wave energy balance according to
Battjes and Janssen (1978) is used. Although this equation gives a good prediction of the
wave heights, the predicted wave dissipation is not very realistic. The wave dissipation is
reacting too fast to bottom changes. The roller equation uses the computed wave dissipation
to determine the dissipation in the roller. Consequently the predicted wave dissipation also
has a negative influence on the resulting dissipation in the roller.

We have adapted the expression for the kinetic energy in the roller according to Okayasu
(1989) and modified this expression with the result that the area of the roller does not have
to be prescribed any more. This expression influences the results positively as could be seen
in the previous chapter. The main difference between the expression for the kinetic energy
according to Okayasu (1993) and Svendsen (1984a) is that the former assumes that the roller
has internal kinetic energy which originates from a spinning motion inside the roller while
the latter assumes that the velocity distribution inside the roller has a constant vertical
velocity distribution. Because of the assumed internal velocity distribution in side the roller
according to Okayasu no shear stresses can be present between the roller and the underlying
wave. It is assumed however that water is transported from the underlying wave to the roller
on the rear side of the roller while water is transported from the front side of the roller to
the underlying wave. This transport of water is assumed to create the shear stress between
the roller and the wave.
In the roller equation also the slope of the wave front appeared (\( \beta \)). This constant was found by Roelvink (1993) to be approximated by 0.10. Some variation however indeed showed that 0.10 was the best value for this constant. We however think that assuming the slope of the wave front to be constant over the complete surf zone is a too coarse assumption. The result of the model can probably be improved if an expression is found in which the slope of the wave front is allowed to vary.

### 8.3 Turbulence model

To incorporate the influence of waves we have implemented new types of boundaries both at the free surface and at the bottom boundary. At the free surface the implemented boundary condition is related to the dissipation in the wave which is assumed to originate from breaking. Furthermore the breaking of waves influences the amount of turbulent kinetic energy and turbulent energy dissipation just below the surface. For this reason we prescribed the amount of both the turbulent kinetic energy and the turbulent energy dissipation.

For the bottom boundary we have developed a new kind of boundary in which the influence of waves is incorporated. The idea behind this new boundary is that there is a not negligible influence of the waves on the turbulent kinetic energy and that the influence of the waves on the turbulent energy can be neglected. With these consideration in mind a new type of boundary is developed which gives viscosity distributions which show agreement with measured profiles. The agreement concerns the shape of the measured and computed viscosity profiles (by comparing the computed viscosity profiles to measured profiles by Mocke et al., 1993). In the physical experiments on which the computations were based no turbulence measurements were made so we can not say to what extent there is quantitative agreement.

### 8.4 Mass flux induced by (breaking) waves

The mass flux is influenced both by the waves and the rollers. In this thesis we derived that the mass flux due to waves can be expressed in terms of the potential energy of the wave according to linear wave theory. The mass flux of the roller can be expressed in terms of the kinetic energy of the roller. The wave energy distribution which was computed with the wave energy balance also includes the potential energy of the roller. Since this energy does not contribute to the mass flux it should be subtracted from the wave energy in order to compute the correct mass flux induced by waves.

In the determination of the mass flux it is assumed that the potential energy is equal to the kinetic energy. This is indeed the case for linear wave theory in which it is assumed that the surface perturbation is small compared to the water depth. In the near shore region this assumption does not hold any more. For this reason the potential energy is corrected and consequently the wave induced mass flux. Still it can be seen from the computations in the previous chapter that the mass flux, especially in the near shore region, is over estimated. This implies that if we want to make a more accurate prediction of the wave induced mass flux this can not be based on linear wave theory. In stead a wave theory should be developed in which higher order terms are included. Various higher order wave models have been developed (e.g. Dean, 1965). A disadvantage however is the fact that this method is not valid
throughout the whole surf zone (the same can be said for all the non linear wave theories which have been developed until now). A solution could be that the correction factor, which has been assumed constant in our case, is varied and depends on the wave height, wave steepness, water depth, etcetera.

8.5 Recommendations

Wave model

The wave model gives reasonable predictions of the wave height distribution. The prediction of the dissipation in the wave is far from realistic. For a better prediction of the dissipation in the roller it therefore essential that the wave model is improved. Maybe improvements can be made by prescribing the maximum dissipation decrease per unit length. This would prevent the large decrease of the dissipation in the wave after bars (see e.g. Fig. 7.6). In the roller equation also some improvements can be made. Especially the empirical factor which denotes the slope of the wave front can be improved. Some expression should be found for this factor since you can expect that as the wave front is travelling to wards the coast it will get steeper just before breaking and less steep after wave breaking.

Turbulence model

The boundary conditions which we have developed in this thesis to take the effects of the waves on the viscosity distribution into account have improved the results considerably. The developed boundary condition however can be improved. We have shown that taking the influence of waves on the turbulent kinetic energy into account together with an increase of the bottom roughness is a method which shows promising results. However more research should be done on this subject.

The effect of streaming is not included in the present model. This can also be seen in the computational results; the model shows good results for cases in which waves are breaking over the total length of the profile (Test IB) while the results of tests in which only breaking occurs in the near shore region (Test IC) is less satisfactory. In order to give the model a more general validity the effect of streaming should be included.

Flow model

In this thesis we have only investigated the influence of waves in a 2Dv situation. For practical application of the model it should be extended into the third dimension. All the formulations derived in this thesis can easily be rewritten to take the third dimension into account. This concerns expressions for the wave model and for the turbulence model. Further research should be conducted on the influence of (breaking) waves on both the undertow in the 3D case and the long shore current.
9 Acknowledgements

First I would like to express my thanks to DELFT HYDRAULICS for giving me the opportunity to make this master’s thesis. Many thanks to all the people that work at DELFT HYDRAULICS who were willing to help me with the problems I encountered.

Many thanks to prof. G.S. Stelling and G. Segal for their professional guidance and useful comments. Furthermore I would like to thank Ad Reniers for all his help with the model and all the inspiring discussions we had.

Besides, I would like to express my gratitude to Dano Roelvink who was my daily supervisor and from whom I learned a lot.

Also many thanks to Martje Loman for caring over us and for all the jokes. Without her it wouldn’t have been the same. Furthermore many thanks to all the Praktikantas Siberias I met during my stay in De Voorst.

Last but certainly not least, thanks to my fellow-worker and friend Maarten Piepers for all his help with the mathematics and the difficult questions because of which my own physical insight is increased.

The flume data were collected during the LIP-11D research Project financed by the European Union within the Large Installation Programme (LIP). The Delta flume belongs to the research laboratory DELFT HYDRAULICS in the Netherlands. Permission to make use of the data was kindly granted by the main investigator, Dr. A. Sánchez-Arcilla (from the Maritime Engineering Laboratory LIM/UPC in Spain) and Dr. J.A. Roelvink (from DELFT HYDRAULICS in the Netherlands). The present research was undertaken as part of the MAST G8 Coastal Morphodynamics research program. It is supported by the Commission of the European Communities, Directorate General for Science, Research and Development, under contract no. MAS2-CT-92-0027.
References

Cramer, C.A.A., 1991. Note on the numerical implementation of the pressure gradient force in TRISULA, DELFT HYDRAULICS.
DELFT HYDRAULICS, 1993. TRISULA, A simulation program for hydrodynamics flows and transports in 2 and 3 dimensions (TRISULA user manual).
Van Kester, J., Stelling, G.S. and Uittenboogaard, R.E., 1989. The $\sigma$-coordinate transformation and the basic equations of TRISULA. Estuaries and Sea Division, DELFT HYDRAULICS.
References (continued)

LIP IID Delta flume experiments
Geometry of Test IA, IB and IC

DELTFT HYDRAULICS
Wave height
Comparison of computed and measured velocities
Dissipation in roller

Energy in roller

Mass flux

--- Roelvink (1993)
--- Present
--- Stive and De Vriend (1994)

Results of wave model
Comparison of various wave models
Dissipation in roller

Energy in roller

Mass flux

Roelvink (1993)  Present

Results of wave model
Comparison of various wave models

DELFT HYDRAULICS
Horizontal velocities
Comparison at 0.1 [m] above bottom

Horizontal velocities
Comparison at 0.2 [m] above bottom

Test IA

Test IB

Test IC

Horizontal Velocity Results
Comparison of computed and measured velocities

DELFT HYDRAULICS
H 1684 FIG. 7.9
Horizontal Velocity Results
Comparison of computed and measured velocities

---

Roelvink (1993)
Present

---

DELFT HYDRAULICS

H 1684  FIG. 7.10
Horizontal Velocity Results
Comparison of computed and measured velocities

Delft Hydraulics

Test IC

H 1684

FIG. 7.11
Rel. Deviation  Test IA  Abs. Deviation

Rel. Deviation  Test IB  Abs. Deviation

Rel. Deviation  Test IC  Abs. Deviation

Roelvink (1993)  Present

Rel. and Abs. deviation per horizontal plane

DELFT HYDRAULICS
Test IA

Test IB

Test IC

△ Roelvink (1993)     * Present

Averaged absolute deviation per vertical

DELFT HYDRAULICS

H 1684  FIG. 7.13
Comparison of measured and computed set-up

DELFT HYDRAULICS
Vector plot of the velocities
Vector plot of the velocities

Bottom profile

MWL

scale 1 cm = 0.5 m/s

DELT HYDRAULICS

TEST IC

H 1684

FIG. 7.18
Dissipation in roller

Energy in roller

Mass flux

Results of wave model
Comparison of various wave models

DELFT HYDRAULICS
Horizontal Velocity Profiles
Comparison of computed and measured velocities

DELFT HYDRAULICS

H 1084

FIG. 7.20

Test IB

position x = 65.0 m

position x = 102.0 m

position x = 115.0 m

position x = 130.0 m

position x = 138.0 m

position x = 145.0 m

position x = 152.0 m

position x = 160.0 m

\[ \beta = 0.07 \]

\[ \beta = 0.10 \]
Turbulence properties

--- Wave induced turbulence included (at the bottom)
------ Without wave induced turbulence (at the bottom)
Turbulence properties at bottom boundary

--- Wave induced turbulence included (at the bottom)
----- Without wave induced turbulence (at the bottom)

DELT HYDRAULICS

Test IC

H 1684  FIG. 7.22
Horizontal Velocity Results
Test of turbulent energy dissipation in free surface boundary

DELFT HYDRAULICS

Test IB

H 1684

FIG. 7.23
Turbulence properties

--- Turbulent energy dissipation included
----- Without turbulent energy dissipation
Horizontal Velocity Results
Comparison of different mass flux expressions
DELFT HYDRAULICS

Comparison of different mass flux expressions

Horizontal Velocity Results

position x = 65.0 m

position x = 102.0 m

position x = 115.0 m

position x = 130.0 m

position x = 138.0 m

position x = 145.0 m

position x = 152.0 m

position x = 160.0 m

position x = 170.0 m

--- Correction for wave induced mass flux included

--- Correction for wave induced mass flux excluded
Horizontal Velocity Results
Comparison of computed and measured velocities

DELFT HYDRAULICS
Horizontal Velocity Results
Comparison of computed and measured velocities
Horizontal Velocity Results
Comparison of computed and measured velocities

DELFT HYDRAULICS

H.1684
FIG. 7.29

Test 1C

\[ \text{dx} = 2 \text{ m} \]
\[ \text{---dx} = 1 \text{ m} \]
Horizontal Velocity Results
Comparison of computed and measured velocities
Horizontal Velocity Results
Comparison of computed and measured velocities
Horizontal Velocity Results
Comparison of computed and measured velocities
Horizontal Velocity Results
Comparison of computed and measured velocities

DELTFT HYDRAULICS
Appendix A

Derivation of the continuity equation, and the inertial terms of the momentum equation
Derivation of the continuity equation, and the inertial terms of the momentum equation

In this Appendix the inertial terms for the 3D case and the continuity equation are derived. Both equations are derived for a control volume which is depicted in Fig. A.1.

The continuity equation and the inertial terms is based on conservation of mass and momentum, respectively.

![Control volume](image)

**Figure A.1 Control volume**

**Continuity equation**

Conservation of mass (in time interval \( dt \)):

\[
\text{mass inflow} - \text{mass outflow} = 0 \tag{A.1}
\]

First the \( x \)-direction:

inflow: \( \rho u dy dz dt \) \( \tag{A.2} \)

outflow: \( \rho u dy dz dt + \left( \rho \frac{\partial u}{\partial x} dy dz dt \right) dx \) \( \tag{A.3} \)

Inflow-outflow gives: \( -\rho \frac{\partial u}{\partial x} dx dy dz dt \) \( \tag{A.4} \)

The same derivation can be made for the \( y \)- and \( z \)-direction, respectively:

\( -\rho \frac{\partial v}{\partial y} dx dy dz dt \) \( \tag{A.5} \)

\( -\rho \frac{\partial w}{\partial z} dx dy dz dt \) \( \tag{A.6} \)

Since the control volume is constant in time \( \frac{\partial (dx dy dz)}{\partial t} = 0 \), the continuity equations is obtained by combining Eqs. A.4, A.5 and A.6:
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]  \hspace{1cm} (A.7)

**Equation of momentum**

The same control volume will be used again. We will derive the momentum equation in the \(x\)-direction, for that purpose we define a momentum density: \(\mu = \rho u\). Now the same routine can be applied again as for the continuity equation.

in \(x\)-direction:

momentum inflow: \(\mu_x u dy dz dt\)  \hspace{1cm} (A.8)

momentum outflow: \(\mu_x u dy dz dt + \frac{\partial (\mu u)}{\partial x} dx dy dz dt\)  \hspace{1cm} (A.9)

Inflow-outflow gives: \(-\frac{\partial (\mu u)}{\partial x} dx dy dz dt\)  \hspace{1cm} (A.10)

The momentum density, \(\mu_x\), through the \(y\)- and \(z\)-planes reads, respectively:

\[-\frac{\partial (\mu u)}{\partial y} dx dy dz dt\]  \hspace{1cm} (A.11)

\[-\frac{\partial (\mu u)}{\partial z} dx dy dz dt\]  \hspace{1cm} (A.12)

The control volume contains an amount of momentum given by: \(\mu_x dx dy dz\). The variation in time per unit of time yields:

\[\frac{\partial \mu_x}{\partial t} dx dy dz dt\]  \hspace{1cm} (A.13)

Combination of Eqs. A.10, A.11, A.12 and A.13 yields:

\[\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} = F\]  \hspace{1cm} (A.14)

where \(F = \) external force.

By applying the chain rule and using the continuity equation Eq. A.14 changes to:

\[\frac{\partial u}{\partial t} + 2u \frac{\partial u}{\partial x} + u \frac{\partial v}{\partial y} + v \frac{\partial u}{\partial y} + u \frac{\partial w}{\partial z} + w \frac{\partial u}{\partial z} = F\]  \hspace{1cm} (A.15)

Inserting the continuity in Eq. A.15 gives the inertial terms in a different form:

\[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = F\]  \hspace{1cm} (A.18)
Appendix B

Shear stress distribution in dissipative waves
Shear stress distribution in dissipative waves

Introduction

In Chapter 2 we derived the shallow water equations and showed how additional terms originate from the Navier-Stokes equations. We distinguished three types of water flows: the averaged flow (e.g. tide, wind driven), wave induced flow (e.g. long shore current, undertow) and turbulent flow. The averaged flow and turbulent flow are incorporated in TRISULA.

In this appendix we will derive relations for the wave induced vertical shear stress distribution. It is then possible to compute the vertical distribution of the time averaged flow by solving the shallow water equations with the wave induced stress as the driving force.

The circulation pattern in the surf zone is of interest, because it is of importance for the on-shore off-shore sediment transport, and thus the development of the coastal profile. It is characteristic for the two-dimensional circulation that the flow directed offshore close to the bed is forming the so called undertow.

The theory outlined in this appendix relies on an imbalance in the vertical distribution of the radiation stress which induces a circulation current.

Review of theoretical development

The radiation stress $S_{ax}$ consists of two parts, one originating from the pressure $S_{ax}^1$ and another originating from the momentum of the orbital motion $S_{ax}^2$ (see e.g. Longuet-Higgins and Stewart, 1964).

![Diagram](image)

Figure B.1 Vertical distribution of radiation shear stress from pressure, $S_{ax}^1$, and momentum, $S_{ax}^2$. 

In linear shallow water waves, the contribution to $S_{xx}$ is located around MWL as sketched in Fig. B.1 and contributes with one third of the total to the radiation stress. The contribution $S_{xx}$ is evenly distributed over the water depth.

After wave-breaking, the wave height decreases in the shoreward direction, creating a gradient in the radiation stress. This will give a slope of the mean water level, defined as the set-up. However, because of the uneven vertical distribution of $S_{xx}$ the gradient in the radiation stress becomes a function of the distance from the bed $z$.

To calculate in detail the consequence of this, a control volume as sketched in Fig. B.2 is considered. The control volume is fixed in time and space.

The shape of the radiation stress gradient is similar to the vertical distribution of $S_{xx}$, shown in Fig. B.1. Because of the set-up, a pressure gradient is introduced, which is uniformly distributed over the depth. In order to obtain a time-averaged force balance on the control volume a resulting shear stress $\tau$ has to be introduced on the bottom of the control volume.

![Control volume](image)

**Figure B.2** Control volume (Eulerian description) on which the momentum equation is applied

The exchange of momentum through the bottom of the control volume of Fig. B.2 is neglected in studies made so far. If this momentum flux is neglected no force balance on the control volume is possible. We will show that taking into account the momentum flux through the bottom of the control volume leads to a shear stress distribution which does not has this inconsistency.

The term \( \bar{uw} \) will be non-zero if there is a phase shift between horizontal and vertical orbital motion. We will show that in the case of non-uniform water waves a phase shift is present between horizontal and vertical orbital motion.

In this appendix all velocities are orbital velocities unless mentioned otherwise. The time-averaging symbols are replaced by an overline for reasons of transparency.
This contribution is evaluated for the two following cases.

(a) Waves with energy dissipation in the boundary layer only (laminar or turbulent).
(b) Waves with energy dissipation due to spilling breakers or broken waves.

It turns out that it is of vital importance to include this $\bar{u}w$ term in order to obtain the correct vertical shear stress distribution. As mentioned in the introduction we will restrict ourselves to considering linear shallow-water wave theory, i.e. hydrostatic pressure distribution.

**Dissipation in the wave boundary layer**

We consider progressive, linear shallow-water waves. The sea bed is horizontal and the only deformation of the waves is assumed to be caused by the energy dissipation in the wave boundary layer. The wave condition is steady, and the only variation in wave height is in the direction of propagation. The derivations made are valid for a turbulent as well as a laminar wave boundary layer.

If the energy dissipation as a first approximation is neglected, the surface elevation $\zeta$ and the horizontal orbital velocity $u_0$ are given by:

\[
\zeta = \frac{H}{2} \cos(kx - \omega t) \quad \text{(B.1)}
\]

\[
u_0 = \frac{Hc}{2d} \cos(kx - \omega t) \quad \text{(B.2)}
\]

in which:

- $H$ = wave height
- $d$ = mean water depth
- $t$ = time
- $x$ = horizontal coordinate
- $\omega$ = wave angular frequency
- $k$ = wave number

The variable $c$ is the wave celerity which for shallow water waves is equal to $\sqrt{gD}$, $g$ being the acceleration of gravity.

The vertical orbital velocity is:

\[
w = \frac{z + d \frac{\partial \zeta}{\partial t}}{d} \quad \text{(B.3)}
\]

where $z$ is the distance from the bed.

**Exchange of energy**

The purpose of the present section is to explain the physical mechanism, which extracts the energy from the outer potential flow and transports it to the wave boundary layer, where it is converted to turbulence and heat.
The time-averaged rate of energy dissipation per unit of the bed, $D$ is:

$$D = \frac{1}{T} \int_0^T u_\tau \mu dt = \rho \bar{u}_0 |u_f|$$

(B.4)

in which:

$t_b$ = the instantaneous bed shear stress

$u_f$ = the shear stress velocity $\sqrt{\tau_b \rho}$

$\rho$ = the density of the fluid

$T$ = the wave period

As shown by Longuet-Higgins (1953) the spatial and temporal gradients in the velocity deficit of the wave boundary layer creates a vertical velocity, which is small compared with the orbital velocities. This additional vertical velocity attains the value $w_m$ outside the wave boundary layer, while it decreases through the boundary layer to be zero at the bed.

$w_m$ can be expressed through the displacement integrated over the entire wave boundary layer, applying the continuity equation:

$$w_m = \frac{1}{c} \int_{-D}^{D} \frac{\partial}{\partial x} (u - u_\tau) dz = \frac{1}{c} \int_{-D}^{D} \frac{\partial}{\partial x} (u - u_\tau) dz$$

(B.5)

in which:

$u_\tau$ = the free stream orbital velocity

$u$ = the horizontal flow velocity in the boundary layer

$\delta$ = the boundary layer thickness

In Eq. B.5 it has been assumed that $u$ is equal to $u_\tau$ at the upper boundary, and the relation between spatial and temporal derivatives has been introduced:

$$\frac{\partial}{\partial x} = \frac{1}{c} \frac{\partial}{\partial t}$$

(B.6)

This is an approximation because the wave height decreases in the x-direction. The approximation Eq. B.6 is therefore only valid for weakly decreasing waves.

The linearized flow equation in the boundary layer reads:

$$\rho \frac{\partial u'}{\partial t} = \frac{\partial \tau}{\partial z} - \frac{\partial p}{\partial x}$$

(B.7)

in which:

$u'$ = $u - u_\tau$

$\tau$ = the shear stress

$p$ = the pressure
Integrating Eq. B.7 over the boundary layer thickness, where the pressure is constant, and inserting Eq. B.5 gives:

\[
 w_n = \frac{\tau_b}{c_p} = -\frac{u_n |u_n|}{c} \tag{B.8}
\]

The wave boundary layer thus acts as a small periodic disturbance which travels along the bed with the phase velocity \( c \) of the wave motion. Such a travelling disturbance will generate a surface wave with increasing wave height in the direction of the propagation. In a later section it is shown how the developing wave causes a reduction of the total wave height in the \( x \)-direction, when it is combined with the basic wave described by Eqs. B.1 and B.2.

In the following calculations we take the vertical velocity generated by the bottom perturbation to be constant to \( w_n \) over the entire depth outside the boundary layer, so the horizontal velocities become equal to zero. This choice can be justified by the following reasons:

Apart from the non-uniform growing wave any small wave can be superposed on the system without interacting with the wave motion at the order under consideration. The wave with \( w = w_n \) is, however, the wave with the minimum energy loss in the wave boundary layer, because no horizontal velocities are generated. Hence, this wave will dominate any other dissipative wave motion after a distance.

It should be noted that the calculations and conclusions made in the following are not affected by superposition of a small surface wave, which then would modify the vertical velocity and cause the horizontal velocity to differ from zero.

\( w_n \) is associated with an additional water surface elevation of:

\[
 \zeta^* = \int w_n \, dt = -\int \frac{1}{c} u_n |u_n| \, dt \tag{B.9}
\]

Longuet-Higgins (1953) used the displacement induced vertical velocity to explain the phenomenon of streaming. It is interesting to note that the perturbation \( \zeta^* \) of the water is not in phase with the horizontal orbital velocity \( u_0 \) (Eq. B.2).

\( \zeta^* \) gives rise to an additional horizontal pressure gradient \( \partial \zeta^* / \partial x \), which for linear shallow water waves is given by:

\[
 \frac{\partial \zeta^*}{\partial x} = \rho g \frac{\partial \zeta^*}{\partial t} = -\frac{\rho g}{c} \frac{\partial u_n}{\partial t} / |u_n| \tag{B.10}
\]

This pressure gradient carries out a resulting work on the horizontal orbital velocity of the wave motion. The time-averaged rate of work per unit bed area is given by:

\[
 W = -\frac{1}{T_0} \int \frac{\partial \zeta^*}{\partial x} u_0 \, dt = -\frac{\rho g |u_n| u_0}{c^2} = -\rho u_n |u_n| u_0 \tag{B.11}
\]
which is of exact the same magnitude as the dissipation in the wave boundary layer. The energy dissipated in the wave boundary layer is thus extracted from the outer wave motion through the work done against the pressure gradient associated with the additional surface elevation, caused by the displacement in the wave boundary layer.

The extraction of energy given by Eq. B.11 is evenly distributed over the depth. As the dissipation takes place in the bottom boundary layer, energy must be transferred from the outerflow to the boundary layer. The downward energy transport is performed by the work done by the hydrostatic pressure on the vertical flow velocity. This time-averaged rate of work is calculated as:

$$ W = -\rho g (\zeta + \zeta') w = \rho g (\zeta + \zeta') \frac{z + d \frac{\partial \zeta}{\partial t} + \omega}{d} $$  \hspace{1cm} (B.12)

Because the waves are periodic $\zeta \frac{\partial \zeta}{\partial t}$ and $w, \zeta'$ are zero, so Eq. B.12 becomes:

$$ W = \rho g \left( \zeta w_\infty + \zeta' z + \frac{D \frac{\partial \zeta}{\partial t}}{D} \right) $$  \hspace{1cm} (B.13)

Substituting Eqs. B.8 and B.9 in Eq. B.13 gives:

$$ W = \rho g \frac{z + d \frac{\partial \zeta}{\partial t}}{c} - \rho g \frac{z + d}{c} \left[ \zeta \frac{\partial}{\partial t} \left( \int \frac{u_j u_j}{c} \, dt \right) - \zeta' \frac{u_j u_j}{c} \right] $$  \hspace{1cm} (B.14)

With

$$ \frac{\partial}{\partial t} \left( \int \frac{u_j u_j}{c} \, dt \right) = \frac{\partial}{\partial t} \zeta (-\zeta) = 0, $$

because of the fact that the waves are assumed periodic, Eq. B.14 reduces to:

$$ W = -\rho g \frac{z + d \frac{\partial \zeta}{\partial t}}{c} - \rho u_j u_j \frac{\partial \zeta}{\partial x} $$  \hspace{1cm} (B.15)

So $W$ is at the bed equal to the work found in Eq. B.11, this implies that the energy extracted from the wave motion is transferred to the wave boundary layer.

**Vertical shear stress distribution**

The time averaged vertical distribution of the shear stress associated with the dissipation of energy in the wave boundary layer is considered. Due to the decay of the wave height in the $x$-direction a slope of the mean water surface, the set-up, is usually found:

$$ S = \frac{\partial \zeta}{\partial x} $$  \hspace{1cm} (B.16)
where $\zeta_f$ is the total water surface elevation:

$$\zeta_f = \zeta + \zeta^* + S_f x$$  \hspace{1cm} (B.17)

In the following no specific value of $S$ is given, but it is assumed to be of second order in terms of the wave height:

$$O\left[ \left( \frac{H}{d} \right)^2 \right]$$

The shear stress distribution is determined from the set-up, the pressure and the momentum of the wave motion including the contributions from the displacement in the wave boundary layer.

The water motion in dissipative waves

First the non-uniform wave field with dissipation in the wave boundary layer is analyzed. The dissipation is assumed to be weak, and the wave field is calculated by perturbation analysis. The effect of the pressure gradient from the displacement in the wave boundary layer is thus first calculated from the unperturbed wave condition (Eqs. B.1, B.2 and B.3) and is then inserted into the flow equation in order to obtain a solution for the dissipative waves.

The streaming-induced pressure gradient given by Eq. B.10 is in the general non-laminar case not varying harmonically. As it is periodic, however, it can be expressed as a Fourier series:

$$\frac{\partial p^*}{\partial x} = \sum_{n=1}^{\infty} \left( a_n \cos[n(kx - \omega t)] + b_n \sin[n(kx - \omega t)] \right)$$  \hspace{1cm} (B.18)

Of all the Fourier components it is only the first ($-a_i$) which can interact with and extract energy from the sinusoidal wave orbital motion (Eq. B.2). Only this component is therefore considered in the following, $a_i$ is found as:

$$a_i = \frac{2}{T} \int_0^T \frac{\partial p^*}{\partial x} \cos(kx - \omega t) dt$$  \hspace{1cm} (B.19)

with Eqs. B.2 and B.10 this becomes:

$$a_i = \frac{2}{T} \int_0^T \rho g \frac{u_j |u_j|}{c^2 \frac{H}{C_U}} \frac{u_0}{Hc/(2d)} dt = \frac{4 \rho g d}{Hc^3} \frac{u_j |u_j| u_0}{u_j |u_j| u_0}$$  \hspace{1cm} (B.20)

Substitution of Eqs. B.2 and B.20 in Eq. B.18 gives a pressure gradient of:

$$\frac{\partial p^*}{\partial x} = a_i \cos(kx - \omega t) = \frac{8 \rho g d^2}{H^2 C_U^4} \frac{u_j |u_j| u_0}{|u_j| u_0} = \epsilon \rho \omega u_0$$  \hspace{1cm} (B.21)
where the dimensionless parameter $\epsilon$ is inverse proportional to $H$ and assumed to be a small quantity:

$$\epsilon = \frac{8pgd^2}{\omega c^4 H^2 u/u_0} < 1$$  \hspace{1cm} (B.22)

Introducing the pressure gradient Eq. B.21 in the flow equation gives:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p_1}{\partial x} = -g \frac{\partial \zeta}{\partial x} - \epsilon \omega u_0$$  \hspace{1cm} (B.23)

where $p_1 = p + p^*$ and in which $u$ is the velocity including the effect of dissipation, while $u_0$ given by Eq. B.2 is the basic solution, neglecting the dissipation. The difference between $u$ and $u_0$ is small and proportional to $\epsilon$: $u = u_0 + O(\epsilon)$, because $u$ approaches $u_0$ for $\epsilon \to 0$. Hence, by neglecting higher-order terms in $\epsilon$ in Eq. B.23, $u_0$ can be replaced by $u$.

The continuity equation reads:

$$\frac{\partial u}{\partial x} = -\frac{1}{d} \frac{\partial \zeta}{\partial t}$$  \hspace{1cm} (B.24)

Eqs. B.23 and B.24, with $u_0 = u$, give the following equation for $\zeta$:

$$gd^2 \frac{\partial \zeta}{\partial x^2} = \frac{\partial \zeta}{\partial t^2} - \epsilon \omega d \frac{\partial u}{\partial x} - u_0 d \frac{\partial \epsilon}{\partial x}$$  \hspace{1cm} (B.25)

or:

$$gd^2 \frac{\partial \zeta}{\partial x^2} = \frac{\partial \zeta}{\partial t^2} - \epsilon \omega d \frac{\partial \zeta}{\partial t} - u_0 d \frac{\partial \epsilon}{\partial x}$$  \hspace{1cm} (B.26)

The last term in the right-hand side of Eq. B.26 is small: $\frac{\partial \epsilon}{\partial x} \sim \epsilon^2$ because the gradient in the $x$-direction is due to dissipation, so $\frac{\partial \epsilon}{\partial x} \approx -\epsilon$. Hence, for a weak dissipation, the term $u_0 d \frac{\partial \epsilon}{\partial x}$ in Eq. B.26 can be cancelled.

Hereby Eq. B.26 attains the form used to describe propagation of tidal wave friction (e.g. Ippen 1966). The solution is:

$$\zeta = \frac{H}{2} \cos(kx - \omega t)$$  \hspace{1cm} (B.27)

$$u = \frac{Hc}{2d} \cos(kx - \omega t - \frac{\epsilon}{2})$$  \hspace{1cm} (B.28)

where the wave height is given by:

$$H = H_0 \exp(-\frac{1}{2}kex)$$  \hspace{1cm} (B.29)
$H_0$ is the wave height at $x = 0$. In the derivation of Eqs. B.27, B.28 and B.29 all terms of order $\epsilon^2$ and higher are neglected.

The vertical flow velocity is found from the continuity equation:

$$w = \frac{z+\partial}{\partial t} \zeta + w_\nu$$  \hspace{1cm} (B.30)

The perturbed solution has two important features:

(a) The wave amplitude is damped in the direction of propagation; it is easily shown that the decay gives a gradient in the energy flux corresponding to the energy dissipation (Eq. B.4).

(b) A phase shift has been introduced between the surface elevation and the horizontal orbital velocity. This means that the horizontal and vertical velocity are not completely out of phase, which is important for determining the shear stress distribution.

**The shear stress distribution**

The shear stress is determined by Eulerian considerations, i.e. by considering the momentum balance for a control volume fixed in space. The shear stress as a function of $z$ is found from the horizontal projection of the momentum equation for the control volume shown in Fig. B.2.

The momentum equation reads:

$$\frac{dB}{dt} = P + F$$  \hspace{1cm} (B.31)

where the left-hand side represents the change in the horizontal momentum including the momentum flux through the boundaries of the control volume, $P$ is the horizontal pressure force and $F$ is the shear force acting on the horizontal bottom of the control body. All quantities are averaged over a wave period.

$$\frac{dB}{dt} = \epsilon \int_A \rho \vec{v} \cdot (\vec{v} \cdot d\vec{A})$$

$$= -\rho uu(\zeta_1 - z) + \rho \left( u + \frac{\partial u}{\partial x} \right)^2 \left( \zeta_1 + \frac{\partial \zeta_1}{\partial x} dx - z \right) - \bar{p} \bar{u} \bar{w} dx$$  \hspace{1cm} (B.32)

in which:

$\epsilon$ = unit vector in the in the normal direction

$\vec{v}$ = velocity vector

$d\vec{A}$ = infinitesimal area of the vector of the surface area $A$ of the control volume

In the right-hand side of Eq. B.32 the first term represents the momentum inflow ($\rightarrow 1$ in Fig. B.3), the second term the momentum outflow ($\rightarrow 2$ in Fig. B.3) and the third term the momentum exchange through the bottom of the control volume due to the organized wave motion ($\rightarrow 3$ in Fig. B.3).
Linearizing Eq. B.32 by maintaining the second order in $H/D$ with:

$$u - O\left[\frac{H}{d}\right], \frac{\partial u}{\partial x} - O\left[\frac{H}{d}\right], \zeta_1 - O\left[\frac{H^2}{d}\right], \frac{\partial \zeta_1}{\partial x} - O\left[\frac{H^2}{d}\right]$$

and neglecting terms containing $(dx)^2$ Eq. B.32 becomes:

$$\frac{\overline{dB}}{dt} = -\rho \left[ 2u - \overline{zdx} + \overline{w dx} \right]$$

(B.34)

The pressure term is found from the hydrostatic forces:

$$\overline{P} = \frac{1}{2} \rho g \left[ (\zeta_1 - z)^2 - \left( \zeta_1 + \frac{\partial \zeta_1}{\partial x} dx - z \right)^2 \right]$$

(B.35)

Linearization, using that $S$ is of order $(H/D)^2$ and again neglecting terms containing $(dx)^2$ gives:

$$\overline{P} = -\rho g \left[ \zeta_1 \frac{\partial \zeta_1}{\partial x} dx - z \frac{\partial \zeta_1}{\partial x} dx \right]$$

(B.36)

$S$, and $\zeta'$ are small compared to $\zeta_1$.

So with $\zeta_1 = \zeta$ and $\frac{\partial \zeta_1}{\partial x} = \frac{\partial \zeta}{\partial x}$ and applying Eq. 16 Eq. B.36 becomes:

$$\overline{P} = -\rho g \frac{\partial \zeta}{\partial x} dx + \rho g S dx$$

(B.37)

The shear force at the bottom of the control body is given by:

$$\overline{P} = -\tau dx$$

(B.38)

where $\tau$ is the shear stress due to viscous effects or turbulent Reynolds stress.
Substituting Eqs. B.34, B.37 and B.38 in Eq. B.31 and dividing by \( \Delta x \) gives the reduced momentum equation:

\[
\frac{z}{\rho} = 2u \frac{\partial u}{\partial x} - g \frac{\partial \zeta}{\partial x} + S_\mu + \bar{ww}
\]  

(B.39)

In the following calculations all quantities are taken at \( x = 0 \). With Eqs. B.8, B.24, B.25 and B.26 at \( x = 0 \):

\[
w = \frac{\tau_b}{c_\rho} = -\frac{u_p |u_p|}{c}
\]

\[
u = \frac{H_c}{2d} \cos(-\omega t - \frac{1}{2} \varepsilon)
\]

\[
H = H_0
\]

\[
w = \frac{z+d}{d} \frac{\partial \zeta}{\partial t} + w_\omega
\]

\( \bar{ww} \) can be found:

\[
\bar{ww} = \frac{1}{T} \int_{t}^{t+T} \frac{H_c}{2d} \cos(-\omega t - \frac{1}{2} \varepsilon) \left( \frac{z+d}{d} \frac{\partial \zeta}{\partial t} + w_\omega \right) dt
\]  

(B.40)

Eq. B.40 can be divided into two term. The two terms between brackets are treated separately in the following. The first term is:

\[
\frac{1}{T} \int_{t}^{t+T} \frac{H_c}{2d} \cos(-\omega t - \frac{1}{2} \varepsilon) \frac{\partial \zeta}{\partial t} dt
\]  

(B.41)

using \( \frac{\partial \zeta}{\partial t} = \frac{\omega H_0}{2} \sin(-\omega t) \) Eq. B.41 can be rewritten as:

\[
\frac{1}{T} \int_{t}^{t+T} \frac{H_c \omega (z+d)}{2d^2} \cos(-\omega t - \frac{1}{2} \varepsilon) \sin(-\omega t) dt
\]

\[
= \frac{1}{2T} \int_{t}^{t+T} \frac{H_c \omega (z+d)}{4d^2} \left[ \sin(-2\omega t - \frac{1}{2} \varepsilon) + \sin(\frac{1}{2} \varepsilon) \right] dt
\]

(B.42)

with \( \varepsilon \ll 1 \) it is assumed that \( \sin(\frac{1}{2} \varepsilon) \approx \frac{1}{2} \varepsilon \), this gives:

\[
\frac{1}{2T} \frac{H_c^2 \omega (z+d)}{4d^2} \frac{1}{2} \varepsilon T = \frac{H_c^2 \varepsilon \omega (z+d)}{16d^2}
\]

(B.43)

The second term of Eq. B.40 is:

\[
\frac{1}{T} \int_{t}^{t+T} \frac{H_c}{2d} \cos(-\omega t - \frac{1}{2} \varepsilon) w_\omega dt
\]

(B.44)

With:

\[
w_\omega = -\frac{u_p |u_p|}{c} = -\frac{c \partial^+}{\rho g} = -\frac{c e \omega}{g} = -\frac{H_c \varepsilon \omega}{2gd} \cos(-\omega t - \frac{1}{2} \varepsilon)
\]

(B.45)
Eq. B.44 becomes:

\[-\frac{1}{T} \frac{H_0^2 c^2 \epsilon \omega}{4gd^2} \int_{t}^{t+T} \cos^2(-\omega t - \frac{1}{2} \epsilon) dt\]

\[= -\frac{1}{T} \frac{H_0^2 c^2 \epsilon \omega}{4gd^2} \int \left( \frac{1}{2} + \frac{1}{2} \cos(-2\omega t - \epsilon) \right) dt\]

\[= \frac{H_0^2 c^2 \epsilon k e}{8d}\]  

So with Eqs. B.43 and B.46 Eq. B.40 can be written as:

\[
\overline{u_w} = \frac{H_0^2 e^2 k e \epsilon}{16d} \left( \frac{z}{d} - 1 \right) \]  

(B.47)

Using \(\frac{dH}{dx} = -\frac{1}{2} k e H_0\) and \(c = \sqrt{gd}\) Eq. B.47 can be rewritten as:

\[
\overline{u_w} = \frac{H_0 g}{8 \frac{dH}{dx}} \left( 1 - \frac{z}{d} \right) \]  

(B.48)

The momentum equation (Eq. B.39) now becomes:

\[
\frac{\tau}{\rho} = 2u \frac{\partial u}{\partial x} - g \frac{\partial z}{\partial x} + s g z + \frac{H_0 g}{8} \frac{dH}{dx} \left( 1 - \frac{z}{d} \right) \]  

(B.49)

For the first term in Eq. B.49 the following derivation can be made:

\[
\overline{u \frac{\partial u}{\partial x}} = \frac{1}{T} \int \overline{u(t) \frac{\partial u}{\partial x}} dt = \frac{H_0^2 c^2}{4d^2} \int \left( k \cos(kx - \omega t - \frac{1}{2} \epsilon) \sin(kx - \omega t - \frac{1}{2} \epsilon) \right) dt
\]

\[= \frac{H_0 \epsilon^2}{4d^2} \frac{dH}{dx} \left( \frac{\cos(kx - \omega - \frac{1}{2} \epsilon)}{d} \right) \]  

(B.50)

the same can be done for the second term in Eq. B.49:

\[
\overline{z \frac{\partial z}{\partial x}} = \frac{1}{T} \int \overline{z(t) \frac{\partial z}{\partial x}} dt = \frac{H_0^2 c^2}{4d^2} \int \left( -H k \cos(kx - \omega t) \sin(kx - \omega t) + \frac{dH}{dx} \cos^2(kx - \omega t) \right) dt
\]

\[= \frac{1}{8} \frac{H_0 \frac{dH}{dx}}{d} \]  

(B.51)

Substituting Eqs. B.50 and B.51 into Eq. B.40 yields for the momentum equation:

\[
\frac{\tau}{\rho} = \frac{1}{4} \frac{g}{d} \frac{dH}{dx} \left( 1 + \frac{1}{8} \frac{H_0 g}{d} \frac{dH}{dx} \left( 1 - \frac{z}{d} \right) \right) + S g z + \frac{1}{8} \frac{H_0 g}{d} \frac{dH}{dx} \left( 1 - \frac{z}{d} \right)
\]

(B.52)
In the intermediate expression of Eq. B.53 the first two terms represent the contributions to the shear stress from the momentum and pressure of the radiation stress gradient. The last term represents the momentum flux through the bottom of the control volume. It is seen that without this term $\tau$ will be different from zero at the water surface. The wave set-up is normally estimated as the mean water surface slope, required to balance the radiation stress gradient. This is because this set-up gives zero mean bed stress, and is therefore assumed to create only a very weak mean flow. From Eq. B.52 it is seen that the shear stress in fact becomes zero over the entire water depth (outside the wave boundary layer) for a wave set-up of:

$$S = \frac{-H}{8} \frac{dH}{dx}$$  \hspace{1cm} (B.53)

which is only one third of the equilibrium set-up commonly attributed to waves with dissipation in the wave boundary layer. It is thus possible to obtain a perfect balance between the radiation stress gradient and a wave set-up without any shear stresses, except in the wave boundary layer.

### Spilling breakers and broken waves

#### The exchange of energy

The present section treats spilling breakers and the inner part of the surf zone. The plunging breakers have been transformed to bore-like waves, denoted broken waves.

In spilling and broken waves the dissipation of energy mainly occurs in the region in and near the surface roller, and in the following the small energy loss in the wave boundary layer is neglected. We will use a description of the spilling breaker and broken waves where the surface rollers are represented by a mass of water which is moving forward with the wave with the celerity $c$. The volume of the surface rollers in the model is taken $0.9H^2$, based on experiments made by Duncan (1981) with waves generated by a towed hydrofoil.

In the following a simple dynamic model is introduced in order to describe the extraction of energy from the wave motion to the region of energy dissipation near the surface. The consideration are based on a mass of water -the surface roller- which is stationary relative to the wave crest.

The wave is sketched in Fig. B.4. The surface elevation without the roller is called $\xi$, while the local thickness of the roller is called $\xi^*$. The water in a surface roller is moving with the wave front without any large vertical accelerations. It is therefore assumed that the pressure in the roller is hydrostatic.
Figure B.4 Definition sketch and forces acting on a surface roller

Assuming hydrostatic pressure in the roller, it is seen that a pressure $p^*$ and a shear stress $\tau_s$ are introduced at the interface between the roller and the water below, which take part in the wave motion. The forces acting on a column of water are shown in Fig. B.5.

The vertical and horizontal force balance gives:

$$p^* = \zeta^* \rho g$$  \hspace{1cm} (B.54)

and:

$$\tau_s \, dx = -p^* \frac{\partial \zeta^*}{\partial x} + \frac{1}{2} \rho g \zeta^* \zeta^* - \frac{1}{2} \rho g \left( \zeta^* + \frac{\partial \zeta^*}{\partial x} \right)^2$$ \hspace{1cm} (B.55)

neglecting terms containing $(dx)^2$ and dividing by $dx$ gives:

$$\tau_s = -\rho g \zeta^* \left( \frac{\partial \zeta^*}{\partial x} + \frac{\partial \zeta}{\partial x} \right)$$ \hspace{1cm} (B.56)

The knowledge on the air content of the water is scarce. It is reasonable to assume that the average density of the roller is close to the density of water e.g. with an air concentration of less than 10-20%. In the following the density of the roller is taken equal to the density of the fluid. None of the main conclusions concerning the shear stress distribution or the energy exchange is affected if a reduced density is assumed.

Analogous to the additional surface elevation caused by the wave boundary layer, $\zeta^*$ creates an additional pressure gradient in the fluid, taking part in the orbital motion below the roller:

$$\frac{\partial p^*}{\partial x} = \rho g \frac{\partial \zeta^*}{\partial x}$$ \hspace{1cm} (B.57)
This pressure gradient performs a work on the fluid, acting on the orbital motion. The time averaged rate of work is given by:

\[ W_1 = \frac{d}{T_t} \int_T^t -u \frac{\partial p^*}{\partial x} dt \]  

(B.58)

With \( dt = \frac{1}{c} \, dx \) and if is assumed that \( cT = L \) Eq. B.63 can be written as:

\[ W_1 = -\frac{d}{Tc} \int_0^{L_x} u \frac{\partial p^*}{\partial x} \, dx \]  

(B.59)

Substituting Eq. B.57 and using Eqs. B.1 and B.2 Eq. B.59 gives:

\[ W_1 = -\frac{d}{Tc} \int_0^{L_x} \frac{\zeta c}{d} \rho g \frac{\partial \zeta^*}{\partial x} \, dx \]

\[ = -\frac{\rho g c}{L} \left( \int_0^{L_x} \zeta^* \frac{\partial \zeta^*}{\partial x} \, dx \right) = \frac{\rho g c}{L} \int_0^{L_x} \zeta^* \frac{\partial \zeta}{\partial x} \, dx \]  

(B.60)

in which \( L_R \) is a certain length scale of the surface roller.

Besides the pressure also the shear stress \( \tau_s \) performs a work on the water outside the roller. If we -as an idealization- consider the roller as a solid body a sketched in Fig. B.5, a boundary layer will be formed below this roller.

The roller is moving with the velocity \( c \) which also must be the flow velocity at the top of the boundary layer.

Therefore the work done on the fluid outside the roller by the shear stress \( \tau_s \) is given by:

\[ W_\tau = -\tau_s c \]  

(B.61)

Figure B.5 Forces acting on vertical column of the roller
The work given by Eq. B.61 is not applied to extract or add energy directly from or to the wave motion, but simply represents energy dissipated in the boundary layer between the roller and the surrounding fluid. In the following it will be shown, how the energy is transported to the surface by the wave motion. By application of the same argument as behind Eq. B.4, it turns out that the dissipation $D$ is given by:

$$ D = \frac{ct_s}{\sqrt{\pi}} $$  \hspace{1cm} (B.62)

Hereby, it is concluded that energy is extracted from the wave motion by the additional pressure gradient in the water with ordinary orbital motion, caused by the surface roller above the wave.

This extraction of energy is evenly distributed over the depth. The dissipation of energy takes place in the intense shear zone of the roller (the dissipation in the bottom boundary layer is neglected), so energy must be transferred up to the roller from the wave motion by the work done by the hydrostatic pressure against the vertical flow velocities.

This work is given by:

$$ W = \overline{wp} $$  \hspace{1cm} (B.63)

where:

$$ w = \frac{z+\delta \zeta}{d \partial t} $$  \hspace{1cm} (B.64)

and:

$$ p = (\zeta + \zeta^*) \rho g $$  \hspace{1cm} (B.65)

Introducing Eqs. B.64 and B.65 into Eq. B.63 gives:

$$ W = \overline{wp} = \frac{z+\delta \zeta}{d \rho g (\zeta + \zeta^*) \frac{\partial \zeta}{\partial t}} $$  \hspace{1cm} (B.66)

Since we are interested in the work performed by the surface roller $\frac{\partial \zeta}{\partial x}$ is omitted, also using $\frac{\partial}{\partial t} = -c \frac{\partial}{\partial x}$ Eq. B.66 becomes:

$$ W = -\frac{z+\delta \zeta}{d \rho g c \zeta^* \frac{\partial \zeta}{\partial x}} = -\frac{z+\delta \zeta}{d \rho g} c \int_{L_0}^{L_s} \zeta^* \frac{\partial \zeta}{\partial x} \, dx $$  \hspace{1cm} (B.67)

which at the surface agrees with Eq. B.60.
Vertical shear stress distribution

In order to calculate the shear stress distribution, it is as in the case of energy dissipation in the wave boundary layer necessary to know the $\bar{U}W$-term originating from the organized orbital motion. The principle is again to determine the phase between $u$ and $\zeta$ from the equation of continuity, Eq. B.24.

The surface elevation of the wave motion is assumed to be periodic in time and is described by a shape function:

$$\zeta = \frac{H(x)}{2} f(kx - \omega t)$$  \hspace{1cm} (B.68)

where $H$ is the local wave height. $f(kx - \omega t)$ can be described by a Fourier series:

$$f(kx - \omega t) = \sum_{n=1}^{\infty} [a_n \cos(n(kx - \omega t)) + b_n \sin(n(kx - \omega t))]$$  \hspace{1cm} (B.69)

The vertical shear stress distribution is determined by the momentum equation Eq. B.31, applied on the control volume shown in Fig. B.7.

The energy loss in the wave implies a gradient in the wave height, which locally can be approximated as:

$$H(x) = H_0 + \frac{dH}{dx} x$$  \hspace{1cm} (B.70)

where $H_0$ is the wave height at $x = 0$ and $\frac{dH}{dx}$ the wave gradient at $x = 0$. The gradient in wave height is assumed to be weak, which is reasonable in the inner zone. Now the orbital velocity is determined from Eqs. B.24, B.68 and B.69 to be:
\[ \frac{\partial u}{\partial x} = -\frac{1}{d} \frac{\partial \zeta}{\partial t} - \frac{H(x) \zeta}{2d} \sum_{n=1}^{\infty} \left[ a_n \eta \sin[n(\xi - \omega t)] - b_n \eta \cos[n(\xi - \omega t)] \right] \]

\[ = \frac{-c}{2d} \left( H_0 + \frac{dH}{dx} \right) \sum_{n=1}^{\infty} \left[ a_n \eta \sin[n(\xi - \omega t)] - b_n \eta \cos[n(\xi - \omega t)] \right] \]

\[ = -\frac{H_0 c}{2d} \sum_{n=1}^{\infty} a_n \left( \eta \sin[n(\xi - \omega t)] + \frac{1}{H_0} \frac{dH}{dx} \eta \sin[n(\xi - \omega t)] \right) \]

\[ + \frac{H_0 c}{2d} \sum_{n=1}^{\infty} b_n \left( \eta \cos[n(\xi - \omega t)] + \frac{1}{H_0} \frac{dH}{dx} \eta \cos[n(\xi - \omega t)] \right) \]  

\hspace{1cm} \text{(B.71)}

---

Integrating with respect to \( x \) using some goniometric properties gives for the orbital velocity \( u \):

\[ u = \frac{H_0 c}{2d} \sum_{n=1}^{\infty} a_n \left( \cos[n(\xi - \omega t)] + \frac{1}{H_0} \frac{dH}{dx} \cos[n(\xi - \omega t)] - \frac{1}{H_0} \frac{dH}{dx} \frac{1}{2} \sin[n(\xi - \omega t)] \right) \]

\[ + \frac{H_0 c}{2d} \sum_{n=1}^{\infty} b_n \left( \sin[n(\xi - \omega t)] + \frac{1}{H_0} \frac{dH}{dx} \sin[n(\xi - \omega t)] + \frac{1}{H_0} \frac{dH}{dx} \frac{1}{2} \cos[n(\xi - \omega t)] \right) \]  

\hspace{1cm} \text{(B.72)}

Both second terms between brackets show the influence of the wave height gradient on the orbital velocity as a function of \( x \). These terms are an amplification factor compared to both first terms. Because we will determine the wave height gradient in every new grid point both second terms can be omitted since \( x = 0 \). Hence, Eq. B.72 becomes:

\[ u = \frac{H_0 c}{2d} \sum_{n=1}^{\infty} a_n \left( \cos[n(\xi - \omega t)] - \frac{1}{H_0} \frac{dH}{dx} \frac{1}{2} \sin[n(\xi - \omega t)] \right) \]

\[ + \frac{H_0 c}{2d} \sum_{n=1}^{\infty} b_n \left( \sin[n(\xi - \omega t)] + \frac{1}{H_0} \frac{dH}{dx} \frac{1}{2} \cos[n(\xi - \omega t)] \right) \]  

\hspace{1cm} \text{(B.73)}

The vertical velocity is given by:

\[ \omega = \frac{z^* d}{d} \frac{\partial \zeta}{\partial t} = \frac{z^* d H_0}{d} \sum_{n=1}^{\infty} \left[ a_n \eta \omega \sin[n(\xi - \omega t)] - b_n \eta \omega \cos[n(\xi - \omega t)] \right] \]  

\hspace{1cm} \text{(B.74)}

---

Figure B.7 Control volume for the momentum equation applied on the roller
\( u \) (Eq. B.78) and \( w \) (Eq. B.79) are known so the term \( \bar{uw} \) can be found:

\[
\frac{\bar{uw}}{T} = \frac{1}{T \int_0^T \overline{\bar{uw}} \, dt}
\]

\[
= \frac{1}{T} \frac{\text{d}H}{4d^2} \frac{\text{d}H}{dx} \sum_{n=1}^{\infty} \left( a_n^2 + b_n^2 \right) n \omega \cos[n(kx - \omega t)] \sin[n(kx - \omega t)] + \frac{1}{H_0} \frac{d}{dx} \frac{\text{d}H}{dx} \frac{1}{k} \sum_{n=1}^{\infty} \left( a_n^2 \sin^2[n(kx - \omega t)] - b_n^2 \cos^2[n(kx - \omega t)] \right) + 2a_n b_n \sin[n(kx - \omega t)] \cos[n(kx - \omega t)] + n \omega a_n b_n \left( \sin^2[n(kx - \omega t)] - \cos^2[n(kx - \omega t)] \right) \, dt
\]  

(B.75)

again using goniometrical properties Eq. B.75 reduces to:

\[
\frac{\bar{uw}}{T} = - \frac{1}{8} \frac{gH_0}{dx} \frac{\text{d}H}{dx} \sum_{n=1}^{\infty} \left( a_n^2 + b_n^2 \right)
\]  

(B.76)

Assuming hydrostatic pressure, the pressure term in Eq. B.31 is still given by Eq. B.35, which can be rewritten to:

\[
\bar{P} = -pg \zeta \frac{\partial \zeta}{\partial x} + pgzS dx = \frac{1}{x} \left[ \frac{1}{2} pg \left( \frac{\partial \zeta}{\partial x} \right)^2 \right] - \frac{1}{2} pg \left( \frac{\partial \zeta}{\partial x} \right)^2 (dx)^2
\]  

(B.77)

Neglecting terms containing \((dx)^2\) gives Eq. B.37 again. However, Deigaard and Fredsøe obtain:

\[
\bar{P} = -pg \zeta \frac{\partial \zeta}{\partial x} + pgzS dx - \frac{1}{2} pg \left( \frac{\partial \zeta}{\partial x} \right)^2 - \frac{1}{2} pg \frac{\partial \zeta}{\partial x} \left( \frac{\partial \zeta}{\partial x} \right) dx
\]  

(B.78)

they stated that: the last two terms describe the gradient in pressure force due to changes in the surface roller, and are argued to be very small (Svendsen, 1984a) and can be omitted. Since the result is identical no more attention is given to this discrepancy.

The average change in momentum is given by Eq. B.32. An additional contribution arises from the roller so, Eq. B.34 now becomes:

\[
\frac{\overline{dB}}{dt} = -p \left[ 2u \frac{\partial \zeta}{\partial x} + \frac{\partial}{\partial x} (c^2 \zeta^2) dx + \bar{uw} dx \right]
\]  

(B.79)

By application of Eqs. B.76, B.78, B.79 and B.31 the shear stress is found to be:

\[
\tau = 2p \bar{u} \frac{\partial \zeta}{\partial x} - \frac{\partial}{\partial x} (c^2 \zeta^2) + \frac{1}{8} \frac{gH_0}{dx} \sum_{n=1}^{\infty} \left( a_n^2 + b_n^2 \right) - \frac{1}{2} pg \zeta \frac{\partial \zeta}{\partial x} + pgzS
\]  

(B.80)

with:

\[
\frac{\partial \zeta}{\partial x} = - \frac{1}{d} \frac{\partial z}{\partial t} \quad and \quad w = \frac{z + d}{d} \frac{\partial \zeta}{\partial t} = \frac{\partial w}{\partial x} = -1 \frac{w}{z + d} \quad so \quad \frac{\partial \zeta}{\partial x} = - \frac{1}{d} \frac{\partial w}{\partial x} = \frac{gH_0}{8d} \sum_{n=1}^{\infty} \left( a_n^2 + b_n^2 \right)
\]  

(B.81)
and:

\[ \frac{\zeta}{dx} = \frac{H_0}{8} \frac{dH}{dx} \sum_{n=1}^{\infty} \left( a_n^2 + b_n^2 \right) \]  

(B.82)

the expression for the shear stress (Eq. B.80) can be written as:

\[ \tau = -\frac{1}{4} \rho g H_0 \frac{dH}{dx} \sum_{n=1}^{\infty} \left( a_n^2 + b_n^2 \right) \left( 1 - \frac{1}{2} \frac{z}{d} \right) - \rho \frac{\partial}{\partial x} \left( c^2 \zeta^* \right) + \rho g S \]  

(B.83)

Using Eqs. B.68 and B.73 a mean energy flux of the wave motion can be defined:

\[ E_f = \rho g d \frac{1}{T} \int_0^{T} \zeta u dt = \rho g d \bar{u} = \frac{\rho g H_0^2 c}{8} \sum_{n=1}^{\infty} \left( a_n^2 + b_n^2 \right) \]  

(B.84)

For \( \zeta^* \) we have:

\[ \bar{\zeta}^* = \frac{1}{T} \int_0^{T} \zeta^* dt = \frac{1}{T c_0} \int_0^{L_R} \zeta^* dx \]  

(B.85)

where \( L_R \) is the horizontal length of the roller (Fig. B.4). Therefore, defining \( A \) as the area of the roller in vertical projection, we get:

\[ \bar{\zeta}^* = \frac{A}{L} ; \quad A = \int_0^{L_R} \zeta^* dx \]  

(B.86)

\( A \) was measured by Duncan (1981) for breaking waves behind a hydrofoil and found by Svendsen (1984a) to be approximated by (see also Fig. B.8):

\[ A \sim 0.9 H^2 \]  

(B.87)

The shear stress can now be written as:

\[ \tau = -\frac{1}{c} \frac{dE_f}{dx} \left( 1 - \frac{z}{2d} \right) - \rho g S (d - z) - \rho \frac{\partial}{\partial x} \left( \frac{Ac}{T} \right) \]  

(B.88)

where \( A \) is the cross sectional area of the surface roller.

At the water surface (\( z=0 \)) Eq. B.88 becomes:

\[ \tau = -\frac{1}{c} \frac{dE_f}{dx} - \rho \frac{d}{T} \frac{d(Ac)}{dx} \]  

(B.89)

For a wave set-up balancing the radiation stress gradient (no mean bed shear stress) the shear stress distribution is given by:

\[ \tau = -\frac{z + \eta}{d} \left( \frac{dE_f}{dx} + \rho \frac{d(Ac)}{d} \right) \]  

(B.90)
Figure B.8  Cross section area A for a roller. Measurements by Duncan (1981)
Appendix C

Discretisation
Discretisation

Introduction

The model used for the computations in this thesis consists out of three modules:

- a wave model;
- conversion program;
- 2DV-model.

In the wave model the wave energy balance and the roller equation are solved numerically. The results of this program are converted by a conversion program which is called GRAD. This conversion program transforms the output of the wave model so that it can be used as input for the 2DV model. The 2DV model finally computes the velocities set-up etcetera.

Since the 2DV model uses a staggered grid it is important to compute the various variables on the correct position in the grid. This section gives a detailed description of the place where the parameters should be prescribed.

The staggered grid

Since the results of the wave program are used as input for the 2DV flow model we start with the staggered grid:

![Staggered Grid Diagram](image)

Figure C.1 The staggered grid

The dashed lines indicate a computational cell.

We will now discuss at which position in a computational cell the various parameters have to be prescribed.

First we will start with the 1D parameters which means that they are only dependent of the horizontal coordinate \( (x) \). So the description of where they are prescribed should be seen horizontally:
• waterdepth \((d)\) is prescribed at vertical velocity point;
• water elevation \(<\xi>\) is prescribed at horizontal velocity point;
• total waterdepth \((h)\) is prescribed at vertical velocity point.

In the 2DV flow model \(<\xi>\) and \(h\) are computed, \(d\) is prescribed as input. This means that the matching x-coordinate gives the position at a vertical velocity point. All the output of the wave model is given at those x-coordinates. Since we will have to prescribe some parameters at a horizontal velocity point an interpolation should be made since there is a shift of a half grid size. First we will elaborate where the shear stress, the gradient of the wave induced mass flux and the production of wave induced turbulence should be prescribed.

![Diagram showing input and output parameters for 2DV model](image)

Figure C.2 Position of the various parameters in the staggered grid

**Interpolation procedure**

As mentioned in the previous section all the output data from the wave model is given at the x-coordinate which at its turn prescribes a vertical velocity point in the 2DV model.

The shear stress, the gradient of the wave induced mass flux and the production wave induced turbulence all have to be prescribed at concentration points. This implies that all variables which are used in the determination of these three parameters have to be interpolated.

As can be seen the figure below a forewards interpolation should be used. The following variables are used in the computation and should be interpolated:

• energy decay of the waves and the rollers;
• percentage of breaking waves \((Q_b)\);
• root mean square waveheight \((H_{rms})\);
• the wave group celerity \((c_g)\).

The interpolation for these variables which is made is shown below:

\[
Diss'(x) = \frac{Diss(x) + Diss(x+dx)}{2}
\]  

(C.1)
In *GRAD* the shear stress and the gradient of the wave induced mass flux are computed. The production of wave induced turbulence is determined in the 2DV model. The output of *GRAD* is written to a file which is used as input for the 2DV model. The file contains the following variables:

- x-coordinate \((x)\) at vertical a velocity point;
- waterdepth \((d)\) at vertical a velocity point;
- shear stress \((\tau)\) at horizontal a velocity point;
- gradient of wave induced mass flux \((dM_x/dx)\) at a horizontal velocity point;
- root mean square waveheight \((H_{rms})\) at a horizontal velocity point;
- dissipation of the wave \((D_w)\) at a horizontal velocity point;
- wave number \((k)\) at a horizontal velocity point.
main office
Rotterdamseweg 185
p.o. box 177
2600 MH Delft
The Netherlands
telephone (31) 15 - 56 93 53
telefax (31) 15 - 61 96 74
telex 38176 hydel-nl

location 'De Voorst'
Voorsterweg 28, Marknesse
p.o. box 152
8300 AD Emmeloord
The Netherlands
telephone (31) 5274 - 29 22
telefax (31) 5274 - 35 73