A SEMI-ANALYTICAL METHOD FOR COMPUTING THIRD-BODY EFFECTS ON EARTH'S AND LUNAR SATELLITE ORBITS

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Nomenclature

Symbols

a  Semi-major axis of osculating orbit

\( a_{ki} \)  Vector; see Eq. (13)

A  Satellite's reference area

b  Semi-minor axis of osculating orbit

\( b_{ki} \)  Vector; see Eq. (13)

c_D  Drag coefficient with respect to reference area A

e  Eccentricity of osculating orbit

G  Universal gravitational constant

h  Altitude above Earth's or lunar surface

\( h \)  Integration time step

i  Inclination of osculating orbit

\( i \)  Unit vector, directed from 0 to the equinox of date

\( j \)  Unit vector from 0, situated in the equatorial plane in accordance with: \( j = k \times i \)

JD  Julian Day

k  Unit vector, directed from 0 to the North Pole

\( l \)  Unit vector from \( O' \), parallel to \( p \)

\( m \)  Unit vector from \( O' \), in accordance with \( m = n \times l \)

\( n \)  Unit vector from \( O' \), parallel to the vector \( \mathbf{r}_{\text{Earth}} \times \mathbf{v}_{\text{Earth}} \) at JD. = 2439134.500

O  Origin of geocentric non-rotating reference frame \( xyz \), situated in the center of the Earth

O'  Origin of selenocentric non-rotating reference frame \( x'y'z' \), situated in the center of the Moon

O[... ]  Order of magnitude

p  Unit vector, directed from 0 to the ascending node of the Moon's osculating orbit about the Earth at JD. = 2439134.500

\( p_{ki} \)  Vector; see Eq. (15)

\(|r|\); see Eq. (22)

\( r \)  Geocentric radius vector

\( r' \)  Selenocentric radius vector

\( r_{ki} \)  Vector from point \( k \) to point \( i \)

\( R \)  Stumpff rest perturbation; see Eq. (10c)

\( R_0 \)  Mean radius of Earth
$S$  Stumpff perturbation; see Eq. (10b)
$t$  Time
$T_{2\pi}$  Orbital period
$UT$  Universal Time
$v$  Velocity
$\frac{v}{r}$  Relative velocity of satellite with respect to the Earth's atmosphere
$x, y, z$  Axes of geocentric reference frame, directed along $i, j, k$
$x', y', z'$  Axes of selenocentric reference frame, parallel to $x, y, z$
$x^*, y^*$  Axes of selenocentric reference frame situated in the osculating plane of the Earth's orbit about the Moon at JD. = 2439134.500; $x^*$ directed along $l$, $y^*$ directed along $m$
$\Delta$  Deviation
$\mu$  Earth's gravitational parameter
$T$  Epoch of last perigee passage
$\omega$  Argument of pericenter of osculating orbit
$\Omega$  Right ascension of ascending node of osculating orbit

**Subscripts**

A  Apocenter
P  Pericenter
S  Satellite
0  Initial state
E  Earth
S  Sun
M  Moon
1. Introduction

This report deals with a semi-analytical method for the computation of satellite orbits which are mainly perturbed by third-body attractions. The method is called semi-analytical because the integration of the perturbing accelerations is for the major part carried out analytically, while for the integration of some minor terms a numerical integration technique is used. So, the method can be seen as a combination of a numerical technique with its stepwise nature, and an analytical technique in which analytical expressions are used for the computation of the perturbations. As a result, the method is for the same accuracy more than 10 times faster than a pure numerical integration method.

With the computer program developed, the following perturbing influences are taken into account:
(1) the gravitational attractions of Sun and Moon,
(2) the perturbing force due to the Earth's oblateness \( (J_2) \),
(3) the perturbing force due to the atmospheric drag at close-Earth passages \( (h_0 < 1000 \text{ km}) \).

For the computation of the satellite's motion, affected by the perturbing forces mentioned under (1), we started from Stumpff's method (Ref. 1) for the determination of the relative motions in the many-body problem. This method is a variant of the well-known Encke special perturbation technique. For Encke's technique the perturbations are defined as the deviations of the satellite's coordinates from those of a Kepler ellipse. These deviations are of the order \( h^2 \), where \( h = t - t_0 \) is the intermediate time beyond \( t_0 \), the epoch of osculation. The Stumpff technique combines several Keplerian orbits to form an intermediate orbit between the osculating ellipse and the actual orbit. The deviations of the actual from the intermediate orbit are termed "rest perturbations". It will be shown that these rest perturbations are of the order \( h^4 \), and are therefore very small for small \( h \). Both Encke's perturbations and Stumpff's rest perturbations can be computed only numerically. However, Stumpff's rest perturbations are so small for small \( h \), that they can be computed sufficiently accurate by Stirling's five-point integration formulas. This is a remarkable advantage over Encke's method where the usage of time-consuming integration procedures cannot be avoided.

Stumpff's method can be applied to all satellite orbits where the satellite stays at remote distances from the Earth for almost its entire lifetime, i.e. for highly-eccentric satellite orbits about the Earth and satellite orbits about the Moon. However, satellites in such orbits can from time to time approach the Earth very close so that the perturbing influences of the Earth's oblateness \( (J_2) \) and the Earth's atmosphere may no longer be neglected. For instance, in the case of highly-eccentric satellite orbits about the Earth, perigee passages at altitudes less than 300 km can occur several times during the satellite's lifetime. But also for satellites which are initially orbiting the Moon, it can happen that they temporary pass the Earth such close that the perturbing influence of the Earth's oblateness can no longer be neglected.

So, for accurate orbit predictions it is necessary to incorporate within the total computing procedure the perturbations due to the influences (2) and (3). In Chapter 2 we will show how the computation of these orbit perturbations can be fitted within Stumpff's method in such a way that the efficiency of the computation method is maintained. In that Chapter it is also shown how Stumpff's method can be improved further by a periodical correction of the predicted positions of Sun and Moon.
For testing the method's accuracy, the evolution of the highly-eccentric orbit of Explorer 28 has been computed for the entire lifetime (about 3 years). The results were compared with the precise results, obtained by the ITEM (Interplanetary Trajectory Encke Method) computer program (Ref. 2).

It was found that the obtained deviations in the predicted values of the osculating elements remain small within almost the entire lifetime. Only during the last 130 days a relatively large increase of the deviations in the predicted values of the osculating elements can be observed. This is probably connected with the severe perturbations that appear in all osculating elements during this period. A small error in the time coordinate will then give rise to relatively large deviations between the predicted and the actual values of the osculating elements. On the contrary, the accuracy of the predicted perigee distance is maintained at a high level throughout the entire lifetime. This is most important as the accuracy of the perigee distance computation determines the accuracy of the satellite's lifetime prediction.

The computer program has further been used in an exploratory study of the effects of Earth's and solar attraction on lunar satellite orbits. Only those lunar satellite orbits are considered for which the osculating planes at the initial epoch, $t_0$, are coinciding with that of the Earth's orbit about the Moon. The satellite orbit's initial aposeelene is for each case chosen in the initial Earth's direction.

It was found that the magnitude of the orbit perturbations is strongly connected with the direction of the satellite's motion about the Moon, especially for orbits at altitudes higher than about 30,000 km. For a satellite orbiting the Moon in direct direction (i.e. in the same direction as the Earth's motion about the Moon), the chance of occurrence of both an impact on the Moon and an escape from the lunar region is much larger than for a satellite in a retrograde lunar orbit (which has the opposite direction of motion).
2. General outline of the method

As mentioned before, the problem we deal with in this Chapter, is to find a computing method for satellite orbits which are mainly perturbed by solar and lunar attraction and only secondary by the Earth's oblateness and atmosphere. This means that we deal in the first place with a restricted four-body problem (Earth, Moon, Sun, satellite), where by the mass of one of the bodies, i.e. the satellite, can be neglected relative to the other masses.

The first part of this Chapter gives a general outline of the method we used for the computation of the satellite's motion. The method is known as Stumpff's method, which is a fast and accurate technique for the determination of the relative motions in the many-body problem. In the second part of this Chapter we show how Stumpff's method can be modified, first by the incorporation of a computing procedure for the secondary perturbations, and secondly by extending the computer program with a subroutine for a more precise prediction of the position of the disturbing bodies (Sun and Moon). These modifications yield a more accurate prediction of the satellite orbit whereas the efficiency of the computing process is almost not affected.

2.1. Stumpff's method for the computation of the relative motions in the many-body problem

Consider in space a system of \( n \) bodies, where it is assumed that the bodies act as point-masses. Let the position of the \( i \)th body with mass \( m_i \) with respect to an inertial reference frame be indicated by the vector \( \mathbf{r}_i \), and the position of an arbitrary other body \( j \) be indicated by the vector \( \mathbf{r}_j \). Let further be \( \mathbf{r}_{ij} \) a vector, defined as:

\[
\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i \quad \text{and} \quad r_{ij} = |\mathbf{r}_{ij}|
\]

Now, according to Newton's second law, for the motion of the body \( m_i \) with respect to the inertial frame, we can write

\[
m_i \ddot{\mathbf{r}}_i = \sum_{j \neq i}^n G \frac{m_i m_j}{r_{ij}^3} \mathbf{r}_{ij}, \tag{1a}
\]

where \( G \) is the universal gravitational constant.

Analogous, the motion of the body \( m_k \) satisfies the equation

\[
m_k \ddot{\mathbf{r}}_k = \sum_{j \neq k}^n G \frac{m_k m_j}{r_{kj}^3} \mathbf{r}_{kj}. \tag{1b}
\]

Equations (1a) and (1b) can be rewritten as

\[
\ddot{\mathbf{r}}_i = G \frac{m_i}{r_{ik}^3} \mathbf{r}_{ik} + \sum_{j \neq i}^n G \frac{m_i m_j}{r_{ij}^3} \mathbf{r}_{ij}, \tag{2}
\]

\[
\ddot{\mathbf{r}}_k = G \frac{m_k}{r_{ki}^3} \mathbf{r}_{ki} + \sum_{j \neq k}^n G \frac{m_k m_j}{r_{kj}^3} \mathbf{r}_{kj}.
\]

Now:

\[
\mathbf{r}_{ki} = \mathbf{r}_i - \mathbf{r}_k \quad , \quad \mathbf{r}_{ik} = \mathbf{r}_k - \mathbf{r}_i \quad , \quad \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j
\]
By substituting Eqs. (3) into Eqs. (2), we obtain

$$\ddot{r}_{ki} = -G \left( \frac{m_i + m_k}{r_{ki}^3} \right) r_{ki} + G \sum_{j \neq k} m_j \left( \frac{r_{kj} - r_{ki}}{r_{kj}^3} \right).$$  \hspace{1cm} (4)

In the following we restrict the problem to the four-body case: Earth (\(E\)), satellite (S), Sun (\(S\)), and Moon (\(M\)). Let each of the subscripts \(i, j, k, l\) stand for one of the symbols \(E, S, \Phi, \Omega\), with the proviso that different subscripts be distinct. For the relative motion of one of the four bodies, \(i\), relative to one of the others, \(k\), we obtain from Eq. (4):

$$\ddot{r}_{ki} = G \left\{ - \left( \frac{m_k + m_i}{r_{ki}^3} \right) r_{ki} + \sum_{j \neq k} m_j \left( \frac{r_{kj}^2}{r_{kj}^3} - \frac{r_{ji}^2}{r_{ji}^3} \right) - m_k \left( \frac{r_{kl}^2}{r_{kl}^3} - \frac{r_{li}^2}{r_{li}^3} \right) \right\}.$$  \hspace{1cm} (5)

The 12 combinations \(k, i\), that can be obtained from \(E, S, \Phi, \Omega\), contain only three linearly independent vectors \(\ddot{r}_{ki}\) (e.g. \(\ddot{r}_{EE}, \ddot{r}_{E\Phi}, \ddot{r}_{E\Omega}\)), as there exist six identities

$$\ddot{r}_{ki} = - \ddot{r}_{ik},$$

and three independent equations of the form

$$\ddot{r}_{ki} + \ddot{r}_{il} + \ddot{r}_{lk} = 0.$$

We assume that for \(t = 0\), \(\ddot{r}_{ki}\) and \(\ddot{r}_{ki}\) are known, which implies that at that time the osculating Keplerian orbits are known. Using the notation \(r_{ki}(0)\) for \(r_{ki}\) at time \(t = 0\) and using square brackets to denote osculating Keplerian orbits, we have at \(t = 0\):

$$[r_{ki}(0)] = r_{ki}(0), \hspace{1cm} [\ddot{r}_{ki}(0)] = \ddot{r}_{ki}(0).$$  \hspace{1cm} (6)

as the conditions of osculation. The osculating Keplerian orbits satisfy the differential equations

$$[\dddot{r}_{ki}] = -G \left( m_k + m_i \right) \frac{[r_{ki}]}{[r_{ki}^3]},$$  \hspace{1cm} (7)

where

$$[r_{ki}] = \left| [r_{ki}] \right|.$$

Equation (7) can be applied to any combination of two different subscripts \(i, j, k\) and \(l\). Now define the vectors \(s_{ki} (= - s_{ik})\) by

$$s_{ki} = \frac{G \left[ \frac{r_{ki}}{[r_{ki}^3 - \frac{r_{ki}^3}{3}] \frac{r_{ki}^3}{3} - \frac{r_{ki}^3}{3}} \right] \dddot{r}_{ki},$$  \hspace{1cm} (8)

Substituting Eq. (8) into Eq. (7) and rearranging, one obtains
\[ G \frac{r_{ki}}{r_{ki}^3} = \frac{1}{m_k + m_i} [\ddot{r}_{ki}] + s_{ki} . \]  

(9)

Using Eq. (9) to eliminate all expressions of this form from the right-hand side of Eq. (5), yields

\[ r_{ki} = \left( \ddot{r}_{ki} \right) + \frac{m_j}{m_k + m_j} [\ddot{r}_{kj}] + \frac{m_j}{m_j + m_i} [\ddot{r}_{ji}] + \frac{m_k}{m_k + m_1} [\ddot{s}_{ki}] + \frac{m_1}{m_1 + m_i} [\ddot{s}_{li}] + \left( m_k + m_i \right) s_{ki} + m_j \left( s_{kj} + s_{ji} \right) + m_1 \left( s_{kl} + s_{li} \right) . \]

(10)

This can be written as

\[ \dot{r}_{ki} = \left( \ddot{r}_{ki} \right) + \frac{s_{ki}}{r_{ki}^3} + \frac{R_{ki}}{r_{ki}^3} , \]

(10a)

where

\[ \frac{s_{ki}}{r_{ki}^3} = \frac{m_j}{m_k + m_j} [\ddot{r}_{kj}] + \frac{m_j}{m_j + m_i} [\ddot{r}_{ji}] + \frac{m_k}{m_k + m_1} [\ddot{s}_{ki}] + \frac{m_1}{m_1 + m_i} [\ddot{s}_{li}] , \]

(10b)

and

\[ \frac{R_{ki}}{r_{ki}^3} = \left( m_k + m_i \right) \frac{s_{ki}}{r_{ki}^3} + m_j \left( s_{kj} + s_{ji} \right) + m_1 \left( s_{kl} + s_{li} \right) . \]

(10c)

It follows from Eq. (10a) that \( \frac{s_{ki}}{r_{ki}^3} \) and \( \frac{R_{ki}}{r_{ki}^3} \) are the components of the perturbing acceleration, where the magnitude of the perturbation expressed by the \( \frac{s}{r^3} \)-term can be computed directly by the theory of the two-body problem. The integration of \( \frac{R_{ki}}{r_{ki}^3} \), however, can be effected only by numerical methods. So, we note the remarkable fact that the integration of the total perturbing acceleration can be divided into an analytical part (the integration of \( \frac{s_{ki}}{r_{ki}^3} \)), and numerical part (the integration of \( \frac{R_{ki}}{r_{ki}^3} \)). This is a great advantage over the well-known classical special perturbation methods of Encke, Covell, or Lagrange, where the total (perturbing) acceleration (Encke and Lagrange: \( \frac{s_{ki}}{r_{ki}^3} \); Covell: \( \frac{R_{ki}}{r_{ki}^3} \)) has to be integrated numerically.

Now, the first and second integrals of Eq. (10a) are

\[ \dot{r}_{ki} = \left( \ddot{r}_{ki} \right) + \frac{s_{ki}}{r_{ki}^3} + \frac{R_{ki}}{r_{ki}^3} + a_{ki} , \]

(11)

\[ r_{ki} = \left[ \dot{r}_{ki} \right] + \frac{s_{ki}}{r_{ki}^3} + \frac{R_{ki}}{r_{ki}^3} + a_{ki} \cdot h + b_{ki} , \]

where \( h = t - t_o \). To determine the constant vectors \( a_{ki} \) and \( b_{ki} \), evaluate Eqs. (11) at \( t_o \), from which follows:

\[ \dot{h}_{ki}(0) + \ddot{h}_{ki}(0) + s_{ki} = 0 \]

(12)

\[ h_{ki}(0) + \ddot{h}_{ki}(0) + b_{ki} = 0 . \]

Without loss of generality, \( \dot{h}_{ki}(0) \) and \( h_{ki}(0) \) can be equated to zero, since non-zero values
can be absorbed by $a_{ki}$ and $b_{ki}$. So,

$$a_{ki} = -\frac{S_{ki}(0)}{a} \quad \text{and} \quad b_{ki} = -\frac{S_{ki}(0)}{b} .$$

Substituting Eqs. (13) into Eqs. (11) yields

$$F_{ki} = [F_{ki}] + r_{ki} + R_{ki} ,$$

$$\dot{F}_{ki} = [F_{ki}] + \dot{r}_{ki} + \dot{R}_{ki} .$$

where

$$\dot{r}_{ki} = \dot{S}_{ki} - \dot{S}_{ki}(0) ,$$

$$\dot{R}_{ki} = S_{ki} - h \cdot \dot{S}_{ki}(0) - S_{ki}(0) .$$

In Eqs. (14), the term $\dot{R}_{ki}$ can be considered as a first-order perturbation. Eq. (8) shows that $\dot{S}_{ki}(0) = 0$, whence, in view of Eq. (10c), $\dot{R}_{ki}(0) = 0$. Similarly $\ddot{R}_{ki}(0) = 0$, as can be seen by differentiation, and hence

$$\dot{R}_{ki} (t_0 + h) = 0 [h] .$$

This means that for sufficiently small values of $h$, $\dot{r}_{ki}$ and $\dot{R}_{ki}$ in Eqs. (14) can be ignored. The resulting approximations are

$$\dot{r}_{ki} = [F_{ki}] + r_{ki} (+0[h]) ,$$

$$\ddot{r}_{ki} = [F_{ki}] + \dot{r}_{ki} (+0[h]) .$$

Now, if we look at Eqs. (10), (10a), and (10b), in combination with Eqs. (17), and if we only consider the satellite's motion about the Earth, perturbed by Sun and Moon, we can replace in (10) the subscripts $k$ by $\theta$, $i$ by $S$, $j$ by $\theta$, and $l$ by $\xi$, respectively. We then obtain

$$F_{\theta S} = [F_{\theta S}] + \frac{m_{\theta}}{m_{\theta} + m_{S}} [F_{\theta S}] + \frac{m_{\theta}}{m_{\theta} + m_{S}} [F_{\theta S}] + \frac{m_{\xi}}{m_{\theta} + m_{S}} [F_{\theta S}]$$

$$+ \frac{m_{\xi}}{m_{\theta} + m_{S}} [F_{\theta S}] + k_{S\theta} + 0 [h^4] ,$$

where

$$k_{S\theta} = -h \cdot \dot{S}_{\theta S}(0) - S_{\theta S}(0)$$

is a constant vector. Expression (17a) can also be written as

$$F_{\theta S} = [F_{\theta S}] + [F_{\theta S}] + [F_{\theta S}] + \frac{m_{\theta}}{m_{\theta} + m_{S}} [F_{\theta S}] + \frac{m_{\xi}}{m_{\theta} + m_{S}} [F_{\theta S}]$$

$$+ k_{S\theta} + 0 [h^4] .$$
Analogous, by replacing the subscripts $k, i, j, l$ by $\theta, \phi, \psi$ and $S$ respectively, we obtain for the Moon's perturbed motion:

$$r_{\theta\psi} = \left[ r_{\theta\psi} \right] + \frac{m_0}{m_\theta + m_\psi} \left[ r_{\theta\psi} \right] + \frac{m_0}{m_\theta + m_\psi} \left[ r_{\theta\psi} \right] + k_{\theta\psi},$$  \hspace{1cm} (17c)

while the expression for the Sun's perturbed motion is obtained by replacing $k, i, j$ and $l$ by $\theta, \phi, \psi$ and $S$ respectively:

$$r_{\theta\phi} = \left[ r_{\theta\phi} \right] + \frac{m_\psi}{m_\phi + m_\theta} \left[ r_{\theta\phi} \right] + \frac{m_\psi}{m_\phi + m_\theta} \left[ r_{\theta\phi} \right] + k_{\theta\phi}.$$  \hspace{1cm} (17d)

The constant vectors $k_{\theta\phi}$ and $k_{\theta\psi}$ in (17c) and (17d) are given by:

$$k_{\theta\phi} = -h \cdot \dot{s}_{\theta\phi}(0) - S_{\theta\phi}(0)$$

$$k_{\theta\psi} = -h \cdot \dot{s}_{\theta\psi}(0) - S_{\theta\psi}(0)$$

Equation (17b) for the relative motion of the satellite about the Earth shows a most remarkable fact, directly following from the Stumpff method: we see that, for small time steps, the actual orbit of the satellite can to a high degree of accuracy be regarded as being composed of the satellite's unperturbed orbit about the Earth, the satellite's unperturbed orbit about the Sun, the satellite's unperturbed orbit about the Moon, the Sun's unperturbed orbit about the Earth, and the Moon's unperturbed orbit about the Earth.

Similarly, the perturbed orbits of Moon and Sun about the Earth, can for small time steps be regarded as a composition of the Moon's and Sun's unperturbed orbits about the Earth, together with their unperturbed orbits about each other. This peculiar fact is quite unknown in celestial mechanics.

If a still higher accuracy than obtained by Eqs. (17) is desired $r_{ki}$ can be approximated by replacing $s_{ki}$ in Eq. (10c) by

$$s_{ki} = G \left[ \frac{G_{ki}}{r_{ki}} \right]^\frac{1}{3} \left[ \frac{G_{ki}}{r_{ki}} \right]^\frac{1}{3}.$$  \hspace{1cm} (18)

$G_{ki}$ and $r_{ki}$ can be obtained by numerical integration, yielding:

$$r_{ki} = r_{ki} + R_{ki},$$

$$\dot{r}_{ki} = \dot{r}_{ki} + \dot{R}_{ki}.$$  \hspace{1cm} (19)

From Eqs. (6), (17), and (18) it follows that

$$s_{ki} - s_{ki} = 0 \left[ h^4 \right].$$

So, the error in $s_{ki}$ is of order $h^4$. Therefore, the error in $r_{ki}$, and thus the error in $r_{ki}$, is of order $h^6$, as can be seen from the expansion for $r_{ki}$:

$$r_{ki} = r_{ki}(0) + \frac{h}{T} \dot{r}_{ki}(0) + \frac{h^2}{2T} \ddot{r}_{ki}(0) + \ldots$$
\[ \begin{align*}
&= 0 + 0 + \frac{h^2}{27} \cdot 0 \left[ h^6 \right] + \ldots \\
&= 0 \left[ h^6 \right].
\end{align*} \tag{20} \]

This error can be neglected completely for small time intervals.

For the numerical integration of \( \mathbf{r}_{k1} \), Stirling's five-point formulas can be used. We then obtain

\[ \begin{align*}
\mathbf{r}_{k1}(t_0 + 4h_o) &= \frac{h_o}{45} \left\{ 14 \mathbf{r}_{k1}(t_0) + 64 \mathbf{r}_{k1}(t_0 + h_o) + 24 \mathbf{r}_{k1}(t_0 + 2h_o) + \\
&\quad 64 \mathbf{r}_{k1}(t_0 + 3h_o) + 14 \mathbf{r}_{k1}(t_0 + 4h_o) \right\}, \\
\mathbf{r}_{k1}(t_0 + 4h_o) &= \frac{h_o^2}{135} \left\{ 168 \mathbf{r}_{k1}(t_0) + 576 \mathbf{r}_{k1}(t_0 + h_o) + 144 \mathbf{r}_{k1}(t_0 + 2h_o) + \\
&\quad 192 \mathbf{r}_{k1}(t_0 + 3h_o) \right\}. \tag{21} \end{align*} \]

To apply these formulas, the time interval \( h \) has to be subdivided into four equally sized steps, where for each step the vector \( \mathbf{r}_{k1}(t_0 + n h_o), n = 1, 2, 3, 4, \) can be computed by the Eqs. (18) and (10c). The vector's \( \frac{\mathbf{r}_{k1}}{t_0}, \frac{\mathbf{r}_{k1}}{t_1}, \) and \( \frac{\mathbf{r}_{k1}}{t_2} \) are known from Eqs. (17).

Substitution of these quantities plus Eqs. (21) in Eqs. (19) yields \( \frac{\mathbf{r}_{k1}}{t_2} \) and \( \frac{\mathbf{r}_{k1}}{t_3} \) at epoch \( t_0 \).

These vectors can be used again as initial vectors for the computation of \( \frac{\mathbf{r}_{k1}}{t_2} \) and \( \frac{\mathbf{r}_{k1}}{t_3} \), where \( t_2 = t_1 + 4 h_o \), and we can proceed to \( t_3 = t_2 + 4 h_o, t_4 = t_3 + 4 h_o, \) etc.

As mentioned before, from the 12 combinations \( k, i \) from the subscripts \( \Theta, S, \Theta, \) and \( S \), we obtain 3 independent vectors \( \mathbf{r}_{\Theta S}, \mathbf{r}_{S S}, \) and \( \mathbf{r}_{SS} \). So, with Eqs. (19) not only the satellite's ephemerides can be predicted, but also the ephemerides of both disturbing bodies Sun and Moon. By this, additional tape-reading of the position of the disturbing bodies is not required, which is another advantage of Stumpff's method.

2.2. Modification of Stumpff's method.

2.2.1. Incorporation of the perturbations due to the Earth's oblateness \( (J_2) \) and atmosphere.

Stumpff's method does not account for the perturbations due to the Earth's oblateness \( (J_2) \) and atmosphere. So, the application of this method is limited to orbit predictions of satellites which stay at remote distances from the Earth for almost the entire lifetime. In many cases, however, the satellite can from time to time approach the Earth such close that the perturbing effects of the Earth's oblateness and possibly the Earth's atmosphere may no longer be neglected. For instance, for many satellites launched into a highly-eccentric orbit, the initial perigee altitude is less than 500 km.

As an example Figure 1 shows the initial orbit of the Explorer 28 satellite, with an eccentricity of 0.95 and a perigee altitude of 200 km. During such close passages the satellite's orbit can be perturbed seriously by the Earth oblateness and the atmosphere drag. However, the magnitude of these perturbations diminishes strongly as the distance between satellite and Earth increases. The perturbing accelerations from the \( J_2 \)-term in the Earth's gravitational potential are proportional to \( r^{-2} \); those due to air-drag are proportional to the air density, which decreases exponentially with increasing altitude above the Earth's
surface. At altitudes above 1000 km, the latter perturbations can be neglected completely.

It should be realised that a satellite in a highly-eccentric orbit will be in the near-perigee region for only a very small part of its orbital period. Consider, for instance, an orbit with the following characteristics:

\[
\begin{align*}
a &= 140,000 \text{ km}, \\
e &= 0.95, \\
r_p &= 7,000 \text{ km}, \\
h &= 600 \text{ km}, \\
r_A &= 273,000 \text{ km}, \\
T_{2n} &= 6.0 \text{ days}
\end{align*}
\]

We then find that the satellite will stay at a distance of less than 30,000 km from the Earth's center for only 2% of its orbital period, and at an altitude of less than 1000 km above the Earth's surface for only 0.1% of this period. Consequently, the underlying idea for the computation method for the perturbing effects from the \( J_2 \)-term and the Earth's atmosphere is that these effects can be computed with a simple integration method, as these perturbations must be of an impulsive type. Figure 2 shows how the osculating elements of the highly-eccentric orbit of Explorer 28 were perturbed by the \( J_2 \)-term during the first perigee passage after launch (\( h_p = 580 \text{ km} \)). The Figure clearly shows that the perturbations in the osculating elements, especially in the semi-major axis, can be severe (\( \Delta a \approx 1700 \text{ km} \)), whereby the duration of all perturbations is of order of 1 hour. The perturbations due to the air-drag are less severe than those due to the \( J_2 \)-term (\( \Delta a_{\text{air-drag}} \) is about 0.17 km during the first perigee passage). For the computation of the impulsive perturbations due to \( J_2 \) and air-drag the Stirling formulas (21) are used in order to maintain a fast total computation procedure.

The vectors \( \mathbf{X}_{J_2} \) can be obtained directly from Eqs. (22), representing the \( J_2 \) perturbing accelerations in \( x, y \) and \( z \) direction, respectively,

\[
\begin{align*}
\ddot{x}_{J_2} &= -\frac{\mu \cdot J_2 \cdot R_0^2}{2 \cdot r^2} (3x^2 - 15xz^2), \\
\ddot{y}_{J_2} &= -\frac{\mu \cdot J_2 \cdot R_0^2}{2 \cdot r^2} (3y^2 - 15yz^2), \\
\ddot{z}_{J_2} &= -\frac{\mu \cdot J_2 \cdot R_0^2}{2 \cdot r^2} (9z^2 - 15z^3).
\end{align*}
\]

(22)

Here, \( \mu \) is the Earth's gravitational parameter and the values of \( x, y, z \) and \( r \) refer to the satellite's position in the intermediate orbit \( \mathbf{X}_0 \) [Eqs. (17)]. In this way, the vector \( \mathbf{X}_{J_2} \):

\[
\mathbf{X}_{J_2} = \mathbf{i} \ddot{x}_{J_2} + \mathbf{j} \ddot{y}_{J_2} + \mathbf{k} \ddot{z}_{J_2}
\]

(23)

for the epochs \( t + n \Delta t \), \( n = 1, 2, 3, 4 \).

(24)

is obtained and the computation of the \( J_2 \) perturbations can be included in the Stumpff program. Here, the vectors \( \mathbf{X}_{\text{deg}} \) in the formulas (21) have to be replaced by the vectors

\[
\mathbf{X}^t_{\text{deg}} = \mathbf{X}_{\text{deg}} + \mathbf{X}_{J_2}.
\]

(25)
The incorporation of the perturbing effects due to air-drag can be accomplished in entirely the same way. For the perturbing accelerations due to the atmospheric drag, we can write

\[ \mathbf{v}_{\text{atm}} = -\frac{1}{2} \frac{c_D A}{m_S} \rho \mathbf{v} \times \mathbf{v} \]  

(26)

In Eq. (26) is \( m_S \) the mass of the satellite, \( A \) a reference area of the satellite, \( c_D \) the drag-coefficient with respect to that reference area, \( \rho \) the local atmospheric density, and \( \mathbf{v} \) the satellite's velocity relative to the atmosphere.

The vectors \( \mathbf{v}_{\text{atm}} \) can be computed for the epochs (24), whereby the satellite's position is also thought to be in the intermediate orbit.

Combination of Eqs. (25) and (26) finally yields

\[ \mathbf{r}_{\text{atm}}' = \mathbf{r}_S + \mathbf{r}_{J_2} + \mathbf{v}_{\text{atm}} \]  

(27)

In Eq. (27), \( \mathbf{r}_{\text{atm}}' \) represents the resulting perturbing acceleration, which acts upon the satellite in the intermediate orbit.

The computational method described above, accounts for the perturbations due to the gravitational attractions of Sun and Moon, the oblateness of the Earth \( (J_2) \), and the atmospheric drag at close-Earth passages. The use of Stirling's integration procedure for the perturbing accelerations of \( J_2 \) and atmospheric drag, however, has been chosen rather intuitively. In order to check the validity of this procedure, a second subroutine has been written, where the perturbing accelerations of \( J_2 \) and air-drag are integrated by means of a Runge-Kutta integration scheme. For the computation of the luni-solar perturbations, this program also used the Stumpff method. With both computer programs the highly-eccentric orbit of Explorer 28 had been computed for the first five perigee passages after launch.

Comparison of the perigee conditions, as predicted by both programs, with the actual values showed no increase at all in the accuracy by using the Runge-Kutta integration method. Besides, the total computing time of the program with Runge-Kutta integration was about the fourfold of that with the Stirling technique. Hence, the Stirling integration method is the obvious technique for the computation of the \( J_2 \) and the atmospheric perturbations, at least if these perturbations have an impulsive behavior.

2.2.2. Improvement of the prediction for the positions of Sun and Moon

As has been shown in Section 2.1., Stumpff's method generates the ephemerides of both satellite and disturbing bodies (Sun and Moon). However, the predicted orbit of the Moon about the Earth is that orbit the Moon would pass through, if only disturbed by the Sun's attraction. Similarly, the predicted orbit of the Sun about the Earth is that orbit the Sun would follow, if only disturbed by the attractional force of the Moon. Consequently, the prediction for the positions of the disturbing bodies does not account for the perturbing effects of other planets in the solar system, especially Venus, Jupiter and Saturn. This can cause considerable inaccuracies in the predicted positions of Sun and Moon if extended periods are considered. Comparison of the predicted positions of the Moon with the actual positions showed that the deviations in the right ascension and declination can reach up to several degrees within a period of one year.

Since satellites in highly-eccentric Earth-centered orbits can reach the Moon at close distances, it was necessary to build in a further modification in the form of a subroutine,
which explicitly computes the positions of Sun and Moon by means of empirically derived formulas (Ref. 3). In the final computer program, this subroutine corrects the positions of Sun and Moon, as predicted by Stumpff’s method, periodically with time steps of 24 hour. Thus, an accurate prediction of the positions of the disturbing bodies is guaranteed, while a time-consuming tape-reading procedure is avoided. Moreover, we can also use the computer program for lunar satellite orbit predictions.
3. Accuracy of the method

With the computer program the highly-eccentric orbit of Explorer 28 has been computed for the entire lifetime (about 3 years). The Explorer 28 orbit is extremely suitable for testing the method's accuracy as the satellite both moves away from the Earth and approaches the Earth's surface very close. This is illustrated in Figure 1 where the osculating orbit is depicted for the initial epoch, $t_0$. For this satellite, all perturbations considered in the foregoing, play a part in the actual evolution of the orbit. Table 1 lists the satellite orbit's initial parameters with respect to a geocentric non-rotating reference frame (x-axis directed to the equinox of date, y-axis situated in the equatorial plane, z-axis directed to the North Pole; see Nomenclature). In table 2 the corresponding parameters are given for both disturbing bodies Sun and Moon.

The predicted evolution of the orbital elements could be compared with precise data, published in Ref. 2. These orbital data were obtained by the ITEM (Interplanetary Trajectory Encke Method) computer program, and serve for checking the accuracy of other computer programs for the computation of highly-eccentric satellite orbits. This is possible because the ITEM data were extensively checked against actual orbit determinations and found to be very accurate.

The Figures 3 and 4 illustrate the gain in accuracy for the predicted values of the osculating elements $\omega$ and $\Omega$, if the positions of Sun and Moon were corrected for each day. Figure 3 shows the magnitudes of the deviations in the predicted values of the argument of perigee ($\omega$) for every 10th perigee passage. Figure 4 shows the corresponding magnitudes for the right ascension of the ascending node ($\Omega$). Both Figures show for the first 1000 days after launch a high gain in accuracy due to the inclusion of the explicit computation of the positions of Sun and Moon. The maximum value of the error observed in $\omega$ decreased from $1.30^\circ$ to $0.48^\circ$; the maximum error observed in $\Omega$ decreased from $1.26^\circ$ to $0.23^\circ$.

Remarkable is the relatively large increase of the deviations obtained in both cases for the last 130 days of the satellite's lifetime. This is probably due to the large variation in the osculating elements during this period. A small error in the time coordinate will then give rise to large deviations between the predicted and the real-time values of the osculating elements.

The developments of the errors in the predicted values of the other osculating elements, and the improvement of the prediction for the first 1000 days after launch, show the same tendency:

<table>
<thead>
<tr>
<th>Maximum error</th>
<th>Without Sun, Moon corrections</th>
<th>With Sun, Moon corrections</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta a$ (km)</td>
<td>1530</td>
<td>655</td>
</tr>
<tr>
<td>$\Delta e$ (-)</td>
<td>$9.10^{-3}$</td>
<td>$7.10^{-3}$</td>
</tr>
<tr>
<td>$\Delta i$ (deg)</td>
<td>2.20</td>
<td>0.97</td>
</tr>
<tr>
<td>$\Delta r_p$ (km)</td>
<td>1620</td>
<td>1080</td>
</tr>
<tr>
<td>$\Delta \tau/\tau(-)$</td>
<td>$1.80.10^{-3}$</td>
<td>$1.51.10^{-3}$</td>
</tr>
</tbody>
</table>

* $\tau$ = epoch of last perigee passage
The evolution of the perigee distance, as predicted by the program with use of the correcting subroutine is represented in Figure 5, where also the ITEM values are plotted for every 10th perigee passage. This Figure clearly indicates the accuracy of the computer program for highly eccentric satellite orbits. We note that the predicted evolution of the perigee distance accurately fits the actual evolution (ITEM values of rₚ). The program predicts the satellite's impact on the Earth just before the 194th perigee passage, which is also in full agreement with the actual flight. If the predicted positions of Sun and Moon were not corrected, the satellite's impact is predicted just before the 193rd perigee passage which means an error in the predicted lifetime of about 6 days. This clearly shows the necessity for the correction process applied for the computation of the positions of Sun and Moon. The increase in computing time caused by the subroutine for the precise computation of the positions of the disturbing bodies amounts about 2 percent of the total computing time. The total computing time for the prediction over 1133 days was about 37 minutes on the IBM-370/158 computer of Delft University. A more extensive discussion of the computation method can be found in Reference 4. In addition this paper also discusses the luni-solar effects on the Explorer 28 orbit.
4. Application to the study of lunar satellite orbit perturbations

This Chapter presents the results of an exploratory study of lunar satellite orbit perturbations due to terrestrial and solar attraction. For the computation of the lunar and solar orbits about the Earth we used the corrected Stumpff method. The gravitational attractions of other planets of the solar system are thus included in the prediction of these orbits. On the other hand, the satellite's motion is considered to be solely determined by the attractional forces of the Earth, Sun and Moon. For all orbits studied the results are reduced to a selenocentric non-rotating reference frame $x'y'z'$ which is parallel to the geocentric reference frame $xyz$ (see Nomenclature).

We have restricted our study to those lunar satellite orbits for which the osculating plane at $t = t_o$ is coinciding with that of the Earth's instantaneous orbit about the Moon. In addition, the satellite orbit's initial aposelene is for each case chosen in Earth's direction, while the satellite's initial orbital position is in its periselene (as viewed from the Earth, the satellite is passing at the back-side of the Moon at $t = t_o$).

The initial epoch, $t_0$, is January 9th, 1968, 0 hr, 0 min, 0 sec UT. (Julian Day 2439134.500).

The initial orbital data of Sun and Earth relative to the selenocentric reference frame $x'y'z'$ are presented in Table 3. As can be seen from this Table, both Sun and Earth are at that time in their respective periselene regions. So their perturbing influences will be at the maximum strength for the initial epoch.

Table 4 presents the orbital data of the 10 lunar satellite orbits which evolutions were traced in this study. As this Table shows, the orbits considered are split up into 5 pairs of increasing aposelene distance. Orbits belonging to the same pair, differ only in the direction of the satellite's orbital motion. For L-named orbits, the satellite is passing through the orbit in a direction corresponding to that of the Earth's motion about the Moon (i.e. in direct or counter-clockwise direction). For R-named orbits the satellite's motion is in the opposite direction (i.e. in retrograde or clockwise direction). The orbits of pair L1, R1 until L4, R4 are all of eccentricity 0.1, whereas the aposelene distance increases from $r_A = 10,000$ km for L1, R1 till $r_A = 40,000$ km for L4, R4. The orbits of the last pair L5, R5 are highly-eccentric ($e = 0.9$) and have a corresponding large value of the aposelene distance: $r_A = 100,000$ km.

To show the effects of the perturbations on the shape and size of the orbits, the evolution of the orbital parameters $a$, $e$, and $r_p$ are traced over periods varying from 50 to 400 days. In addition, for interesting cases the projection of the actual satellite orbit on the osculating plane of the initial Earth orbit about the Moon is also plotted. These projections can be regarded as highly accurate representations of the actual orbit developments as the maximum angle between the orbital planes of Earth and satellite is for all evolutions considered only of the order of $0.3^\circ$.

4.1. Results

4.1.1. The perturbations in the semi-major axis

Figures 6, 7 and 10 show the perturbations in the semi-major axis for the 8 low-eccentricity orbits from Table 4. The corresponding perturbations for the 2 highly-eccentric orbits L5 and R5 are left out of consideration as these orbits are already
hyperbolic after about 3 days.

Figure 6 shows the effect of increasing aposelene distance on the fluctuations in a for the retrograde orbits R1-R4. These fluctuations seem to have a very regular pattern, while the magnitude of the fluctuations increases progressively with increasing aposelene distance. For orbits with an initial aposelene distance less than about 20,000 km, the Earth's perturbing influence is almost negligible: \( \frac{\Delta a}{a} \approx 0.02 \). However, for an orbit with \( r_{\text{A init}} \approx 40,000 \text{ km} \), the value of \( \frac{\Delta a}{a} \) can already reach up to about 0.10. Note that no secular or long-periodic effects can be observed in the mean values of the semi-major axis. For each case, the semi-major axis tends to fluctuate around a specific mean value which stays remarkably constant throughout the considered evolution period.

Figure 7 shows the perturbations in the semi-major axis for the direct orbits L1-L4. The change in the direction of motion appears to have a significant effect on the evolution pattern for orbits where \( r_{\text{A init}} > 30,000 \text{ km} \) (L3, R3 and L4, R4). For orbits with \( r_{\text{A init}} < 20,000 \text{ km} \) (L1, R1 and L2, R2), however, the effect of the change in the direction of motion is almost negligible. An interesting development of the semi-major axis can be observed for L4 (\( r_{\text{A init}} = 40,000 \text{ km} \)). This orbit is perturbed such severely, that its shape becomes hyperbolic after already 31.2 days. Note the rather abrupt way in which the transition into the hyperbolic phase takes place, whereas during the preceding 31 day's period no clear secular or long-periodic effect can be noticed. Another interesting development of a can be observed in L3's evolution (\( r_{\text{A init}} = 30,000 \text{ km} \)). Hereby, a type of 'resonance period' occurs between \( t \approx 25 \) days and \( t \approx 50 \) days. During this period the amplitude of the fluctuations in a rises up to about twice its normal magnitude. The second 'resonance period' starts at \( t \approx 100 \) days (see Figure 10). This time, the increase in the fluctuation's amplitude is considerably larger. Finally, at \( t = 165 \) days a transition in the shape of L3 takes place from elliptic to hyperbolic. This happens - as for orbit L4 - in a rather abrupt way, whereas the fluctuations in a have been around a constant mean value throughout the entire elliptical phase.

Summarizing the perturbations in a, we note that there is no evidence of a secular change in the average value of the semi-major axis. This is a common phenomenon in space dynamics, although its generality is still an open question (Reference 5).

4.1.2. The perturbations in the eccentricity

Figures 8, 9, and 11 show the evolutions of the eccentricity for the 8 low-eccentricity orbits from Table 4. The corresponding evolutions for the 2 highly-eccentric orbits L5 and R5 are left out of consideration again.

Figures 8 and 9 show that beside a short-term perturbation, also a long-term perturbation with \( T_{2m} = 14 \) to 15 days appears in e. Further, the eccentricity appears to be more sensitive to the perturbations than the semi-major axis, as for e the difference in direction of motion manifests itself already in the developments of the lowest altitude orbits L1 and R1. Here, the long-term evolution pattern is clearly different. The difference in the long-term evolution pattern is even more apparent in the orbits R2 and L2. For orbit R2, the value of the eccentricity fluctuates around a mean value \( \bar{e} = 0.11 \), whereas this mean value is 0.06 for orbit L2. In the developments of the higher-altitude orbits R3, L3, R4, and L4 the long-term effects can no longer be observed because of the increased amplitude of the short-term perturbations. However, also for these short-term perturbations we can see large differences depending on the direction of motion of the satellite; direct orbits are perturbed significantly more severe in the eccentricity than their retrograde counterparts. For example, in the case of the direct orbit L3 the eccentricity can - during the first
"resonance period" - temporary rise up to $e = 0.63$, whereas for the 10,000 km higher, but retrograde orbit $R4$, the eccentricity does not exceed the value $e = 0.22$ during the evolution period considered. We further note the very severe perturbations that appear in $e$ for both orbits $L3$ and $L4$ during the last 30 to 50 days of their elliptical phase. During this pre-transition phase the value of the eccentricity can easily change an amount $\Delta e \approx 0.8$ within 1 to 1.5 revolution (6 to 8 days).

4.1.3. The perturbations in the periselene distance

The development of the periselene distance is connected with both that of the semi-major axis and that of the eccentricity:

$$r_p = a(1 - e)$$  \hfill (28)

Let $a$ and $e$ be perturbed by an amount $\Delta a$ and $\Delta e$, respectively. Then, for the perturbation $\Delta r_p$, we can write, neglecting second-order terms,

$$\Delta r_p = (1 - e) \Delta a - a \Delta e$$  \hfill (29)

Now, comparing the evolutions of $a$ and $e$ with each other for each of the orbits $L3$, $R3$, $L4$ and $R4$, we note that large increases, respectively decreases, in the semi-major axis are coupled with large decreases, respectively increases, in the eccentricity. So, according to Eq. (29) both perturbations $\Delta a$ and $\Delta e$ amplify each other's contribution to $\Delta r_p$, which can lead to very severe perturbations in the periselene distance.

The most interesting periselene distance evolutions occur, of course, in the direct satellite orbits. These evolutions are shown in Figure 12. The curves show a strong progressive increase of the fluctuations in $r_p$ as the orbit altitude increases. Especially for orbit $L4$, the periselene distance development is most interesting. Note the almost catastrophic decrease in the periselene distance to $r_p = 1900$ km, which occurs about 17 days before the orbit becomes hyperbolic. At this epoch, $t = 15.2$ days, the satellite's orbital position is also in the periselene. So, orbit $L4$ had nearly developed into a lunar impact orbit instead of a lunar escape orbit. In the periselene distance evolutions of orbits $L2$ and $L3$ the separate contributions of the fluctuations in $a$ and $e$ can clearly be noticed. For small to moderate perturbations, the frequency of the fluctuations in $a$ is about twice the frequency of the fluctuations in $e$, which follows directly from comparing Figures 7 and 9. Combining both fluctuations in accordance with Eq. (29) yields the typical two-topped fluctating pattern in the development of $r_p$. This is especially apparent for orbit $L2$.

For orbit $L3$ the $r_p$-fluctuations are for $t < 10$ days mainly a result of the fluctuations in $a$, as the fluctuations in $e$ are very small during this period. The two-topped fluctuating pattern starts at $t = 10$ days, though it can hardly be observed during the "resonance period". This is due to the temporary large fluctuations in $a$, which have about the same frequency as the fluctuations in $e$. After the "resonance period", however, the two-topped pattern can clearly be observed again. During the first "resonance period" the periselene distance decreases considerably. Hereby, a minimum value of $r_p = 9070$ km is reached at $t = 44$ days. During the second "resonance period" (not pictured for $r_p$) the periselene distance will even decrease to $r_p = 3840$ km ($t = 123$ days). An possible further decrease of $r_p$ during a next "resonance period" could not be traced for $L3$, as this orbit becomes hyperbolic at the end of its second "resonance period". In view of the strong decreases which can periodically appear in $r_p$. 

however, the probability seems most likely that there can exist lunar satellite orbits in
which the satellite will, in course of time, impact on the Moon, solely as a result of this
type of resonance effects.

4.1.4. Strongly-perturbed orbits

This section gives an illustration of the enormous orbit perturbations which can
result for lunar satellites in high-altitude orbits. For this purpose we have traced the
orbit developments of 3 strongly-perturbed lunar satellite orbits, namely L4, L5 and R5. The
results are given in the form of plots, showing the projection of the satellite orbit on the
plane of the Earth's (or satellite's) initial orbit about the Moon. These plots can be
regarded as very precise representations of the actual trajectories, as the osculating planes
of both orbits (satellite and Earth) remain for all cases almost coinciding over the entire
periods considered (Section 4). Tables 3 and 4 show the initial values of the orbital
parameters of satellite, Sun and Earth with respect to the selenocentric non-rotating
reference frame x'y'z'. The x^\text{x}- and y^\text{x}-axis used in Figures 13-17 represent a fixed two-
dimensional reference frame situated in the projection plane. For the specification of the x^\text{x}-axis
and y^\text{x}-axis, see Nomenclature.

The evolution of orbit L4

As was shown in the Figures 7, 9, and 12, it takes about 31 days for a satellite in
orbit L4 to reach the lunar escape velocity. In Figure 13, the trajectory L4 is depicted for
the first 30 days. During this period the shape and size of L4 appear to be subjected to
radical changes, often resulting in a complete alteration of the evolution pattern. For the
first 1.5 revolutions (t < 12 days) the satellite seems to be receding more or less
gradually form the Moon. After about 1.5 revolutions, however, a rapid increase in the
eccentricity appears, and for the following days the satellite finds itself on a path,
almost straight towards the Moon. This ends in a skimflight over the lunar surface
(h_s ≈ 160 km) at t = 15.2 days. During the first part of the 3rd revolution (15 days < t <
18 days) the orbital shape remains highly eccentric so that the satellite-Moon distance
rapidly increases. At t ≈ 18 days this distance has grown to about 55,000 km. The next 1.5
revolutions (18 days < t < 28 days) are characterized by the satellite's spiral motion
towards the Moon. The final satellite-Moon distance reached is about 3300 km (t = 28 days).
After this, within 0.5 revolution, another radical change in the development pattern takes
place, this time resulting in a change of the orbital shape from elliptic to hyperbolic.
The satellite escapes from the lunar space and enters in a high-altitude low-eccentricity
orbit about the Earth.

L4's evolution for 30 days < t < 80 days is shown in Figure 14. During this period the
satellite completes about 3 revolutions about the Earth while it stays at remote distances
from Earth and Moon. We have traced L4's development further up to 400 days. Within this
period the satellite continues its revolutions about the Earth. The Moon, however, is
reached several times at distances less than 70,000 km. The minimum satellite-Earth
distance, reached within this period is, about 73,000 km.

The evolutions of orbits L5 and R5

The final pair of lunar satellite orbits that we will discuss here are the highly-
eccentric orbits L5 and R5, for each of which $e_{\text{init}} = 0.9$ and $r_{\text{init}} = 100,000 \text{ km}$. We note that L5 has a direct motion (i.e. in accordance with the Earth's motion about the Moon), whereas orbit R5 is passed through in retrograde sense (i.e. opposite to the Earth's motion about the Moon).

The evolution of L5 is depicted in Figure 15. As this Figure shows, the satellite will leave the lunar space already at the beginning of its first orbit about the Moon. The lunar escape velocity is reached at $t \approx 3 \text{ d}$. At this epoch the satellite enters in a high-altitude low-eccentricity orbit about the Earth with $e \approx 0.1$ and $a \approx 175,000 \text{ km}$. The evolution pattern of L5 is quite similar to that of L4, shown in Figure 14.

Figure 15 shows that the satellite will reach the Moon so close at $t \approx 80 \text{ days}$ that its flight will change rather abruptly from an Earth-centered motion into an almost rectilinear flight towards the Moon. This ends at $t = 83.1 \text{ days}$ with the satellite's impact on the Moon.

Orbit R5 gives an even more interesting evolution pattern. This is shown in the Figures 16 and 17. Similarly as for L5, the transition from Moon-centered to Earth-centered motion already starts at the beginning of the satellite's first revolution about the Moon. In the case of R5, however, the transition is accompanied by a change in the orbital motion from retrograde to direct, as can be seen from the close-up of the first part of R5 ($t < 4 \text{ days}$), depicted in Figure 17. Once orbiting the Earth, the satellite passes through a low-eccentric type of orbit, for which $e \approx 0.1$ and $a \approx 320,000 \text{ km}$. The satellite's orbit about the Earth is thus quite similar to the lunar orbit about the Earth (which has an eccentricity $e = 0.05$ and a semi-major axis $a = 384,400 \text{ km}$). As the satellite's motion is initially directed outwards, this implies that it will take a considerable time for the satellite to reenter Earth-Moon space, but also that the satellite will be able to reach the Moon very close when its orbital path finally comes within Earth-Moon space. The first conclusion follows directly from the fact that the orbital periods of both the satellite orbit about the Earth and the Earth orbit about the Moon, are of the same order. This is clearly illustrated in Figure 16. We further note from Figure 16 that the satellite completes about 3 revolutions around the Earth in 73 days, when being outside the Earth-Moon space.

We may expect that the satellite's impact on the Moon will already take place the first time that the satellite's flight path comes within the Earth-Moon space. This is indeed in agreement with the actual flight, but, as Figures 16 and 17 show, the last part of R5 ($t > 80 \text{ days}$) is not simply a rectilinear motion towards the Moon. On the contrary, the satellite first passes through various capricious curves before the impact takes place. Hereby, the revolving direction of orbital motion changes several times. The first change takes place at $t \approx 90 \text{ days}$. At this epoch the orbital motion becomes retrograde again. The next change takes place at $t \approx 99 \text{ days}$. From $t \approx 99 \text{ days}$ until $t \approx 103 \text{ days}$, the satellite passes through a direct loop. Note that the Earth's position at $t = 100 \text{ days}$ is almost on the negative $x^2$-axis. Finally, from $t \approx 103 \text{ days}$ the satellite finds itself in an almost rectilinear motion towards the Moon. The impact takes place at $t = 104.6 \text{ days}$.

4.2. Summary of the results

From the foregoing Sections, some important conclusions can be summarized:

It was found that the direction of motion of high-altitude satellites about the Moon
is closely related to the extent in which the orbit will be perturbed. Direct orbits are perturbed significantly more strongly than their retrograde counterparts. Orbits at altitudes less than about 10,000 km, however, are perturbed only very little by third-body attraction, irrespective of their direction of orbital motion.

There is no evidence of a secular variation in the mean value of the semi-major axis (as long as the orbit's shape remains elliptic). The transition to lunar escape and the "pre-collision stage" at the end of the orbital evolution, seem to be more or less abrupt phenomena, which are accompanied by large fluctuations in the orbit's shape and size.

The perturbations in both the semi-major axis and the eccentricity appeared to be connected in such a way that they amplify the variations in the periselene distance. This makes the periselene distance far more sensitive to third-body perturbations than either the semi-major axis or the eccentricity.

The periselene distance can decrease considerably during "resonance periods". It seems most likely that there can exist lunar satellite orbits in which the satellite will, in course of time, impact on the Moon, solely as the result of these resonance effects.
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Table 1: Initial orbital parameters for
the Explorer 28 satellite relative to
the geocentric reference frame.
Epoch = 2438910.00486 JD.

<table>
<thead>
<tr>
<th>STATE VECTOR</th>
<th>OSCULATING ELEMENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>x (km) = 6099.584</td>
<td>a (km) = 138572.885</td>
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<td>y (km) = 602.051</td>
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<td>z (km) = 2409.161</td>
<td>ω (deg) = 135.727</td>
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<td>ṩ (km/s) = 1.104753</td>
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<td>ẏ (km/s) = 9.855613</td>
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<td>r_A (km) = 270565.065</td>
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Table 2: Initial orbital parameters of Sun and Moon relative to the geocentric reference frame.
Epoch = 2438910.00486 JD.

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<th>OSCULATING ELEMENTS</th>
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<td>M</td>
<td>x (km) = 245889.82</td>
<td>a (km) = 384402.92</td>
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<td>y (km) = 265302.91</td>
<td>e (°) = 0.055</td>
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<tr>
<td>O</td>
<td>z (km) = 99437.19</td>
<td>ω (deg) = 105.161</td>
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<td>O</td>
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<td>r_A (km) = 405506.74</td>
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<td>a (km) = 149592835</td>
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<td>e (°) = 0.017</td>
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Table 3: Initial orbital parameters of Earth and Sun used for the lunar satellite orbit study, relative to the selenocentric reference frame.

Epoch = 2439134.500 JD.

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<td>a (km) = 384402.91</td>
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<td>y' (km) = -236316.50</td>
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<td>T</td>
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<td>θ (deg) = 350.914</td>
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Table 4: Initial orbital parameters for some lunar satellites, relative to the selenocentric reference frame. Epoch = 2439134.500 JD.

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<th>ORBIT</th>
<th>STATE VECTOR</th>
<th>OSCULATING ELEMENTS</th>
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<td>( x' ) (km) = -5414.762</td>
<td>a (km) = 9090.909</td>
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<td>( \Omega ) (deg) = 10.348</td>
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Fig. 1: The osculating orbit of Explorer 28 at epoch $t_0$. 

A = apogee
F = focus
P = perigee

$e_0 = 0.953$

$t_0 = 0$ hr
$t_6 = 140$ hr

$t_1 = 24$ hr
$t_2 = 48$ hr
$t_3 = 72$ hr
$t_4 = 96$ hr
$t_5 = 120$ hr

$a_o = 138573$ km
$b_o = 42196$ km

Earth + atmosphere ($h_{atm} = 1000$ km)
Fig. 2: The J2 perturbations in the osculating elements of Explorer 28 during the first perigee passage.
Fig. 3: The magnitude of the errors in the predicted argument of perigee for Explorer 28

Fig. 4: The magnitude of the errors in the predicted right ascension of the ascending node for Explorer 28
Fig. 6: The perturbation in the semi-major axis for orbits R1–R4.
Fig. 7: The perturbation in the semi-major axis for orbits L1 - L4.
Fig. 8: The perturbation in the eccentricity for orbits R1 - R4.
Fig. 9: The perturbation in the eccentricity for orbits L1-L4.
Fig. 10: The perturbation in the semi-major axis for orbit L3 ($t>90$ days)

Fig. 11: The perturbation in the eccentricity for orbit L3 ($t>90$ days)
Fig. 12: The perturbations in the periselene distance for orbits L1-L4
Fig. 13: The evolution of orbit L4 (t < 30 days)
Fig. 14: The evolution of orbit L4 (30 days < t < 80 days)
Fig. 15: The evolution of orbit L5

$r_{A_{\text{init}}} = 100,000 \text{ km}$

$e_{\text{init}} = 0.9$

Earth's position at $t = n$ days

Satellite's position at $t = n$ days
Fig.16: The evolution of orbit R5

\[ r_{A_{\text{init}}} = 100,000 \text{ km} \]
\[ e_{\text{init}} = 0.9 \]
Fig. 17: Close-up of first and last part of orbit R5