TWO- AND THREE-DIMENSIONAL SIMULATION OF
A RISING BUBBLE AND FALLING DROPLET
USING LEVEL SET METHOD

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Key words: Level Set Method, Rising Bubble, Droplet, Runge-Kutta Scheme, WENO

Abstract. The numerical simulation of a rising bubble in liquid and falling droplet is presented. The two- and three-dimensional incompressible Navier-Stokes equations are used to solve the gas-liquid two phases flow system with free surfaces. A sharp interface separates fluids of different density and viscosity. The interface between gas and liquid is described as the zero level set of a smooth function. In order to maintain the level set function as a smooth signed distance function from the interface, a reinitialization operation is required. We also apply a constraint to improve the conservation of mass significantly. A WENO scheme for space derivatives and third-order Runge-Kutta scheme for the time discretization are used to solve the level set equation. The velocity field can be obtained by second-order explicit Adams-Bashforth scheme, while the pressure is solved. The staggered grid system is used in the numerical model. The model allows us to simulate a wide range of flow regimes and surface tension effect can be consideration. The effect of deformation and motion of a riging bubble is investigated by changing the Reynolds numbers and Weber numbers. A droplet falls into a tank of a quiescent fluid is also simulated. Complex numerical simulations show the capability of our numerical model. Our model can be used to simulate a number of problems and to investigate many physical phenomena involving multi-fluid flows that consider the free surfaces.

1 INTRODUCTION

In this work, we present a incompressible two-phase flow models. The purposes are to simulate high density and viscosity ratio flows with complex interface changes, such as flows with constantaneous occurrence of bubbles and droplets. The motion of bubbles and droplets may be very complex due to high density and viscosity ratios as well as topology changes.
Computational fluid dynamic (CFD) is a powerful tool can simulate the details of bubble and droplet motion. There are many numerical methods and experiments have been studied. Ryskin and Leal developed a numerical method to compute the steady motion of a bubble rising in the liquid. The numerical result was compared with the experimental data by Hnat and Buckmaster show excellent. Bhaga and Weber determined the shape and terminal velocities of bubbles rising in viscous liquids by experiment. Most popular numerical methods for interface tracking are Volume-of-Fluid (VOF) method and Level Set method. VOF method is proposed by Hirt and Nichols. They were introduced a VOF function as the volume fraction in each cell to represent the interface. The advantage of this method is preserved mass conservation, but the disadvantage is complicated geometric calculations for the interface reconstruction. Chen and Li were using a modified VOF method to simulate two-phase flows with a varying density. In the Level Set method, is devised by Osher and Sethian, is using a smooth level set function $\phi$ to track the interface. The interface can be found when $\phi = 0$. This method allows computing for two-phase flows with high density and viscosity ratios. One of the advantages of this method is that we can easily track and present the interface. But, this method has a disadvantage that mass is not really conservation. Sussman and Smereka were studied three-dimensional axisymmetric free boundary problems using level set method including a gas bubble rising, burst at a gas-liquid interface, and drop impact on a pool of water. The computational results were compared with experimental data and literatures show good agreement. Recently, Sussman and Puckett were presented a coupled level set/volume-of-fluid (CLSVOF) method to combine the advantages and improve the disadvantages in VOF method and Level Set method. Ohta et al. were used CLSVOF method to simulate three-dimensional numerical motion of a gas bubble rising in viscous liquids. Liao and Chen were used the level set method to simulate the motion of a rising bubble in liquid in Cartesian coordinate system.

In this paper, we combine level set method and two- and three-dimensional incompressible Navier-Stokes equations to simulate the motion of a rising bubble and a falling droplet. Surface tension is also considered. The effect of deformation and motion of a riging bubble is investigated by changing the Reynolds numbers and Weber numbers. We were also simulated a droplet falls into a tank of a quiescent fluid of the same phase. Complex numerical simulations show the capability of our method.

2 GOVERNING EQUATIONS

2.1 Level Set Method

The level set method for moving interfaces was used in Osher and Sethian. An application of the level set formulation to incompressible two-phase flow was used in Sussman and Smereka. The original ideas of level set method is to define a smooth function $\phi(x,t)$ that represents the interface at $\phi(x,t) = 0$. The interface can be captured at any time by locating
the zero level set, $\phi < 0$ in the gas phase and $\phi > 0$ in the liquid phase. Since the interface moves with the fluid particles the evolution of $\phi$ in flow field is given by

$$\phi + \dot{U} \cdot \nabla \phi = 0$$  \hspace{1cm} (1)

where $\dot{U}$ is the velocity of fluid. The numerical oscillation in the interface may occur due to large density ratio of gas and liquid phase. In order to avoid this numerical instability, the density and viscosity of fluids are replaced by

$$\rho(\phi) = H(\phi) + \left( \frac{\rho_g}{\rho_l} \right) (1 - H(\phi))$$  \hspace{1cm} (2)

$$\mu(\phi) = H(\phi) + \left( \frac{\mu_g}{\mu_l} \right) (1 - H(\phi))$$  \hspace{1cm} (3)

The subscript $g$ and $l$ represent in gas and in liquid, respectively and $H(\phi)$ is the smooth Heaviside function given by

$$H(\phi) = \begin{cases} 0 & \text{if } \phi < -\varepsilon \\ \frac{1}{2} \left( 1 + \frac{\phi}{\varepsilon} + \frac{1}{\pi} \sin \left( \frac{\pi\phi}{\varepsilon} \right) \right) & \text{if } |\phi| \leq \varepsilon \\ 1 & \text{if } \phi > \varepsilon \end{cases}$$  \hspace{1cm} (4)

The physical concept is that the zero thickness interface expands to 2$\varepsilon$ width. In our numerical model, we take $\varepsilon = \alpha \Delta h$, where $\Delta h$ is the minimum grid size and $\alpha$ is a parameter of thickness of the interface, it usually takes 1.5~3.0. We use $\alpha=1.0$ in our model.

### 2.2 Governing Equations for Fluids

Using the Level Set Method to solve two-phase flow problems, the properties change across the gas-liquid interface. In Eulerian Coordinate System, for incompressible Newtonian fluids with constant of density and viscosity both in gas and liquid, the equations of motion are given by the incompressible Navier-Stokes equations

$$\nabla \cdot \dot{U} = 0$$  \hspace{1cm} (5)

$$\frac{\nabla \cdot \dot{U}}{\rho(\phi)} + \nabla \cdot \nabla p + \nabla \cdot \frac{\nabla \phi}{\rho(\phi)} = \frac{\nabla \cdot (2\mu(\phi)D)}{\rho(\phi)} - \frac{\sigma \kappa(\phi) \nabla \phi \delta(\phi)}{\rho(\phi)} - \vec{F}$$  \hspace{1cm} (6)

where $U = (u, v, w)$ is the velocity; $p$ is the pressure; $\phi$ is the level set function; $\rho$ and $\mu$ is the density and the viscosity of the fluid, respectively, which is the function of $\phi$ and can be obtained from equation (2) and (3); $\sigma$ is the surface tension coefficient; $D = \frac{1}{2} \left( (\nabla U) + (\nabla U)^T \right)$ is the rate of deformation tensor; $\kappa(\phi) = \nabla \cdot \left( \frac{\nabla \phi}{|\nabla \phi|} \right)$ is the curvature of interface; $\delta(\phi)$ is the Dirac delta function that is defined as
\[
\delta(\phi) = \frac{dH(\phi)}{d\phi} = \begin{cases} 
\frac{1}{\varepsilon} \left(1 + \cos \left(\frac{\pi \phi}{\varepsilon}\right)\right) & \text{if } |\phi| < \varepsilon \\
0 & \text{otherwise}
\end{cases}
\]

\[
F = (0, 0, g) \text{ is the body force, } g \text{ is the gravity.}
\]

2.3 Initialization of Level Set Function

When the initial interface for a fluid problem is much too complicated to hard to define the initial value of level set function. A simple modification method was proposed in Sussman and Fatemi\textsuperscript{11}, called a first-time only redistance step to obtain the initial level set function. We applied the method to decide the initial value of \( \phi \). This method is executed only once for the duration of the entire computed process. First, we assume \( \phi = +1 \) in the liquid region and \( \phi = -1 \) in the gas region. Then we solve equation (7), and compute to \( t = L \), where \( L \) is the length of computational domain. We can get the initial value of level set function of flow problem.

2.4 Re-initialization of Level Set Function

While \( \phi \) is initialized as a distance function, but it doesn’t ensure \( \phi \) as a distance function as time proceeds by equation (1). Thus, we need a re-initialization process to ensure the condition of \( |\nabla \phi| = 1 \). We use an iterative procedure proposed by Sussman and Fatemi to re-initialize the level set function at each time step to maintain the level set function as a distance function. The procedure is solved the hyperbolic equation to steady state as follows

\[
\frac{\partial d}{\partial \tau} = \text{sign}(\phi_0)(1 - |\nabla d|), \quad d(x, 0) = \phi_0(x)
\]

where \( d \) is the normal distance from \( x \) to the interface, \( \tau \) is the artificial time, \( \text{sign}(\phi_0) \) is the smoothed sign function defined as

\[
\text{sign}(\phi_0) = 2 \left(H(\phi) - 0.5\right)
\]

The steady state solution of this problem is the signed distance function from the boundary of \( \phi_0 = 0 \). That is

\[
|\nabla d| = 1 \quad \text{for } |d| \leq \varepsilon
\]

2.5 Mass Conservation

In equation (7), because \( \text{sign}(0) = 0 \) theoretically, the free surface captured by the zero level set doesn’t change the position during the re-initialization procedure. But this is not ensured in numerical computation. When doing re-initialization procedure, it may be induced mass error. In order to maintain the mass conservation, the equation (7) is modified as

\[
\frac{\partial d}{\partial \tau} = \text{sign}(\phi_0) \left(1 - |\nabla d|\right) + \lambda f(\phi) \equiv L(\phi_0, d) + \lambda f(\phi)
\]
where \( \lambda = \frac{\int_\Omega H'(\phi) L(\phi, d)}{\int_\Omega H'(\phi) f(\phi)} \) and \( f(\phi) \equiv H'(\phi)|\nabla \phi| \).

This correct factor makes the liquid mass conservation Sussman and Fatemi.

### 2.6 Numerical Procedures

The outlines of our numerical model are as follows:

1. Set initial condition and boundary conditions of problem.
2. Use WENO method and Runge-Kutta method to solve \( \phi^{n+1} \) from equation (1).
3. Compute re-initialization and mass conservation by solving equation (10).
4. Solve \( P^{n+1} \) from the pressure Poisson equation.
5. Compute velocity \( u^{n+1}, v^{n+1}, w^{n+1} \) by second order Adams-Bashforth method from equation (6).
6. Repeat step 2 to step 5 until the desired time.

### 3 COMPUTATIONAL EXAMPLES

#### 3.1 Two Dimensional Bubble

The computed conditions of bubble A depict in table 1. Other paramets are \( \rho_g/\rho_l = 0.0011, \mu_g/\mu_l = 0.0085 \), and the dimensionless quantities are \( \text{Re} = 9.8, \text{Fr} = 0.6227 \), and \( \text{We} = 7.6 \). The fluid domain is \( 3 \times 12 \) and the grid size is \( 64 \times 256 \). Results are shown in Figure 1. We observe that the shapes of the bubble are similar to Sussman and Smereka. The dimensionless rise speed should be 1. But, in this case, the dimensionless rise speed just have 0.5–0.6, since we are not using the axisymmetric coordinate system. For this reason, we extend our model to truly three-dimensional formulaitons.

#### 3.2 Three Dimensional Simulation of Bubble And Droplet

##### 3.2.1 Steady-state Results

We test the steady-state cases proposed by Hnat and Buckmaster experimentally with rising spherical cap air bubbles. The numerical calculation also has been proposed by Sussman and Smereka. In our computation, the domain is \( 3D \times 3D \times 12D \) (D is the diameter of the bubble) and the grid size is \( 32 \times 32 \times 128 \). In the initial setting, a spherical bubble was imposed at the bottom of the domain and velocity was zero at \( t = 0 \). We use closed computational domain and non-slip boundary condition on all walls. We take bubble A and B from Table 1 of Hnat and Buckmaster as our numerical simulations. The properties of the gas and the liquid are \( \rho_g = 0.001 \text{g/cm}^3, \rho_l = 0.8755 \text{g/cm}^3, \mu_g = 0.01 \text{P}, \mu_l = 1.18 \text{P} \). The coefficient of surface tension of the liquid is \( \sigma = 32.2 \text{dyn/cm} \). The details of the properties of bubbles are shown in Table 1.
The volume of bubble is $V$. They were observed that $V$ is steady rising speed. So the effective diameter of the bubble is $D = (6V/p)^{1/3}$. The dimensional parameters can also know from Table 1. In numerical computation, the dimensionless rise speed should be 1. In our computational results, we find that the bubble A and B reach a steady state with final speed of 1.03-1.06. Figure 3 displays the time evolution of the bubble A. Figure 4 shows the time history of computed rise speed of bubble A and B. The numerical results are also shown in good agreement with experimental results of Hnat and Buckmaster.

### 3.2.2 Bubble Shapes

In this section, we use the experimental data of Bhaga and Weber. The properties of the liquid are $\mu = 0.82 - 28.0 \text{ Pa}$, $\rho = 1.314 \text{ g/cm}^3$, $\sigma = 76.9 \text{ dyn/cm}$. Computational domain and grid size are same as last section. The bubble volume and rise speed were used bubble A from Table 1. Table 2 is the details of computed cases. Figure 5 shows shape regime map for bubbles in liquids presented by Bhaga and Weber. The dimensionless parameters of Morton (Mo), Eotvos (Eo), and Reynolds (Re) number are given by (see Bhaga and Weber)

$$Mo = \frac{g\mu^4}{\rho \sigma^3}, \quad Eo = \frac{gD^3 \rho}{\sigma}$$

In numerical simulations, we used a fixed density ratio and compared the effect of viscosity on bubble shape. When we fixed density ratio, the Eo also must be fixed. So there just compare the effect of Mo. From Table 2, at low Re (high Mo), the bubble is nearly an oblate sphericity. Figure 6 shows the final steady-state bubble shape. We compare the shapes of the numerical results with the diagram of Grace. This comparison enables us to conclude that our numerical method is in a good agreement with experimental predictions.

### 3.2.3 Droplet Impact

In this case, a droplet falls into a tank of a quiescent fluid of the same phase. This is a good example to show the changes of the free surface. The nondimensional parameters were given by: $Re = 30.0$, $Fr = 1.0$, $We = 50.0$. And we set $\rho_s/\rho_l = 0.5$, and $\mu_s/\mu_l = 0.5$. The fluid domain is $3 \times 3 \times 12$ and grid size is $32 \times 32 \times 64$. The droplet located at $(0, 0, 5.0)$ is stationary when $t = 0$. Figure 6 shows the changes of the free surface at different time. In this figure, we are also able to observe the surface tension effects acting over the free surfaces. In the further work, we can simulate the impact of the droplet on different planes, and investigate more information about droplet impact process and splashing.

### 4 CONCLUSIONS

We presented the Level Set Method and two- and three-dimensional incompressible Navier-Stokes equations for simulating the motion of the rising bubble and the falling droplet. Our computational results compare with experimental data and other numerical models show a good agreement. We prove the ability to simulate the flow including the complex free surface problems. In the future, we would like to improve our numerical method to make
computations easily. And we will study two-phase flows with immersed boundary.

ACKNOWLEDGEMENT

This study was supported by the National Science Council of the Republic of China, under grant number NSC-93-2211-E-035-005, it is gratefully appreciated.

REFERENCES


Table 1: The properties of bubbles shown in Figure 1 and Figure 2.

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<th>Bubble</th>
<th>$V$(ml)</th>
<th>$\rho$(g/ml)</th>
<th>$\mu$(P)</th>
<th>$\sigma$(dyn/cm)</th>
<th>$V_s$(cm/s)</th>
<th>$D$(cm)</th>
<th>Fr</th>
<th>We</th>
<th>Re</th>
<th>Eo</th>
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<td>B</td>
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<td>0.8755</td>
<td>1.18</td>
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Table 2: The properties of computed bubbles in section 3.2.2.

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<th>$\mu$(P)</th>
<th>Fr</th>
<th>We</th>
<th>Re</th>
<th>Eo</th>
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Figure 1: Two dimensional simulation of a rising bubble with $Re = 9.8$, $Fr = 0.6227$, $We = 7.6$. 
Figure 1: (Continue)

Figure 2: (a) Simulated positions and shape variations of a rising bubble A in gas-liquid flow and time increment is 1.0;(b)-(e) cross sectional velocity distribution of the bubble A at four time instants shown in (a).
Figure 3: The time history of the rise speed of the top of the bubble A and B.

Figure 4: Shape regime map for bubbles in liquids by Bhaga and Weber, the dot is case of this study.
Figure 5: Simulated final shape of six rising bubbles in gas-liquid flow from Table 2.

Figure 6: Evolution of a falling droplet impact the free surface.
Figure 6: (Continue)