A real time Michelson interferometer for quantitative refractive index profile measurements

Design, validation and utilization
1. Een fasegestapte real time interferometer is een zeer geschikt instrument om de actuele fasetap in een tijdverschoven fasegestapte interferometer te bepalen, zelfs als de real-time interferometer niet optimaal is uitgelijnd (toegepast in hoofdstuk 4 van dit proefschrift).

2. De benodigde lineariteit van het piezo-element in een tijdverschoven fasegestapte interferometer is aanzienlijk lager indien tegelijkertijd met ieder interferogram ook een over $\pi/2$ radiaal verschoven interferogram wordt vastgelegd.

3. De risico’s van wetenschappelijk onderzoek op een universiteit dienen te worden gedragen door de universiteit en niet persoonlijk door de uitvoerende promovendus. Derhalve is een tijdelijk contract ongeschikt voor een promovendus op een praktisch onderzoek waarvan de resultaten pas in de eindfase mogen worden verwacht.

4. De strenge kledingvoorschriften bij promotieplechtigheden voor met name de promovendus suggereren ten onrechte dat ceremonie minstens even belangrijk is als wetenschappelijke inhoud, ze dienen daarom te worden afgeschaft.

5. Zonder sturing zal het kapitalistische systeem als gevolg van een concentratie van de macht zeker ten onder gaan.

6. Het is effectiever voor de verkeersveiligheid om snelheidsovertredingen onmiddellijk te bestraffen met een tijdelijke wielklem dan met een geldboete zes weken na dato.

7. Reductie van het aantal verkeersslachtoffers tot nul betekent reductie van de reissnelheid tot nul.

8. Men koopt geen produkt maar een produkt met een geschiedenis. De bijbehorende verantwoordelijkheid wordt te vaak vergeten.

Deze stellingen worden verdedigbaar geacht en zijn als zodanig goedgekeurd door de promotor prof.dr.ir. P.G. Bakker.
Propositions belonging to the dissertation entitled

**A real time Michelson interferometer for quantitative refractive index profile measurements**

**Design, validation and utilization**

by Juup Nijholt

1. A phase stepping real time interferometer is a very suitable tool to determine the actual phase step (shift) in a temporal phase stepping (shifting) interferometer, even if the real time interferometer is not aligned perfectly (applied in Chapter 4 of this thesis).

2. The required linearity of the piezo-electric transducer in a temporal phase stepping (shifting) interferometer is reduced substantially if simultaneously to each interferogram also an interferogram is recorded which is shifted over $\pi/2$ radians.

3. The risks of scientific research at an university should be taken by the university and not by the performing promovendus. Therefore, a temporary labour contract is unsuitable for a promovendus doing a practical research of which the results are expected to appear at the final stage.

4. The stringent dressing instructions at Ph.D. defence ceremonies for especially the promovendus suggest wrongly that ceremony is at least as important as scientific knowledge, hence they should be given up.

5. Without management the capitalist system will go down due to a concentration of power.

6. The road safety is served more effectively if exceeding of the speed limit is punished by a temporary but immediately fixed wheel lock than by a fine six weeks later.

7. Reducing the number of traffic victims to zero means reducing the speed limit to zero.

8. When buying a product one also buys the past of a product. The responsibility that comes with purchasing this past is forgotten too often.

These propositions are considered defendable and as such have been approved by the supervisor prof.dr.ir. P.G. Bakker.
A real time Michelson interferometer for quantitative refractive index profile measurements

Design, validation and utilization

Juup Nijholt
A real time Michelson interferometer for quantitative refractive index profile measurements

Design, validation and utilization

Proefschrift

ter verkrijging van de graad van doctor
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Summary

A real time Michelson interferometer for quantitative refractive index profile measurements
Design, validation and utilization

This thesis describes the design, the validation and the utilization of a real time Michelson interferometer for quantitative measurements to 2D refractive index profiles. The three channel Michelson interferometer is usable in different fields of application, but the eventual application in the future is measuring density profiles in compressible flows in a wind tunnel. The study is a feasibility study with special attention to the attainable measuring accuracy for the case that the test object may assumed to be a (locally) linearly stratified medium bounded by windows. The time resolution is kept low (video rate). The utility of the interferometer is shown in another field of application than compressible flows: the jet induced homogenization after a jet of a dilute sucrose solution is injected into an aqueous bulk fluid in a small cuvette.

In Chapter 1 existing optical techniques used for measuring flow are discussed. The principle of flow measurement divides them in two groups: techniques that detect moving particles or molecules in the flow and techniques that measure the density in the flow. Interferometry belongs to the second group. Common interferometric set-ups like a one channel Mach-Zehnder interferometer, a one channel Michelson interferometer and a holographic interferometer can all be used for repetitive measurements. However, the
number of recorded interferograms per measurement is limited to one and thus their measuring accuracy is lower than that of a phase shifting real time interferometer that records more interferograms per measurement.

Chapter 2 treats the minimization of the refraction effects in a Michelson interferometer by imaging. The refraction of light rays in a non-homogeneous refractive index profile leads to phase errors in the interferogram. These phase errors can be minimized by imaging a convenient (virtual) object plane in the test object on the interferogram in the recording plane. The position of the virtual object plane to keep the phase error zero appears to be a weak function of the refractive index gradient if the medium is a (locally) linearly stratified medium. Hence, the phase error can never be exactly zero over a whole gradient interval where an interferometer is designed for. However, the maximum refractive index gradient is generally small enough to neglect the dependence of the position of the virtual object plane on the refractive index gradient so that the phase error in the interferogram is determined by the attained accuracy with which the correct (virtual) object plane is imaged. The allowed position of the virtual object plane is calculated as a function of the maximum refractive index gradient and the allowed phase error for two test object configurations: a wind tunnel configuration and a cuvette configuration. A distance of 1mm between the positions of the virtual object plane and the optimum virtual object plane leads to a maximum phase error of 1.0% of $2\pi$ radians for a dimensionless refractive index gradient up to 0.0018. The relative phase error induced by a Michelson interferometer is twice the relative phase error induced by a Mach-Zehnder interferometer. The distance over which a refractive index gradient must be approximately constant so that a medium may be assumed locally linearly stratified is at least four times the distance in a Mach-Zehnder interferometer.

Chapter 3 presents the design of the real time interferometer (RTI). There is started with the choice of the phase calculation method. Several methods are discussed. Finally, a phase stepping method is preferred that needs three interferograms with a mutual phase step of $\pi/2$ rad. This method is robust, can deal with the typical interferograms obtained in compressible flow and the complexity of a three channel set-up is not extremely high. Next, the experimental set-up is presented. Apart from the RTI, also a temporal phase shifting interferometer (TPSI) is built in in the set-up to validate the RTI. The alignment procedure and the fringe analysis process for the two interferometers are given. Next, an extended accuracy analysis is given for the RTI and a brief one for the TPSI. Finally, an optimum rms phase error is calculated for the RTI and the TPSI, i.e., a rms phase error introduced by quantization and noise, non-linear camera response, misalignment of the cameras, and phase step errors. This optimum rms phase error lies between 1.5% and 3.8% of $2\pi$ radians for the RTI and is about 0.5% of $2\pi$ radians for the TPSI.

In Chapter 4 the RTI is experimentally validated by performing some 'tilted mirror' experiments. These tilted mirror experiments are more or less simulations of
measurements to a linear stratified medium. To find the measuring accuracy of the RTI, the phase distributions obtained by the RTI are compared to those obtained by the TPSI and to fitted surfaces. The rms phase error and the maximum absolute phase error induced by the RTI are determined as a function of the phase gradient during the actual measurement, provided that the phases during the actual and the reference measurement are homogeneously distributed over [0,2π) and there is only a small gradient during the reference measurement. The most important error sources are determined. The rms phase error induced by the RTI is at most about 2.8%-4.3% of 2π radians larger than that induced by the TPSI. The maximum phase difference is about 12.1% of 2π radians. The total phase error introduced by the RTI depends on the object configuration. Provided that the medium is locally linearly stratified and the phase gradient at the position of the cameras is less than 58.10^{-3}rad/m, the rms phase error varies from about 4.6% to 13.1% of 2π radians for a wind tunnel configuration and from about 4.6% to 9.7% of 2π radians for a cuvette configuration. The maximum absolute phase error varies from about 11.0% to 27.5% of 2π radians and from about 11.0% to 22.3% of 2π radians. The experimental found rms phase error agrees with the theoretically expected rms phase error, provided that a deformation is removed from the experimentally obtained phase distributions.

In Chapter 5 the RTI is used to study the homogeneity of mixtures in immuno assays. The jet induced homogenization is measured as a function of the time after a jet of a dilute sucrose solution is injected into an aqueous bulk in a small cuvette. The fractional standard deviation is used as a measure for the large-scale homogeneity, the fractional rms concentration gradient is used to detect the presence of local concentration gradients. The measurements show that the mixtures become homogeneous within 18s after the jet injection if the rate of the jet volume and the bulk volume exceeds 0.67. Local gradients are certainly present until 78s after the jet injection.

Preceding to the experiments an extended accuracy analysis of the concentration measurements is presented. After that, the reliability of the RTI for concentration measurements is shown by measuring the diffusion process of sucrose in water and comparing the results to theoretical results based on Fick’s second law in one dimension. A reasonable value for the diffusion coefficient is extracted from the results even though the initial condition of the diffusion process is not well-defined. The fractional standard deviation is shown to be a good measure for large-scale homogeneity and the rms concentration gradient is shown to be a convenient measure to detect large local concentration gradients.

A summary of the conclusions and some suggestions for a further development of the real time interferometer are presented in the final Chapter 6.
## Contents

Summary ........................................... v

Contents ........................................... ix

1. Optical measurement techniques for refractive index profiles and flow ........................................ 1
   1.1 Introduction ................................... 1
   1.2 Techniques to detect particles or molecules ................................... 3
   1.3 Techniques to detect refractive index profiles ................................... 7
   1.4 Aim and outline ................................... 14
   References ........................................... 15

2. Minimization of refraction effects in a Michelson interferometer by imaging ................................... 21
   2.1 Introduction ................................... 21
   2.2 The ray trajectory ................................... 25
   2.3 The phase error ................................... 29
   2.4 Application ................................... 36
   2.5 The Mach-Zehnder interferometer ................................... 42
   2.6 Discussion and conclusions ................................... 46
   References ........................................... 51
3. Design of the real time interferometer and accuracy analysis ........................................... 53
   3.1 Introduction .................................................................................................................. 53
   3.2 Choice of the phase calculation method ...................................................................... 54
       3.2.1 A review of the phase calculation methods ....................................................... 54
       3.2.2 Choice of the phase calculation method ............................................................. 59
   3.3 The experimental set-up ............................................................................................... 61
       3.3.1 Description of the set-up ...................................................................................... 61
       3.3.2 Alignment of the set-up ...................................................................................... 63
   3.4 Fringe analysis ............................................................................................................. 66
       3.4.1 The real time interferometer .............................................................................. 66
       3.4.2 The temporal phase shifting interferometer ....................................................... 70
       3.4.3 Phase unwrapping ............................................................................................... 72
   3.5 Phase errors due to several error sources ................................................................... 73
       3.5.1 Phase errors due to a limited imaging accuracy .................................................. 73
       3.5.2 Phase errors due to quantization and noise ......................................................... 74
       3.5.3 Phase errors due to a nonlinear camera response .............................................. 75
       3.5.4 Phase errors due to the finite pixel surfaces ....................................................... 77
       3.5.5 Phase errors due to vibrations and air turbulence ............................................ 79
   3.6 Phase errors due to misalignment of the cameras ......................................................... 81
   3.7 Phase errors due to the optical the optical elements .................................................... 94
   3.8 Phase errors due to reflections ..................................................................................... 104
   3.9 Phase errors due to a wavefront aberration ............................................................... 113
   3.10 Phase errors in the TPSI ........................................................................................... 117
   3.11 Discussion and conclusions ....................................................................................... 123
   References ....................................................................................................................... 127

4. Experimental validation .................................................................................................... 133
   4.1 Introduction .................................................................................................................. 133
   4.2 Validation of the measuring accuracy ........................................................................ 135
       4.2.1 Theory ................................................................................................................ 136
       4.2.2 Relative accuracy .............................................................................................. 139
       4.2.3 Total accuracy ................................................................................................... 142
       4.2.4 Effects of filtering ............................................................................................. 145
       4.2.5 Discussion ......................................................................................................... 146
   4.3 Validation of the stability ............................................................................................. 152
   4.4 Application of the results ............................................................................................ 158
   4.5 Conclusions ................................................................................................................. 164
   References ....................................................................................................................... 167
5. Jet injection into liquid in a cuvette: real time detection and quantization of the concentration homogeneity ........................................ 169
5.1 Introduction .............................................................................. 169
5.2 Relevance of homogenization in immuno assays ...................... 172
5.3 Interferometric measurements and analysis ......................... 173
  5.3.1 Interferometric concentration measurements ......................... 173
  5.3.2 Homogeneity analysis ......................................................... 174
  5.3.3 Calculation of the diffusion coefficient ............................... 176
5.4 Materials and processing ...................................................... 177
  5.4.1 Materials ........................................................................... 177
  5.4.2 Processing ......................................................................... 178
5.5 Accuracy of the concentration measurements ..................... 179
  5.5.1 Qualitative consideration .................................................... 179
  5.5.2 Quantitative consideration ................................................ 181
  5.5.3 Consequences for the measurements ............................... 185
5.6 Validation of the application .................................................. 187
  5.6.1 Concentration measurements .......................................... 188
  5.6.2 Determination of the diffusion coefficient ......................... 189
  5.6.3 Comparison to 1-D diffusion theory ................................. 191
  5.6.4 Homogeneity ..................................................................... 193
  5.6.5 Conclusions ...................................................................... 194
5.7 Homogeneity after jet injection ............................................. 195
  5.7.1 The experiments ................................................................. 195
  5.7.2 Results and conclusions .................................................... 196
5.8 Discussion .............................................................................. 199
Figures ....................................................................................... 201
References .................................................................................. 215

6. Conclusions and suggestions .................................................. 219
  6.1 Conclusions .......................................................................... 219
  6.2 Suggestions .......................................................................... 221

Appendix ....................................................................................... 225
Samenvatting ............................................................................... 231
Nawoord ...................................................................................... 235
Curriculum vitae .......................................................................... 237
1.1 Introduction

Flow is often accompanied by a variation in the fluid density, for example, in the fields of heat transfer, mixing of two fluids of different densities, combustion and compressible gas flow. Hence, researchers are interested in measuring density fields. Since variations in the fluid density implies a variation in the refractive index, interferometry is a convenient tool for measuring such spatially extended density fields and, therefore, this optical technique is often applied. Especially holographic interferometry has proven to be a good quantitative measuring tool. Because it is robust and it needs only one recording per measurement which can be realized momentary in time, it is suitable for measuring unsteady flows. One of the main disadvantages of holographic interferometry is, however, that it is unsuitable for repetitive measurements so that the time dependence of an unsteady flow cannot be studied. This is the reason why a new interferometer has been developed for real-time quantitative refractive index profile measurements.

Experimental aerodynamic flow research has the objective to provide insight into the properties of air in motion, i.e., pressure, density and velocity. Traditionally, one of these properties is measured by placing probes in the air flow, for example, Pitot tubes to measure pressure and hot wires to measure velocity. However, there are three main disadvantages attached to these measuring techniques, some of which can be avoided by
the application of a convenient optical flow visualization technique like interferometry. Firstly, the probe disturbs the original flow, so the measured properties are not the desired properties of the original flow but the properties of the disturbed flow. Secondly, due to the finite size of the probes, they are less suitable for measuring high spatial gradients. Thirdly, the measuring techniques deliver information about one position in the flow only. To gather data on the complete flow field, the probe has to be traversed through the field. Apart from the fact that it is practically almost impossible to move the probe through the whole field, the data are gathered sequentially in time, which means that only statistical properties of the flow fields can be measured and virtually no information can be obtained on the instantaneous structure (coherence) of the unsteady flow fields.

Several optical techniques can be used for flow visualization. The principle by which the flow is visualized places them in two groups. The first group comprises techniques characterized by the detection of (moving) particles or molecules in the flow. These particles or molecules may originally be present in the flow or they may be seeded in the flow. Although the assumption that the movement of the (seeded) particles is equal to the movement of the fluid is not exactly true, the techniques give good results in stationary flows. However, in compressible flow the accuracy decreases. Techniques that rely on molecule detection may thus be preferred. Some of these techniques are discussed in section 1.2.

The second group includes techniques which are based on the detection of the refractive index profile in the flow field. The quantity of interest, i.e., the density profile, can easily be calculated from this refractive index profile. The required condition for applying these techniques in aerodynamic research is the existence of a density profile in the flow, i.e., the flow must be in the compressible regime. The effect of the compressibility of air increases with the flow velocity. Practically, an air flow can be treated as compressible at velocities larger than Mach 0.3–0.4. This means that, in general, the flow visualization in this group is inconvenient for low speed air flows. Compensationally, however, these techniques are not limited to flow visualization, but they can be used for general refractive index profile visualization and thus applied in other fields of research. Some techniques are discussed in section 1.3.

The aim of this thesis is the study of the development of an interferometer for real-time quantitative refractive index profile measurements. Although the interferometer may be used in many applications, the intended application in the future is experimental aerodynamic flow research. Then, it introduces a method to the second group of flow visualization techniques. The method is non-intrusive, it covers an spatially extended flow field and it measures refractive index profiles quantitatively and real-time. It is a convenient tool for visualization of two dimensional compressible flows, while it does not have the disadvantages of many of the more traditional flow measurement techniques.
1.2 Techniques to detect particles or molecules

Several optical techniques for quantitative flow visualization are based on the detection of particles or molecules in the flow, their motion or their number density in order to infer the local flow velocity or density. They make use of light scattering by the particles or molecules, which may be elastic as well as inelastic. The particles or molecules may naturally present in the flow or otherwise they may be seeded. A required condition for a proper flow measurement is that the particles follow the flow with a high degree of confidence. This condition may be problematic for transonic flows, supersonic flows and highly turbulent flows where the velocity changes are rapid and the particles may lag the fluid motion. Therefore, in these cases techniques which uses gas molecules as scatterers are more convenient, although the signal is much lower.

Below, some of the most common optical techniques are discussed. All these techniques can be applied as a laser-sheet technique. Some associated techniques can be found in literature\textsuperscript{5,6}. A laser-sheet technique illuminates a plane cross section of the flow by a light sheet, and the light scattered by present particles is detected by the CCD-camera in the direction perpendicular to the sheet. In contrary to point measurement techniques, the resulting image provides 2D information about the flow in the illuminated cross section.

*Laser Doppler velocimetry* (LDV), or *laser Doppler anemometry* (LDA), measures the velocity of moving particles. For this, the Doppler effect is employed which appears when light interacts with moving particles. As soon as monochromatic laser light is scattered by a particle in motion, it is shifted in frequency. This frequency shift $\Delta \nu$, which is called Doppler shift, is proportional to the particle's velocity and is given by\textsuperscript{7-10}:

\[
\Delta \nu = \frac{2|v|}{\lambda} \sin \frac{\theta}{2} \cos \alpha,
\]  

(1.1)

where $v$ is the velocity vector of the particle, $\lambda$ is the wavelength of the incident light, $\theta$ is the angle between the wave vectors $k_0$ and $k$, of the incident light respectively the scattered light and $\alpha$ is the angle between $(k - k_0)$ and $v$, see Figure 1.1a. The Doppler shift is measured by applying heterodyne detection, whereby the scattered light is mixed on a square-law detector with light of a slightly different frequency. Due to interference the detector signal oscillates with the difference frequency.

Two methods of heterodyne detection are often applied. The first method is the *reference beam method*. In this method a particle scatters light from one single beam. The scattered light is mixed up with the incident light, so the frequency of the detector signal is equal to the Doppler shift of Equation 1.1. The second method is the *dual beam method*, or the *differential method*. In this method two light beams intersect, and when a particle passes this intersection it scatters light from both beams, see Figure 1.1b. This light interferes on the square-law detector, which results in a detector signal oscillating with a frequency $\Delta \nu$ given by:
\[ \Delta v = \frac{2|v|}{\lambda} \sin \frac{\varphi}{2} \cos \alpha, \]

(1.2)

where \( \varphi \) is the angle between the wave vectors \( k_0 \) and \( k_1 \) of the intersecting beams and \( \alpha \) is the angle between \( (k_1-k_0) \) and \( v \).

An important advantage of the dual beam method is that the measurement is independent of the scattering direction. Therefore, the aperture of the detector may be larger than in the reference beam method, so that more scattered light can be collected and the totally required scattered intensity to keep the signal to noise ratio acceptable is lower. Because this implies that the particle density may be low, the dual beam method is convenient for aerodynamic flow research\(^7\,\,8\). The reference beam method, however, is preferable in fluids with a higher particle density. Hence, this method is, for example, applied for flow measurement in water that contains many (natural) particles and blood\(^10\,\,11\). A great disadvantage of LDV is the fact that it only provides point measurements. A rather new Doppler technique is, however, capable to measure the velocity distribution simultaneously over a whole cross-section. This technique called Planar Doppler velocimetry (PDV) or Doppler Global Velocimetry (DGV) is a great promise to provide accurate measurements, especially in high speed flows. In this technique, the scattered light is filtered by a molecular filter. Since the transmission of the filter is a known function of the light frequency and thus of the Doppler shift, the transmitted intensity is a measure for the flow velocity\(^12\,\,9\,\,13\).

Particle image velocimetry (PIV) is inherently a laser-sheet technique and hence it provides information about a total cross section of the flow\(^14\,\,15\,\,16\,\,9\). The technique analyzes the 2D displacement of particles in a short time interval. For this, a plane cross section of the flow is illuminated by a light sheet and the particles in the sheet are imaged on a recording plane. The particle displacement is detected by making two images in rapid succession. In general, the seeding density in the flow is such that the individual particles can be distinguished in each image, but their individual displacements between the two
images cannot be recognized. To determine a local particle displacement, a spatial cross-correlation function of a small interrogation area is calculated. This interrogation area must be sufficient small so that the flow can be assumed to be uniform. Then, the cross-correlation gives a clear peak which corresponds to the average displacement of the particles and thus the particle velocity can be calculated if the time interval between the two image recordings is known. By systematically moving the interrogation area over the whole image, a vector filed of the flow velocity is obtained. Much effort has been paid to the development of more advanced interrogation techniques to improve the accuracy, the spatial resolution and the dynamic range of PIV. Applications of PIV in aerodynamic flows can be found in the literature\textsuperscript{17,18,19,20,21}.

If a flow is seeded so dense that the individual particles tend to overlap in the image, in principle, the same method for the particle displacement calculation can be applied. If the light in the light sheet is coherent, interference will appear and a speckle pattern arises in the image. Because this speckle pattern moves along with the particles in the flow, it can be used for the displacement calculation. The technique is now called \textit{laser speckle velocimetry} (LSV). The drawback of LSV is that, due to the high seeding density, the particles tend to follow the flow badly.

In the case that the seeding density is so low that the particle pairs can be distinguished in the image, the algorithm for the calculation of the particle movement can be simplified. The technique reduces to \textit{particle tracking velocimetry} (PTV). The drawback of PTV is that the displacement can only be calculated at the positions where a particle happens to be present. Because these positions may be relatively rare, the spatial resolution is low.

\textit{Laser-induced fluorescence} (LIF) is a quantitative flow measurement technique which uses inelastic light scattering. Here, light of an appropriate wavelength excites molecules in the flow to a higher energy level, after which they decay under emission of fluorescence radiation. Because the decay occurs in two steps, a radiationless decay to an intermediate energy level followed by a radiating decay to the initial energy level, the wavelength of the induced fluorescence differs from the wavelength of the exciting light. The intensity is proportional to the molecule density and the temperature. However, due to occurrence of collisional fluorescence quenching, it is also sensitive to physical conditions within the flow, especially at high pressures.

In many applications fluorescing molecules are seeded in the flow like iodine and nitric oxide. These materials have a narrow absorption linewidth so that, additionally, the flow velocity can be determined by the detection of the Doppler shift of the fluorescence. However, both materials are hazardous and iodine is also corrosive, so the usage of them must be accompanied with precautions.

In unseeded air flows oxygen is a convenient fluorescing molecule. The short lifetime of this molecule in the excited state implies that there occurs no collisional fluorescence quenching at pressures of a few atmospheres or below. However, it also implies that the absorption lines are broad compared to a possible Doppler shift and hence an accurate velocity measurement is not possible. Therefore, the application of LIF from oxygen in
aerodynamics is mainly focused on density and temperature measurements. Because the intensity of the induced fluorescence is proportional to both density and temperature, for quantitative density and temperature measurements LIF is often used in combination with another technique like the detection of Rayleigh scattering\textsuperscript{22} and Raman scattering\textsuperscript{23}.

*Rayleigh scattering* is a relatively weak, elastic scattering by particles that are much smaller than the wavelength of the light, like molecules. The scattered intensity is proportional to the density of the scatterers, the Doppler shift is proportional to their velocity. Because Rayleigh scattering is proportional to $\lambda^4$, a deep UV laser is the most convenient light source.

Rayleigh scattering is often used for density measurements in gasses. Since Rayleigh scattering is molecule dependent but, in contrast to LIF and Raman, not molecule specific, a condition is that the mixture of the components in the gas remains homogeneous. A serious problem in measuring Rayleigh scattering is the occurrence of reflections somewhere in the test section or the occurrence of Mie scattering by larger particles such as dust or vapor droplets. Because the wavelengths of the reflections and the Mie-scattering are equal to the wavelength of Rayleigh scattering, they cannot be filtered out and the can dominate the weak Rayleigh scattering. The presence of a very low concentration of water in air flow (parts per million) can influence the Rayleigh signal strongly\textsuperscript{24}. Examples of measurements in supersonic flows, turbulent flows and other flows can be found in literature\textsuperscript{25-29}.

*Raman scattering* does not have the disadvantages of both LIF and Rayleigh scattering. The scattering is inelastic, so it can be spectrally separated from reflections and Mie scattering. It is molecule specific, so there is no disturbance by scattering from other molecules. The signal is proportional to the density of the molecules and there is no distortion by molecular collisions. The drawback is, however, that the scattering efficiency is extremely low, even much lower than that of Rayleigh scattering. Hence, the number of applications in the field of aerodynamics is limited. However, since the advantages of Raman scattering are serious, rather successful Raman measurements are found in literature\textsuperscript{30,31,32,23,29}.

*Molecular tagging velocimetry* (MTV) is a technique that has shown significant advances in the past ten years\textsuperscript{33,34}. MTV relies on molecules that can be used as tracers after they have been 'tagged' by a laser pulse of an appropriate wavelength, that is, after they are excited (e.g., photochromic or phosphorescent molecules) or created. At a specific time $t$, a spatially well-defined grid of molecules in the flow is tagged. This grid is next deformed by the flow and after a suitable period of time it is probed by using a light sheet or one of the earlier mentioned techniques, for example, LIF. The difference between the probed grid and the tagged grid contains information about the flow velocity. Rather recently, MTV techniques have been developed for aerodynamic flows where no seeding of molecules is required. The tagging is based on Raman excitation of $O_2$\textsuperscript{35,36},
photodissociation of $O_2^{37,38}$, and photochemical creation of NO-molecules from $N_2$ and $O_2^{39,40}$

1.3. Techniques to detect refractive index profiles.

In general, there are three optical techniques for detection of refractive index profiles that can be distinguished. They are the shadowgraph technique, the schlieren technique and interferometry. All these techniques can be used for general refractive index profile measurement as well as for visualization of compressible flows. The techniques are non-intrusive, so the flows remain undisturbed.

![Figure 1.2 Basic shadowgraph set-up.](image)

The simplest technique for refractive index profile detection is the *shadowgraph technique*. It relies on the deflection of light when it traverses a refractive index gradient. The optical set-up, which is shown in Figure 1.2, is very simple. A parallel light beam is generated by placing a point light source in the focal point of a lens. This parallel light beam traverses the medium containing the unknown refractive index profile. The intensity distribution of the beam is recorded in a recording plane at distance $l$ behind the medium. In cases that there are no refractive index gradients in the medium, the intensity distribution on the recording plane is equal to the intensity distribution in the beam just behind the lens. However, if there is a refractive index gradient in the medium, the rays in the beam are deflected in the direction of the highest refractive index. The intensity distribution on the recording plane will then contain maxima in the areas where the rays converge and minima in the areas where the beams are diverge. This intensity distribution may now be written as:

$$\frac{\Delta I(x,y)}{I} = l \int_{x_i}^{x_f} \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] \ln n(x,y,z) \, dz,$$

where $I(x,y)$ represents the intensity distribution without refraction, $\Delta I(x,y)$ the deviation
from this intensity due to refraction, \( n(x,y,z) \) the refractive index profile in the medium, \((x,y,z)\) the rectangular coordinate system in which the beam propagates in the \(z\)-direction, \(z_1\) and \(z_2\) are the \(z\)-coordinate of the points where the beam enters and leaves the medium. The equation shows that the intensity deviation is sensitive to the second order spatial derivative of the refraction index. This implies that the capacity of the shadowgraph technique for quantitative refraction index measurements is very limited because the necessary double integration of the intensity deviation leads to an amplification of the errors in the measurements. Although attempts to obtain quantitative data can be found in literature\(^{41}\), the shadowgraph technique is particularly convenient for qualitative measurements. Nevertheless, the technique can be very sensitive, especially to refractive index profiles where the second order derivative changes drastically, like it happens in shock waves and turbulent flow fields.

The addition of a few extra optical components changes the shadowgraph set-up into a schlieren set-up. Schlieren\(^{42}\) is an often applied technique for optical visualization in aerodynamics and thermodynamics. The optical arrangement is still quite simple and the resolution is relatively high. With this technique, the light beam leaving the medium is focused by a second lens. See Figure 1.3. A third lens is used to image a plane in the medium on the recording plane, thereby minimizing shadow effects. If there is no refractive index gradient in the medium, the point light source is imaged in the focal plane of the second lens. However, if there is a local refractive index gradient in the medium, some rays in the beam will be deflected and as a consequence they will be imaged in a slightly displaced way in the focal plane. By placing a spatial filter (called schlieren filter) in the focal plane, the deflected rays can be made visible. The simplest filter is the knife-edge filter. This filter cuts off a part of the light source image and this, in turn, results in a reduction of the intensity in the recording plane. Deflection of rays in a direction perpendicular to the knife edge leads to a change in this reduced intensity. This deviation is measured. In Figure 1.3 the knife-edge is parallel to the \(x\)-axis. This implies that the set-up is convenient for measuring refractive index gradients in the \(y\)-direction. The deviation \(\Delta I(x,y)\) in the intensity is given by\(^1\):

\[
\frac{\Delta I(x,y)}{I(x,y)} = f_2 \int_{z_1}^{z_2} \frac{1}{n(x,y,z)} \frac{\partial n(x,y,z)}{\partial y} \, dz, \tag{1.4}
\]

where \(I(x,y)\) is the intensity in case there is no refraction, \(a\) is the reduced height of the source image and \(f_2\) is the focal length of the second lens. High sensitivity requires a second lens with a large focal distance. However, since the signal is proportional to the first spatial derivative of the refractive index and thus there must be integrated to find the refractive index, the set-up is less convenient for refractive index profile measurements. Several attempts have been reported to obtain a calibrated schlieren set-up, for example, by using a variously colored beam and a line grid filter\(^{43}\), by using a rainbow filter\(^{44}\), or by using a calibrated knife-edge filter\(^{45}\). Promising techniques are Calibrated Color
Schlieren (CSS)\textsuperscript{46} and Background Oriented Schlieren\textsuperscript{47,48,46}. Both techniques are capable of measuring two-dimensional refractive index gradients. In CCS the knife-edge filter is replaced by a color filter with a gradual variation in its color. The color hue in the image determines the refractive index gradient. In BOS a random pattern background is imaged through the medium of interest. The refraction of the light in the medium causes a distortion of the pattern in the image. The local pattern shift with respect to the original pattern is proportional to the integrated refractive index gradient. It can be analyzed by similar cross-correlation methods as used in PIV.

A more direct technique for quantitative refractive index profile measurement is interferometry. This technique relies on the retardation of light when it traverses a medium. This retardation is dependent on the refractive index on the path of the ray. Therefore, in the case of a light beam traversing an inhomogeneous refractive index profile, all the rays in the beam experience a different retardation. This inhomogeneous retardation of the rays leads to a deformation of the wavefront of the beam, see Figure 1.4a. However, in cases where a beam traverses a homogeneous refractive index the rays experience equal retardations and therefore the wavefront of the beam remains undeformed, see Figure 1.4b. Interferometry is a technique which visualizes the differences between the wavefronts of two beams. For this purpose, these two beams are combined to interfere. In the recording plane of the interferometer a modified intensity distribution arises which is called 'fringe pattern' or 'interferogram'. This intensity distribution $I(x,y)$ is given by\textsuperscript{49,50}:

$$I(x,y) = I_1(x,y) + I_2(x,y) + 2\sqrt{I_1(x,y)I_2(x,y)} \cos \phi(x,y),$$

where $I_1(x,y)$ and $I_2(x,y)$ represent the intensity distributions in the two beams and $\phi(x,y)$ represents the phase difference between the two beams. This phase difference is given by:
1. Optical measurement techniques for refractive index profiles and flow

\[ \phi(x, y) = \frac{2\pi}{\lambda} \int_{\text{path}_1} n_1(x, y, z) \, dz - \frac{2\pi}{\lambda} \int_{\text{path}_2} n_2(x, y, z) \, dz, \quad (1.6) \]

where \( n_1(x, y) \) and \( n_2(x, y) \) represent the refractive indices experienced by the two beams and which are integrated over the total paths of the beams. The equation shows that the phase difference \( \phi(x, y) \) is sensitive to the difference in optical path lengths through the refractive index profiles.

There are two classes of interferometers in view of the way in which the phase difference is related to the refractive index profile of interest. The first class is called shearing interferometer. A shearing interferometer is characterized by the fact that both beams traverse the refractive index profile while the beams are displaced along a small distance \( d \) with respect to each other. If the beams are displaced in \( y \)-direction, the phase difference \( \phi(x, y) \) of equation (1.6) reduces to:

\[ \phi(x, y) = \frac{2\pi}{\lambda} d \int_{\text{path}} \frac{\partial}{\partial y} n(x, y, z) \, dz. \quad (1.7) \]

This equation implies that the shearing interferometer is, just like the schlieren set-up, sensitive to the gradient of the refractive index profile.

The second class of interferometers is called reference beam interferometer. It is characterized by the fact that one of the beams does not traverse the refractive index profile of interest. This beam, which is called the reference beam, propagates very often through the surrounding air. Due to the (assumed) constant refractive index \( n_o \) of the surrounding air the wavefront of the reference beam remains undeformed. The other beam, which is called the test beam, traverses the refractive index profile of interest \( n(x, y, z) \) through what results in a deformed wavefront. The interferogram is now given by:
Figure 1.5 The basic set-up of a Mach-Zehnder interferometer.

\[ I(x,y) = I_t(x,y) + I_r(x,y) + 2\sqrt{I_t(x,y)I_r(x,y)} \cos \phi(x,y), \]  
\[ \phi(x,y) = \frac{2\pi}{\lambda} \int_{\text{path}} (n(x,y,z) - n_0) \, dz, \]

where \( I_t(x,y) \) and \( I_r(x,y) \) are the intensity distributions in the test beam respectively the reference beam. The phase difference is now given by:

providing that the lengths of the paths traversed by the reference beam and the test beam are the same. This phase difference, which is also called the phase of the interferogram, is now proportional to the line integral of the refractive index along the path through the test section. Therefore, the reference beam interferometer is sensitive to the real refractive index and not to its gradient. This implies that the reference beam interferometer is a suitable tool for quantitative refractive index profile measurements.

The most common reference beam interferometer is the Mach-Zehnder interferometer. Its basic set-up is shown in figure 1.5. The light from a coherent light source, a point light source for example, is made parallel with a lens. The resulting beam is split into a test beam and a reference beam by a beam splitter. The test beam traverses the medium containing the refractive index profile of interest while the reference beam propagates along a path outside the medium. Both beams are recombined to interfere by a second beam splitter. A lens system, which images a plane in the medium, realizes the interferogram on the recording plane.

A more sensitive set-up is the Michelson interferometer, as in this set-up the test beam traverses the medium twice. To that end, the test beam is reflected by a plane mirror just behind the medium, see Figure 1.6. Also the reference beam is reflected by a mirror. The same beam splitter which splitted the original beam into a test beam and a reference beam
now recombines both beams to interfere. A lens system realizes the desired interferogram on the recording plane.

A reference beam interferometer which is often applied in flow visualization is the **holographic interferometer**. Unlike the Mach-Zehnder interferometer and the Michelson interferometer, in this interferometer the test beam and the reference beam are not generated simultaneously. The information contained by both beams is stored on a holographic plate separately. The basic set-up of the interferometer is shown in Figure
1.7. A beam originating from a coherent light source, here a laser, is split by a beam splitter into two beams which are called the holographic test beam and the holographic reference beam. The term 'holographic' is added in the name to avoid confusion with the test beam and the reference beam of the interferometer. The holographic test beam is expanded, it traverses the medium containing the refractive index profile as a parallel beam and finally it strikes the holographic plate. The holographic reference beam propagates outside the medium, is expanded and also strikes the holographic plate. Exposure of the holographic plate with the holographic test beam and the holographic reference beam simultaneously leads, after photographic processing, to a hologram. This hologram has the property that if it is illuminated only by the holographic reference beam, the holographic test beam will be reconstructed although the real holographic test beam has been removed\(^5\). In general, two exposures are recorded on the same holographic plate (double exposure method): one exposure in absence of the refractive index profile in the medium, i.e., storage of the reference beam, and one exposure in presence of it, i.e., storage of the test beam. By illumination of the hologram with the holographic reference beam, the test beam as well as the reference beam are reconstructed and they interfere.

A convenient aligned lens system realizes the desired interferogram on the recording plane, see Figure 1.7. The main advantage of the holographic interferometer when compared to the other interferometers is its insensitivity to imperfections in the optical components. Because the reference beam and the test beam traverse the same optical path, wave front deformations due to imperfections cancel out.

All the interferometer set-ups described above are convenient for real-time measurements, i.e., generation of interferograms in sequence in time. During real-time measurements, the time dependent refractive index profile is positioned in the test section of the interferometer and the time dependent interferogram is recorded continuously by a CCD-camera in the recording plane. Also the holographic interferometer is suitable for real-time measurements. In that case, only the reference beam is generated by illumination of the hologram with the holographic reference beam, while the test beam is generated directly by the beam which traverses the time dependent refractive index profile.

The disadvantage of using these set-ups for real-time measurements is that they only record one interferogram per measurement. Under some conditions, the registration of one single interferogram is can be enough for a quantitative analysis of a phase distribution, see section 3.2.1. However, the results are less accurate than the results found by using phase shifting methods\(^6\). These phase shifting methods require the recording of at least three interferograms which are phase shifted with a fixed phase step with respect to each other.

The simplest way to introduce an additional phase shift in an interferogram is to introduce a controlled path length change in the reference beam of the interferometer. For this purpose, for example, the mirror in the reference beam of the Michelson interferometer can be mounted on a piezo-electric transducer, which can generate controlled displacements of the mirror to introduce the desired path length change. In this kind of
set-ups, the phase shifted interferograms are generated in sequence in time. This implies that the set-ups are only suitable for accurate quantitative measurements of time independent refractive index profiles.

A set-up which is convenient for the registration of phase shifted interferograms of time dependent refractive index profiles is the holographic interferometer. The test beam and the reference beam store an instantaneous realisation of the refractive index profile on the holographic plate, while the phase shift in the reference beam can be introduced in the post-processing phase. For this purpose, a slightly modified version of the holographic interferometer of Figure 1.7 is necessary\textsuperscript{52}. In this version the reference beam and the test beam have to be reconstructed by separate holographic reference beams. The phase shift in the interferogram is introduced by phase shifting the holographic reference beam which reconstructs the reference beam, while the holographic reference beam which reconstructs the test beam remains unshifted.

Common holographic interferometers are inconvenient for repetitive quantitative measurements because of the necessary photographic processing of the holographic plate between the recordings. Therefore, in more advanced holographic interferometers, the holographic plate is replaced by a recording medium which is selfdeveloping and which is reusable like photorefractive crystals, photo polymers and films of photochromic materials. Such systems have been used rather successfully for real-time measurements\textsuperscript{53-54}. However, the systems are rather complex and their time resolution of about 1Hz is rather low.

1.4 Aim and outline

The aim of the study presented in this thesis is the development of a real time interferometer for quantitative measurements of two dimensional refractive index profiles. The interferometer has to be generally usable in different fields of application, but the design and method of operation must be particularly convenient for measuring density profiles in compressible aerodynamic flows such as two-dimensional boundary layers along curved walls. The major tasks involved are designing the interferometer, building an experimental set-up (including software), validating the measuring accuracy, and demonstrating its utility. The study is a feasibility study in which the attainable measuring accuracy is determined. In order to reduce the costs, the time resolution is kept relatively low (video rate) which limits the demands on the hardware. Hence, the experimental set-up is only suitable for estimations of large scale fluctuations in a flow. Therefore, an actual application of the interferometer for aerodynamic flow measurements is not aimed at this thesis. Instead, the operation of the interferometer is tested on a number of slow mixing processes.

The real time interferometer as presented here is a spatial phase stepping interferometer based on a Michelson set-up. Its measuring accuracy is evaluated extensively. Over the
last 10 years, some other spatial phase stepping interferometers have been used for refractive index profile measurements, for example, for measuring diffusion fields\textsuperscript{55,56}, concentration fields\textsuperscript{57} and flow\textsuperscript{54,58}. However, in contrary to the study presented in this thesis, they make use of a Mach-Zehnder and a holographic set-up, while the accuracy analysis of these systems has been addressed only briefly\textsuperscript{59,60}.

This thesis is organized into 6 chapters. Chapter 2 deals with the theory of refraction effects which occur if a strong gradient is present in the refractive index profile. There, it is deduced which plane in the profile must be imaged on the recording plane to minimize these effects and the phase error is calculated for the case that a slightly shifted plane is imaged. In Chapter 3, the experimental set-up of the real time interferometer is presented, its alignment procedure and an extended theoretical accuracy analysis. In Chapter 4, the measuring accuracy and the stability of the set-up are experimentally validated by carrying out some tilted mirror experiments. The utility of the real time interferometer is shown in Chapter 5. Here, a measurement of the diffusion process of sucrose in an aqueous bulk fluid demonstrates the reliability of the results obtained by the interferometer. After that, on behalf of a study into the homogeneity of mixtures in immuno assays, the jet induced homogenization is measured as a function of the time after a jet of a dilute sucrose solution is injected into an aqueous bulk fluid in a cuvette. Finally, conclusions and suggestions are given in Chapter 6.

References


1. Optical measurement techniques for refractive index profiles and flow
Minimization of refraction effects in a Michelson interferometer by imaging

2.1 Introduction

When an interferometer is used for refractive index profile measurements, the rays of the test beam are supposed to propagate along straight trajectories through a medium in the test section. Only then the measured phase is a linear function of the refractive index and, since the paths of the test and reference beam are known, this function is given by Equation 1.6. However, as soon as there is a refractive index gradient in the medium, refraction of the rays occurs. As a consequence, the rays traverse a curved trajectory and their phase will be different from when they would have traversed a straight trajectory. The result is a phase error in the interferogram. In general, this phase error depends on the choice of the object plane which is imaged on the recording plane. A single passage of the test beam through a 2D medium requires an image of a plane at 2/3 of the medium width to minimize the error\textsuperscript{1,2,3}. An axisymmetric medium requires the recording plane to be focused on the plane containing the symmetry axis\textsuperscript{4,5}.

The real time interferometer (RTI) described in this thesis is based on a Michelson set-up. The double passage of the test beam through the test section makes the RTI relatively easy to install around big objects like wind tunnels because almost all the optical elements can be positioned at one side of the object. The set-up is relatively compact and stable because no passage of the reference beam around the object is needed. However, the double passage of the test beam through the object increases the refraction effects when compared
to the single passage in a Mach-Zehnder interferometer. The minimization of the phase error in a Michelson interferometer by imaging a suitable plane in a 2D medium is the topic of this chapter. For this, the light propagation through an object is studied. The configuration of the object proposed in the analysis corresponds to the objects to be tested by the RTI, i.e., a compressible flow in a wind tunnel and a mixing/diffusion process of a dilute sucrose solution in an aqueous buffer liquid in a square cuvette. These objects are a medium containing a refractive index profile bounded by two parallel glass windows. In the analysis, the medium is supposed to be linearly stratified, i.e., a medium in which the refractive index varies linearly and only in one direction, a direction perpendicular to the propagation direction of the test beam. Although the linear stratification is not completely satisfied in practice, the results remain approximately valid as long as the medium is locally linearly stratified, that is, the medium is linearly stratified for each individual ray trajectory.

Figure 2.1 shows an object (medium length $L$, window thickness $t$, refractive index of the windows $n_d$) placed in the test section of a basic Michelson interferometer. A laser beam is expanded by the combination of the lenses L0 and L1. After that, the beam is split into a test beam and a reference beam by beam splitter BS. The test beam traverses the object, it is reflected by the flat mirror M0 and it traverses the object for the second time. The reference beam is reflected by the flat mirror M1. The test beam and the reference beam are recombined by the beam splitter after which they interfere. The interferogram is imaged on the recording plane by lens L2.

Figure 2.2 schematically shows the ray propagation through the test section of a Michelson interferometer and towards the recording plane. In the coordinate system, $z=0$ at the border between the first window and the medium. The reflection of the rays by
Figure 2.2 Propagation of a refracted ray and a virtual straight ray through the object and towards the recording plane in a Michelson interferometer.

mirror M0 and the second passage through the object is modelled by a transmission through the mirror and a passage through a second object. The objects are supposed to contain the same linearly stratified medium with the refractive index gradient in the positive y-direction, i.e., the refractive index increases with y. A refracted and a straight ray are shown. They both enter the first medium parallel to the z-axis and they are both refracted by lens L2 towards point S in the interferogram in the recording plane. The refracted ray represents the real ray that enters the medium at $y=y_r$. It follows the trajectory ABCDEFGHH'S. The straight ray is the virtual ray which is assumed to exist in the interferogram analysis. It traverses the test section at the projection height $y=y_p$ and it follows the trajectory IJKLMPQRR'S.

The refracted ray enters the first glass window parallel to the z-axis at ($y=y_r$, $z=-t$) and it next propagates along the trajectory AB. At $z=0$ the ray enters the first medium where it is refracted along the curve BC. At $z=L$ it is incident upon the second window at an angle $\theta_e$ to the normal. The ray is refracted by the window surface and, after that, it propagates along the straight trajectory CD at an angle $\theta_2$ to the normal. At $z=L+t$ the ray enters the surrounding air and it is refracted by the refractive index difference between air and glass. It propagates through the air along a straight trajectory at an angle $\theta_e$. In reality the ray is next reflected by a mirror at $z=L+t+a/2$ back to the object. In the scheme of Figure 2.2 this is described by a transmission of the ray through the mirror and a passage through a second object at $z=L+t+a$. So after traversing the air slab along the straight trajectory DE, the ray is incident upon the first window of the second object at
an angle \( \theta_s \) to the normal. It traverses the window along the straight trajectory EF at an angle \( \theta_s \) and it enters the medium at an angle \( \theta_{L2} \). The ray traverses the medium along curve FG and is incident upon the second window at an angle \( \theta_{L3} \). It passes the window along the straight trajectory GH at an angle \( \theta_{s2} \) to the normal and it enters the surrounding air at an angle \( \theta_{a2} \). Finally, the ray is refracted by lens L2 in point H’ and it strikes the recording plane in point S.

The virtual straight ray enters the first window at \((y=y_p, z=-t)\). The ray is not refracted in the medium. After refraction by lens L2 in point R’ it strikes the recording plane in point S.

The interferogram in the recording plane is not influenced by refraction in point S if the refracted ray is in phase with the virtual straight ray. Only then the assumption of the straight rays in the analysis will lead to a correct refractive index profile. The \( y \)-coordinate \( y_p \) of the virtual straight ray is the height where the ray appears to cross the refractive index profile when it is imaged in S. This coordinate is determined by the position of the virtual object plane (virtual plane of focus), i.e., the plane which would be imaged on the recording plane if the medium and the windows would have the same constant refractive index of the surrounding air. The term 'virtual' is added to the term 'object plane' because in practice there are refractive index differences in the medium and between the medium, the windows and the surrounding air and thus the term 'object plane' is no longer valid. The position of the virtual object plane is denoted by \( z'_f \), where the prime distinguishes this position from the position of \( z_f \) of the true object plane. A shift in the position of the virtual object plane leads to a change in the height \( y_p \) of the virtual straight ray which is imaged in S. In case of a linearly stratified medium, this implies a change in the refractive index traversed by the virtual straight ray and thus in its phase in S. Hence, the position of the virtual object plane determines the phase difference between the virtual straight ray and the refracted ray, and can thus be used to make this difference equal to zero. This is further explained below.

A refracted ray and a virtual straight ray coincide in the recording plane if both rays appear to originate from the same point O in the virtual object plane. This implies that the position \( z'_f \) of the virtual object plane has to coincide with the crossing of the virtual straight ray and the backward extension of the refracted ray leaving the window behind the second medium, see Figure 2.2. To satisfy phase equality in point S, the optical path lengths traversed by the refracted ray and the virtual straight ray must be the same. Up to \( z=0 \) the two rays are parallel and in phase. Hence a difference in optical path length originates from beyond \( z=0 \). In Figure 2.2, HR is a circle arc with center O. Since the optical path length traversed by a ray going from an object plane to a recording plane via a thin lens is independent of the trajectory, the optical path lengths of the trajectories HH’S and RR’S are equal. Hence, the phase difference between the refracted ray and the virtual straight ray at point S is determined by the optical path lengths of the trajectories BCDEFGH and JKLMPQR. The entrance position \( y_p \) of the virtual straight ray increases with \( z'_f \). An increasing value of \( y_p \) leads, under the assumption of a linearly
stratified medium, to an increase of the optical path length along the trajectories JK and NP and to a decrease of the optical path length along the trajectory QR. Hence, the total optical path length of the trajectory followed by the virtual straight ray is dependent on $z_f'$ and thus a correct image can lead to a zero phase error in the interferogram. That is, the error can be removed by choosing the optimum virtual object plane.

Section 2.2 describes the trajectories of the refracted rays through the object under the condition that the medium is linearly stratified. In Section 2.3 the value of $z_f'$ is calculated to keep the phase error due to refraction equal zero. However, since this value appears to be slightly dependent on the refractive index gradient, an exact errorfree interferogram over a gradient interval cannot be realized. The induced phase error due to an imperfect image is calculated as a function of the (dimensionless) refractive index gradient. Section 2.4 describes the results for two object configurations, the configurations intended to be tested by the real time interferometer, i.e., a compressible aerodynamic flow in a wind tunnel and a diffusion/mixing process of a dilute sucrose solution in an aqueous bulk in a square cuvette. In Section 2.5 the ray trajectories, the required value of $z_f'$ to keep the interferogram error free and the phase error due to an imperfect image are calculated for a Mach-Zehnder interferometer. Finally, Section 2.6 presents a discussion about the relevance of the results, the consequences for the real time interferometer and a comparison to a Mach-Zehnder interferometer.

2.2 The ray trajectory

Figure 2.3 depicts the propagation of a light ray through a non-homogeneous medium. In this figure $r(s)$ represents the position vector of a point on the ray and $s$ is the length of the ray measured from some fixed point. The relation between the ray trajectory and the refractive index distribution $n(x,y,z)$ in a medium is given by:

$$\frac{d}{ds} \left\{ n \frac{dr}{ds} \right\} = \nabla n. \quad (2.1)$$

![Figure 2.3 Propagation of a light ray through a non-homogeneous medium.](image)
\[ n_G = n_F \left( \frac{y_{FG} \left( \frac{1}{2} \left( 2L + 2t + a \right) + 1 \right)}{y_{FG} \left( \frac{1}{2} \left( L + 2t + a \right) + 1 \right)} \right)^\frac{1}{2}. \] (2.10c)

For the trajectories is found:

\[ y_{BC}(z) = y_e + \frac{n_e}{n'} \cosh \left( \frac{n_e}{n'_e} z \right) - \frac{n_e}{n'}, \] (2.11a)

\[ y_{CD}(z) = y_{BC}(L) + (z-L)\tan \theta_g \]
\[ = y_{BC}(L) + (z-L)\tan \arcsin \left( \frac{n_C}{n_g} \sin [\arctan y_{BC}(L)] \right), \] (2.11b)

\[ y_{DE} = y_{CD}(L+t) + (z-L-t)\tan \theta_a \]
\[ = y_{CD}(L+t) + (z-L-t)\tan \arcsin \left( \frac{n_e y_{BC}'(L)}{n_g} \right), \] (2.11c)

\[ y_{EF} = y_{DE}(L+t+a) + (z-L-t-a)\tan \theta_g \]
\[ = y_{DE}(L+t+a) + (z-L-t-a)\tan \arcsin \left( \frac{n_e y_{BC}'(L)}{n_g} \right), \] (2.11d)

\[ y_{FG}(z) = y_{EF}(L+2t+a) + \]
\[ \frac{n_F}{n'} \cosh \left( (z-L-2t-a) n' \sqrt{y_F'^2 + 1} + \arccosh \left( \sqrt{y_F'^2 + 1} \right) \right) - \frac{n_F}{n'}, \] (2.11e)

\[ y_{GH}(z) = y_{FG}(2L+2t+a) + (z-2L-2t-a)\tan \theta_g = y_{FG}(2L+2t+a) + \]
\[ (z-2L-2t-a)\tan \arcsin \left( \frac{n_F}{n'_e} \sinh \left( L \frac{n' \sqrt{y_F'^2 + 1}}{n_F} + \arccosh \sqrt{y_F'^2 + 1} \right) \right), \] (2.11f)
where $y'_F$ is defined as:

$$y'_F = y'_F(L+2t+a) = \tan^{-1} \left( \frac{n_e}{n_F y'_{BC}(L)} \right), \quad (2.12)$$

and $y'_{BC}(z)$ is given by:

$$y'_{BC}(z) = \frac{dy_{BC}(z)}{dz} = \sinh \left( \frac{n'}{n_e} z \right). \quad (2.13)$$

An important measure is the distance $y-y_e$ over which the ray is refracted since it entered the first medium at $z=0$, further called the bending. The maximum bending of a ray in the second medium is equal to $y_{FG}(2L+2t+a)-y_e$, the maximum bending of a ray in the second object (medium and windows) is equal to $y_{GH}(2L+3t+a)-y_e$. These bendings can be calculated Equations 2.11e,f. For small values of $n'$ they become:

$$y_{FG}(2L+2t+a)-y_e = \left( \frac{2L}{n_e} + \frac{2t}{n_g} + \frac{a}{n_a} \right) n'L + O(n'^3), \quad (2.14a)$$

$$y_{GH}(2+3T+A)-y_e = \left( \frac{2L}{n_e} + \frac{4t}{n_g} + \frac{a}{n_a} \right) n'L + O(n'^3). \quad (2.14b)$$

### 2.3 The phase error

The phase error in an interferogram at a point S in an interferogram (Figure 2.2) is equal to the phase difference between the refracted ray and the assumed virtual straight ray at that point. Since the optical path lengths over the trajectories HH’S and RR’S are equal, and the refracted ray at point B is in phase with the virtual straight ray at point J, the phase difference in S is determined by the phase shift of the refracted ray over trajectory BCDEFGH and the phase shift of the virtual straight ray over JKLMNPQR. The phase shift $\phi$ is related to the optical path length $\Phi$ along a trajectory by$^6,7$:

$$\phi = \frac{2\pi \Phi}{\lambda}. \quad (2.15)$$

This section starts with the calculation of the optical path lengths along the two trajectories in case of a linearly stratified medium bounded by windows. The requirement that the optical path lengths must be the same leads to the dimensionless position of the virtual object plane for which the phase error in the interferogram is zero. A deviation of this position leads to a phase error. This phase error is calculated at the end of this section.
The optical path length along trajectory BCDEFGH

The optical path length $\Phi$ of a light ray along a trajectory is given by:

$$\Phi = \int n(s)\,ds,$$

(2.16a)

where the refractive index $n(s)$ is integrated over the whole trajectory. Substitution of the Equations 2.3 and 2.6 leads to the optical path length for a refracted ray in a linearly stratified medium, which is given by:

$$\Phi = \frac{n_0}{\sqrt{y_0'^2 + 1}} \int (y'^2 + 1)\,dz.$$

(2.16b)

Here, $n_0$ is the refractive index of the medium at the point of entrance and $y_0'$ is the first order derivative of the trajectory with respect to $z$ at the point of entrance. Substitution of the first order derivative of Equation 2.11a, and the boundary conditions $n_0=n_e$ and $y_0'=0$ leads to the optical path length along trajectory BC (Figure 2.2). The optical path length $\Phi_{BC}$ is found by integration from $z=0$ to $z=L$:

$$\Phi_{BC} = n_e \int_0^L \left( 1 + \sinh^2 \left[ \frac{n'_e}{n_e} z \right] \right) \,dz = \frac{1}{2} n_L + \frac{n_e^2}{4n'_e} \sinh \left[ 2 \frac{n'_e}{n_e} L \right].$$

(2.17a)

Similarly, substitution of the first order derivative of Equation 2.11e and the boundary conditions $n_0=n_F$ and $y_0'=y_F'$ in Equation 2.16b gives, after integration from $z=L+2t+a$ to $z=2L+2t+a$, the optical path length $\Phi_{FG}$ along trajectory FG:

$$\Phi_{FG} = \frac{n_F}{\sqrt{y_F'^2 + 1}} \int_{L+2t+a}^{2L+2t+a} \sinh \left[ \left( z-L-2t-a \right) \frac{n'_F}{n_F} \sqrt{y_F'^2 + 1} + \arccosh \sqrt{y_F'^2 + 1} \right] + 1 \,dz =$$

$$= \frac{n_F}{\sqrt{y_F'^2 + 1}} \left( \frac{n_F}{2} \sqrt{\frac{2n'_L}{n_F} \sqrt{y_F'^2 + 1} + 2 \arccosh \sqrt{y_F'^2 + 1} } - \sinh \left[ 2 \arccosh \sqrt{y_F'^2 + 1} \right] \right) + \frac{L}{2}. $$

(2.17e)

The optical paths $\Phi_{CD}$, $\Phi_{DE}$, $\Phi_{EF}$ and $\Phi_{GH}$ along the trajectories CD, DE, EF and GH, respectively, are calculated by multiplying the constant refractive index with the length of the straight path:

$$\Phi_{CD} = n_0 \sqrt{1 + \tan^2 \theta_g} = n_0 \sqrt{1 + \tan^2 \arcsin \left[ n_e y_{BC}(L)/n_e \right]},$$

(2.17b)
\[ \Phi_{DE} = n_a a \sqrt{1 + \tan^2 \vartheta_a} = n_a a \sqrt{1 + \tan^2 \text{arcsin}\left[n_e y_{BC}^\prime (L)/n_a\right]}, \quad (2.17c) \]

\[ \Phi_{EF} = \Phi_{CD} = n_g f \sqrt{1 + \tan^2 \text{arcsin}\left[n_e y_{BC}^\prime (L)/n_g\right]}, \quad (2.17d) \]

\[ \Phi_{GH} = n_g f \sqrt{1 + \tan^2 \vartheta_2} \]

\[ \left[ 1 + \tan^2 \text{arcsin} \left( \frac{n_F}{n_g \sqrt{y_F^2 + 1}} \sinh \left( \frac{n'L}{n_F} \sqrt{y_F^2 + 1} + \text{arccosh}\sqrt{y_F^2 + 1} \right) \right) \right]^\frac{1}{2} \]. \quad (2.17f)

**The optical path length along trajectory JKLMPQR**

The optical path length along the straight trajectory is dependent on \( y_p \), the entrance position of the straight trajectory in the medium along the \( y \)-axis. The value of \( y_p \) is determined by the position \( z_f' \) of the virtual object plane of lens L2 in Figure 2.2. In the figure, it is easy to see that the relation between \( y_p \) and \( z_f' \) is given by:

\[ y_p = y_{GH}(2L + 3t + a) - (2L + 3t + a - z_f') \tan \vartheta_{a2}, \quad (2.18) \]

where \( \vartheta_{a2} \) is represented by:

\[ \vartheta_{a2} = \text{arcsin} \left( \frac{n_F}{n_a} \frac{1}{\sqrt{y_F^2 + 1}} \sinh \left( \frac{n'L}{n_F} \sqrt{y_F^2 + 1} + \text{arccosh}\sqrt{y_F^2 + 1} \right) \right). \quad (2.19) \]

The optical path lengths \( \Phi_{JK} \) and \( \Phi_{NP} \) along the straight trajectories JK and NP through the medium are:

\[ \Phi_{JK} = \Phi_{NP} = n_L + n'L(y_p - y_s) = n_L + n'L(y_{GH}(2L + 3t + a) - y_s - (2L + 3t + a - z_f') \tan \vartheta_{a2}). \quad (2.20a) \]

where \( \vartheta_{a2} \) is given by Equation 2.19. The optical path length \( \Phi_{KL}, \Phi_{MN}, \Phi_{PQ} \) and \( \Phi_{LM} \) along the trajectories KL, MN, and PQ through the windows and trajectory LM through the air slab can be derived straightforward and they are given by:
\[ \Phi_{KL} = \Phi_{MN} = \Phi_{PQ} = n_a t, \quad (2.20b) \]
\[ \Phi_{LM} = n_a a. \quad (2.20c) \]

The optical path length \( \Phi_{QR} \) along the trajectory QR through the surrounding air is given by:

\[ \Phi_{QR} = n_a(2L + 3t + a - z') \left( \sqrt{1 + \tan^2 \theta_a} - 1 \right) = \]
\[ n_a(2L + 3t + a - z') \cdot \left( \frac{1 + \tan^2 \text{arc} \sin \left[ \frac{n_F}{n_a} \sqrt{\frac{y_F^2}{y_F^2 + 1}} \right]}{\sinh \left[ \frac{n'L}{n_F} \sqrt{\frac{y_F^2}{y_F^2 + 1}} + \text{arccosh} \frac{y_F^2}{y_F^2 + 1} \right]} \right)^{\frac{1}{2}} - 1 \cdot \quad (2.20d) \]

Minimization of the phase error

The error in the interferogram due to refraction is zero if the refracted ray and the virtual straight ray are in phase at point S (Figure 2.2). This means that the optical path lengths along the trajectories BCDEFGH and JKLMPQR are equal, i.e., the difference between the optical path lengths is zero. This difference \( \Delta \Phi \) is given by:

\[ \Delta \Phi = (\Phi_{BC} + \Phi_{CD} + \Phi_{DE} + \Phi_{EF} + \Phi_{FG} + \Phi_{GH}) - (\Phi_{IK} + \Phi_{KL} + \Phi_{LM} + \Phi_{MN} + \Phi_{NP} + \Phi_{PQ} + \Phi_{QR}). \quad (2.21) \]

Since \( \Phi_{BC}, \Phi_{CD}, \Phi_{DE}, \Phi_{EF}, \Phi_{FG}, \Phi_{GH}, \Phi_{IK}, \Phi_{NP} \) and \( \Phi_{QR} \) are functions of \( n' \), and \( \Phi_{IK}, \Phi_{NP} \) and \( \Phi_{QR} \) are also functions of \( z'_j \), \( \Delta \Phi \) is a function of \( n' \) and \( z'_j \). The condition that \( \Delta \Phi = 0 \) leads to a relation between \( z'_j \) and \( n' \) for which the phase error is zero. The value of \( z'_j \) is now denoted by \( z'_{j,\text{opt}} \) and described by:

\[ z'_{j,\text{opt}} = \]
\[ \frac{\Phi_{BC} + \Phi_{CD} + \Phi_{DE} + \Phi_{EF} + \Phi_{FG} + \Phi_{GH} - \Phi_{KL} - \Phi_{LM} - \Phi_{MN} - \Phi_{PQ}}{2n'L \tan \theta_a - n_a (\sqrt{1 + \tan^2 \theta_a} - 1)} - \frac{n_a(2L + 3t + a - z') \left( \sqrt{1 + \tan^2 \theta_a} - 1 \right) + 2n'L + 2nJ}{2n'L \tan \theta_a - n_a (\sqrt{1 + \tan^2 \theta_a} - 1)} \quad (2.22) \]

where

\[ \gamma = y_{GH}(2L + 3t + a) - y_c - (2L + 3t + a) \tan \theta_a \quad (2.23) \]

and \( \theta_a \) is given by Equation 2.19. The plane at position \( z'_{j,\text{opt}} \) is called the optimum virtual
object plane. The value of \( z'_{f,op} \) appears to be slightly dependent on \( n' \) and thus the virtual object plane in an interferometer cannot be positioned such that the phase error is exactly zero over a range of \( n' \). The error can only be minimized. This is further discussed in Section 2.4. For small values of \( n' \), however, Equation 2.22 can be approximated by:

\[
z'_{f,op} = 2L - \frac{2}{3} \frac{n_a}{n_e} L + \left( 3 - \frac{3}{2} \frac{n_a}{n_e} \right) t + \frac{3}{4} a + O(n'^2),
\]

and is the dependence \( z'_{f,op} \) on \( n' \) of the second order.

**The phase error due to misimaging**

A deviation of the position of the virtual object plane imaged by lens L2 (actual virtual object plane) from the position of the optimum virtual object plane leads to a phase error in the interferogram. Hence, a limited positioning accuracy of lens L2 leads to a phase error due to misimaging. Below, the relation between the phase error and the relative positions of the actual and the optimum virtual object plane is derived.

In Figure 2.4, a refracted ray leaving the object in the test section of the Michelson interferometer in Figure 2.2 is represented. In point H at \( z=z_a=2L+3t+a \) it leaves the last window along the straight trajectory HH' at an angle \( \theta_a \). It is refracted by lens L2 and it strikes two image planes (planes that are possibly used as recording planes): it strikes an image plane at a distance \( s \) behind the lens in point S and it strikes an image plane at a distance \( s_{op} \) behind the lens in point \( S_{op} \). The straight ray trajectory HH' of the refracted ray is backward extended in the object (dotted line). This backward extension strikes two virtual straight rays. The virtual straight ray at \( y=y_p \) crosses the extension at \( z=z'_p \). The plane at \( z=z'_p \) is the virtual object plane of lens L2 if the image plane at distance \( s \) would be the recording plane. The virtual straight ray and the refracted ray strike this image plane in point S. The virtual straight ray at \( y=y_{p,op} \) crosses the backward extended refracted ray trajectory at \( z=z'_{f,op} \). The plane at \( z=z'_{f,op} \) is the virtual object plane of lens L2 if the image plane at a distance \( s_{op} \) behind the lens is the recording plane. The rays strike each other in point \( S_{op} \) in the image plane.

Let's assume that the plane at \( z=z'_{f,op} \) is the optimum virtual object plane resulting in a zero phase difference between the refracted ray and the virtual straight ray at point \( S_{op} \) in the image plane at a distance \( s_{op} \) behind the lens. Line \( HR_{op} \) is a circle arc with its centre at \( (y=y_{op}, z=z'_{f,op}) \). This implies that the phase difference between the refracted ray at H and the virtual straight ray at \( R_{op} \) is equal to the phase difference between the refracted ray and the virtual straight ray in \( S_{op} \). Since the phase difference in \( S_{op} \) is zero, this means that the virtual straight ray in \( R_{op} \) is in phase with the refracted ray in point H. The pase \( \phi \), of the refracted ray at point H can now be written as:
2. Minimization of refraction effects in a Michelson interferometer by imaging

![Diagram](image)

Figure 2.4 Propagation of a refracted ray and two virtual straight rays via lens L2 towards two image planes.

\[
\phi_r = \frac{2\pi}{\lambda} \Phi_{rop} = \frac{2\pi}{\lambda} (2n_{op}L + 3n_{f}t + n_{d}a + n_{r}f). \tag{2.25}
\]

Here is \(\Phi_{rop}\) the optical path length traversed by the virtual straight ray in point \(R_{rop}\), \(n_{op}\) is the refractive index in the medium at \(y=y_{op}\), and \(r_{op}\) is the distance between the last window of the object and point \(R_{rop}\). If the recording plane is mispositioned, i.e., it is not positioned at a distance \(s_{op}\) behind the lens but at a distance \(s\), the actual virtual plane is not at \(z=z_{rop}'\) but at \(z=z'\). Now the refracted ray interferes with a virtual straight ray in point S. This straight ray traverses the object at \(y=y_p\) and not at \(y=y_{op}\). In consequence, the phase difference between the refracted and the virtual straight ray in S is unequal to zero. Because HR is a circle arc with its centre in \((y=y_p, z=z')\), the same non zero phase difference is present between the refracted ray at point H and the virtual straight ray at point R. The phase \(\phi_r\) of the refracted ray at point H is given by Equation 2.25. The phase \(\phi\) of the virtual straight ray in point R is given by:

\[
\phi = \frac{2\pi}{\lambda} \Phi_R = \frac{2\pi}{\lambda} \left(2(n_{op}+n'_{op}(y_p-y_{op}))L + 3n_{f}t + n_{d}a + n_{r}r\right). \tag{2.26}
\]

where \(\Phi_R\) is the optical path length traversed by the virtual straight ray in point R and \(r\) is the distance between the last window of the object and point R. In the interferogram analysis, the virtual straight rays are assumed to exist, so the phase error is equal to the phase difference between the refracted ray and the virtual straight ray. By using Equations 2.25 and 2.26 for this phase error \(\Delta\phi\) is found:
\[ \Delta \phi = \phi_r - \phi = \frac{2\pi}{\lambda} \left[ 2n' (y_{op} - y_p) L + n_a (r_{op} - r) \right]. \quad (2.27) \]

It follows from Figure 2.4 that for \( r_{op}, r \) and \( (y_{op} - y_p) \) can be written:

\[ r_{op} = \left( z_u - z_{f,op}' \right) \left( \sqrt{\tan^2 \theta_{a2} + 1} - 1 \right), \quad (2.28) \]

\[ r = \left( z_u - z_f' \right) \left( \sqrt{\tan^2 \theta_{a2} + 1} - 1 \right), \quad (2.29) \]

\[ y_{op} - y_p = \left( z_{f,op}' - z_f' \right) \tan \theta_{a2}, \quad (2.30) \]

where \( \theta_{a2} \) is given by Equation 2.19. Substitution of Equations 2.28, 2.29 and 2.30 in Equation 2.27 gives:

\[ \frac{\Delta \phi}{2\pi} = \frac{-2n' L \tan \theta_{a2} + n_a \left( \sqrt{\tan^2 \theta_{a2} + 1} - 1 \right)}{\lambda} \Delta z_f', \quad (2.31) \]

where \( \Delta z_f' \) represents

\[ \Delta z_f' = z_f' - z_{f,op}'. \quad (2.32) \]

Equation 2.31 describes the relation between the phase error and the distance over which the virtual object plane is misimaged. The phase error is positive or negative. For the absolute phase error follows:

\[ \frac{|\Delta \phi|}{2\pi} = \left| \frac{-2n' L \tan \theta_{a2} + n_a \left( \sqrt{\tan^2 \theta_{a2} + 1} - 1 \right)}{\lambda} \right| |\Delta z_f'|. \quad (2.33) \]

For small values of \( n' \), Equation 2.33 reduces to:

\[ \frac{|\Delta \phi|}{2\pi} = 2 \frac{n'^2 L^2}{n_a \lambda} |\Delta z_f'| + O(n'^4). \quad (2.34) \]

Here \( \theta_{a2} \) is approximated by:

\[ \theta_{a2} = 2 \frac{n' L}{n_a} + O(n'^3). \quad (2.35) \]

Equations 2.33 and 2.34 imply that the phase error in the interferogram is a linear function of the shift between the actual virtual object plane with respect to the optimum virtual object plane. For small \( n' \), the phase error becomes proportional to \( n'^2 L^2 \).
2.4 Application

An interferometer that is proposed to make (sets of) interferograms of a profile with a refractive index gradient up to $n_{\text{max'}}$ must be aligned such that for every gradient in the interval $[0,n_{\text{max'}}]$ the phase error is sufficient small. This means that the position of the actual virtual object plane must lie sufficiently close to the position of the optimum virtual object plane for every $n'$. In this section, the allowed position of the actual virtual object plane in a Michelson interferometer to keep the absolute phase error smaller than a value $|\Delta \phi|$ is calculated for two object configurations. The calculations are made under the condition that the medium between the windows of the object is at least locally linearly stratified. That is, the medium contains a one-dimensional or a two-dimensional refractive index profile (with its gradient perpendicular to the propagation direction of the test beam) which can locally be treated, for every individual ray, as being linearly stratified. Then, the theory developed in the Sections 2.2 and 2.3 remains valid. Local linear stratification implies that a range of refractive index gradients within the interval $[0,n_{\text{max'}}]$ may appear simultaneously in the profile, but that the allowed second order derivative of the refractive index is limited. The different gradients may also appear in a dynamic profile as a function of time. In that case, the phases in succeedingly recorded (sets of) interferograms are accurately related to each other within an absolute phase error $|\Delta \phi|$. The wavelength of the test beam in the calculations is 632.8nm and the refractive index $n_a$ of the surrounding air is 1. The object configurations are described by:

1. a compressible aerodynamic flow in a wind tunnel: medium length $L=150\text{mm}$, windows with thickness $t=20\text{mm}$ and refractive index $n_g=1.5$, entrance refractive index $n_a=1$;
2. a diffusion/mixing process of a dilute sucrose solution in an aqueous bulk liquid in a square cuvette: medium length $L=10\text{mm}$, windows with thickness $t=1.25\text{mm}$ and refractive index $n_g=1.5$, entrance refractive index $n_a=1.33$.

For these objects the range of allowed positions of the actual virtual object plane as a function of $n_{\text{max'}}$ and $|\Delta \phi|$ is calculated.

An important measure beside the refractive index gradient $n'$ is the dimensionless refractive index gradient $n'L$. The dimensionless refractive index gradient has two advantages over the refractive index gradient. Firstly, in a first approximation where $z_{\text{f,op}}$ does not depend on $n'$ (Equation 2.24) and the phase error is described by Equation 2.34 $n'L$ determines the relation between the phase error and the distance over which the virtual object plane is misimaged. Secondly, the phase gradient in the recording plane is related to $n'L$ and thus also the number of fringes in this plane. Provided that the actual virtual object plane lies sufficiently close to the optimum virtual object plane, for a Michelson interferometer the phase gradient $\phi'$ in the recording plane is given by:

$$\phi' = \frac{4\pi}{\lambda G} n'L,$$  
(2.36)
Figure 2.5a,b The bending of the ray trajectories in the wind tunnel configuration for $a=0$ (a) and $a=1$ (b).

Figure 2.6a,b The bending of the ray trajectories in the cuvette configuration if $a=0$ (a) and $A=L$ (b).

where $G$ is the magnification of the imaging optics. The disadvantage of the dimensionless refractive index gradient with respect to the refractive index gradient is that it is not an independent parameter in the exact mathematical description of the ray trajectories (Equation 2.11), the position of the optimum virtual object plane (Equation 2.22) and the
phase error (Equation 2.33). Apart from $n' L$ also $L$ appears in the equations. Hence, in the calculations a change in $n' L$ must always originate from a change in $n'$ while $L$ is kept constant. For the sake of clearness, the results of the exact calculations will be presented as a function of both $n'$ and $n' L$.

The assumption of locally linearly stratification in the calculations implies that the second order derivative of the refractive index is limited. Or better, the refractive index gradient is approximately constant over a distance equal to the bending of the ray at point $G$. This bending is approximated by Equation 2.14a. For completeness, the exact bending based on Equation 2.11 is presented as a function of $z$ in Figures 2.5a,b for the wind tunnel configuration and in Figures 2.6a,b for the cuvette configuration. The bendings correspond to $n' L = 0.01$, $n' L = 0.006$, $n' L = 0.003$ and $n' L = 0.0015$. The Figures 2.5a and 2.6a show the bendings for $a = 0$ (mirror M0 is positioned against the window of the object, see Figure 2.1), the Figures 2.5b and 2.6b show the trajectories for $a = L$ (the distance between mirror M0 and the window is $L/2$). The figures show that the distance over which the refractive index gradient must be approximately constant depends on $n'$ and is of the order of a few millimeters for the wind tunnel configuration and of the order of a few tenth of a millimeter in the cuvette configuration, provided that $n' L \leq 0.01$. Further, this distance increases linearly with length $a$ of the air slab.

The position of the optimum virtual object plane $z_{vop}'$ is a weak function of $n'$, see
Equation 2.24. Figures 2.7a,b show the shift of $z_{f,op}'$ as a function of $n'$ for the wind tunnel configuration and the cuvette configuration, i.e., $z_{f,op}'(n')$-$z_{f,op}'(0)$. The shift increases slightly with $a$, and is considerable larger in the wind tunnel configuration than in the cuvette configuration in case of equal $n'L$. The allowed distance $|\Delta z'|$ between the actual and the optimum virtual object plane to keep the absolute phase error smaller than $|\Delta \phi|$ is given by Equation 2.33. It increases linearly with $|\Delta \phi|$ and it is almost proportional to $(n'L)^2$ (Equation 2.34). The increase of $z_{f,op}'$ with $n'$ and the decrease of $|\Delta z'|$ with $n'$ implies that it is impossible to keep the absolute phase error in the interferogram smaller than any arbitrary value $|\Delta \phi|$ for every $n'$ in $[0,n_{max}')$, even if the actual virtual object plane could be chosen with an infinite accuracy. The choice of the value $|\Delta \phi|$ is coupled to a maximum value for $n_{max}'$, whereby $n_{max}'$ increases with $|\Delta \phi|$. This is elucidated in Figure 2.8. The curve represents $z_{f,op}'$ as a function of $n'$ for the wind tunnel configuration and $a=0$. The error bars represent $|\Delta z'|$ for different values of $n'$, whereby $|\Delta \phi| \leq 1\%$ of $2\pi$ radians. To keep the absolute phase error smaller than 1% of $2\pi$ radians for any value of $n'$, the actual virtual object plane must be positioned somewhere between the upper bound and the lower bound given by the error bars belonging to that $n'$. To keep the absolute phase error smaller than 1% of $2\pi$ radians for all the values of $n'$ in the interval $[0,n_{max}')$, the actual virtual object plane must be positioned somewhere in the overlap area of the error bars. Figure 2.8 shows the overlap area for $n_{max}'=0.097m^{-1}$. It also shows that there is no overlap area if $n_{max}' > 0.118m^{-1}$. Hence, the theoretical maximum value for $n_{max}'$ is equal to 0.118m$^{-1}$ if $|\Delta \phi| \leq 1\%$ of $2\pi$ radians. An increase of the allowed absolute phase error leads to an increase of $|\Delta z'|$ and also to an increase of the overlap area and the maximum $n_{max}'$. Since $z_{f,op}'$ increases with $n'$ and $|\Delta z'|$ decreases with $n'$, the lower bound of the overlap
area (as long there is any overlap) is determined by the lower error bar at \( n' = n_{\text{max}}' \). The upper bound is determined by the upper error bar which corresponds to the lowest value of \( z_j' \). In Figure 2.8, this is the upper error bar at \( n' = 0.076 \text{m}^{-1} \). In order to keep the absolute phase error smaller than \(|\Delta \phi|\) over the whole interval \([0, n_{\text{max}}']\), this implies that the actual virtual object plane must be positioned at a position \( z_j' \) given by:

\[
z_j'_{\text{op}}(n=n_{\text{max}}') - |\Delta z_j'(n' = n_{\text{max}}', |\Delta \phi|) \leq z_j' \leq \text{Min}\left\{z_j'_{\text{op}}(n') + |\Delta z_j'(n', |\Delta \phi|)\right\}; \quad n' \in [0, n_{\text{max}}'].
\]  

(2.37)

As soon as the term on the left hand side of Equation 2.47 exceeds the term on the right hand side, it is impossible to make interferograms with an absolute phase error smaller than \(|\Delta \phi|\) over the whole interval \([0, n_{\text{max}}']\).

Figure 2.9a shows the upper bound and the lower bound of \( z_j' \) as a function of \( n_{\text{max}}' \) with \(|\Delta \phi|\) being parameter for the wind tunnel configuration and \( a = 0 \). Figure 2.9b shows the same for the cuvette configuration.

In practical measurements, the value of \( n_{\text{max}}' \) is often much lower than its theoretical maximum value belonging to the desired maximum phase error \(|\Delta \phi|\). Then, the upper
Figure 2.10: The upper and lower bounds of $z_f$ for $|\Delta \phi| \leq 1\%$ of $2\pi$ rad and $|\Delta \phi| \leq 10\%$ of $2\pi$ rad: exact and approximated.

and lower bound of $z_f$ are only of interest for low values of $n_{max}$. As a consequence, the dependence of $z_{f,op}$ on $n'$ can be neglected and $|\Delta z_f|$ can be assumed to be proportional to $(n'L)^2$. Equation 2.37 can now be approximated by:

$$z_{f,op} + |\Delta z_f(n' = n_{max}', |\Delta \phi|)| \leq z_f \leq z_{f,op}(n' = n_{max}') + |\Delta z_f(n' = n_{max}', |\Delta \phi|)|,$$

(2.38)

where $z_{f,op}$ is approximated by Equation 2.24 and $|\Delta z_f(n', |\Delta \phi|)|$ is approximated by Equation 2.34. The allowed distance between the optimum virtual object plane and the actual virtual object plane to keep the absolute phase error smaller than $|\Delta \phi|$ has become a true function of the independent parameter $n'L$ and thus dependent of the configuration of the object. Only the position of the optimum virtual object plane is dependent on the object configuration and approximated by the value corresponding to $n' = 0$, i.e., $z_{f,op}(0)$. Figure 2.10 shows the upper and lower bound of the allowed distance $z_f(n') - z_{f,op}(0)$ as a function of $n'L$ in order to keep $|\Delta \phi| \leq 1\%$ of $2\pi$ radians and $|\Delta \phi| \leq 10\%$ of $2\pi$ radians. Presented are the exact calculations for the wind tunnel configuration and the cuvette configuration based on Equation 2.37 ($a = 0$) and the approximation based on Equation 2.38. The figure shows that the bounds following from the approximation and the exact calculation coincide for small values of $n'L$. For larger values of $n'L$ they show a discrepancy, finally resulting in no prediction of a maximum value for $n'$ by the approximation. Under the assumption that the approximation satisfies when the deviation between the approximated and calculated bounds is less than $10\%$ of the distance between the upper and lower bound, the approximation satisfies for the wind tunnel configuration for $n'L$ up to 0.0076 ($|\Delta \phi| \leq 1\%$ of $2\pi$ rad) and 0.0135 ($|\Delta \phi| \leq 10\%$ of $2\pi$ rad). For
the cuvette configuration this is up to 0.0167 ($|\Delta \phi| \leq 1\%$ of $2\pi$ rad) and 0.0297 ($|\Delta \phi| \leq 10\%$ of $2\pi$ rad).

In the interpretation of the presented results for practical usage there must be kept in mind that in the calculations of the upper bound and lower bound of $z'$ the value of $n_e$ is kept constant. However, in practical measurements $n'$ is unequal to zero and thus $n_e$ is a weak function of the entrance position of the rays in the medium. Moreover, if a dynamic process is measured, $n_e$ is also a function of the time. This implies that $n_e$ varies in space and time over an interval $[n_e - \Delta n_e, n_e + \Delta n_e]$. Now, the required position of the virtual object plane has become a function of the position in the interferogram and the time. However, the Figures 2.10a,b remain a good approximation as long as the shift in the position of the virtual object plane due to a change in $n_e$ remains much smaller than the allowed position interval in the figure. For small values of $n_{\text{max}}'$ follows by using Equations 2.24 and 2.34:

$$\frac{dz'_{\text{op}}}{dn_e} \Delta n_e \leq |\Delta z'| \iff \Delta n_e \leq \frac{3}{2} \frac{n_e^2}{n_e L} |\Delta z'| \iff \Delta n_e \leq \frac{3}{4} \frac{n_e^2 \lambda}{n_e^2 L^3} \frac{|\Delta \phi|}{2\pi}. \quad (2.39)$$

Within one single interferogram the maximum variation in $n_e$ is given by $\Delta n_e \leq n' h/2$, where $h$ is the height of the imaged part of the object. Then, Equation 2.39 gives a limitation to the height over which the phase can be measured within a phase error $|\Delta \phi|$, i.e.,

$$h \leq \frac{3}{n_e n' L} |\Delta z'| \iff h \leq \frac{3}{2} \frac{n_e^2 \lambda}{(n' L)^2} \frac{|\Delta \phi|}{2\pi}. \quad (2.40)$$

### 2.5 The Mach-Zehnder interferometer

The double passage of the test beam through the object makes a Michelson interferometer more sensitive to refraction effects than a Mach-Zehnder interferometer. A similar imaging accuracy of the virtual object plane on the recording plane leads in the Michelson interferometer to a larger absolute phase error in the interferogram. In order to compare the measuring accuracy of a Michelson interferometer and a Mach-Zehnder interferometer (see the discussion in Section 2.6), the ray trajectories, the position of the optimum virtual object plane and the absolute phase error introduced by misimaging are calculated for the Mach-Zehnder interferometer in this section.

In Figure 2.11 the object from Section 2.1 (a linearly stratified medium with length $L$ bounded by parallel windows with thickness $t$) is positioned in the test section of a basic Mach-Zehnder interferometer. The ray propagation in the test section and towards the recording plane is presented in Figure 2.12. The coordinate system is equal to that in
Figure 2.11 An object placed in the test section of a basic Mach-Zehnder interferometer.

Figure 2.12 Propagation of a refracted ray and a virtual straight ray through the object and towards the recording plane in a Mach-Zehnder interferometer.
Figure 2.2. A refracted ray and a virtual straight ray traverse the object and are finally refracted by lens L2 towards the recording plane. The refracted ray enters the first window at \((y=y_e, z=-t)\) and follows the trajectory ABCDD'S, the virtual straight ray enters the first window at \((y=y_e, z=-t)\) and follows the trajectory IJKLRR'S. Similar to the arc circle HR in Figure 2.2, DR is a circle arc with its center in point O. Point O is the crossing of the backward extension of the refracted ray leaving the window (trajectory DD') and lies in the virtual object plane of lens L2 at \(z=z_f'\). The phase difference in S is equal to the phase difference between the refracted ray in point D and the virtual straight ray in point R. Since the two rays are in phase up to \(z=0\), the phase difference originates from a difference in optical path length along the trajectories BCD and JKL.

The trajectories BC and CD of the refracted ray through the medium and the window are equal to those in the Michelson interferometer and thus they are described by Equation 2.11a,b. Their bendings can for small \(n'\) be approximated by:

\[
y_{bc}(L) - y_e = \frac{n'^2L^2}{2n_e} + O(n'^3)
\]  \hspace{1cm} (2.41a)

and

\[
y_{cd}(L+t) - y_e = \left[ \frac{L}{2n_e} + \frac{t}{n_e} \right] n' L + O(n'^3).
\]  \hspace{1cm} (2.41b)

The optical path lengths along the trajectories are given by the Equation 2.17a,b.

The optical path length along trajectory KL of the virtual straight ray through the medium is found in a similar way as the optical path length along the corresponding trajectory in the Michelson interferometer. From Figure 2.12 follows that the entrance height \(y_p\) is related to the position \(z_f'\) of the virtual object plane by:

\[
y_p = y_{cd}(L+t) - (L+t-z_f')\tan\theta_a,
\]  \hspace{1cm} (2.42)

where \(\theta_a\) is given by:

\[
\theta_a = \arcsin \left[ \frac{n_e y'_e}{n'_a y_{bc}(L)} \right],
\]  \hspace{1cm} (2.43)

and \(y_{bc}'(z)\) is given by Equation 2.13. For the optical path length \(\Phi_{KL}\) follows:

\[
\Phi_{KL} = n_e L + n' L (y_p - y_e) = n_e L + n' L \left( y_{cd}(L+t) - y_e - (L+t-z_f')\tan\theta_a \right).
\]  \hspace{1cm} (2.44)

The optical path length \(\Phi_{KL}\) along trajectory KL through the window is equal to that in the
Michelson interferometer and is simply represented by Equation 2.20b. The optical path length along trajectory LR through the surrounding air is unequal to the optical path length QR in the Michelson interferometer. From Figure 2.12 follows for the optical path length \( \Phi_{LR} \):

\[
\Phi_{LR} = n_a(L+t-z'_f) \sqrt{1+\tan^2 \vartheta_a} - 1
\]

\[
= n_a(L+t-z'_f) \cdot \left(\sqrt{1+\tan^2 \arcsin \left[ n_e y_{BC}(L)/n_a \right]} - 1 \right).
\]

(2.45)

The phase error in the interferogram is determined by the difference in the optical path lengths along the trajectories BCD and JKLR, which is described by:

\[
\Delta \Phi = (\Phi_{BC} + \Phi_{CD}) - (\Phi_{JK} + \Phi_{KL} + \Phi_{LR}).
\]

(2.46)

Substitution of the Equations 2.44 and 2.45 under the requirement that \( \Delta \Phi = 0 \) leads to the next equation for the position \( z_{f,op}' \) of the optimum virtual object plane:

\[
z_{f,op}' = \frac{\Phi_{BC} + \Phi_{CD} - \Phi_{KL} - n_a(L+t)(\sqrt{1+\tan^2 \vartheta_a} - 1) - \gamma n'L - n_aL}{n'L \tan \vartheta_a - n_a(\sqrt{1+\tan^2 \vartheta_a} - 1)},
\]

(2.47)

where \( \gamma \) represents

\[
\gamma = y_{CD}(L+t) - y'_e - (L+t) \tan \vartheta_a.
\]

(2.48)

For small values of \( n' \), \( z_{f,op}' \) can be approximated by:

\[
z_{f,op}' = L - \frac{1}{3} \frac{n_a}{n_e} L + \left(1 - \frac{n_e}{n_a}\right) t + O(n'^2).
\]

(2.49)

This position of the optimum virtual object plane is in agreement with the position found earlier by Lanen\(^1\) and it implies that an object plane at \( z_f = 2L/3 \) must be imaged for a minimum phase error.

Misimaging by lens L2 introduces a phase error in the interferogram. In the same way as it was done for the Michelson interferometer in Section 2.3, the relation between the absolute phase error \( |\Delta \Phi| \) and the distance between the actual virtual object plane and the optimum virtual object plane \( \Delta z_f' \) is derived. For this relation is found:
\[
\frac{|\Delta \phi|}{2\pi} = \left| -n'L \tan \theta_a + n_a \left( \frac{\tan^2 \theta_a + 1}{\lambda} \right) \right| |\Delta z'_f|, \tag{2.50}
\]

where

\[
\Delta z'_f = z'_f - z_{f,opt}. \tag{2.51}
\]

For small values of \( n' \), Equation 2.50 can be approximated to the second order with respect to \( n' \) by:

\[
\frac{|\Delta \phi|}{2\pi} \approx \frac{1}{2} \frac{n'^2 L^2}{n_a \lambda} |\Delta z'_f| + O(n'^4). \tag{2.52}
\]

Here is made use of the first order approximation of Equation 2.43 given by:

\[
\theta_a = \frac{n'L}{n_a} + O(n'^3). \tag{2.53}
\]

### 2.6 Discussion and conclusions

**Relevance of the results**

Often, a refractive index profile is not a true locally linearly stratified medium. The second order derivative of the refractive index is too large or the refractive index varies also in the \( z \)-direction and thus the refractive index has become three dimensional. Lanen\(^1\) and Beach\(^2\) studied the ray propagation through an object in the test section of a Mach-Zehnder interferometer in presence of a large second order gradient. They found that in order to minimize the phase error due to refraction effects the virtual object plane must be the same as in the case of linear stratification. It is plausible that this is also valid for the Michelson interferometer. The phase error, however, will be larger than in case of local linear stratification. In asymmetric three dimensional refractive index profiles the position of the virtual object plane is mainly determined by the avoidance of ray crossing in the interferogram and can deviate from the position under linear stratification\(^8\).

The assumption of a locally linearly stratified medium in the wind tunnel configuration and the cuvette configuration are approximations. A compressible aerodynamic flow (e.g., a boundary layer flow) in a wind tunnel is never exactly two-dimensional since there will always be a boundary layer along windows and thus a refractive index gradient in the \( z \)-direction. Moreover, large density changes can lead to a serious second order derivative in the refractive index. A diffusion process of a dilute sucrose solution in an aqueous liquid bulk in a cuvette can give rise to a locally linearly stratified medium, but active
mixing results in a three dimensional refractive index profile. However, as long as the medium is almost locally linearly stratified it is plausible that the phase error is minimum if the virtual object plane is chosen such as the medium would be locally linearly stratified.

The expound in Section 2.4 shows that even if a Michelson interferometer would be perfect (perfect alignment, infinite small pixel size, perfect optical elements, no noise) and the medium is locally linearly stratified, there will always be a phase error due to refraction effects. This error originates from the fact that the optimum virtual object plane is a weak function of \( n' \) and appears as a spatially varying phase error over the interferogram and as a relative phase error between succeedingly recorded interferograms. A maximum absolute phase error is coupled to a maximum value of \( n' \) and a maximum value of \( n' \) is coupled to a fixed maximum phase error. In practise, the maximum measurable refractive index gradient will be smaller than presented in Section 2.4. Often, the maximum value of \( n' \) is determined by other aspects than the shift of the optimum virtual object plane, e.g., the distance over which there is misimaged and the required minimum number of pixels per fringe in the interferogram. In general, the maximum value of \( n' \) in a refractive index profile is much smaller than the theoretically measurable gradient belonging to the desired maximum phase error. Then, the required position of the actual virtual object plane can be found by using the approximations for the position of the optimum virtual object plane and the allowed deviation, i.e., Equations 2.24 and 2.34. This implies that the position of the virtual object plane depends on the object configuration, but the allowed deviation is a function of \( n'L \) and thus a function of the phase gradient and independent of the object configuration.

The potential measuring accuracy of an interferometer can best be characterized by using a (locally) linearly stratified medium, provided that \( n' \) is small. Then, no high imaging accuracy is required to keep the phase error due to refraction effects small and thus possible errors in the phase measurements are induced by the interferometer and not by the refractive index profile. The phase error generated during an utilization when the medium is not locally linearly stratified is the sum of the errors induced by the interferometer and the refractive index profile. This last error can possibly be approximated by using a ray tracing scheme.1

Consequences for the real time interferometer

The real time interferometer is a Michelson interferometer that operates at a wavelength \( \lambda = 632.8 \text{nm} \) and that is designed for measuring dimensionless refractive indices up to \( n_{\text{max}}L = 0.0018 \). The object configurations intended to be tested are a wind tunnel configuration and a cuvette configuration. The results of Section 2.4 lead to several consequences for the measurements performed by real time interferometer. They are
discussed below for the case that the medium is linearly stratified.

The bending of the ray trajectory in an object determines:
- the distance over which the refractive index gradient must be approximately constant in order to satisfy local linear stratification,
- the extension of a dark area around an opaque object in a medium where the phase cannot be measured.

From Equation 2.14a follows that the bending of the rays at the entrance of the last window is minimized if \( a \) is minimized, preferably set equal to 0. The bending is equal to 588\( \mu \)m in the wind tunnel configuration and equal to 30\( \mu \)m in the cuvette configuration. This implies that the refractive index gradient in the wind tunnel configuration must remain approximately constant over a much larger distance than in the cuvette configuration.

The bending of the rays can also introduce a dark area around an opaque object in the medium (e.g., a wall) where the phase cannot be measured. It appears if the refractive index increases perpendicular to an opaque object and the rays are refracted from its surface. Apart from the bending, the extension of the area is also dependent on the position of the virtual object plane. Figure 2.13 shows the ray propagation along an opaque object that is extended along the whole length of the medium. The ray that enters the medium nearest to the surface of the opaque object at \( y = y_e \) is refracted from that surface and is imaged at the position S in the interferogram in the recording plane belonging to the virtual straight ray at \( y = y_p \). The interferogram is dark in the area belonging to the virtual straight rays between \( y = y_e \) and \( y = y_p \) because the corresponding refracted rays are removed by the opaque object. The distance \( y_p - y_e \) can easily be
approximated for small \( n' \). If the actual and optimum virtual object plan coincide, for a Michelson interferometer follows from Equations 2.18, 2.14b, 2.24 and 2.35 that \( y_{p} y_{c} = (2L/3n_{c} + t/n_{c} + a/2n_{c})n' L \). Provided that \( a = 0 \), that is a maximum dark area of 204\( \mu \)m in the wind tunnel configuration and a maximum dark area of 11\( \mu \)m in the cuvette configuration.

The value \( n_{\text{max}}'L = 0.0018 \) is small enough to describe \( z_{f,\text{op}}' \) and \( |\Delta z_{j}'| \) by their approximations, i.e., Equations 2.24 and 2.34. The relation between \( |\Delta z_{j}'| \) and the phase error is independent of the object configuration. An imaging accuracy whereby \( |\Delta z_{j}'| \leq 1\text{mm} \) leads to a maximum phase error of 1.0% of \( 2\pi \) radians, an imaging accuracy whereby \( |\Delta z_{j}'| \leq 0.5\text{mm} \) leads to a maximum phase error of 0.5% of \( 2\pi \) radians. However, these accuracies can only be obtained if Equations 2.39 and 2.40 are fulfilled. For \( |\Delta z_{j}'| \leq 0.5\text{mm} \) follows for the wind tunnel configuration \( \Delta n_{c} \ll 0.005 \) and \( h \ll 0.8\text{m} \). For the cuvette configuration follows \( \Delta n_{c} \ll 0.1 \) and \( h \ll 1.5\text{m} \). These conditions are generally fulfilled, although the variation in \( n_{c} \) with the time during aerodynamic measurements depend on the character of the measurements. The height \( h \) of the imaged part of the object is in the real time interferometer about 14\( \text{mm} \). The variation in \( n_{c} \) during sucrose diffusion/mixing experiments is smaller than 0.007 provided that the sucrose concentration remains smaller than 50g/l.

**Comparison between a Michelson and a Mach-Zehnder interferometer**

Minimization of the phase error in 2D refractive index profile measurements requires a minimum deviation of the position of the actual virtual object plane from the optimum virtual object plane. That is, the position of the actual virtual object plane must lie sufficiently close to the position given by Equation 2.22 for the Michelson interferometer and to the position given by Equation 2.47 for the Mach-Zehnder interferometer. An increasing deviation leads to an increasing phase error. A potential relative measuring accuracy of a Michelson interferometer with respect to a Mach-Zehnder interferometer can be calculated by using the results found in the preceding sections. The term 'potential' is added here, since not all of the required conditions will be satisfied practically. These conditions are:

- there is no other error source than refraction and misimaging,
- the medium in the object is locally linearly stratified,
- the refractive index gradient lies in the interval \([0, n_{\text{max}}']\), and \( n_{\text{max}}' \) is small enough to keep \( z_{f,\text{op}}' \) independent of \( n' \) and to let the distance \( \Delta z_{j}' \) between the actual and the optimum virtual object plane be fully determined by the limited alignment accuracy (and possible imperfections) of the optical elements in the interferometer,
- \( \Delta z_{j}' \) is the same in the Michelson interferometer and the Mach-Zehnder interferometer. Under these conditions, the absolute phase errors induced by the Michelson interferometer and the Mach-Zehnder interferometer are given by Equation 2.34 and Equation 2.52,

3

Design of the real time interferometer and accuracy analysis

3.1 Introduction

Two dimensional interferometry is based on the extraction of the phase in every point of a digitized interferogram. There are several methods for the extraction of a so-called phase distribution, each with its own advantages and disadvantages. The number of interferograms needed for the phase calculation differ per method. Some methods need only one single interferogram while other methods require some additional, phase shifted interferograms.

Most interferometers have only one optical channel in which interferograms can be recorded. If they do not operate in real time, the required number of phase shifted interferograms can be recorded sequentially in time. Then, the phase shift can be introduced by piezo-electric transducer mounted on the mirror in the reference beam or an optical element in the reference beam, for example, a tiltable glass plate or a movable diffraction grating\(^1\). However, in a real time operation no phase shift can be introduced and applicable phase calculation methods are limited to those who need only one interferogram.

In a real time interferometer all the phase shifted interferograms belonging to one measurement have to be recorded simultaneously. The number of the interferograms depends on the phase calculation method. It implies that the real time interferometer must be equipped with a separate optical channel for each interferogram and that the phase
steps must be introduced by using polarization optics or a diffraction grating. Hence, the complexity of the optical set-up increases with the required number of interferograms and thus the design of the interferometer is strongly dependent on the chosen method for phase calculation. A design process of a real time interferometer must, therefore, be started with an analyses of the possible phase calculation methods. The final choice of a method is dependent on the characteristic properties of the expected interferograms and the required accuracy of the phase calculation.

The various phase calculation methods are discussed in section 3.2 and finally one method, a spatial phase stepping method, is preferred to be applied in the RTI. The design of the optical set-up is presented in Section 3.3 together with its alignment procedure. The fringe analysis is discussed in Section 3.4. An extended accuracy analysis is given in the Sections 3.5-3.9. Here, the phase errors due to different error sources are calculated. In Section 3.10 the optimum accuracy of the real time interferometer (RTI) is compared to the optimum accuracy of a temporal phase shifting interferometer (TPSI). This TPSI is built in in the set-up of the RTI and it is used in Chapter 4 for the validation of the RTI. Finally, Section 3.11 presents theoretical conclusions.

### 3.2 Choice of phase calculation method

The optical design of the RTI is based on the method of phase calculation. Therefore, this section presents a brief review of known phase calculation methods with their specific advantaged and disadvantages. This review is followed by a discussion about the application of one of these methods, the phase stepping method, in this thesis.

#### 3.2.1 A review of the phase calculation methods

The intensity distribution \( I(x,y) \) within an interferogram is given by Equation 1.6. The phase distribution \( \phi(x,y) \) is related to the quantity being measured, for example, a refractive index profile \( n(x,y) \), and therefore this phase distribution has to be extracted. For the sake of clearness, Equation 1.6 can be rewritten as:

\[
I(x,y) = I_b(x,y) + I_m(x,y) \cos \phi(x,y),
\]

(3.1)

where \( I_b(x,y) \) is the bias intensity (the background intensity) given by:

\[
I_b(x,y) = I(x,y) + I_s(x,y)
\]

(3.2a)

and \( I_m(x,y) \) is the modulation intensity (the contrast) and is given by:

\[
I_m(x,y) = 2 \sqrt{I(x,y)I_s(x,y)}.
\]

(3.2b)
Here $I_t(x,y)$ and $I_r(x,y)$ represent the intensities in the test beam and the reference beam, respectively. Calculation of the phase $\phi(x,y)$ from Equation 3.1 is impossible since it contains, apart from $\phi(x,y)$, two additional unknowns, i.e., the bias intensity $I_b(x,y)$ and the modulation intensity $I_m(x,y)$. The phase calculation methods overcome this problem, each in its own specific way. Here will be distinguished the phase stepping method, the fringe counting method, several spatial carrier methods, the phase-locked loop method and heterodyne interferometry. The phase stepping method requires at least three interferograms to calculate the phase distribution, heterodyne interferometry a continuous phase shift. The other methods need only one single interferogram.

In the phase stepping method multiple interferograms are recorded which are phase shifted with respect to each other over a phase step $\gamma(x,y)$. This phase step must be fixed for corresponding points $(x,y)$ in the different interferograms, but it is not necessarily required to be equal for each point in the interferogram. The intensity distribution in the phase stepped interferograms are now given by:

$$I_i(x,y) = I_b(x,y) + I_m(x,y)\cos(\phi(x,y) + i\gamma(x,y)) \quad i=0,1,2,\ldots$$

(3.3)

where $I_i(x,y)$ is the intensity distribution in interferogram number $i$. If the phase step $\gamma(x,y)$ is known, the minimum number of interferograms to solve the set of Equations 3.3 is 3. However, if the phase step is an additional unknown, at least 4 phase stepped interferograms are required. Several phase stepping algorithms are known for the calculation of $\phi(x,y)$ from Equations 3.3. Most of them require the phase step to be equal to a pre-fixed value. Some other, the so called self-calibrating algorithms, do not have this restriction.

An important advantage of the phase stepping method is that the set of Equations 3.3 is solved for each point $(x,y)$ separately. Therefore the phase stepping method is insensitive to spatial deviations in the bias intensity, the modulation intensity and, possibly, the phase step.

The phase stepping method can be subdivided in a temporal phase stepping method and a spatial phase stepping method. In the temporal phase stepping method the phase stepped interferograms are recorded in sequence in time in a single channel of an interferometer. A high measuring accuracy can be achieved, i.e., 1%-2% of $2\pi$ radians. The method is very convenient to be applied in holographic interferometry, for example, for measuring density profiles in supersonic flows$^{4,5}$. In the field of speckle interferometry the method is used for deformation measurements$^{6-9}$. The spatial phase stepping method records the phase stepped interferograms simultaneously in different channels of an interferometer. Due to this simultaneous recording, several errors due to mechanical instabilities and turbulence of the surrounding air may be reduced. However, the optical set-up is more complex and several additional errors are introduced. These errors are caused by the fact that each interferogram is imaged in a different channel, each with a finite alignment accuracy, slightly different optical elements and a different (part of a) camera. Also the bias intensity and the modulation intensity may differ slightly in the different channels.
High quality optics and high quality cameras in combination with adjusted software will reduce the errors. Nevertheless, the total accuracy of the spatial phase stepped method remains lower than the accuracy of the temporal phase stepped method\textsuperscript{10}. The advantage of the spatial phase stepped method is, however, its convenience for repetitive measurements. A typical time resolution is video rate. This rather low resolution limits the application to rather slow processes. Several proposals for a spatial phase stepping interferometer can be found in the literature\textsuperscript{11-18}. Some of them are used in the field of refractive index profile measurements, for example, for measuring flow\textsuperscript{19,20,21}, concentration fields\textsuperscript{22} and diffusion fields\textsuperscript{23,24}.

The \textit{fringe counting method} is a method which extracts the phase distribution from one single interferogram. It makes use of the fact that the phase difference between the maxima (or minima) of two neighbouring fringes is equal to $2\pi$ radians. The method starts with determining the positions of the maxima (or minima) in the interferogram, often by applying a fringe tracking procedure or a skeletonizing procedure. After that, the phase at the positions between the fringe maxima (or minima) is determined by interpolation. Several (semi-) automatic fringe calculation algorithms have been developed\textsuperscript{25-28}. However, fringe counting has several disadvantages. Firstly, some prior information about the increase and the decrease of the phase in the field is needed, because the sign of the phase change cannot be extracted. Secondly, the method is very sensitive to noise and thus pre-filtering is necessary. Thirdly, a small gradient in the refractive index profile leads to widely spread fringes. Since the method is based on interpolation between fringe maxima, the spatial resolution is low for small gradients. Finally, the position of the fringe maxima is influenced by non-uniformities in the bias intensity and the modulation intensity so that the phase difference between two maxima is not exactly equal to $2\pi$ radians and an error may occur in the analysis. Although the effects of the disadvantages may be overcome or may be reduced by adding carrier fringes to the interferogram\textsuperscript{10}, the accuracy remains rather low.

The existence of carrier fringes characterizes a so-called finite-fringe interferogram. Methods based on the calculation of the phase distribution from one single finite-fringe interferogram are called \textit{spatial carrier methods}. Several spatial carrier methods exist. In general, linear carrier fringes are introduced by tilting the wavefront of the reference beam with respect to the wavefront of the test beam. In the interferogram the phase $\phi(x,y)$ of the test beam is visible in the deviations of the fringes from parallel straight lines. If the carrier fringes are parallel to the y-axis, the intensity distribution in the finite-fringe interferogram is given by:

$$I(x,y) = I_b(x,y) + I_m(x,y) \cos(2\pi f_0 x + \phi(x,y)),$$

where $f_0$ is the spatial carrier frequency in the x-direction. An extraction of the phase $\phi(x,y)$ from this equation without sign ambiguity and without magnitude ambiguity requires several conditions to be fulfilled. Firstly, the total phase must be a monotonic
function. Therefore, the phase gradient $2\pi f_0$ introduced by the carrier fringes must be higher than the maximum phase gradient in $\phi(x,y)$. Secondly, the individual fringes must be visible. Hence the carrier frequency must be less than half of the sampling frequency (Nyquist condition). Finally, the variations in the bias intensity and the modulation intensity must be slow when compared to the carrier frequency.

Typical spatial carrier methods are the **Fourier transform method**, the **spatially synchronous fringe analysis**, the **spatial carrier phase stepping method** and the **phase locked loop method**.

The **Fourier transform method** is a spatial carrier method in which the processing is performed in the frequency domain\textsuperscript{29,10,30,31}. The digitized finite-fringe interferogram is Fourier transformed by using a fast Fourier transform-algorithm. If $I_6(x,y)$, $I_7(x,y)$ and $\phi(x,y)$ are slowly varying functions compared to $2\pi f_c x$, the Fourier spectrum will exhibit three distinct lobes. One of the lobes is located around the origin and is related to the bias intensity. The other two lobes are complex conjugates, they are located around the carrier frequency $\pm f_0$ and they are related to the modulation intensity and the phase. One of these two lobes is filtered out, is translated to the origin of the Fourier spectrum and is, after that, inversely Fourier transformed. From the resulting complex function the phase can be calculated by taking the arctangent of the ratio of the imaginary part and the real part. Optimum results can be achieved with the Fourier transform method if the carrier frequency is about 4 pixels per fringe\textsuperscript{32,33}. However there are several error sources\textsuperscript{29}. Firstly, there are the errors due to the discrete nature of the fast Fourier transform. Here one can think of aliasing, if the sampling frequency does not exceed the Nyquist condition; the picket fence effect, if there are frequencies in the interferogram unequal to the discrete frequencies; and an error in the translation of the lobe to the origin of the spectrum, if the carrier frequency is not exactly equal to a discrete frequency\textsuperscript{34}. Secondly, spurious fringes and random noise will lead to errors. Spatial frequencies lower than the maximum fringe frequency cannot be filtered out. Thirdly, and most importantly, energy leakage will appear. Due to the finite extension of the sample window side ripples will appear in each lobe of the Fourier spectrum. These ripples will interfere with each other and cause a leakage of energy from one frequency into adjacent frequencies. Significant errors arise, mainly at the boundaries of the sample window\textsuperscript{35}.

**Spatially synchronous fringe analysis** is a method equivalent to the Fourier transform method. The difference is that the filtering process is not performed in the frequency domain but in the space domain. Several algorithms are proposed in the literature\textsuperscript{36}.

The **spatial carrier phase stepping method**\textsuperscript{40,29} is an application of the spatial phase stepping method where the intensities belonging to the different phase steps are extracted from one single finite-fringe interferogram. In the method the bias intensity, the modulation intensity and the gradient of the phase $\phi(x,y)$ are assumed to be constant within a period of the spatial carrier. Then, the total phase over one period will vary
linearly and the phase difference between adjacent pixels is constant. The phase is
calculated by substituting the intensities of several pixels in line within a small interval
around the pixel of interest in the phase stepping algorithm. After correction for the phase
due to the spatial carrier, the phase $\phi(x,y)$ is found. The applied phase stepping algorithm
determines the restrictions for this phase. In general, the gradient of $\phi(x,y)$ will be
unequal to zero over the interval and will vary over the interferogram. Hence the phase
step will vary over the interferogram and, therefore, a self-calibrating algorithm is
preferred\textsuperscript{37,38}.

The phase-locked loop method\textsuperscript{39,40,29} for the extraction of the phase from a finite-fringe
interferogram is based on the well known phase-locked loop technique in electronic
communications for demodulation of electrical signals. Now, the phase-locked loop is not
a piece of electronics but it is implemented in software. The input signal is a line scan of
the interferogram. The basic building blocks for the phase-locked loop is shown in Figure
3.1. The phase modulated input signal is multiplied with the output of a voltage-controlled
oscillator (VCO). This signal passes a first-order low pass filter. The output of this filter
is the input for the VCO. In consequence, the output follows the phase of the input signal
continuously as long as the input phase does not have large discontinuities. The output
phase is, in contrary to other methods, an unwrapped phase. The signal to noise ratio is
rather high because the low pass filter rejects high frequencies.

The most accurate method for quantitative phase measurement is heterodyne
interferometry\textsuperscript{41,29}. In this technique the relative phase between the reference beam and the
test beam is made to increase linearly in time. Hence the interference pattern is time
dependent and exhibits heterodyning beats. The relative phase of these beats in a point in
the interference pattern compared to a reference signal is equal to the wavefront
deformation. An accuracy of 0.1% of $2\pi$ radians can be obtained. Fast measurements can
also be achieved, because the beat frequency may be up to 1MHz. However, high
experimental requirements and a high speed detector are required\textsuperscript{41,29}. 

![Figure 3.1 Building blocks for the phase-locked loop.](image-url)
3.2.2 Choice of the phase calculation method

The phase calculation method preferred to be applied in the real time interferometer must satisfy three criteria, i.e.,
1. it is capable to real time measurements,
2. the corresponding experimental set-up is a practical tool for experimental usage,
3. it can deal with typical interferograms obtained in compressible flows.

The spatial phase stepping method, the fringe counting method and the spatial carrier methods satisfy the conditions 1 and 2. A preference of one of these methods is dependent on criterion 3.

In interferograms obtained in compressible aerodynamic flows the following typical situations may occur:
1. the existence of obscured areas due to the presence of opaque objects like a model or a wall,
2. the fringe-order is unknown,
3. the fringe spacing varies strongly, broad cloudlike fringes and closely spaced fringes exist in the same interferogram,
4. the fringes are discontinuous due to the existence of shock waves.

The fringe counting method gives serious problems in the situations 2 and 3. The method requires prior knowledge about the phase gradient to determine the fringe-order and it will be very inaccurate if the fringe frequency is low. The spatial carrier methods are most convenient for measuring relatively low phase gradients because the phase gradient due to the carrier fringes may not be exceeded. Situation 4 means that also high phase gradients are expected and thus errors are introduced. The Fourier transform method introduces an additional error in situation 1. Just like the sample window the existence of obscure areas will lead to energy leakage. Errors will appear in the (possible) interesting regions around the present object. The spatial synchronous fringe analysis is, for reasons of computational efficiency, often applied in a relatively inaccurate way. In a more accurate application, the method tends to be equivalent to the Fourier transform method and exhibits the same problems. The spatial carrier phase stepping method requires the phase step to be constant over a number of pixels, or, at least, a constant gradient. Chan et al. showed that the expected maximum phase error will be about 1.4% of $2\pi$ radians when a 3 step algorithm is used to calculate a quadratic phase increase to $3\pi$ radians over 525 pixels. This error may be expected to increase linearly with the gradient. The phase locked loop method, finally, introduces errors because it goes out of lock if there are discontinuities in the interferogram.

In contrast to the methods discussed above, the spatial phase stepping method is robust and can deal with all the mentioned situations in the interferograms. In fact, it can deal with any interferogram as long as the sampling frequency exceeds the Nyquist frequency. There is no sign ambiguity in the resulting phase distribution and, due to the absence of carrier fringes, rather high phase gradients are measurable. Finally, bad pixels and pixels with a low modulation depth can be excluded from the evaluation process. A reasonable
approximation for the phase in these pixels can be found by interpolation within the phase distribution.

All these reasons make the spatial phase stepping method a very convenient method to be applied in a real time interferometer for quantitative refractive index profile measurements and, especially, in a real time interferometer used in compressible flow experiments. Hence, the new developed interferometer is based on this method and is thus a multi-channel interferometer. Remains the choice of the number of channels and the phases step between the interferograms.

The choice of the number of channels and the phase step between the interferograms depends on:
1. the accuracy of the corresponding phase stepping algorithm,
2. the size and complexity of the practical set-up.

Several phase stepping methods are known for calculating the phase distribution from a set of phase stepped interferograms\(^{29,1,43}\), i.e., for solving Equation 3.3. Each algorithm requires its own specific number of phase stepped interferograms. Their accuracy is limited by several error sources like, for example, miscalibration of the phase step, non-linearities in the phase step, distortion of the interferogram due to multiple reflections and non-linearities in the camera response. Miscalibration of the phase step leads to quasi-sinusoidal error in the phase distribution with a frequency equal to twice the fringe frequency. Other error sources may also lead to errors which involve first order or higher order terms. The sensitivity to the different error sources varies for the different algorithms. An algorithm can reduce one error while it expasnes the other. For example, an algorithm may correct some higher order errors while it is sensitive to miscalibration of the phase step and vice versa. Theoretically, an increase of the number of phase stepped interferograms may lead to a decrease of the total error in the phase distribution\(^{44-50}\). Practically, however, this reduction of the phase error can mainly be achieved in the temporal phase stepping method. In the spatial phase stepping method it is very difficult to improve the accuracy by increasing the number of interferograms\(^{10}\). The size and the complexity of the experimental real time interferometer increases with the number of channels. A high complexity hinders a quick alignment and thus a practical utility of the tool. Since the increase of the measuring accuracy with the number of interferograms is limited in the spatial phase stepping method, there is chosen to equip the real interferometer with the minimum number of channels, that is three. In order to attain a further reduction of the complexity of the set-up, the phase step is chosen to be \(\pi/2\) radians.
3.3. The experimental set-up

Although there are many applications for a real time interferometer, the final application is that of experimental compressible flow research. This implies that the interferometer must be installed rather easily around a large object like a wind tunnel. Therefore a system based on a Michelson set-up is preferred to a system based on a Mach-Zehnder set-up. In a Michelson set-up the light emission part as well as the detection part can be positioned at one side of the wind tunnel so that it is not necessary to lead the reference beam around the tunnel. The optical elements at the other side of the tunnel are (almost) limited to one single mirror. This section describes the experimental real time Michelson interferometer and its alignment.

3.3.1 Description of the set-up

The layout of the real time, three channel Michelson interferometer is shown in Figure 3.2. The light source is a plane polarized HeNe laser (wavelength 632.8nm, extinction ratio 500:1) with a maximum output power of 17.0mW. The polarization direction of the beam, which is determined by polarization filter P0 (extinction ratio 10000:1), is about 45° to the x-axis (which lies in the plane of drawing). Half wave plate HWPO rotates the polarization direction of the beam and therefore its orientation regulates together with polarization filter P0 the effective intensity of the beam. After expansion by lens combination L0 and L1, the e² beam waist is now 3.0mm, the beam is split by polarizing beam splitter cube PBS0 into a test beam and a reference beam. The intensities of both beams are (almost) equal and their polarization directions are perpendicular. The test beam is expanded by lens combination L2 and L3, it traverses the test section twice and it is shrunk again to its original waist. The double passage of quarter wave plate QWPO, whose fast axis is 45° to the x-axis, rotates the polarization direction so the test beam is transmitted by PBS0. The reference beam is reflected by mirror M1. The double passage of quarter wave plate QWP1 rotates the polarization direction to obtain the right polarisation for reflection by PBS0. After recombination of the test beam and the reference beam by PBS0, their polarisation directions are still perpendicular. Both beams are expanded by lens combination L4 and L5 and split up by the non-polarizing beam splitter cube BS0.

The beams transmitted by BS0 pass quarter wave plate QWP2, whose fast axis is 45° to the x-axis. The test beam and the reference beam are now opposite circularly polarized. After transmission or reflection by polarizing beam splitter cube PBS1, only one linear polarization direction remains what makes the test beam and the reference beam interfere. Due to the original opposite circular polarization of the beams, the phase difference between the beams is shifted with a constant factor. This factor depends on the remaining polarization direction. The light transmitted by PBS1 is polarized in x-direction. Therefore the interferogram on CCD-camera CCD2 has a +π/2 phase shift. The light reflected by
PBS1 is polarized in y-direction so that the interferogram on CCD-camera CCD0 is shifted \(-\pi/2\) in phase.

The test beam and the reference beam which are reflected by BS0 interfere on CCD-camera CCD1 without an additional phase shift. Therefore the beams pass half wave plate HWP1 and polarizing beam splitter PBS2. Because PBS2 transmits light polarized in x-direction only, the adjustment of HWP1 determines the relative intensity of the test beam compared to the reference beam in the interferogram on CCD-camera CCD1. This relative intensity is made equal to the relative intensities on CCD0 and CCD2. With this, errors due to the polarization dependency in the transmission and the reflection by BS0 are corrected in an optical way.

A high level of accuracy requires a high perfection of the optical elements in the set-up. This is further discussed in Section 3.7. The elements used in the set-up are described in Table 3.1.

The extension of the test volume in the interferometer is determined by the beam waist D2 in the test section. Therefore it is determined by the relative focal lengths of the lenses L2 and L3. In all the experiments presented in this thesis the focal lengths of lens L2 and lens L3 are equal to 100mm and 30mm, and thus the e\(^2\) beam waist is equal to 10mm.
Table 3.1 The elements in the real time interferometer

In wind tunnel experiments, which may be carried out in the future, the focus length of L2 can chosen to be higher, what results in an increased $e^2$ beam waist and, hence, in an increased test volume.

The three synchronized black and white CCD-cameras are grabbed by an 24 bits RGB frame grabber (Matrox Corona). Each camera is connected to one of the entrances (R, G or B) and is grabbed in 8 bits. The effective size of the imaging area on the CCD-chips is $8.2\text{mm} \times 6.3\text{mm}$, the pixel size is $11\mu\text{m} \times 11\mu\text{m}$ and the effective number of pixels is $756 \times 571$. The cameras record the images non-interlaced with a frequency of 25Hz. The exposure time of one image is $1/10000$ s.

In the experimental set-up of the real time interferometer a temporal phase shifting interferometer is build in. Therefore the mirror M1 in the reference beam is mounted on a piezo-electric transducer. The interferograms produced by this piezo-interferometer are recorded by CCD1.

3.3.2 Alignment of the set-up

Accurate measurements require an optimum alignment of the optical elements in the set-up. For the alignment of most elements, like lenses, wave plates and beam splitters, common optical methods satisfy. The alignment of the CCD-cameras, however, is more complex.

Each CCD-camera has six degrees of freedom to be aligned, i.e., three translations and three rotations along and about the x-axis, the y-axis and the z-axis (see Figure 3.3). Errors in the alignment lead to deformation of the interferograms and to mutual
3. Design of the real time interferometer and accuracy analysis

Figure 3.3 The six degrees of freedom of a CCD-chip.

Figure 3.4 A moiré pattern as a result of the sum of two line patterns.

decorrelation. A convenient alignment procedure is developed which is partly similar to a method found in the literature\textsuperscript{15}. Below, the principle of this procedure is discussed stepwise. During the procedure the reference beam is obstructed and the test beam is intensified.

1. Coarse alignment
A flat mirror with a grid of thin wire (diameter of 80\textmu m) on its surface is placed in the (virtual) image plane in the test section. The three camera images are represented in red, green and blue in the same color picture in the computer. All the cameras are adjusted for translation in the z-direction until their images appear to be sharp. They are adjusted for translation in the x- and y-direction and for the rotation about the z-axis until the images coincide coarsely.

2. Alignment of the rotations about the x- and y-axis
The orientation of the CCD-chip is visualized by the reflection of an in waist reduced test beam. The rotations about the x-axis and the y-axis are adjusted until the beam is reflected perpendicularly with a deviation of less than 0.5°.

3. Further in plane alignment
Step 1 is repeated for the translations in the x- and y-direction and rotation about the z-axis, now with an accuracy of about 1 pixel.

4. Further alignment of the translation in the z-direction
While the flat mirror with the grid is positioned exactly in the (virtual) image plane, the three CCD-cameras are adjusted for translation in the z-direction until the grid is imaged sharply.

5. Choice of the reference camera
One CCD-camera is chosen to be the reference camera in the further alignment procedure. This camera is aligned correctly now by definition and the other two camera must be aligned with respect to this camera. Theoretically, CCD1 would be the optimum choice
to be reference camera, see section 3.6.1. This camera has, however, the disadvantage that its illuminating beam traverses from BS0 an other trajectory than the illuminating beams of CCD0 and CCD2. These last two beams traverse the same trajectory until they are split by PBS1 and, hence, they pass the same optical elements over a longer distance and possible distortions are the same for both beams. Therefore, the correlation between CCD1 and the other cameras is relatively low, and thus it is better to choose CCD0 or CCD2 to be reference camera (here CCD2).

6. Alignment of the rotation about the z-axis
A grating is positioned near the (virtual) image plane in the test section such that it is imaged sharply on the CCD-cameras. The grating contains 12.5 horizontal lines per mm, resulting in dark and bright lines on the CCD-chips with a thickness of about 2 pixels. To orient CCDi (i=0,1) with respect to CCD2, a method based on a moiré technique is applied\textsuperscript{15,56}. In this method the intensity pattern on CCDi is added to the intensity pattern on CCD2. If the orientation of the two cameras are unequal, the recorded lines are not parallel and the result is a moiré pattern with low frequency lines perpendicular to the grating lines, see Figure 3.4. The mutual distance $D$ of these low frequency lines is directly related to the angle $\varphi$, a relation which is given by:

$$\sin \frac{\varphi}{2} = \frac{p}{2D},$$  \hspace{1cm} (3.5)

where $p$ is the period of the grating lines. The alignment is carried out stepwise:
- CCDi is oriented such that only two low frequency lines are visible in the moiré pattern, their mutual distance is $D$.
- CCDi is rotated in the right direction to let the low frequency lines disappear and reappear. The rotation is stopped as soon as the mutual distance is equal to $D$ again.
- CCDi is rotated in the opposite direction over half the angle it was rotated in the previous step, so that the dark and bright lines are parallel and thus CCDi and CCD2 are equally oriented.

The accuracy of the alignment is limited to the accuracy the distance $D$ can be determined with and is about $0.02^\circ$.

7. Alignment of the translations
The alignment of CCDi with respect to CCD2 for the translations in the $x$-, $y$- and $z$-direction is an iterative process in which the different translations are adjusted alternately. During the process, the grating is positioned near the (virtual) image plane in the test section. The lines are oriented horizontally throughout the alignments in the $y$- and $z$-direction and they are oriented vertically throughout the alignments in the $x$-direction. The translations are adjusted by optimizing the correlation between the images on the two cameras. Therefore, the correlation coefficient is calculated, which is defined by:
\[ |\mu(I_i, I_o)|^2 = \frac{\langle I_i I_o \rangle - \langle I_i \rangle \langle I_o \rangle}{\sqrt{\langle I_i^2 \rangle - \langle I_i \rangle^2 \sqrt{\langle I_o^2 \rangle - \langle I_o \rangle^2}}} \]  \hspace{1cm} (3.6)

where is averaged over the total image. The iterative alignment process comes to an end as soon as any adjustment of any translation leads to a decrease of the correlation coefficient. At the end of the process a correlation of 0.99 (horizontal lines) and 0.89 (vertical lines) has been obtained between CCD0 and CCD2. The correlation between CCD1 and CCD0 is somewhat lower, i.e., 0.97 (horizontal lines) and 0.89 (vertical lines). The alignment accuracy is about 0.5 \( \mu \text{m} \) for the translations in the \( x \)- and \( y \)-direction and about 200 \( \mu \text{m} \) for the translation in the \( z \)-direction.

3.4 Fringe analysis

The two interferometers in the experimental set-up, i.e., the real time interferometer (RTI) and the temporal phase shifting interferometer (TPSI), record a set of phase stepped interferograms per measurement in digitized images. The phase distribution is extracted from two sets of interferograms: the set belonging to the actual measurement and the set belonging to the reference measurement. The actual measurement is made in presence of the refractive index profile of interest. The corresponding phase distribution includes two contributions: the phase distribution of interest and an undesired phase distribution induced by imperfect elements and (possible) windows around the refractive index profile. The reference measurement is made in absence of the refractive index profile of interest, it contains the undesired phase distributions only, and it is used to correct the actual measurement. The RTI and the TPSI use different phase calculation algorithms. Hence, the number of interferograms per set and the phase step between them are different. Both algorithms are based on an arctangent function. This implies that the phase distribution is calculated modulo \( 2\pi \) radians. The continuous phase distribution is obtained by applying a phase unwrapping algorithm.

3.4.1 The real time interferometer

Per measurement the RTI records a set of three phase stepped interferograms simultaneously. The phase step is \( \pi/2 \) radians. The fringe analysis is based on the calculation of the phase distribution by a three step algorithm.

The three step algorithm

Suppose that there is a set of three phase stepped interferograms recorded during an actual
measurement. The intensity distributions of the interferograms are described by:

\[ I_i(x,y) = I_0(x,y) + I_m(x,y) \cos \left( \phi(x,y) + (i-1)\frac{\pi}{2} \right) \]  

(3.7)

where \( I_i(x,y) \) is the intensity distribution in interferogram number \( i \). The bias intensity \( I_0(x,y) \) and the modulation intensity \( I_m(x,y) \) are given by the Equations 3.2a and 3.2b. The set of three stepped equations can be solved for the phase distribution \( \phi(x,y) \) by using two different algorithms. The first algorithm was proposed by Wyant et al. and is written as\(^{31,29}\):

\[ \phi(x,y) = \tan^{-1} \left( \frac{I_2(x,y) - I_1(x,y)}{I_0(x,y) - I_1(x,y)} \right) - \frac{\pi}{4}. \]  

(3.8)

The second algorithm is given by\(^{29}\):

\[ \phi(x,y) = \tan^{-1} \left( \frac{-I_0(x,y) + 2I_1(x,y) - I_2(x,y)}{I_0(x,y) - I_1(x,y)} \right) - \frac{\pi}{2}. \]  

(3.9)

The phase distribution \( \phi(x,y) \) is built up by two contributions: the phase distribution of interest \( \alpha(x,y) \) and an undesired phase distribution \( \beta(x,y) \), that is:

\[ \phi(x,y) = \alpha(x,y) + \beta(x,y). \]  

(3.10)

To extract the phase distribution \( \alpha(x,y) \) from \( \phi(x,y) \), the phase distribution \( \beta(x,y) \) must be determined by a reference measurement. The intensity distributions \( I_{0,i}(x,y) \) in the interferograms of the reference set are described by:

\[ I_{0,i}(x,y) = I_0(x,y) + I_m(x,y) \cos \left( \beta(x,y) + (i-1)\frac{\pi}{2} \right) \]  

(3.11)

Similar to Equation 3.8 and Equation 3.9, the phase distribution \( \beta(x,y) \) can be calculated from these intensity distribution by applying one of the following algorithms:

\[ \beta(x,y) = \tan^{-1} \left( \frac{I_{0,2}(x,y) - I_{0,1}(x,y)}{I_{0,0}(x,y) - I_{0,1}(x,y)} \right) - \frac{\pi}{4} \]  

(3.12)

or

\[ \beta(x,y) = \tan^{-1} \left( \frac{-I_{0,0}(x,y) + 2I_{0,1}(x,y) - I_{0,2}(x,y)}{I_{0,0}(x,y) - I_{0,2}(x,y)} \right) - \frac{\pi}{2}. \]  

(3.13)

The algorithm for the calculation of the phase distribution of interest \( \alpha(x,y) \) is found by
3.4.2 The temporal phase shifting interferometer

Per measurement the TPSI records a set of four interferograms in sequence in time in the same channel of the set-up and by the same CCD-camera. The TPSI differs from a temporal phase stepping interferometer in the recording method of the interferograms. In the phase stepping interferometer the interferograms are recorded after the phase is shifted over a discrete step. During the recording the phase remains constant. In the phase shifting interferometer, however, the interferograms are recorded while the phase changes linearly in time. Because the recorded interferogram is an integration over a short time interval, it is now an integration over a small phase interval. The effect of this integration is only a reduction of the bias intensity\(^1\). Therefore, the interferograms obtained by the TPSI in the experimental set-up are similar to those obtained by a temporal phase stepping interferometer.

The algorithm of Carré

The phase calculation algorithm used in the TPSI is the self-calibrating algorithm of Carré\(^3\). This algorithm is self-calibrating and it does not require a pre-fixed phase step. This implies that the algorithm is insensitive to miscalibration of the piezo-electric transducer and tilt effects of the mirror during the displacements\(^3\). Moreover, the algorithm of Carré is less sensitive to variations in the phase step between the interferograms than other phase calculation algorithms\(^4\). The algorithm needs four interferograms which are phase shifted over a rather arbitrary phase step \(\gamma(x,y)\). If their intensity distributions are given by:

\[
I_i(x,y) = I_0(x,y) + I_m(x,y) \cos \left[ \phi(x,y) + \left( i - \frac{3}{2} \right) \gamma(x,y) \right] \tag{3.19}
\]

then the algorithm is described by:

\[
\phi(x,y) = \tan^{-1} \left[ \frac{\sqrt{(I_0-I_3+I_1-I_2)(3(I_1-I_2)-(I_0-I_3))}}{|(I_1+I_2)-(I_0+I_3)|} \right] \tag{3.20}
\]

This equation calculates the phase on an interval \([0, \pi/2]\). By taking into account the sign of the nominator and the denominator, the inverse tangent function calculates the phase on the interval \([0,2\pi]\). Then Equation 3.20 becomes\(^4\):

\[
\phi(x,y) = \tan^{-1} \left[ \frac{\text{sign}(I_0-I_3+I_1-I_2)\sqrt{(I_0-I_3+I_1-I_2)(3(I_1-I_2)-(I_0-I_3))}}{(I_1+I_2)-(I_0+I_3)} \right] \tag{3.21}
\]
The algorithm does not correct for the contribution of imperfect optical elements etc. Therefore, \( \phi(x,y) \) is calculated for the actual measurements (in presence of the phase of interest) and for the reference measurement. The results are subtracted to find \( \alpha(x,y) \).

Fringe analysis process

Similar to a phase distributions obtained by using the RTI, a phase distribution obtained by the TPSI describes the phase difference between an actual measurement and a reference measurement. In the TPSI all the phase shifted interferograms needed to calculate the phase distribution are recorded in the same channel and by the same CCD-camera. Hence, normalization of the interferograms is not necessary. The choice of the channel and the camera to be used in the set-up is rather arbitrary, here CCD1. The phase step is chosen to be approximately 1.9 radians. Although the algorithm of Carré does not require a pre-fixed phase step, a phase step of 1.9 radians lead to an optimum accuracy\(^{53,4,52}\).

The TPSI is used for the validation of the RTI. In these experiments each measurement consist of the recording of four phase stepped interferograms sequently in time by each camera. The TPSI extracts the phase distribution from the interferograms recorded by CCD1 and the RTI extracts the phase distribution from every second interferogram recorded by CCD0, CCD1 and CCD2. This way of processing induces a phase difference between the phase distributions found by the two interferometers as soon as the phase step in the TPSI is not exactly the same during the actual measurement and the reference measurement. This phase difference \( \Delta \alpha(x,y) \), is given by:

\[
\Delta \alpha(x,y) = \frac{\gamma(x,y) - \gamma_0(x,y)}{2},
\]

where \( \gamma(x,y) \) and \( \gamma_0(x,y) \) are the phase step in the TPSI during the actual and the reference measurement. The phase distribution found by the temporal phase shifting interferometer must be corrected with this value.

The phase step between two interferograms recorded by the TPSI is found by using the complete sets of three simultaneously recorded interferograms (by CCD0, CCD1 and CCD2) and the fringe analysis process of the RTI. The firstly recorded set of interferograms is used as reference set and the secondly recorded set is used as actual set. The mean phase over the resulting phase distribution is adopted as being the (spatially constant) phase step. Although it is not proved, it is at least plausible that this method of phase step determination is rather accurate, provided that fringes in the interferograms are homogeneously distributed their number is high enough. As will be shown in the following sections, the phase error generated in the RTI depends on the modulo \( 2\pi \) phase and the phase gradient during in the measurements. The phase step actually is the phase difference between two phase distributions \( \phi \) and \( \beta \), each with the same phase gradient and probability distribution of the phase. Hence, the mean error in the two distributions
will be equal, and thus the mean error in the phase step is (almost) zero. This method of phase step determination has two advantages over other methods. Firstly, it needs only two phase stepped interferograms recorded by the TPSI. Since this includes only one phase step, the result is insensitive to possible variations in the phase step in the TPSI. Standard methods require more phase stepped interferograms recorded by the TPSI and the calculated phase step becomes an average of the different phase steps if the step is not constant. Secondly, the method is not so sensitive to spatial variations in the bias and modulation intensity, provided that the phase is homogeneously distributed in the interferograms. Recently proposed phase step calculation methods are possibly more sensitive to these variations.

3.4.3 Phase unwrapping

Phase calculation algorithms are generally based on an arctangent function like Equation 3.14 or 3.20. Hence, the phase is determined modulo $2\pi$ radians and artificial $2\pi$ steps may arise in the phase distribution. To extract a continuous phase distribution, these $2\pi$ steps must be removed. This process of removing the $2\pi$ steps is called phase unwrapping. In this process a multiple of $2\pi$ radians is added or subtracted as soon as the phase difference between two adjacent pixels exceeds $\pi$ radians. Condition for a reliable unwrapping is that the difference of the true measured phase in two adjacent pixels is less than $\pi$ radians, i.e., the sampling frequency is required to be at least 2 pixels per fringe (Nyquist condition) and errors in the measured phase must be sufficient small. If this condition is not fulfilled, the resulting unwrapped phase distribution is inconsistent and is dependent on the path along which the unwrapping is performed.

Many algorithms have been developed for two dimensional phase unwrapping. Some of them can only handle consistent phase distributions, while others are designed for unwrapping phase distributions containing discontinuous phase jumps, phase distributions containing high frequency noise or even subsampled phase distributions. In this thesis an algorithm based on the method of Bone is applied, which is also used by Timmerman. This method is developed to deal with discontinuous phase jumps as will appear in experimental flow research due the existence of shock waves, and is relatively insensitive to high frequency noise.

The input of the algorithm is, apart from a wrapped phase distribution, also a binary mask. In this mask the pixels in the phase distribution are registered which must be circumvented during the unwrapping process to prevent that a path is followed that could lead to inconsistencies. The input mask mainly contains pixels with a low modulation depth. The algorithm adds pixels to the mask where the second differences in the locally unwrapped phase exceeds some preset value. By that, regions in the phase distribution containing discontinuities are isolated and excluded from the unwrapping process. Provided that either there is a non-integer fringe shift across a discontinuity or the phase
gradient differs sufficiently on both sides of the discontinuity, a consistent unwrapped phase distribution can be realized.

3.5 Phase errors due to several error sources

The measuring accuracy of the real time interferometer is determined by several aspects. There can be distinguished the alignment accuracy of the CCD-cameras, the quality and alignment of the optical elements, the existence of reflections in the set-up, the presence of aberrations in the test beam, the imaging accuracy in the set-up, the quantization of the interferograms in combination with the addition of noise, non-linearities in the camera response, the finite surface of the pixels and the presence of turbulence in the surrounding air. Each of these aspects may give rise to an error source.

The accuracy of the interferometer is characterized by the root mean square (rms) phase error over all the pixels in the phase distribution and by the maximum absolute phase error. Under the assumption of uniform probability distributions of the phases $\alpha$ and $\beta$ over the interval $[0,2\pi)$ within a phase distribution, the rms phase error $\Delta\alpha_{ms}$ is defined as:

$$\Delta\alpha_{ms}^2 = \left \langle (\alpha_m - \alpha)^2 \right \rangle_{\alpha,\beta}$$

(3.23)

where $\alpha$ is the true phase and $\alpha_m$ is the measured phase which is calculated by using Equation 3.14. There is averaged over $\alpha$ and $\beta$ over the interval $[0,2\pi)$.

Below, the real time interferometer is characterized theoretically by an evaluation of the phase error due to the different error sources. Extended analyses of the phase error due to the finite alignment accuracy of the CCD-cameras, due to the finite quality and finite alignment accuracy of the optical elements, due to the existence of reflections and due to aberrations in the test beam are presented in the following sections. In this section brief analyses are given of the phase error due to the other error sources.

3.5.1 Phase errors due to a limited imaging accuracy

An extended analysis of the phase error induced by the limited accuracy with which the optimum virtual object plane can be imaged on the cameras is presented in Chapter 2. There, the phase error is calculated as a function of the refractive index gradient $n'$ for the case that the object is a linearly stratified medium bounded by glass windows. For the two object configurations intended to be tested by the RTI, i.e., a compressible flow in a wind tunnel and a diffusion/mixing process of a dilute sucrose solution in an aqueous bulk liquid in a square cuvette, the required position of the actual virtual object plane to
keep the phase error smaller than a certain value is presented in the Figures 2.9a,b. For small \( n' \), the relation between the phase error and the distance between the actual and the optimum virtual object plane becomes a function of the dimensionless refractive index gradient \( n' L \) and independent of the object configuration, see Equation 2.34. A phase error arises in both, the actual measurement and the reference measurement. In general, however, \( n' \) is small during the reference measurement and thus the induced phase error. Hence, the maximum phase error in the phase distribution of interest \( \alpha(x,y) \) is determined by the maximum value of \( n' \) in the actual measurement.

The RTI is a Michelson interferometer which is designed for phase gradients up to \( 60 \cdot 10^3 \text{rad/m} \) at the position of the cameras (\( \approx 78 \) fringes over the chip surface). For the set-up with a optical magnification \( G = 0.6 \), equation 2.36 shows that this corresponds to a dimensionless refractive index gradient \( n' L \) up to 0.0018. If the phase gradient is negligible in the reference measurement, Equation 2.36 can be written as:

\[
G \alpha' = \frac{4\pi}{\lambda} n' L,
\]  

(3.24)

where \( \alpha' \) is the gradient in \( \alpha \). Substitution of this equation into Equation 2.34 leads to the relation between \( \alpha' \) and the distance \( \Delta z' \) between the actual and the optimum virtual object plane, that is:

\[
|\Delta \alpha| = \frac{\lambda G^2}{4\pi n_a} \alpha'^2 |\Delta z'|,
\]  

(3.25)

where \( n_a \) is the refractive index of the surrounding air. For the RTI where \( \lambda = 632.8 \text{nm} \), \( G = 0.6 \), \( n_a = 1 \) and \( \alpha' \leq 60 \cdot 10^3 \text{rad/m} \) follows a maximum phase error of 1.0% of \( 2\pi \) radians if \( |\Delta z'| \leq 1 \text{mm} \).

3.5.2 Phase errors due to quantization and noise

The recording of the interferograms introduces errors due to the quantization of the signal and the existence of noise. The quantization error is the result of the analog-to-digital conversion by which the analog video signal from the CCD-camera is rounded to discrete quantized levels. A quantization into 8 bits, i.e., a quantization into 256 levels, leads to a negligible phase error.\textsuperscript{60,61} Condition is that the intensity distribution in the interferogram must cover the dynamic range of the CCD-camera as much as possible. Noise induces a random fluctuation of the signal. There can be distinguished electronic noise and photon noise. Electronic noise is caused by the CCD-chips and the amplifiers like there is thermal noise, dark current noise and 1/f noise. It is present even in the absence of light on the chip. Photon noise is introduced by the statistical behaviour of photons. Hence, it appears only in presence of light and especially if the number of photons is relatively small.
In the presence of light the total noise in the system is rather high, i.e., about \( \pm 2 \) levels. The phase error introduced by this noise is strongly dependent on the effective number of quantized levels within an interferogram. This effective number of levels corresponds to twice the number of levels related to the modulation intensity and is in the real time interferometer about 200.

To find a convenient approximation of the phase error, the interferograms on the CCD-chips are assumed to be the result of interference between a test beam and a reference beam whose intensity distributions are equal and Gaussian with a \( e^2 \)-waist of 6mm. The centers of the beams coincide with the centers of the CCD-chips and the maximum possible intensity in the interferograms corresponds to 200 levels. The rms phase error due to quantization and noise is calculated by a numerical evaluation of Equation 3.23. However, since the modulation intensity of the interferograms decrease towards the borders of the CCD-chip and thus the phase error increases towards the border, there is not only averaged over \( \alpha \) and \( \beta \) but also over the chip surface. In the calculation \( \alpha_m \) is given by Equation 3.14. After evaluation of one of the intensities \( I_{00}, I_{01}, I_{02}, I_0, I_1 \) or \( I_2 \), its value is rounded to one of the quantization levels (simulation of the quantization) and a random number of levels between -2 and +2 is added (simulation of noise). Also the normalization of the cameras is simulated (see section 3.4.1). Here is assumed that the Gaussian normalization intensity would be equal on each camera in absence of noise. The noise, which is basically equal to the noise in the interferograms, is reduced by averaging it over 10 registrations. The result is an rms phase error of 0.6% of \( 2\pi \) radians (0.038 rad).

### 3.5.3 Phase errors due to a nonlinear camera response

A nonlinear camera response leads to a deformation of the stored interferograms. Only if the nonlinear relationship between the camera’s electric output and the optical intensity on the CCD-chip is exactly known, a compensation can be made. Usually, this relationship is not known and a phase error appears. The extension of this error is dependent on the used algorithm. In general, a higher number of buckets leads to a lower phase error\(^ {62,44,45} \).

The cameras in the experimental set-up are commercial cameras. The output of such cameras deviates typically not more than 5% from the linear response. It is important to know how this non-linearity influences the phase measured by the real time interferometer.

Figure 3.5 shows the phase error as a function of the phase due to a detector non-linearity of maximal 5%. Presented are the error due to a second order non-linearity and the error due to a third order non-linearity in the phase calculated from one single set of interferograms by the three camera algorithm of Equation 3.8. Since the real time interferometer calculates (by using Equation 3.14) the phase difference between the phases
3. Design of the real time interferometer and accuracy analysis

![Graph showing phase error as a function of phase for second and third order non-linearities](image)

Figure 3.5 Phase error as a function of the phase, due to second and third order non-linearities, for the algorithm of Equation 3.8.

Following from two sets of interferograms, the maximum phase error corresponds to the peak-to-peak values of the curves in Figure 3.5. These values equal to 0.048 rad for the second order non-linearity and equal to 0.072 rad for the third order non-linearity.

The deformation of an interferogram due to the non-linearities increases with the part of the dynamic range of the camera that is used. This implies not only that the deformation is low in the minima and high in the maxima, but also that the deformation decreases with a decreasing bias and modulation intensity. The latter means that, if the interferogram is a result of the interference between a Gaussian test beam and a Gaussian reference beam with their centers coinciding with the center of the CCD-chip, the deformation decreases towards the borders of the chip. Hence, the phase error decreases towards the borders of the chip. To obtain an approximation for the rms phase error in the phase distribution found by the real time interferometer from two sets of interferograms, the same interferograms are assumed to be at the CCD-chips as in section 3.5.2. The maximum non-linearity of 5% appears if the full dynamic range is used, i.e., in a potential maximum in the center of the interferogram. Averaging of Equation 3.23 over the total chip surface while \( \alpha_m \) is given by Equation 3.14 results in the rms phase error. For the second order non-linearity follows \( \Delta \alpha_{rms} = 0.18\% \) of \( 2\pi \) radians (0.011 rad) and for the third order follows \( \Delta \alpha_{rms} = 0.21\% \) of \( 2\pi \) radians (0.013 rad). This implies that the maximum phase error may differ for second order and third order non-linearity, but the rms errors are almost equal. Although the maximum phase error may be up to 1.1% of \( 2\pi \) radians for third order non-linearity, the rms is much smaller and almost negligible.
3.5.4 Phase errors due to the finite pixel surfaces

To enable computer processing, the CCD-chip samples the interferograms with a rectangular grid of $m \times n$ pixels. Since the rectangular sensitive surfaces of these pixels are not infinitely small, the discrete output of the CCD-chip corresponds to local intensity distributions which are integrated over the pixel surfaces. In the image processing, however, the discrete output is assumed to correspond to the intensities in the center of each pixel. This is actually not true and, hence, a phase error is introduced.

In the real time interferometer a set of three phase stepped interferograms are recorded by three different CCD-cameras simultaneously. The discrete intensity distribution of an interferogram detected CCDi is given by:

$$I(x,y) = \frac{w/2}{-h/2} \int \frac{w/2}{-w/2} I_b(x,y) + I_m(x,y) \cos \left( \phi(x+X,y+Y) + (i-1) \frac{\pi}{2} \right) \, dx \, dy. \quad (3.26)$$

Here, the coordinates $(x,y)$ have discrete values corresponding to the positions of the centers of the pixels. These positions are assumed to be equal for each CCD-chip. The variables $X$ and $Y$ are continuous and they describe the relative position in $x$- and $y$-direction with respect to the center of the pixels. This implies that, although $(x,y)$ has discrete values, the phase $\phi(x+X,y+Y)$ is a continuous function. The bias intensity and the modulation intensity are assumed to be constant over a pixel surface. Under the condition that the phase $\phi(x+X,y+Y)$ is a smooth function of $X$ and $Y$ over the pixel surface, a second order Taylor expansion of $\cos(\phi(x+X,y+Y) + (i-1)\pi/2)$ can be made. Substitution of this expansion in Equation 3.26 followed by an integration over $X$ and $Y$ leads to:

$$I(x,y) = A_0(x,y) + A_1(x,y) \cos \left( \phi(x,y) + (i-1) \frac{\pi}{2} \right) + A_2(x,y) \sin \left( \phi(x,y) + (i-1) \frac{\pi}{2} \right)$$

$$= (3.27)$$

with

$$A_0(x,y) = I_b(x,y)hw, \quad (3.28a)$$

$$A_1(x,y) = I_m(x,y) \left[ hw - \frac{w^3h}{24} \left( \frac{\partial \phi(x,y)}{\partial x} \right)^2 - \frac{wh^3}{24} \left( \frac{\partial \phi(x,y)}{\partial y} \right)^2 \right], \quad (3.28b)$$

$$A_2(x,y) = -I_m(x,y) \left[ \frac{w^3h}{24} \frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{wh^3}{24} \frac{\partial^2 \phi(x,y)}{\partial y^2} \right]. \quad (3.28c)$$

These equations show that the size of the pixel surface reduces the modulation depth and adds a disturbing term proportional to $\sin(\phi(x,y) + (i-1)\pi/2)$. The reduction of the
modulation depth is quadratically dependent on the first order derivatives of the phase distribution, the amplitude of the disturbing term is proportional to the second order derivatives. Although the parameters $A_1(x,y)$ and $A_2(x,y)$ are dependent on the first and second order derivatives of the phase distribution, none of the parameters $A_0(x,y)$, $A_1(x,y)$ and $A_2(x,y)$ are dependent on the phase step. Hence, they are equal for all the interferograms in one set.

The phase distribution measured by the real time interferometer is the result of the registration of two sets of interferograms. Equations 3.27 and 3.28a,b,c describe the interferograms belonging to the actual measurement. Likewise, the interferograms belonging to the reference measurement can found to be:

$$I_{0}(x,y) = A_{0,0}(x,y) + A_{0,1}(x,y) \cos \left( \beta(x,y) + (i-1) \frac{\pi}{2} \right) + A_{0,2}(x,y) \sin \left( \beta(x,y) + (i-1) \frac{\pi}{2} \right)$$

(3.29)

with

$$A_{0,0}(x,y) = I_{s}(x,y)hw,$$

(3.30a)

$$A_{0,1}(x,y) = I_{m}(x,y) \left[ hw - \frac{w^3 h}{24} \left( \frac{\partial^2 \beta(x,y)}{\partial x^2} \right)^2 - \frac{wh}{24} \left( \frac{\partial \beta(x,y)}{\partial y} \right)^2 \right],$$

(3.30b)

$$A_{0,2}(x,y) = -I_{m}(x,y) \left[ \frac{w^3 h}{24} \frac{\partial^2 \beta(x,y)}{\partial x^2} + \frac{wh}{24} \frac{\partial^2 \beta(x,y)}{\partial y^2} \right].$$

(3.30c)

For the calculation of the phase distribution of interest $\alpha(x,y) = \phi(x,y) - \beta(x,y)$ Equations 3.27 and 3.29 are substituted in Equation 3.14. The result, however, does not lead to a value equal to $\alpha(x,y)$, which would be the case if the pixels were infinitely small, but to a sum of $\alpha(x,y)$ and an error $\Delta \alpha(x,y)$. For this error follows:

$$\Delta \alpha(x,y) = \tan^{-1} \left( \frac{A_1(x,y)A_{0,2}(x,y) - A_2(x,y)A_{0,1}(x,y)}{A_1(x,y)A_{0,1}(x,y) + A_2(x,y)A_{0,2}(x,y)} \right).$$

(3.31)

This error is independent of $\alpha(x,y)$, $\beta(x,y)$ and $\phi(x,y)$, and only dependent on derivatives of $\beta(x,y)$ and $\phi(x,y)$. If the phase distribution during the reference measurement is almost flat so that the first and second order derivative of $\beta(x,y)$ may assumed to be zero, follows:
\[
\Delta \alpha(x,y) = \tan^{-1} \left\{ \frac{w^2 \frac{\partial^2 \phi(x,y)}{\partial x^2} + h^2 \frac{\partial^2 \phi(x,y)}{\partial y^2}}{24 - w^2 \left( \frac{\partial \phi(x,y)}{\partial x} \right)^2 - h^2 \left( \frac{\partial \phi(x,y)}{\partial y} \right)^2} \right\} .
\] (3.32)

This phase error is the arctangent of \(A_2(x,y)/A_1(x,y)\), i.e., the quotient of the amplitude of the disturbance due to the existence of a second order derivative and the modulation depth during the actual measurement. The real time interferometer is designed to measure phase gradients up to \(60 \cdot 10^3 \text{rad/m}\). The pixel size is \(w \times h = 11 \mu \text{m} \times 11 \mu \text{m}\). Substitution of these values in Equation 3.32 leads, under the condition that the error is small, to a maximum phase error equal to:

\[
\Delta \alpha(x,y) \leq 5.2 \cdot 10^{-12} \left( \frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} \right) .
\] (3.33)

In this phase error the error due to the reduction of the modulation depth is excluded. This reduction is, however, small and the resulting phase error is negligible. Equation 3.33 shows that if the second order derivatives of the phase distribution remains smaller than \(6 \cdot 10^8 \text{rad/m}^2\), the introduced phase error remains smaller than 0.1% of \(2\pi\) radians. A second order derivative of \(6 \cdot 10^8 \text{rad/m}^2\) means a change of the first derivative of 6600 rad/m over 1 pixel, i.e., a change of the most negative phase gradient of \(-60 \cdot 10^3 \text{rad/m}\) to the most positive phase gradient of \(+60 \cdot 10^3 \text{rad/m}\) over 18 pixels. In general, the second order derivative will be much smaller and, hence, the phase error will be negligible.

### 3.5.5 Phase errors due to vibrations and air turbulence

Problematic phase errors can arise due to the existence of mechanical vibrations and air turbulence\(^6\) in the set-up. Hence, precautions are necessary like placing the interferometer on a vibration-isolated table, turning off air ducts and shielding the beam paths for air flow. For the real time interferometer, however, the shielding requirement may be relieved to some extend. Since the test beam and the reference beam traverse the same trajectory through the interferometer over a long distance, air turbulence in this trajectory disturbs the phase of both beams (almost) equally and, therefore, it does not influence the interferograms.

Some insight in the influence of vibrations may be gained by assuming that the vibrations introduce a small sinusoidal phase distortion with amplitude \(F\), frequency \(\nu\) and phase offset \(\tau\). If the integration time of the CCD-camera is \(T\), the detected intensity distribution of the interferogram on \(CCDi\) in the real time interferometer is given by\(^6\) :
$$I(x, y) = \int_{T/2}^{T} I_b + I_m \cos \left( \phi + (i-1) \frac{\pi}{2} + F \cos(2\pi \nu t + \tau) \right) \, dt$$

$$\approx \int_{T/2}^{T} I_b + I_m \cos \left( \phi + (i-1) \frac{\pi}{2} \right) - I_m F \cos(2\pi \nu t + \tau) \sin \left( \phi + (i-1) \frac{\pi}{2} \right) \, dt$$

$$= A_0 + A_1 \cos \left( \phi + (i-1) \frac{\pi}{2} \right) + A_2 \sin \left( \phi + (i-1) \frac{\pi}{2} \right)$$

(3.34)

with

$$A_0 = I_b T,$$  \hspace{1cm}  (3.35a)

$$A_1 = I_m T,$$  \hspace{1cm}  (3.35b)

$$A_2 = -I_m T F \cos \tau \sinc(\nu T).$$  \hspace{1cm}  (3.35c)

where the \((x, y)\)-dependence has been omitted for notational reasons. Since the RTI records a set of interferograms simultaneously, the amplitude, frequency and phase offset of the vibration are equal for each interferogram in the same set. Hence, apart from \(A_0\) and \(A_1\), also \(A_2\) is equal for each interferogram in one set. The distortion of the interferograms is similar to the distortion due to the finite pixel surface. Equation 3.34 is equal to Equation 3.26 and the coefficients are equal for all interferograms in one set. This implies that the phase error based on two sets of interferograms is given by Equation 3.31. In the case of the existence of vibrations with a small amplitude this error results in:

$$\Delta \alpha(x, y) = \tan^{-1} \left( \frac{F \cos \tau \sinc(\nu T) - F_0 \cos \tau_0 \sinc(\nu_0 T)}{1 + F \cos \tau \sinc(\nu T) \cdot F_0 \cos \tau_0 \sinc(\nu_0 T)} \right)$$

(3.36)

$$\approx F \cos \tau \sinc(\nu T) - F_0 \cos \tau_0 \sinc(\nu_0 T) + O(F^3),$$

where \(F, \nu, \tau\) and \(F_0, \nu_0, \tau_0\) represent the amplitude, the frequency and the phase offset during the actual measurement and the reference measurement, respectively. This equation shows that:
- the phase error is zero if \(F=F_0, \nu=\nu_0\) and \(\tau=\tau_0\),
- the phase error is zero if \(\nu\) and \(\nu_0\) is an integer times \(1/T\),
- the phase error reduces quickly with the frequency.

Practically, the amplitude, frequency and phase offset are unequal during the actual measurement and the reference measurement, and a phase error given by Equation 3.36 must be taken into consideration. However, since the phase error is independent of the phase, it is constant over the whole phase distribution if the amplitude, frequency and phase offset of the vibration are independent of the coordinates \((x, y)\). Then, the phase error does not deform the phase distribution but it adds only a constant. In general, the
vibration encompasses a frequency spectrum and the distortion of the phase distribution must be determined experimentally.

3.6 Phase errors due to misalignment of the cameras

Each CCD-camera in the set-up has six degrees of freedom: three translations and three rotations along and about the x-axis, y-axis and z-axis respectively, see Figure 3.3. Errors in the alignment of the translations along the x-axis and the y-axis as well as an error in the rotation about the z-axis lead to a decorrelation between the various interferograms. Errors in the translation along the z-axis and the rotations about the x-axis and the y-axis lead to phase errors within the interferograms. The influence of these errors on the measured phase is described in this section.

To simplify the analytical calculations, the undesired phase $\beta$ is assumed to be zero. Substitution of Equation 3.14 in Equation 3.23 under this condition leads to:

$$\Delta \alpha_{rms} = \left(\left\langle (\phi_m - \phi)^2 \right\rangle_{\phi} \right)_{\alpha} = \left(\left\langle (\alpha'_m - \alpha)^2 \right\rangle_{\alpha} \right),$$

(3.37)

where $\phi_m$ is the measured value of $\phi$ which is given by Equation 3.8. Because $\alpha'_m$ is the measured value of the phase of interest in the case that $\beta = 0$, Equation 3.37 implies that for the calculation of $\Delta \alpha_{rms}$ Equation 3.8 with $\phi = \alpha$ can be used instead of Equation 3.14. The phase errors due to the misalignment will appear to be mainly determined by the phase gradient. Because the gradient in the phase of interest $\alpha$ will be much higher than the gradient in the undesired phase $\beta$, the assumption will not lead to considerable errors in the analysis. In the calculations the phase $\alpha$ describes the phase at the position of the cameras and not in the test section.

Camera misalignments lead to a change of the phase, the bias intensity and the modulation intensity in the recorded interferograms. Since these changes differ per camera, they result eventually in an error in the measured phase. Generally, the test beam and the reference beam are Gaussian distributed and their waists are sufficient large to keep the errors due to the changes in the bias intensity and the modulation intensity negligible. Then, the phase error is fully determined by the change in phase per interferogram. All the camera misalignments can now be converted to a spatial translation of the interferogram whereby the bias and modulation intensity may be assumed spatially constant.

The spatial translation of the interferogram is not necessarily the same for each pixel, as shown in Figure 3.6. Here, interferogram number $i$ is imaged on CCD$i$ in presence of a misalignment of the translations in the x- and y-direction and of the rotation about the z-axis (perpendicular to the chip-surface). As a consequence of the misalignment, the assumed coordinate axes in the interferogram analysis (those joint to the chip-surface) do not coincide with coordinate axes joint to the interferogram. Hence, in the interferogram
The values of $\Delta x_i$ and $\Delta y_i$ and thus the value of $\Delta \alpha_i$ generally differ per camera, but often one of the cameras is aligned perfectly by definition. For this so-called reference camera holds $\Delta x_i = \Delta y_i = 0$ and thus $\Delta \alpha_i = 0$ so that the interferogram on this camera does not contain a phase error. Equation 3.41b shows that the phase error is maximum if for the other two cameras holds that $\Delta \alpha_i$ is maximum (thus $(\partial \alpha/\partial x)\Delta x_i$ and $(\partial \alpha/\partial y)\Delta y_i$ have the same sign, and $\Delta x_i$ and $\Delta y_i$ have their extreme values) and the signs of $\Delta \alpha_i$ are equal for both cameras. For (the upper bound of) the rms phase error then follows:

$$\Delta \alpha_{rms}^2 = \frac{c}{8} \left\langle \Delta A^2 \right\rangle_{x,y}, \tag{3.41c}$$

where $\Delta A$ is the maximum phase error as a function of $(x,y)$ in one of the interferograms. The constant $c$ depends on which camera is the reference and the phase error $\Delta \alpha_i$ in the interferograms on the other cameras. The value $c=3$ corresponds to CCD1 being the reference camera, $\Delta \alpha_0 = \Delta \alpha_2 = \Delta A$. The value $c=5$ corresponds to CCD0 or CCD2 being the reference with $\Delta \alpha_j = \Delta \alpha_i = \Delta A$ ($j = 2$ or $j = 0$). The value $c=7$ describes the phase error if CCD1 is the reference with $\Delta \alpha_0 = \Delta \alpha_2/2 = \Delta A$ or $\Delta \alpha_0/2 = \Delta \alpha_2 = \Delta A$, i.e., if CCD2 or CCD0 is aligned with half accuracy with respect to the other camera. The value $c=8$ describes the case that there is no reference camera and $\Delta \alpha_0 = \Delta \alpha_1 = \Delta \alpha_2 = \Delta A$.

Below, the phase errors induced by the misalignment of the cameras are discussed per degree of freedom. Upper bounds of $\Delta \alpha_{rms}$ are found for two types of media in the test section: a linearly stratified medium, i.e., $\partial \alpha/\partial x$ and $\partial \alpha/\partial y$ are constant over the phase distribution, and a locally linearly stratified medium, i.e., $\partial \alpha/\partial x$ and $\partial \alpha/\partial y$ vary over the phase distribution but $\partial^2 \alpha/\partial x^2$ and $\partial^2 \alpha/\partial y^2$ are negligible.

**Translation along the $x$-axis and $y$-axis**

Misalignment of camera $CCDi$ for the translations along the $x$-axis and the $y$-axis lead to constant values of $\Delta x_i$ and $\Delta y_i$ over the whole chip-surface. Equation 3.42 shows that this results in a spatially constant phase error $\Delta \alpha_i$ in the interferogram as long as $\partial \alpha/\partial x$ and $\partial \alpha/\partial y$ are spatially constant (linearly stratified medium), a phase error which can differ per camera because $\Delta x_i$ and $\Delta y_i$ may differ. By using Equation 3.41c follows for (the upper bound of) the rms phase error:

$$\Delta \alpha_{rms} = \frac{\sqrt{c}}{\sqrt{8}} \left| \left( \frac{\partial \alpha}{\partial x} \Delta x + \frac{\partial \alpha}{\partial y} \Delta y \right) \right| = \frac{\sqrt{c}}{\sqrt{8}} |(\Delta x \cos \Theta + \Delta y \sin \Theta)|. \tag{3.43a}$$

The value $c=3$ corresponds to CCD1 being the reference camera, $\Delta x_0 = \Delta x_2 = \Delta x$ and $\Delta y_0 = \Delta y_2 = \Delta y$. The value $c=5$ corresponds to CCD0 or CCD2 being the reference with $\Delta x_j = \Delta x_i = \Delta x$ and $\Delta y_j = \Delta y_i = \Delta y$ ($j = 2$ or $j = 0$). The value $c=7$ describes the phase error if CCD1 is the reference with $\Delta x_0 = \Delta x_2/2 = \Delta x$ and $\Delta y_0 = \Delta y_2/2 = \Delta y$ or $\Delta x_0/2 = \Delta x_2 = \Delta x$ and $\Delta y_0/2 = \Delta y_2 = \Delta y$. If the cameras are positioned with an accuracy of $\Delta x = \Delta y = 0.5 \mu m$
follows:

\[
\Delta \alpha_{\text{rms}} = 1.77 \cdot 10^{-7} \sqrt{c} \left| \frac{\partial \alpha}{\partial x} + \frac{\partial \alpha}{\partial y} \right| = 1.77 \cdot 10^{-7} \sqrt{c} \alpha' \left| (\cos \Theta + \sin \Theta) \right|. \tag{3.43b}
\]

The rms phase error becomes maximum if the angle \( \Theta \) between the phase gradient and the x-axis is 45° or 225°, so that:

\[
\Delta \alpha_{\text{rms}} \leq 2.50 \cdot 10^{-7} \sqrt{c} \alpha' \tag{3.44}
\]

This maximum also holds if \( \partial \alpha/\partial x \) and \( \partial \alpha/\partial y \) are not spatially constant but the phase gradient does nowhere exceed \( \alpha' \) (locally linearly stratified medium).

**Rotation about the z-axis**

A misalignment of a CCD-camera for the rotation about the z-axis lead to a position dependent translation in the x- and y-direction, see Figure 3.8. If the angle of rotation \( \rho_{z,i} \) of CCDi is small, then the translation of a point \( P' \) with respect to point \( P \) is given by:

\[
\Delta x_i = \rho_{z,i} r \sin \theta + O(\rho_{z,i}^3) = \rho_{z,i} y + O(\rho_{z,i}^3) \tag{3.45a}
\]

and

\[
\Delta y_i = -\rho_{z,i} r \cos \theta + O(\rho_{z,i}^3) = -\rho_{z,i} x + O(\rho_{z,i}^3), \tag{3.45b}
\]

where \( r \) and \( \theta \) are the polar coordinates of point \( P' \). Substitution of these translations into Equation 3.42 leads to a phase error in interferogram number \( i \) equal to:

\[
\Delta \alpha_i = \frac{\partial \alpha}{\partial x} \rho_{z,i} y - \frac{\partial \alpha}{\partial y} \rho_{z,i} x = \alpha' \rho_{z,i} r \sin \theta \cos \Theta - \alpha' \rho_{z,i} r \cos \theta \sin \Theta, \tag{3.46}
\]

which becomes maximum if \( \rho_{z,i} \) is maximum. In the case that \( \partial \alpha/\partial x \) and \( \partial \alpha/\partial y \) are constant over the chip surface (linearly stratified medium), substitution of Equation 3.46 into Equation 3.41c leads to the following (upper bound of the) rms phase error:

\[
\Delta \alpha_{\text{rms}} = \frac{\sqrt{c} \rho_z}{\sqrt{8}} \left\{ \left( \frac{\partial \alpha}{\partial y} \right)^2 \right\}^{1/2}.
\]

\[
= \frac{\sqrt{c} \rho_z}{\sqrt{24}} \left[ \left( \frac{\partial \alpha}{\partial x} \right)^2 H^2 + \left( \frac{\partial \alpha}{\partial y} \right)^2 B^2 \right]^{1/2} \tag{3.47a}
\]

\[
= \frac{\sqrt{c} \rho_z \alpha'}{\sqrt{24}} \sqrt{H^2 \cos^2 \Theta + B^2 \sin^2 \Theta}.
\]
Here, $B$ is the half chip length in the $x$-direction and $H$ is the half chip length in the $y$-direction. Again, $c=3$ and $c=7$ correspond to CCD1 being the reference. For $c=3$ holds $\rho_{z0}=\rho_{x2}=\rho_\ast$, for $c=7$ holds $\rho_{z0}=\rho_{z2}/2=\rho_\ast$ or $\rho_{y0}/2=\rho_{x2}=\rho_\ast$. The value $c=5$ corresponds to CCD0 or CCD2 being the reference while $\rho_{z1}=\rho_{z2}=\rho_\ast (j=2 \text{ or } j=0)$. The rotation can be adjusted within an accuracy of $3.5\cdot10^{-4}$. If $B=4.1\text{mm}$ and $H=3.15\text{mm}$ follows a rms phase error equal to:

$$\Delta \alpha_{\text{rms}} = 2.93\cdot10^{-7} \sqrt{c} \left[ 0.59 \left( \frac{\partial \alpha}{\partial x} \right)^2 + \left( \frac{\partial \alpha}{\partial y} \right)^2 \right]^{1/2} = 2.93\cdot10^{-7} \sqrt{c} \alpha' \sqrt{0.59 \cos^2 \Theta + \sin^2 \Theta}. \quad (3.47b)$$

The phase error on CCD$i$ becomes maximum if $\alpha'$ is maximum and everywhere on the chip-surface perpendicular to the displacement of the chip due to the rotation, i.e., $\Theta = \theta \pm \pi/2$. Substitution of Equation 3.46 under these conditions into Equation 3.41c leads to:

$$\Delta \alpha_{\text{rms}} \leq \frac{\sqrt{c} \rho_\ast \sqrt{\langle r^2 \rangle}}{\sqrt{8}} = \frac{\sqrt{c} \rho_\ast \alpha'}{\sqrt{24}} \sqrt{B^2 + H^2}, \quad (3.48a)$$

where for $c$ the same conditions hold as in Equation 3.47. If $\rho_\ast = 3.5\cdot10^{-4}$, $B=4.1\text{mm}$ and $H=3.15\text{mm}$ follows:

$$\Delta \alpha_{\text{rms}} \leq 3.69\cdot10^{-7} \sqrt{c} \alpha'. \quad (3.48b)$$

This upper bound remains also valid if the phase gradient is not constant over the phase distribution, provided that it does not exceed $\alpha'$ (locally linearly stratified medium).
Translation along the z-axis

A phase error due to the misalignment of the CCD-cameras for translation in the z-direction arises because, unlike the straight rays of the reference beam, the refracted rays of the test beam do not strike the CCD-chip perpendicularly, see Figure 3.9. Hence, the phase difference between the reference beam and the test beam is correct in case of no misalignment but it will contain an error in case there actually is a misalignment. This phase error contains two contributions. Firstly, there is an error caused by the difference in path lengths between the refracted rays in the test beam and the straight rays in the reference beam over the distance that the CCD-camera is mispositioned. Secondly, there is an error due to a ray displacement: there is a distance between the position where the refracted ray actually strikes the CCD-chip and the position where it should strike the CCD-chip in case of no misalignment of CCDi. Under the assumption that the phase distribution is locally linear, the two contributions cancel each other partly.

If a light ray traverses a linearly stratified medium in the test section of a Michelson interferometer, the total angle of refraction is given by Equation 2.19. In case of a low refraction this angle can be approximated by Equation 2.35. Since the imaging optics in the RTI consist of couples of lenses with a mutual distance equal to the sum of their focal lengths, the angle of incidence $\varphi$ of the ray on the CCD-chips may be approximated by:

$$
\varphi = \frac{\lambda}{2\pi} \left[ \left( \frac{\partial \alpha}{\partial x} \right)^2 + \left( \frac{\partial \alpha}{\partial y} \right)^2 \right]^{1/2}.
$$

(3.49a)

Here, $\partial \alpha/\partial x$ and $\partial \alpha/\partial y$ are the phase gradients on the CCD-chip in the x- and y-direction, $\lambda$ is the wavelength of the light and the refractive index of the surrounding air is assumed to be 1.0. If the angle of incidence is written in terms of its components $\varphi_x$ and $\varphi_y$ in the positive x- and y-direction, see Figure 3.9, Equation 3.49a becomes:

$$
\varphi_x = \frac{\lambda}{2\pi} \frac{\partial \alpha}{\partial x},
$$

(3.49b)

$$
\varphi_y = \frac{\lambda}{2\pi} \frac{\partial \alpha}{\partial y},
$$

(3.49c)

The distances $\Delta x_i$ in the x-direction and $\Delta y_i$ in the y-direction between the positions where the ray should strike the CCD-chip and it actually does if the chip is translated are given by:

$$
\Delta x_i = \varphi_x \Delta z_i + O(4) \approx \frac{\lambda}{2\pi} \frac{\partial \alpha}{\partial x} \Delta z_i
$$

(3.50a)

and
3. Design of the real time interferometer and accuracy analysis

Figure 3.9 A refracted ray from the test beam and a straight ray from the reference beam strike CCDi which is misaligned for the translation in the z-direction.

\[ \Delta y_i = \varphi_y \Delta z_i + O(4) = \frac{\lambda}{2\pi} \frac{\partial \alpha}{\partial y} \Delta z_i, \quad (3.50b) \]

where \( \Delta z_i \) is the distance over which CCDi is misaligned in the z-direction. The term \( O(4) \) denotes fourth and higher order multiplicands of \( \varphi_x, \varphi_y, \) and \( \Delta z_i \). The total phase error \( \Delta \alpha_i \) in the interferogram on CCDi is given by:

\[
\Delta \alpha_i = \frac{\pi}{\lambda} (\varphi_x^2 + \varphi_y^2) \Delta z_i - \left[ \frac{\partial \alpha}{\partial x} \Delta x_i + \frac{\partial \alpha}{\partial y} \Delta y_i \right] + O(4)
\]

\[
= -\frac{\lambda}{4\pi} \left( \frac{\partial \alpha}{\partial x} \right)^2 + \left( \frac{\partial \alpha}{\partial y} \right)^2 \Delta z_i, + O(4) \quad (3.51)
\]

\[
= -\frac{1}{2} \left( \frac{\partial \alpha}{\partial x} \Delta x_i + \frac{\partial \alpha}{\partial y} \Delta y_i \right), + O(4).
\]

Here, the first term of the second part of the first line describes the phase error due to the path length difference and the second term describes the phase error due to the ray displacement. The second and third line follow by substitution of the Equations 3.49b,c and 3.50a,b. The total phase error, may be interpreted as a phase error due to a translation of the interferogram over a distance \( -\Delta x_i/2 \) in the x-direction over a distance \( -\Delta y_i/2 \) in the y-direction. Hence, the phase error in the measurement due to misalignment of the translation in the z-direction may be calculated by using the formalism for the misalignment in the x- and y-direction. Also for the translation in the z-direction one of the cameras can be denoted to be the reference camera. This camera, however, does not necessarily be aligned correctly for this translation, but a possible error in the interferogram can be accepted as being an error due to the limited imaging accuracy. Substitution of \( -\Delta x_i/2 \) and \( -\Delta y_i/2 \) (Equation 3.50a,b) in Equation 3.41b shows that the rms
phase error becomes maximum if for the other two cameras holds that \( \Delta z_j \) have their extreme values and also the same sign. If \( \partial \alpha / \partial x \) and \( \partial \alpha / \partial y \) are constant over the phase distribution (linearly stratified medium) follows for (the upper bound of) the rms phase error by using Equation 3.41c:

\[
\frac{\Delta \alpha_{rms}}{4\pi \sqrt{8}} \left( \left( \frac{\partial \alpha}{\partial x} \right)^2 + \left( \frac{\partial \alpha}{\partial y} \right)^2 \right) |\Delta z| = 0.028\sqrt{c} \lambda \alpha'^2 |\Delta z|.
\]  

(3.52a)

The values \( c=3 \) and \( c=7 \) correspond to CCD1 being the reference whereby \( \Delta z_0 = \Delta z_2 = \Delta z \) and \( \Delta z_0 = \Delta z_2 / 2 = \Delta z \) or \( \Delta z_0 / 2 = \Delta z_2 = \Delta z \). The value \( c=5 \) implies that CCD0 or CCD2 is reference and \( \Delta z_j = \Delta z_i = \Delta z \) (\( j=2 \) or \( j=0 \)). Substitution of the wavelength \( \lambda = 632.8 \text{nm} \) and an upper bound of the displacement \( \Delta z = 200 \mu \text{m} \) in Equation 3.52 gives:

\[
\Delta \alpha_{rms} = 3.56 \cdot 10^{-12} \sqrt{c} \left( \left( \frac{\partial \alpha}{\partial x} \right)^2 + \left( \frac{\partial \alpha}{\partial y} \right)^2 \right) = 3.56 \cdot 10^{-12} \sqrt{c} \alpha'^2.
\]  

(3.52b)

Equations 3.52a,b show that \( \Delta \alpha_{rms} \) depends on \( \alpha' \) but is independent of the direction of the gradient. Hence, the presented rms phase error is also the upper estimate in the case that the phase gradient is not constant over the phase distribution (locally linearly stratified medium), provided that the gradient does not exceed \( \alpha' \).

**Rotation about the x-axis and y-axis**

Misalignment of the CCD-cameras for rotations about the x-axis and the y-axis leads to a phase error due to two reasons. Firstly, there is a position dependent translation of the chip surface in the z-direction. Secondly, the distortion in the chip orientation leads to a different orientation of the assumed x- and y-axis and the true x- and y-axis. Suppose that due to a misalignment the chip surface shows a small rotation \( \rho_x \) about the x-axis and a small rotation \( \rho_y \) about the y-axis as defined in Figure 3.10a. Further, the angles of incidence of a ray are given by \( \varphi_x \) and \( \varphi_y \) as defined in Figure 3.9. Then the phase error \( \alpha_i \) on CCD\( i \) is given by:

\[
\Delta \alpha_i = -\frac{\lambda}{4\pi} \left( \left( \frac{\partial \alpha}{\partial x} \right)^2 + \left( \frac{\partial \alpha}{\partial y} \right)^2 \right) \left( \rho_{y,x} \rho_{x,y} + \rho_{x,y} \rho_{y,x} \right) - \frac{\partial \alpha}{\partial x} \rho_{y,x} \rho_{x,y} - \frac{1}{2} \frac{\partial \alpha}{\partial x} \rho_{x,y}^2 - \frac{1}{2} \frac{\partial \alpha}{\partial y} \rho_{x,y}^2 + O(4),
\]  

(3.53a)

where \( O(4) \) denotes fourth and higher order multiplicands of \( \varphi_x, \varphi_y, \rho_{x,y} \) and \( \rho_{y,x} \). The first term describes the phase error induced by the translation of the chip surface in the z-direction and is found by substitution of \( \Delta z = (\rho_{y,x} x + \rho_{x,y} y) + O(3) \) in Equation 3.51. The
Figure 3.10a,b The chip-surface of CCDi in misalignment for the rotation about the x-axis and the y-axis: side-view (a) and top-view (b).

other terms are a consequence of the different orientation of the x- and y-axis and they follow from a geometrical analysis of the Figures 3.10a and 3.10b. Rewriting of Equation 3.54 gives:

\[
\Delta \alpha_i = -\frac{\partial \alpha}{\partial x} \left( \frac{\lambda}{4\pi} \frac{\partial \alpha}{\partial x} \left( \rho_{y,x} x + \rho_{x,y} y \right) + \frac{\rho_{x,x}^2}{2} x + \rho_{x,y} \rho_{y,y} y \right) - \frac{\partial \alpha}{\partial y} \left( \frac{\lambda}{4\pi} \frac{\partial \alpha}{\partial y} \left( \rho_{y,x} x + \rho_{x,y} y \right) + \frac{\rho_{x,x}^2}{2} y \right)
\]

(3.53b)

By this, the phase error can be interpreted as being a phase error introduced by a translation in the x-direction and a translation in the y-direction and thus Equation 3.41c can be used for the calculation of the rms phase error. The rotation about the x-axis and
the $y$-axis lead to a disorientation of all three cameras and thus, unlike the foregoing misalignments, there is no reference camera. The rms phase error becomes maximum if $\partial \alpha / \partial x$, $\partial \alpha / \partial y$ and the rotations $\rho_{x,i}$ and $\rho_{y,i}$ of all three cameras are positive. If $\rho_{x,0} = \rho_{x,1} = \rho_{x,2} = \rho_x$ and $\rho_{y,0} = \rho_{y,1} = \rho_{y,2} = \rho_y$ and $\partial \alpha / \partial x$ and $\partial \alpha / \partial y$ constant (linearly stratified medium) follows by using Equation 3.41c with $c = 8$ for (the upper bound of) the rms phase error:

$$\Delta \alpha_{\text{rms}}^2 = \frac{\lambda^2}{48 \pi^2} \left( \left( \frac{\partial \alpha}{\partial x} \right)^2 + \left( \frac{\partial \alpha}{\partial y} \right)^2 \right) \left( \rho_x B^2 + \rho_y H^2 \right) + \frac{\lambda}{12 \pi} \left( \left( \frac{\partial \alpha}{\partial x} \right)^3 + \frac{\partial \alpha}{\partial x} \left( \frac{\partial \alpha}{\partial y} \right)^2 \right) \rho_x B^2 + \frac{\lambda}{12 \pi} \left( \left( \frac{\partial \alpha}{\partial y} \right)^3 + \left( \frac{\partial \alpha}{\partial x} \right)^2 \frac{\partial \alpha}{\partial y} \right) \rho_y H^2 + \frac{\lambda}{6 \pi} \left( \left( \frac{\partial \alpha}{\partial x} \right)^3 + \frac{\partial \alpha}{\partial x} \left( \frac{\partial \alpha}{\partial y} \right)^2 \right) \rho_x \rho_y H^2 + \frac{1}{3} \left( \frac{\partial \alpha}{\partial x} \right)^2 \left( \frac{\partial \alpha}{\partial y} \right)^2 \rho_x \rho_y H^2 + \frac{1}{3} \frac{\partial \alpha}{\partial x} \frac{\partial \alpha}{\partial y} \rho_x \rho_y H^2,$$

(3.54a)

where $B$ is the half of the chip length in the $x$-direction and $H$ is the half of the chip length in the $y$-direction. If $\lambda = 632.8 \text{nm}$, $B = 4.1 \text{mm}$, $H = 3.15 \text{mm}$ and $\rho_x = \rho_y = 0.5^\circ$, the rms phase error is found to be:

$$\Delta \alpha_{\text{rms}}^2 = 1.72 \times 10^{-24} \left( \frac{\partial \alpha}{\partial x} \right)^2 + \left( \frac{\partial \alpha}{\partial y} \right)^2 + 4.09 \times 10^{-19} \left( \frac{\partial \alpha}{\partial x} \right)^3 + 1.11 \times 10^{-19} \left( \frac{\partial \alpha}{\partial y} \right)^2 + \frac{\partial \alpha}{\partial x} \left( \frac{\partial \alpha}{\partial y} \right)^2 + 1.11 \times 10^{-19} \left( \frac{\partial \alpha}{\partial y} \right)^3 + 2.73 \times 10^{-14} \left( \frac{\partial \alpha}{\partial x} \right)^2 + 1.92 \times 10^{-14} \frac{\partial \alpha}{\partial x} \frac{\partial \alpha}{\partial y} + 4.80 \times 10^{-15} \left( \frac{\partial \alpha}{\partial y} \right)^2$$

(3.54b)

Special circumstances arise if the phase distributions obtained by the RTI are compared to those obtained by the TPSI. This TPSI uses one of the cameras of the RTI and thus this camera is used as reference, independent of its orientation is correct or not. The other two cameras are rotated with respect to the reference camera over $2\rho_{x,i}$ and $2\rho_{y,i}$. Substitution of $\rho_{x,i}=2\rho_x$ and $\rho_{y,i}=2\rho_y$ for two cameras and $\rho_{x,i}=\rho_{y,i}=0$ for the third in Equation 3.53a,b and substitution of the resulting value for $\Delta \alpha_{\text{rms}}^2$ and in Equation 3.41c leads to an upper estimation of the phase difference between the RTI and the TPSI, i.e.,
\[
\frac{\Delta \alpha_{rms}^2}{c} = \frac{\lambda^2}{96\pi^2} \left( \left( \frac{\partial \alpha}{\partial x} \right)^2 + \left( \frac{\partial \alpha}{\partial y} \right)^2 \right)^2 \left( \rho_x^2 B^2 + \rho_y^2 H^2 \right) + \frac{\lambda}{12\pi} \left( \left( \frac{\partial \alpha}{\partial x} \right)^3 + \frac{\partial \alpha}{\partial x} \left( \frac{\partial \alpha}{\partial y} \right)^2 \right) \rho_x B^2 + \frac{\lambda}{6\pi} \left( \frac{\partial \alpha}{\partial y} \right)^3 + \frac{\partial \alpha}{\partial y} \left( \frac{\partial \alpha}{\partial x} \right)^2 \rho_y H^2 + \frac{\lambda}{6\pi} \left( \frac{\partial \alpha}{\partial x} \right)^3 + \frac{\partial \alpha}{\partial x} \left( \frac{\partial \alpha}{\partial y} \right)^2 \rho_x^2 \rho_y H^2 + \frac{2}{3} \frac{\partial \alpha}{\partial x} \left( \frac{\partial \alpha}{\partial y} \right)^2 \rho_x \rho_y^2 H^2 + \frac{2}{3} \frac{\partial \alpha}{\partial x} \frac{\partial \alpha}{\partial y} \rho_x \rho_y H^2.
\]

(3.55a)

Here, \(c=3\) if CCD1 is the reference and \(c=5\) if CCD0 or CCD2 is the reference. Substitution of \(\lambda=632.8\) nm, \(B=4.1\) mm, \(H=3.15\) mm and \(\rho_x=\rho_y=0.5^\circ\) gives:

\[
\frac{\Delta \alpha_{rms}^2}{c} = 8.60 \times 10^{-25} \left( \left( \frac{\partial \alpha}{\partial x} \right)^2 + \left( \frac{\partial \alpha}{\partial y} \right)^2 \right)^2 + 4.09 \times 10^{-19} \left( \frac{\partial \alpha}{\partial x} \right)^3 + 1.11 \times 10^{-19} \left( \frac{\partial \alpha}{\partial y} \right)^3 + 1.11 \times 10^{-19} \left( \frac{\partial \alpha}{\partial x} \right)^2 \frac{\partial \alpha}{\partial y} + 4.94 \times 10^{-14} \frac{\partial \alpha}{\partial x} \frac{\partial \alpha}{\partial y} + 9.59 \times 10^{-15} \left( \frac{\partial \alpha}{\partial y} \right)^2
\]

(3.55b)

In order to find an upper bound of \(\Delta \alpha_{rms}\) if \(\partial \alpha/\partial x\) and \(\partial \alpha/\partial y\) are not constant over the chip-surface (locally linearly stratified medium) Equation 3.53a is rewritten again:

\[
\Delta \alpha_i = -\frac{\lambda}{4\pi} \alpha^2 (\rho_x x + \rho_y y) - \frac{\partial \alpha}{\partial x} \rho_{x,i} \rho_y y - \frac{1}{2} \frac{\partial \alpha}{\partial x} \rho_x x - \frac{1}{2} \frac{\partial \alpha}{\partial y} \rho_y y + O(4), \quad (3.53c)
\]

Equation 3.53c shows that the first term does not depend on the direction of the phase gradient while the second, third and fourth term do. A maximum phase error appears if the direction of the phase gradient varies over the chip-surface such that the combination of the second, third and fourth term maximally intensifies the first term, i.e., the size of the combination of terms is maximum and its sign is same as that of the first term for every \((x,y)\). The combination of terms can be interpreted as being a translation over \(\rho_x \rho_y y + \rho_x x/2\) in the \(x\)-direction and a translation over \(\rho_y y/2\) in the \(y\)-direction which results in a total translation over \(\sqrt{(\rho_x \rho_y y + \rho_x x/2)^2} + \rho_y y^2/4\). A maximum size of the combination of terms appears if the direction of the total translation coincides with the direction of the phase gradient. Then Equation 3.53c can be rewritten as:
\[ \Delta \alpha_i = -\frac{\lambda}{4\pi} \alpha^2 (\rho_{x,i} x + \rho_{y,i} y) - \text{sign}(\rho_{x,i} x + \rho_{y,i} y) \alpha \sqrt{\left(\rho_{x,i} \rho_{y,i} y + \rho_{y,i} x / 2\right)^2 + \rho_{y,i}^4 y^2 / 4} + \text{O}(4), \]  

(3.53d)

where the function \( \text{sign}(\rho_{y,i} x + \rho_{x,i} y) \) ensures that the two terms in Equation 3.53d have the same sign. Substitution of Equation 3.53d in Equation 3.41c under the assumption that \( \rho_{x,0} = \rho_{x,1} = \rho_{x,2} = \rho_{y,0} = \rho_{y,1} = \rho_{y,2} = 0.5^\circ \), \( B = 4.1 \text{mm} \), \( H = 3.15 \text{mm} \), \( \lambda = 632.8 \text{nm} \) and \( c = 8 \) (no reference camera) leads to:

\[ \Delta \alpha_{\text{rms}} \leq \sqrt{1.72 \cdot 10^{-24} \alpha^4 + 4.45 \cdot 10^{-19} \alpha^3 + 3.21 \cdot 10^{-14} \alpha^2} \]  

(3.56)

**Total phase error**

In general, the phase errors introduced by the misalignments of the translations and rotations of the cameras show statistical dependence and to obtain (an upper bound of) the total induced rms phase error the rms phase errors induced by the misalignment of the translation along the x-axis and y-axis, the rotation about the z-axis, the translation along the z-axis and the rotation about the x-axis and y-axis must be added, i.e.,

\[ \Delta \alpha_{\text{rms}} = \Delta \alpha_{\text{rms}}[\text{trans.}x,y] + \Delta \alpha_{\text{rms}}[\text{rot.}z] + \Delta \alpha_{\text{rms}}[\text{trans.}z] + \Delta \alpha_{\text{rms}}[\text{rot.}x,y]. \]  

(3.57a)

However, in the special case of a linearly stratified medium the phase errors induced by the translation along the x-axis and y-axis and the translation along the z-axis are statistically independent of the errors induced by the rotation about the x-axis and the y-axis and the rotation about the z-axis. Hence, the total rms phase error is equal to:

\[ \Delta \alpha_{\text{rms}} = \sqrt{(\Delta \alpha_{\text{rms}}[\text{trans.}x,y] + \Delta \alpha_{\text{rms}}[\text{trans.}z])^2 + (\Delta \alpha_{\text{rms}}[\text{rot.}x,y] + \Delta \alpha_{\text{rms}}[\text{rot.}z])^2}. \]  

(3.57b)

Figure 3.11 shows (the upper bound of) the total phase error as a function of the phase gradient \( \alpha' \) for various \( c \) if \( \Delta x = \Delta y = 0.5^\circ \), \( \Delta z = 200 \mu\text{m} \), \( \rho_{x,i} = 0.02^\circ \), \( \rho_{x,1} = \rho_{x,2} = 0.5^\circ \). In case of a linearly stratified medium, the rms phase error is calculated by substitution of the Equations 3.43b, 3.47b, 3.52b and 3.54b in Equation 3.57b. Presented are the results for a gradient parallel to the x-axis (vertical fringes), a gradient parallel to the y-axis (horizontal fringes) and a gradient under 45° with the x-axis (fringes 45°). Further, the maximum phase error is presented in case of a locally linearly stratified medium, which is based on the substitution of Equations 3.44, 3.48b, 3.52b and 3.56 in Equation 3.57a. The figure shows that the rms phase errors increase with \( \alpha' \) and \( c \). In case of a linearly stratified medium, the phase error is largest if the fringes are sloping. In case of a locally linearly stratified medium, the rms phase error can be up to 1.1% of 2π radians larger than in case of a linearly stratified medium, whereby the exact increase depends on \( \alpha' \) and \( c \). However, there must be noted that the maximum rms phase error is a theoretical upper
Figure 3.11 The total rms phase error due to camera misalignment as a function of the phase gradient $\alpha'$ and the number of fringes over 8.8mm length of the CCD-chip.

bound which cannot be reached practically because Equations 3.44, 3.48b and 3.56 are based on different phase distributions in order to maximize the rms phase induced by each individual misalignment. Further, the rms phase error is calculated by using Equation 3.57a which assumes a fully statistical dependence between the errors induced by the individual misalignments, an assumption that is not true in practise.

3.7 Phase errors due to the optical elements

The optical elements in the interferometer introduce phase errors due to two reasons. Firstly, the elements have a limited perfection and, secondly, the alignment accuracy of the elements in the set-up is finite. Two groups of optical elements may be distinguished: the imaging elements and the polarization elements.

The imaging elements, i.e., the combination of lenses which images the object plane in the test section on the CCD-cameras, introduce wavefront deformations due to Seidel aberrations and, hence, a slight deformation and a slight disturbance of the interferograms. The extension of the deformation and the disturbance is dependent on the imaging optics, but it is usually small because the effective numerical aperture in an interferometer is small. In the real time interferometer the disturbance is reduced by using achromatic lenses. The error induced by the imaging elements can be divided into two parts: an error induced by an aberrated wavefront in the test section and an error induced
by a distorted image of the object plane on the CCD-cameras. The first error is discussed extensively in Section 3.9, the second error is determined experimentally.

The group of polarization elements includes the polarization filter, the wave plates and the polarizing beam splitters. Also the non-polarizing beam splitter belongs to this group because its transmission and its reflection are not fully polarization independent.

Several error sources arise due the limited perfection of these elements. Imperfections in the retardation in the wave plates lead to cross-talk and, hence, to phase errors.\cite{66,67} This implies not only that the retardation accuracy of the wave plates must be sufficiently high, but also that the incidence of the rays must be perpendicular. Therefore, the wave plates are placed at positions in the interferometer where the beam is (almost) parallel. However, if a non-homogeneous refractive index profile is present in the test section, refraction of the rays will appear and a non-perpendicular incidence cannot be avoided. To keep the phase error, nevertheless, negligible, zero order wave plates have been applied. Another advantage of zero order wave plates when compared to multiple order wave plates is the low temperature dependency of the retardation.

Just like imperfections in the retardation in the wave plates, imperfections in the extinction ratio of the polarizing beam splitters and the polarization filter lead to cross-talk and to phase errors. Although cross-talk in the polarizing beam splitters lead to much higher phase errors than cross-talk in the polarization filter, all extinction ratios are chosen to be sufficient high, i.e., 1000:1 for both the transmitted and the reflected beam of the polarizing beam splitters and 10000:1 for the polarization filter.

Another error source arises due to the fact that the refractive indices of the beam splitters (cubes) and the wave plates do not match the refractive index of the surrounding air and thus refraction of the light rays appears. The result is a displacement of the image plane and an erroneous additional phase shift of the refracted rays in the image plane when compared to the corresponding straight rays. The effects are, however, rather unimportant. The shift of the image plane contains two parts, a part which is independent of the angle of incidence of the ray on the beam splitter (or wave plate) and a part which is dependent of the angle of incidence. The first part can be compensated by shifting the detection planes accordingly. The second part is a second order function of the angle of incidence and, hence, it can be ignored in case of small numerical apertures. The additional phase shift of the refracted rays is a high order function of the angle of incidence and can also be neglected.

A final error source caused by imperfections in the optical elements is the fact that the transmission and the reflection by the non-polarizing beam splitter BS0 is not fully polarization independent. Hence, the relative intensities of the test beam and the reference beam on CCD1 deviates from the relative intensities on CCD0 and CCD2. However, this effect can be corrected by a suitable adjustment of half wave plate HWP1 and, hence, the introduced phase error is negligible.

Apart from the limited perfection of the optical elements, misalignment of the wave plates and the polarizing beam splitters is an important source for phase errors. Mainly
misorientations of the quarter wave plate QWP2, the half wave plate HWPI and the polarizing beam splitters PBS1 and PBS2 can lead to large errors. Misorientation of the fast axis of QWP2 leads to an elliptical polarization of the beam instead of a circular polarization. Misorientation of PBS1 introduces an error in the phase steps between the interferograms on the different cameras. A misorientation of the fast axis of HWP1 or a misorientation of PBS2 implies a deviation in the relative intensities of the test beam and the reference beam on CCD1 compared to CCD0 and CCD2.

It is difficult to calculate the contributions to $\Delta \alpha_{\text{rms}}$ due to all imperfections and misalignments of the polarizing elements separately because their effects are often coupled. Therefore, the calculation is made for a typical set of previously determined misalignments and imperfections. Use is made here of Equation 3.23. Condition for the calculation of $\Delta \alpha_{\text{rms}}$ is that the measured phase $\alpha_m$ must be calculated and, hence, the actual intensities on the CCD-cameras must be evaluated including both the desired intensities and the disturbing intensities. This requires a calculation of all the electric field vectors everywhere in the interferometer whereby all the imperfections and misalignments are taken into consideration.

In the calculations the electric field is represented as a complex vector field and the polarizing elements are described in the Jones formalism. The lenses are not considered and all rays are assumed to be parallel and perpendicular to the polarizing elements. The local electric field vectors and the local coordinate systems everywhere in the set-up are defined in Figure 3.12. The beam always propagates in the positive z-direction. The $x$-axis is parallel to the plane of incidence of polarizing beam splitter PBS0, the $y$-axis is orthogonal to this plane. A reflection of the beam is described by a change in the direction of the $z$-axis.

The wave plates in the set-up, each with their own exact value for the retardation $\chi$ and with their own angle between the fast axis and the positive $x$-axis, are described by $M_{\text{ret}}(\chi, \varepsilon)$:

$$
M_{\text{ret}}(\chi, \varepsilon) = \begin{bmatrix} \cos \varepsilon & -\sin \varepsilon \\ \sin \varepsilon & \cos \varepsilon \end{bmatrix} \begin{bmatrix} \cos(\frac{\chi}{2}) & -i\sin(\frac{\chi}{2}) \\ -i\sin(\frac{\chi}{2}) & \cos(\frac{\chi}{2}) \end{bmatrix} \begin{bmatrix} \cos \varepsilon & \sin \varepsilon \\ -\sin \varepsilon & \cos \varepsilon \end{bmatrix} \tag{3.58}
$$

$$
= \cos(\frac{\chi}{2}) + \sin(\frac{\chi}{2})M(\varepsilon)e^{-i\frac{\pi}{2}}.
$$

Here, the central transformation matrix on the first line describes a wave plate with the fast axis under $\pi/4$ radians with the $x$-axis. The other two transformation matrices describe a forward and a backward rotation of the coordinate system over an angle $\varepsilon$ so that the total angle between the fast axis and the $x$-axis is equal to ($\varepsilon + \pi/4$). In the following calculations is $M_{\text{ret}}(\chi, \varepsilon)$ described by the second line in Equation 3.58, where the
transformation matrix \( M(\varepsilon) \) is only dependent on the angle \( \chi \) and is given by:

\[
M(\varepsilon) = \begin{bmatrix}
-\sin 2\varepsilon & \cos 2\varepsilon \\
\cos 2\varepsilon & \sin 2\varepsilon
\end{bmatrix}.
\]

This matrix contains only real components and, hence, it introduces no phase shift in the electric field. This implies that the phase shift introduced by the wave plate is fully described by the term \( e^{i\pi/2} \) in the second line of Equation 3.58.

The linear polarization filter in the set-up is described by the transformation matrix \( M_{\text{pol}}(t_x, t_y, \varepsilon) \). If the electric field vector of the transmitted light makes an angle \( \varepsilon \) with the \( x \)-axis this transformation matrix is described by:

\[
M_{\text{pol}}(t_x, t_y, \varepsilon) = \begin{bmatrix}
\cos \varepsilon & -\sin \varepsilon \\
\sin \varepsilon & \cos \varepsilon
\end{bmatrix} \begin{bmatrix}
t_x & 0 \\
0 & t_y
\end{bmatrix} \begin{bmatrix}
\cos \varepsilon & \sin \varepsilon \\
-\sin \varepsilon & \cos \varepsilon
\end{bmatrix}
= \begin{bmatrix}
t_x \cos^2 \varepsilon + t_y \sin^2 \varepsilon & \frac{t_x - t_y}{2} \sin 2\varepsilon \\
\frac{t_x - t_y}{2} \sin 2\varepsilon & t_x \sin^2 \varepsilon + t_y \cos^2 \varepsilon
\end{bmatrix}.
\]

Figure 3.12 Electric field vectors in the experimental real time Michelson interferometer.
The second matrix on the first line describes a linear polarization filter which transmits light polarized parallel to the x-axis. The amplitude transmission coefficient for light polarized in that direction is equal to \( t_x \), the amplitude transmission coefficient for light polarized perpendicular to it is equal to \( t_y \). The third and the first matrices describe a forward and a backward rotation of the coordinate system over an angle \( \varepsilon \). The total transformation matrix is represented in the second line. Since this matrix contain only real components, there is no introduction of a phase shift.

The polarizing beam splitter transmits linearly polarized light with its electric field vector parallel to the plane of incidence (p-polarized) and it reflects light with the electric field vector orthogonal to the plane of incidence (s-polarized). The transmission as well as the reflection may be described by the same formalism as the linear polarization filter. The coefficients \( t_x \) and \( t_y \), however, must be replaced by the amplitude transmission coefficient (or the amplitude reflection coefficients) for p-polarized light and s-polarized light while \( \varepsilon \) is the angle between the plane of incidence and the x-axis.

The non-polarizing beam splitter may also be described by Equation 3.60. The coefficients \( t_x \) and \( t_y \) are now equal to the amplitude transmission coefficients (or amplitude reflection coefficients) of the beam splitter for light polarized in the x-direction and the y-direction, respectively. The angle \( \varepsilon \) is equal to 0 radians.

The calculation of the electric field vectors all through the set-up, as defined in Figure 3.12, requires a step by step approach whereby one electric field is extracted from one or more other electric fields. Below, this step by step approach is presented.

1. The electric field of the laser beam is given by \( E_1 \) as presented by:

\[
E_1 = \begin{bmatrix} E_{\text{LASER} \cos \theta_{\text{LASER}}} \\ E_{\text{LASER} \sin \theta_{\text{LASER}}} \end{bmatrix} e^{-i(\alpha - \psi)},
\]

(3.61)

where \( E_{\text{LASER}} \) is the electric field amplitude of the laser beam, \( \theta_{\text{LASER}} \) represents the angle between the polarization direction of the laser and the x-axis, \( \alpha \) represents the angular frequency of the field and \( \psi \) represents the phase of the field. The z-dependent part \( e^{ikz} \) (with \( k \) the propagation number \( 2\pi/\lambda \)) is omitted. Since all the light traverses the same optical path length between the optical elements in the set-up, this term will not introduce a relative phase shift between the different components of the total field vector.

2. After passage through half wave plate HWPO with retardation \( \chi_{\text{HWPO}} \) and with the fast axis under \( \theta_{\text{HWPO}} \) radians with the x-axis, the field vector \( E_1 \) is transformed to:

\[
E_2 = M_{\text{ret}}(\chi_{\text{HWPO}}, \theta_{\text{HWPO}} - \frac{\pi}{4}) E_1
\]

\[
= \cos\left(\frac{\chi_{\text{HWPO}}}{2}\right) E_1 + \sin\left(\frac{\chi_{\text{HWPO}}}{2}\right) M(\theta_{\text{HWPO}} - \frac{\pi}{4}) E_1 e^{-i\frac{\pi}{2}}
\]

(3.62)

\[
= E_{2,1} + E_{2,2} e^{-i\frac{\pi}{2}}.
\]
Here use is made of Equation 3.58. The resulting field vector $E_2$ contains two components which are shifted in phase over $\pi/2$ radians with respect to each other, i.e., $E_{2,1}$ and $E_{2,2} e^{i\pi/2}$. Since the matrix $M$ has only real components, the phases of $E_{2,1}$ and $E_{2,2}$ are equal.

3. The transmission of the field by polarization filter $P_0$ is described by Equation 3.60 where $t_{x} = t_{x,P_0}$ and $t_{y} = t_{y,P_0}$. The resulting field vector $E_3$ is given by:

$$
E_3 = M_{pol}(t_{x,P_0}, t_{y,P_0}, \theta_{P_0}) E_2 = E_{3,1} + E_{3,2} e^{-i\pi/2}.
$$

(3.63)

4. The reflection by polarizing beam splitter PBS0 extracts the electric field vector of the test beam $E_4$ from $E_3$. By definition of the coordinate system, the angle between the plane of incidence and the $x$-axis is equal to zero and, hence, the electric field in the test beam is:

$$
E_4 = M_{pol}(r_{p,PBS0}, r_{s,PBS0}, 0) E_3 = E_{4,1} + E_{4,2} e^{-i\pi/2}.
$$

(3.64)

Here, $r_{p,PBS0}$ and $r_{s,PBS0}$ are the amplitude reflection coefficients of the $p$-polarized and the $s$-polarized light in PBS0.

5. The electric field $E_4$ passes quarter wave plate QWP0, it is reflected by mirror M0 and, after that, it passes quarter wave plate QWP0 for the second time. The transformation of the electric field in this path is the same as if the field would have passed one single half wave plate. Hence, the electric field $E_5$ which returns to PBS0 is equal to:

$$
E_5 = M_{ret}(2\chi_{QWP0}, \varepsilon_{QWP0}) E_4
$$

$$
= \cos(\chi_{QWP0}) E_4 + \sin(\chi_{QWP0}) M(\varepsilon_{QWP0}) E_4 e^{-i\pi/2}
$$

$$
= \cos(\chi_{QWP0}) E_{4,1} + \sin(\chi_{QWP0}) M(\varepsilon_{QWP0}) E_{4,2}
$$

$$
+ (\cos(\chi_{QWP0}) E_{4,2} + \sin(\chi_{QWP0}) M(\varepsilon_{QWP0}) E_{4,1}) e^{-i\pi/2}
$$

$$
= E_{5,1} + E_{5,2} e^{-i\pi/2},
$$

(3.65)

where $\chi_{QWP0}$ is the retardation in QWP0 and $\varepsilon_{QWP0}$ is the misorientation of the fast axis from $\pi/4$ radians with respect to the $x$-axis. Since the matrix $M(\varepsilon_{QWP0})$ is real, the vectors $E_{5,1}$ and $E_{5,2}$ are in phase with each other.

6. The electric field $E_6$ in the reference beam is extracted from $E_3$ by the transmission through polarizing beam splitter PBS0, i.e.,

$$
E_6 = M_{pol}(t_{p,PBS0}, t_{s,PBS0}, 0) E_3 = E_{6,1} + E_{6,2} e^{-i\pi/2},
$$

(3.66)

where $t_{p,PBS0}$ and $t_{s,PBS0}$ are the amplitude transmission coefficients of PBS0 for $p$-polarized and $s$-polarized light. Due to the definition of the coordinate system the angle between the plane of incidence and the $x$-axis is equal to 0 radians.
7. A passage through quarter wave plate QWP1 in combination with a reflection by mirror M1 and a second passage through QWP1 transforms electric filed vector $E_6$ into $E_7$. This transformation is, analogous to a single passage of a half wave plate, presented by:

$$
E_7 = M_{\text{rot}}(2\chi_{\text{QWP1}}, e_{\text{QWP1}}) E_6
$$

$$
= \cos(\chi_{\text{QWP1}}) E_6 + \sin(\chi_{\text{QWP1}})M(e_{\text{QWP1}}) E_6 e^{-i\frac{\pi}{2}}
$$

$$
= \cos(\chi_{\text{QWP1}})E_{6,1} - \sin(\chi_{\text{QWP1}})M(e_{\text{QWP1}}) E_{6,2}
$$

$$
+ (\cos(\chi_{\text{QWP1}})E_{6,2} + \sin(\chi_{\text{QWP1}})M(e_{\text{QWP1}}) E_{6,1}) e^{-i\frac{\pi}{2}}
$$

$$
= E_{7,1} + E_{7,2} e^{-i\frac{\pi}{2}}. \tag{3.67}
$$

Here, $\chi_{\text{QWP1}}$ is the retardation in QWP1 and $e_{\text{QWP1}}$ is the misorientation of the fast axis with respect to an angle of $\pi/4$ radians with the $x$-axis.

8. The test beam and the reference beam are recombined by polarizing beam splitter PBS0. Therefore, it transmits electric field vector $E_5$ and it reflects electric field vector $E_7$. Since the optical path lengths traversed by the test beam and the reference beam are unequal, the relative phase of $E_5$ is shifted over $\phi$ with respect to $E_7$. Hence, the total electric field vector after recombination $E_8$ is equal to:

$$
E_8 = M_{\text{pol}}(r_{p,\text{PBS0}}, t_{s,\text{PBS0}}, 0) E_7 + M_{\text{pol}}(t_{p,\text{PBS0}}, r_{s,\text{PBS0}}, 0) E_5 e^{i\phi}
$$

$$
= M_{\text{pol}}(r_{p,\text{PBS0}}, r_{s,\text{PBS0}}, 0) E_{7,1} + M_{\text{pol}}(r_{p,\text{PBS0}}, r_{s,\text{PBS0}}, 0) E_{7,2} e^{-i\frac{\pi}{2}}
$$

$$
+ M_{\text{pol}}(t_{p,\text{PBS0}}, t_{s,\text{PBS0}}, 0) E_{5,1} e^{i\phi} + M_{\text{pol}}(t_{p,\text{PBS0}}, t_{s,\text{PBS0}}, 0) E_{5,2} e^{i(\phi - \frac{\pi}{2})} \tag{3.68}
$$

$$
= E_{8,1} + E_{8,2} e^{-i\frac{\pi}{2}} + E_{8,3} e^{i\phi} + E_{8,4} e^{i(\phi - \frac{\pi}{2})},
$$

where $r_{p,\text{PBS0}}$ and $r_{s,\text{PBS0}}$ are the amplitude reflection coefficients of p-polarized and s-polarized light in PBS0, and $t_{p,\text{PBS0}}$ and $t_{s,\text{PBS0}}$ are the amplitude transmission coefficients of p-polarized and s-polarized light. The angle between the plane of incidence and the $x$-axis is zero by definition.

9. Non-polarizing beam splitter BSO splits the beam to CCD1 from the beam to CCD0 and CCD2. The electric field vector in this separated beam $E_9$ is given by:

$$
E_9 = M_{\text{pol}}(r_{x,\text{BSO}}, r_{y,\text{BSO}}, 0) E_8 = E_{9,1} + E_{9,2} e^{-i\frac{\pi}{2}} + E_{9,3} e^{i\phi} + E_{9,4} e^{i(\phi - \frac{\pi}{2})}. \tag{3.69}
$$

where $r_{x,\text{BSO}}$ is the amplitude reflection coefficient of the $x$-polarized light and $r_{y,\text{BSO}}$ is the amplitude reflection coefficient of the $y$-polarized light. Because these coefficients are assumed to be related to the $x$-axis and the $y$-axis and not to the plane of incidence, $e$ in this Equation is assumed to be zero.

10. Transmission of $E_9$ by half wave plate HWP1 with retardation $\chi_{\text{HWP1}}$ and with its fast axis under an angle of $\theta_{\text{HWP1}}$ with the $x$-axis leads to the electric field vector $E_{10}$, i.e.,
\[
E_{10} = M_{\text{rot}}(X_{\text{HWP}}, \theta_{\text{HWP}}, -\frac{\pi}{4}) E_9 \\
= \cos\left(\frac{X_{\text{HWP}}}{2}\right) E_9 + \sin\left(\frac{X_{\text{HWP}}}{2}\right) M(\theta_{\text{HWP}}, -\frac{\pi}{4}) E_9 e^{-\frac{i\pi}{2}} \\
= \left\{ \cos\left(\frac{X_{\text{HWP}}}{2}\right) E_{9,1} - \sin\left(\frac{X_{\text{HWP}}}{2}\right) M(\theta_{\text{HWP}}, -\frac{\pi}{4}) E_{9,2} \right\} e^{-\frac{i\pi}{2}} \\
+ \left\{ \cos\left(\frac{X_{\text{HWP}}}{2}\right) E_{9,2} + \sin\left(\frac{X_{\text{HWP}}}{2}\right) M(\theta_{\text{HWP}}, -\frac{\pi}{4}) E_{9,1} \right\} e^{i\phi} \\
+ \left\{ \cos\left(\frac{X_{\text{HWP}}}{2}\right) E_{9,3} - \sin\left(\frac{X_{\text{HWP}}}{2}\right) M(\theta_{\text{HWP}}, -\frac{\pi}{4}) E_{9,4} \right\} e^{i(\phi - \frac{\pi}{2})} \\
= E_{10,1} + E_{10,2} e^{-\frac{i\pi}{2}} + E_{10,3} e^{i\phi} + E_{10,4} e^{i(\phi - \frac{\pi}{2})}. \tag{3.70}
\]

11. Transmission of \(E_{10}\) by polarizing beam splitter PBS2 with the plane of incidence under an angle of \(\varepsilon_{\text{PBS2}}\) with the x-axis results in the electric field vector \(E_{11}\) which will strike CCD1. This vector is given by:

\[
E_{11} = M_{\text{pol}}(t_{p,\text{PBS2}}, t_{s,\text{PBS2}}, E_{\text{PBS2}}) E_{10} = E_{11,1} + E_{11,2} e^{-\frac{i\pi}{2}} + E_{11,3} e^{i\phi} + E_{11,4} e^{i(\phi - \frac{\pi}{2})}. \tag{3.71}
\]

Here, \(t_{p,\text{PBS2}}\) and \(t_{s,\text{PBS2}}\) are the amplitude transmission coefficients of PBS2 for p-polarized and s-polarized light. The intensity on CCD1 is a result of interference between the different electric field components \(E_{11,1}, E_{11,2}, E_{11,3}\) and \(E_{11,4}\). The \(x\)-components of \(E_{11,2}\) and \(E_{11,4}\) are relatively large and they describe mainly the desired electrical fields due to the reference beam and the test beam. The \(y\)-components of \(E_{11,2}\) and \(E_{11,4}\) are, just like both components of \(E_{11,1}\) and \(E_{11,3}\), rather weak and they contain only undesired contributions to the total electric field. The total intensity on CCD1 is given by:

\[
I_1 = E_{11} E_{11}^* \\
= E_{11,1} E_{11,1}^* + E_{11,2} E_{11,2}^* + E_{11,3} E_{11,3}^* + E_{11,4} E_{11,4}^* + 2\left(E_{11,1} E_{11,3}^* + E_{11,2} E_{11,4}^*\right) \cos \phi + 2\left(E_{11,1} E_{11,4}^* - E_{11,2} E_{11,3}^*\right) \sin \phi, \tag{3.72}
\]

where \(^*\) denotes a complex conjugate. In this equation is made use of the fact that the relative phase of the electric field vectors \(E_{11,1}, E_{11,2}, E_{11,3}\) and \(E_{11,4}\) are described by external exponents so that the vectors themselves are in phase and all the intensity components \(E_{11,i} E_{11,j}^*\) \((i,j=1,2,3,4)\) are real and \(E_{11,i} E_{11,j}^* = E_{11,j} E_{11,i}^*\).

12. Electric field \(E_{12}\) is a result of the transmission of \(E_9\) by non-polarizing beam splitter.
BS0. This field is described by:

\[
E_{12} = M_{\text{pol}}(t_x, BS0, t_y, BS0, 0) E_6 = E_{12,1} + E_{12,2} e^{-i \frac{\pi}{2}} + E_{12,3} e^{i \phi} + E_{12,4} e^{i(\phi - \frac{\pi}{2})}, \quad (3.73)
\]

where \(t_x, BS0\) and \(t_y, BS0\) are the amplitude transmission coefficients of BS0 for \(x\)-polarized and \(y\)-polarized light.

13. The electric field vector \(E_{12}\) is transformed to electric field vector \(E_{13}\) by quarter wave plate QWP2, whose fast axis is approximately under an angle of \(\pi/4\) radians with the \(x\)-axis. This transformation is described by:

\[
E_{13} = M_{\text{ret}}(\chi_{\text{QWP2}}, \varepsilon_{\text{QWP2}}) E_{12}
= \cos(\frac{\chi_{\text{QWP2}}}{2}) E_{12,1} + \sin(\frac{\chi_{\text{QWP2}}}{2}) M(\varepsilon_{\text{QWP2}}) E_{12} e^{-i \frac{\pi}{2}}
= \left\{ \cos(\frac{\chi_{\text{QWP2}}}{2}) E_{12,1} - \sin(\frac{\chi_{\text{QWP2}}}{2}) M(\varepsilon_{\text{QWP2}}) E_{12,2} \right\}
\begin{aligned}
&+ \left\{ \cos(\frac{\chi_{\text{QWP2}}}{2}) E_{12,2} + \sin(\frac{\chi_{\text{QWP2}}}{2}) M(\varepsilon_{\text{QWP2}}) E_{12,3} \right\} e^{-i \frac{\pi}{2}} e^{i \phi} \\
&+ \left\{ \cos(\frac{\chi_{\text{QWP2}}}{2}) E_{12,3} - \sin(\frac{\chi_{\text{QWP2}}}{2}) M(\varepsilon_{\text{QWP2}}) E_{12,4} \right\} e^{i(\phi - \frac{\pi}{2})}
\end{aligned} \quad (3.74)
= E_{13,1} + E_{13,2} e^{-i \frac{\pi}{2}} + E_{13,3} e^{i \phi} + E_{13,4} e^{i(\phi - \frac{\pi}{2})}.

Here, \(\chi_{\text{QWP2}}\) denotes the retardation in QWP2 and \(\varepsilon_{\text{QWP2}}\) denotes the angle of misorientation of the fast axis.

14. The reflection by polarizing beam splitter PBS1 extracts the electric field vector \(E_{14}\) which will strike CCD0 from \(E_{13}\). The extracted field is described by:

\[
E_{14} = M_{\text{pol}}(r_p, PBS1, r_s, PBS1, s_{PBS1}) E_{13} = E_{14,1} + E_{14,2} e^{-i \frac{\pi}{2}} + E_{14,3} e^{i \phi} + E_{14,4} e^{i(\phi - \frac{\pi}{2})} \quad (3.75)
\]

with \(r_p, PBS1\) and \(r_s, PBS1\) the amplitude reflection coefficients of PBS1 for \(p\)-polarized and \(s\)-polarized light, and \(s_{PBS1}\), the angle between the plane of incidence and the \(x\)-axis. Since \(r_s, PBS1 \gg r_p, PBS1\) and \(s_{PBS1}\) is small, the light on CCD0 is mainly polarized in the \(y\)-direction. The desired electric field caused by the reference beam is contained in the \(y\)-component of \(E_{14,1}\), and the desired electric field caused by the test beam is contained the \(y\)-component of \(E_{14,4}\). The other components contain only undesired contributions to the total electric field. Under the same conditions as the intensity on CCD1 has been calculated follows for the intensity on CCD0:
\[ I_0 = E_{14}^* E_{14} \]
\[ = E_{14,1}^* E_{14,1} + E_{14,2}^* E_{14,2} + E_{14,3}^* E_{14,3} + E_{14,4}^* E_{14,4} + 2(E_{14,1}^* E_{14,3} + E_{14,2}^* E_{14,4}) \cos \phi + 2(E_{14,1}^* E_{14,4} - E_{14,2}^* E_{14,3}) \sin \phi. \] (3.76)

15. The transmission by polarizing beam splitter PBS1 extracts the electric field vector \( E_{15} \) which will strike CCD2 from \( E_{13} \). This vector is given by:

\[ E_{15} = M_{pol}(t_{p,PBS}, t_{s,PBS}, e_{PBS}) E_{13} = E_{15,1} + E_{15,2} e^{-i \frac{\pi}{2}} + E_{15,3} e^{i \phi} + E_{15,4} e^{i(\phi - \frac{\pi}{2})}. \] (3.77)

where \( t_{p,PBS} \) and \( t_{s,PBS} \) are the amplitude transmission coefficients of PBS1 for p-polarized and s-polarized light. Because \( t_{p,PBS} \gg t_{s,PBS} \) and \( e_{PBS} \) is small, the light on CCD0 is mainly polarized in the x-direction. The x-component of \( E_{15,2} \) contains the desired contribution of the reference beam and the x-component of \( E_{15,3} \) contains the desired contribution of the test beam. All the other components contain only undesired contributions. The total intensity on CCD2 is given by:

\[ I_2 = E_{15}^* E_{15} \]
\[ = E_{15,1}^* E_{15,1} + E_{15,2}^* E_{15,2} + E_{15,3}^* E_{15,3} + E_{15,4}^* E_{15,4} + 2(E_{15,1}^* E_{15,3} + E_{15,2}^* E_{15,4}) \cos \phi + 2(E_{15,1}^* E_{15,4} - E_{15,2}^* E_{15,3}) \sin \phi. \] (3.78)

A typical combination of actual values for the retardation in the wave plates, the extinction ratio of the polarizer and the polarizing beam splitters, and the power transmission and reflection coefficient of the non-polarizing beam splitter is presented in Table 3.2. Also typical angles of orientation of the half wave plates and typical angles of misorientation of the quarter wave plates, polarizers and polarizing beam splitters are presented. This set of values is used to calculate a typical value for the rms phase error due to imperfections and misalignment of the polarizing elements. Special attention must be paid to the orientation of HWP0, PBS0, HWP1 and PBS2. The orientation angle of HWP0 has a rather arbitrary value of -30°. Since HWP0 is used to regulate the total power of the laser beam, its actual value is not approximately known and may differ for different measurements because it is dependent on the light absorption in the set-up. However, due to the high value of the extinction coefficient, the orientation of HWP0 hardly influences the rms phase error. The angle of misorientation of PBS0 is zero because the coordinate system is defined by the orientation of this polarizing beam splitter. The optimum orientation angle of HWP1 is dependent on the values of the reflection coefficients in BS0. The reflection coefficients presented in Table 3.2 imply that an orientation angle of 22.7° corresponds to a misorientation of 0.8°. The effect of this misorientation is strongly dependent on the misorientation of PBS2. If PBS2 is rotated over the double angle, the effect vanishes. Hence, the angle of misorientation of HWP1 may be interpreted as a kind of relative misorientation with respect to PBS2 and the misorientation of PBS2 may be
3. Design of the real time interferometer and accuracy analysis

<table>
<thead>
<tr>
<th>Element</th>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
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<tr>
<td>LASER</td>
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<tr>
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<td></td>
<td>retardation ($\chi_{\text{HWPO}}$)</td>
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</tr>
<tr>
<td></td>
<td>extinction ratio</td>
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</tr>
<tr>
<td>PBS0</td>
<td>angle of misorientation</td>
<td>$0^\circ$</td>
</tr>
<tr>
<td></td>
<td>extinction ratio transmission</td>
<td>1000:1</td>
</tr>
<tr>
<td></td>
<td>extinction ratio reflection</td>
<td>1000:1</td>
</tr>
<tr>
<td>QWPO</td>
<td>angle of misorientation ($\epsilon_{\text{QWPO}}$)</td>
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</tr>
<tr>
<td></td>
<td>retardation ($\chi_{\text{QWPO}}$)</td>
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</tr>
<tr>
<td>QWP1</td>
<td>angle of misorientation ($\epsilon_{\text{QWP}}$)</td>
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</tr>
<tr>
<td></td>
<td>retardation ($\chi_{\text{QWP}}$)</td>
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</tr>
<tr>
<td>BS0</td>
<td>refl. coëf. x-pol.:</td>
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</tr>
<tr>
<td></td>
<td>refl. coëf. y-pol.:</td>
<td>0.51</td>
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<tr>
<td></td>
<td>extinction ratio reflection</td>
<td>1000:1</td>
</tr>
</tbody>
</table>

Table 3.2 Typical properties of the optical elements in the set-up.

assumed to be zero.

The rms phase error is calculated by using Equation 3.23 while $\alpha_m$ is given by Equation 3.14. The values of $I_0$, $I_1$, $I_2$ and $I_{01}$, $I_{02}, I_{20}$ are found by using Equations 3.78, 3.72, and 3.76 with $\phi = \alpha + \beta$ and $\phi = \beta$. Each intensity is normalized as described in section 3.4.1, so possible errors in the normalization are negotiated. The resulting rms phase error is 1.1% of $2\pi$ radians (0.072 rad). The maximum phase error is about 2.4% of $2\pi$ radians (0.15 rad).

3.8 Phase errors due to reflections

In an interferometer set-up the surfaces of a test object as well as the surfaces of the optical elements give rise to spurious unwanted reflections. These reflections lead to multiple beam interference on the CCD-chips and, hence, to errors in the interferograms and eventually in the measured phase distribution. Therefore, reflections in the direction of the CCD-cameras must be prevented. However, it is impossible to reduce the
reflections to absolute zero.
In the real-time interferometer the reflections are reduced by using anti-reflection coated elements. Besides, most first order reflections propagate finally in the direction of the laser and, hence, they will not disturb the interferograms. Only first order reflections which appear between QWP0 and M0 and between QWP1 and M1 will strike the cameras, i.e., reflections at the back surfaces of QWP0 and QWP1 and reflections at the surfaces of a test object in the test section. The latter reflections will dominate because the surfaces of a test object are often not anti-reflection coated. However, especially these reflections can be filtered out rather easily. By positioning the object slightly tilted in the test section, the reflected rays propagate under an angle with the optical axis. If this angle is larger than the maximum angle of refraction of the rays in the test section, the reflections are filtered out after a suitable adjustment of the adjustable spatial filter. The most reflections in the direction of the cameras are of the second and higher order. It is impossible to filter them all out by the adjustable spatial filter. Fortunately, their intensities are low and the disturbance of the interferograms remain relatively small.

The existence of reflections in the real-time interferometer leads to errors in the measured phase due to two reasons. Firstly, the multiple beam interference disturbs the intensity distribution within the interferograms so that it is no longer exactly described by Equation 3.15. Secondly, the intensity distribution in the test beam which is used for the normalization of the cameras contains interference fringes. In order to estimate the phase errors due to reflections, the exact intensity distributions in presence of reflections must be derived previously.
Reflections appear in the test beam as well as in the reference beam. Hence, both beams must be represented as a sum of the effective beam, i.e., the original beam minus the reflections, and a number of coherent reflections. Interference of these two beams whereby the test beam is shifted in phase over \( \phi(x,y) \) with respect to the reference beam leads to an interferogram whose intensity distribution can be calculated straightforward. The total electric field in the interferogram is equal to the sum of the electric fields of the effective test beam, the effective reference beam and their phase shifted reflections. Averaging of the square of this field over time leads to the intensity distribution of the disturbed interferogram, which is given by:

\[
I(x,y) = A_0(x,y) + A_1(x,y) \cos \phi(x,y) + A_2(x,y) \sin \phi(x,y) \tag{3.79}
\]

with

\[
A_0(x,y) = I_t + I_r + 2 \sqrt{I_r} \sum_k R_k \cos \rho_k + 2 \sqrt{I_t} \sum_j T_j \cos \gamma_j + \sum_{j,d} T_j T_i \cos (\gamma_j - \gamma_i) + \sum_{k,m} R_k R_m \cos (\rho_k - \rho_m), \tag{3.80a}
\]
\[ A_1(x,y) = 2 \left( \sqrt{I_r} + \sqrt{I_r} \sum_k R_k \cos \rho_k + \sqrt{I_r} \sum_j T_j \cos \gamma_j + \sum_{j,k} T_j R_k \cos(\gamma_j - \rho_k) \right), \]  
\[ A_2(x,y) = 2 \left( \sqrt{I_r} \sum_k R_k \sin \rho_k - \sqrt{I_r} \sum_j T_j \sin \gamma_j - \sum_{j,k} T_j R_k \sin(\gamma_j - \rho_k) \right). \]  

Here, \( I_r \) and \( I_r \) describe the intensity distributions in the effective test beam and the effective reference beam. \( T_j \) describes the square root of the intensity of reflection number \( j (j=0,1,2,\ldots) \) in the test beam and \( \gamma_j \) describes the additional phase shift with respect to \( \phi(x,y) \). Likewise, \( R_k \) is the square root of the intensity of reflection number \( k (k=0,1,2,\ldots) \) in the reference beam while \( \rho_k \) describes its additional phase shift. A sum over \( j \) or \( l \) adds all the contributions due to all the reflections in the test beam and a sum over \( k \) or \( m \) adds all the contributions due to reflections in the reference beam. The parameters \( I_r, I_r, T_j, R_k, \gamma_j \) and \( \rho_k \) are all a function of \((x,y)\). The intensity distribution in the disturbed interferogram contains three terms, i.e., a bias intensity \( A_0(x,y) \), a modulation described by \( \cos \phi(x,y) \) with amplitude \( A_1(x,y) \) and a modulation described by \( \sin \phi(x,y) \) with amplitude \( A_2(x,y) \). It is important to notice that \( A_0(x,y), A_1(x,y) \) and \( A_2(x,y) \) are functions of \( T_j, R_k, \gamma_j \) and \( \rho_k \). Consequently, their values vary for different interferograms unless the reflections and their additional phase shifts are equal during the different registrations of the interferograms.

The intensity distribution in the test beam used for the normalization of the cameras is actually an interference pattern of the effective test beam with its coherent reflections. This normalization intensity distribution \( I_n(x,y) \) can be calculated straightforward and is given by:

\[ I_n(x,y) = I_r(x,y) + 2 \sqrt{I_r} \sum_j T_j \cos \gamma_j + \sum_{j,l} T_j T_l \cos(\gamma_j - \gamma_l) \]

\[ j,l=0,1,2,\ldots, \]

where \( j \) is summed over all the reflections. The intensity distribution is dependent on the intensity distributions \( T_j \) of the reflections and their relative phase shifts \( \gamma_j \). Hence, an errorless normalisation of the cameras is only possible if the reflections are equal for each camera.

In the real-time interferometer two types of reflections can be distinguished. The first type of reflections originate from positions in the set-up before the light passes non-polarizing beam splitter BSO. These reflections are characterized by the fact that their influence is equal for all the cameras, i.e., their intensity distributions and their additional phase shifts are equal on all cameras. The second type of reflections originate from positions somewhere between BSO and one of the CCD-cameras. The intensity distributions of these reflections, their additional phase shifts and also their number will differ on each camera.
The first type of reflections give only rise to phase errors due to the distortion of the interferograms. Normalization of cameras will in this case not lead to phase errors. Since the reflections towards the cameras contain equal intensity distributions and additional phase shifts, the normalization intensity distributions $I_0(x,y)$ are equal on each camera and, hence, a correct normalization is realized.

The measurement of a refractive index profile requires the registration of two sets of interferograms: one set resulting from the actual measurement and one set resulting from the reference measurement. Although the influence of the reflections is equal for each interferogram within one set, it may vary between different sets. This variation is introduced by the reflections in the test beam. In contrary to the reflections in the reference beam, the reflections in the test beam are influenced by the actual refractive index profile in the test section. Their additional phase shift, and to less extend also their intensity, is determined by the size of the refraction of the light in the test section and, in case of reflections passing through the test section, also by the actual refractive index profile.

Several authors studied the influence of the first type of reflections\textsuperscript{39,50,68,2}. In the analysis below their method is followed roughly. However, in contrary to them, this analysis is not based on the calculation of the phase error resulting from one single set of interferograms but on the calculation of the phase error resulting from two sets of interferograms, an actual measurement and a reference measurement. Besides, not one single reflection is assumed but multiple reflections are assumed in both the test beam and the reference beam.

Suppose the real-time interferometer has registered two sets of interferograms. The reflections in the reference beam are characterized by $R_k$ and $\rho_k$ ($k=0,1,2,\ldots$) for both the reference measurement and the actual measurement. The reflections in the test beam are characterized by $T_j$ and $\gamma_j$ ($j=0,1,2,\ldots$) during the reference measurements and they are characterized by $T'_j$ and $\gamma_j + \Delta \gamma_j$ during the actual measurement. Here, $\Delta \gamma_j$ represents the position dependent increase of the additional phase shift during the actual measurement with respect to the reference measurement. The intensity distributions each set of interferograms is similar to Equation 3.79, but the interferograms in one set are shifted in phase with respect to each other. For the interferogram $I(x,y)$ on CCD$i$ during the actual measurement follows:

\[
I(x,y) = A_0(x,y) + A_1(x,y) \cos \left( \phi(x,y) - (i-1) \frac{\pi}{2} \right) + A_2(x,y) \sin \left( \phi(x,y) - (i-1) \frac{\pi}{2} \right),
\]

(3.82)

where $\phi(x,y)$ is the phase difference between the test beam and the reference beam. Similar to Equations 3.80a,b,c $A_0(x,y)$, $A_1(x,y)$ and $A_2(x,y)$ are given by:
\[ A_0(x,y) = I_0 + I_r + 2\sqrt{I_r} \sum_k R_k \cos \rho_k + 2\sqrt{I_r} \sum_j T'_j \cos(\gamma_j + \Delta \gamma_j) \]
\[ + \sum_{j,k} T'_j T'_k \cos(\gamma_j + \Delta \gamma_j - \gamma_k - \Delta \gamma_k) + \sum_{k,m} R_k R_m \cos(\rho_k - \rho_m), \quad (3.83a) \]

\[ A_1(x,y) = 2 \left( \sqrt{I_r} \sum_k R_k \cos \rho_k + \sqrt{I_r} \sum_j T'_j \cos(\gamma_j + \Delta \gamma_j) + \sum_{j,k} T'_j R_k \cos(\gamma_j + \Delta \gamma_j - \rho_k) \right), \quad (3.83b) \]

\[ A_2(x,y) = 2 \left( \sqrt{I_r} \sum_k R_k \sin \rho_k - \sqrt{I_r} \sum_j T'_j \sin(\gamma_j + \Delta \gamma_j) - \sum_{j,k} T'_j R_k \sin(\gamma_j + \Delta \gamma_j - \rho_k) \right). \quad (3.83c) \]

Likewise, for the intensity distributions \(I_{0,i}\) of the interferogram on CCD\(i\) during the reference measurement may be written:

\[ I(x,y) = A_{0,0}(x,y) + A_{0,1}(x,y) \cos \left( \beta(x,y) - (i-1)\frac{\pi}{2} \right) + A_{0,2}(x,y) \sin \left( \beta(x,y) - (i-1)\frac{\pi}{2} \right), \quad (3.84) \]

where \(\beta(x,y)\) represents the phase difference between the test beam and the reference beam in absence of the refractive index profile of interest. The parameters \(A_{0,0}(x,y)\), \(A_{0,1}(x,y)\) and \(A_{0,2}(x,y)\) are given by:

\[ A_{0,0}(x,y) = I_0 + I_r + 2\sqrt{I_r} \sum_k R_k \cos \rho_k + 2\sqrt{I_r} \sum_j T_j \cos \gamma_j \]
\[ + \sum_{j,k} T_j T_k \cos(\gamma_j - \gamma_k) + \sum_{k,m} R_k R_m \cos(\rho_k - \rho_m), \quad (3.85a) \]

\[ A_1(x,y) = 2 \left( \sqrt{I_r} \sum_k R_k \cos \rho_k + \sqrt{I_r} \sum_j T_j \cos \gamma_j + \sum_{j,k} T_j R_k \cos(\gamma_j - \rho_k) \right), \quad (3.85b) \]

\[ A_2(x,y) = 2 \left( \sqrt{I_r} \sum_k R_k \sin \rho_k - \sqrt{I_r} \sum_j T_j \sin \gamma_j - \sum_{j,k} T_j R_k \sin(\gamma_j - \rho_k) \right). \quad (3.85c) \]

For the calculation of the phase of interest \(\alpha(x,y) = \phi(x,y) - \beta(x,y)\) the Equations 3.82 and 3.84 are substituted in Equation 3.14. The result, however, does not lead to a value equal \(\alpha(x,y)\), which would be the case if there was no distortion, but to a sum of \(\alpha(x,y)\) and an error \(\Delta \alpha(x,y)\). For this error follows:
\[ \Delta \alpha(x,y) = \tan^{-1} \left\{ \frac{A_1(x,y)A_{0,2}(x,y) - A_2(x,y)A_{0,1}(x,y)}{A_1(x,y)A_{0,1}(x,y) + A_2(x,y)A_{0,2}(x,y)} \right\} \]

\[ = \tan^{-1} \left\{ \frac{\sum_j \frac{T_j'}{\sqrt{I_t}} \sin(\gamma_j + \Delta \gamma_j) - \sum_i \frac{T_i}{\sqrt{I_t}} \sin \gamma_i + \sum_{ij} \frac{T_j'T_i}{I_t} \sin(\gamma_j + \Delta \gamma_j - \gamma_i)}{1 + \sum_j \frac{T_j'}{\sqrt{I_t}} \cos(\gamma_j + \Delta \gamma_j) + \sum_i \frac{T_i}{\sqrt{I_t}} \cos \gamma_i + \sum_{ij} \frac{T_j'T_i}{I_t} \cos(\gamma_j + \Delta \gamma_j - \gamma_i)} \right\} . \]

(3.86)

In the equation is assumed that for the effective reference beam and its reflections holds:

\[ I_r + 2\sqrt{I_r} \sum_k R_k \cos \rho_k + \left( \sum_k R_k \cos \rho_k \right)^2 + \left( \sum_k R_k \sin \rho_k \right)^2 \neq 0. \]  

(3.87)

This is a condition which is fulfilled automatically as long as the total reflected intensity in the reference beam remains small.

Equation 3.86 shows that the phase error is only dependent on the reflections in the test beam and that it is independent of the reflections in the reference beam. The error disappears as soon as the intensity distributions and the additional phase shifts of the reflections in the test beam during the actual measurement and the reference measurement are equal. In the special case that there is only one reflection during the actual measurement and no reflections during the reference measurement, Equation 3.86 reduces to the equations found in literature \(^{29,30,68}\).

The calculation of the rms phase error \( \Delta \alpha_{rms} \), necessary to characterize the interferometer, requires a further study of the phase error and its parameters \( \gamma_j \), \( \Delta \gamma_j \), \( T_j^{a2}/I_t \) and \( T_j^2/I_t \), with \( j = 0, 1, 2, \ldots \). The additional phase shift \( \gamma_j \) is a parameter which is highly dependent on the exact positions of the surfaces at which the light is reflected. The exact value is unknown. However, a small tilt of the surfaces makes \( \gamma_j \) be spatial varying over the interval \([0, 2\pi]\). Hence, for the calculation of \( \Delta \alpha_{rms} \) the parameter \( \gamma_j \) \((j = 0, 1, 2, \ldots)\) is averaged over this interval. The increase \( \Delta \gamma_j \) of the additional phase shift is dependent on the change in (the gradient of) the refractive index profile between the reference measurement and the actual measurement. A small change in (the gradient of) the refractive index profile leads to a small value of \( \Delta \gamma_j \), and, hence, to a small phase error, see Equation 3.86. Because of this dependency, \( \Delta \gamma_j \) \((j = 0, 1, 2, \ldots)\) are treated as independent parameters in the calculation of \( |\Delta \alpha|_{ave} \). In general, the intensity gradient in the test beam is not too high and the contribution of \( \Delta \gamma_j \) to the phase error dominates the contribution due to the change of the reflected intensity from \( T_j^{a2}/I_t \) to \( T_j^2/I_t \). Therefore, to \( T_j^{a2}/I_t \), may be assumed to be equal to \( T_j^2/I_t \) in the calculation of \( \Delta \alpha_{rms} \). The values of \( T_j^{a2}/I_t \) \((j = 0, 1, 2, \ldots)\) remain as independent parameters.

The rms phase error has been calculated for the cases that there is one reflection
normalization intensity is given by Equation 3.81. The interferograms recorded by the other CCD-cameras are undisturbed and given by Equation 3.7 and 3.11. The normalization intensities are equal to $I_t$. The disturbed value of $\alpha$ is found after substitution in Equation 3.14. The mean square phase error is found after averaging the square phase error over $\gamma_j$, $\rho_j$, $\alpha$ and $\beta$ ($j,k=0,1,2...$).
Figure 3.15a shows the rms phase error introduced by one single reflection in the test beam and one single reflection in the reference beam to one camera under the condition that $R_0 = T_0 = T_0'$ and $I_r = I_r'$. Presented is the rms phase error as a function of $\Delta \gamma_0$, the increase of the additional phase shift in the test beam during the actual measurement with respect to the reference measurement. Parameters are the camera number (0,1,2) and $T_0^2/I_r$ (0.003, 0.005, 0.010). The figure shows that in case of equal reflected intensities, the rms phase errors introduced by CCD0 and CCD1 are the same while CCD1 introduces a higher rms phase error. This last error is equal to the rms phase error introduced by CCD0 or CCD2 if the reflected intensity is doubled.

Figure 3.15b presents the rms phase error as a function of $\Delta \gamma_0$ if there are reflections on all three cameras with equal intensities and equal values $\Delta \gamma_0$. The additional phase shift $\gamma_0$ varies over the detection surface and per CCD-camera. The curves are calculated for $T_0^2/I_r$ equal to 0.003, 0.005 and 0.010 from the corresponding curves in Figure 3.15a under the assumption that the errors introduced by the separate cameras are statistical independent. This is, the curves belonging to CCD0, CCD1 and CCD2 are summed quadratically and then the square root has been calculated. The condition that $\Delta \gamma_0$ is equal for each camera can, in general, hardly be fulfilled and, hence, the curves are of low physical relevance. However, their maxima and their minima correspond to the upper and lower limit of the rms phase error.

The figure shows that in case of one single reflection in the test beam and in the reference beam to every camera, a rms phase error may be expected equal to 1.5-2.6% of $2\pi$ radians if $T_0^2/I_r = 0.003$, equal to 2.0-3.6% of $2\pi$ radians if $T_0^2/I_r = 0.005$ and equal to 2.8-4.8% of $2\pi$ radians if $T_0^2/I_r = 0.010$. Corresponding reflections of the first type lead to phase errors of 0-1.2% of $2\pi$ radians ($T_0^2/I_r = 0.003$), 0-1.6% of $2\pi$ radians ($T_0^2/I_r = 0.005$) and 0-2.3% of $2\pi$ radians ($T_0^2/I_r = 0.010$), see Figure 3.13. This means that the reflections of the second type introduce a rms phase error of a size of 1 to 2 times the maximum rms phase error introduced by a corresponding reflection of the first type.

### 3.9 Phase errors due to a wavefront aberration

Accurate measurements of refractive index profiles require a flat wavefront entering the test section. In practice, however, the flatness of the wavefront may be limited due to aberrations induced by optical elements (lenses) in the set-up. An aberrated wavefront at the entrance surface of the refractive index profile reduces the accuracy because the spatially displaced rays are not in phase with each other and, moreover, they traverse the profile under different angles. This makes the local phase in the interferograms not only dependent on the local refractive index, but also on the ray that is imaged. Since the refractive index gradient is not the same during the actual and the reference measurement, rays are imaged on different positions in the interferograms. Hence, the error distributions in the actual set of interferograms and reference set are not equal and the error will not cancel out in the calculation of the phase distribution. The remaining error depends on the
position of the optimum virtual object plane in case $\theta = 0$ and $n'$, i.e., $z_{f}'$ is given by
Equation 2.24, then follows for $F(\theta)$:

$$F(\theta) = \frac{2\pi}{\lambda} \Delta \Phi = \frac{2\pi}{\lambda} \left( \frac{2 n_e^2}{3} + \frac{5}{8} \frac{n_e}{n_g} t + \frac{3}{4} \frac{n_a}{a} \right) \theta^2 + O(\theta^4, \theta^3 n', \theta^2 n'^2, \theta n'^3, n'^4)$$

(3.95)

For $y_w$ follows under the same conditions:

$$y_w = y_p - \left( \frac{2 L}{3 n_e} + \frac{t}{n_g} + \frac{a}{2 n_e} \right) n'L - \left( \frac{4 n_a L}{3 n_e} + \frac{5 n_a t}{2 n_g} + \frac{3 a}{4} \right) \theta + O(\theta^3, \theta^2 n', \theta n'^2, n'^3).$$

(3.96)

Substitution of $F(\theta)$ and $y_w$ in Equation 3.95 gives:

$$\Delta \alpha(y_p) = \frac{2\pi}{\lambda} \left[ \left( \frac{4 n_a L}{3 n_e} + \frac{5 n_a t}{2 n_g} + \frac{3 a}{4} \right) W''(y_p) + 1 \right] \left[ \frac{2 L^2}{3 n_e} + \frac{5 L t}{2 n_g} + \frac{3 a L}{2 n_e} \right] W'(y_p) \Delta n',$n

(3.97)

where $W \geq 0$. Equation 3.93 and thus Equation 3.97 represent the maximum phase error as long as Equation 3.92 is satisfied. That is, by using Equation 3.96, as long as:

$$1 \gg \left( \frac{4 n_a L}{3 n_e} + \frac{5 n_a t}{2 n_g} + \frac{3 a}{4} \right) W''(y_w).$$

(3.98)

Under the assumption that $W''(y_w) \approx W''(y_p)$ a further simplification of Equation 3.97 is allowed. Since $dF(\theta)/d\theta = -dy_b/d\theta$, the term between brackets can be approximated by 1 and thus $\Delta \alpha(y_p)$ becomes:

$$\Delta \alpha(y_p) = \frac{2\pi}{\lambda} \left( \frac{2 L^2}{3 n_e} + \frac{5 L t}{2 n_g} + \frac{3 a L}{2 n_e} \right) W'(y_p) \Delta n'.$$

(3.99)

The phase error appears to be a linear function of $W'(y_p)$ and is fully determined by the ray shift $y_w - y_w, \beta$ as a consequence of the difference in $\Delta n'$ between the refractive index gradients during the actual and the reference measurement. The error is independent of a possible oblique trajectory as long as Equation 3.98 is satisfied. Under the special condition that $n'=0$ and thus $\Delta n' \approx n'$ follows by using equation 3.24:
\[ \Delta \alpha(y_p) = \left( \frac{1}{3} \frac{L}{n_e} + \frac{t}{2n_g} + \frac{1}{4} \frac{a}{n_a} \right) W'(y_p) G\alpha'(Gy_p), \]  

(3.100)

where \( \alpha'(Gy_p) \) is the phase gradient at the position of the chip-surface and \( G \) is the magnification of the imaging optics. The phase error thus increases linearly with the gradients in the wavefront aberration and the phase.

The configurations to be tested in the RTI are a compressible flow in a wind tunnel and a diffusion/mixing process of a dilute sucrose solution in an aqueous bulk liquid in a cuvette. For the details is referred to Section 2.4. In both cases the object is positioned in the test section against the mirror \( (a=0) \) and the magnification \( G \) is 0.6. Then, the maximum phase error is given by \( 2040W'(y_p) \) (wind tunnel configuration) and \( 105W'(y_p) \) (cuvette configuration), i.e., the phase error if \( \alpha' \) has its maximum value of \( 60 \cdot 10^3 \text{rad/m} \). This means that as long as \( W'(y_p) \leq 1.7 \cdot 10^{-4} \) \( (\theta \leq 0.01^\circ) \) the maximum error is less than 5.7% of \( 2\pi \) radians for the wind tunnel configuration, but less than 0.3% of \( 2\pi \) radians for the cuvette configuration.

In general, a wavefront aberration is a combination of the primary Seidel aberrations\(^{69,70,71,29} \). The induced rms phase error depends on the contribution of the different the Seidel aberrations and varies possibly after each alignment of the beam expanding optics. In this thesis, the rms phase error is assumed to be 0.66 times the maximum phase error if the medium in the test section is linearly stratified and 0.86 times the maximum phase error if the medium is locally linearly stratified. In the Appendix is shown that this is a reasonable assumption for the aberration detected in the test beam of the RTI.

### 3.10 Phase errors in the TPSI

A temporal phase shifting interferometer is built in the set-up of the real time interferometer. The TPSI records the phase shifted interferograms in sequence in time by using CCD1, while the piezo-electric transducer mounted on mirror M1 introduces the phase shift. It is used in Chapter 4 to verify the RTI experimentally. In these verifications phase distributions obtained by using RTI are compared to corresponding phase distributions obtained by using the TPSI. To realize a convenient interpretation of the experimental results, the theoretical accuracies of both interferometers must be known. Most error sources in the TPSI are similar to the error sources in the RTI. Since the TPSI uses the same optical set-up, corresponding error sources may even lead to exactly the same phase errors. Below, the phase errors due to the different error sources are discussed briefly.
in case of Equation 3.72, are real and equal for each interferogram. Hence, the interferograms recorded by the TPSI may be described by the Equations 3.101 and 3.102 in the special case that $A_0 = A_{0,0}$, $A_1 = A_{0,1}$ and $A_2 = A_{0,2}$. Equation 3.31 shows that the phase error is equal to zero.

Contrary to the polarizing elements, the imaging elements can introduce a phase error. However, since the TPSI and the RTI make use of the same lenses, this phase error may be expected to be equal for both interferometers.

An important error source is the existence of reflections in the optical set-up. Unlike the RTI, only one type of reflections appear in the TPSI. Since the TPSI records all the interferograms in the same channel of the optical set-up, the influence of the reflections is equal for each interferogram in the same set, provided that the tilt of mirror M1 which may appear during its displacement is small enough to keep the reflections in the reference beam unchanged. Hence, all reflections are of the first type as described in Section 3.8. The intensity distributions in the interferograms belonging to the measurement set and the reference set are now described by Equation 3.101 and 3.102. The coefficients $A_0, A_1, A_2$ and $A_{00}, A_{01}, A_{02}$ are exactly equal to the corresponding coefficients in the interferograms obtained by the RTI in presence of reflections of the first type and they are given by Equations 3.83abc and 3.85abc. The resulting phase error is given by Equation 3.86. This means that a reflection of the first type in the set-up leads to the same phase error in the TPSI and the RTI. A reflection of the second type leads to different phase errors.

An additional error source in the TPSI is a variation in the phase step, for example, introduced by a nonlinear response of the piezo-electric transducer. Although the algorithm of Carré does not need a predefined phase step, a variation in the phase step may lead to a serious phase error. In this case the phase shifts between succeeding interferograms in each set are unequal.

Let's assume that the phase step between interferogram numbers 0 and 1, between interferogram numbers 1 and 2, and between interferogram numbers 2 and 3 are denoted by $\gamma_{0,1}$, $\gamma_{1,2}$ and $\gamma_{2,3}$. If the phase step $\gamma_{1,2}$ is used as reference, then the phase steps $\gamma_{0,1}$ and $\gamma_{2,3}$ can be written as:

$$
\gamma_{0,1} = \gamma_{1,2} + \Delta \gamma_{0,1},
$$

(3.103a)

$$
\gamma_{2,3} = \gamma_{1,2} + \Delta \gamma_{2,3}.
$$

(3.103b)

Figure 3.18 shows the phase error as a function of the phase $\phi$ in case that the phase is calculated from one set of interferograms. Presented are the errors if $\gamma_{0,1}$ and $\gamma_{2,3}$ deviates from $\gamma_{1,2} = 1.9$ rad. The error appears to be periodic at twice the fringe frequency with discontinuities around $\phi = 0$ rad and $\phi = \pi$ rad. The error oscillates around zero or it (almost) hits zero, its mean value is generally unequal to zero. An exchange of $\gamma_{0,1}$ and $\gamma_{2,3}$ leads to a sign change of the error and a reflection about $\phi = \pi$.

Figures 3.19a, b, c represent the mean phase error, the rms phase error and the maximum
absolute phase error as a function of $\Delta\gamma_{0,1}$ for various functional relations between $\Delta\gamma_{0,1}$ and $\Delta\gamma_{2,3}$ if $\gamma_{1,2}=1.9$ rad. In the calculations the discontinuities are removed and replaced by a linear interpolation in an area $\pm 0.1$ around $\phi=0$ rad and $\phi=\pi$ rad. Figure 3.19a shows that the mean phase error is approximately proportional to $\Delta\gamma_{2,3} \cdot \Delta\gamma_{0,1}$. The Figures 3.19b,c show that both the rms phase error and the maximum absolute phase error are smaller if $\Delta\gamma_{0,1}$ and $\Delta\gamma_{2,3}$ have the same sign than if they have an opposite sign. If $\Delta\gamma_{2,3}=0$ while $\Delta\gamma_{0,1} \neq 0$, the phase error is smaller than if $\Delta\gamma_{0,1} = -\Delta\gamma_{2,3}$ but it is larger than if $\Delta\gamma_{0,1} = \Delta\gamma_{2,3}$. An exchange of the values of $\Delta\gamma_{0,1}$ and $\Delta\gamma_{2,3}$ leads to the same results for the rms phase error and the maximum absolute phase error. A deviation of maximal 0.1 radians in $\gamma_{1,2}$ leads to a maximum deviation of 0.03% of $2\pi$ radians in the rms phase error and a maximum deviation of 0.06% of $2\pi$ radians in the absolute maximum phase error.

The periodic error generated by the variation in the phase step can be reduced by using a two-camera TPSI. In this two-camera TPSI, the phase distribution is extracted independently from the sets of interferograms recorded by CCD1 and CCD2. The eventual phase distribution is the average of these two phase distributions. Then, the phase step of $\pi/2$ radians between the two sets of interferograms results in an optimum reduction of the periodic error. However, a spatially constant error and some discontinuities remain. If the phase distributions are extracted from an actual set of interferograms and a reference set of interferograms, also this spatially constant error disappears, provided that the variation in the phase step is equal during the actual and the reference measurement. In this thesis, a two-camera TPSI is not used. The experimental set-up exhibits some instability so that the variation in the phase step is not the same during the actual measurement and the reference measurement. The result would be a spatially constant error in the phase distribution, but the information about its size is lost. In a one-camera
Figure 3.19a,b,c  The mean phase error (a), the rms phase error (b) and the maximum absolute phase error (c) introduced by the algorithm of Carré as a function of the difference $\Delta \gamma_{0,1}$ between the first and second phase step if the second phase step $\gamma_{1,2}$ is equal to 1.9 radians. The errors are presented various functional relations between $\Delta \gamma_{0,1}$ and the difference $\Delta \gamma_{2,3}$ between the second and third phase step.

TPSI, the error oscillates around the correct phase. Then, a possible error can be approximated by using this oscillation, provided that the shape of the phase distribution is known beforehand.
3.11 Discussion and conclusions

A three channel real time Michelson interferometer has been designed which is convenient
to perform quantitative 2D refractive index profile measurements. Its measuring accuracy
is limited by several error sources. They are discussed separately in the preceding
sections. Of importance is, of course, the total measuring accuracy of the RTI and,
especially, the total accuracy compared to the total accuracy of a more commonly used
TPSI.

It is difficult to calculate the total accuracy of a real time interferometer a priori. Since
the accuracy depends on the presence of vibrations and air turbulence, and higher order
derivatives of the refractive index profile, it is more or less determined by the
environmental circumstances at the moment of the measurement. Also the extent to which
reflections are filtered out cannot be estimated beforehand. Therefore, an optimum total
accuracy will be calculated here, which appears if vibrations, air turbulence and
reflections are neglected. Also errors introduced by the imaging optics, by misimaging of
the virtual object plane and an aberrated test beam are not taken into consideration. These
last errors are not specific errors generated by the RTI, but they are also generated by the
TPSI.

The optimum rms phase error is constituted by the phase errors quantization and noise,
nonlinear camera response, misalignment of the cameras, and limited perfection and
alignment accuracy of the polarizing elements. Apart from quantization and noise, the
various error sources lead to statistically dependent phase errors, i.e., the averaged cross
products of the errors are not necessarily equal to zero. Hence, (the upper bound of) the
optimum rms phase error is given by:

$$
\Delta \alpha_{rms} = \sqrt{\Delta \alpha_{rms}[q+n] + (\Delta \alpha_{rms}[cam.\,respo.])^2 + (\Delta \alpha_{rms}[cam.\,al.])^2 + (\Delta \alpha_{rms}[pol.\,comp.])^2}.
$$

(3.104)

Figure 3.20 shows the optimum rms phase error as a function of the phase gradient for
various values of \(c\). Presented are the rms phase error in case of a linearly stratified
medium (vertical fringes, horizontal fringes, fringes under 45\(^\circ\)) and a maximum rms phase
error which can serve as an upper bound in case of a locally linearly stratified medium.
The errors are computed by using the results found in Section 3.5.2, Section 3.5.3 and
Section 3.6 (Figure 3.11). This implies that uniform probability distributions of the phases
\(\alpha\) and \(\beta\) over \([0,2\pi]\) are assumed, while the gradient of \(\beta\) is low (approximately 0). An
overview of the accepted properties of the polarizing elements is presented in Table 3.2,
the accepted noise, camera non-linearity and alignment accuracy is given in Table 3.3.
The error induced by the limited alignment accuracy of the cameras makes the optimum
rms phase error depend on \(c\) and the fringe orientation.

The figure shows that the gradient independent part of the optimum rms phase error is
3. Design of the real time interferometer and accuracy analysis

- noise: ±2 levels
- nonlinear camera response: 5%
- misalignment cameras
  - translation along x,y-axis: 0.5μm
  - translation along z-axis: 200μm
  - rotation about x,y-axis: 0.5°
  - rotation about z-axis: 0.02°

Table 3.3 Accepted noise, camera non-linearity and camera misalignment.

![Number of fringes over 8.2mm length](image)

**Figure 3.20** The optimum rms phase error as a function of the phase gradient $\alpha'$ and the number of fringes over 8.8mm length of the CCD-chip.

About 1.5% of $2\pi$ radians. This part is dominated by the contributions of the polarizing elements (rms 1.1% of $2\pi$ radians) and the noise (rms 0.6% of $2\pi$ radians). The gradient dependent part depends on the fringe orientation and varies in case of a linearly stratified medium between 0 and 1.0% of $2\pi$ radians if $c=3$, between 0 and 1.2% of $2\pi$ radians if $c=5$, and between 0 and 1.4% of $2\pi$ radians if $c=7$. In case of a locally linearly stratified medium this is between 0 and 1.6% of $2\pi$ radians if $c=3$, between 0 and 2.0% of $2\pi$ radians if $c=5$, and between 0 and 2.3% of $2\pi$ radians if $c=7$. 
The results in Figure 3.20 are based on the assumptions that the phase distribution \( \alpha \) in the actual measurement and the phase distribution \( \beta \) in the reference measurement have uniform probability distributions over \([0, 2\pi]\) and that the gradient of \( \beta \) is approximately 0. However, these assumptions will not always be satisfied practically and thus the optimum rms phase error can deviate from the curves presented in the figure. Three extreme situations can be distinguished.

In the first situation \( \alpha \) and \( \beta \) are both almost constant over the chip-surface. Now, the probability distributions are not uniform while their gradients are approximately zero. As a consequence, the rms phase error due to quantization and noise and due to a non-linear camera response increase or decrease and the phase error due to the misalignment of the cameras reduces to almost zero. Effectively, this means that the optimum rms phase error can increase with up to about 2.9\% of \( 2\pi \) radians. However, since the phase is almost constant, the spatial variation in the error is limited. As soon as the number of almost uniformly spread fringes exceeds 4 in \( \alpha \) and \( \beta \), the deviation from the curves in Figure 3.20 is reduced to less than about 0.2\% of \( 2\pi \) radians.

In the second situation only \( \beta \) is constant over the surface. Now only the probability distribution in \( \beta \) is non-uniform, what leads to a change in the rms phase error due to quantization and noise and due to a non-linear camera response. The maximum increase of the optimum rms phase error is less than in the first situation because there is averaged over \( \alpha \): 0.7\% of \( 2\pi \) radians.

In the third situation the gradient of \( \beta \) is not approximately 0. By this, a phase error due to a misalignment of the cameras is introduced which was negligible before. The rms error can now be found by applying Figure 3.20 to both the actual and the reference measurement: an upper bound of the rms phase error is equal to the sum of the total rms phase error during the actual measurement and the gradient dependent part of the rms phase error during the reference measurement. If the gradient in \( \beta \) goes up to 60-10^3 rad/m, the maximum increase of the optimum rms phase error is 2.3\% of \( 2\pi \) radians.

The curves in Figure 3.20 must be interpreted as a global approximation because some parameters are more or less estimated in the calculations. Moreover, the phase error due to the polarizing elements has been calculated for one particular situation and with constant imperfections in the retardation over the surface of the wave plate. This will not occur in an actual situation.

The optimum rms phase error in the one-camera TPSI is independent of the phase gradient and is given by:

\[
\Delta \alpha_{\text{rms}} = \sqrt{\Delta \alpha_{\text{rms}}^2[q+n] + (\langle \Delta \phi[s] \rangle - \langle \Delta \beta[s] \rangle)^2 + (\Delta \alpha_{\text{res}}[\text{cam.resp}])^2 + \Delta \phi_{\text{rms}}[s] - \langle \Delta \phi[s] \rangle^2 + \Delta \beta_{\text{rms}}[s] - \langle \Delta \beta[s] \rangle^2}.
\]

(3.105)

where \( \Delta \phi[s] \) and \( \Delta \beta[s] \) denote the error due to a variation in the phase step during the
actual measurement and the reference measurement, respectively. If the variation in the phase step is negligible, the optimum rms phase error is determined by the quantization and noise, and by the non-linear camera response. Under the conditions that the gradient in $\beta$ is small and $\beta$ is homogeneously distributed over $[0,2\pi)$, there follows from the results in Section 3.10 that the optimum rms phase error is 0.5% of $2\pi$ radians. This implies that in case of a linearly stratified medium, the rms error induced by the RTI is maximal about 1.0%-2.0 of $2\pi$ radians ($c=3$), 1.0%-2.2% of $2\pi$ radians ($c=5$) or 1.0%-2.4% of $2\pi$ radians ($c=7$) higher than the optimum rms phase error induced by the TPSI, dependent on the gradient in $\alpha$. In case of a locally linearly stratified medium this is 1.0-2.6% of $2\pi$ radians ($c=3$), 1.0%-3.0% of $2\pi$ radians ($c=5$) or 1.0%-3.3% of $2\pi$ radians ($c=7$). These differences are mainly caused by the imperfections and misalignment of the polarizing elements and by the misalignment of the cameras.

The total rms phase error of the RTI and the TPSI is also influenced by reflections, vibrations and air turbulence, misimaging and Seidel aberrations induced by the imaging optics. These contributions possibly have the same extent as the optimum rms phase error and, hence, they can influence the measuring accuracy substantially. Although most first order reflections are filtered out, a remaining reflection of 0.3% in the test and reference beam can lead to a rms phase error up to 1.2% of $2\pi$ radians in the TPSI and to a rms phase error up to 2.6% of $2\pi$ radians in the RTI. Dependent on the position of the reflection in the set-up, the resulting phase error is equal in the RTI and TPSI or different. The influence of vibrations and air turbulence must be determined experimentally, but the distortion of the phase obtained by the RTI may expected to be smaller and more constant over the whole distribution. Misimaging leads to exactly the same phase errors in both interferometers. In case of a linearly stratified refractive index profile resulting in a maximum phase gradient of $60\times10^{3}$rad/m at the position of the cameras, a misimaging of about 1mm leads to a rms phase error of maximal 1.0% of $2\pi$ radians. A wavefront aberration in the test beam leads also to an equal error in the RTI and the TPSI. This error becomes serious if the propagation length of the beam trough the medium is large and the refractive index is low.

Non-linearly stratified refractive index profiles lead to larger phase errors than presented above. The induced additional phase error is dependent on the exact shape of the profile. To avoid the influences of the shape, a linearly stratified refractive index profile is more convenient for testing interferometric set-ups. Practically, the linearly stratified refractive index profile can be replaced by tilted mirror at the position of the virtual object plane. Then, the effective numerical aperture of the optics as a function of the phase gradient is approximately equal and, hence, the phase error is comparable. Only errors introduced by misimaging and aberrations in the test beam are absent.

In summary, the theoretical analyses in this chapter lead to the following conclusions:
- A three channel real time Michelson interferometer has been designed for quantitative
refractive index profile measurements. The time resolution is video rate and the integration time per measurement is $10^4$ seconds. The interferometer can deal with the typical interferograms obtained in experimental flow research.

- Under the assumptions that the probability distributions of $\alpha$ and $\beta$ are uniform over $[0,2\pi)$, the gradient in $\beta$ is small and the gradient in $\alpha$ is less than 60 $10^3$ rad/m at the position of the cameras, the optimum rms phase error lies approximately between 1.5% and 2.9% of $2\pi$ radians if the medium is linearly stratified and between 1.5% and 3.8% of $2\pi$ radians if the medium is locally linearly stratified. This is about 1%-2.4% of $2\pi$ radians and 1%-3.3% of $2\pi$ radians worse than the optimum rms phase error in the TPSI. If the gradient in $\beta$ is not small, the rms phase error is 1.5%-4.3% and 1.5%-6.1% of $2\pi$ radians. If $\beta$ is constant over the detection surface, then the phase error is about 0.8%-3.6% of $2\pi$ radians and 0.8%-4.5% of $2\pi$ radians.

- The phase error due to reflections, aberrations, vibrations and air turbulence must be determined experimentally. They can reduce the measuring accuracy considerably.

- Misimaging of the (virtual) object plane leads to an equal phase error in the RTI and the TPSI. Misimaging of the virtual image plane over 1mm results in a maximum error of 1.0% of $2\pi$ radians.

- Especially wind tunnel measurements require a high degree of flatness of the wavefront in the test section. In the RTI, a gradient in the wavefront of $1.7\cdot10^4$ leads to a maximum phase error of 5.7% of $2\pi$ radians. Measurements to a cuvette configuration are less sensitive to the wavefront aberration: the maximum error is 0.3% of $2\pi$ radians.

- If all possible errors are taken into account, the total rms phase error generated by the RTI in case of a linearly stratified refractive index profile is a few percent of $2\pi$ radians. A rms phase error of less than 10% of $2\pi$ radians can certainly be realized.

- The difference in total accuracy between the RTI and the TPSI will be of the size of a few percent of $2\pi$ radians.

References


3. Design of the real time interferometer and accuracy analysis


3. Design of the real time interferometer and accuracy analysis


4

Experimental validation

4.1 Introduction

The reliability of the RTI depends on its measuring accuracy and its stability. The measuring accuracy determines the phase error within an interferogram, the stability determines the reproducibility of an interferogram and the relative phase error between succeeding recorded interferograms.

The theoretical accuracy of the RTI is discussed extensively in Chapter 3. Misalignment and limited perfection of the optical elements, quantization and noise, misalignment of the cameras, non-linearity in the camera response, the existence of reflections, the presence of turbulence in the surrounding air and misimaging are all error sources which limit the measuring accuracy. Also the fact that the interferograms are recorded in different channels by different cameras contributes to this limitation. An optimum rms phase error can be calculated beforehand, see Section 3.11. However, several error sources are not taken into account and, hence, the optimum rms phase error must be interpreted as being a lower bound. The actual value of the rms phase error will generally be higher and must be determined experimentally.

Instabilities are caused by (low frequency) vibrations of the elements in the set-up, turbulence of the surrounding air and also by laser instabilities if the optical path lengths of the test beam and the reference beam are not sufficiently balanced. They introduce phase variations which appear to have a large constant component over the whole phase
distribution and a smaller spatially varying component. This means that the measured phase profile is hardly disturbed, but the relative phase with respect to previously measured phase distributions actually is. A correction for the constant component is, however, possible. The constant component and the spatially varying component of the variation between different phase distributions have to be measured experimentally.

This chapter describes the validation of the RTI, i.e., the experimental determination of its measuring accuracy and its stability. For this, some 'tilted mirror' experiments are performed. In these 'tilted mirror' experiments a two-dimensional phase distribution with a constant gradient over the whole surface is generated by the presence of a tilted mirror. This mirror is positioned in the virtual object plane so that the reflected rays propagate along (almost) the same trajectories through the imaging optics as the refracted rays would do in case of the existence of a linearly stratified medium. The angles at which the rays are leaving the test section of the interferometer are equal to the first order to those in case of a linearly stratified medium for all phase gradients (Equations 2.35 and 3.24 lead to $\theta_a = \lambda G \gamma / 2 \pi n_a$). Hence, the effective numerical aperture of the imaging optics is equal, just like the angle of incidence of the rays on the CCD-chips (Equation 3.49). This implies that the error in the measured phase distributions may be expected to be similar. However, the 'tilted mirror' experiments are, unlike refractive index profile measurements, insensitive to a limited imaging accuracy and wavefront aberrations developed before the test beam enters the test section. Misimaging in the 'tilted mirror' experiments leads to a linear shift in the phase gradient which will be interpreted as being an additional mirror tilt. The effects of an aberrated wavefront are eliminated by the fact that the mirror is positioned in the virtual object plane, resulting in equal errors during the actual measurement and the reference measurement which cancel in the phase calculation. This implies that in order to find the total phase error, these two phase errors must be supplemented to the phase error found in the 'tilted mirror' experiments.

As described in Section 3.4, in the 'tilted mirror' experiments a reference measurement is followed by one or more actual measurements. During the reference measurements the mirror is perpendicular to the test beam or is tilted slightly, depending on the desired reference phase profile. During the actual measurements the mirror is tilted more and in different directions. In the case that the mirror is perpendicular to the test beam, the interferograms show about one circularly shaped fringe. This is caused by a small aberration of the wavefront in the test beam with respect to the wavefront in the reference beam, possibly induced by an unequal collimation of the two beams. It implies that, as soon as the mirror is tilted, the fringes show some deviation from straight lines. The phase distribution, however, remains almost undisturbed because the deviation appears in the interferograms of both the reference measurement and the actual measurements. During the validation vertical, sloping and horizontal fringes will be distinguished whereby the small deviation from straight lines will be ignored.

The phase distributions are often measured by using both the RTI and the TPSI. All the interferograms are processed according to the description in the Sections 3.4.1 and 3.4.2,
and they are unwrapped as described in Section 3.4.3. Unlike the image processing during the utilization of the RTI (Chapter 5), pixels with a low modulation depth are not circumvented. Since the images do not contain any pixels which are lying in the shadow of the object, the threshold level as described in Section 3.4.1 cannot be determined. This implies that, apart from the pixels with a reasonable modulation depth, also some pixels with a low modulation depth and thus a high degree of inaccuracy are processed. Hence, the phase errors found in this chapter must be interpreted as being an upper bound.

Section 4.2 presents the experimental validation of the measuring accuracy, Section 4.3 the experimental validation of the stability. The total phase error in two-dimensional refractive index profile measurements, i.e., the experimentally found phase error enlarged with the phase errors due to a limited imaging accuracy and a wavefront aberration in the test beam, is discussed in Section 4.4. This is followed by the conclusions in Section 4.5.

### 4.2 Validation of the measuring accuracy

The accuracy of the RTI is fully characterized by the induced rms phase error and maximum absolute phase error. Since these errors are functions of the gradient in the phase distribution $\beta$ during the reference measurement and the gradient in the phase distribution $\phi$ during the actual measurement, the accuracy of the RTI must be verified for suitable values of these gradients. In the utilization experiments the gradient in $\beta$ may expected to be small but unequal to zero because usually there is a small refractive index gradient in the windows bounding the refractive index profile of interest. The gradient in $\phi$ will vary over the whole design range of the RTI, i.e., about 0 to $60\cdot10^5$ rad/m (≈ 78 fringes over the total chip length of 8.2mm). Therefore, to characterize the measuring accuracy of the RTI similar phase gradients will be tested in the 'tilted mirror' experiments.

To characterize the RTI, some 'tilted mirror' experiments are performed whereby the phase distributions are measured by the RTI and the TPSI. Each experiment is composed of a reference measurement followed by several actual measurements. During the reference measurement there is a small vertical phase gradient of about 5100 rad/m or 7700 rad/m. During the different actual measurements the phase gradient varies in size and direction. Each measurement includes the temporal recordings of four phase shifted interferograms by all three cameras in the set-up. The phase step between the interferograms is generated by the piezo-electric transducer and is about 1.9 radians. It is verified as described in Section 3.4.2. The TPSI extracts the phase distribution from the four interferograms recorded by CCD1 and the RTI extracts the phase distribution from every second recording by CCD0, CCD1 and CCD2. As will be shown in Section 4.3, the experimental set-up exhibits a slight instability that results in a slight variation in
the different phase step in the TPSI, within a measurement and between the actual measurement and the reference measurement. The rms phase error remains, however, less than 1.8% of \(2\pi\) radians. The correction of the phase distribution obtained by the TPSI with respect to that obtained by the RTI (Equation 3.22) is based on the mean phase step between the secondly and thirdly recorded interferogram per measurement \(I_1\) and \(I_2\).

The accuracy analysis is presented in five sections. In Section 4.2.1 the theory is described. After that, the experimental results are presented. Section 4.2.2 describes the relative accuracy of the RTI with respect to the TPSI, Section 4.2.3 the total measuring accuracy of the RTI. The effects of filtering the unwrapped phase distributions with a linear filter are studied in Section 4.2.4. Finally, all the results are discussed in Section 4.2.5.

### 4.2.1 Theory

A good method for a quantitative comparison between the phase distributions obtained by the RTI and the TPSI appears to be the calculation of the rms differences between them. As will be shown below, the result is a value for the rms phase error induced by the RTI, acquired with an accuracy equal to the rms phase error induced by the TPSI. In the special case of the 'tilted mirror' experiments, however, the accuracy of the calculation can be increased because the shape of the phase distribution is known a priori.

Let the phase error introduced by the RTI be described by \(\Delta\alpha_{\text{RTI}}\) and the phase error introduced by the TPSI be described by \(\Delta\alpha_{\text{TPSI}}\). They are both functions of the positions \((x,y)\) in the phase distributions, but this dependence will be omitted for notational reasons. The rms phase difference \(\delta_{rms}\) between the two phase distributions is then given by:

\[
\delta_{rms} = \sqrt{\langle (\Delta\alpha_{\text{RTI}} - \Delta\alpha_{\text{TPSI}})^2 \rangle} = \sqrt{\langle \Delta\alpha_{\text{RTI}}^2 \rangle + \langle \Delta\alpha_{\text{TPSI}}^2 \rangle - 2 \langle \Delta\alpha_{\text{RTI}} \Delta\alpha_{\text{TPSI}} \rangle} \tag{4.1}
\]

Here the average takes place over all the pixels in the phase distributions. The term to be calculated is the mean square (ms) phase error introduced by the RTI \(\langle \Delta\alpha_{\text{RTI}}^2 \rangle\), the terms assumed to be known are \(\delta_{rms}\) (is calculated from the phase distributions) and \(\langle \Delta\alpha_{\text{TPSI}}^2 \rangle\) (ms phase error introduced by the TPSI). The term \(\langle \Delta\alpha_{\text{RTI}} \Delta\alpha_{\text{TPSI}} \rangle\) is unknown because the phase errors introduced by the RTI and the TPSI are not statistically independent. An upper bound of the absolute value of this term, however, can be derived, i.e.,

\[
| \langle \Delta\alpha_{\text{RTI}} \Delta\alpha_{\text{TPSI}} \rangle | \leq \sqrt{\langle \Delta\alpha_{\text{RTI}}^2 \rangle \langle \Delta\alpha_{\text{TPSI}}^2 \rangle} \tag{4.2}
\]

Substitution of Equation 4.2 in Equation 4.1 in case \(\langle \Delta\alpha_{\text{RTI}} \Delta\alpha_{\text{TPSI}} \rangle \geq 0\) leads to:

\[
\delta_{rms} \geq \sqrt{\langle \Delta\alpha_{\text{RTI}}^2 \rangle} - \sqrt{\langle \Delta\alpha_{\text{TPSI}}^2 \rangle} \tag{4.3a}
\]
as long as the term on the right and side remains positive, i.e., the rms phase error induced by the RTI is larger than that induced by the TPSI. Substitution of Equation 4.2 in Equation 4.1 in case \( \langle \Delta \alpha_{RTI} \Delta \alpha_{TPSI} \rangle \leq 0 \) gives:

\[
\delta_{rms} \leq \sqrt{\langle \Delta \alpha_{RTI}^2 \rangle} + \sqrt{\langle \Delta \alpha_{TPSI}^2 \rangle} \quad (4.3b)
\]

Calculation of the rms phase error of the RTI, \( \Delta \alpha_{rms,RTI} \), from Equations 4.3a and 4.3b gives:

\[
\delta_{rms} - \Delta \alpha_{rms,TPSI} \leq \Delta \alpha_{rms,RTI} \leq \delta_{rms} + \Delta \alpha_{rms,TPSI} \quad (4.4)
\]

where \( \Delta \alpha_{rms,TPSI} \) describes the rms phase error introduced by the TPSI. Equation 4.4 shows that the rms phase error can be determined with an error of \( \pm \Delta \alpha_{rms,TPSI} \). The extreme values of \( \Delta \alpha_{rms,RTI} \) appear if

\[
\Delta \alpha_{rms,RTI}(x,y) = c \Delta \alpha_{rms,TPSI}(x,y) \quad (4.5)
\]

with \( c \) a constant between -1 and 1. If \( c \) is positive, i.e., \( \Delta \alpha_{RTI}(x,y) \) increases linearly with \( \Delta \alpha_{TPS}(x,y) \) over the surface, the rms phase error is maximum and equal to \( \delta_{rms} + \Delta \alpha_{rms,TPSI} \). If \( c \) is negative, i.e., \( \Delta \alpha_{RTI}(x,y) \) increases linearly with \( \Delta \alpha_{TPS}(x,y) \) over the surface but with the opposite sign, the rms phase error is minimum and equal to \( \delta_{rms} - \Delta \alpha_{rms,TPSI} \). In general, these extreme conditions do not appear and the phase error lies somewhere between the two extremes. The value of \( \delta_{rms} \) can be interpreted as the maximum difference between the rms phase error induced by the RTI and the TPSI.

The error sources are divided into four groups. Group 1 contains the error sources which lead to different errors in the RTI and the TPSI. These error sources are quantization and noise, nonlinear response of the CCD-cameras, vibration and air turbulence and reflections of the second type. Group 2 contains the error sources which lead to equal errors in the RTI and the TPSI. In the 'tilted mirror' experiments these sources are the deformation due to aberrations induced by the imaging optics and reflections of the first type (see Section 3.8). Group 3 is the group of error sources that introduce errors in the RTI only. Practically, these are the errors introduced by the imperfections and the misalignment of the polarizing elements, and misalignment of the CCD-cameras with respect to each other. Finally, group 4 is the group of error sources that introduces only a phase error in the TPSI, i.e., the error induced by a variation in the phase step, for example, as a result of a non-linear response of the piezo-electric transducer.

The existence of the different error groups has consequences for the Equations 4.4 and 4.5. The existence of Group 2 implies that Equation 4.5 cannot be fulfilled for \( c \) negative and, hence, the lower bound of \( \Delta \alpha_{rms,RTI} \) given in Equation 4.4 is underrated. The existence of the Groups 1,3 and 4 implies that Equation 4.5 cannot be fulfilled either for \( c \) positive so that the upper bound of \( \Delta \alpha_{rms,RTI} \) given in Equation 4.4 is overrated. In case of the 'tilted mirror' experiments, however, a better approximation for both the lower bound and the upper bound, can be found because there can be made use of the a priori
knowledge of the shape of the unwrapped phase distributions: a linear surface.
To obtain the lower bound, flat surfaces are fitted through the unwrapped phase
distributions generated by the RTI by applying the least squares method\(^1\). Although the
fitted surfaces may deviate from the true surfaces, they are oriented by definition such that
an absolute minimum rms phase difference between the measured phase distribution and
the fitted flat surface is obtained. This rms phase difference \(\delta_{\text{rms}}\) is the new approximation
of the lower bound of \(\Delta\alpha_{\text{rms,RTI}}\). If the fitted phase surface and the true phase surface do
not coincide, the true rms phase difference between the true phase surface and the
measured phase distribution is always larger than this value.
To make a reasonable approximation of the upper bound of the rms phase error induced
by the RTI, the true phase distribution must be determined with a well approximated
accuracy. This true phase distribution must be extracted from the phase distribution
generated by the TPSI since the phase error introduced by the TPSI is smaller than that
introduced by the RTI. Taking into account a possible deformation due to the aberrations
induced by the imaging elements, a flat surface must be fitted through that part of the
phase distribution where the deformation is minimum, i.e., a relatively small area around
the optical axis. Because the deformation is minimum here, the gradient of the fitted
surface corresponds as good as possible to the gradient in the true phase distribution. Due
to a probably uniform sign of the deformation all over the phase distribution and also
within the small area, a translation of the fitted surface is possibly necessary to obtain an
optimum coincidence with the true phase distribution. However, the size and direction of
this translation can be unknown. Therefore, the surface is translated in both directions,
to higher and to lower phases, such that the surface just hits the phase distribution in the
small area. This is elucidated in Figure 4.1. The true phase surface lies somewhere
between these two surfaces and is practically parallel to them. Next, the rms phase
differences between the two translated surfaces and the unwrapped phase distribution generated by the RTI are calculated. The maximum of these two values, \( \delta_{s,\text{rms}} \), is equal to the new approximation of the upper bound of \( \Delta\alpha_{\text{rms,RTI}} \). This value is an upper estimate because the error in the phase distribution measured by the TPSI does not only contain contributions due to deformation, but also contributions due to quantization and noise, nonlinear camera response etc.. Hence, the translation of the fitted surface is probably too large, which results in a too large value of \( \delta_{s,\text{rms}} \).

The new approximations for the lower bound and the upper bound lead to the following estimation of \( \Delta\alpha_{\text{rms,RTI}} \):

\[
\delta_{s,\text{rms}} \leq \Delta\alpha_{\text{rms,RTI}} \leq \delta_{s,\text{rms}}
\]

(4.6)

The maximum absolute phase error induced by the RTI can be approximated by using the same reference surfaces as for the calculation of the rms phase error. If \( |\delta|_{s,\text{max}} \) is the maximum absolute phase error with respect to the least squares fit through the phase distribution obtained by the RTI and \( |\delta|_{s,\text{rms}} \) is the maximum absolute phase error with respect to the translated surfaces obtained by using a small part of the phase distribution generated by the TPSI, for the maximum absolute phase error generated by the RTI, \( |\Delta\alpha|_{\text{max,RTI}} \), is found to be:

\[
|\delta|_{s,\text{max}} \leq |\Delta\alpha|_{\text{max,RTI}} \leq |\delta|_{s,\text{max}}
\]

(4.7)

### 4.2.2 Relative accuracy

A typical phase distribution generated by the RTI is shown in Figure 4.2a. This phase distribution is extracted from an actual measurement in which the mirror is tilted and the interferograms contain almost straight fringes and a reference measurement in which the mirror is perpendicular to the test beam and the interferograms contain about one circular fringe. For the sake of clearness, the phase distribution is presented in interval ranges from black to white in which a smooth variation from black to white represents a phase increase of \( 2\pi \) radians. Figure 4.3a shows the same phase distribution, but now generated by the TPSI.

A representation of the errors in the two phase distributions is given in the Figures 4.2b and 4.3b. A surface has been fitted through the phase distribution by using a least squares method and the figures show the discrepancy between the phase distribution and the fitted surface. Here, the interval range from black to white represents a discrepancy (error) of \(-\pi/4\) radians to \(+\pi/4\) radians. Figures 4.2b and 4.3b visualize similar periodic phase errors, but the amplitude of the these errors are much smaller in the TPSI than in the RTI. The first of the three errors is an error varying at twice the fringe frequency in the interferograms belonging to the reference measurement. Since the fringes during the reference measurement are circular, the phase error varies in the radial direction and it is visible as circles. The error is a consequence of a phase step error between the
interferograms and in the RTI caused by imperfections and misalignment of the polarizing optics and by misalignment of the cameras. In the TPSI the error is caused by a phase step variation, probably due to a non-linear response of the piezo-electric transducer. The second error varies from the upper left to the lower right and is a consequence of a phase step error between the interferograms belonging to the actual measurement. Since the fringes are almost straight, the error is visible as almost straight lines. The third error varies from the upper right to the lower left. This error is most likely introduced by reflections of the first type and visible as straight lines.

To characterize the measuring accuracy of the RTI with respect to the TPSI quantitatively, the difference in the rms phase error induced by the two interferometers must be
determined. As described in Equations 4.3a and 4.4, an upper estimate of this difference is given by $\delta_{rms}$ and, hence, this value is used for the characterization. Therefore, the corresponding wrapped phase distributions obtained by the RTI and the TPSI during the 'tilted mirror' experiments are subtracted pixel wise. In the resulting phase difference distributions, the pixels with a value higher than $\pi$ radians or a value lower than $-\pi$ radians corrected with a factor $2\pi$. After that, the rms phase difference $\delta_{rms}$ is calculated. In Figure 4.4 the results are presented as a function of the gradient in the phase distribution $\phi$ by the large markers. The results originate from two experiments performed with time intervals of several weeks and between which the CCD-cameras are realigned for translation in the x- and y-direction. In the figure, the gradients in $\phi$ are distinguished for the orientation of the resulting fringes: vertical, sloping, horizontal and in case of no gradient circular. The values of $\delta_{rms}$ increase from 2.8% of $2\pi$ radians for small gradients in $\phi$ to 4.3% of $2\pi$ radians for a gradient in $\phi$ of $58 \times 10^3$rad/m. It appears that the orientation of the fringes hardly influences $\delta_{rms}$.

Apart from the difference in the rms phase error induced by the RTI and the TPSI, a full relative characterization also requires the difference in the maximum absolute phase error. An upper estimate of this difference is the maximum absolute phase difference between the corresponding phase distributions obtained by the RTI and the TPSI. Unfortunately, in the 'tilted mirror' experiments pixels with a low modulation depth are not circumvented.
during the image processing. Hence, the largest phase differences occur due to disfunctioning of some pixels which result in differences of the order π radians. In order to get a reasonable impression of the maximum value of the absolute phase difference due to optical, alignment and phase calculation effects, \( |\delta|_{99} \) has been calculated. \( |\delta|_{99} \) Is defined as a value larger or equal to the absolute phase difference in 99% of all the pixels while the remaining 1% of the pixels have a higher absolute difference. Figure 4.4 shows the results by the small markers. It appears that \( |\delta|_{99} \) increases with the gradient in \( \phi \) from about 7.1% of 2π radians for a small gradient to 12.1% of 2π radians for a gradient of 58·10^3 rad/m.

### 4.2.3 Total accuracy

Utilization of the RTI requires knowledge of the total induced rms phase error and the total maximum absolute phase error in the phase distribution. Apart from the errors introduced by the error sources which are discussed extensively in Chapter 3, the phase distributions show also some deformation. In the 'tilted mirror' experiments, the measured phase distributions deviate slightly from flat surfaces, especially if the phase gradient is large. A small curvature arises in the direction perpendicular to the fringes. This deformation appears similarly in the phase distributions obtained by the RTI and the TPSI and, most likely, it arises due to Seidel aberrations induced by the imaging optics\(^3\,\,^4\).

The total rms phase error induced by the RTI, \( \Delta \alpha_{rms,RTI} \), has been derived from the 'tilted mirror' experiments by the application of Equation 4.6. The size of the phase distributions is 746×561 pixels, that is a reduction of the original size of 756×571 pixels because some pixels are lost during the image processing. Per experiment \( \delta_{rms} \) and \( \delta_{rms} \) have been calculated as described in section 4.2.1. For the calculation of \( \delta_{rms} \), there is made use of a small area of 256×256 pixels in the phase distribution obtained by the TPSI around the optical axis. To find \( \delta_{rms} \), the fitted surface is translated to higher and lower phases over a distance equal to three times the rms phase difference in the area: it appears that then the surface just hits the measured phase distribution. Only in case of gradient equal to zero there is made use of the whole phase distribution obtained by the TPSI in stead of a small area. A large area is now necessary to obtain a reasonable fit because the actual measurement contains only one circular fringe and thus the phase distribution contains a spatially low frequency periodic phase error. The deformation is negligible if the phase gradient is zero and will not disturb the fit.

Figure 4.5a shows the rms phase error \( \Delta \alpha_{rms,RTI} \) as a function of the phase gradient in \( \phi \). The upper bound of the error bars corresponds to \( \delta_{rms} \) and the lower bound corresponds to \( \delta_{rms} \). This means \( \Delta \alpha_{rms,RTI} \) lies in the range from 2.5%–4.8% of 2π radians for small phase gradients to 4.9%–10.1% of 2π radians for a phase gradient up to 58·10^3 rad/m. The range in which \( \Delta \alpha_{rms,RTI} \) may lie in the Figure 4.5a is rather large, i.e., up to 5.2% of 2π radians for phase gradients of 58·10^3 rad/m. The upper bound overrates the value
Figure 4.5a,b  The rms phase error $\Delta\alpha_{\text{rms,RTI}}$ (a) and the maximum absolute phase error $|\Delta\alpha|_{99}$ (b) as a function of the gradient in the phase distribution $\phi$ during an actual measurement (error bars). The bold markers represent an improved approximation. The thin markers represent the lower bound of $\Delta\alpha_{\text{rms,RTI}}$ after filtering by a linear filter with a window size of $7 \times 7$ pixels (only a).
of $\Delta \alpha_{\text{RTI}}$ and, hence, an improved approximation of $\Delta \alpha_{\text{RTI}}$ is desirable. Such an improved approximation of $\Delta \alpha_{\text{RTI}}$ is possible under the assumption that the low frequency fluctuations in the unwrapped phase distributions obtained by the TPSI are caused by the deformation and that the high frequency fluctuations are caused by the other error sources and have a known averaged value, most likely equal to zero. Under these assumptions, the phase errors in the 'tilted mirror' experiments induced by the deformation and the other error sources can be separated by using a low pass filter. Here, a linear filter is used to separate the errors in the phase distributions obtained by the TPSI, i.e., a filter that fits a linear surface locally to the unwrapped phase distribution by using a least squares method$^3$. The window size of $99 \times 99$ pixels has been chosen such that all the high frequency fluctuations, i.e., fluctuations with a spatial frequency up to the fringe frequency in $\beta$, are just removed. The filtered phase distribution is assumed to contain, apart from the phase distribution due to the mirror tilt, only a spatially varying contribution due to the deformation and a spatially constant contribution due to a variation in the phase step. This last contribution is corrected by the theoretical mean phase error based on the measured mean phase step. Next, $\delta_{\text{flat,RTI}}$ is calculated from this filtered phase distribution and the unfiltered phase distribution obtained by the RTI. The results are presented by the small bold markers in Figure 4.5a. They may only be interpreted as being a global indication of $\Delta \alpha_{\text{RTI}}$ because, since the window size of the filter is rather large, some distortion of the phase error induced by the deformation cannot be excluded. Moreover, the mean value of the error introduced by reflections is not exactly equal to zero and also the correction for the mean phase error due to a variation in the phase step can contain a small error. This results in an overall shift of the filtered phase distribution and thus of the surfaces to determine $\delta_{\text{flat,RTI}}$. Especially the improved approximation at a phase gradient equal to zero can show a deviation. Errors introduced during the actual measurement have a low spatial frequency and they will mix up with the error introduced by the deformation. Besides, as a consequence of the inhomogeneous probability density of the phase, the phase step during the actual measurement cannot be determined very accurately and thus the correction for the variation in the phase step is not accurate either. The approximated values for $\Delta \alpha_{\text{RTI}}$ vary from 3.2% of $2\pi$ radians for a small phase gradient to 6.3% of $2\pi$ radians for a phase gradient of about $58 \cdot 10^3$ rad/m. They lie close to the lower bound for small gradients and move more to the upper bound for large gradients. This is in agreement with the theoretical expectation: the deformation increases with the phase gradient and, hence, the measured phase distribution deviates more from the flat surface on which the lower bound is based.

To find a value of the maximum absolute phase error generated by the RTI $|\Delta \alpha|_{\text{RTI}}$ Equation 4.7 is used. The values of $|\Delta \alpha|_{\text{RTI}}$, $|\delta|_{s,\text{max}}$ and $|\delta|_{s,\text{min}}$ are approximated by $|\Delta \alpha|_{99,\text{RTI}}$, $|\delta|_{s,99}$ and $|\delta|_{s,99}$. Similarly to $|\delta|_{99}$, $|\delta|_{s,\text{max}}$ and $|\delta|_{s,\text{min}}$ are defined as values larger or equal to the absolute phase error in 99% of all the pixels while the remaining 1% of the pixels have a larger absolute error. The resulting values for $|\Delta \alpha|_{99,\text{RTI}}$, i.e., all the values between the maximum and the minimum, are presented by error bars in
Figure 4.5b. They increase from 6.8%-9.7% of $2\pi$ radians for small gradients to 13.7%-22.9% of $2\pi$ radians for a gradient up to $58\cdot10^3$ rad/m. The small bold markers in the figure represent the results of an improved approximation, in which the same reference surface has been used as in the further approximation of $\Delta\alpha_{\text{rms,RTI}}$. This improved approximation varies with the phase gradient from 7.5% of $2\pi$ radians to 17.5% of $2\pi$ radians.

4.2.4 Effects of filtering

In order to suppress spatial noise and, hence, to increase the measuring accuracy, the unwrapped phase distributions are commonly filtered. In the utilization experiments presented in Chapter 5, a linearly spatial filter with a window size of $7\times7$ pixels is used. This filter is based on fitting a linear surface to the phase distribution within the window by applying the least squares method. Apart from suppressing spatial noise, it is also used to restore the phase in the pixels which are circumvented during the phase unwrapping by interpolation. Filters based on this method are used before, for example, by Lanen and Maas.

To investigate the reduction of the relative phase error between the RTI and the TPSI, the filter has been applied to the unwrapped phase distributions obtained during the 'tilted mirror' experiments. After that, $\delta_{\text{rms}}$ and $|\delta|_{99}$ have been calculated over $746\times561$ pixels. The results are presented as a function of the gradient in $\phi$ in Figure 4.6. They have to be compared to the unfiltered results in Figure 4.4. The value of $\delta_{\text{rms}}$ (large markers) varies after filtering between 1.8%-2.3% of $2\pi$ radians, which implies a reduction of 0.8%-2.2% of $2\pi$ radians. The values for $|\delta|_{99}$ (small markers) vary between 4.7%-5.6% of $2\pi$ radians, that is a reduction of 2.4%-6.8% of $2\pi$ radians. The reduction is, in general, more for higher gradients than for lower gradients. The figure shows that, after filtering, the relative phase error tends to be independent of the phase gradient.

To study the effect of filtering to the total phase error induced by the RTI, $\delta_{\text{rms}}$ has been calculated from the phase distributions after filtering. The values of $\delta_{\text{rms}}$ are presented in Figures 4.5a by the small thin markers. They have to be compared to the lower bounds of the also presented error bars. It appears that filtering reduces the rms phase error with about 0.9%-1.6% of $2\pi$ radians. This reduction must be interpreted as being an upper estimate of the error reduction by filtering. The reduction of the maximum absolute phase error is about 2-4% of $2\pi$ radians.
4. Experimental validation

![Graph showing number of fringes over 8.2mm length vs phase gradient (10^3 rad/m)](image)

**Figure 4.7** The rms phase error $\Delta \alpha_{\text{rms,RTI}}$ as a function of the gradient in the phase distribution $\phi$ during an actual measurement in absence of deformation. The large thin markers flanked by the error bars represent the results based on Equation 4.4, the small bold markers represent the results calculated directly from the phase distributions obtained by the RTI.

Deformation is not a typical error of the RTI and it can possibly be reduced by, for example, an exchange of one or more lenses, the potential accuracy of the experimental set-up in absence of the deformation is of interest. This potential accuracy includes only phase errors introduced by the typical elements of the RTI like the (polarizing) beam splitters, the wave plates and the multi camera system. This accuracy can be extracted from the 'tilted mirror' experiments.

Figure 4.7 shows the potential values of $\Delta \alpha_{\text{rms,RTI}}$ as a function of the gradient in $\phi$. The results are based on Equation 4.4. The large thin markers represent $\delta_{\text{rms}}$, whose values are equal to those in Figure 4.2 because the deformation in the RTI is equal to the deformation in the TPSI. The error bars represent $\Delta \alpha_{\text{rms,TPSI}}$. To obtain these last values, the phase distribution generated by the TPSI is filtered by a linear filter with a window size of 99×99 pixels. The filtered phase distribution is assumed to describe the phase introduced by the tilted mirror which is distorted by the deformation only. Next, $\Delta \alpha_{\text{rms,TPSI}}$ is approximated by the rms phase difference between the unfiltered and the filtered phase distribution. The use of a linear filter implicitly assumes that all the phase errors except those induced by the deformation vary spatially with a relative high frequency (compared to the window size) and have an averaged value equal to zero. Practically, this implies that phase errors due to vibrations and turbulence in the surrounding air must be negligible, that the phases $\beta$ and $\phi$ must have a homogeneous probability distribution, the reflections must be small and their additional phase shift $\gamma_i$ homogeneously distributed,
and the variation in phase step may not lead to an overall error. Most of these conditions seem to be fulfilled in the 'tilted mirror' experiments, except for the measurement where the phase gradient is approximately zero. The TPSI induces a small overall error due to a variation in the phase step, but this hardly influences the value found for $\Delta \alpha_{\text{rms,TPSI}}$: it will be underrated at most about 0.2% of $2\pi$ radians. The figure shows that the potential value of $\Delta \alpha_{\text{rms,RTI}}$ varies from 1.8%-3.7% of $2\pi$ radians for a small phase gradient to 3.0%-5.6% of $2\pi$ radians for a phase gradient up to $58 \cdot 10^3 \text{rad/m}$. The small bold markers represent the potential values of $\Delta \alpha_{\text{rms,RTI}}$ which are calculated directly from the phase distributions generated by the RTI, in a similar way as the values of $\Delta \alpha_{\text{rms,TPSI}}$ were calculated. Also in the RTI the required conditions for the phase errors are expected to be fulfilled. The results coincide within the error bars and, hence, it supports the presented potential accuracy.

For a theoretical analysis of the experimentally obtained potential accuracy of the RTI, the experimental results based on Equation 4.4 from Figure 4.7 are presented again in Figure 4.8. The large thin markers represent $\delta_{\text{rms}}$ and the long error bars represent the experimentally found values of $\Delta \alpha_{\text{rms,TPSI}}$. The short error bars represent the optimum rms phase error given by Equation 3.105, which is based on the error sources quantization and noise, the non-linear camera response and the variation in the phase step. In accordance with the experimental values of $\Delta \alpha_{\text{rms,TPSI}}$, overall phase errors are set equal to zero in the optimum value of $\Delta \alpha_{\text{rms,TPSI}}$, i.e., $\langle \Delta \phi[s] \rangle = \langle \Delta \beta[s] \rangle = 0$ in Equation 3.105. Apart from the experimental values of $\Delta \alpha_{\text{rms,RTI}}$, the figure presents also the theoretical optimum rms phase error (Equation 3.104) based on the properties in the Tables 3.2 and 3.3. These theoretical curves deviate slightly from the curves in Figure 3.20 ($c=7$) because the assumed phase error due to the misalignment of the cameras differs slightly. The curves in Figure 3.20 are based on the assumption that all three cameras have an error of $0.5^\circ$ in the rotations about the x- and y-axis and thus the rms phase error is given by Equation 3.54b, while the curves in Figure 4.8a are based on the assumption that CCD1 is oriented correctly for these rotations and the other cameras show an error up to $1^\circ$ so that the rms phase error is given by Equation 3.55b. This is more in agreement with the experimental circumstances. However, the adjustment leads to a small change in $\Delta \alpha_{\text{rms,RTI}}$: maximal 0.16% of $2\pi$ radians for a gradient up to $58 \cdot 10^3 \text{rad/m}$. The difference between the rms phase errors introduced by vertically and horizontally oriented fringes is negligible (less than 0.05% of $2\pi$ radians).

In Figure 4.8a one can see that the increase of the rms phase error with the gradient in $\phi$ is described correctly by the optimum rms phase error. The absolute rms phase error is, however, a little underrated. The theoretical curves just hit the lower error bars of the experimental results. Some small reflections in the set-up let the experimental rms phase error increase. They let also the rms phase error in the TPSI increase so that the experimentally obtained error can be considerably larger than the theoretical optimum error.
Figure 4.8a,b The rms phase error $\Delta \alpha_{rms,RTI}$ as a function of the gradient in the phase distribution $\phi$ during an actual measurement in absence of deformation found by using an one-camera TPSI (a). Explanation of the markers, error bars and lines (b).

Recordings of the intensity distributions in the test beams to the three CCD-cameras show a small reflection of the first type which is less than 1%. Moreover, CCD2 shows also a reflection of the second type which is less than 0.3%. The first reflection induces a phase error in the RTI and the TPSI, while the second reflection induces an error in the RTI only. A reflection of the first type leads to exactly the same errors in the RTI and the TPSI. Hence, the rms phase error in the RTI can be calculated from the experimental and the theoretical optimum rms phase error in the TPSI. Since this error is independent of $\phi$ and $\beta$ and its mean value is almost zero, it is statistically independent on the other error sources and thus its mean square (ms) value is equal to the difference between the experimental ms phase error and the theoretical ms phase error. The resulting rms phase
error appears to vary between 0.7% and 1.1% of $2\pi$ radians. This is in agreement with
the calculated values in Figure 3.13 for a reflection of 1%. Nevertheless, the values must
be accepted as being approximations, An possible uncertainty in the actual phase steps in
the TPSI (see Section 3.10) lead to an uncertainty in the rms phase error and, hence, in
the error found for the reflection.

Addition of the ms phase error due to the reflection of the first type to the optimum ms
phase error of the RTI lead the rms phase errors which are presented by the lower
‘♦’-markers in Figure 4.8a. They lay, generally, better within the error bars than the
original optimum rms phase error.

The reflection of 0.3% of the second type on CCD2 leads to a rms phase error
somewhere between 0.76% and 1.31% of $2\pi$ radians. This error depends on $\phi$ and $\beta$ and,
however, it is not statistically independent of the errors induced by the non-linear camera
response, the misalignment of the cameras, and the limited perfection and alignment
accuracy of the polarizing elements. Addition of an extra term of 1.31% of $2\pi$ radians
between the brackets in Equation 3.104 leads, if the reflection of the first type is taken
into account as well, to the theoretical values which are given by the upper ‘♦’-markers
in Figure 4.8a. These theoretical values lie within the error bars and they are
approximately equal to the experimentally obtained value of $\delta_{\text{rms}}$.

The foregoing shows that the experimentally obtained accuracy of the RTI can be
theoretically explained. However, there must be remembered that the theory describes the
worst case whereby the rms phase errors induced by several statistically dependent error
sources (non-linear camera response misalignment of the cameras, limited alignment and
perfection of the polarizing elements, and reflection of the second type) are simply added.
Since the worst case description is needed to describe the experimental results, it cannot
be excluded that some additional error sources give also essential contributions to the total
phase error. These additional error sources are, for example, the bad modulation depth
at some pixels, the existence of some small particles of dust in the imaging optics
resulting in a local diffraction pattern, an erroneous pixelwise sensitivity correction and,
possible but not likely, the presence of some air turbulence. Nevertheless, there may be
concluded that the theory gives an acceptable description of the experimental results.

A maximum rms phase error of 10.1% of $2\pi$ radians and a maximum absolute phase error
of 22.9% of $2\pi$ radians are rather large. A reduction of the deformation can, however,
reduce these errors substantially. The deformation is probably the result of aberrations
induced by the imaging optics, i.e., the combination of the lenses L2, L3, L4 and L5
(Figure 3.2). Since lenses with a short focal distance induce more aberrations than lenses
with a longer focal distance, the deformation is mainly introduced by lens L3. An
exchange of the combination of lenses L3 and L4 with a combination with longer focal
distances can reduce the deformation.

The necessity of the reduction of the deformation depends on the application. Small phase
gradients do not lead to the maximum rms phase error of 10.1% of $2\pi$ radians and the
maximum absolute phase error of 22.9% of $2\pi$ radians. Also, if $\beta = \phi$ and both gradients
are large, the errors are smaller than these values. The maximum phase errors arise only if the gradient in $\beta$ is approximately zero and the gradient in $\phi$ is large. In that case, it is good to realize that the maximum rms phase error and the maximum absolute phase error are equivalent to only 0.13% respectively 0.29% of the total measuring range of the RTI (78 fringes over the whole surface). These errors are acceptable if the total range is used.

4.3 Stability

Instabilities in the experimental set-up introduce time dependent phase errors in the interferograms. Because of this time dependency, the errors are different during the actual measurement and the reference measurement through which they do not cancel in the calculation of the phase distributions. Moreover, the phase errors vary in successively measured phase distributions.

To study the phase error introduced by instabilities in the RTI, a new 'tilted mirror' experiment is carried out. In this experiment, the mirror is tilted such that about 13 vertical fringes appear on the cameras. After that, several sets of interferograms are recorded in sequence in time while the tilt of the mirror remains unchanged. In the processing, the first set is the reference measurement and the following sets are actual measurements. Since the phase difference between the actual measurement and the reference measurement in a stable (and noise-free) set-up is spatially constant and equal to zero, the result of the processing is a sequence of phase error distributions as a function of the time between the reference measurement and the actual measurement. The error distributions originate from a combination of stability determining error sources (like vibrations and turbulence in the surrounding air) and noise. Besides, phase shifts due to instabilities can also give rise to phase dependent errors like those induced by the non-linear camera response, the misalignment of the cameras, and the limited perfection and alignment accuracy of the polarizing components. Hence, the phase error distribution is not only determined by the instabilities but also by the measuring accuracy. The phase error distributions must, therefore, be interpreted as an upper limit of the phase error due to instabilities. To reduce the influence of noise, the phase distributions obtained in the 'tilted mirror' experiments are filtered by linear spatial filter with a window size of $7 \times 7$ pixels before they are analyzed.

In the phase error distributions two statistically independent types of phase errors can be distinguished. The first type is a constant error over the whole profile and is given by the averaged error in the phase error distribution. This error does not disturb the phase profile but influences the relative phase between successive phase distributions. The second type is a spatially varying error which distorts the phase profile, for example, mirror tilt or air turbulence. The rms value is given by the standard deviation in the phase error
Figure 4.9a,b  The phase error as a function of the time $t$ between the reference measurement and the actual measurement, uncorrected and corrected: $t<80$s (a), $t<500$s (b).

distribution.

The experimentally found errors as a function of the time between the reference measurement and the actual measurement are presented in Figure 4.9a,b. The '•'-markers represent the first type error (the averaged error in the phase error distribution over $746 \times 561$ pixels), the error bars represent the rms second type error (standard deviation). Figure 4.9a shows the results for a short time period (0-80s) and Figure 4.9b shows the
Several contributions to the total phase error depend on the values of the phase in the reference measurement and the actual measurement. Or more precisely, they depend on the phase during the reference measurement and the phase difference between the actual measurement and the reference measurement. This concerns also the phase error distribution in the reference area. It makes the averaged phase error dependent on the phase in the reference measurement and, hence, also the correction of the whole phase error distribution. To study the possible variation in the total phase error after the
Figure 4.12a,b The total rms phase error and the total maximum absolute phase error (correction based on a 15 pixels wide reference area) and their maximum values as a function of the time $t$: $t<80s$ (a), $t<500s$ (b).

correction, the averaged phase error is calculated for 65 reference areas. These reference areas are all positioned at the left side of the phase error distribution, they extend over the full height, they are 1 pixel wide and they are shifted over 1 pixel with respect to each other. This means that the averaged phase in the different reference areas during the reference measurement varies over the interval $[0,2\pi]$. The averaged phase error in each reference area is corrected for tilt effects and, after that, compared to the averaged phase
error in the 15 pixels wide reference area. The possible change varies per measurement, but it appears to be maximal 2 digitized levels. This means a possible change of the first type phase error. Although the first type phase error changes per measurement, its absolute value appears to remain smaller than 1.1% of $2\pi$ radians. The maximum value of the total rms phase error is compared to the total rms phase error based on a 15 pixel wide reference area in the Figures 4.12a,b for $t<80$ and $t<500$. It appears that the total rms phase error increases in some measurements, but it remains smaller than 1.4% of $2\pi$ radians. Also in the figure, there is the maximum value of the total maximum absolute phase error compared to the total maximum absolute phase error found by using a 15 pixel wide reference area. The error increases with a value less than 0.78% of $2\pi$ radians through which it is better than 3.5% of $2\pi$ radians.

Summarized, if $t<440$ there may be concluded:
- the error in the profile (second type phase error):
  \[
  \text{rms}<1.0\% \text{ of } 2\pi \text{ rad} \\
  \text{max. abs. } <2.1\% \text{ of } 2\pi \text{ rad (in general)} \\
  \text{max. abs. } <2.6\% \text{ of } 2\pi \text{ rad (certain)}
  \]
- the constant error (absolute value of the first type phase error):
  \[
  <6.4\% \text{ of } 2\pi \text{ rad (uncorrected)} \\
  <1.1\% \text{ of } 2\pi \text{ rad (corrected)}
  \]
- the total error:
  \[
  \text{rms}<6.5\% \text{ of } 2\pi \text{ rad (uncorrected)} \\
  \text{max. abs. } <9.0\% \text{ of } 2\pi \text{ rad (uncorrected)} \\
  \text{rms}<1.4\% \text{ of } 2\pi \text{ rad (corrected)} \\
  \text{max. abs. } <3.5\% \text{ of } 2\pi \text{ rad (corrected)}
  \]

### 4.4 Application of the results

If the RTI is used for refractive index profile measurements, there are five sorts of contributions to the total phase error which can be distinguished by their origin:

1. the limited measuring accuracy (Section 4.2),
2. the limited stability of the set-up (section 4.3),
3. a wavefront aberration in the test beam at the beginning of the test section (absent in the 'tilted mirror' experiments),
4. the limited imaging accuracy of the object plane on the CCD-cameras (absent in the 'tilted mirror' experiments),
5. non-linearities in the refractive index profile.

The contributions 1 and 2 are fully characterized by the experimental set-up and they are discussed extensively in the Sections 4.2 and 4.3. The total rms phase error due to these set-up determined contributions are presented in Figure 4.13a for the case that the phase
Figure 4.13a The rms phase error induced due to a combination of a limited accuracy and stability (error bars), due to misimaging over 1mm and due to an aberrated wavefront (approximated).

Figure 4.13b The maximum absolute phase error due to a combination of a limited accuracy and stability (error bars), due to misimaging over 1mm and due an aberrated wavefront.

distributions are corrected by using a 15 pixels wide reference area. The upper bounds of the error bars are found by adding the upper bounds in Figure 4.5a and the maximum rms phase error of 1.4% of $2\pi$ radians due to instabilities. This implies that the
contributions are treated as being totally statistically dependent and that the upper bounds are true upper bounds. The lower bounds of the error bars are equal to the lower bounds in Figure 4.5a. The assumption of no error induced by instabilities implies that the presented lower bounds are true lower bounds. The figure shows also an improved approximation which is found by adding the improved approximation in Figure 4.5a and the maximum rms phase error induced by the instabilities. The improved approximation appears to vary from 4.6% of $2\pi$ radians for small gradients to 7.7% of $2\pi$ radians for a gradient of about $58 \cdot 10^3 \text{rad/m}$.

Figure 4.13b shows the total maximum absolute phase error. Here, the upper bound, the lower bound and the improved approximation are calculated in the same way from the results in Figure 4.5b and the maximum absolute phase error of 3.5% of $2\pi$ radians due to instabilities. The values for the further approximation varies from 11.0% to 21.0% of $2\pi$ radians.

The contributions 3 and 4 to the total phase error depend on the refractive index profile under consideration. The phase error induced by a wavefront aberration in the test beam at the beginning of the test section is discussed in Section 3.9 for the case that the object is a linearly stratified medium bounded by parallel windows. An upper estimate is given by Equation 3.100. A global measurement with a lateral shear interferometer learns that in the RTI the wavefront aberration in the test beam is a combination of a spherical aberration and defocusing (see Appendix) in which the maximum wavefront gradient is approximately $1.7 \cdot 10^4$. Substitution of $W'=1.7 \cdot 10^4$ and $G=0.6$ in Equation 3.100 under the assumption that $\phi' = \alpha'$ leads to the maximum phase error as a function of the gradient $\phi'$, provided the gradient in phase distribution $\beta$ is small. Figure 4.13b shows this relation for the wind tunnel configuration and the cuvette configuration for the case that the objects are positioned against the mirror ($a=0$). A phase gradient up to $58 \cdot 10^3 \text{rad/m}$ leads to a maximum phase error of 5.7% of $2\pi$ radians for the wind tunnel configuration and to a maximum phase error of 0.3% of $2\pi$ radians for the cuvette configuration. Figure 4.13a shows the rms phase error for the case that the medium is linearly stratified, i.e., the rms phase error is 0.66 times the maximum phase error, see Appendix.

The limited imaging accuracy lead in a linearly stratified medium to a constant phase error which is given by Equation 3.25. The rms value and the maximum absolute value are the same. The Figures 4.13a,b show this value as a function of the phase gradient at the position of the cameras if $G=0.6$ and $\Delta z'=1\text{mm}$. For a phase gradient smaller than $58 \cdot 10^3 \text{rad/m}$, the phase error is smaller than 1.0% of $2\pi$ radians.

Under the conditions that the medium is linearly stratified and the phase distribution $\beta$ is linear with a small gradient, the total rms and the total maximum absolute phase error induced by the RTI can be extracted from the Figures 4.13a,b by adding the different contributions. The results are presented in Table 4.1 for both the wind tunnel experiments and the cuvette experiments. Presented are the minimum, the maximum and the improved approximated errors for the phase gradients $\phi'=0\text{rad/m}$, $\phi'=22 \cdot 10^3 \text{rad/m}$,
<table>
<thead>
<tr>
<th></th>
<th>rms error</th>
<th></th>
<th>max. abs. error</th>
<th></th>
</tr>
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<td></td>
<td>min.</td>
<td>max.</td>
<td>app.</td>
<td>min.</td>
</tr>
<tr>
<td>wind tunnel conf., φ' = 0 rad/m</td>
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<td>4.6°</td>
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<td>10.0</td>
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<td>4.6°</td>
<td>6.8</td>
</tr>
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<td>8.7</td>
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<td>cuvette conf., φ' = 43·10^3 rad/m</td>
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<td>11.4</td>
<td>7.5</td>
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<tr>
<td>cuvette conf., φ' = 58·10^3 rad/m</td>
<td>6.0</td>
<td>12.7</td>
<td>8.9</td>
<td>14.9</td>
</tr>
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</table>

* depends on the phase

Table 4.1 The total rms phase error and the total maximum absolute phase error (minimum, maximum, improved approximation) as a function of the phase gradient φ' under the assumption that the wind tunnel configuration and the cuvette configuration contain a linearly stratified medium (in % of 2π radians).

<table>
<thead>
<tr>
<th>correction to the rms phase error</th>
</tr>
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<tbody>
<tr>
<td>φ' = 0 rad/m</td>
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<tr>
<td>φ' = 22·10^3 rad/m</td>
</tr>
<tr>
<td>φ' = 43·10^3 rad/m</td>
</tr>
<tr>
<td>φ' = 58·10^3 rad/m</td>
</tr>
</tbody>
</table>

Table 4.2 Estimated correction to the rms phase error as a function of the phase gradient φ' if the medium is locally linearly stratified and there are no more or less circular fringes about the optical axis (in % of 2π radians).

φ' = 43·10^3 rad/m and φ' = 58·10^3 rad/m, whereby the rms phase errors for φ' = 0 rad/m contain an uncertainty due to the phase dependence. The improved approximation of the rms phase error increases with φ' from 4.6% to 12.3% of 2π radians for the wind tunnel configuration and from 4.6% to 8.9% of 2π radians for the cuvette configuration. The maximum absolute error varies from 11.0% to 27.5% of 2π radians and from 11.0% to 22.3% of 2π radians, respectively. The errors in Table 4.1 may be interpreted as being upper estimates since (1) they are the result of adding the different contributions, i.e., the contributions are assumed to be statistically dependent which is not totally true, (2) the phase errors found for the limited accuracy and the limited stability can contain common
contributions which are counted twice and (3) pixels with a low modulation depth have also been processed.

The phase errors in Table 4.1 are valid as long as the refractive index profile is a linearly stratified medium with its gradient perpendicular to the propagation direction of the beam, the probability distributions are uniform over [0,2) and there is only a small gradient in β. In case of a locally linearly stratified medium, i.e., a medium in which each individual ray traverses an approximately one dimensional linear refractive index profile, the rms phase error and the maximum absolute phase error are smaller than those errors corresponding to the maximum gradient in the phase distribution. However, since the phase gradient now varies over the phase distributions and the fringes are no longer straight and equidistant, the rms phase errors in the Table 4.1 need a small correction. Contribution 1 to the rms phase error possibly increases because the phase error due to the misalignment of the cameras increases and thus the optimum rms phase error. This increase is caused by the fact that the phase errors induced by the translations and the rotations are no longer statistically independent: the rms phase error is no longer described by Equation 3.57b but by Equation 3.57a. Moreover, due to a change of the shape of the fringes, the rms phase error induced by the rotations possibly increase. In Table 4.2 a correction is estimated by the difference in Figure 3.20 between the maximum optimum rms phase error and the optimum rms phase error for vertical fringes (fringes under 45° if φ' = 58·10⁴ rad/m) for c = 7.

One-dimensional and two-dimensional refractive index profiles where the test beam enters perpendicular to the refractive index gradient can satisfy the condition of locally linear stratification. However, there must be kept in mind that the rms phase error probably exceeds the values given in the Tables 4.1 and 4.2 if the fringes encircle the optical axis, i.e., the fringes become more less circular about the optical axis. Then, the effects of the deformation and the aberration in the wavefront of the test beam (see Appendix) increase. A global calculation learns that this error can be up to some percents of 2π radians, mainly caused by the deformation. In wind tunnel en cuvette experiments circularly shaped fringes are not expected and thus the rms phase errors given in the tables may assumed to be sufficient accurate.

As soon as the condition of the locally linear stratification is not fulfilled, an additional error arises (contribution 5). Lanen and Beach et al. showed, by using a numerical ray tracing scheme, that if the same object plane is imaged on the cameras as in the case that the condition would be satisfied, this additional phase error remains minimal.

In the experimental flow research, a property of interest is the air density ρ(x,y). Therefore, it is important to know the accuracy with which this density can be determined in the wind tunnel by using the RTI.

The density is related to the refractive index according to the Gladstone-Dale relation:
\[ \rho(x,y) = \frac{n(x,y)-1}{K} \]  \hspace{1cm} (4.8)

Here is \( K \) the Gladstone-Dale constant, which is a weak function of the wavelength of the light and almost independent of the temperature and the pressure. For \( T=288K \) and \( \lambda=632.3\text{nm} \), \( K \) is about \( 0.2256 \times 10^{3}\text{m}^3/\text{kg} \). Equation 4.8 means that the density can be determined with an accuracy \( \Delta \rho(x,y) \) which is given by:

\[ \Delta \rho(x,y) = \frac{\Delta n(x,y)}{K} \]  \hspace{1cm} (4.9)

Here \( \Delta n(x,y) \) is the error in the measured refractive index. Substitution of the relation between the error in the refractive index and the phase error \( \Delta \alpha(x,y) \) in a Michelson interferometer, i.e.,

\[ \Delta \alpha(x,y) = \frac{4\pi}{\lambda} \Delta n(x,y)L \]  \hspace{1cm} (4.10)

leads to:

\[ \Delta \rho = \frac{\lambda}{4\pi KL} \Delta \alpha(x,y) \]  \hspace{1cm} (4.11)

This equation implies that measurements in the wind tunnel \( (L=150\text{mm}) \) by using the RTI \( (\lambda=632.8\text{nm}) \) with a maximum rms phase error \( \Delta \alpha_{\text{max,RTI}} = 13.1\% \) of \( 2\pi \) radians lead to a rms density error of \( 1.2\text{g/m}^3 \). The maximum absolute phase error of \( |\Delta \alpha|_{\text{max,RTI}} = 27.5\% \) of \( 2\pi \) radians lead to a maximum absolute density error of \( |\Delta \rho|_{\text{max,RTI}} = 2.6\text{g/m}^3 \). These errors are equal to 0.10\% respectively 0.21\% of the air density of 1.21g/l at standard conditions \( (T=288K, p=0.1\text{MPa}) \).

In the experiments of Chapter 5, a sucrose concentration in an aqueous bulk fluid in a cuvette is measured as a function of the time. This sucrose concentration \( c(x,y) \) is related to the refractive index \( n(x,y) \) by:

\[ c(x,y) = \frac{n(x,y)-n_w}{dn/dc} \]  \hspace{1cm} (4.12)

where \( n_w \) is the refractive index of sucrose free water and \( dn/dc \) is the first derivative of the refractive index with respect to the sucrose concentration, i.e., about \( 0.0001371/\text{g} \) (see Chapter 5). This equation implies that the sucrose concentration can be determined with an accuracy \( \Delta c(x,y) \) equal to:

\[ \Delta c(x,y) = \frac{\Delta n(x,y)}{dn/dc} \]  \hspace{1cm} (4.13)

where \( \Delta n(x,y) \) is the error in the measured refractive index. Substitution of Equation 4.10 leads to:
\[ \Delta c(x,y) = \frac{\lambda}{4\pi L \frac{dn}{dc}} \Delta \alpha(x,y) \]  

(4.14)

This equation means, under the condition that the temperature distribution is sufficiently stable, that the experiments to the sucrose solution in the cuvette \((L = 10\text{mm})\) by using the RTI \((\lambda = 632.8\text{nm})\) with a maximum rms phase error of 9.7\% of 2\(\pi\) radians leads to a rms error in the concentration of \(2.2 \cdot 10^{-3}\text{g/l}\). The maximum absolute phase error of 22.3\% of 2\(\pi\) radians implies a maximum absolute concentration error of \(5.2 \cdot 10^{-3}\text{g/l}\).

4.5 Conclusions

The RTI has been validated experimentally by performing some 'tilted mirror' experiments. The linear phase distributions measured in these experiments are used to verify the accuracy and the stability. The rms phase error and the maximum absolute phase error have been analyzed as a function of the phase gradient during the actual measurement (maximal \(58 \cdot 10^3\text{rad/m}\) at the position of the cameras) while during the reference measurement the phase distribution is linear with a small gradient. To find the relative measuring accuracy, the phase distributions obtained by the RTI are compared to those obtained by the TPSI. To find the total accuracy, the results are compared to fits. The stability has been verified by comparing temporally registered phase distributions. The experimental results are completed with some theoretical results for two additional phase errors which do not appear in the 'tilted mirror' experiments but which will appear in refractive index profile measurements to linearly stratified media, i.e., the error introduced by a wavefront aberration in the test beam and the error introduced by the limited imaging accuracy of the object plane on the CCD-cameras. For measurements to refractive index profiles that satisfy the weaker condition of locally linear stratification, a theoretical correction is made for a possible change in the optimum rms phase error. It leads to the following conclusions:

1. **RTI versus TPSI**

Under the assumption of the same stability error, the rms phase error induced by the RTI is at most about 2.8\%-4.3\% of 2\(\pi\) radians larger than the phase error induced by the TPSI. The exact difference in the rms phase error depends on the phase gradient during the actual measurement but is almost independent of the fringe orientation. The maximum absolute phase difference between the RTI and the TPSI is about 12.1\% of 2\(\pi\) radians. The application of a linear filter with a window size of \(7 \times 7\) pixels to the unwrapped phase distributions measured by the RTI and the TPSI reduces the difference in rms phase error to 1.8-2.3\% of 2\(\pi\) radians and the maximum absolute phase difference to about 5.6\% of 2\(\pi\) radians.
2. *Phase error introduced by the RTI*

*a.* The rms phase error induced by the limited measuring accuracy of the RTI is, in absence of instabilities, about 2.5%-10.1% of $2\pi$ radians. The exact value depends mainly on the phase gradient during the actual measurement. The maximum absolute phase error varies between about 6.8% of $2\pi$ radians and 22.9% of $2\pi$ radians. An improved approximation of these errors leads to a maximum rms phase error of about 6.3% of $2\pi$ radians and a maximum absolute phase error of about 17.5% of $2\pi$ radians.

*b.* Instabilities in the set-up lead to a rms phase error within the phase distribution of less than 1.0% of $2\pi$ radians, the maximum absolute phase error is about 2.1% of $2\pi$ radians. In case of succeedingly registered phase distributions, the rms phase error is less than about 6.5% of $2\pi$ radians and the maximum absolute phase error is less than about 9.0% of $2\pi$ radians. These errors are mainly the result of a spatially constant phase shift between the phase distributions. After a correction for this spatially constant phase shift, the rms phase error and the maximum absolute phase error are about 1.4% respectively 3.5% of $2\pi$ radians.

*c.* Refractive index profile measurements give rise to two additional phase errors, i.e.:  
- a phase error due to a limited imaging accuracy which increases with the phase gradient and can be up to 1.0% of $2\pi$ radians if the virtual object plane is mismaged over 1mm;  
- a phase error due to a wavefront aberration developed in the test beam before it enters the test section which is maximal 5.7% of $2\pi$ radians in experiments to a wind tunnel configuration and maximal 0.3% of $2\pi$ radians for experiments to a cuvette configuration.

*d.* The total phase error in the refractive index profile measurements contains contributions of all the error sources mentioned in conclusions 2a, 2b and 2c, and depends on the phase gradient and some properties of the medium. For experiments to a wind tunnel configuration (propagation length of the beam is 150mm) the total rms phase error and the total maximum absolute phase error varies approximately from 4.6% to 12.3% of $2\pi$ radians and from 11.0% to 27.5% of $2\pi$ radians for a phase gradient up to $58 \cdot 10^3$ rad/m. For the cuvette configuration this is from 4.6% to 8.9% of $2\pi$ radians and from 11.0% to 22.3% of $2\pi$ radians, respectively. These phase errors are valid under the assumption of a linearly stratified medium, a small gradient during the reference measurement and a homogeneous phase distribution during the measurements over the interval $[0, 2\pi)$.

*e.* If the medium satisfies the weaker condition of locally linear stratification, the rms phase errors need only a small correction (up to 0.8% of $2\pi$ radians) provided that the phase distributions do not show more or less circularly shaped fringes about the optical axis. They now become 4.6% to 13.1% of $2\pi$ radians for experiments to a wind tunnel configuration and 4.6% to 9.7% of $2\pi$ radians for experiments to a cuvette configuration.

3. *Application*

Under the conditions mentioned in conclusion 2e, wind tunnel experiments by using the RTI lead to a maximum rms density error of 1.2g/m$^3$, that is 0.10% of the air density at
standard conditions. The maximum absolute error is 2.6g/m³ or 0.21% of the air density at standard conditions. The cuvette experiments result, under the condition of a sufficient temperature stability, in a rms error in the sucrose concentration of $2.2 \cdot 10^{-2}$g/l and a maximum error of $5.2 \cdot 10^{-2}$g/l.

4. **Important error sources**
The most important error sources in the RTI are the limited alignment perfection of the polarizing elements (rms $\sim 1.1\%$ of 2π radians), the deformation (only for large gradients, rms $<4.8\%$ of 2π radians in absence of more or less circular fringes), reflections (rms $<2\%$ of 2π radians), the misalignment of the cameras (only for large gradients, rms $<2.3\%$ of 2π radians), the limited stability (rms $<1.4\%$ of 2π radians if corrected) and the limited flatness of the wavefront (only in the wind tunnel experiments, maximum $<5.7\%$ of 2π radians)

5. **Filtering**
The application of a linear filter with a window size of 7×7 pixels to the unwrapped phase distributions measured by the RTI reduces the rms phase error maximally with about 0.8-2.2% of 2π radians. The reduction depends on the phase gradient.

6. **Realignment**
The cameras in the experimental set-up of the RTI can be realigned after which the measuring accuracy is reproduced within a few tenths of a percent of 2π radians.

7. **Potential phase error**
If the deformation in the phase distributions should be removed, the potential rms phase error due to the limited measuring accuracy in the RTI varies approximately from 1.8%-3.7% of 2π radians for a small phase gradient to 3.0%-5.6% of 2π radians for a phase gradient (at the position of the cameras) of $58 \cdot 10^{3}$rad/m. This means that the removal of the deformation gives rise to a reduction of 0.7%-4.6% of 2π radians in the total rms phase error.

8. **Comparison to theory**
The experimentally found rms phase errors agree, after a removal of the phase error due to the deformation, with the theory presented in Chapter 3. The rms phase error is, however, larger than the optimum rms phase error due to some additional error sources.
References


Jet injection into liquid in a cuvette: real time detection and quantization of the concentration homogeneity

5.1 Introduction

Over the past decades, solid phase immuno assays have become a wide spread technology to demonstrate extremely low concentrations of specific compounds in biologically or environmentally relevant fluids. Basically, the technology is based on isolation from these molecules from their original matrix (generally a water based fluid) by binding them to a highly affine, specifically pre-activated solid phase, followed by removal of the original fluid matrix and subsequent specific binding of a labelled compound to the captured molecule. More modern assay concepts combine the reaction of sample fluid and labelled compound in a single incubation step, see Figure 5.1.
Often applied labels are enzymes\(^1\) or microparticles\(^2\), both enhance the visibility of the captured agent.

Typical examples of the use of this technology are clinical diagnostic assays, where bodily fluids are investigated for antigens or antibodies against harmful agents like the HIV virus or malaria, or assays where trace concentrations of less than \(10^8\) virus particles or molecules per milliliter can be demonstrated in ground water. To give an idea on the extreme sensitivity of such assays, concentrations as low as pico or femto moles per liter are detectable from sub-milliliter volumes within only one hour of reaction time, by binding only a few million of the relevant molecules.

It is easy to understand that an essential requirement for such assays to be as sensitive as
possible, is the highest degree of reproducibility of all handling and reaction steps that are performed: temperature settings\(^3\), timing, washing of excess reagents\(^4\), chemical reaction conditions, etc.

As indicated above, the solid-liquid interphase of the assay has been activated with capture molecules that are able to react at high affinity with molecules that may be present in the fluid phase. If present there, and after some reaction period, a fraction of the latter molecules will thus be removed from the fluid phase and bound onto the solid phase. If the local reaction rate exceeds diffusion from the bulk of the fluid phase, the liquid becomes locally depleted and the effective reaction becomes diffusion limited. This particular problem has been studied extensively\(^5\) and practical solutions to enhance transport towards the solid phase have been presented\(^6\) and implemented by industry\(^8\). A related problem, that has however hardly got any industrial attention, is the effect of anisotropic concentration distributions on the reaction rate. This problems typically occurs after two different reactive fluids have been added into the reaction vessel without adequate mixing. These fluids may differ in chemical composition (pH, ionic strength,...) but also in their physical constitution (viscosity, density). The combined effect of anisotropic concentration, reaction conditions and mass transport conditions is that the cumulative binding of analyte and label is governed by the sum of all local binding rates under sub-optimum conditions. One extreme example is the use of so called homogeneous assays, in which reaction conditions have been optimized for simultaneous incubation of sample fluid (containing the analyte) and the labelled compounds. Because both molecules not only interact once bound to the solid phase but also in solution, the optimum amount of captured sample and label molecules follows from a delicately tuned set of reaction conditions. In its most extreme form one simply lays both liquids on top of each other and compares the assay results to those obtained from perfectly pre-mixed reactions. Even though the pre-mixing needs only several seconds, the final assay signal level after 90 minutes of incubation differs as much as 30% from the non-pre-mixed reaction, see Section 5.2. This results does show the relevance of adequate pre-homogenisation of all reactants on the performance of such assays, and the need for a tool to quantify homogenization for people developing processing conditions for such assays.
A widespread format for immuno assay applications is the micro(titer)plate, a polymer rack containing a matrix of 8 by 12 cylindrical wells. Before the incubation of the assay starts, the user has to fill the wells with all the relevant fluids, typically the sample that is to be tested and one or more liquid reagents. On a small scale, this can be done manually by using a single channel pipet for the sample and a multichannel pipet for the reagents. When the assay is performed in larger numbers, users generally use an automated pipetting robot with bar-code identification of fluids that is connected to the laboratory's LIMS. In all cases, the fluid enters the well as a free falling jet through which some rate of mixing is induced. In order to study aspects of mixing, video imaging of mixing dyed solutions has been described. Yet, these experiments did not allow quantization due to the variable path length through the cavity and due to the optical distortions. Also, these experiments did not include a major aspect of all applications: a density difference between subsequently injected fluids. Most literature relates to mixing in half infinite cavities and thus typically lacks both the density difference and the limited dimensions of the problem, making the prediction or understanding of the basic characteristics less likely. Another possible approach, using numerical simulation techniques, could not meet the demands because commercially available finite element methods cannot deal with a free surface, a free jet of liquid and a time-dependent volume in the cavity. In the study presented here, the experimental problems in microplate cavities are mimicked as good as good as possible, but a rectangular cavity is used instead of a cylindrical one and a 10 times larger total volume.

The aim of the experiments is to study the mixing after a jet of a dilute sucrose solution is injected in a bulk of sucrose free fluid in a small cuvette. Concentration measurements performed by using the RTI have to lead to a quantitative determination of the large-scale homogeneity as a function of the time and a detection of local inhomogeneities. Previous to these experiments, the suitability of the RTI for quantitative concentration measurements is demonstrated.

The chapter starts in Section 5.2 with the description and the results of a previously performed experiment that demonstrates the relevance of homogenization in immuno assays. After that, the theory about interferometrical concentration measurements, the determination of the homogeneity in a solution and the calculus of the diffusion coefficient of a solute are discussed in Section 5.3. Section 5.4 describes the materials used in the experiments and the details of the processing. Section 5.5 presents an accuracy analysis of the concentration measurements performed by the RTI later in the chapter. In a qualitative consideration, a mixing process recorded by the RTI is compared to a similar mixing process visualized by adding a dye. After that, a extended quantitative consideration is given. Section 5.6 concerns the validation of the RTI to be a convenient tool for quantitative concentration measurements. For this, the diffusion process of sucrose in a buffer is studied. The measured concentration distributions are analyzed, the diffusion coefficient is determined and the homogeneity is calculated as a function of the time. Section 5.7 describes the true measurements of the homogeneity of a solution after a jet
is injected into a bulk in a small cuvette, followed by the results. At the end of the chapter, in Section 5.8, there is a discussion about the interpretation of the results.

5.2 Relevance of homogenization in immuno assays

For demonstration of the relevance of homogenisation of sample and reagents prior to incubation, bioMérieux’ HIV 5.0 microplate assay was used\textsuperscript{11}. This assay has been designed to demonstrate the presence of antibodies against the human immunodeficiency virus (HIV) in human serum or plasma, the assay has been designed for screening of donated blood in blood banks.

In its prescribed operation, each well of the microplate contains a small, freeze dried and thus porous sugar bead that contains the labelled conjugate and additional buffer components. As soon as serum or plasma has been added, the sugar bead dissolves and the reaction between antibodies and conjugate can start. One important experimental complication of this format is that the dissolution process of the bead is not very reproducible, in cases one may even end up with a foam layer on the fluid caused by the release of enclosed air from the entrapped porous sugar bead. The assay’s processing prescription therefore includes a mixing step using for instance an orbitally rotating plate mixer.

The assay was then performed as follows: 50\(\mu\)l of the dissolved beads was pipetted into each well of the microplate. Then 100\(\mu\)l of

![Figure 5.2](image.png)

**Figure 5.2** The absorbance at 450nm as a function of the concentration of antibodies against HIV in the serum if the mixture of dissolved bead material and serum is homogenized and not homogenized.
'negative' serum (not containing the target antibody) or 100μl of positive serum, containing a known and low concentration of the target antibody was added. The serum was pipetted slowly on top of the dissolved bead material, in order to prevent mixing by pipetting only. Half the wells were then thoroughly homogenized using an orbital plate shaker (Sarstedt TPM-2) at 900RPM for 30 seconds. The other wells were not homogenized at all. From then on the plate was stored at ambient temperature to prevent thermal convection to homogenize the well content, thus letting diffusion doing the mixing. After 90 minutes of incubation the wells were washed using a bioMérieux plate washer (washer-400, bioMérieux BV, Boxtel, Netherlands), 100μl of TMB substrate was added and incubated at AT for 30 minutes after the enzymatic reaction was stopped by the addition of 100μl of 1 mole/l H₂SO₄. Of each well the absorbance at 450nm was measured using a microplate reader. Figure 5.2 shows the absorbance as a function of the concentration of antibodies against HIV in the serum for both the shaken and the not shaked wells. It appears that the homogenization of the mixture of dissolved bead material and serum leads to a 30% larger absorbance.

5.3 Interferometric measurements and analysis

This chapter presents the theory of interferometric concentration measurements. The conversion of a phase distribution into a concentration distribution, the analysis of the concentration homogeneity in a sucrose solution and the calculation of the diffusion coefficient of sucrose in a solution.

5.3.1 Interferometric concentration measurements

The phase measured by an interferometer is proportional to the mean refractive index over the propagation length of the test beam through the test object. If the phase is extracted from a combination of an actual measurement and a reference measurement, it is proportional to the difference in the refractive index between the two measurements. In the case that the object is a cuvette filled with a non-homogeneous dilute sucrose solution in the actual measurement and a cuvette filled with a homogeneous dilute sucrose solution in the reference measurement, the phase $\alpha(x,y)$ measured by a Michelson interferometer is given by:

$$\alpha(x,y) = \frac{4\pi}{\lambda} (n(x,y) - n_s(x,y))L$$

(5.1)

where $n(x,y)$ is the refractive index profile in the non-homogeneous dilute sucrose solution and $n_s(x,y)$ is the constant refractive index of the homogeneous sucrose solution, both enlarged with the refractive index profile in the windows. The length of the cuvette in the propagation direction of the test beam is $L$ and the wavelength of the light is $\lambda$. In dilute
sucrose solutions, the refractive index varies approximately linearly with the concentration \( c(x,y) \). Then, the difference between the sucrose concentration during the actual measurement and the homogeneous sucrose concentration \( c_h \) during the reference measurement is given by:

\[
c(x,y) - c_h = \left( \frac{dn(x,y)}{dc} - n(x,y) \right) = \frac{\lambda}{4\pi L \cdot dn/dc} \alpha(x,y)
\]

(5.2)

Here, \( dn/dc \) is the first order derivative of the refractive index with respect to the sucrose concentration which is equal to 0.000137l/g at room temperature (see Section 5.4.1). The term on the right hand side follows by substitution of Equation 5.1. The equation reduces to Equation 4.12 if sucrose free water is used as reference. It must be taken into consideration that the phase \( \alpha(x,y) \) is determined modulo \( 2\pi \) radians. After the unwrapping, the phase distribution is fully determined except for an offset. As a consequence, \( c(x,y) - c_h \) is determined apart from an overall constant and, since \( c_h \) is spatially constant, also \( \alpha(x,y) \). In order to find \( c(x,y) \), the concentration distribution must be normalized by adding an offset such that the mean concentration corresponds to the true mean concentration. Then \( c(x,y) \) is given by:

\[
c(x,y) = \langle c(x,y) \rangle + \frac{\lambda}{4\pi L \cdot dn/dc} \left( \alpha(x,y) - \langle \alpha(x,y) \rangle \right)
\]

(5.3)

Note that the spatial change in \( c(x,y) \) is still proportional to the spatial change in \( \alpha(x,y) \). If the concentration distribution is one of a sequence, it is not necessary to normalize the other recorded concentration distributions in the same way. If the image recording rate is sufficiently high, they can be normalized under the assumption that the shift between two successively recorded concentration distributions in a slowly varying area is smaller than \( \lambda/(4L \cdot dn/dc) \), what corresponds to a phase shift of less than \( \pi \) radians in a phase distribution.

### 5.3.2 Homogeneity analysis

A good measure for the deviation of a dilute sucrose solution from true homogeneity is the fractional standard deviation \( D_v \). Although \( D_v \) is presented as a function of volume concentrations of two mixing liquids in the literature\(^1\), here it is used as a function of the sucrose concentration \( c \). Then the fractional standard deviation becomes:

\[
D_v = \frac{\sigma_c}{\langle c \rangle}
\]

(5.4)

where \( \sigma_c \) is the standard deviation in the sucrose concentration and \( \langle c \rangle \) is the averaged concentration in the solution. The fractional standard deviation is independent of the total amount of sucrose in the solution and it tends from a finite value to zero if the solution becomes more homogeneous. The advantage over an other often applied measure, the
intensity of segregation\textsuperscript{13,14,12}, is that it is also sensitive to the rate of homogeneity of the unmixed state, i.e., the state that the jet of concentrated sucrose solution is present as a blob in the sucrose free buffer. The presence of a voluminous blob of a lower concentration is seen as being a more homogeneous state than the presence of a small blob of a higher concentration, provided that the total amount of sucrose and the total volume are equal.

In experiments like done in this study, the value of \(\langle c \rangle\) is generally known beforehand. To find \(D_v\), the value of \(\sigma_e\) must be extracted from a measured concentration distribution. In case of interferometry, this concentration distribution is a two-dimensional projection. The standard deviation is calculated over all the pixels in the distribution. However, since the concentration is an average over the propagation length of the test beam, the concentration variation in that direction is flattened out. Hence, the resulting value of \(D_v\) is an under estimation.

Mixing of fluids is a process in which one fluid stretches and folds throughout an other\textsuperscript{15,16}. A lamellar structure is the result. In the case of mixing of two fluids with a different sucrose concentration, fluid compartments of high concentration and compartments of low concentration alternate. A better mixing results in smaller structures. The further homogenization is a process of diffusion.

In the experiments presented in this chapter whereby a jet of a dilute sucrose solution is injected into a cuvette filled with sucrose free phosphate buffer, apart from the lamellar structure, there is a large-scale concentration gradient. Then, the fractional standard deviation is hardly sensitive to the presence of a lamellar structure, especially if it is extracted from a two-dimensional projection like it is done by using interferometry. Also the scale of segregation, proposed by Danckwerts to be a measure of the compartment size\textsuperscript{13}, cannot be calculated. Therefore, an other measure is adopted here.

It makes sense to assume that the thinner the compartments of high an low concentration, the faster the diffusion drives mass flux from a high concentration area to a low concentration area. The period of time in which this mass flux exists depends on the thickness of the compartments, but it is much shorter than the period of time needed by the total mass flux to resolve the large-scale concentration gradient. Following this line of thought, the rms concentration gradient is a measure for the integral mass flux and, hence, it is a convenient tool to detect the presence of a lamellar structure. Here, in order to make the measure independent of the integral concentration, its concentration normalized equivalent is used. This is the fractional rms concentration gradient \(F\), which is defined by:

\[
F = \frac{\sqrt{\langle (\nabla c)^2 \rangle}}{\langle c \rangle} \tag{5.5}
\]

In the experiments \(\langle c \rangle\) is known beforehand and \(\sqrt{\langle (\nabla c)^2 \rangle}\) is extracted from a measured phase distribution by averaging over all the pixels. Since the measured phase distribution
by interferometry is an average over the propagation length of the test beam, the value of \( F \) is dependent on the orientation of the lamellar structure. As long as this orientation is random over the measuring field, \( F \) is proportional to the concentration normalized mass flux. There must be kept in mind, however, that the measuring accuracy of the interferometer is reduced if there is a three-dimensional concentration field, since the refractive index profile is not locally linearly stratified. Nevertheless, \( F \) is a practical measure to detect the presence of local inhomogeneities.

### 5.3.3 The calculation of the diffusion coefficient

Suppose the diffusion of a dilute solution in a cuvette is 1-dimensional with the concentration gradient varying along the \( y \)-axis. The diffusion coefficient \( D \) is independent of the concentration and the diffusion process is described by Fick's second law\(^{17} \), i.e.,

\[
\frac{\partial c(y,t)}{\partial t} = D c''(y,t)
\]

where \( t \) represents the time, \( c(y,t) \) represents the 1-dimensional concentration distribution and \( c''(y,t) = \frac{\partial^2 c(y,t)}{\partial y^2} \). Then, the diffusion coefficient can be deduced from two successively recorded concentration distributions, one recorded at time \( t \) and one recorded at \( t + \Delta t \). Suppose that the two distributions are digitized and stored in an array of \( M \) pixels, whereby the concentration stored in pixel number \( i \) (\( i = 0,1,\ldots,M-1 \)) corresponds to the discrete position \( y_i \) in the cuvette. Then, Equation 5.6 is valid for every discrete position \( y_i \). If the period of time \( \Delta t \) between the two concentration distributions is not too long, so \( \frac{\partial c(y_i,t)}{\partial t} \) can be approximated by \( \{ c(y_i,t+\Delta t) - c(y_i,t) \} / \Delta t \), follows:

\[
\frac{c(y_i,t+\Delta t) - c(y_i,t)}{\Delta t} = D c''(y_i,t)
\]

The terms \( c(y_i,t) \), \( c(y_i,t+\Delta t) \) and \( c''(y_i,t) \) can be deduced from the two concentration distributions and, if \( \Delta t \) is known, the equation can be solved for \( D \). To reduce the influence of local concentration errors there may be summed over \( i \). This is allowed since \( D \) is assumed to be independent of the concentration and, hence, also of \( y_i \). After summation over \( N \) pixels in the interval \( i \in [p,p+N-1] \) \((0 \leq p < M-N)\), Equation 5.7 becomes:

\[
\sum_{i=p}^{p+N-1} \frac{c(y_i,t+\Delta t) - c(y_i,t)}{\Delta t} = D \sum_{i=p}^{p+N-1} c''(y_i,t)
\]

The condition for the calculation of \( D \) is that the interval is chosen appropriately and that \( N \) is not too large \((N \ll M)\), so the summations in Equation 5.8 do not become approximately zero.

A problem arises if there is an overall error in the concentration distribution and this error
is not the same at time $t$ and time $t+\Delta t$. Then, an error appears on the left hand side of equation 5.8. This problem can be overcome by subtracting from Equation 5.8 an identical sum over another pixel interval, e.g. the interval $i\in[n,n+N-1]$ ($0\leq n<M-N, n\neq p$). That is:

$$\sum_{i=p}^{p+N-1} \frac{c(y_i,t+\Delta t)-c(y_i,t)}{\Delta t} - \sum_{i=n}^{n+N-1} \frac{c(y_i,t+\Delta t)-c(y_i,t)}{\Delta t} =$$

$$D\left\{ \sum_{i=p}^{p+N-1} c''(y_i,t) - \sum_{i=n}^{n+N-1} c''(y_i,t) \right\}$$

(5.9)

The most accurate value for $D$ is found if the interval $[p,p+N-1]$ lies in a part of the concentration distribution where $c''(y_i,t)$ is maximally positive (i.e., the concentration increases with the time) and the interval $[n,n+N-1]$ lies in a part where $c''(y_i,t)$ is maximally negative (i.e., the concentration decreases with the time). Note that the linear relation between the concentration and the phase given by Equation 5.2 implies that $c(y_i,t)$ in Equation 5.8 may be replaced by $\alpha(y_i,t)$. Hence, the value found for $D$ is independent of $\lambda$, $L$ and $dn/dc$.

### 5.4 Materials and processing

The basic element of a microplate immuno assay is a cylindrical cavity with radius 3.5mm and height in the order of 7mm. In this cavity the user typically injects a free flow jet up to a total of 100 to 200$\mu$l of liquid. In the experiments, the problems in microplate cavities are mimicked as good as good as possible, but a rectangular cavity is used instead of a cylindrical one and a 10 fold larger total volume. The rectangular cavity is provoked by the experimental capability of the interferometric set-up. The aspect ratio of the front view of the liquid (width:height), the relative volumes of both liquids, the density differences and the injection speed were selected to be similar to those used in the microplates. Below, the materials used in the experiments and the data processing are described.

#### 5.4.1 Materials

The cavity used in the experiments is a quartz cuvette (Quartz Silicate) with a bottom area of $9.2 \times 10$ mm$^2$ and a height of 45mm. The pathlength $L$ is 10mm, the window thickness is 1.25mm. The cuvette is positioned in the test section of the RTI against the mirror (Figure 3.2). It is slightly tilted over about 0.5°, in order to let the spatial filter remove reflections on the windows and the walls. The virtual object plane lies, according to Equation 2.31 under the condition that the fluid is aqueous ($n_e=1.33$, $n_g=1.5$) at $z=1.75L$ (Figure 2.2), with a deviation of less than 0.5mm. The measuring volume is about 900$\mu$l.
Two liquids are used: a bulk fluid and a jet fluid. The bulk fluid is a 150mM phosphate buffer which is basically milli-Q water with some salts to fix the pH-value. The jet fluid is bulk fluid to which sucrose is added, 25g/l or 50 g/l. This fluid is intended to mimic the dissolved reagent beat as described in Section 5.2. At the same time, the sucrose is the tracer for detection by the interferometer. The refractive index of a dilute sucrose solution is approximately a linear function of the concentration. To obtain a reliable value of \( \frac{dn}{dc} \), the refractive index of the sucrose solution is measured as a function of the concentration by using a refractometer (wavelength 589.3nm, \( T = 20^\circ C \)) over the range 0-50g/l. The value of \( \frac{dn}{dc} \) is determined by the calculation of the gradient of a linear fit through the data. After that, \( \frac{dn}{dc} \) is determined in a similar way from data found in the literature\(^{18} \) where the solvent is pure water. The final value of \( \frac{dn}{dc} \) and its error is chosen such that both values lie within the error bands. Considering a possible temperature uncertainty in the range 17.5°C-22.5°C, there is found \( \frac{dn}{dc} = 0.0001371/g \) with an error of 2.5%.

The jet is injected from an altitude of approximately 2cm into the free surface of the bulk fluid in the cuvette. In the main experiments, this was done by using an Hamilton Microlab 500 diluter/dispenser pump (Hamilton Bonaduz, Switzerland) equipped with a 1ml syringe that is operated at a flow rate of 1ml/second. In some preliminary measurements, the jet was injected manually by using a flow disposable tip pipet (FlowLab, Finland) or a plastic squirt (Pastette LW4060, Alpha Laboratories Ltd, UK) at an (estimated) equivalent flow rate.

### 5.4.2 Processing

The phase stepped interferograms are recorded by the RTI in a 756×573 pixel format. After an off-line data processing, the format is 756×571 pixels. The reference measurement is made when the cuvette is filled with the bulk but before the jet is injected, or after the actual measurements when the solution is carefully homogenized by stirring. The phase distributions are extracted from the interferograms as described in Section 3.4.1. An offset is added such that the mean phase in a thin vertically oriented area (about 25×10³ pixels) at the left side of the phase distribution (where the light rays do not traverse the cuvette) is kept constant. A preselected serie of phase distributions is unwrapped as described in Section 3.4.3 and filtered by a linear filter with a window size of 7×7 pixels (Section 4.2.4). The phase of the pixels with a modulation depth in the interferograms below the threshold level (Section 3.4.1) are also reconstructed by this filter. Pixels which cannot be reconstructed by a linear interpolation in the window of 7×7 pixels are circumvented in the further calculations. The position of the walls and the bottom of the cuvette is determined from the map of the modulation depth: as a consequence of the transition in refractive index between the solution and the walls the rays here are refracted and filtered out. The concentration distributions are calculated by using Equation 5.2. To find the absolute concentration distribution, one of the concentration distributions per
experiment (called the reference concentration distribution) is normalized with respect to the theoretical mean concentration (Equation 5.3). For this, the reference concentration distribution is expanded at the top by a linear extrapolation of the upper 51 rows of pixels in the measuring volume (using a window of $51 \times 51$ pixels) over such an area that the total volume agrees with the true volume of the dilute sucrose solution. Here, the existence of a meniscus is ignored. The mean concentration in the total volume is made equal to the theoretical mean concentration by adding an offset. After that, the other concentration distributions are normalized with respect to the reference concentration distribution by adding a suitable offset as described in Section 5.3.1. Here, use is made of all the recorded phase/concentration distributions, also those that are not unwrapped. The linear extrapolation of the reference concentration is only an approximation of the true concentration distribution through which an offset error can be introduced in the results.

5.5 Accuracy of the concentration measurements

The accuracy analysis of the interferometric concentration measurements is divided into three parts. In the first part, a qualitative consideration is given about the global features of the concentration distribution in the cuvette as a function of the time after a jet is injected. In the second part, the errors introduced by several error sources are analyzed in a quantitative consideration. In the third part, the consequences for the measurements are discussed, i.e., the accuracy of the measured concentration distributions and the resulting values of $D_v$ and $F$.

5.5.1 Qualitative consideration

A qualitative visualization of the mixing process of a jet in the bulk fluid in a cuvette is presented in Figure 5.13. A jet of 100µl sucrose solution (25g/l) is injected into 900µl water in the cuvette. The jet is colored by Levafix Brilliant Rot dye. The cuvette is illuminated by an more or less diffuse light source, a two-dimensional projection is recorded by a video camera. The injection speed of the jet is kept low (0.1ml/s) to avoid a rapid mixing and a mixing of a high degree so that concentrations differences are visible. The figure shows the dye concentration (and thus the sucrose concentration) as a function of the time, whereby the concentration is higher if the image darkens. At the top, the meniscus is visible. The dark color of the meniscus is partly caused by refraction of the light and possibly also by some remainder of the jet.

At $t=0.2s$ after (the start of) the jet injection, the jet hits the bottom of the cuvette. At $t=0.4s$ the jet is reflected. After that, convective mixing appears until about $t=3s$. A concentration distribution with relatively large local gradients remains until $t=8s$, indicating a lamellar structure. After $t=8s$, only a large-scale inhomogeneity remains,
whereby the concentration at the top of the cuvette is lower than at the bottom. It reduces slowly by diffusion but is still present for $t > 38s$. Remarkable are the areas of low concentrations near the left and right wall of the cuvette if $t \leq 18s$. It is evident that the mixing is relatively poor in these areas.

Figure 5.14 shows a mixing process as studied in this chapter and which is recorded by using the RTI. A jet of $100\mu l$ sucrose solution (25g/l) is injected in $900\mu l$ bulk fluid in the cuvette with a speed of 1ml/s. The interferograms and the resulting phase distributions are processed as described in Section 5.4.2. The normalization is performed with the concentration distribution at $t = 558s$ as reference. Only the phase distribution at $t = 8s$ is not normalized since it is too chaotic for a reliable normalization. For $t < 8s$, the phase distributions are too chaotic to be unwrapped. Figure 5.14 shows the concentration distribution as a function of the time. For the sake of clearness, the concentration is presented in interval ranges from black to white, whereby a smooth variation from black to white represents a concentration increase of 0.23g/l (corresponding to a phase increase of $2\pi$ radians).

The mixing process is qualitatively similar to the process visualized by the dyed jet. The time-scale differs as a consequence of the increased injection speed (and thus the increased duration of the convective mixing) and the higher sensitivity of interferometry with respect to the dyed jet method. At $t = 8s$, the concentration distribution is chaotic and it contains large local gradients which are visualized by high fringe frequencies. This becomes even better visible if the range from black to white corresponds to a variation from the minimum concentration to the maximum concentration, see Figure 5.3. At $t = 18s$, a large-scale inhomogeneity arises, whereby the concentration at the top of the cuvette is lower than at the bottom. Local gradients remain until about $t = 58s$. Until $t > 58s$, a similar concentration distortion is visible near the walls as in the process visualized by the dyed jet until $t = 18s$. Especially near to the right wall the fringes bend downwards, indicating a reduction of the concentration. At $t = 198s$ the distortion near the walls is disappeared. It is clear that the two experiments show qualitatively a similar development in the
process. The advantage of interferometry is, however, the possibility to obtain accurate quantitative results and without adding a dye to the jet, provided that the concentration distributions are locally linearly stratified in a sufficient degree. Moreover, interferometry is more sensitive to relatively small concentration differences: a mixing process that cannot be analyzed by using the dyed jet method because of a rapid mixing and a mixing to a high degree (injection speed 1ml/s) can be visualized by interferometry.

5.5.2 Quantitative consideration

Several error sources which are not directly intrinsic to the specific measuring capacities of the RTI reduce the measuring accuracy of the sucrose concentrations in the cuvette. Hence, the error in the concentration distribution deviates from the error which would be expected by Table 4.1 in combination with Equation 4.14. Two types of error sources can be distinguished: errors induced by the cuvette and errors induced by physical properties of the sucrose solution. The first type of error sources include the presence of a serious refractive index in the windows of the cuvette, instabilities in the position of the cuvette, the tilt of the cuvette to remove reflections and the uncertainty in the pathlength between the windows. The resulting phase errors are relatively small and they are partly cancelled by the reduction of the phase error due the application of a linear filter. More severe are the errors of the second type, i.e., temperature variations in the solution, a deviation in \( dn/dc \) from 0.0001371/g and concentration inhomogeneities through which the dilute sucrose solution is not a linearly stratified medium.

Errors induced by the cuvette

An interferogram of a reference measurement whereby there is no sucrose concentration in the water is show in Figure 5.4. A more or less linear phase distribution is visible with about 28 fringes per 8.2mm chip length \( \beta' = 21\cdot10^3 \text{rad/m at the position of the chip} \). This gradient is possibly induced by a refractive index gradient in the windows of the cuvette. Then, it introduces an additional phase error for two reasons. The first reason is the increase of \( \beta' \) with respect to the tilted mirror experiments where \( \beta' \) at the position of the CCD-chip is 5100-7700 rad/m. Under the assumption that the phase error increases similarly with \( \beta' \) as with \( \phi' \), the additional rms phase error can be approximately extracted from Figure 4.5a and the additional maximum absolute phase error from Figure 4.5b. The increase of the improved approximations of the rms phase error between \( \phi' = 22\cdot10^3 \text{rad/m} \) and \( \phi' = 43\cdot10^3 \text{rad/m} \) is about 6.2\( \cdot10^3 \% \) of 2\( \pi \) radians per rad/m. This implies that an increase of \( \beta' \) from 5100 rad/m towards 21\( \cdot10^3 \text{rad/m} \) leads to an increase of 1\% of 2\( \pi \) radians in the rms phase error (\( \sim 2\cdot10^3 \text{g/l} \)). Similarly, for the increase of the maximum absolute phase error follows 1.6\( \cdot10^4 \% \) of 2\( \pi \) radians per rad/m. This results in an increase of 2.5\% of 2\( \pi \) radians (\( \sim 6\cdot10^3 \text{g/l} \)) if \( \beta' \) increases from
5. Jet injection into liquid in a cuvette.

<table>
<thead>
<tr>
<th>$t$ (s)</th>
<th>$\nabla c^*$ (g/l cm$^{-1}$)</th>
<th>RTI (g/l)</th>
<th>temp. (g/l)</th>
<th>$L$ (g/l)</th>
<th>$\frac{dn}{dc}$ (g/l)</th>
<th>total (g/l)</th>
<th>$D_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>4.9</td>
<td>0.015</td>
<td>0.016</td>
<td>0.005$\sigma_r$</td>
<td>0.025$\sigma_r$</td>
<td>0.031+0.030$\sigma_r$</td>
<td>0.033/(c)+0.082$D_r$</td>
</tr>
<tr>
<td>18</td>
<td>9.5</td>
<td>0.019</td>
<td>0.016</td>
<td>0.005$\sigma_r$</td>
<td>0.025$\sigma_r$</td>
<td>0.035+0.030$\sigma_r$</td>
<td>0.037/(c)+0.082$D_r$</td>
</tr>
<tr>
<td>18</td>
<td>12.8</td>
<td>0.022</td>
<td>0.016</td>
<td>0.005$\sigma_r$</td>
<td>0.025$\sigma_r$</td>
<td>0.038+0.030$\sigma_r$</td>
<td>0.040/(c)+0.082$D_r$</td>
</tr>
<tr>
<td>$\geq$178</td>
<td>4.9</td>
<td>0.015</td>
<td>0.008</td>
<td>0.005$\sigma_r$</td>
<td>0.025$\sigma_r$</td>
<td>0.023+0.030$\sigma_r$</td>
<td>0.024/(c)+0.082$D_r$</td>
</tr>
<tr>
<td>$\geq$178</td>
<td>9.5</td>
<td>0.019</td>
<td>0.008</td>
<td>0.005$\sigma_r$</td>
<td>0.025$\sigma_r$</td>
<td>0.027+0.030$\sigma_r$</td>
<td>0.028/(c)+0.082$D_r$</td>
</tr>
<tr>
<td>$\geq$178</td>
<td>12.8</td>
<td>0.022</td>
<td>0.008</td>
<td>0.005$\sigma_r$</td>
<td>0.025$\sigma_r$</td>
<td>0.030+0.030$\sigma_r$</td>
<td>0.032/(c)+0.082$D_r$</td>
</tr>
</tbody>
</table>

Table 5.1 The four contributions to the error in $\sigma_c$ (the limited measuring accuracy of the RTI, the temperature differences and instabilities, the tolerance in $L$, and the error in $\frac{dn}{dc}$), the total error in $\sigma_c$ and the error in $D_r$ as a function of the maximum gradient $\nabla c^*$ during the actual measurement for $t=18$s and $t \geq 178$s.

The error in the relative concentration between one distribution and the other belonging to the same experiment is described by the error in the mean concentration. In case of a theoretically constant mean concentration, this is the deviation of the measured mean concentration from this theoretical mean concentration. This error is mainly influenced by error source 3. As shown in section 5.5.2, the temperature differences and the instabilities can lead to large errors, i.e., of the order 38.3% of $2\pi$ radians (0.09 g/l). Since the error is strongly dependent on the relative temperature and the relative volume of the jet and the bulk, a reliable approximation cannot be made beforehand. However, an estimation can be made after the experiment by the calculation of the change in the mean concentration as a function of the time.

The error within an individual concentration distribution is described by the error in the standard deviation $\sigma_c$. This error is influenced by the error sources 2, 3, 4 and 5. Table 5.1 presents the four contributions and the total error for the case that the dilute sucrose solution is a locally linearly stratified medium. It is presented as a function of the maximum phase gradient during the actual measurement (represented by $\nabla c^*$ in terms of a concentration gradient) for $t=18$s and $t > 178$s.

The dependence on $\nabla c^*$ is introduced by the limited measuring accuracy of the RTI. The error introduced by the RTI is calculated from the Tables 4.1 and 4.2. It is an upper estimate for the case that the medium satisfies the condition of locally linear stratification, provided that $\nabla c^*$ in the table corresponds to the maximum gradient during the actual measurement.

The dependence of the total error in $\sigma_c$ on the time is introduced by the temperature differences and instabilities. The contribution of this error source in Table 5.1 is based on the experiment in Section 5.5.2 whereby a jet of 200$\mu$l is injected into a bulk of 1000$\mu$l. Since the volume of the jet is small compared to the volume of the bulk, the mixing is relatively poor (see Section 5.7.2) and a large standard deviation is the result.
Hence, the standard deviation due to temperature differences and instabilities as presented in Figure 5.5 may assumed to be an upper estimate, provided that the influence of the exact way of injection and the relative temperature between the jet and the bulk is the same. However, these restriction is not necessarily fulfilled and the error must be interpreted as a global estimation, especially for $t<18$ s. For large $t$, the rate in which the restrictions must be satisfied becomes less important. Figure 5.5 shows that for $t>178$ s the error in the standard deviation becomes even almost independent whether there actually was a jet injection or not.

The error in the concentration distribution induced by the tolerance in $L$ is described in Section 5.2.2. The error in $\sigma_c$ follows by a straight forward calculation. Under the condition that $c_h=\langle c(x,y) \rangle$ it leads to a maximum error equal to $\Delta L/100L\%$ for both the spatially constant part and the spatially varying part. When the value of $\Delta L$ and $L$ from Section 5.5.2 are used, that is 0.45% and 0.05%, respectively. In a similar way, the uncertainty in $dn/dc$ of 2.5% leads also to a maximum error in $\sigma_c$ of 2.5%.

The total error in the standard deviation $\sigma_c$ contains two terms, one proportional to $\sigma_c$ and one dependent on the gradient $\nabla c^*$ during the actual measurement. Which of these terms dominates depends on $\sigma_c$.

The fractional standard deviation $D_c$ is equal to the quotient of the standard deviation and the mean concentration, see Equation 5.4. In the experiments presented in this chapter, the error in the mean concentration is maximal 5%. Hence, the error in $D_c$ is equal to the quotient of the error in the standard deviation and the mean concentration after correction for the error in the mean concentration. This error is presented as a function of the gradient $\nabla c^*$ in the most right column of Table 5.1 for $t=18$ s and $t>178$ s.

The error introduced in the rms concentration gradient $F$ is not of importance. It is used to detect the presence of local gradients, even if the medium is not linearly stratified. Of more importance is the noise level. To approximate this level, the rms phase gradient is calculated as a function of the time for the experiment from Section 5.5.2 whereby there is no jet injected. After conversion of the maximum found value to a rms concentration gradient, the noise level in $F$ is found by dividing it by the mean concentration. The noise level appears to be $1.4\langle c \rangle$, where $\langle c \rangle$ is expressed in g/l.

### 5.6 Validation of the application

In order to validate the RTI to be a convenient tool for real time concentration profile measurements and in order to demonstrate that interferometry is capable to determine the homogeneity of dilute sucrose solutions, a diffusion process of sucrose in water is studied. For this, $200\mu l$ sucrose solution is added carefully to $1000\mu l$ water in the cuvette. After sedimentation of the sucrose solution towards the bottom of the cuvette and damping of the induced convection (inevitable attended by some mixing), diffusion moves the sucrose
upwards. An approximately 1-dimensional concentration distribution appears in which the concentration increases towards the bottom. The concentration distribution is measured as a function of the time and analyzed quantitatively and qualitatively. The utility of the successively measured concentration distributions is demonstrated by the extraction of the diffusion coefficient of sucrose in water. In order to demonstrate the capability of interferometry for homogeneity analysis and the suitability of the proposed measures, the fractional standard deviation and the fractional rms concentration gradient are determined as a function of the time.

5.6.1 Concentration measurements

The cuvette filled with 1000μl water is positioned in the test section of the RTI. The measuring volume extends upwards from the bottom of the cuvette and it contains 900μl of the water. The meniscus lies outside the measuring volume. Under these conditions, the reference measurement is made. After that, the jet of 200μl sucrose solution (50g/l) is injected slowly. Every 5 minutes an actual measurement is carried out until \( t = 395 \text{min} \). Later, an additional actual measurement is made at \( t = 874 \text{min} \). The modulo \( 2\pi \) radians phase distributions are calculated for each measurement. Some of these phase distributions are unwrapped and further processed as described in Section 5.4.2. The concentration distributions are obtained by using the concentration distribution at \( t = 395 \) as reference. Figure 5.17 shows the concentration distributions (phase distributions) as a function of the time. For the sake of clearness, it is represented in interval ranges varying from black to white which represent a concentration increase of 0.23g/l (2\( \pi \) radians). Figure 5.18 shows the quasi 3-dimensional plots of the concentration distributions.

The Figures 5.17 and 5.18 show that the concentration distributions are approximately 1-dimensional. The concentration at the bottom of the cuvette decreases with the time and the concentration at the top increases. This is in agreement with Fick's first law\(^{24,17} \), which describes a mass flux proportional to the concentration gradient from the area with a high concentration towards the area with low concentration. Some discrepancies are shown at the bottom of the cuvette and on the right hand side near the wall. Here, the concentration distribution is not exactly 1-dimensional. At the bottom, the concentration increases towards the left. Possibly, there is more sucrose solution sedimented at the left than at the right.

The mean concentration in the measuring volume increases between \( t = 5 \text{min} \) and \( t = 395 \text{min} \) with 1.2g/l. This is 14.4% of the mean concentration. Expected, however, was a slight decrease of the concentration because some mass is expected to diffuse into the region outside the measuring volume. This increase must be caused by distortions in the measurements, like there is evaporation of water, a temperature decrease and instabilities.
5.6.2 Determination of the diffusion coefficient

In a lot of studies into diffusion processes in liquids and gels, interferometry is used to determine the diffusion coefficient. In particular, the holographic interferometer is used\textsuperscript{25}, that is, the real time holographic interferometer\textsuperscript{26-33}, the double exposure holographic interferometer\textsuperscript{34,35,27,29} and, more recently, the sandwich holographic interferometer\textsuperscript{36,37,38}. Further, an electronic speckle interferometer\textsuperscript{39,40}, a multiple beam interferometer\textsuperscript{41}, a real time phase shifting interferometer\textsuperscript{42} and a scanning beam interferometer\textsuperscript{43} are rather recently proposed. The studies have in common that there is made use of an unsteady state method to find the diffusion coefficient. For this, the initial condition in the concentration distribution is required to be a well-defined step function, e.g. \( c = C \) for \( y < 0 \) and \( c = 0 \) for \( y > 0 \). Fick’s second law in one dimension is solved mathematically\textsuperscript{17} and the solution is used to extract the diffusion coefficient.

In the diffusion experiment presented here, there is no well defined initial condition. Hence, the mathematical solution of Fick’s second law is not known. However, in contrary to the most previously mentioned studies, the experimentally found phase (concentration) distributions are continuous. These continuous distributions are used to extract the diffusion constant directly from Fick’s second law.

The diffusion process is assumed to be 1-dimensional and the diffusion coefficient \( D \) is assumed to be independent of the concentration. The diffusion coefficient is extracted from a pair of successively recorded concentration distributions. From these concentration distributions there is an area used in which the profile is really one-dimensional, that is the area between pixel number \( i = 333 \) and \( i = 432 \) in the \( x \)-direction and between pixel number \( j = 0 \) and \( j = 480 \) in the \( y \)-direction (see Figure 5.6). The diffusion coefficient is calculated by using Equation 5.9. This implies that the result is independent of differences in the overall concentration error between the two concentration distributions. The process is carried out in 4 steps.

1. Extraction of two real one dimensional concentration distributions from the 1-dimensional areas in the two measured concentration distributions by averaging of the concentration over \( x \).

2. Calculation of \( \partial^2 c/\partial y^2 \) for each value of \( y_j \) in the one dimensional phase distribution belonging to the first recording (\( y_j \) is the discrete position in the measuring volume in the \( y \)-direction belonging to pixel number \( j \) in the one-dimensional concentration distribution). For this, a parabola is fitted through the concentrations in the discrete position interval \([y_{j-10}, y_{j+10}]\).

3. Calculation of the averaged value of \( \partial^2 c/\partial y^2 \) for every interval of 20 successive discrete positions in the measuring volume. The maximum averaged value corresponds to the position interval \([y_p, y_{p+19}]\), the minimum averaged value corresponds to the position interval \([y_n, y_{n+19}]\).

4. Calculation of the diffusion coefficient by the application of Equation 5.9 whereby \( N = 20 \).
5. Jet injection into liquid in a cuvette.

Figure 5.6 The area in the concentration distributions used for the calculation of the diffusion coefficient.

<table>
<thead>
<tr>
<th>Conc. distributions</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t=25\text{ min}$</td>
<td>$t=35\text{ min}$</td>
</tr>
<tr>
<td>$t=35\text{ min}$</td>
<td>$t=45\text{ min}$</td>
</tr>
</tbody>
</table>

Table 5.2 The experimentally found values for the diffusion coefficient.

Figure 5.7 The diffusion coefficient for sucrose in water $D$ as a function of the temperature $T$ for $c=0\text{ g/l}$ and $c=10\text{ g/l}$. The diffusion coefficient is calculated from the diffusion coefficient $D_{T=298.15}$ at $T=298.15\text{K}$ by^{24}: $D=\frac{(T/\mu)(\mu/T)}{D_{T=298.15}}$ where $T$ is the temperature and $\mu$ is the viscosity of water. The values of $D_{T=298.15}$ originate from American Institute of Physics Handbook^{44} and the value of $\mu$ as a function of the temperature from CRC handbook of Chemistry and Physics^{45}.
The diffusion coefficient is calculated for two pairs of concentration distributions, i.e., the concentration distributions at $t=25\text{min}$ and $t=35\text{min}$, and the concentration distributions at $t=35\text{min}$ and $t=45\text{min}$. These concentration distributions show clear position intervals in the 1-dimensional area where $\partial^2c/\partial y^2$ is maximum and minimum. The difference between these maxima and minima is relatively large, what makes the value found for $D$ more accurate. The phase gradient at the position of the cameras during the actual measurements does not exceed the maximum gradient where the RTI is designed for too much, it is maximal about $77\cdot10^3\text{rad/m}$. Moreover, the period of time of 10 minutes between the phase distribution is not so large and thus there is made a rather good approximation of $\partial c/\partial t$. The values found for $D$ are presented in Table 5.2.

The difference between the experimentally found values of $D$ is 2.6%. This is less than the difference between the differences found by Gabelmann-Gray and Fenichel[35] who measured the diffusion constant of sucrose in water (10% sucrose solution diffusing in water) every 11 minutes by using an unsteady state method and a holographic interferometer. Figure 5.7 shows the theoretical value of the diffusion constant of sucrose in water as a function of the temperature with the concentration being parameter. In the concentration distributions used for the experimental determination of $D$, the concentration in the used discrete position intervals $[y_p,y_{p+15}]$ and $[y_n,y_{n+10}]$ are of the order 8-12g/l. Let's assume that these concentrations my be approximated by 10g/l. Then, the experimental found values of $D=4.31\cdot10^{-10}\text{m}^2/\text{s}$ and $D=4.42\cdot10^{-10}\text{m}^2/\text{s}$ correspond to a water temperature of about 292K. This is room temperature, what means that the experimentally found values for $D$ are very reasonable.

The foregoing demonstrates the utility of the successively recorded concentration distributions.

### 5.6.3 Comparison to 1-D diffusion theory

It is difficult to compare the measured concentration distributions to those expected by the diffusion theory because the initial conditions are not controlled in the experiments, a part of the liquid lies outside the measuring volume, the influence of the meniscus is unknown and the rate of evaporation can only be estimated. However, the global change in the measured concentration distribution as a function of the time can be verified. For this, the measured concentration distributions are made one-dimensional by averaging in the $x$-direction over the pixels between $i=333$ and $i=432$. Figure 5.8 shows the one-dimensional concentration distributions for $t=15\text{min}$, $t=45\text{min}$, $t=95\text{min}$, $t=245\text{min}$ and $t=395\text{min}$ (straight lines). These concentration distributions are normalized by adding an offset such that the mean concentration corresponds ,after a linear extrapolation at the top over the liquid volume outside the measuring volume (whereby the existence of the meniscus is ignored, but an evaporation of 200$\mu$l/24hr is taken into account), to the
5. Jet injection into liquid in a cuvette.

Figure 5.8 The sucrose concentration $c$ as a function of the distance to the bottom of the cuvette: normalized measurements and calculations.

Theoretical mean concentration. These one-dimensional concentration distributions are compared to those expected by a simplified one-dimensional model.

The theoretical model is based on the solution of Fick's second law in one dimension (Equation 5.6), a solution that is also presented by Carslaw and Jeager to solve a heat diffusion problem\textsuperscript{46}, i.e.,

$$c(y, t+\Delta t) = \int_{-\infty}^{\infty} \frac{c(\varepsilon, t)}{\sqrt{4\pi D \Delta t}} \exp \left[ -\frac{(y-\varepsilon)^2}{4D \Delta t}\right] \, d\varepsilon \quad (5.11)$$

This equation relates the concentration distribution at time $t+\Delta t$ to that at time $t$. In the model there is assumed that:
- the initial condition is equal to the measured (and extrapolated) concentration distribution at $t=15\text{min}$;
- $D$ is equal to the averaged value found in Section 5.4.2, i.e., $D=4.36 \cdot 10^{-10}\text{m}^2/\text{s}$;
- The rate of evaporation is equal to 200$\mu\text{l}/24\text{hr}$.

The concentration distributions are numerically found by an iterative application of Equation 5.11, whereby $\Delta t=1\text{s}$. To avoid a rapid mass transfer towards regions outside the liquid, the concentration distributions are expanded with 110 pixels long areas at the
top and at the bottom of constant concentrations. Every iteration step, the concentration here is made equal to that at the top and at the bottom of the concentration distribution. Mass diffused into these areas is transferred back towards the borders of the concentration distribution each step. The calculated concentration distributions for $t=45\text{min}$, $t=95\text{min}$, $t=245\text{min}$ and $t=395\text{min}$ are presented in Figure 5.8 (dotted lines).

The figure shows that the global change in the measured and calculated concentration distribution as a function of the time agree. As long as $t<45\text{min}$ the curves coincide. After that, a discrepancy arises. At the bottom of the cuvette a flattening or a slight decrease in concentration is measured where the theory expects a slight increase. This is possibly caused by two-dimensional effects or a distortion of the measurement. At the top of the cuvette, the theory predicts a lower concentration than measured. Possibly, this is the result of the existence of the meniscus that hinders the mass diffusion what is not included in the theory. Apart from that, it can be the result of an offset error because the shape of the measured and the theoretical distributions are almost equal at the top of the cuvette.

### 5.6.4 Homogeneity

To demonstrate the capability of interferometry for a homogeneity analysis, the homogeneity of the dilute sucrose solution during the diffusion process is determined as a function of the time. For this, the fractional standard deviation $D_{v}$ and the fractional rms concentration gradient $F$ are calculated, whereby $\sigma_{c}$ and $\sqrt{\langle (\nabla c)^2 \rangle}$ are extracted from the measured concentration distributions and $\langle c \rangle$ is set equal to the theoretical value of 8.333$g/l$. In the analysis, only those pixels are taken into account which show a well defined concentration in all the concentration distributions. Figure 5.9 presents $D_{v}$ as a function of the time and Figure 5.10 presents $F$ as a function of the time.

The fractional standard deviation obtained by interferometry appears to be a good measure for homogeneity: $D_{v}$ decreases almost exponentially with the time from $D_{v}=0.27$ at $t=5\text{min}$ to $D_{v}=0.016$ at $t=874\text{min}$ and it tends to decrease to zero for larger values of $t$. The fractional rms concentration gradient tends also to decrease to zero for large values of $t$, but it appears to be more sensitive to the presence of large gradients. These large gradients are present if $t$ is small, in this experiment not due to the existence of a lamellar structure but due to the existence of a high concentration at the bottom of the cuvette. In contrary to $D_{v}$, $F$ as a function of $t$ shows a clear change in the gradient between $t=35\text{min}$ and $t=95\text{min}$. Hence, $F$ is a convenient measure to detect the presence of large concentration gradients and, more important, to detect the moment of disappearance of these gradients.
Table 5.3 The values of $V_{\text{jet}}/V_{\text{bulk}}$, $V_{\text{jet}}$ and $V_{\text{bulk}}$ used in the experiments and the mean concentration after the jet injection.

<table>
<thead>
<tr>
<th>$V_{\text{jet}}/V_{\text{bulk}}$</th>
<th>$V_{\text{jet}}$ (µl)</th>
<th>$V_{\text{bulk}}$ (µl)</th>
<th>(c) (g/l)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.11</td>
<td>100</td>
<td>900</td>
<td>2.5</td>
</tr>
<tr>
<td>0.25</td>
<td>200</td>
<td>800</td>
<td>5</td>
</tr>
<tr>
<td>0.67</td>
<td>400</td>
<td>600</td>
<td>10</td>
</tr>
<tr>
<td>1.5</td>
<td>600</td>
<td>400</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>800</td>
<td>200</td>
<td>20</td>
</tr>
</tbody>
</table>

measurement is made per experiment after the actual measurements, when the solution is homogenized by carefully stirring. The interferograms are processed as described in Section 5.4.2. A number of the concentration distributions is calculated, whereby the concentration distribution at $t=558$ s is used as reference. Figure 5.14 shows the concentration distributions as a function of the time for the case that $V_{\text{jet}}/V_{\text{bulk}}=0.11$. For completeness, there is an increase of the measured mean concentration as a function of the time established in all five experiments. This increase differs per measurement and is possibly caused by temperature effects and instabilities (Section 5.5.2). It is maximal 6%.

5.7.2 Results and conclusions

The concentration distributions are used to analyze the homogeneity. For this, the fractional standard deviation $D_v$ and the fractional rms concentration gradient are calculated by the application of the Equations 5.4 and 5.5, whereby there is only made use of those pixels that have a well defined concentration in all the concentration distributions of the experiment.

Figure 5.11 presents $D_v$ as a function of $V_{\text{jet}}/V_{\text{bulk}}$ for $t=18$ s and $t=558$ s. At $t=18$ s, the dilute sucrose solution has become homogeneously enough to assume that a linearly stratified medium is not a bad approximation. Then, the measured concentration distribution is reliable and, hence, also $D_v$. At $t=558$ s the latest recording is made and the homogenization has certainly become a diffusion process induced by a large-scale concentration gradient. The figure shows that at $t=18$ s $D_v$ decreases as a function of $V_{\text{jet}}/V_{\text{bulk}}$ from $D_v=0.39$ for $V_{\text{jet}}/V_{\text{bulk}}=0.11$ until $D_v=0.016$ for $V_{\text{jet}}/V_{\text{bulk}}=0.67$. For larger values of $V_{\text{jet}}/V_{\text{bulk}}$, $D_v$ remains more or less constant. Only if $V_{\text{jet}}/V_{\text{bulk}}<0.67$ there is a small reduction of $D_v$ visible as a function of the time.

Figure 5.12 presents $F$ as a function of the time with $V_{\text{jet}}/V_{\text{bulk}}$ being the parameter. For all the values of $V_{\text{jet}}/V_{\text{bulk}}$, $F$ appears to be a decreasing function of the time and the curves show a clear change in gradient between $t=78$ s and $t=198$ s. After $t=198$ s, $F$ becomes only a weak function of the time. For $V_{\text{jet}}/V_{\text{bulk}} \geq 0.67$, $F$ even disappears in the background of
Figure 5.11 The fractional standard deviation $D_v$ as a function of the relative jet volume $V_{jet}/V_{bulk}$ for $t=18s$ and $t=558s$. The numbers represent the error based on Table 5.1.

Figure 5.12 The fractional rms concentration gradient as a function of the time for different values of the relative jet volume $V_{jet}/V_{bulk}$.

1.4/(c). This means that the local concentration gradients and the large-scale gradient are disappeared. For $t<78s$, there must certainly be taken into account the existence of local concentration gradients.

The combination of the results of the Figures 5.11 and 5.12 lead to the following characteristics for the dilute sucrose solution after a jet injection:
- if $V_{jet}/V_{bulk} \geq 0.67$, the sucrose solution is almost homogeneous at $t=18$s, but local gradients are certainly present until $t=78$s;
- if $V_{jet}/V_{bulk} < 0.67$, the dilute sucrose solution is not homogeneous and it must become homogeneous by a large-scale diffusion process; local gradients remain certainly present until $t=78$s.

To understand these characteristics, the concentration distributions for the different values of $V_{jet}/V_{bulk}$ at $t=18$s, $t=78$s, $t=198$s and $t=558$s are presented in the Figures 5.19-5.23. Here, a smooth variation from black to white corresponds to a increase in concentration of $0.23$g/l (corresponding to a phase increase of $2\pi$ radians in the phase distribution). There must be noted that the mean sucrose concentration is proportional to $V_{jet}/(V_{jet}+V_{bulk})$. That is, since $V_{jet}+V_{bulk}$ is equal in all the experiments, the mean concentration is proportional to $V_{jet}$. Hence, the number of fringes (smooth variation from black to white) in the concentration distribution belonging to the same degree of homogenization, i.e., the same values of $D_v$ and $F$, increases with $V_{jet}/V_{bulk}$.

The figures show that if $V_{jet}/V_{bulk}=0.11$ or $V_{jet}/V_{bulk}=0.25$, the concentration distribution at $t=18$s contains a vertical large-scale gradient which is locally distorted. Moreover, there is an additional distortion near the walls of the cuvette. Although these distortions hardly influence the standard deviation (this is determined by the large-scale concentration gradient) and $D_v$, there are locally large gradients which result in a high value of $F$. Under the assumption that the convection is damped, these large local gradients induce a mass flux by diffusion. Since the mass transport is relatively large and only over a short distance, the local distortions are disappeared long before the large-scale gradient is disappeared. At $t=78$s, the local distortion is disappeared and the large-scale gradient presents itself by almost straight horizontal fringes. Only near the walls a distortion remains which is visible in the bending of the fringes. As a consequence, $F$ is reduced. At $t=198$s, also the distortion near the walls is disappeared. The further homogenization is now a diffusion process induced by the large-scale gradient. This process is very slow and the concentration distribution hardly changes between $t=198$s and $t=558$s. $F$ becomes almost constant. Since the change of the large-scale gradient is small during the whole experiment, the total decrease of $D_v$ is small.

If $V_{jet}/V_{bulk}=0.67$, $V_{jet}/V_{bulk}=1.5$ or $V_{jet}/V_{bulk}=4$ there is a reasonable mixing during and just after the jet injection. Only a small large-scale gradient arises if $V_{jet}/V_{bulk}=0.67$ and $V_{jet}/V_{bulk}=1.5$. Here must be kept in mind that the mean concentration is high, so the solution is almost homogeneous. An even more homogeneous solution is obtained if $V_{jet}/V_{bulk}=4$. A distortion near the walls is visible in all the concentration distributions until $t=78$s. Just like in the cases that $V_{jet}/V_{bulk}=0.11$ and $V_{jet}/V_{bulk}=0.25$, this distortion vanishes between $t=78$s and $t=198$s. This results in a decreasing value of $F$ as a function of the time until $t \approx 198$s after which $F$ becomes (almost) zero. Because the solution is almost homogeneous since $t=18$s already, $D_v$ remains constant as a function of the time.
The results described above lead to the conclusion that to obtain a reasonable homogeneous concentration distribution at \( t = 18 \)s, it is required that \( V_{\text{jet}}/V_{\text{bulk}} \geq 0.67 \). However, some distortion near the walls is present which disappears after a course of time between 78s and 198s. Figures 5.19-5.23 show that the distortion spans maximally about 5 fringes for all values of \( V_{\text{jet}}/V_{\text{bulk}} \geq 0.67 \). This corresponds to a decrease in concentration of maximal 1.1g/l, that is 11.5% of the mean concentration if \( V_{\text{jet}}/V_{\text{bulk}} = 0.67 \), 7.7% if \( V_{\text{jet}}/V_{\text{bulk}} = 1.5 \) and 5.7% if \( V_{\text{jet}}/V_{\text{bulk}} = 4 \). There must be taken into consideration, however, that the phase distribution (and so the concentration distribution) obtained by the RTI is an average over the propagation length of the test beam through the cuvette. Hence, the distortion may actually not be extended along the whole wall. It is conceivable that there is only a distortion in the edges between the walls and the windows of the cuvette. This local distortion will be considerable larger than the errors mentioned above.

### 5.8 Discussion

The experiments in this chapter showed that the RTI is a suitable tool for measuring medium rate concentration distribution changes as a function of the time. By demonstrating the possibility to calculate the diffusion coefficient of a dissolved compound from two successively measured concentration distributions, it was not only shown that the data do describe the physics quite well, but it was also shown that the long term stability of the set-up is appropriate for quantitative measurements. This statement is further supported by the fact that the measured concentration distributions in the diffusion experiment agree, after normalization, quite well with the simplified theory, especially if the measuring period is shorter than 30min. The shift in offset during a sequence of measurements may be expected to originate mainly from temperature effects, effects that must be controlled if necessary.

After the injection of a jet of dilute sucrose solution into a sucrose free bulk in a cuvette, there are two phases of homogenization: first a phase of rapid, flow induced homogenization, then a much longer phase of slow, diffusion induced homogenization. The first phase goes on for a period of time of the order of 20s, the second period depends on the large-scale gradient and can last as long as several hours. If \( V_{\text{jet}}/V_{\text{bulk}} \) is small, the flow induced homogenization is small (\( D_{x} = 0.39 \) if \( V_{\text{jet}}/V_{\text{bulk}} = 0.11 \)) and the further homogenization is a diffusion process. However, as soon as \( V_{\text{jet}}/V_{\text{bulk}} \geq 0.67 \), the flow causes almost a total homogenization (\( D_{y} < 0.02 \)). Only (in some regions) near the walls a larger deviation in the concentration is visible until about 3 minutes after the injection, that is, a mean deviation up to 11.5%.

Trying to interpret the consequences of the findings in terms of the microplate format that is used for immuno assays, there may generally be assumed that the time scales of flow induced homogenization and diffusion induced homogenization are similar to those found
in the experiments. Also the degree of homogenization may be expected to be similar, provided that the densities and the viscosity of the jet fluid and the bulk fluid are similar. This implies that for a good jet flow induced homogenization $V_{\text{jet}}/V_{\text{bulk}}$ must be larger than about 0.67. Possibly, the degree of homogenization near the walls is better than found in the experiments because the wells in the microplate are not rectangular but cylindrical, and thus this area is better accessible for flow. However, the influence of the lower concentration near the walls on the result of an immuno assay will be limited. The total time scale of such assays, generally between 30 and 90 minutes, is much longer than the period of about 3 minutes that the deviation in concentration exists.

In the experiments, a jet of high density is injected into a bulk of low density. In immuno assays, the density difference between jet and bulk may vary from positive to negative whereas the density of patient material may differ from patient to patient, and the density of reagents may vary from assay to assay. Even though there were no experiments performed here in which the density of the jet is lower than the density of the bulk, the result can be guessed: at high injection speeds the momentum will be high enough to initiate mixing, at low speeds the low density fluid will lay down for a great part on top of the high density fluid.

In the use of immuno assays, the injection flow rates are relatively low in order to prevent 'splashing' and thereby prevent potential contamination of nearby other wells. This will result in relatively poor mixing. Further research into this problem is recommended.

In summary: the interferometer does allow quantitative analysis of mixing of fluids in industrially relevant time and length scale. It clearly shows the possibilities and the limitations of jet based homogenization. In terms of immuno assay applications, the study once more confirmed the need of homogenization. A homogenization that can be realized by a suitable choice of $V_{\text{jet}}/V_{\text{bulk}}$ if the density of the jet is higher than the density of the bulk.
Figures

Figure 5.13 A visualization of the mixing process induced by the injection of a jet of 100μl dyed sucrose solution (25g/l) into a bulk of 900μl. Injection speed: 0.1ml/s.
Figure 5.13 continued.
Figure 5.14 The concentration distribution (phase distribution) as a function of the time after an injection of a jet of 100μl sucrose solution (25g/l) into a bulk of 900μl. Injection speed: 1ml/s. Interval range: 0.23g/l (2π rad).
Figure 5.15 The phase distribution as a function of the time if no jet is injected (shifted over π rad). Interval range: 2π rad.
Figure 5.16 The phase distribution as a function of the time after the injection of a sucrose free jet of 200μl into a bulk of 1000μl. Interval range: $2\pi$ rad.
5. Jet injection into liquid in a cuvette.

$t=5\text{min}$

$t=15\text{min}$

$t=25\text{min}$

$t=35\text{min}$

$t=45\text{min}$
Figure 5.17 (left and right page) The concentration distribution (phase distribution) as a function of the time induced by the diffusion of sucrose. Interval range: 0.23g/l (2\pi rad).
5. Jet injection into liquid in a cuvette.

$t=5\text{min}$

$t=15\text{min}$

$t=25\text{min}$

$t=35\text{min}$

$t=45\text{min}$
Figure 5.18 (left and right page) Quasi-3-dimensional plots of the concentration distributions in Figure 5.17.
Figure 5.19 The concentration distributions (phase distributions) at $t=18s$, $t=78s$, $t=198s$ and $t=558s$ after the injection of a jet of sucrose solution (25g/l) into a bulk, whereby $V_{jir}=100\mu l$ and $V_{bulk}=900\mu l$ ($V_{jir}/V_{bulk}=0.11$). Interval range: 0.23g/l ($2\pi$ rad).
Figure 5.20 The concentration distributions (phase distributions) at $t=18s$, $t=78s$, $t=198s$ and $t=558s$ after the injection of a jet of sucrose solution (25g/l) into a bulk, whereby $V_{jet}=200\mu l$ and $V_{bulk}=800\mu l$ ($V_{jet}/V_{bulk}=0.25$). Interval range: 0.23g/l (2\pi rad).
Figure 5.21 The concentration distributions (phase distributions) at $t=18s$, $t=78s$, $t=198s$ and $t=558s$ after the injection of a jet of sucrose solution (25g/l) into a bulk, whereby $V_{jet}=400\mu l$ and $V_{bulk}=600\mu l$ ($V_{jet}/V_{bulk}=0.67$). Interval range: 0.25g/l (2$\pi$ rad).
Figure 5.22 The concentration distributions (phase distributions) at $t=18s$, $t=78s$, $t=198s$ and $t=558s$ after the injection of a jet of sucrose solution (25g/l) into a bulk, whereby $V_{\text{jet}}=600\mu l$ and $V_{\text{bulk}}=400\mu l$ ($V_{\text{jet}}/V_{\text{bulk}}=1.5$). Interval range: 0.23g/l ($2\pi$ rad).
5. Jet injection into liquid in a cuvette.

Figure 5.23 The concentration distributions (phase distributions) at $t=18s$, $t=78s$, $t=198s$ and $t=558s$ after the injection of a jet of sucrose solution ($25g/l$) into a bulk, whereby $V_{\text{jet}}=800\mu l$ and $V_{\text{bulk}}=200\mu l$ ($V_{\text{jet}}/V_{\text{bulk}}=4$). Interval range: $0.23g/l$ ($2\pi$ rad).
References


Conclusions and suggestions

6.1 Conclusions

A real time Michelson interferometer (RTI) has been developed for quantitative measurements of 2-dimensional refractive index profiles. It concerns an experimental set-up based on the spatial phase stepping method, whereby three phase stepped interferograms are recorded in three separate channels by three separate cameras. The phase step of \(\pi/2\) radians between the interferograms is performed by using the polarization properties of light. The time resolution is video rate and the integration time is 10 ms.

Experiments with the RTI consist of a reference measurement followed by a sequence of actual measurements. The reference measurement corrects the phase distribution for undesired contributions due to refractive index profiles in the windows bounding the medium of interest. This makes the RTI a convenient tool for measuring a compressible flow in a wind tunnel or a diffusion process in a cuvette. The phase stepping method is robust, accurate and it can deal with the typical interferograms obtained in high speed aerodynamic research where the fringe order is unknown, the fringe spacing varies strongly, the fringes are probably discontinuous and which contain obscure areas due to the presence of an object. Although the time resolution is too low to follow turbulent processes, it is a suitable tool for measuring large-scale fluctuations.
The measuring accuracy of the RTI depends on the phase gradient during the actual measurement and the reference measurement. It decreases if one of these gradients increases. The RTI is validated for the case that the phase distributions are one-dimensional and linear, containing a small gradient during the reference measurement and a gradient up to 58·10^3 rad/m at the position of the cameras (corresponding to about 75 fringes over the chip length). The phase distributions are generated by a tilted mirror at the position of the virtual object plane in the test section. Then, the reflected rays traverse approximately the same trajectories through the imaging optics as they would in case of a linearly stratified medium. The validation leads to the following conclusions:

1. The rms phase error induced by the RTI is maximal 2.8%-4.3% of 2π radians larger than the rms induced by the temporal phase shifting interferometer which is also build in in the set-up. The absolute maximum phase difference is 12.1% of 2π radians. After filtering by a linear filter with a window size of 7×7 pixels, the difference in the rms phase error is reduced to 1.8%-2.3% of 2π radians and the maximum absolute phase difference is reduced to 5.6 of 2π radians.

2. Instabilities in the set-up lead to a standard deviation in the profiles of less than 1% of 2π radians (maximum absolute phase difference 2.1% of 2π radians). If succeeding phase distributions are not normalized, the total rms phase error is less than 6.5% of 2π radians (maximum absolute phase error 9.0%). After normalization, this is reduced to 1.4% of 2π radians (3.5% of 2π radians).

3. The total rms phase error in a typical wind tunnel experiment (entrance refractive index \( n_e = 1.0 \), window thickness \( t = 20 \text{mm} \), medium length \( L = 150 \text{mm} \)) is about 4.6%-13.1% of 2π radians (maximum absolute phase error 11.0%-27.5% of 2π radians) as long as the medium is locally linearly stratified and the fringes are homogeneously distributed and not more or less circular about the optical axis. This corresponds to a maximum rms density error of 1.2 g/m³, that is 0.10% of the air density at standard conditions.

4. For the cuvette measurements (entrance refractive index \( n_e = 1.33 \), window thickness \( t = 1.25 \text{mm} \), medium length \( L = 10 \text{mm} \)), the total rms phase error is about 4.6%-9.7% of 2π radians (maximum absolute phase error 11.0%-22.3% of 2π radians). Under the condition of a perfect temperature stability, this corresponds to a rms error in the sucrose concentration up to 2.2·10^{-2} g/l (maximum absolute 5.2·10^{-2} g/l).

5. The application of a linear filter with a window size of 7×7 pixels reduces the rms phase error with about 0.9-1.6% of 2π radians.

6. After a realignment of the cameras in the set-up, the accuracy of the RTI is reproduced within a few tenths of a percent of 2π radians.

In the case of a large gradient during the actual measurement, the phase distributions show some deformation. A slight curvature arises in the direction perpendicular to the fringes. Removal of this deformation gives rise to a reduction in the rms phase error, possibly up to 4.8% of 2π radians (in case of more or less circular fringes more). The remaining rms phase error of 1.8%-5.6% of 2π radians (exclusive the phase errors due to the limited imaging accuracy, aberrations in the test beam and instabilities) is in agreement with the theoretically expected phase error.
The most important error sources in the RTI are the limited perfection of the alignment of the polarizing elements (rms 1.1% of 2\pi radians), the deformation (only for large gradients, rms < 4.8% of 2\pi radians in absence of more or less circular fringes), reflections (rms < 2% of 2\pi radians), the misalignment of the cameras (only for large gradients, rms < 2.3% of 2\pi radians), the limited stability (rms < 1.4% of 2\pi radians if normalized), the limited flatness of the wavefront (if propagation length is large, e.g. in wind tunnel experiments < 5.7% of 2\pi radians) and possible non-linearities in the refractive index profile.

The RTI has proven to be a convenient tool for repetitive quantitative refractive index profile measurements in an industrially relevant time scale. This is demonstrated by measuring the diffusion process of sucrose in a phosphate buffer: the recorded concentration distributions are sufficient accurate to abstract a very reasonable diffusion coefficient even if the initial condition of the process is unknown.

The RTI is applied to study jet based homogenization after a jet of sucrose solution is injected into a bulk of phosphate buffer in a small cuvette. This in order to mimic the homogenization in immuno assays. At a jet velocity of 1ml/s, a reasonable homogenization (fractional standard deviation D_v < 0.02, measured by interferometry) within a period of 18s is realized if the relative jet volume V_{jet}/V_{bulk} ≥ 0.67. In a square cuvette, a local concentration near the walls remains certainly until 78s after the jet injection. However, this will hardly influence the result of an immuno assay since the time scale of these assays is generally 30-90min.

6.2 Suggestions

Considerable contributions to the total phase error induced by the RTI originate from the deformation in the phase distributions and the wavefront aberration in the test beam (if L is large and/or \(n_v\) is small). These contributions can tried to be reduced. However, this is only necessary if an application requires a better measuring accuracy than presented in this thesis. Often, this will not be the case. If, nevertheless, a better accuracy is desired, then it is recommended to replace the lenses with a short focal length by lenses with a longer focal length, since highly curved lenses introduce greater Seidel aberrations. This means the replacement of the lenses L2 and/or L0, and, in order to keep the optical magnification unchanged, also the lenses L3 and/or L1. A disadvantage is, however, that the spatial extension of the experimental set-up increases since an imaging system of weak lenses is more voluminous. A dominant aberration is expected to be distortion because this aberration is proportional to the aperture. It results in a misshaping of the image with respect to the object by a radially displacement of each image point. In interferometry, it results in a similar distortion of the phase distribution. It can be overcome by a suitable correction in the computer analysis.
During the alignment of the RTI, a lot of time is spent by the fine adjustment of the CCD-cameras. Especially the alignment for the translations in the x-, y- and z-direction is time consuming. Moreover, the alignment of the translations in the x- and y-direction must be repeated in the beginning of every day of experiments. Therefore, it is recommended that this process is computerized as long as the RTI is still an experimental set-up. For this, the manually operated micrometers in the translation stages must be replaced by computer controlled piezo-electric transducers. The grating used during the fine alignment must be replaced by a glass plate containing a pattern which makes the correlation between the images on the different cameras dependent on the translation in all three directions, for example, a suitable grid or a suitable dotted structure. The iterative process can now be computerized whereby all translations of two cameras are adjusted with respect to the third camera such that the value of the correlation coefficients are optimal.

To let the time resolution increase, the RTI can be equipped with higher speed cameras. High speed cameras with a framing rate up to 1Mhz are available. If not a too high framing rate is desired, CCD or CMOS-cameras can be used, for example. CMOS-cameras are cheaper than CCD-cameras and they have a higher framing rate. However, their light sensitivity is lower and they are more susceptible to noise. Typical framing rates are up to 1kHz for a full frame and up to over 32kHz for a reduced frame. The minimum integration time lies between 2μs and 10μs. As a consequence of a short integration time, the power of the laser beam must be increased. If the integration time is 10μs, the required power is, dependent on the exact sensitivity of the cameras, of the order 35mW or more.

To visualize the instantaneous structure of an unsteady (turbulent) aerodynamic flow, the integration time should be sufficiently small to record the flow without image blur. For this, Δt should be of the order $l_{min}/v$, where $l_{min}$ is the minimum resolvable length scale in the flow and $v$ is the convection velocity. The value of $l_{min}$ is limited by the smallest physical scale present in the flow and the resolution of the measurement technique. The latter sets a practical limit on $l_{min}$ since the resolution can never be better than the pixel size. Hence $l_{min} \geq p/G$ where $p$ is the pixel distance and $G$ is the optical magnification in the set-up. Combining the above results, an instantaneous visualization requires an integration time of the order $p/vG$. For a flow with a Mach number 2, the typical convection velocity is 500m/s and there follows for $p = 12\mu m$ that $\Delta t < 40\mu s$ if $G = 0.6$ and that $\Delta t < 400\mu s$ if $G = 0.06$. Integration times that satisfy these conditions can be generated by using a pulsed laser. However, there must be noticed that the replacement of the HeNe-laser by a pulsed laser of a different wavelength has to be accompanied by the replacement of other elements in the set-up. The measuring accuracy depends on a high extinction coefficient of the polarizing beam splitters. Polarizing beam splitters which split the beam into two perpendicularly oriented beams both with an extinction ratio of 1000:1 are only available for a few wavelengths.
The real time interferometer is an experimental set-up. This means that the set-up is voluminous and weighty, and that the set-up cannot be moved without distortion of the alignment. Possibly, a displaceable set-up can be realized in the future so that it becomes easier to install the interferometer around large objects like wind tunnels. Especially a further mechanical development is needed. The measuring accuracy of such a displaceable interferometer depends strongly on the realized mechanical stability and accuracy, and also on the allowed effort needed to realign the set-up after a displacement. Alternatively, a set-up as presented in this thesis can be placed somewhere permanently. It can be used for measurements to relatively small objects which can be moved to the RTI, test objects for diffusion experiments, for example.
6. Conclusions and suggestions
Appendix

Spherical aberration and defocussing in the test beam

The RTI shows a small aberration in the test beam, seaming a combination of spherical aberration and defocussing (outside the focus). Such an aberration is axisymmetric about the optical axis and is given by\(^1\):

\[ W(r) = Ar^4 + Dr^2, \quad (A.1) \]

where \( r^2 = x^2 + y^2 \). \( A \) is the spherical aberration coefficient (positive sign) and \( D \) is the defocussing coefficient (negative sign). The aberration is shown in Figure A.1a. In Figure A.1b the aberration and its gradient in radial direction \( \partial w/\partial r \) are presented as a function of \( r \). The aberration is characterized by a maximal negative gradient \( \partial W/\partial r = -W^\prime_0 \) at \( r = r_0 = \sqrt{-D/6A} \), whereby \( W^\prime_0 \) is given by:

\[ W^\prime_0 = \sqrt{-8D^3/27A}. \quad (A.2) \]

At \( r = R_0 = \sqrt{-2D/3A} \) the gradient is equal to that at \( r = r_0 \) but with opposite sign, i.e., \( \partial W/\partial r = + W^\prime_0 \).

The phase error in Equation 3.99 depends only on the ray shift \( y_w - y_w^* \) introduced by the change in the refractive index gradient between the actual and the reference measurement.
in presence of a wave front aberration and is independent of a possible oblique ray trajectory. Hence, it can also be used for a calculation of two-dimensional phase error distributions provided that the wavefront aberration $W(x,y)$ and the refractive index gradients $n'(x,y)$ and $n_a'(x,y)$ (and thus $\Delta n'(x,y)$) are known. Under the assumption that $n_a'(x,y) \approx 0$ and the size $n'(x,y)$ is spatially constant (not necessarily its direction) follows:

$$\Delta \alpha(x,y) = \frac{2\pi}{\lambda} \left[ \frac{2}{3} \frac{L^2}{n_e} + \frac{tL}{n_g} + \frac{1}{2} \frac{aL}{n_a} \right] W'(x,y)n' ,$$  \hspace{1cm} (A.3)$$

where $W(x,y)$ is the gradient of $W(x,y)$ in the direction of $n'$. For the rms phase error $\Delta \alpha_{rms}$ and the maximum absolute phase error $|\Delta \alpha|_{max}$ follows:

$$\Delta \alpha_{rms} = \frac{2\pi}{\lambda} \left[ \frac{2}{3} \frac{L^2}{n_e} + \frac{tL}{n_g} + \frac{1}{2} \frac{aL}{n_a} \right] W_{rms}'n' ,$$  \hspace{1cm} (A.4)$$

and

$$|\Delta \alpha|_{max} = \frac{2\pi}{\lambda} \left[ \frac{2}{3} \frac{L^2}{n_e} + \frac{tL}{n_g} + \frac{1}{2} \frac{aL}{n_a} \right] |W'|_{max}n' .$$  \hspace{1cm} (A.5)$$

Now, the calculation of $\Delta \alpha_{rms}$ and $|\Delta \alpha|_{max}$ is reduced to the calculation of $W_{rms}'$ and $|W'|_{max}$, i.e., the rms gradient and the maximal absolute gradient in the wavefront
aberration in the direction of \( n' \).

Let's calculate \( W'_{\text{rms'}} \) and \( |W|_{\text{max}} \) for the aberration of Equation A.1 for the cases that \( n' \) is directed in the radial direction (resulting in circular fringes), in the \( x \)-direction (resulting in vertical fringes) and in the \( y \)-direction (resulting in horizontal fringes). The measuring field is presented in Figure A.2. It is extended from \( x = -B_r \) to \( x = +B_r \) and from \( y = -H_r \) to \( y = +H_r \). For a gradient in the wavefront aberration in the radial direction, \( x \)-direction and the \( y \)-direction holds:

\[
W'(r) = \frac{\partial W(r)}{\partial r} = 4Ar^3 + 2Dr, \tag{A.6a}
\]

\[
W'(x,y) = \frac{\partial W(x,y)}{\partial x} = 4Ax^3 + 4Ay^2x + 2Dx \tag{A.6b}
\]

and

\[
W'(x,y) = \frac{\partial W(x,y)}{\partial y} = 4Ay^3 + 4Ax^2y + 2Dy, \tag{A.6c}
\]

respectively. A straightforward averaging of \( W'^2(x,y) \) in the radial direction, the \( x \)-direction and the \( y \)-direction over the measuring field yields:

\[
w'_{\text{rms'}} = 54w_r^2 \left\{ \left[ \frac{1}{3}B^2 \right] - \left[ \frac{1}{5}B^2H^2 \right] \left[ \frac{-D}{\lambda} \right]^2 + \left[ \frac{1}{7}B^2 \right] \left[ \frac{1}{5}B^2H^2 \right] \left[ \frac{-D}{\lambda} \right]^2 + \frac{1}{12} \left[ B^2 + H^2 \right] \left[ \frac{-D}{\lambda} \right]^3 \right\}. \tag{A.7a}
\]
\[ w_{rs} = 54 w_0^2 \left[ \frac{1}{7} B_1^2 + \frac{2}{T_5} B_1 H_1^2 + \frac{1}{T_3} B_1 H_1 \right] \left[ \frac{D}{\lambda} \right]^3 - \left[ \frac{1}{3} B_1^2 + \frac{1}{9} B_1 H_1 \right] \left[ \frac{D}{\lambda} \right]^2 + \frac{1}{7} B_1 \left[ \frac{D}{\lambda} \right]^{-1} \]  

(A.7b)  

\[ w_{x} = 54 w_0^2 \left[ \frac{1}{7} H_1^2 + \frac{2}{T_5} H_1 B_1^2 + \frac{1}{T_3} H_1 B_1 \right] \left[ \frac{D}{\lambda} \right]^3 - \left[ \frac{1}{3} H_1^2 + \frac{1}{9} H_1 B_1 \right] \left[ \frac{D}{\lambda} \right]^2 + \frac{1}{7} H_1 \left[ \frac{D}{\lambda} \right]^{-1} \]  

(A.7c)  

respectively, where \( w_0 \) is given by Equation A.2. The equations show that the rate \( W_{rms} / W_0 \) depends only on the field size and the rate \(-D/A\). Figure A.3 shows \( W_{rms} / W_0 \) as a function of \(-D/A\) for the case that \( B_f = 6.83 \text{mm} \) and \( H_f = 5.25 \text{mm} \) (corresponding to the measuring field of the RTI). Also \( |W'\max|/W_0 \) is presented, which is found by a numerical evaluation of the Equations A.6abc after substitution of Equation A.2. In order to facilitate the interpretation, also the values of \( r_0^2 \) and \( R_0^2 \) which are proportional to \(-D/A\) are presented.

For the interpretation of the curves in Figure A.3, let’s take a look at \( W_{rms} / W_0 \) and \( |W'\max|/W_0 \) for circular fringes. For small values of \(-D/A\) the rate \( W_{rms} / W_0 \) is large. Then, \( R_0 \) is small and the large gradients near the border of the aberrated wavefront lie inside the measuring field. The rate \( W_{rms} / W_0 \) decreases if \(-D/A\) increases because \( R_0 \) increases and thus more of the large gradients lie outside the measuring field. This decrease comes to an end if the border of the measuring field lies somewhere between \( r_0 \) and \( R_0 \). Next, \( W_{rms} / W_0 \) increases slightly because small gradients are leaving the measuring field. After that, relative large gradients leave the measuring field and thus \( W_{rms} / W_0 \) decreases again. The rate \( |W'\max|/W_0 \) is also large for small values of \(-D/A\). It decreases with \(-D/A\) until \( R_0 \) lies outside the measuring field. Then \( |W'\max|/W_0 \) becomes equal to 1 because \( |W'\max| \) is determined by the gradient at \( r_0 \) which lies still inside the measuring field. As soon as \(-D/A\) is further increased and also \( r_0 \) lies outside the measuring field \( |W'\max|/W_0 \) further decreases.

The interpretation for the vertical and horizontal fringes is similar, only the essential values of \(-D/A\) are slightly shifted. The values of \( W_{rms} / W_0 \) are considerably lower because the values for \( W(x,y) \) are lower at several positions \( (x,y) \). The curve lies lower for horizontal fringes than for vertical fringes. Possible sloping fringes would lead to curve somewhere between these two curves.

Figure A.3 shows that if \(-D/A > 1.1 \cdot 10^4 \text{m}^2\), \( |W'\max| \leq W_0 \), and \( W_{rms} \leq 0.86 W_0 \) (circular fringes), \( W_{rms} \leq 0.66 W_0 \) (vertical fringes) and \( W_{rms} \leq 0.55 W_0 \) (horizontal fringes). This means by using Equation A.4 and A.5 that \( |\Delta \alpha| \max \leq 0.86 \Delta \alpha_{rms} \) (circular fringes), \( |\Delta \alpha| \max \leq 0.66 \Delta \alpha_{rms} \) (vertical fringes) and \( |\Delta \alpha| \max \leq 0.55 \Delta \alpha_{rms} \) (horizontal fringes). In case of a linearly stratified medium the rms phase error is maximum if the fringes are vertically oriented and thus holds \( |\Delta \alpha| \max \leq 0.66 \Delta \alpha_{rms} \). In case of a locally linearly
stratified medium also circularly shaped fringes can appear and thus holds $|\Delta \alpha|_{\text{max}} \leq 0.86 |\Delta \alpha|_{\text{refr}}$. A global measurement with a lateral shear interferometer learns that in the RTI $A=90m^3$ and $D \approx -0.021m^1$, and thus $W_0' = 1.7 \cdot 10^4$ and $-D/A \approx 2.3 \cdot 10^4 m^2$. Substitution of these values in Equations A.2 and A.5 lead to an estimated values of $|\Delta \alpha|_{\text{max}}$. For the case that the test object is a wind tunnel configuration or a cuvette configuration (for the details is referred to Section 2.4), the object is positioned against the window ($a=0$) and the wavelength is $\lambda = 632.8 nm$, the maximum absolute phase errors are equal $|\Delta \alpha|_{\text{max}} = 29.5 n'$ radians and $|\Delta \alpha|_{\text{max}} = 0.10 n'$ radians, respectively.

Reference

**Samenvatting**

**Een real time Michelson interferometer voor kwantitatieve brekingsindexprofiel metingen**  
**Ontwerp, validatie en utilisatie**

Dit proefschrift beschrijft het onwerp, de validatie en de utilisatie van een real time Michelson interferometer voor kwantitatieve metingen aan 2D brekingsindexprofielen. De Michelson interferometer heeft drie kanalen en is bruikbaar in diverse toepassingsgebieden, maar de uiteindelijke toepassing in de toekomst is het meten van dichtheidsprofielen in compressible stromingen in een windtunnel. De studie is een haalbaarheidsstudie gericht op de verkrijgbare meetnauwkeurigheid voor het geval dat het testobject mag worden beschouwd als een (lokaal) lineair gestratificeerd medium begrensd door ramen. De tijdresolutie is laag (video snelheid). De bruikbaarheid van de interferometer is aangetoond in een ander toepassingsgebied dan compressible stromingen: de door de jet geïnduceerde homogenisatie nadat een jet van een verdurde sucrose-oplossing is geïjecteerd in een waterige bulkvloeistof in een klein cuvet.

In hoofdstuk 1 worden de bestaande optische technieken bediscussieerd die voor het meten van stromingen worden gebruikt. De wijze waarop de stroming wordt gemeten verdeelt ze in twee groepen: technieken die bewegende deeltjes of moleculen in de stroming detecteren en technieken die de dichtheid in de stroming meten. Interferometrie behoort tot de tweede groep. Gebruikelijke interferometrische opstellingen zoals een
Mach-Zehnder interferometer met één kanaal, een Michelson interferometer met één kanaal en een holografische interferometer kunnen allemaal worden gebruikt voor repeteterende metingen. Echter, het aantal geregistreerde interferogrammen per meting is beperkt tot één waardoor hun meetnauwkeurigheid lager is dan die van een fasegestapte real time interferometer die meerdere interferogrammen registreert per meting.

Hoofdstuk 2 behandelt de minimalisatie van de afbuigings-effecten in een Michelson interferometer door het maken van een geschikte afbeelding. De afbuiging van lichtstralen in een inhomogene brekingsindexprofiel leidt tot fouten in het interferogram. Deze fouten kunnen worden geminimaliseerd door een geschikt (virtueel) voorwerpvlak in het testobject af te beelden op het interferogram in het beeldvlak. De positie van het voorwerpvlak behorend bij een fasefout gelijk aan nul blijkt in lichte mate afhankelijk te zijn van de brekingsindexgradient in een (lokaal) lineair gestratificeerd medium. Hierdoor kan de fasefout nooit exact nul zijn over een geheel gradieent interval waarvoor een interferometer is ontworpen. Echter, de maximale brekingsindexgradient is over het algemeen klein genoeg om de afhankelijkheid van de positie van het virtuele voorwerpvlak van de brekingsindexgradient te kunnen verwaarlozen waardoor de fasefout in het interferogram wordt bepaald door de nauwkeurigheid waarmee het correcte (virtuele) voorwerpvlak is afgebeeld. De toegestane positie van het virtuele voorwerpvlak is berekend als een functie van de maximale brekingsindexgradient en de toegestane fasefout voor twee configuraties van het testobject: een windtunnel-configuratie en een cuvet-configuratie. Een afstand van 1mm tussen de posities van het virtuele voorwerpvlak en het optimale virtuele voorwerpvlak leidt tot een maximale fout van 1.0% van 2\pi radiaal indien de dimensieloze brekingsindexgradient is maximaal 0.0018. De relatieve fasefout geïdeeducreerd door een Michelson interferometer is twee keer zo groot als de relatieve fasefout geïdeeducreerd door een Mach-Zehnder interferometer. De afstand waarover een brekingsindex bij benadering constant moet zijn om een lokaal lineair medium te mogen veronderstellen is tenminste vier keer die afstand in een Mach-Zehnder interferometer.

Hoofdstuk 3 toont het ontwerp van de real time interferometer (RTI). Er wordt begonnen met de keuze van de methode waarmee de fase wordt berekend. Meerdere methodes worden besproken. Uiteindelijk wordt er gekozen voor een fasegestapte methode die gebruik maakt van drie interferogrammen met een onderlinge fases stap van π/2 rad. Deze methode is robuust, is geschikt voor de typische interferogrammen die worden verkregen in een compressibele stroming en de complexheid van een opstelling met drie kanalen is niet extreem hoog. Naast de RTI is er ook een tijdverschoven fasegestapte interferometer (TPSI) ingebouwd in de opstelling om de RTI te valideren. De uitlijningprocedure en het analyse proces van de fringes in de twee interferometers is beschreven. Vervolgens wordt er een uitgebreide nauwkeurigheidsanalyse gegeven voor de RTI en een korte voor de TPSI. Uiteindelijk wordt er een optimale rms fasefout berekend voor de RTI en de TPSI, dat is een rms fasefout geïntroduceerd door kwantisatie en ruis, een niet-lineaire camera
respons, misuitlijning van de cameras en fouten in de fase stap. Deze optimale rms fase fout ligt tussen 1.5% en 3.8% van $2\pi$ radiaal voor de RTI en is ongeveer 0.5% van $2\pi$ radiaal voor de TPSI.

In hoofdstuk 4 wordt de RTI op experimentele wijze gevalideerd aan de hand van een aantal 'kantelende spiegel' experimenten. Deze 'kantelende spiegel' metingen zijn min of meer simulaties van een lineair gestratificeerd medium. Om de meetnauwkeurigheid van de RTI te bepalen zijn faseverdelingen verkregen met de RTI vergeleken met die verkregen met de TPSI en met gefitte vlakken. De rms fase fout en de maximale absolute fase fout die door de RTI worden geïnduceerd zijn bepaald als functie van de fase gradient gedurende de actuele meting onder de condities dat fasen gedurende de actuele en de referentie meting homogeen verdeeld zijn over $[0,2\pi)$ en dat er slechts een kleine gradient is gedurende de referentie meting. De belangrijkste foutenbronnen zijn bepaald. De rms fase fout geïnduceerd door de RTI is maximaal ongeveer 2.8%-4.3% van $2\pi$ radiaal groter dan die geïnduceerd de TPSI. Het maximale fase verschil is ongeveer 12.1% van $2\pi$ radiaal. De totaal door de RTI geïnduceerde fase fout is afhankelijk van de configuratie van het testobject. Als het medium lokaal lineair gestratificeerd is en de gradient in de fase ter plekke van de camera's is minder dan $58 \times 10^3$ rad/m, dan varieert de rms fase fout ongeveer van 4.6% tot 13.1% van $2\pi$ radiaal voor een windtunnel-configuratie en ongeveer van 4.6% tot 9.7% van $2\pi$ radiaal voor een cuvet-configuratie. De maximale absolute fase fout varieert ongeveer van 11.0% tot 27.5% van $2\pi$ radiaal en van 11.0% tot 22.3% van $2\pi$ radiaal. De experimenteel gevonden rms fase fout komt overeen met de theoretisch verwachte rms fase fout nadat er een vervorming is verwijderd uit de experimenteel verkregen faseverdelingen.

In hoofdstuk 5 wordt de RTI gebruikt om de homogeniteit van mengsels in immuno assays te bestuderen. De door de jet geïnduceerde homogenisatie is gemeten als functie van de tijd nadat een jet van een verdunne sucrose-oplossing is geïjecteerd in een waterige bulk vloeistof in een klein cuvet. De gefractioneerde standaarddeviatie is gebruikt als maat voor de homogeniteit op grote schaal, de fractionele rms concentratie gradient is gebruikt om de aanwezigheid van lokale concentratie gradienten te detecteren. De metingen tonen aan dat de mengsels homogeen worden binnen 18s na de jetinjectie als de verhouding van het jetvolume en het bulkvolume groter is dan 0.67. Lokale gradienten blijven zeker aanwezig tot 78s na de jetinjectie.

Voorafgaand aan de experimenten is er een uitgebreide nauwkeurigheidsanalyse. Daarna is de betrouwbaarheid van de RTI voor concentratiemetingen aangetoond door het diffusies proces van sucrose in water te meten en de resultaten te vergelijken met theoretische resultaten op basis van de tweede wet van Fick in eendimensionale vorm. Er is experimenteel een redelijke waarde voor de diffusiecoefficient gevonden ondanks dat de begincoconditie van het diffusieprocess niet duidelijk is gedefinieerd. Van de fractionele standaarddeviatie is aangetoond dat het een goede maat is voor de homogeniteit op grote schaal en van de rms concentratie gradient is aangetoond dat het een goede maat is om
grote lokale concentratiegradienten te detecteren.

Een samenvatting van de conclusies en enige suggesties voor een verdere ontwikkeling van de real time interferometer worden gegeven in het afsluitende hoofdstuk 6.
Nawoord

Toen ik in december 1994 aan mijn promotie-onderzoek begon was de opdracht globaal beschouwd 'het ontwerpen en bouwen van de optiek van een real time interferometer en deze vervolgens valideren en utiliseren'. Aanvankelijk gingen de ontwikkelingen snel. Waren de aanwezige middelen eerst beperkt tot een optische tafel en een kleine HeNe-laser, zo'n anderhalf jaar later was er een ontwerp en een eerste optische set-up. Helaas werd deze snelle ontwikkeling getemperd door een langdurig ziekteverlof. Toen ik terug keerde werd i.v.m. het vertrek van een collega de opdracht uitgebreid: niet langer alleen de verzorging van de optiek maar ook het schrijven van de benodigde software. Dit was weliswaar een nieuwe uitdaging maar het bracht wel heel veel extra werk met zich mee. Het uitlijnen van de optiek en de camera's en het schrijven van software bleek een tijdverdrogende bezigheid. Pas toen de framegrabber defect raakte en er maanden gewacht moest worden op de levering van een nieuwe was er weer tijd voor een verdere theoretische verdieping. In 1999 was de real time interferometer operatief. Echter als gevolg van een verbouwing werd mijn laboratorium in dat zelfde jaar afgebroken en moest de interferometer worden verplaatst. Pas in 2000 was het systeem opnieuw uitgelijnd en kon het worden gevalideerd en geëxporteerd. De interferometer heeft honderden interferogrammen opgeslagen. Een groot aantal van deze interferogrammen zijn gebruikt in de analyses in de hoofdstukken 4 en 5 van dit proefschrift.
Natuurlijk hadden de real time interferometer en dit proefschrift nooit kunnen worden gerealiseerd zonder de directe bijdrage van velen. Eric de Keizer, Nico Lam, Peter Duyndam en Nico van Beek leverden de technische ondersteuning bij de bouw van de interferometer. Frits Donker Duyvis assisteerde bij het uitlijnen van de interferometer en de verwerking van de vele interferogrammen. De utilisatie is geschied in samenwerking met Tom Beumer. Hij heeft hiermee een essentiële bijdrage geleverd aan de totstandkoming van hoofdstuk 5 van dit proefschrift. Brenda Timmerman was vaak bereid tot vakinhoudelijke discussies, ook nadat ze niet meer aan de TU was verbonden. Prof. Bakker, Brenda Timmerman, Bas van Oudheusden en Henk Pietersma hebben één of meer stukken die moesten leiden tot dit proefschrift kritisch bekeken. Al deze mensen wil ik heel hartelijk danken!


Juup Nijholt
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