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Sensitivity analysis in passive vibration control

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Aiming for insight in the dynamic behaviour of mechatronic systems

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Abstract

The achievable dynamic performance in high-precision systems by applying light and stiff design principles is reaching its limits. To further increase performance, better insight is needed in the cause and effect of the systems modeshapes and forced vibrations. In this way, transfer functions between actuator/excitation sources and sensor locations/point of interest can be optimized specifically, instead of just pushing for the highest first resonance frequency on system level.

The purpose of this study is to enhance the understanding of the dynamic behavior of a structure and to provide methods for improving the dynamic response by applying sensitivity analysis. This is realized by first investigating the existing methods to derive the sensitivity values of structural response functions that characterize the dynamic behavior.

With this knowledge on computational methods, the investigation proceeds with a focus on design variables. It is shown that the element-wise design variables can be assembled into a vector. This vector then describes a global design change. By doing so, the number of design variables is effectively decreased without losing the ability to reach an optimal solution. Reducing both the design space and the solution space has led to the development of an estimation procedure for 1D structures. With this method the required structural change that leads to an intended modeshape modification can be estimated.

In three case studies sensitivity analysis was employed in the evaluation of a conceptual design of a mechatronic system. The application of sensitivity analysis in the first case study resulted in a significant improvement for the rejection of floor vibrations. However, it was also shown that such an improvement could not be obtained for a coordinate direction in which a controller is present.

In the second and third case study, a method for connecting two components was developed. The method uses the nodal points of one component as connection locations. Combining both the static and dynamic nodal points led to a minimization of the energy transfer between the two. The development of a demonstrator must validate the concept. By studying the sensitivity of the nodal points two tuning parameters were identified, namely force distributions and addition of lumped masses. They created the possibility to show the effect of the optimization that uses the nodal points. Furthermore, the method of tuning for optimal dynamic response could be the next step in mastering dynamic behavior.
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“The concept of ‘model simple, think complicated’ brings us to the idea that models are ‘tools for thinking’. It would be wrong to interpret that phrase as being ‘tools to replace thinking’. Instead they are tools to support and extend the power of thinking. Thus, a complicated model which is poorly employed may be worse than a simple model used as a tool for careful thought.”

— M. Pidd

“Don’t confuse symmetry with balance.”

— T. Robbins
Chapter 1

Introduction

Structural dynamics entails the study of the dynamic response of a mechanical structure, i.e. the system. The focus of this thesis will be on mechatronic systems which combine a mechanical structure with electronics to perform the required function [20]. The dynamic interaction between the mechanics and electronics has been an area of research for many years. This research is driven by the constantly increasing complexity of mechatronic systems and more demanding requirements. To be able to comply with the requirements, the understanding of the system dynamics becomes increasingly important. Rankers [28] made a large contribution in this field with the work on machine dynamics in mechatronic systems. The importance of simplified modeling is exemplified and the emphasis is put on understanding.

The dynamics of a mechatronic system combine both desired motions of for example a stage and unwanted dynamics created by for example reaction forces or floor vibrations. These unwanted dynamics or often referred to as vibrations. Without focusing on mechatronic systems, Mead [19] covers the topic of reducing mechanical vibration for general mechanical systems. The work is restricted to passive solutions, meaning that no external energy source is used for suppressing vibration.

This thesis will focus on vibrations (unwanted dynamics) of mechatronic systems. Similar to the work of Mead it is restricted to the passive domain however, within the area of application that is covered by Rankers. New methods are investigated that should increase the insight in the dynamic behavior of mechatronic systems. Insight is thought to be created by efficiently gathering and visualizing the information obtained from a limited number of models. Sensitivity analysis is a tool that has the potential to aid in this process of retrieving more information from existing models.

In the field of optimization, sensitivity analysis has seen much attention with the aim of speeding up computations. This has resulted in sensitivity formulations that have potential to provided extra insight into the dynamics of the model on top of the knowledge gained by modal analyses with very little computational effort [23]. Furthermore, it is widely accepted that sensitivity analysis is a tool in its own right, apart from the application in optimization [4]. Commercial analysis packages incorporate sensitivity analysis for several purposes among which are so called 'what if' analysis, pretest analysis and model updating. 'What if' analysis entails the study of the effect of modeling assumptions, e.g. dimensional uncertainties, on a response. Pretest analysis focuses on the prediction of the effect that sensors and actuators will have on experimental test results. Model updating aims at combining the experimental results with the numerical model to improve the correlation between the model and the real world behavior.

This thesis discusses the sensitivity analysis for 'what if' scenarios. By studying the effect of parameter changes on the response function the direction of optimization is retrieved. Consequently the fundamental properties of the model become more comprehensible by observation of these optimal directions.
The work is divided into two main parts. A theoretical part and an implementation part. The first part addresses passive vibration control, modeling and the theoretical aspects of sensitivity analysis for a selection of response functions. The second part contains three case studies on passive vibration control.

Chapter 2 on modeling and vibration control is mostly based on existing literature concerning these two subjects. The field of vibration control is characterized by with the work of Mead [19] and it clarifies the position of this thesis. A section on modeling is added to relate the aim of more insight to the common problems of modeling a complex system.

Chapter 3 starts with the definition of the response functions that will be used throughout this thesis. These functions will be used as a basis to discuss the other fundamentals of sensitivity analysis, the design variables and the method of calculation.

The implementation part starting from Chapter 5 contains three case studies on passive vibration control which are derived from a project at TNO Optomechatronics. The project concerns the development of a mechanical structure that has stringent requirements on the vibration levels. Details of the application are left out of the discussion for commercial reasons. The first case concerns the assessment of a conceptual design and its performance regarding the effects of floor vibrations. The second and third case study involves an investigation into the possibilities to reduce the vibrations created by an actuator.

The final chapter reflects on the results of the case studies. The yield of the sensitivity analysis on the retrieved insight is discussed and recommendations are made for future research.

The envisioned goal of this thesis is to develop insight into methods of approach and analysis to solve vibration related problems. The emphasis is not on a final design of a system but on the process that is used to get there. It therefore contains a combination of mathematical derivations, practical applications and a whit of "philosophy" on the topic of modeling and insight. This results in observations that are partly based on mathematical certainties as well as empirically derived statements. It should lead to interesting discussions and hopefully some careful thought (see quote page).
Chapter 2

Modeling and passive vibration control

This chapter introduces the field of passive vibration control, which includes all the actions that could be taken to suppress (control) vibration without adding energy to the system. In the following section the different options that could fulfill this task are discussed. After characterizing passive vibration control a second section discusses modeling.

Since the goal of this thesis is to explore the potential of sensitivity analysis with the aim of increasing insight, some remarks on modeling seem appropriate. Especially the complexity of models is relevant since it could be falsely assumed that a complexer model will describe reality in a better way and thereby increase the insight. In addition, the strength of simple models is recognized in all fields of engineering but despite the increasing computational power of computer systems the time spend on modeling remains approximately equal.

For example, a model that requires one night of computations was found to be acceptable ten years ago and still is today. This seems strange since the computational power has increased tremendously. The section on modeling discusses related literature that covers the question why the complexity of models follow the increase in computational power and how this could be avoided.

2-1 Passive vibration control

The book on passive vibration control gives an in-depth description of all aspects of the subject at hand [19]. It is not the intent to repeat the content of the book but to highlight some of the interesting concepts that strongly relate to the work presented in this thesis. Table 2-1 summarizes the different solution directions that exist to solve vibration related problems in a passive sens. The solution space is divided in four directions including structural design, localized additions, added damping an resilient isolation.

Structural design includes methods to reduce vibrations that require adaptations in the design. This could for example be the width or hight of a beam. How to add damping material to the structure is covered by the direction of localized additions. The topic of added damping introduces auxiliary systems that will absorb vibration energy from the main structure. The last topic of resilient isolation refers to the often used vibration isolations systems on which for example optical tables are placed.

The work of this thesis is strongly related to the class of vibration control by structural design. The items written in italic refer to solution paths that directly relate to the solutions of the three case studies discussed in Chapter 5, Chapter 6 and Chapter 7. These concepts will be briefly explained in the following subsection.
Table 2-1: The field of passive vibration control divided into the four main approaches. The items written in italic refer to topics that relate to the work of this thesis.

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**Added damping**
- The simple mass or spring as an auxiliary system.
- The single damper attached to the main system.
- The vibration neutralizer attached to the main system.
- Wide bandwidth vibration suppressors.

**Resilient isolation**
- The isolation of a rigid body on a inelastic foundation.

**De-tuning**

De-tuning the structure implies that it is designed or modified in such a way that its resonance frequency does not coincide with the harmonic excitation frequencies. The sensitivity analysis presented in Section 3-3 is directly applicable to achieve such an optimization. In mechatronic systems an eigenfrequency of 50 Hz is very undesirable since it equals the frequency of the main power supply. In practice this 50 Hz and 100 Hz prove to be the frequencies that show a peak in the disturbance spectrum. Preventing a structural eigenfrequency on these disturbance peaks is a typical example of de-tuning.

**Reducing the number of responding modes**

This concept aims for a structural change that pushes a large part of the mode content outside the bandwidth that determines the overall performance. This is a typical approach to prevent controller instability. The first pole/zero combinations in the transfer function from actuation point to sensor location determine for a large part the achievable bandwidth of a controller. These pole/zero combinations are created by mechanical resonances and the combination of actuator and sensor location. Mechanical resonances therefore directly affect the performance of a mechatronic system. The classical approach to achieve the highest possible bandwidth is by light and stiff design reducing the modal density in the low frequency region. As was already stated in the introduction of this thesis this approach is reaching its limits with respect to the total system performance. Light and stiff design is limited by the applicable materials, the static strength requirements and the thermal requirements. The ultimate goal is to push beyond this limiting factor by a improved insight in dynamic behavior.

**Nodalizing**

The concept of nodal points or lines is introduced in Section 3-1-3. The term nodalizing refers to using such a nodal point as a sensor or actuator location. Mead states two interesting observations for the behavior of nodal points on a beam structure:

- **There is a consistent tendency for the added mass to draw the nodal points towards itself. It is always found that the nodes which are influenced most, are those closest to the mass, while those which are more remote are hardly affected.**
- **A small region of reduced flexural stiffness draws the nodal point towards itself.**
Optimal use of nodal points for modes and nodal points of the full response (which are not the same) is of key importance to increase dynamic performance.

Apart from nodalizing components that are already defined in the base-line system, extra Nodalized elements could be added to improve performance. This technique has seen successful implementation in the rotor construction of helicopters ([33] [37] [44]).

**De-coupling**

De-coupling is a term often used in engineering communities, however not always with the same definition. Mead combines the possible meanings in three different processes.

*De-coupling different types of motion.* In this case de-coupling refers to the removal of an unwanted motion component in a static or dynamic modeshape. For instance when a bending motion induces a torsional motion at a different location of the system.

*Separating the natural frequencies of different structural components.* When one modeshape primarily shows bending and a second modeshape exhibits mostly torsion, an increase in frequency difference will lead to a de-coupling of the two types of motion.

*Physically de-coupling different substructures.* This type of de-coupling focuses on the flow of energy between two parts and prevention of it. This could be created by either a very compliant connection reducing the transmitted forces, or by creating a very stiff connection which should minimize movement at the connection.

**Optimizing the structural geometry**

This topic relates to all previous methods in a sense that they can be the desired objective which is achieved by changing the structural geometry. This thesis does not discuss optimization in detail however, sensitivity analysis is an essential tool to make it computationally efficient. All the introduced sensitivities are based on geometry changes which translate into changes in the mass and stiffness matrix.

**2-2 Modeling**

Modeling is used to evaluate a systems response and reduce the risk in the development of mechatronic systems. The creation of a model often requires simplification of reality to increase computational efficiency. How and when to simplify is a question that every analyst faces in the process of model building. This section on modeling discusses related literature that covers the aspects of model complexity and simplification.

**Complexity of simulation models**

There is no general definition of what the complexity of a model entails. Definitions from [9] describe complexity as a cognitive aspect that determines the difficulty of understanding the system that has been modeled. Herbert [34] describes complexity with the number of parts and elements that it contains. A common point of view described by Stewart [30] is that complexity is related to the 'level of detail' not to be confused with 'scope'. Scope divines what is included and consequently the level of detail is related to how it is included. Chwif et al. [6] summarizes that no standard measure of complexity is widely accepted and there is a room for improvement.

Chwif et al. [6] continues with the observation that models tend to become more complex and they describe technical and non-technical reasons for this.
Non-technical: "show off" (impress manager), "include all" syndrome (usually insecure inexperienced modeler) and possibility factor (available computational resources). Technical factors: lack of understanding of the real system, inability to model the problem correctly, lack of simulation software knowledge and unclear simulation objective.

Especially inexperienced modelers have a tendency to develop more complex models than an expert modeler mostly due to the "include all" syndrome. Expert modelers tend to concentrate their modeling efforts mostly at the conceptual phase of modeling resulting in simpler models [45].

Determining when a model is adequate for the job is a subjective question and as computing power increases there is a shift in when a model is deemed adequate. This is eloquently described by Venkataraman et al. [40] (Structural optimization complexity: what has Moore’s law done for us?). The ultimate goal in optimization is described as combining the maximum in analysis complexity, model complexity and optimization complexity. The reality is that current technology does not allow the combination of all three. Maximizing one requires dropping down in the other areas. This situation remains unchanged due to the constant recalibration on what is the maximum within the three axes.

Verification and validation

Wall [41] emphasizes that a model has no value as a decision support until it is judged valid. The model can however support other purposes such as increasing understanding of the system. The author continues by stating that verification and validation suffer from the same problem as the concept of complexity. The definition of the concepts are neither consistent in practice nor in literature. Verification is often defined as the process of determining if the model works as intended, meaning that the coding and implementation is done correctly. Validation is often defined as a measure of usefulness in relation to the investigation objectives, and could be divided into model validation, data validation and operational validation. Model validation entails the checking of the assumptions that were made in the development of the model. Data validation includes a correctness check on the data that is necessary to run and build the model. Operational validation must determine if the model can describe the physical phenomena of interest.

The fidelity of a model indicates to what extend it describes the real world system that is modeled. Wall [41] proposes to judge a model on the combination of fidelity and validity. In project management risk is quantified as the product of the probability of occurrence and its consequences. In the same way a model’s performance might be quantified as the lowering of risk for the decision maker. In this case the risk is determined by the product of fidelity and validity of the model. The performance of the model will be high when the risk for the decision maker is low.

Modeling is generally an iterative process, where the model is updated according to the gained experience of the analyst. Therefore the validity increases through iteration. There will always be a trade-off between the costs of a model and its combination of fidelity and validity. The model should therefore be as good as necessary, not as good as possible. Knowing when to stop the iteration is crucial to reduce the cost. In practice it seems that iteration is stopped after the last step has proven to be a step too far. By reducing the step size the cost of this final nonproductive step could be minimized. Sensitivity analysis could be seen as a method to extract the maximal information from a model without excessive computational effort before taking the next iterative step. Furthermore the analysis will provide the optimal direction in which a step should be taken.

Five simple principles of modeling

The last section in this chapter describes five simple guidelines which should prevent some of the problems described in the previous sections. These guidelines are known as Pidd’s principles for model building. The main message is summarized in the enumeration below. A full explanation can be found in [25].
1. **Model simple, think complicated.** Models are not just built, they are also used. The combination of the model and the user must match the variety of the system being modeled. Not either on of the two. Therefore complicated models have no divine right of acceptance. The idea of "model simple, think complicated" develops to the notion that models are tools for thinking and not tools to replace thinking.

2. **Be parsimonious, start small and add.** Develop models that are initially too simple for the task at hand and gradually refine or replace.

3. **Divide and conquer, avoid mega-models.** The best way to face up to complexity is to break it down into manageable pieces. In the field of structural dynamics this is often called decomposition or sub-structuring.

4. **Do not fall in love with data.** The model should drive the data collection and not vice versa. This means that the analyst should try to develop some ideas of the model and its parameters and from this determine what type of data is needed.

5. **Model building may feel like modeling through.** Because a simulation model is the result of an attempt to represent some part of reality to increase insight and understanding, it might be thought that model building is a linear and highly rational process. There is little research done in this area but the few results that are published suggests that modeling is not a linear nor a smooth process. Instead people seem to "muddle through", making use of insights and taking time away from modeling, trying to look at things from a different perspective.
Chapter 3

Sensitivity analysis

The introduction of this thesis describes that sensitivity analysis can be categorized by the intention with which it is employed. Focusing on the "what if" analysis, the predictability of the final product and the insight gained are the two main drivers behind the application of sensitivity analysis. This chapter will introduce the mathematical derivation of several sensitivity functions.

Sensitivity analysis is a very broad topic with a large collection of applications. For linear systems in all fields of application an extensive review is given in [39]. Calculating a sensitivity involves the declaration of a response function and a design variable. The response function defines the property or relation that quantifies the objective. Sensitivity analysis must indicate how such a response function will change as a result to a change in design variable. The design variable is declared by the analyst and defines the property that is varied. The topic of design variables is covered extensively and it is therefore treated separately in Chapter 4.

This thesis is restricted to the response functions that characterize the dynamic performance of a mechanical system. Even more specific only the steady state linear behavior of mechanical structures, also known as linear time-invariant systems, is covered.

3-1 Response functions

In the following sections six response functions will be introduced. These functions define the performance of the system that either needs to be improved or just simply understood. After the introduction the following sections will discuss the methods to calculate the sensitivity for each function.

3-1-1 Eigenfrequencies and eigenmodes

The eigenfrequencies of a mechanical system are one of the most important performance indicators. The eigenfrequencies are directly linked to the corresponding eigenmodes. Both are derived from the general equation of motion,

\[ \mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}. \]  

Eq. (3-1) includes the mass matrix \( \mathbf{M} \), the stiffness matrix \( \mathbf{K} \) and the damping matrix \( \mathbf{D} \). The vector \( \mathbf{q} \) contains \( N \) degrees of freedom and \( \mathbf{f} \) represents the applied load. The matrices \( \mathbf{M}, \mathbf{D} \) and \( \mathbf{K} \) are of the dimension \( N \times N \). Both matrices \( \mathbf{M} \) and \( \mathbf{K} \) are symmetric positive definite. \( \mathbf{C} \) is a constant.
When the damping is neglected and there is no applied force the equation reduces to the undamped free vibration equation of motion,
\[ M\ddot{q} + Kq = 0. \] (3-2)
Writing the motion of a degree of freedom in a spatial depended component \( x \) and a time depended component results in:
\[ q(t) = x(\alpha \cos(\omega t) + \beta \sin(\omega t)). \] (3-3)
Substituting Eq. (3-3) into Eq. (3-1) results in an eigenvalue problem Eq. (3-4) where \( \lambda_r = \omega_r^2 \)
\[ (K - \lambda_r M)x_r = 0 \quad \text{where} \ r = 1 \ldots N. \] (3-4)
Solving the eigenvalue problem results in a set of eigenvalues \( \lambda_r \) and eigenvectors \( x_r \) for \( r = 1 \ldots N \).
\[ x_r^T M x_p = \delta_{pr} \] (3-5)
\[ x_r^T K x_p = \omega_p^2 \delta_{pr} \] (3-6)
The eigenvectors have an arbitrary amplitude and in this case mass normalization is assumed according to Eq. (3-5) where \( \delta_{pr} \) represents the Kronecker delta.

### 3-1-2 Frequency response function

The frequency response function (FRF) describes the response of a system for a range of input frequencies. By using a linear model it is assumed that the system will respond at the same frequency as the applied load. The gain and phase of the response with respect to the input are determined by the system and are shown in the magnitude and phase plot respectively.

There are different types of FRF’s and depending on the application each has its own merits. The difference lies in nature of the load and response quantity that is calculated or measured. This work mostly deals with displacement as a result of an applied force. The FRF is derived from the equation of motion Eq. (3-1) which is first transformed to the frequency domain. Assuming equilibrium at the trivial solution and all initial conditions to be zero results in,
\[ \left(-\omega^2 M + j\omega D + K\right)q(\omega) = f(\omega). \] (3-6)
The dynamic stiffness \( Z \) is defined as the force required to produce a unit displacement,
\[ \frac{f(\omega)}{q(\omega)} = Z(\omega). \] (3-7)
The receptance \( Y \) is defined as the displacement required to produce a unit force,
\[ \frac{q(\omega)}{f(\omega)} = Y(\omega) \] (3-8)
The FRF of the receptance can also be written in modal coordinates.
\[ q = y e^{j\omega t} \] (3-9)
\[ f = s e^{j\omega t} \] (3-10)
\[ y = \sum_{s=1}^{n} \alpha_s x_s. \] (3-11)
By substituting Eq. (3-9) and Eq. (3-10) into Eq. (3-1) and assuming a lightly damped system the mode superposition according to Eq. (3-11) can be applied which leads to:
\[ y = \sum_{s=1}^{n} \frac{1}{\omega_0^2 s^2 - \omega^2 + 2j\zeta_0 \omega_0 s} \frac{x(s)x_T(s)}{\mu_s} s, \] (3-12)
excluding rigid body modes. In Eq. (3-12) \( \zeta_s \) represents the modal damping.
3-1-3 Nodal location

The degree of freedom of a structure that exhibits no movement in its response is called a nodal point or nodal line. Such a property can be described in the total response of a system or in its vibration modes. The presence of nodal points in the total response will be discussed in Section 3-1-4. Nodal points or lines for modeshapes are not dependent on the applied load however, vibration modes do not necessarily exhibit these points or lines. For beam and plate like structures the lowest symmetric modes are usually without nodal points or lines. In passive vibration control it can be of high importance to know the location of nodal points since it could be an interesting location for sensors or actuators. In principal the nodal location is defined as the degree of freedom $i$ in mode $r$ that exhibits $x_i^r = 0$. In practice a designer is interested in locations where all translational degrees of freedom are zero or at least the degrees of freedom which could be observed by a sensor or actuated by an actuator.

3-1-4 Anti-resonance frequency

An anti-resonance frequency (ARF) is typically defined as a property of a response function. It indicates a frequency where the magnitude of the response tends to zero. Only in the case of an undamped system will the response be truly zero. In this case the ARF indicates a nodal point in the total response of the system. In practice an ARF exhibits the same behavior as a damped resonance peak where the height of the peak is governed by the damping. In the case of an anti-resonance the depth of the magnitude dip is determined by the damping.

The ARF is a response property that is dependent on the actuation and sensor location, the actuation frequency and the dynamics of the system. When the movement of a system is completely described by one mode the eigenfrequency of this mode will also be the ARF if the sensor is measuring at a nodal point. In practice the movement is never described by one mode and therefore the summation of all modes must combine to create an ARF. To emphasize how the ARF and the nodal points of the modes and the total response are related, an example is given for a cantilever beam after an explanation on the method of computation.

The ARF for an undamped system is derived by starting at the forced equation of motion,

$$\mathbf{M} \ddot{q} + \mathbf{K} q = b e^{j\omega t}. \quad (3-13)$$

Wang [42] describes a method to transforms this equation to a new modified eigenproblem. The transformation matrices are computed with the null space of the load vector $\mathbf{b}$,

$$\mathbf{B}_N = \text{Null}(\mathbf{b}^T). \quad (3-14)$$

and the null space of the sensor location $\mathbf{c}$,

$$\mathbf{C}_N = \text{Null}(\mathbf{c}). \quad (3-15)$$

The modified system matrices are given by:

$$\tilde{\mathbf{K}} = \mathbf{B}_N^T \mathbf{K} \mathbf{C}_N \quad (3-16)$$

$$\tilde{\mathbf{M}} = \mathbf{B}_N^T \mathbf{M} \mathbf{C}_N \quad (3-17)$$

The resulting modified eigenproblem can be solved to retrieve the anti-resonance frequencies $\mu$ and the corresponding modeshapes $\phi$

$$\left( \tilde{\mathbf{K}} - \mu \tilde{\mathbf{M}} \right) \phi = 0 \quad (3-19)$$

In general the matrices $\tilde{\mathbf{K}}$ and $\tilde{\mathbf{M}}$ are not symmetric and the eigenvalue $\mu$ can be positive, negative or even complex. For each positive $\mu$ there is a corresponding ARF. The modeshape $\phi$ is arbitrarily scaled. The response can be written as,

$$\mathbf{q}_a = \mathbf{S} \mathbf{C}_N \phi_a. \quad (3-20)$$
Substituting this into Eq. (3-21) result in Eq. (3-22)

\[
(K - \lambda M) q = b 
\]

(3-21)

\[
(b^T Z C_N \phi ) S = b^T b 
\]

(3-22)

This equation can be solved for the scaling factor $S$ where $Z$ represents the dynamic stiffness matrix.

### 3-1-5 Example for ARF

A cantilever beam with the first four vibration modes is shown in Figure 3-1. The modes are plotted at the position on the y-axis that corresponds with their natural frequency. The ARF computed according to Eq. (3-19) for a force applied at the free end of the beam, are plotted as blue lines. At low frequencies the structure does not contain an ARF. As the actuation frequency increases an ARF exists at the free end of the beam. This point moves to the left for increasing frequency. At the frequency corresponding to the second eigenmode the nodal point of $x_2$ and the ARF of the global structure are almost equal. Since damping has been neglected the amplitude of $x_2$ would be infinite yet the total response is described by all the modes and correspondingly the ARF is almost identical to the nodal point.

The fundamental difference between an ARF and a nodal point of a mode is the fact that an ARF describes a nodal point of the total system response. Since the response of a system will never be described by a single mode the nodal point of a mode will be similar yet not exactly the same as that of the total response. Furthermore, in the presence of damping the nodal point of the total response will not be truly zero.

![Figure 3-1: The position of ARF's as a function of excitation frequency. At the frequency that corresponded to an eigenfrequency the mode shape is plotted. This illustrates how the ARF of the full response intersect with the nodal points of a mode at the eigenfrequency](image)
3-2 Sensitivity classes

Before the sensitivities of the previously discussed response functions are introduced, the different types of sensitivity analysis are presented. The difference lies in the method of calculation. All the presented sensitivity functions in this thesis can be divided in the following classes.

Finite difference method

The finite difference method originates from a Taylor series expansion to approximate the sensitivity and is undoubtedly the simplest method to implement [38]. However, the accuracy is not guaranteed and is for example depended on the step size. Choosing the correct step size can be problematic since it is bounded by numerical round-off errors when the step size becomes too small and affected by truncation errors when the step size becomes too large. A step size sweep is an appropriate method to determine for which sizes the derivative remains constant. Finite difference can be subdivided in the forward, backward and central finite difference. An example of forward finite difference is stated in Eq. (3-25).

Continuum and discrete sensitivity analysis

Sensitivity analysis can be applied for both an continuum or a discrete model. This thesis will only cover sensitivities that are applied after the discretization of a model since continuum models are seldom used in engineering practice. This is mainly due to the fact that engineering applications are often too complex to be described by an analytical function with an exact solution.

Semi-analytical

The semi-analytical method refers to a sensitivity analysis where the first steps comprise out of analytical derivations and the proceeding steps include finite difference computations. Several examples of this method will be discussed.

Direct and adjoint method

Two different methods to calculate sensitivities are the direct and adjoint method each with its own benefits. The difference between the two methods is created by the computational efficiency. The adjoint method requires the solution of one adjoint problem for every response function. The direct method requires the solution of a pseudo load problem for each design variable \( s \). If the number of response functions exceeds the number of design variables, the direct method is preferable, and vice versa. A detailed explanation of the two methods is given in Section 3-5 were the sensitivity of an FRF is derived. These sensitivities have a tendency to be computationally expensive and for this reason efficiency becomes important.

Scaled sensitivities

When comparing sensitivities with respect to different design variables it is important to scale them according to

\[
\frac{\partial s q}{\partial s} = s \frac{\partial q}{\partial s}
\]  

(3-23)

This results in a fair comparison when, for example, all values are perturbed by 1% [5].
The logarithmic derivative [38] written as,
\[ \frac{\partial L}{\partial s} = \frac{\partial \ln q}{\partial \ln s} = s \frac{q}{q} \frac{\partial q}{\partial s} \] (3-24)
is essential when comparing the sensitivity of a design parameter for different response functions. Both
the value of \( s \) and \( q \) are used for scaling. When \( s \) or \( q \) are zero the scaling can not be applied.

### 3-3 Sensitivity of eigenvalues

In Section 3-1-1 the eigenfrequency was introduced. For the following chapters the eigenvalue \( \lambda_r \)
which is equal to \( \omega_r^2 \) will be used. For this reason the section is called sensitivity of eigenvalues. The
eigenvalues that stem from the equation of motions as defined in Eq. (3-2) exhibit properties that
 correspond to the symmetric positive definite system matrices. In general not all eigenvalue problems
are derived from such a system. The following derivations are only valid for eigenvalues that arise from
the previously described equation of motion.

Since 1842 a lot of work is done on computing the sensitivities of both the eigenvalue and the eigenvector.
This section aims to introduce the eigenvalue sensitivity and the different methods of calculation.

#### 3-3-1 Global finite difference

Since sensitivity analysis is in essence an indication how the response function depends on the design
variable \( s \), the finite difference method is a rudimentary tool to clarify this dependence.
\[ \frac{\partial \lambda_r}{\partial s} = \frac{\lambda_r + \Delta \lambda - \lambda_r}{\Delta s} \] (3-25)

Eq. (3-25) describes the difference of the original eigenvalue and that of a modified structure divided
by the step size. This method is computationally expensive since an eigenvalue problem has be solved
for each design modification.

#### 3-3-2 Rayleigh’s method

The Rayleigh quotient is known as a method to accurately predict the eigenfrequency of a system
when the system matrices are known. Furthermore an estimation of corresponding modeshape is also
required. The quotient, calculate with
\[ R(x_r) = \frac{x_r^T K x_r}{x_r^T M x_r} = \lambda_r \] (3-26)

reduces an error in the the modeshape estimation of the order \( \varepsilon \) to an error in the eigenfrequency of
the order \( \varepsilon^2 \) [29].

If a new eigenvalue \( \lambda_q \) is created by a small design modification,
\[ \lambda_q = (\lambda_r + \Delta \lambda) = \frac{x_r^T (K + \Delta K) x_r}{x_r^T (M + \Delta M) x_r} \] (3-27)

it is a summation of the old and a \( \Delta \lambda \). Rewriting leads to,
\[ \Delta \lambda x_r^T (M + \Delta M) x_r = x_r^T (K + \Delta K) x_r - \lambda_r x_r^T (M + \Delta M) x_r. \] (3-28)

It is assumed that for small modifications the mode shape does not change and the product of \( \Delta \lambda \Delta M \)
is small.
\[ \Delta \lambda x_r^T (M) x_r = x_r^T (K - \lambda_r M) x_r + x_r^T (\Delta K - \lambda_r \Delta M) x_r. \] (3-29)
The final result Eq. (3-31) is an expression for the relative change in an eigenvalue due to a structural modification. This derivation is referred to as Rayleigh’s method [4]

$$\Delta \lambda = \frac{x_r^T (\Delta K - \lambda_r \Delta M) x_r}{x_r^T (M) x_r}$$ (3-30)

$$\frac{\Delta \lambda}{\lambda_r} = \frac{x_r^T (\Delta K - \lambda_r \Delta M) x_r}{x_r^T (K) x_r}$$ (3-31)

### 3-3-3 Modal method

The derivative of the eigenvalue with respect to a design variable $s$ can be computed by differentiating the eigenvalue problem Eq. (3-4) resulting in

$$\left( \frac{\partial K}{\partial s} - \lambda_r \frac{\partial M}{\partial s} \right) x_r - \frac{\partial \lambda_r}{\partial s} M x_r + (K - \lambda_r M) \frac{\partial x_r}{\partial s} = 0.$$ (3-32)

Multiplying Eq. (3-32) with $x_r^T$ leads to,

$$x_r^T \left( \frac{\partial K}{\partial s} - \lambda_r \frac{\partial M}{\partial s} \right) x_r - \frac{\partial \lambda_r}{\partial s} x_r^T M x_r + x_r^T (K - \lambda_r M) \frac{\partial x_r}{\partial s} = 0.$$ (3-33)

Using the fact that the eigenvalue problem equals 0, Eq. (3-33) can be simplified into

$$\frac{\partial \lambda_r}{\partial s} = \frac{x_r^T \left( \frac{\partial K}{\partial s} - \lambda_r \frac{\partial M}{\partial s} \right) x_r}{x_r^T M x_r}.$$ (3-34)

Through the years this derivation is often used however, for one of the first publications the reader is referred to [46]. Eq. (3-34) is often used with a semi analytical approach where the derivatives of the mass and stiffness matrix are determined by means of finite difference. This approach is computational very efficient compared to the global finite difference method.

### 3-3-4 Elastic and kinetic energy

The Rayleigh quotient as presented in Eq. (3-26) can also be written as a balance between elastic and kinetic energy

$$\frac{1}{2} x_r^T K x_r = \frac{1}{2} \omega_r^2 x_r^T M x_r.$$ (3-35)

These results are readily available in most commercial FE codes. Combining this expression with the sensitivity formulation of the previous paragraph leads to an interesting conclusion. If an eigenfrequency of a system needs to be modified the effect of stiffness and mass changes can be calculated separately. The resulting equations for the derivative of the eigenfrequency with respect to the Young’s modulus $E$

$$\frac{\partial \omega_r^2}{\partial E} = x_r^T \frac{\partial K}{\partial E} x_r,$$ (3-36)

or the material density $\rho$

$$\frac{\partial \omega_r^2}{\partial \rho} = -\omega_r^2 x_r^T \frac{\partial M}{\partial \rho} x_r,$$ (3-37)

show a similar form as the elastic and kinetic energy terms in the Rayleigh quotient. Instead of $K$ and $M$ the equations now contain their derivatives. Since these matrices are only linear depended on $E$ and $\rho$ the difference between the derivative $(\partial M)/(\partial \rho)$ and $M$ or the difference between $(\partial K)/(\partial E)$ and $K$ is a constant that is equal to half the density or half the Young’s modulus respectively.

This leads to the conclusion that the elastic energy distribution of $x_r$ can be interpreted as the sensitivity distribution of the $\omega_r$ with respect to the stiffness $E$. The kinetic energy distribution can be interpreted as the sensitivity distribution of the eigenfrequency with respect to the density $\rho$.

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3-4 Sensitivity of eigenmodes

Continuing with the derivative of the eigenproblem stated in Eq. (3-32) it can be rewritten into,

\[
(K - \lambda r M) \frac{\partial x_r}{\partial s} = - \left( \frac{\partial K}{\partial s} - \lambda r \frac{\partial M}{\partial s} \right) x_r + \frac{\partial \lambda r}{\partial s} M x_r.
\]  

(3-38)

This results cannot be solved for \( \frac{\partial x_r}{\partial s} \) since the matrix \((K - \lambda r M)\) is singular by definition. Several methods will be presented to overcome this problem starting with the modal method.

3-4-1 Modal method

Fox et al.\cite{7} introduced a method to overcome the singular problem by stating that the derivative of eigenvector \( x_r \) can be represented by a superposition of all the eigenvectors of the system. The \( N \)-dimensional space spanned by the \( N \) eigenvectors is used as a basis for the derivative of \( x_r \). This is also known as Fox’s or the modal method and is defined by

\[
\frac{\partial x_r}{\partial s} = Xc_r,
\]  

(3-39)

where \( c_r \) is a vector containing the amplitudes with which each eigenmode \( x_p \) contributes to \( (\partial x_r)/(\partial s) \). Substituting Eq. (3-39) into Eq. (3-32) and pre-multiplying with \( x^T_p \) \( (p \neq r) \) results in

\[
c_r(p) = \begin{cases} 
\frac{x^T_p(-\frac{\partial K}{\partial s} - \lambda r \frac{\partial M}{\partial s} - \frac{\partial \lambda r}{\partial s} M x_r)}{(\lambda p - \lambda r)} & p \neq r \\
-\frac{1}{2} x^T_r \frac{\partial M}{\partial s} x_r & p = r
\end{cases}
\]  

(3-40)
as the expression for \( c_{p \neq r} \). The solution for \( p = r \) can be derived by substituting Eq. (3-39) into the derivative of the mass orthogonality relationship (Eq. (3-5)) which is given by

\[
x^T_r \frac{\partial M}{\partial s} x_r + 2 x^T_r M \frac{\partial x_r}{\partial s} = 0.
\]  

(3-41)

This method is computational very efficient since the eigenvectors are already computed. The downside of this method is the fact that it is only an exact solution when all the modes of the system are used. This is a problem for large models. If needed the modes can be truncated but the results will no longer be exact and according to Nelson such a reduction often leads to poor results \cite{21}. Methods to reduce the negative effect of modal truncation are known as modified modal methods but will not be discussed in this thesis. For more details the reader is referred to \cite{43} \cite{36}.

3-4-2 Nelson’s method

An alternative method to overcome the problem of the singular matrix in Eq. (3-38) is presented by Nelson \cite{21}. This method solves equation (3-32) exactly by supplying an extra boundary condition to the singular matrix \((K - \lambda r M)\). The benefit of this method is the fact that only the eigenmode of interest is required. This can be critical for large system. The drawback of this method is the fact that a matrix factorization has to be performed for every derivative. This causes the computational cost to increase very rapidly for multiple response functions. The method proposes to write the sensitivity of an eigenvector according to

\[
\frac{\partial x_r}{\partial s} = v_r + c_r x_r.
\]  

(3-42)

Substituting Eq. (3-42) into Eq. (3-32) results in

\[
(K - \lambda r M)v_r = f
\]  

(3-43)

\[
f = - \left( \frac{\partial K}{\partial s} - \lambda r \frac{\partial M}{\partial s} - \frac{\partial \lambda r}{\partial s} M \right) x_r
\]  

(3-44)
This equation is still singular but the procedure described in Table 3-1 overcomes this to retrieve the desired result.

<table>
<thead>
<tr>
<th>Table 3-1: Nelson’s method to determine the sensitivity $\frac{\partial \lambda}{\partial s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Derive $\frac{\partial \lambda_r}{\partial s}$</td>
</tr>
<tr>
<td>(2) Define $f = - \left( \frac{\partial K}{\partial s} - \lambda_r \frac{\partial M}{\partial s} - \frac{\partial \lambda_r}{\partial s} M \right) x_r$</td>
</tr>
<tr>
<td>(3) Define $A = K - \lambda_r M$</td>
</tr>
<tr>
<td>(4) Find the degree of freedom $p$ in mode $x_r$ that has the maximal amplitude</td>
</tr>
<tr>
<td>(5) Replace row and column $p$ of $A$ with zeros except for the diagonal term which should become 1</td>
</tr>
<tr>
<td>(6) Replace row $p$ of $f$ with a zero</td>
</tr>
<tr>
<td>(7) Solve $Av_r = f$</td>
</tr>
<tr>
<td>(8) Differentiate the normalization condition to retrieve Eq. (3-41)</td>
</tr>
<tr>
<td>(9) Substitute Eq. (3-42) into Eq. (3-41) to retrieve $c_r$</td>
</tr>
<tr>
<td>(10) Compute $\frac{\partial x}{\partial s}$</td>
</tr>
</tbody>
</table>

### 3-4-3 An algebraic method

The final method for the computation of the eigenmode sensitivity is an improvement of Nelson’s method. The implementation is by far the simplest of all presented methods. The benefits & downsides are similar to that of Nelson’s method where only the eigenmode of interest is required and a matrix factorization has to be performed for every objective.

The method starts from Eq. (3-38) and rewrites it into

$$(K - \lambda_r M) \frac{\partial x_r}{\partial s} - \lambda_r \frac{\partial M}{\partial s} x_r = - \left( \frac{\partial K}{\partial s} - \lambda_r \frac{\partial M}{\partial s} \right) x_r.$$  \hspace{1cm} (3-45)

Simply combining this with the derivative of the normalization condition (Eq. (3-41)) into one matrix equation leads to,

$$\begin{bmatrix} (K - \lambda_r M) & -M x_r \\ -x_r^T M & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial x_r}{\partial s} \\ \frac{\partial \lambda_r}{\partial s} \end{bmatrix} = \begin{bmatrix} (\lambda_r \frac{\partial M}{\partial s} - \frac{\partial K}{\partial s}) x_r \\ 0.5 x_r^T \frac{\partial M}{\partial s} x_r \end{bmatrix}.$$  \hspace{1cm} (3-46)

This equation can be solved to arrive at the sensitivity of the eigenvalue and eigenmode in one step.

Besides the methods presented here iterative methods described in [2] could be interesting alternatives. These methods can be very efficient depending on the level of access to the finite element program.

### 3-4-4 Eigenvector normalization

Comparing the different methods it becomes apparent that the eigenvector normalization is always involved in the solutions steps. This is due to the fact that the result depend on the normalization that is used. Since the amplitude of a mode shape is undefined a normalization is chosen to fix this degree of freedom. Smith et al. [35] has shown the effect on the results for three different normalizations. Hooijkamp [12] used these results to investigate possible applications of this dependence on the normalization. The author shows that shape and amplitude changes of an eigenvector can be studied separately depending on the normalization that is used.

### 3-5 Sensitivity of frequency response function

In the design of mechatronic systems the FRF is an invaluable tool to assess the performance of a system. Sensitivity analysis of this response is therefore as interesting since it indicates what the effect of a design change will be. This section will present three methods of computation. First the direct and adjoint approach will be discussed for a system written according to Eq. (3-6). The third method will describe the sensitivity for a system written in modal coordinates.
3-5-1 Direct and adjoint approach

To describe the difference between the direct and the adjoint approach a formal declaration of the response function is written as

\[ r(s,\omega) = g(q(s,\omega),s), \]  

where \( r(s,\omega) \) is the response function that is dependent on the design variable \( s \) and frequency \( \omega \). The response is written as a function \( g(q(s,\omega)),s) \) which is implicitly and explicitly dependent on \( s \) and it is a function of the displacement \( q \). In the case of an FRF \( g \) is written as

\[ g(q(s,\omega)) = l \cdot q(s,\omega), \]  

were \( l \) is a vector that indicates the sensor location. The total derivative of the response \( r \) with respect to a design variable \( s \) is written as

\[ \frac{dr}{ds} = \frac{\partial g}{\partial q} \frac{\partial q}{\partial s} + \frac{\partial g}{\partial s}, \]  

were

\[ \frac{\partial g}{\partial q} = l \quad \text{and} \quad \frac{\partial g}{\partial s} = 0. \]  

**Direct method** From Eq. (3-49) it becomes apparent that the derivative of the response \( r \) requires the evaluation of the derivative \((dq)/(ds)\). Starting from Eq. (3-6) the first partial derivative is computed resulting in

\[ \left( -\omega^2M + j\omega D + K \right) \frac{\partial q}{\partial s} = \frac{\partial f}{\partial s} - \left( -\omega^2 \frac{\partial M}{\partial s} + j\omega \frac{\partial D}{\partial s} + \frac{\partial K}{\partial s} \right) q. \]  

The direct method presented in [11] creates a term on the right hand side of Eq. (3-51) that can be interpreted as a pseudo load. If this load is applied to the system and solved, the derivative is retrieved. Depending on the level of access to the FE code the factorization of the left hand side of Eq. (3-51) could be used for both the calculation of \( q \) and \((dq)/(ds)\). Taking this into account the entire procedure requires forward and backward substitutions at every frequency for each load or pseudo load. Each design variable creates a specific pseudo load and therefore the computational effort is directly related to the bandwidth and the number of design variables. The computational effort is described by

\[ \text{cost} = n_\omega \cdot n_s. \]  

Obviously the cost of the substitution steps is dependent on the size of the model however, this affects all presented methods equally. If there are multiple response functions \( r \) which all dependent on \((dq)/(ds)\) the effort does not significantly increase since only \((\partial g)/(\partial q)\) might change. This makes the direct method a suitable choice when the number of response function is larger then the number of design variables.

In the design of mechatronic systems the FRF often describes the relation between actuator and sensor. The number of response functions that need further evaluation for improvement is not likely to exceed the number of design variables. Therefore an alternative method based on the addition of an adjoint variable is presented. The method eliminates the \((dq)/(ds)\) term from Eq. (3-49) to reduce the computational cost created by \( n_s \).

**Adjoint method** Starting from Eq. (3-47) an additional term is added resulting in

\[ r(s,\omega) = g(q(s,\omega),s) + \xi (Zq - f) \]  

were \( \xi \) is the adjoint variable.
were the dynamic stiffness notation is used. The adjoint variable $\xi$ is arbitrary since the added term equals zero independent of $\xi$. Differentiating this with respect to $s$ results in

$$
\frac{dr}{ds} = \frac{\partial g}{\partial q} \frac{dq}{ds} + \frac{\partial g}{\partial s} + \frac{\partial \xi}{\partial s} (Zq - f) + \xi \left( \frac{\partial Z}{\partial s} q + Z \frac{\partial q}{\partial s} - \frac{\partial f}{\partial s} \right). \tag{3-54}
$$

Eliminating $\frac{\partial \xi}{\partial s} (Zq - f)$ from Eq. (3-54) since it equals zero and rewriting the result leads to

$$
\frac{dr}{ds} = \frac{\partial g}{\partial s} + \xi \left( \frac{\partial Z}{\partial s} q - \frac{\partial f}{\partial s} \right) + \frac{\partial q}{\partial s} \left( \frac{\partial g}{\partial q} + \xi Z^T \right). \tag{3-55}
$$

Since $\xi$ can be chosen freely the last part of Eq. (3-55) is used to eliminate $(\partial q)/(\partial s)$ by creating an adjoint problem,

$$
Z \xi = - \frac{\partial g}{\partial q}. \tag{3-56}
$$

The adjoint problem in Eq. (3-56) has to be solved for every function $g$. The resulting sensitivity of the response function which has been reduced to

$$
\frac{dr}{ds} = \frac{\partial g}{\partial s} + \xi \left( \frac{\partial Z}{\partial s} q - \frac{\partial f}{\partial s} \right), \tag{3-57}
$$

can now be efficiently obtained without forward and backward substitution procedures. Therefore the computational cost is primarily dependent on the number of adjoint problems that need to be solved and the bandwidth of the response.

$$
cost = n_\omega \cdot n_r. \tag{3-58}
$$

### Sensitivity of FRF magnitude

The presented results contain complex valued sensitivities of the frequency response. For structural modifications a designer is mostly interested in the change of magnitude. The magnitude of the FRF is defined as

$$
m = \sqrt{\Re\{q\}^2 + \Im\{q\}^2}. \tag{3-59}
$$

Differentiating this expression leads the sensitivity of the response magnitude

$$
\frac{\partial m}{\partial s} = \frac{1}{2} \frac{1}{\sqrt{\Re\{q\}^2 + \Im\{q\}^2}} \cdot 2 \left( \Re\{q\} \Re\{\frac{\partial q}{\partial s}\} + \Im\{q\} \Im\{\frac{\partial q}{\partial s}\} \right). \tag{3-60}
$$

Simplification results in

$$
\frac{\partial m}{\partial s} = \frac{1}{m} \left( \Re\{q\} \Re\{\frac{\partial q}{\partial s}\} + \Im\{q\} \Im\{\frac{\partial q}{\partial s}\} \right). \tag{3-61}
$$

### 3-5-2 Modal method

Yoshimura [47] and Sharp [32] introduce the sensitivity analysis of an FRF. By starting with an alternative form of Eq. (3-12), Yoshimura [47] uses the receptance notation as a starting point. Splitting the response $y$ into the receptance matrix and a force vector $y = Ys$. The receptance matrix is written as

$$
Y = \sum_{m=1}^{n} \frac{C_m}{1 - \left( \frac{\omega}{\omega_{m,\omega}} \right)^2 + 2j \zeta_{m,\omega} \frac{\omega}{\omega_{m,\omega}}}, \tag{3-62}
$$

where $C_m$ represents the modal flexibility matrix written as

$$
C_m = \frac{x_{(m)}^T x_{(m)}^T}{x_{(m)}^T K x_{(m)}}. \tag{3-63}
$$
Consequently the derivative of the separate terms is calculated.

\[
\frac{\partial Y}{\partial s} = \frac{\partial C_m}{\partial s} \left( 1 - \left( \frac{\omega}{\omega_{0m}} \right)^2 + 2j\zeta_m \frac{\omega}{\omega_{0m}} \right) - C_m \frac{\partial}{\partial s} \left( 1 - \left( \frac{\omega}{\omega_{0m}} \right)^2 + 2j\zeta_m \frac{\omega}{\omega_{0m}} \right)
\]

(3-64)

\[
\frac{\partial C_m}{\partial s} = -\frac{1}{\omega_m^4} x_{(m)}^T x_{(m)}^T \frac{\partial K}{\partial s} x_{(m)} + \frac{1}{\omega_m^2} \frac{\partial x_{(m)}}{\partial s} x_{(m)}^T + \frac{2}{\omega_m^4} x_{(m)} \frac{\partial x_{(m)}}{\partial s} - \frac{2}{\omega_m^4} x_{(m)}^T x_{(m)}^T K \frac{\partial x_{(m)}}{\partial s}
\]

(3-65)

\[
\frac{\partial}{\partial s} \left( 1 - \left( \frac{\omega}{\omega_{0m}} \right)^2 + 2j\zeta_m \frac{\omega}{\omega_{0m}} \right) = 2 \frac{\omega^2}{\omega_{0m}^2} \frac{\partial \omega_m}{\partial s} + 2j \frac{\partial \zeta_m}{\partial s} \frac{\omega}{\omega_{0m}} - 2j \zeta_m \frac{\omega}{\omega_{0m}} \frac{\partial \omega_m}{\partial s}
\]

(3-67)

\[
\zeta_m = \frac{\omega_m x_m^T D x_m}{2x_m^T K x_m}
\]

(3-68)

\[
\frac{\partial \zeta_m}{\partial s} = \frac{1}{x_m^T K x_m} \left( \frac{1}{4\omega_m} \frac{\partial \omega_m^2}{\partial s} x_m^T D x_m + \omega_m \frac{\partial x_m^T}{\partial s} D x_m + \frac{1}{2} \omega_m \frac{\partial D}{\partial s} x_m - 2 \zeta_m \frac{\partial x_m^T}{\partial s} K x_m - \zeta_m x_m^T \frac{\partial K}{\partial s} x_m \right)
\]

(3-69)

It is clear that the implementation of the method requires a high number a vector and matrix algebra. Qu et al.[27] describes methods to reduce the number of included modes and thereby focusing the calculation on a specific frequency region. Han [11] chooses to reduce the model with modal order reduction to increase efficiency. It has a similar effect as reducing the number of included modes since it limit’s the accuracy to a specified bandwidth.

The modal method allows for a direct or adjoint approach. These derivations are not presented here. When computing the FRF sensitivity for large models and efficiency is of key importance it is advisable to use these methods. By applying the modal coordinates both the response and it’s derivative can be estimated by modal truncation. Combining these formulations with the concept of modal acceleration, results in efficient approximations of the exact derivatives presented in Section 3-5-1.

### 3-6 Sensitivity of nodal location

Pritchard et al.[26] introduces the sensitivity of a nodal point for one and two dimensional structures. Figure 3-2 shows a mode of a cantilever beam and dotted line representing the modified mode. The movement of the nodal point is the described by

\[
y_{np}^* = y_{np} + \frac{\partial y_{np}}{\partial s} ds.
\]

(3-70)

The sensitivity of the nodal location is based on a Taylor expansion on the nominal nodal point. Neglecting the higher order terms leads to

\[
\mathbf{x} \left( y_{np} + dy_{np}, s + ds \right) = \mathbf{x} \left( y_{np}, s \right) + \frac{\partial \mathbf{x}}{\partial y} \bigg|_{y_{np}, s} dy_{np} + \frac{\partial \mathbf{x}}{\partial s} \bigg|_{y_{np}, s} ds.
\]

(3-71)

Using the fact that a node has zero amplitude and rewriting the previous result produces

\[
\frac{\partial \mathbf{x}}{\partial y} \bigg|_{y_{np}, s} dy_{np} + \frac{\partial \mathbf{x}}{\partial s} \bigg|_{y_{np}, s} ds = \left( \frac{\partial \mathbf{x}}{\partial y} \bigg|_{y_{np}, s} \frac{dy_{np}}{ds} + \frac{\partial \mathbf{x}}{\partial s} \bigg|_{y_{np}, s} \right) ds = 0.
\]

(3-72)
Since $ds$ can be any arbitrary value the equation can be solved for the sensitivity of the nodal location resulting in

$$\frac{dy_n}{ds} = -\left. \frac{\partial \mu_a}{\partial x} \frac{\partial x}{\partial y} \right|_{y_{np},s}.$$ (3-73)

### Alternative approach

An alternative method to manipulate nodal locations is to define an error function. This function is based on the local amplitude of a vibration mode which should be zero to become a node. The sensitivity of the local amplitude can be used to apply design changes that will create nodes at the desired location. Two examples of such an approach are given in [48] and [14] respectively.

### 3-7 Sensitivity of anti-resonance frequency

The sensitivity of an ARF can be computed in a similar fashion as the eigenvalue problems presented in Section 3-3. Wang [42] presents two methods similar to the modal and algebraic method.

#### 3-7-1 Modal method

Since $\mu_a = \omega_a^2$ the sensitivity can be written as

$$\frac{\partial \omega_a}{\partial s} = \frac{\partial \mu_a}{\partial s} \frac{\omega_a}{2\omega_a}.$$ (3-74)

The solution from the modal method can now be used for $\mu_a$ with a small difference. Due to the possible asymmetry of $\tilde{K}$ and $\tilde{M}$ the left and right eigenvectors are used

$$\frac{\partial \mu_a}{\partial s} = \eta^T \left( \frac{\partial \tilde{K}}{\partial s} - \mu_a \frac{\partial \tilde{M}}{\partial s} \right) \phi \frac{\eta^T \tilde{M} \mu}{\eta^T \tilde{M} \mu}.$$ (3-75)

#### 3-7-2 Algebraic method

A second method starts from the forced response at anti resonance frequency

$$Z \eta_a = b \quad \text{with} \quad Z = K - \omega_a^2 M.$$ (3-76)
$$c \eta_a = 0.$$ (3-77)
Differentiating Eq. (3-76) with respect to a design variable $s$ results in
\[
\frac{\partial Z}{\partial s} q_a + Z \frac{\partial q_a}{\partial s} = \frac{\partial b}{\partial s}.
\] (3-78)

Differentiating Eq. (3-77) with respect to a design variable $s$ results in
\[
c \frac{\partial q_a}{\partial s} = -\frac{\partial c}{\partial s} \omega_a.
\] (3-79)

Rewriting the sensitivity of $Z$ into the separate components yields
\[
\frac{\partial Z}{\partial s} = \frac{\partial K}{\partial s} - \frac{\partial \omega_a^2}{\partial s} M - \omega_a^2 \frac{\partial M}{\partial s}.
\] (3-80)

Combining Eq. (3-79) and Eq. (3-80) into a matrix equation provides
\[
\begin{bmatrix} Z & -Mq_a \\ c & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial q_a}{\partial s} \\ \frac{\partial \mu_a}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{\partial b}{\partial s} + (\mu_a \frac{\partial M}{\partial s} - \frac{\partial K}{\partial s}) q_a \end{bmatrix}.
\] (3-81)

### 3-8 Summary on sensitivity analysis

This chapter combines a set of response functions with their sensitivity derivations. The included functions such as eigenvalues and frequency response functions are typically used to assess the dynamic behavior of a system.

The presented sensitivities combine to only a small fraction of all the developed methods. This collection should provide a good starting point for the investigation into the application. When efficiency is important most of the presented methods have alternative derivations that are preferable. Since the following chapters will focus on the usage of the results this is of less importance for this thesis. The sensitivity derivations provide a knowledge base for the case studies in Chapter 5 till Chapter 7. One aspect of sensitivity analysis remains to be discussed which is the design variable. This is covered in the next chapter.
For most applications the goal of sensitivity analysis is to find the structural properties that have either a large or minimal effect on a certain response. By selecting a set of design variables the user reduces the design space. In most cases this is no problem however a designer should be aware that the choice of design variable already requires some fundamental knowledge on the sensitivity of the response.

In general design variables can include structural, boundary, actuation and sensor parameters. Structural parameters can include Young’s modulus, material density and width or height of an element. Design variables can also affect the boundary conditions of a system. Actuation parameters can include the amplitude of the applied force but also the spatial distribution and the frequency content. Sensor parameters could include the absolute or relative position. This chapter will only cover design variables that belong to the structural domain.

4-1 Design variables for beams and plates

In the field of sensitivity analysis beams and plates are the most practical elements to use. This is due to the fact that there parameters such as width and height are directly related to geometrical property of the global structure. This is not necessarily the case for solids as will be discussed in the following section.

Besides the geometrical parameters all element types contain the Young’s modulus and the material density as possible design variables. However, the combination of both changes in stiffness and mass that is provided by the width and height can give crucial insight as will be shown in the following example.

Eigenvalue sensitivity for a cantilever beam

The cantilever beam in Figure 3-2 is drawn in its second vibration mode. If the eigenfrequency needs to be de-tuned (see Section 2-1) there are multiple candidates for a design variable. As stated all element types can easily be differentiated with respect to a Young’s modulus or material density. The result for the sensitivity analysis for these two variables are shown in Figure 4-1a and Figure 4-1b. Apart from the fact that a change in either $E$ or $\rho$ will be hard to achieve in practice the results lack the coupling between changes in mass and stiffness. Two variables that are available for a beam can be either the width or the height. Engineering judgment leads to the idea that the height will mostly affect stiffness and the width will linearly influence the mass and the stiffness. However it is to be expected that an increase in height will not be effective near the free edge of the beam. Sensitivity analysis indicates the cross over point where the effect of stiffness increase is smaller then the addition of mass.
Figure 4-1: Eigenvalue sensitivity of a cantilever beam for several design variables. The height and the
width combine a change in stiffness and mass resulting in more meaningful results for structural optimization.

Figure 4-1c and Figure 4-1d show the resulting sensitivity plots for both variables. As expected the
height has a mostly positive sensitivity apart from an area near then tenth element and at the free edge
of the beam. The width of the beam shows a linear combination of the stiffness and mass change. Near
the clamping the stiffness change dominates the response and near the free end the mass determines
the effect on the response function.

4-2 Design variables for solids

For large complex structures the previous design variables will be available when the model consists
out of beam and plate elements. When the model is build with solid elements these variables are not
readily available and it could take a lot of effort to define them in a large model. The most practical
design variables for a solid are the material density and the Young’s modulus. In Section 3-3-4 the
similarity between the kinetic and elastic energy of a modeshape and the sensitivity of the corresponding
eigenvalue is discussed. It is shown that the distribution of the two energies per element is equal to the
sensitivity per element.
4-3 Reduction of design space by modification vectors

The number of described design variables for beams and plates increases linearly with the number of elements. This can quickly lead to efficiency problems for the calculation of the sensitivities. The

**Parameter free shape optimization**

As its name already predicts this method is not dependent on the declared parameters but uses the nodes of a mesh as design variables. By selecting a set of exterior nodes and perturbing them over a small distance orthogonal to the surface, the derivative of the system matrices can be determined by finite difference. This method requires very little implementation effort by the user for an existing code. Commercial software packages seem to be using this method but in a less general form (Femtools). The software that allows the sensitivities to be plotted requires the user to create a mesh around the existing mesh. This user defined mesh is then used to perform the perturbations. This reduces the amount of design variables but also requires quite some effort from the user. The more general approach with only the existing mesh of the model is in development however, this is still prone to computation errors.

**4-3 Reduction of design space by modification vectors**

The number of described design variables for beams and plates increases linearly with the number of elements. This can quickly lead to efficiency problems for the calculation of the sensitivities. The

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(a) Kinetic energy of the second modeshape of cantilever beam. The distribution is equal to the sensitivity of the corresponding eigenvalue with respect to the material density $\rho$.

(b) Elastic energy of the second modeshape of cantilever beam. The distribution is equal to the sensitivity of the corresponding eigenvalue with respect to the Young’s modulus $E$.

**Figure 4-2:** Kinetic and elastic energy of second modeshape for a cantilever beam. These results are readily available in ANSYS and are strongly related to the eigenvalue sensitivities.
computational cost can be decreased by reducing the design space. A designer might for example only be interested in symmetrical design changes for a symmetric structure.

Hooijkamp [13] applies a reduction in design space by using the thermal modes as a basis to describe design changes. The word 'basis' refers to a mathematical concept where a set of vectors forms the basis of a space. Any vector within this space can be described by a combination of the vectors that form the basis. In this case the "space" would represent the design space of allowable geometry changes. In the application of Hooijkamp the thermal modes indicate the shape change of the entire structure. These modes have no relevance for structural dynamics and therefore this section will investigate alternatives. The section starts with the mathematical description of the sensitivity with respect to a geometry modification described by a vector.

4-3-1 Sensitivity with respect to a vector

The geometry change of a structure as a function of vectors can be described by

$$\Delta s = \sum_k \Phi_k \sigma_k,$$  \hspace{1cm} (4-1)

where $\Phi_k$ is the thermal mode in the case of [13] and $\sigma_k$ the participation factor to the total design change. The sensitivity of such a design modification follows from the product rule of differentiation. For example the sensitivity of eigenmode $x_i$ with respect to a design change according to mode $\Phi_k$ is written as

$$\frac{\partial x_i}{\partial \sigma_k} = \frac{\partial x_i}{\partial s_l} \frac{\partial s_l}{\partial \sigma_k},$$  \hspace{1cm} (4-2)

where $s_l$ is the element-wise design variable for example the height. The term $\frac{\partial s_l}{\partial \sigma_k}$ is determined by

$$\frac{\partial s_l}{\partial \sigma_k} = \Phi_l^k.$$  \hspace{1cm} (4-3)

All the presented sensitivity functions from the previous chapter depend on the derivatives of the system matrices. For example the modal method for the eigenvector derivative (Eq. (3-40)). These derivatives can now be written in terms of the vector shape change according to

$$\frac{\partial K}{\partial \sigma_k} = \sum_{l=1}^{n} \frac{\partial K}{\partial s_l} \Phi_l^k,$$  \hspace{1cm} (4-4)

and

$$\frac{\partial M}{\partial \sigma_k} = \sum_{l=1}^{n} \frac{\partial M}{\partial s_l} \Phi_l^k,$$  \hspace{1cm} (4-5)

where $n$ is the number of elements. By selecting the thermal modes the designer can influence the possible solution directions in an efficient way. Due to the orthogonal basis a small selection of modes can describe a large variety in shapes. Thermal modes are a practical basis for thermo mechanical optimization but they have no relevance for structural dynamics. The next section will discuss possible bases that are applicable for structural dynamics.

4-3-2 Applicable modification basis for structural dynamics

The following subsection will discuss four different bases that could be used for design modifications. Benefits and downsides are discussed such as the implementation effort and the relevance of the structural change that the basis can describe. The different bases are linked to the mechanical structure and thereby implicitly contain information on for example sensitivity. By linking the basis to general properties of a mechanical structure the method remains applicable for different models.
Modal basis

The first basis of structural change is described by the modal basis of the system. This requires the transformation of the modal movement to a structural design change. In general the number of nodes is not equal to the number of elements. To link the node movement to an element the amplitude of the nodes that belong to element \( l \) are interpolated to retrieve one value. The movement in the three translational directions is summed to one value. Details on how to transform the modes are described in Section 7-6-1.

The limitation of this basis lies in the fact that it can only describe structural change at locations where there is considerable modal movement. However stiffness changes can be very effective at locations that show little movement. Think of a cantilever beam where the stiffness change near the clamped end of the beam has the highest influence on the eigenfrequency. If the structural design variable affects mostly stiffness there is no physical reason why the modal basis would be very efficient.

Elastic en kinetic energy distribution

The basis of the elastic and kinetic energy distribution is selected to overcome the limitations of the previous modal basis. Since the elastic energy is directly linked to the stiffness this basis can describe parameter changes at locations where stiffness has a large influence on the eigensolution. The implementation of the this basis is quite simple since the number of elements is equal to the number of corresponding indexes in the energy vector. Either the elastic or kinetic energy can directly be implemented as a scaling vector for the element sensitivity of Eq. (4-4) or Eq. (4-5).

Limitations of this basis lie in the fact that all the basis vectors are symmetric for a symmetric design. This prevents asymmetrical design perturbations which could be seen as a limitation. Apart from the symmetry aspect the distributions do not contain a sign change. This results in design perturbations that are either positive or negative for all elements. The modal basis does include these sign changes.

Eigenvalue sensitivity distribution

In Section 3-3-4 it was discussed that the elastic and kinetic energy distribution have a direct relation to the sensitivity of an eigenvalue. The energy basis is equivalent to a basis of eigenvalue sensitivities with respect to the Young's modulus and the material density. The energy basis can thus be seen as an eigenvalue sensitivity basis with the difference that no actual sensitivity needs to be calculated for the energy vectors.

From Eq. (4-2) it becomes clear that the derivatives of the mass and stiffness matrix per element are always needed. Assuming that \( (\partial M)/(\partial s_l) \) and \( (\partial K)/(\partial s_l) \) are calculated irrespective of the selected basis the eigenvalue sensitivity distribution can be retrieved efficiently. In the case of a geometrical parameter the eigenvalue sensitivity results supply an alternative basis for structural modification. Obviously in the case of de-tuning the sensitivity distribution corresponding to the eigenvalue that needs to be altered is the most efficient design perturbation. Therefore the reduction doesn’t offer much for eigenvalue sensitivity. However modeshape and transfer sensitivities are greatly reduced in design variables and thereby the optimization becomes more efficient.

The eigenvalue sensitivity basis combines the benefits of the energy and the modal basis since it can both describes addition as well as removal of material. Furthermore the method is easy to implement since the number of derivatives equals the number of elements.

4-4 Simplification of the design space

Besides the efficiency aspect there are alternative arguments why the reduction of the design space could be useful. By restricting the design changes, an estimation could be made on the expected
effect. This section describes a modification basis that creates the possibility to estimate the required structural change for a targeted modeshape modification of a 1D beam structure.

### 4-4-1 Modal difference basis

In the case of modeshape optimization a specific modeshape is targeted for a change in its response. In Section 3-4-1 it is shown that the shape change of a mode is described by all the other modes of the system. For now it is assumed that the required improvement can be described by one dominant mode \( x_p \).

By focusing on \( x_p \) the value of \( c_r(p) \) should be as high as possible. Since \( x_r \) and \( x_p \) are known, only the structural change matrix is left as an unknown. When the sensitivity of the mass and stiffness matrix with respect to a certain design variable is captured in one matrix \( B \) as

\[
B = \frac{\partial K}{\partial s} - \lambda_r \frac{\partial M}{\partial s},
\]

the product of

\[
c_r(p) = -x_r^T B x_r \quad \frac{\lambda_r - \lambda_p}{(\lambda_p - \lambda_r)}
\]

needs to be maximized. To achieve this the numerator should be as large as possible and the denominator as small as possible. The denominator indicates that modes that are closely spaced to \( x_r \) in the frequency domain will mostly describe the derivative.

In general there are infinite possibilities to maximize an arbitrary vector matrix vector product in the numerator of Eq. (4-7). In the case of modeshapes the vectors \( x_r \) and \( x_p \) satisfy certain conditions. They are continuous and smooth and typically show a wave like behavior. For a double clamped beam as shown in Figure 4-3a the vectors could be represented graphically by Figure 4-3b.

The next step is to assume that the structural changes are creating a diagonal \( B \) matrix. This could for example be created by adding lumped masses to the nodes. With this assumption it becomes apparent that the multiplication would be optimal if the sign of certain elements of \( B \) would be negative. This is illustrated in Figure 4-4a. The amplitude of the structural change is undetermined but taking in account the effectiveness of mass change for a specific mode the shape of \( x_p \) seems a logical choice. This results in figure Figure 4-4b. The fact that mode \( x_p \) could be used as a basis for design change to create a shape change that corresponds with this mode is only true when the mode \( x_r \) does not contain a sign change. An example where this is not the case is shown in figure 4-5. Note that the vector on the diagonal of the \( B \) matrix can still very easily be computed with equations (4-8) till (4-9).

\[
X_{\text{unit}} = \frac{X_{\text{translation}}}{|X_{\text{translation}}|}
\]

\[
b_{\text{diag}} = x_{\text{unit}}^r \cdot x_{\text{unit}}^p
\]
4-4 Simplification of the design space

(a) Graphical representation of an optimal diagonal $B$ matrix to maximize the product in Eq. (4-7).

(b) Combining sign and amplitude of mode $x_p$ to describe the diagonal structural change matrix. This leads to a very simple shortcut to determine the required structural change.

Figure 4-4: Graphical representation of weighting factor that determines the contribution of mode $x_p$ to the derivative of mode $x_r$.

This vector is constructed only using the translation Dof in the axis corresponding to the modal movement. The length of vector $b_{diag}$ is equal to the number of unconstrained nodes and can therefore be visualized in the mesh of the model. For instance the color of the node can indicate a plus or minus sign. Result per node could then visually be interpolated to determine the required change for an element.

Amplitude of change When the designer has identified whether the change will primarily change stiffness or mass the amplitude of change needs to be determined. If the design variable will mostly create stiffness change the elastic energy of mode $x_r$ and $x_p$ could be used to determine where a high amplitude would be most efficient. When mass change is most important the kinetic energy can provide insight into the most efficient structural change.

4-4-2 Limitations of Modal difference basis

In the previous paragraph a modal difference basis is introduced to estimate the structural change that is required for a desired modeshape modification. This seems to work well for 1D structures since the matrices are generally block diagonal. 2D or 3D structures tend to be less banded due to the interconnection of several elements at one node. Such a connection creates indices in the system matrices that lie far from the diagonal. This renders the assumption on the diagonal $B$ invalid. Several experiments have showed a poor predictive behavior for models that are not described by block diagonal system matrices.

Figure 4-5: When both modes contain sign changes the vector $b_{diag}$ is not equal to either mode $x_r$ or mode $x_p$. The required diagonal is computed with Eq. (4-9)
4-5 Summary on design variables

This chapter introduces some of the common design variables for beam, plate and solid elements. By calculating the eigenvalue sensitivity for a cantilever beam the potential benefits of geometrical design variables are exemplified. Solid elements typically lack a direct link with the geometry of the model and therefore the calculation of sensitivities is more labor intensive. The material properties such as the Young’s modulus and the material density are available for all element types. Section 4-2 shows how the elastic and kinetic energy distributions are equivalent to the eigenvalue sensitivities distributions with respect to $E$ and $\rho$.

The third section discusses a method to reduce the design space by application of modification vectors. These vectors that describe a potential shape change of the structure are constructed from solution results. By using solution results such as the modeshapes or elastic and kinetic energy distributions the method is applicable for a wide variety of models.

By combining a reduction in design variables with the sensitivities of modeshapes Section 4-4 introduces an approximation method. The method assumes that a specific eigenvector $x_r$ needs to be changed in a shape that is described by a different vector $x_p$. Furthermore, the method assumes that the required shape modification creates a block diagonal sensitivity matrix. This method works for 1D models but is not generally applicable for 2D and 3D structures.
In the conceptual design phase of a mechatronic system the performance is tested for several load cases. Motion control dynamics, floor vibration and acoustic disturbance are typical dynamic load cases. This chapter includes the analysis of a conceptual mechatronic system with the focus on the effect of floor vibrations. In the first section the floor vibration levels that will be used throughout the remainder of the analysis are defined.

Section 5-2 introduces a 1D model of a conceptual design for a mechatronic system. Details of the system will not be discussed due to commercial interests. The 1D model is used to characterize the response in $z$ direction. Sensitivity analysis is employed to determine possible improvements of the concept. Such an improvement is found and a detailed investigation describes the underlying principle. Furthermore an improved model is introduced and the advantages and disadvantages are discussed.

The same system can also be modeled for its response in $x$ direction. The basic layout of this second model is similar to the model describing the response in $z$ direction yet it incorporates a controller. This 1D model in $x$ direction is introduced in Section 5-3 after which the effect of the controller on the applicability of the previously found improvement is investigated. By designing a controller for both a baseline system and an "improved" system a comparison is made and the presence of potential problems is confirmed.

### 5-1 Floor vibration

The levels of floor vibration are typically denoted by the so called VC-curves. Gordon [10] describes the different curves that are developed over time. The level of floor vibrations that the mechatronic system of this case study will be subjected to is defined by a specific curve. This curve is based on the known vibration levels at the location where the system must operate. The power spectral density is shown in Figure 5-1a and the corresponding cumulative amplitude spectrum left and right are shown in Figure 5-1b. The plots indicate that the largest contribution lie in the frequency range from 20 till 300 Hz.

### 5-2 1D model in $z$ direction

The system being studied is modeled with a simple lumped mass model which describes the motion in $z$ direction. The model consists out of three mass spring/damper combinations. In this case the model
Case study: passive vibration control for conceptual design

(a) Power spectral density of floor vibration specified at plant location

(b) Cumulative amplitude spectrum integrated from the left and right

Figure 5-1: Properties of floor vibration disturbance signal for the 1D model

(a) 1D model of mechatronic system in z direction. $M_1$, $M_2$ and $M_3$ represent a wafer stage, a frame and a base respectively. The base supports the main function of the system.

(b) Modeshapes of the 1D model. The first mode shows a piston movement of all masses. In the second mode, $M_1$ moves out of phase with $M_2$ and $M_3$. The third mode describes movement of the base with respect to the frame.

Figure 5-2: 1D model in z direction and the corresponding modeshapes

represents a system where $M_1$ is a wafer stage, $M_2$ is the frame and $M_3$ is the base that supports the main function of the device. To include the acceleration of the floor a parameter $x_0$ is defined.

The model in z direction described by Figure 5-2a includes a spring damper $k_1$&$c_1$ that represents the stiff direction of a leaf spring. The second spring damper represents the floor vibration isolation system with high damping and low stiffness. The third spring damper represents a stiff connection between the frame and the base. The performance is determined by the relative movement of $M_1$ and $M_3$ given by

$$z_p = z_1 - z_3.$$  \hspace{1cm} (5-1)

The parameters of the model are given in Table 5-1.

5-2-1 Modeshapes and response to floor vibrations

With the information from Section 5-2 the eigenfrequencies and modeshapes can be calculated. The frequencies are stated in Table 5-1 and the modeshapes are drawn in Figure 5-2b.
In the case of floor vibrations, an acceleration will be applied to the ground. To calculate the transfer from this so called base excitation a slightly different equation of motion compared to the usual mass spring equations is required. Formulation of the equations of motion results in a right hand term that depends on $\ddot{x}_0$ and the lumped masses,

$$
\begin{bmatrix}
M_1 & 0 & 0 \\
0 & M_2 & 0 \\
0 & 0 & M_3 \\
\end{bmatrix}
\begin{bmatrix}
\ddot{z}_1 \\
\ddot{z}_2 \\
\ddot{z}_3 \\
\end{bmatrix}
+
\begin{bmatrix}
c_1 & -c_1 & 0 \\
-c_1 & c_1 + c_2 + c_3 & -c_3 \\
0 & -c_3 & c_3 \\
\end{bmatrix}
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\end{bmatrix}
+
\begin{bmatrix}
k_1 & -k_1 & 0 \\
-k_1 & k_1 + k_2 + k_3 & -k_3 \\
0 & -k_3 & k_3 \\
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
\end{bmatrix}
=
\begin{bmatrix}
M_1\ddot{x}_0 \\
M_2\ddot{x}_0 \\
M_3\ddot{x}_0 \\
\end{bmatrix}
(5-2)
$$

Transforming the equation to the frequency domain results in:

$$
\left(-\omega^2 M + j\omega C + K\right)\mathbf{z}(\omega) = m\ddot{x}_0(\omega).
(5-3)
$$

This leads to the receptance formulation of the base excitation

$$
\frac{\mathbf{z}(\omega)}{\ddot{x}_0(\omega)} = \frac{m}{\left(-\omega^2 M + j\omega C + K\right)}.
(5-4)
$$

The resulting transfer function between floor disturbance ($x_0$) and the performance indicator $z_p$ is given in Figure 5-3a. This transfer function can be considered as the response function which should be minimized to suppress the effect of floor vibrations.

### 5-2-2 Sensitivity analysis of the response to floor vibration

The FRF given in Figure 5-3a supplies a designer with a clear response which should be minimized. The parameter change that will lead to a minimization can be determined by calculating the derivatives of the FRF. How to compute such a derivate is discussed in Section 3-5.

Figure 5-3b shows the derivative of the FRF with respect to the spring stiffnesses. The derivatives are scaled with the design variable ($s$) according to Section 3-2. In this case the design variables are the three spring stiffnesses. The derivatives with respect to damping or mass could also be plotted in the same manner. In this case these design variables did not result in additional insight so they are omitted from the figure for the sake of comprehensibility.

The derivative with respect to the third spring stiffness ($k_3$) shows interesting behavior. A large part of the frequency domain shows a positive derivative meaning that the spring can be weakened resulting in a decrease of the FRF. In the conceptual design stage most structural parameters are often estimated to create the lowest mass and stiffest construction. This is partly due to the notion that motion control is limited by the lowest resonance frequencies of a system. By minimizing the mass and maximizing the stiffness the resonances are pushed as high as possible. For this reason, a sensitivity result which indicates that an improvement could be made by lowering a stiffness is considered interesting. The following paragraph investigates the explanation for the observed direction of improvement.

### 5-2-3 Clarification of the FRF sensitivity results

To clarify the FRF sensitivity results the modeshapes are investigated in more detail. Inspection of the modal derivatives with respect to the third spring stiffness should create more insight into the suggested improvement of the FRF.

#### Table 5-1: Parameters for 1D model in $z$ direction of mechatronic system

<table>
<thead>
<tr>
<th>Number</th>
<th>M [kg]</th>
<th>k [N/m] (z)</th>
<th>$\zeta$ (%) (z)</th>
<th>Freq [Hz] (z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>22</td>
<td>83.45e6</td>
<td>0.01</td>
<td>1.94</td>
</tr>
<tr>
<td>2</td>
<td>50</td>
<td>13107</td>
<td>0.25</td>
<td>358.6</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>2.12e8</td>
<td>0.01</td>
<td>778.1</td>
</tr>
</tbody>
</table>
(a) Base excitation response $z_p$ for floor vibrations $x_0$ of the 1D model in $z$ direction. The three modeshapes of the system are all observable in the response.

(b) Double plot of the FRF combined with the sensitivity results. The left y-axis correspond with the FRF. The right y-axis indicate the scaled sensitivity values.

Figure 5-3: FRF from floor vibrations to $z_p$ and the sensitivity plot with respect to stiffness

The modeshape derivatives for weakening of $k_3$ are shown in Figure 5-4. The figure shows that the shape change of the second mode ($x_2$) could make it unactuated. This is derived by observing that the amplitude $z_2$ will be decreased for a weakening of $k_3$. Reanalysis indicates that $z_2$ could be made zero with a proper choice of $k_3$.

Applying such a change to $k_3$ will also affect the other modeshapes. The shape change of the third mode ($x_3$) appears to be beneficial for both modes since it will make it unobservable. This can be derived by observing that the sensitivity indicates that the relative motion between $z_1$ and $z_3$ will be reduced. Reanalysis indicates that the value of $k_3$ which renders $x_2$ unactuated, makes the third mode ($x_3$) unobservable. This implies that the amplitude of $z_1$ and $z_3$ in mode $x_3$ are exactly equal.

In summery, by tweaking the spring stiffness $k_3$ all of the modes will be removed in the FRF from floor vibration to $z_p$. Therefore the system becomes insensitive to disturbances acting on the frame or originating from its connection to the ground. The above derivation is a very complicated way to come to a very simple and obvious conclusion. If the ratio between $M_1$ and $k_1$ is equal to the ratio of $M_3$ and $k_3$ they will both have the same displacement and thereby create zero error. The system is thus balanced in such a way that the relative position difference remains zero. Note that this is only true for disturbances acting on the frame or its

Figure 5-4: Modeshape sensitivity of the three structural modes with respect to the spring stiffness. The dotted arrows indicate the direction of amplitude change for the corresponding dof in the mode.
connection to ground. However in practice the solutions always seem trivial and obvious after they have been determined. The model of hand is simple and therefore the resulting optimization is also simple. With the addition of several more masses, interconnecting springs or including more degrees of freedom, a potential for improvement is less trivial to find. The FRF sensitivity will however indicate these potentials independently of the model complexity.

### 5-2-4 Introduction of a balanced model

In the previous section it was determined that the baseline model of the system can be improved by balancing the first and third mass spring combination. Either the spring stiffness $k_3$ or the mass $m_3$ can be altered to achieve balancing. By reducing $k_3$ the increase in amplitude will not lead to an increase in spring forces. However, the introduced flexibility will create a higher sensitivity with respect to disturbances acting on $m_3$. The opposite is true when $m_3$ would be increased to obtain balancing.

For this case study $k_3$ is slightly reduced and $m_3$ approximately doubled to obtain a balanced model in $z$ direction. The new parameters are stated in Table 5-2.

For these values the transfer between floor vibrations and $z_p$ is zero. However, disturbance forces can act on all masses. The following figures compare the FRF’s of the balanced model with respect to the original baseline for alternative disturbance sources. Figure 5-5a plots the transfer function of a disturbance forces acting on the stage of the original system and the balanced version. Figure 5-5b shows the results for a disturbance on the base. It depends on the objective what could be considered an improvement. Disturbances on the stage create a similar static gain but with less fluctuations in gain and phase. A disturbance on the base creates a slightly larger error since the stiffness is reduced by a small amount to achieve a balanced system.

**Table 5-2:** Parameters for balanced 1D model in $z$ direction

<table>
<thead>
<tr>
<th>Number</th>
<th>$M$ [kg]</th>
<th>$k$ [N/m] ($z$)</th>
<th>$\zeta$ % ($z$)</th>
<th>$Freq$ [Hz] ($z$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>22</td>
<td>15780</td>
<td>0.01</td>
<td>1.837</td>
</tr>
<tr>
<td>2</td>
<td>50</td>
<td>13107</td>
<td>0.25</td>
<td>309.97</td>
</tr>
<tr>
<td>3</td>
<td>26.36</td>
<td>1e8</td>
<td>0.01</td>
<td>434.77</td>
</tr>
</tbody>
</table>
Case study: passive vibration control for conceptual design

By using $m_3$ for balancing, there is no negative effect on the disturbance rejection for forces acting on the stage or the base. The mass is increased by a factor 2.3 which will result in the same increase of the spring forces in $k_3$ as a result of floor vibrations. An additional analysis in a more detailed phase of development must point out whether this is acceptable. When the increased forces create an excessive amount of deformations the balancing could still be achieved by reducing the stiffness $k_3$. However, this will deteriorate the disturbance rejection for forces acting on the stage and the base.

5-2-5 Sensitivity to imperfect balancing

The perfectly balanced system has a zero transfer function for disturbances acting on the frame or originating from the ground to the performance indicator $z_p$. When either $k_3$ or $m_3$ deviate from the balanced value the resulting FRF will be a percentage of the original unbalanced FRF. Sensitivity analysis confirms this linear behavior. The resulting sensitivities are plotted in Figure 5-6. Mass and stiffness perturbation have a similar effect. A 1% perturbation of either mass or stiffness will create a static gain of approximately $2 \cdot 10^{-9}[m/(m/s^2)]$. Unbalanced damping will have a significantly less effect since damping forces are much smaller than the inertial and spring forces.
5-3 1D mechanical model that contains a controller

In the previous section it was shown that a balanced system creates an ideal disturbance suppression for floor vibrations in z direction. The following section will investigate the potential gain of mechanical balancing in x direction where a controller is active.

5-3-1 1D model in x direction

Figure 5-7 describes the system in x direction were the spring damper $k_1$&$c_1$ represents a leaf spring that has minimal stiffness. The second spring damper represents the floor vibration isolation system with high damping and low stiffness. The third spring damper represents a stiff connection between the frame and the base. A collocated controller between the stage and the frame creates the required motion control for the function of the system. However, the performance is defined as the relative motion between $M_1$ and $M_3$ given by,

$$ x_p = x_1 - x_3. $$

(5-5)

To model the disturbance rejection a controller should be included since it acts as a virtual spring/damper when applied in a collocated configuration. Due to this behavior of the controller the idea arises that this virtual spring/damper might also be balanced with $k_3$&$c_3$.

To investigate this possibility two systems are defined. System 1 is the original baseline model and System 2 is a model that is prepared for balancing with a controller that has a 100 Hz bandwidth. This means reducing the stiffness $k_3$ or increasing the mass $m_3$ with the appropriate amount to create a frequency close to the 100 Hz.

The system will be balanced when $m_1$ (the stage) and the virtual spring from the controller create a single mass spring system that is equivalent to the a mass spring system of $m_3$ and $k_3$. In this case study the mass $m_3$ must be increased to 400 [kg] to achieve the required frequency. This perturbation is considered too large, resulting in the necessity to both increase $m_3$ and decrease $k_3$. The parameters of both systems are given in Table 5-3. The eigenfrequencies presented in Table 5-3 correspond to the system without the controller.

5-3-2 Controller design

This section describes the controller design for both systems. The controller for the balanced system (2) is developed in several steps from a version that creates perfect balancing to a controller that is
Table 5-3: Parameters for 1D model in $x$ direction of mechatronic system 1 and 2

<table>
<thead>
<tr>
<th>Number</th>
<th>M [kg]</th>
<th>k [N/m] (x)</th>
<th>Freq [Hz]</th>
<th>M [kg]</th>
<th>k [N/m] (x)</th>
<th>Freq [Hz]</th>
<th>$\zeta$ % (x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>22</td>
<td>15780</td>
<td>1.93</td>
<td>22</td>
<td>15780</td>
<td>1.94</td>
<td>0.01</td>
</tr>
<tr>
<td>2</td>
<td>50</td>
<td>13107</td>
<td>5.14</td>
<td>50</td>
<td>15363</td>
<td>4.98</td>
<td>0.25</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>1.59e8</td>
<td>668.35</td>
<td>25.29</td>
<td>1e7</td>
<td>122.84</td>
<td>0.01</td>
</tr>
</tbody>
</table>

(a) Control loop for general plant. The reference signal $r$ is controlled by feedback of the sensor signal $y$. The disturbance $x_0$ affects the performance $x_p$ both directly and via the control loop.

(b) Transfer function $G_{uy}$ from actuator force to actuator displacement $y$ for two systems. System 2 has a reduced third resonance frequency which should ultimately lead to balancing in combination with the closed loop controller.

**Figure 5-8:** General plant and mechanical FRF of the baseline system (1) and a second system that is prepared for balancing.

consistent with the rules of thumb as they are known from controller design [20].

To design a controller the actuator and sensor location have to be defined. From the baseline design concept it is determined that the motion control of the stage will be achieved with a collocated controller between the stage and the frame ($x_1$ and $x_2$). Therefore the sensor signal is written as,

$$y = x_1 - x_2. \tag{5-6}$$

The performance is determined by the movement between the stage and the base Eq. (5-5). The control loop is defined in Figure 5-8a. The mechanical plant is represented by a standard General plant notation to take into account that the measured output is not the performance indicator [8]. The different mechanical transfer functions shown in the figure are declared in Eq. (5-7) till Eq. (5-10).

$$G_{uy} = L_y(sI - A)^{-1}B_u \tag{5-7}$$
$$G_{xyy} = L_y(sI - A)^{-1}B_{x_0} \tag{5-8}$$
$$G_{uxp} = L_{xp}(sI - A)^{-1}B_u \tag{5-9}$$
$$G_{x_0xp} = L_{xp}(sI - A)^{-1}B_{x_0} \tag{5-10}$$

With the General plant formulation and the mechanical transfer function, the closed loop transfer functions of the entire system can be determined. The transfer function between the set point ($r$) and the performance indicator ($x_p$) is given by

$$\frac{x_p}{r} = \frac{CG_{uxp}}{1 + CG_{uy}}. \tag{5-11}$$
5-3 1D mechanical model that contains a controller

(a) Open loop transfer function for multiple controller set-ups of system 1 and 2.

(b) Nyquist plot for multiple controller set-ups of system 1 and 2.

**Figure 5-9:** Open loop behavior of the controller combined with system 1 and 2. System 2 PD, system 2 PD tamed, system 2 PD large and system 2 PID describe the evolution of the controller from a version that creates perfect balancing to a controller that is designed according to rules of thumb for controller design.

The transfer function between a disturbance \(x_0\) and the performance \(x_p\) is given by

\[
\frac{x_p}{x_0} = G_{x_0x_p} - G_{ux_p} \left( \frac{CG_{x_0y}}{1 + CG_{uy}} \right).
\] (5-12)

The mechanical transfer function required for the controller design \(G_{uy}\) of both systems is shown in Figure 5-8b. Based on this transfer function the controller parameters can be determined to create a bandwidth of 100 Hz.

To assess the effect of the controller four different set-ups are created for System 2. The first consist out of a controller where the parameters are defined in such a way that it is the equivalence of a spring damper with respect to disturbances. This set-up is called "System 2 PD" and is creating perfect balancing. The next controller uses a tamed damper but still keeps all the values the same ("System 2 PD tamed"). The introduction of the tamed damper is the first source of imbalance. The third system incorporates a higher damping value that is more in the range of the values that would be chosen when designing by the rules of thumb ("System 2 PD large"). This damping value significantly disturbs the balance despite the fact that the sensitivity with respect to damping is generally small. The final controller is designed completely according to the rules of thumb including an integrator ("System 2 PID"). All de different open loop transfer functions are shown in Figure 5-9a. Complementary to the open loop transfer functions the Nyquist plots are shown in Figure 5-9b.

The open loop bode plot and Nyquist plot of both systems show they have a 100 Hz bandwidth. Due to the zero/pole combination there is no theoretical limitation to the height of the crossover frequency, however for this example the 100 Hz bandwidth is given as a constraint.

The Nyquist plot of "System 2 PD" shows that there is a very small phase margin as it approaches the -1 point. In practice a controller can never be designed with so little margin. Furthermore the lack of damping will negatively influence the close loop behavior since it directly affects the settling time of the system. This will be discussed in the next section.

### 5-3-3 Closed loop behavior

Figure 5-10a shows the closed loop behavior of the previously defined set-ups. The resulting step response is shown in Figure 5-10b. It is clear that the balancing will not improve motion control or tracking performance. This is above all caused by the lack of damping which creates an unacceptable
settling time. The results for System 2 with a PID controller are still poor compared to the results of System 1. This is caused by the proximity of the third resonance frequency to the cross-over frequency. This is required for balancing. Considering the fact that the balancing is primarily performed to improve the disturbance rejection this is the focus of the next paragraph.

5-3-4 Disturbance rejection

The potential improvement that was suggested by the sensitivity analysis of the transfer function for floor vibrations in \(z\) direction was the starting point of this chapter. Consequently the concept of balancing in combination with a controller is investigated. In the previous section it becomes apparent that the closed loop behavior suffers from the lack of damping which is required for balancing. Next the effect on the disturbance rejection is investigated to determine if there are any benefits for mechanical balancing in combination with a controller.

Starting with the effects of a disturbance acting on the stage, Figure 5-11a shows the transfer function from stage disturbance to the error \(x_p\). The transfer function is plotted for several set-ups. The original unbalanced system with a PID controller is used as a baseline. From the figure it can be observed that perfect balancing leads to poor results since there is no integrator action to suppress the low frequency errors. The balanced system with PID controller shows similar behavior as the original system (1). Next the transfer functions for floor disturbances acting on the frame are investigated. Figure 5-11b shows that perfect balancing with a PD controller leads to a transfer function that is zero (it is not visible). The 'PD tamed' and 'PD large' transfer function indicate the effect of a more realistic damping which is required for phase margin. In Section 5-2-5 it was already shown that imperfect damping effects the quality of balancing. The transfer function of the 'PD large' system shows that the phase margin requirements for control purposes are canceling all the benefits of balancing. Especially the PID controller set-up needs the large damping for stability since the phase starts at -90 due to the integrator action. System 2 with PID control shows a higher static gain then the PID controlled System 1. This is due to the fact that the integrating action can only suppress the error between stage and frame (\(y\)) and not between stage and base (\(x_p\)). To balance system 2 the eigenfrequency corresponding to the base has been lowered. This directly affects it’s static gain for an applied acceleration.

The third disturbance source is a force acting on the base. Figure 5-11c shows the corresponding transfer functions. Since the actuator and sensor location are between the stage and the frame but the real error is determined by the difference between the stage and the base, the PID controllers are
not able to suppress the low frequency error. System 2 that is prepared for balancing has a weakened spring connection between the base and the frame and this is creating the higher static amplitude in the error for System 2.

5-3-5 Conclusion on balancing a 1D model that incorporates a controller

Taking the disturbance rejection and motion control results into account, the balancing of a mechanical system with a controller connected as described, is not worth the effort. A realistic controller needs a very high damping compared to the mechanical damping and this destroys the balancing. If the damping problem could be reduced there are still negative effects for disturbances coming from different sources then the floor. Furthermore the primary function of the controller is the motion control of the stage. Disturbance rejection is important however, in this case it could be seen as secondary with respect to motion control. Therefore an adjustment that deteriorates the prime performance but improves secondary objectives is unlikely to be implemented.

5-4 Summary on balancing

This chapter describes the application of sensitivity analysis for two models. The first model represented a passive system that was subjected to floor vibrations. The sensitivity analysis pointed towards a solution of mechanically balancing the model. This resulted in a system that was completely insensitive to floor vibrations. By selecting either a mass or a stiffness as the balancing parameter the potential disadvantage could be shifted to either an increase in spring forces or an increase in sensitivity with respect to other disturbance forces.

The second part of this chapter investigated the applicability of the same method on a model that incorporated a controller. This proofed to be unrealistic since the balancing requirements were in direct conflict with the motion control requirements.
(a) Disturbance rejection for forces acting on the stage. A integrator is necessary to reduce the DC error.

(b) Disturbance rejection of accelerations originating from the floor. Perfect balancing (System 2 PD) has a zero transfer function. The requirements for motion control undo the potential benefits of balancing.

(c) Disturbance rejection for forces acting on the base. The reduced stiffness of the base that is required for balancing has resulted in a higher DC error.

Figure 5-11: Disturbance rejection of system 1 and 2 for different sources. System 2 PD, system 2 PD tamed, system 2 PD large and system 2 PID describe the evolution of the controller from a version that creates perfect balancing to a controller that is designed according to rules of thumb for controller design.
Chapter 6

Case study: connecting a balanced actuator to a compliant frame

This chapter is dedicated to studying a dynamic problem that could occur in mechanical systems that contain actuators and moving components. The problem is derived from a real machine which will be left out of the discussion due to commercial interests. This machine has been realized as a first prototype. One of the key features that determines the performance of the system is the use of a balanced actuator. During the detailed design phase it was found that nodalization of the balanced actuator pair improved the performance substantially. This chapter starts with illustrating the relevance of nodalizing for the prototype by means of simple lumped mass models.

Section 6-1 first introduces the requirement that determines the performance of the system. Next a conceptual balanced layout is presented which should create satisfactory results with respect to the requirements. By increasing the fidelity of the model a fundamental problem is revealed that limits the performance of the concept. By application of sensitivity analysis the problem is thoroughly studied and the necessity of nodalization becomes evident. In Section 6-2-5 the application in the prototype is discussed by means of a simple 3D model.

The performance of the prototype and the dependence on nodalization is a starting point for the development of a demonstrator. This will be discussed in Chapter 7

6-1 Requirement definition

A common source of machine vibrations are the actuation and reaction forces that create the necessary acceleration of a payload. Figure 6-1a visualizes such a situation were a force \( f \) should position the payload (P) but inevitably introduces a motion of the frame (F). In this study the model is a subpart of the system and therefore the connection via \( k_2 \) to the ground resembles the connection to a larger machine structure. The payload resembles a small structure with low mass while the frame is a more compliant structure with high mass. The numerical data belonging to this model and the models that will be introduced in the following sections are combined in table A-1 that is included in Appendix A-1.

Due to the large mass of the frame the first free vibration mode is described by a piston motion of the frame with a very small relative movement of the payload with respect to the frame. The second mode is mostly described by movement of the payload. In vectors both mass normalized modes are described by

\[
\begin{align*}
\mathbf{x}_1 &= \begin{bmatrix} 0.2285 \\ 0.2177 \end{bmatrix} \\
\mathbf{x}_2 &= \begin{bmatrix} 0.9735 \\ -0.0511 \end{bmatrix}.
\end{align*}
\]
Case study: connecting a balanced actuator to a compliant frame

(a) Lumped mass model 1 of a typical system containing a payload \( P \) that is positioned by a force. The reaction force acts on a frame \( F \) that is compliantly connected to the ground.

(b) The transfer function from \( f \) to absolute displacement \( x \) of the model in Figure 6-1a. The response shows a pole-zero combination to the left of the payload resonance. A requirement is placed on the height of this gain fluctuation.

Figure 6-1: Model 1 of simple lump mass system

In the FRF from actuation force \( f \) to the absolute movement of the payload \( x \) the first piston mode results in a small pole-zero combination in the bandwidth between zero Hz and the second eigenmode. This can be observed in Figure 6-1b. In this case study these gain fluctuations created by the first mode should comply with a strict requirement. The fluctuations are not allowed to be larger than 1%. The requirement statement is as follows:

In the transfer function from actuation force to absolute movement of the payload a bandwidth is defined where gain fluctuations of no more than 1% are allowed. The bandwidth is defined as the region between zero [Hz] and the eigenfrequency corresponding to the main actuation mode. The main actuation mode is the mode that has the highest similarity with the static deflection shape (MAC value). The gain fluctuation created by the actuated mode is not restricted by the requirement. It only applies to modes that correspond to eigenfrequencies below the frequency of the actuated mode.

The MAC value or modal assurance criterion [1], is used to determine the correlation between two modeshapes. The MAC value is computed by

\[
\text{MAC} = \left( \frac{|x_r^T x_p|}{\|x_r\| \|x_p\|} \right)^2.
\]

6-1-1 Estimating the performance with respect to the requirement

The amplitude of gain fluctuation created by modes with a resonance frequency below that of the actuated mode can be estimated using the modal data. The full response is calculated using mode superposition as shown in Figure 6-2. The contribution of each mode to the static gain is computed with the modal receptance formulation from Eq. (3-12) at zero Hz resulting in

\[
\frac{x}{f}_{st} = \frac{x_r^T x_r}{\omega_r^2} s,
\]

were \( s \) the the vector that determines the action and reaction point of the force \( f \). The receptance...
of both modes sums to the full static response. As shown in Figure 6-2 the receptance for mode 1 results in a value of $x_f = 5.2233 \times 10^{-8} [m/N]$. This is the contribution of the first mode to the static displacement. The height of the resonance peak can now be determined by taking into account the damping. The modal damping is assumed at $\zeta = 0.01$. Rewriting this into the $Q$-factor results in

$$Q = \frac{1}{2\xi} = 50.$$  \hfill (6-4)

The $Q$-factor indicates that the resonance peak will be 50 times higher than the static gain. Combining this with the receptance of the second mode an estimate of the gain fluctuation is retrieved

$$\delta \frac{y}{F} = \left( \sqrt{(50 \cdot \frac{y_1}{F})^2 + (\frac{y_2}{F})^2} \cdot 100\% = 193\% \right).$$  \hfill (6-5)

Since mode one and two are $90^\circ$ out of phase at resonance the sum is calculated as the RSS value. The result is an estimate since the spring lines are not completely horizontal meaning that the gain of the second mode at the resonance frequency $\omega_1$ is not equal to its static gain but a little bit higher. The real fluctuation turns out to be 197% confirming the validity of the approximation.

### 6-1-2 Sensitivity analysis on performance estimate

The height of the gain fluctuation is affected by two variables. First of all the amplitude of the resonance peak in the receptance of the first mode. This height is determined by both the static gain of the first mode and the $Q$-factor. The second variable is the distance between the two modal static gains, meaning that an increase in static gain of the second mode will reduce the relative contribution of the first mode. Therefore, the performance of this system is directly related to the height of the first mode resonance and the corresponding amplitude of the second mode at that frequency. The last is estimated by its static gain which is a good approximation in the bandwidth well below the resonance frequency of the second mode.

The amplitude of the first mode resonance is estimated with the $Q$-factor and the static gain. The ratio between the two is used as the response function

$$r = \frac{Q_1 \frac{y_1}{F}_{st}}{\frac{y_2}{F}_{st}}.$$  \hfill (6-6)
Deriving the gain ratio with respect to a design variable $s$ results in

$$\frac{\partial r}{\partial s} = \left( \left( \frac{\partial Q_1 y_1}{\partial s} F + Q_1 \frac{\partial}{\partial s} \left( \frac{y_2}{F} \right) \right) \frac{y_2}{F} - \frac{Q_1 y_1 \frac{\partial}{\partial s} \left( \frac{y_2}{F} \right) }{F} \right)^2$$

(6-7)

The resulting expression contains the sensitivity of the static gain that was introduced earlier in Eq. (3-67). The resulting sensitivity values for the spring stiffnesses, damping and masses are summarized in Table 6-1. A negative sensitivity means that the increase of the respective design variable will lead to a reduction of the gain ratio and thereby a reduction of the gain fluctuation. An interesting result is the effect of an increase in stiffness for $k_1$. At first the weakening of this spring should lead to a more compliant structure and thereby making the static gain of the second mode higher. However it also effects the first mode and apparently this dominates the final ratio. This is due to the $Q$-factor which puts more weight on the movement of the static gain of mode 1. It should be noted that an increase of $k_1$ will result in a proportional drop of the static gain of the global transfer function. This will require an increase of the applied force to achieve the same displacement. The sensitivities clearly indicate that the focus should lie on either increasing the mass ratio or increasing the damping of the first piston mode.

Assuming that all the design variables are chosen in an optimal sense with respect to these results the performance will be limited by other requirements. A quick investigation determines that either the damping, the mass of the payload or the weight of the frame has to be changed by a factor 100. This does not lie in the region of permissible design changes and therefore other options have to be investigated.

### 6-2 Conceptual solution

One method to overcome the gain fluctuation problem is the application of a balancing actuator depicted in Figure 6-3a. By balancing the actuator force with a dummy mass (D) the scan head (S) does not move due to reaction forces. The scan head resembles an auxiliary frame to support both the payload and the dummy mass as well as the corresponding actuator pair. The resulting transfer function from applied force $f$ to displacement of the payload $x$ in Figure 6-1b is free of unwanted gain fluctuations. This method is obviously very depended on the quality of the balancing. The FRF sensitivity can give an important insight into the allowable tolerances for fabrication.

#### 6-2-1 Sensitivity analysis for balanced concept

In the current configuration there are two modes in the bandwidth of operation that could become observable if asymmetry is present. The first mode is the piston mode of the entire structure and the second is the piston mode of the scan head assembly with respect to the frame.

The FRF sensitivity results are combined in Figure 6-4. The effect of asymmetry in damping is approximately two orders lower compared with stiffness or mass perturbations since the damping forces are much smaller.

To investigate the to be expected amplitude fluctuation the sensitivity value for $k_1$ at the second mode is retrieved from Figure 6-4a as $s(\partial x)/(\partial k_1) = 2.666 \cdot 10^{-6}[m/N]$. A 1% perturbation of stiffness
results in a $\Delta FRF$ of $2.666 \times 10^{-8}[m/N]$. At the corresponding amplitude this equals an amplitude variation of approximately 2%. Although this value cannot be linked to a structural design variable in this conceptual model it does indicate that this is no easy fix. The balanced system might perform within spec but it will require constant monitoring of the production tolerances for all critical parts.

### 6-2-2 Increasing the fidelity of the model

By increasing the fidelity of the model potential problems of the conceptual solution can be revealed. Two slightly more complex models will show that there is an intrinsic problem related to conceptual solution. In the second model shown in Figure 6-3a the scan head is connected to the frame with one spring. The connection to the scan head is not moving due to the balancing. A new model shown in Figure 6-5a creates a deformable scan head with the connection to the frame at locations that will exhibit movement. The internal flexibility of the scan head is modeled to be stiff compared to the connection of the payload. If the model is completely symmetric and perfectly balanced most of the modes remain un-actuated for the given applied force. Nevertheless one symmetric mode of the scan head can be observed in the transfer function shown in Figure 6-5b. As stated in the performance definition this mode does not create a problem since it is not in the critical frequency range. However, the actuation of this symmetric mode indicates the potential problem created by symmetric modes. The frame might also exhibit such a symmetric mode which will be lower in frequency and could therefore create a problematic gain fluctuation.

The fourth model increases the complexity by including flexibility in the frame. The internal frame stiffness is modeled to be weaker then the scan head. The high mass of the frame and relative low stiffness create a problematic mode that can be observed in the transfer of Figure 6-6a. Not taking into account the amplitude of the pole-zero it becomes clear that even with a perfect symmetric and balanced design the heavy frame will create an observable mode in the transfer from actuation force to the displacement of the payload.

The values where chosen to demonstrated that even with perfect balancing, cross-talk can still create problems. The problem decreases as the stiffness of the connection decreases or increases and as the ratio of the payload mass and that of the frame increases.
Figure 6-4: FRF sensitivity results for the second model. The results indicate the presence of two unobservable modes in the frequency range below 100 Hz. The sensitivity results indicate which design variable perturbation will create the largest gain fluctuation by these modes.

(a) FRF sensitivity with respect to spring stiffness.  
(b) FRF sensitivity with respect to damping. Note that the y-axis scale is different compared to the other two results.

(c) FRF sensitivity with respect to mass.

Figure 6-5: Model 3 with the corresponding FRF. The introduced flexibility in the scan head creates a gain fluctuation in the bandwidth to the right of the resonance peak.

(a) Lumped mass model of an balanced actuator configuration. The scan head (S) contains internal flexibility creating a non-stationary connection to the frame.

(b) The transfer function from $f$ to $x$ of the third model shows a small zero-pole combination after the first resonance frequency created by the compliance of the scan head.
6-2 Conceptual solution

(a) Lumped mass model of an balanced actuator configuration. Both the scan head (S) and the frame (F) contain flexibility. This increases the modal density in the bandwidth below the actuated mode.

(b) The transfer function from f to x of the fourth model shows a small pole-zero combination before the first resonance frequency created by the compliance of the scan head and the frame.

Figure 6-6: Model 4 with the corresponding FRF. The introduced flexibility in the scan head and the frame creates a gain fluctuation in the bandwidth to the right and left of the resonance peak.

In practice asymmetry in the frame creates substantially more problems. Therefore the sensitivity analysis will be repeated to assess the required design tolerances on stiffness, damping and mass.

6-2-3 Sensitivity analysis for a model with higher fidelity

The transfer function shown in Figure 6-6b contains 4 modes in the bandwidth of operation of which one is already visible as a gain fluctuation. In principle all interesting modes need evaluation on their potential to create a gain fluctuation. For this example the evaluation is limited to the sensitivity with respect to stiffness. Similar to the previous investigation the sensitivities with respect to damping are approximately two orders lower and the mass sensitivity is similar in amplitude to that of the stiffness.

The first mode has an eigenfrequency of 14 Hz and resembles the piston movement of all masses with respect to mass 6. This is due to the fact that the internal stiffness of the frame is modeled as compliant. Figure 6-7a plots the sensitivities of the transfer function zoomed in on the frequency of the first mode. The sensitivity of the FRF with respect to the stiffness indicates that $k_1, k_8, k_5$ and $k_6$ are the most critical. Since it is expected that the frame will not be symmetric the maximal values of $\frac{\partial \text{FRF}}{\partial k_5}$ and $\frac{\partial \text{FRF}}{\partial k_6}$ are the most interesting. The sensitivities indicate that a 1% stiffness change will result in a gain increase of $1.794 \cdot 10^{-8}[m/N]$. This equals a 1.6% gain fluctuation.

The second mode has an eigenfrequency of 62 Hz and resembles an anti-piston movement of the frame masses with respect to the center mass. This is the mode that is already creating a gain fluctuation in the FRF. The third mode is very close at 68 Hz and resembles the symmetric version of mode 2. The sensitivity is plotted in Figure 6-7b. Since the fluctuation is already present in the magnitude plot the sensitivity loses its usefulness. The sensitivity values are determined by the shifting of the resonance peak and the relative fluctuation difference is obscured by this. The movement of the eigenfrequency is most sensitive to changes in the stiffness of the connection between the scan head and the frame.

The fourth mode has an eigenfrequency of 100 Hz and resembles the symmetric version of the main actuated mode. Figure 6-7c plots the sensitivities of the transfer function zoomed in on the frequency of the fourth mode. This mode is most sensitive to stiffness perturbations of $k_1, k_8$ and $k_4, k_7$. The last two represent the connection to the frame and a perturbation of 1% results in a gain fluctuation of 1.4%. The sensitivity to perturbations in $k_1$ and $k_8$ is approximately twice as high resulting in a gain fluctuation of 3%.
(a) FRF sensitivity with respect to spring stiffness. Zoomed in on the bandwidth of the first mode that is unobservable in the balanced system.

(b) FRF sensitivity with respect to spring stiffness. Zoomed in on the bandwidth of the second and third mode. The second mode is already observable in the balanced system.

(c) FRF sensitivity with respect to spring stiffness. Zoomed in on the bandwidth of the fourth mode that is unobservable in the balanced system.

**Figure 6-7:** FRF sensitivity results for the fourth model. The results focus on four modes in the bandwidth of operation. The sensitivity results indicate which design variable perturbation will create the largest gain fluctuation as a result of the introduced observability or actuation of a mode.
6-2-4 Dependence of the results on the movement of the connection

The results show that both the frame and the scan head need to be balanced for the optimal performance. In this case study it is known on forehand that the frame will not be balanced and therefore the lumped models indicate that the requirements will not be met.

This conclusion would not be valid if the connection between the scan head and the frame would remain stationary. The frame could be highly asymmetric without creating gain fluctuations if the connection at the scan head would be exactly between the two applied forces $f$. The current model parameters result in a movement ratio of the payload and the connection point at the scan head of approximately one order. By increasing $k_2$ and $k_3$ the resulting displacement of the connection point as a function of the applied force reduces linearly.

The sensitivity analysis is repeated for a modified model were $k_2$ and $k_3$ are increased with a factor 10. This should allow the verification of the effect on the gain fluctuations. In the previous model the sensitivities values for the first mode where approximately 2%. These values have remained the same for the $k_1$ and $k_8$. The sensitivity for $k_5$ and $k_6$ reduce significantly by approximately one order. This indicates that by reducing the movement of the connection points the sensitivity for frame asymmetry reduces.

Reducing the movement of the connection is equivalent to reducing the energy transfer between the scan head and the frame. Ideally these points remain stationary for all frequencies and applied loads. If these stationary points are not present nodalization could be an effective method to achieve the reduction of movement at the connection points of the scan head. In the developed prototype there were no stationary points in the area where the connection could be placed.

6-2-5 Realization of balanced scan head design

Section 6-2 introduced a balanced actuator configuration that enables the actuation of a payload without exciting the frame dynamics. The analysis showed that such a ideal situation can only be achieved when the connection between the scan head and the frame is placed in the center of the scan head. If the connection on the scan head shows residual movement it could trigger a dynamic response in an asymmetric frame.

This section discusses the implementation of the scan head in the prototype. Since details of the machine can not be discussed an abstract interpretation of the scan head is given. As stated earlier the scan head resembles a structure that contains the balanced actuator pair and a connection point for both the payload and the dummy mass.

Figure 6-8 shows the lumped mass model of the scan head and its translation into a 2D model. The gray area in the 2D model indicates the surface were connections to the frame could not be placed. The modal deformation of the actuated mode is shown in the same figure. The modeshape includes four nodal points were both the translation in $x$ and $y$ direction are zero. In the design of the prototype these points were selected as the location for the connection with the frame. By connecting the scan head on the nodal points of the actuated mode the transfer function from actuation force to the absolute displacement of the payload showed acceptable gain fluctuations.

6-3 Summary of the insight gained in the analysis of the prototype

Section 6-1 introduced a requirement on gain fluctuations in a frequency response. These requirements could not be satisfied with the baseline model. Balancing the actuator force by addition of a dummy mass proved to be a promising solution.

A detailed investigation showed that perfect balancing did not necessarily solve the problem since symmetric modes of the frame could still be visible in the transfer function. The sensitivity of the FRF to asymmetry in the design was in the order 2% gain fluctuation for a 1% design perturbation.
These sensitivities have to be checked for a more detailed design but for now the values are acceptable and do not depreciate the concept.

The sensitivity with respect to asymmetry in the frame proved to be dependent on the relative movement of the connection of the scan head. A reduction in movement of the scan head connections will directly reduce the gain fluctuations on the FRF from actuation force to payload position.

Nodalization could be an effective method to achieve such a reduction of movement. As shown in Section 6-2-5 the concept of nodalization has already been successfully applied in a prototype.

The performance of the prototype is satisfactory however, it is unclear to what extent it could be improved. Nodalization typically affects performance at small frequency regions near eigenfrequencies. As stated in the problem definition the performance is determined by the gain fluctuations from zero Hz up to the first actuated mode. Therefore the analysis team wants to better understand the potential gain of nodalizing. The development of a demonstrator should enable this research for increased insight in nodalization. Chapter 7 introduces such a demonstrator concept and defines an analysis method that leads to an optimal performance.

**Figure 6-8:** 1D lumped mass model of the scan head plus payload and dummy mass, combined with a 2D translation of the implemented scan head in the prototype and its actuated modeshape. The gray areas in the scan head are not accessible for a connection point to the frame. The indicated nodal points have been used as a connection location between the scan head and the frame of the developed prototype.
Chapter 7

Case study: designing with nodal points

This chapter discusses the design and analysis method for the development of a demonstrator. The goal of this demonstrator is introduced for the first time in Section 6-3. A short summary will be given on the relevant information from the previous chapter.

Chapter 6 starts with the introduction of a requirement on gain fluctuations in a FRF. This requirement led to the necessity of balancing the actuator forces. Furthermore it is shown that irrespective of balancing asymmetry in a supporting frame will still affect the performance. It is concluded that the sensitivity with respect to asymmetry in the frame is highly dependent on the movement of the connection point of the scan head. In other words, if the connection points remain stationary the dynamics of the frame do not disturb the transfer function of the scan head.

A research project has been initiated to specifically demonstrate the effectiveness of nodalization in the conditions as encountered in the development of the prototype of Chapter 6. This will be achieved by the realization of a demonstrator which should be able to show "before and after" behavior with regards to nodalizing. In this way a live demonstration should show the effectiveness of the method.

To achieve the "before and after" behavior the demonstrator should incorporate tuning capability. The tuning should also create the possibility to compensate for modeling and fabrication uncertainties.

This chapter investigates the design/analysis method that uses nodal points in an optimal way. Besides the use of nodal points, methods of tuning are also included in the analysis. In Section 7-1 a brief summery will be given on the related work on nodalization in the field of structural dynamics. Section 7-2 start with the introduction of a conceptual demonstrator after which Section 7-4 defines the location of nodal lines in both the static and the dynamic deformation. Section 7-5 investigates the possibility to manipulate these nodal lines to achieve optimal performance. Section 7-6-1 creates such an optimal scan head which is connected to a frame in Section 7-8. The final section of this chapter states the recommendations and conclusions for the development of the demonstrator.

7-1 Related work on the topic of nodalizing

Apart from describing the essence of nodalizing Mead [19] gives several examples on successful implementations. Nodalizing can in some cases be achieved by modifications of the structural modeshapes. Peters [24] illustrates how the modeshape modifications can be achieved. In the field of mechatronic systems nodalization has been used for the optical pick-up units of cd players [16]. A second example of nodalizing in high precision positioning systems is given in Seki et al.[31] where the effect on the control properties are also included. Okubo et al.[22] demonstrates the usefulness of sensitivity analysis with the intend to solve vibration related problems. Several cases are discussed of which many combine concepts of nodalization and sensitivity analysis.
Yu et al.\cite{48} optimizes the thickness distribution of a violin top plate to achieve the desired nodal line position. The sensitivity analysis used in this optimization is discussed in Section 3-6. Inzarulfaisham et al.\cite{14} demonstrates nodalizing for chassis like frame structures. Another sensitivity related nodal optimization is discussed in \cite{26}. This paper discusses the application of nodalization for wing type structures. The manipulation of anti-resonance frequencies (which are closely related to nodes and nodal lines) is discussed in \cite{15}. The work describes the importance of the knowledge on anti-resonance frequencies in vibration testing. It demonstrates this relevance with the analysis of the clamping plate that is used for experimental vibration tests.

### 7-2 Introduction of demonstrator concept

The demonstrator is divided into two main components: the frame and the scan head. The 3D model of the frame and scan head are shown in Figure 7-1. For this case study the frame is considered a "black box" which exhibits a certain dynamic response.

#### 7-2-1 Scan head

The scan head of the demonstrator is the same as the one that is introduced in Section 6-2-5. The location of the applied force and the payload are indicated in the figure. The payload and the dummy mass are omitted from the model for simplification. What remains is a scan head on which a payload can be mounted and positioned by the applied force. The geometry of the scan head is very simple but it contains an important feature. The rectangular design creates nodal points in the corners that show very little movement in both the $x$ and $y$ direction.

Since the function of the scan head is to move the payload the design might seem overly stiff in the direction of movement. However a situation must be created where the scan head frequencies are
well above those of the frame. Furthermore a design could be made to reduce movement in other parts of the scan head. This is intentionally left out since the goal is to demonstrate the effectiveness of nodalization. A design with for example leaf springs inside the scan head would greatly reduce the unwanted movement at possible connection points. However, movement would still be there but maybe two orders lower. Nodalization could then still improve the performance but it will be tough to measure. It is for this reason that the design is poor in the respect that it shows excessive movement at most parts of the scan head.

7-2-2 The frame

For this case study the frame is considered a "black box". There are two important features of this box. The frame is modeled to contain a high modal density in the frequency range of operation. The frame is asymmetric and unbalanced with respect to the four connection locations. The combination of asymmetry and high modal density within the operating bandwidth creates a considerable challenge with respect to the gain fluctuations requirement.

7-3 Nodal points/lines of the scan head

If the connection to the scan head is at a location that remains stationary there will be no transfer of energy to the frame. A stationary point is a location in the model where all the degrees of freedom remain constant for an applied load. However, there can also be locations where the translational degrees of freedom are constant but the node does show rotation. These point or lines are typically called nodal point or nodal lines. These nodal locations can be stationary in translation or rotation but usually show movement in some of the six’s degrees of freedom.

The stationary point and nodal lines of the actuated mode are shown in Figure 6-8. From this deformation plot there are two main solution paths regarding the connection to the frame.

- Connect the scan head at stationary points that remain stationary in both static and dynamic load cases.
- Connect the scan head at nodal points or lines that are stationary at one specific frequency.

In the current configuration there is one stationary point at the center of the scan head model. Due to the balance of applied force this point remains stationary for both static and dynamic load cases. In Section 6-2-5 this location is stated to be unusable due to design restrictions. The model contains nodal lines if the deformations in $x$ and $y$ direction are plotted separately. In this case the nodal lines represent the first actuated mode. The static deflection shape has a similar modeshape however the corresponding nodal lines are slightly shifted.

7-3-1 Static and modeshape deformation of scan head model in ANSYS

Given an applied load the static deformation can be split up into $x$ and $y$ components. Figure 7-2 shows the static deflections at the top left quarter of the scan head for an applied load as shown in Figure 7-1. The colors indicate positive or negative deformation. At the line where positive and negative meet there is a line of zero deformation in the corresponding direction. The zero deformation line in the $x$ and $y$ plot intersect near the corner of the structure which creates a nodal point.

The modeshape deformation shown in Figure 7-3 is very similar to the static deformation. Close comparison of the two deformations shows that the lines of zero deformation are not at the same location. It appears that they are approximately 1 mm apart.

It should be noted that the nodal points or lines in the static deformation are nodal locations for the total response of the system. These points or lines in the modeshape are not representative of the total response which is described by the summation of the modes. The difference between the two can be confusing which is why it is extensively discussed in Section 3-1-4 and Section 3-1-5.
(a) The static deformation in $x$ direction. The blue (b) The static deformation in $y$ direction. The blue and red color indicates negative and positive deformation respectively.

Figure 7-2: The top-left corner of the scan head is modeled in ANSYS. By applying symmetric boundary conditions the results are similar to a completely symmetric model. The difference lies in the fact that asymmetric modes are not described by this model.

(a) The modal deformation in $x$ direction of the first actuated mode. The blue and red color indicates negative and positive deformation respectively. (b) The modal deformation in $y$ direction of the first actuated mode. The blue and red color indicates negative and positive deformation respectively.

Figure 7-3: The top-left corner of the scan head is modeled in ANSYS. By applying symmetric boundary conditions the results are similar to a completely symmetric model. The difference lies in the fact that asymmetric modes are not described by this model.

7-4 Relation between modal and static nodal point

It appears that the static nodal lines and the corresponding lines belonging to the first actuated mode are very close together. It is expected that there should be a transition in the frequency domain between the two. To investigate this relation a grid of points is created in the model of the scan head. Figure 7-4a illustrates the position of these points. The relative distance between the nodal lines is exaggerated for visualization. Seven points lie between the static and modal nodal line for the deformation in $y$-direction. Figure 7-4b plots the resulting FRF from the actuation force to the displacement of the nine points. Since point one is to the left of the static line and point two to the right the static zero is in between these two results. The zero slowly moves towards the modal line as the excitation frequency increases. As the output location approaches the modal line the resonance
7-4 Relation between modal and static nodal point

(a) Location of 9 sample points in the upper left corner of the scan head model in ANSYS. The points span the area between the static and modal nodal line. The figure is not to scale!

(b) FRF from actuation force to the displacement at point 1 till 9. For increasing frequency the stationary line moves from the static towards the modal line.

Figure 7-4: Movement of the nodal line as a function of frequency of the applied load. By observing the FRF at nine different locations the frequency dependent behavior of the zero deformation line is visualized.

peak decreases. At point eight the amplitude of the resonance is at a minimum.

From Figure 7-4b it becomes apparent that the frequency region in which the reduction is highest, is a function of the position between the static and modal nodal line. In summary:

If the nodal point in the static deformation shape would be used, an FRF from actuation force to the displacement of the connection would start with a +2 slope in the bandwidth between 0 Hz and the first actuated modeshape. By selecting this point as a connection location the low frequency transmission is greatly reduced.

If the nodal point in the modeshape is used as a connection location the emphasis of reduction is placed near the frequency band of the corresponding eigenfrequency.

These results lead to the following questions:

How can the relative position of these lines be affected.

Can the lines be made to coincide and will this result in an optimal performance?

These questions are discussed in Section 7-5.

7-4-1 Verification of the model

The results of eigenfrequencies is generally mildly sensitivity to mesh size. The location sensitivity of the nodal lines is unknown. Therefore a mesh size sweep was performed to check convergence of the model. Table 7-1 summarizes the results from the mesh sweep. As expected the eigenfrequency of the tenth mode converges rapidly. The position of the static and modal nodal line seems to be very mesh insensitive however closer inspection indicates that the results are scattered. As the number of elements increases the prediction of the location should become more accurate but there is a high dependency on the shape of the mesh and not so much the number of elements. The following optimization requires exact positioning of the connection point on the nodal line. A relative dense mesh on the surface of the model should therefore provide sufficient accuracy in the prediction of nodal points.
Table 7-1: Result for mesh sweep on scan head model. Both the convergence of the eigenfrequency and the location of the nodal lines is checked

<table>
<thead>
<tr>
<th>Mesh size [m]</th>
<th>Elements</th>
<th>Frequency [Hz]</th>
<th>Static ST-line x [m]</th>
<th>Modal ST-line x [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>108</td>
<td>2053.8</td>
<td>-0.0939</td>
<td>-0.0945</td>
</tr>
<tr>
<td>0.01</td>
<td>648</td>
<td>2026.9</td>
<td>-0.09412</td>
<td>-0.0943</td>
</tr>
<tr>
<td>0.005</td>
<td>5184</td>
<td>2018.5</td>
<td>-0.0939</td>
<td>-0.0944</td>
</tr>
<tr>
<td>0.0025</td>
<td>41472</td>
<td>2014.9</td>
<td>-0.0939</td>
<td>-0.0945</td>
</tr>
<tr>
<td>0.001</td>
<td>674392</td>
<td>2012.9</td>
<td>-0.0941</td>
<td>-0.0942</td>
</tr>
</tbody>
</table>

7-5 Manipulating the static and modal deformation

From Figure 7-4b it becomes apparent that the selection of the connecting location between the static and modal deformation line can be used to either focus on low or high frequency suppression. However, the relative position of these lines can be influenced. For example the static line could be changed by either changing the applied force vector or by changing the local stiffness. The modal line could be influenced by mass or stiffness changes.

By changing the force or adding lumped masses both lines can be manipulated separately. Consequently the question arises, what would happen if the lines are brought closer together and could they be made to intersect. To study the effect of force and mass changes a Matlab model is introduced in Section 7-5-1. This model allows for fast reanalysis and describes the dependency of the nodal lines in $y$ direction on the force and mass distribution correctly. The following analysis is restricted to the manipulation of the nodal lines in the $y$ deformation for the sake of clarity.

7-5-1 Beam model of the scan head in Matlab

The ANSYS model shown in Figure 7-1 is translated to a Matlab beam model that consist out of Euler-beam elements, Figure 7-5a. Since the beam elements have two nodes per element the nodal line can not be described by the degrees of freedom present in the model. To overcome this problem four whiskers are added at the corners of the model. The location of a nodal line could also be directly interpolated from the translation and rotation of the corner node. However, this does not create the opportunity to connect the model at this point. Therefore the whiskers are added to interpolate the translation and rotation and to create a possible connection point on the scan head.

To verify the behavior of the model some characteristic responses are compared to the ANSYS model. Table 7-2 contains the resulting data comparison. The results of the Matlab model are deemed good enough since they describe the relative and absolute position of the nodal line with a difference of 0.1%.

Table 7-2: Comparison of several response function evaluations for the Matlab and ANSYS model

<table>
<thead>
<tr>
<th>Type of response</th>
<th>ANSYS model</th>
<th>Matlab beam model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Static deflection at $x,y=0,0.15$ for point load [m]</td>
<td>1.4539e-5 [m]</td>
<td>1.4689e-5 [m]</td>
</tr>
<tr>
<td>Static deflection at $x,y=0,0.15$ for distributed load [m]</td>
<td>9.338e-6 [m]</td>
<td>9.1384e-6 [m]</td>
</tr>
<tr>
<td>x-coordinate of static nodal line</td>
<td>-0.0939 [m]</td>
<td>-0.0940 [m]</td>
</tr>
<tr>
<td>x-coordinate of modal nodal line</td>
<td>-0.0945 [m]</td>
<td>-0.0944 [m]</td>
</tr>
<tr>
<td>Frequency of the first actuated mode</td>
<td>2014.9 [Hz]</td>
<td>2004.2 [Hz]</td>
</tr>
</tbody>
</table>
Scan head model in matlab constructed with beam elements. The whiskers at the corners create a substitute for the interior nodes of the ANSYS model. This allows for a connection at the nodal line. The dotted lines indicate the thickness of the modeled beam.

Incremental distribution change of the applied load. The load is increasingly broadened while adjusting the amplitude to keep the maximal deflection constant. The node that is used for the addition of lumped masses is also indicated.

Figure 7-5: Matlab model of the scan head combined with an overview of the tuning parameters that influence the modal and static nodal line position.
7-5-2 Changing the force distribution

To determine the effect of the force distribution on the position of the nodal lines in the static deformation the amount of loaded nodes are increased incrementally. Figure 7-5b shows how the load is applied for the first node and then incrementally expanded. To compare the results the applied load is scaled in such a manner that the maximal deflection remains the same. The load is changed in ten steps from a single point load on the center node to a completely distributed load over all the 20 nodes of the top and bottom beams.

Figure 7-6: Response of the nodal line position to load and mass changes. By combining the two tuning parameters the nodal lines can be made to coincide at the same location.

Figure 7-6a shows the absolute position in y-direction of both the static and the modal nodal line as a function of the load. The modal line remains fixed since it is not dependent the applied load. The static line moves from the right to the left of the modal line and thereby crossing it. The fact that they cross means that for the corresponding load distribution the modal and static line lie at the same location. This could be an interesting configuration because it could combine the high and low frequency suppression. Since the load distribution can only be changed incrementally per node in the model the distribution required at the crossing can not be created exactly. The amplitude of each force term could also be adjusted however, this option is omitted for now.

7-5-3 Adding lumped masses

The shape of the mode and thereby the position of the nodal line could be adjusted by adding lumped masses to the system. Starting from first load configuration shown in Eq. (7-5b) the top and lower beam are actuated by a point load. In this configuration the modal line lies to the left of the static line. Mead [19] describes that modal lines are pulled towards the location where lumped masses are added. Therefore lumped masses are added at the location where the force is applied. The added mass is incrementally increased and the resulting positions are shown in Figure 7-6b. The movement of the modal line shows an asymptotic behavior which does not result in a crossing between the two lines.

7-5-4 Combining both methods

Combining the two methods the load distribution is chosen such that the static line remains just to the right of the modal line. The modal line can now be pulled to the right by the addition of lumped
7-6 Optimization of scan head geometry for static nodal point location

(a) Evolution of the FRF from actuation force to static nodal line as a function of mass tuning. By the addition of 0.16 [kg] the first resonance peak is removed from the FRF

(b) Evolution of the FRF from actuation force to modal nodal line as a function of mass tuning. By the addition of 0.16 [kg] the static gain is reduced in the FRF. Note that the observer follows the movement of the modal line towards the static nodal line

Mass tuning results in a FRF that has a low static gain and a unobservable actuated mode.

The results show that it is possible to combine the modal and static nodal line which combines the positive properties of both. It appears that the removal of the first mode in the FRF to the static nodal line is less meaningful then the reduction of the static gain at the position of the modal nodal line. This is of course enlarged by the logarithmic plot. The removal of the resonance peak in the FRF of Figure 7-7a will only lead to an improved performance when the frame has a specific resonance at the corresponding frequency. Multiple numerical experiments with the coupling of the scan head to a frame has supported the notion that the use of the static nodal line has the largest contribution to the performance improvement.

7-6 Optimization of scan head geometry for static nodal point location

The previous results have shown that it is possible to combine the static and modal nodal line at the same location. The following challenge will be to place the static nodal line exactly at a node which allows for the connection to a frame. Mass addition should then pull the modal line to the same location.

An alternative optimization could directly place the static and modal line at the required position. However, this is computationally more expensive since it needs both the static solution and the modeshape results. By only optimizing the static solution and using mass tuning to relocate the modal nodal line an efficient optimization is achieved. Furthermore, the structural geometry is adjusted to move the static nodal line to the appropriate position. This leaves the force modifications free for tuning in a latter stage of development.

This section starts with the definition of the design variables that are used for the optimization. In Section 4-3-2 the concept of reducing the number of design variables by selecting an appropriate basis
was introduced. This concept will now be used to test different sets of design variables. After the introduction of the design variables the optimization settings and results will be discussed.

7-6-1 Design variables for the scan head beam model

In the optimization of the static nodal line the width of each beam element is considered a design variable. The results should be symmetric which translates into a number of design variables that is equal to a quarter of the number of elements.

Different bases will be tested on their performance which will be rated by comparison with the results of using a quarter of all the design variables. The following paragraphs will introduce the bases that will be compared.

Modal basis

To be able to translate the modeshapes of the structure into a basis of structural design changes some choices have to be made on how to interpret the available information. For this case the translational movement of the modes is considered as the useful information. Therefore the following data processing will be performed on the $x, y, z$ degrees of freedom per node.

The second property of the modes that has to be discussed is the sign of the amplitude (plus or minus). By including this sign difference into the basis a design change can incorporate both addition and subtraction of the design variable at hand. Therefore this property should be preserved in the translation from modeshape to structural design basis.

For a 1D beam structure shown in Figure 3-2 the definition of positive and negative is very simple since it directly corresponds to the amplitude sign. However the scan head beam model is a 3D structure where a symmetric modeshape has negative amplitude at the lower beam and positive amplitude for the upper beam for an outward motion. For this structure positive is defined as an outward movement. To create this translation the position of a node in the global axis can be used. A negative amplitude for a node that exhibits a negative position in the reference system is transformed to a positive value. In general this is formulated according to

$$x_{n}^{\text{sign}} = x_{n}^{x,y,z} \cdot \frac{q_{x,y,x}^{n}}{|q_{x,y,z}^{n}|} n = 1...\text{number of nodes}, \quad (7-1)$$

where $n$ equals one of the two nodes that belong to the current element. The following steps sums the $x, y, z$ amplitude of the $x_{n}^{\text{sign}}$ vector for the two nodes that belong to the element and averages this value. This results in a final value that forms the modal basis vector $m$

$$m = \frac{x_{1}^{\text{sign}} + x_{2}^{\text{sign}}}{2} \quad (7-2)$$

The number of indexes in this vector equals the number of elements in the model. The modal basis vector is normalized to length 1 by applying

$$||m|| = 1.$$

This equalizes the total effect on the system for different vectors. The resulting modal basis translation for the first six’s symmetric modes is shown in Figure B-1.

Energy distribution basis

The energy distribution basis consisting out of the elastic and kinetic energy can be easily derived with the Rayleigh quotient Eq. (3-26). The basis is split up into vectors $e$ containing the elastic energy distributions and the vector $k$ containing the kinetic energy. The vectors are length normalized like the modal basis vectors. The resulting basis for the same six’s modes is shown in Figure B-2 and Figure B-3.
7-7 Optimization results

Eigenvalue sensitivity basis

The eigenvalue sensitivity basis can be constructed from the results obtained with one of the methods from Section 3-3. The choice of design variable is free but it seems sensible to use the design variable that will be effected by the basis. In this case that is the width of the beams since a change there will result in the most effective change of stiffness. Normalizing the vectors for a unit length results in Figure B-4

7-6-2 Optimization settings and constrained functions

To achieve the optimization the Matlab \textit{fmincon} function is used and to prevent unfeasible designs a constraint function is added. This constraint function defines a maximal allowable thickness change of half the beam width. Furthermore the maximal allowable function evaluations and iterations is set at 10,000. The convergence criteria requires a a change in function value smaller then $1e^{-8}$ before stopping the iteration.

7-7 Optimization results

In the previous section three possible sets of design basis vectors where introduced that could be used for optimization routines. There are numerous combinations that could be used from using one single vector of for example the modal basis till using multiple vectors or multiple bases. The results in this study are limited to an optimization comparison of single vectors from the modal basis and a comparison of all the introduced bases using six's vectors from each set.

7-7-1 Single vector optimization

Six's vectors from the modal basis are selected Figure B-1. Each vector is used separately in an optimization routine to achieve a static nodal line at the node of interest. The resulting scan head after optimization is shown in Figure C-1. The corresponding FRF from actuation force to the displacement of the node is shown in Figure 7-8a. The results show that not all of the routines were successful in achieving the objective. This is based on the fact that not all the FRF results show a low static gain at zero Hz. Using mode 55 or mode 68 as a basis for the design change did not lead to a satisfactory result. However the results also show that several other modifications did create the desired result and that there are multiple design perturbations that lead to the required response.

7-7-2 Multiple vector optimization

For this optimization six's vectors from each basis are selected. The vectors are shown in Figure B-1 till Figure B-4. The optimization is performed for each basis plus an extra run which combines the kinetic and elastic to one energy basis. Therefore this optimization uses twelve design variables. The resulting scan head for each basis is shown in Figure C-2 and the corresponding FRF from actuation force to displacement is shown in Figure 7-8b. The results indicate that all runs where successful in achieving the zero gain at zero Hz while each basis achieved this in a different way.

The optimization that uses the Kinetic energy basis has almost achieved the second optimization step which is removing the first mode from the FRF. This result will only require a very small mass modification to place both the static and the modal nodal line at the intended node.

The results also contain a basis called "Symmetric". This represents an optimization where a quarter of all elements are given as design variables separately. This leaves 37 design variables for optimization. This basis is used as a comparison to check the effect of the design space reduction on the optimization results. Figure 7-8b show that the reduction has not affected the optimization since the optimum found with the symmetric basis is similar to the results obtained with the reduced sets of design variables.
Case study: designing with nodal points

7-7-3 Combing modal and static line at the same location

With the results from Figure 7-8b the optimization process is completed by tuning the system to remove the first resonance peak from the response. This could be done with the addition of lumped masses. In principal all results could be optimized to omit the first resonance, however the scan head that was obtained with the Kinetic energy basis is already close to the final objective. Therefore this result is selected to ease the further refinement.

7-7-4 Alternative objective function

The previous results are optimized for the objective to create a modal and static nodal line at the node that will become the connection to the frame. However, this formulation is not necessarily the one that will lead to the best results. Therefore an alternative objective is formulated that reduces the average gain between zero Hz and the actuated mode of the scan head. This average is determined by linear sampling in the bandwidth. The design variables are chosen similarly to the previous optimization where six’s vectors from each basis are used as sets of design variables. The resulting scan heads after optimization are shown in Figure C-3 and the corresponding FRF’s are plotted in Figure 7-9a.

7-7-5 Optimized scan head

The scan head that has been optimized for its modal and static nodal line location will be compared with the scan head that has the lowest average gain in Figure 7-9a. This is the model obtained with the symmetric basis.

The resulting FRF for optimal nodal line location, the lowest average FRF and a non-optimized original scan head are shown in Figure 7-9b. These three scan heads will be coupled to the frame in the following section.

Figure 7-8: Optimization results for multiple optimizations with one design variable and multiple optimizations with six design variables.

(a) FRF result for six optimized scan heads. Each scan head geometry is modified with one modification vector described by a modeshape. Only two vectors were not able to achieve a static nodal point at the desired location.

(b) FRF result for six optimized scan heads. Each scan head geometry is modified with six modification vector described by the modal, elastic, kinetic, energy, and eigenvalue sensitivity basis respectively. Each optimization uses six vectors from the corresponding basis except for the energy basis which combines the elastic and kinetic distributions.
7-8 Connecting the scan head on the frame

This section describes the coupling of the scan head and the frame. First the frame model will be introduced after which it will be coupled to the scan head. The resulting performance will be discussed for three scan head variations.

7-8-1 Frame model in Matlab

The frame that will support the scan head was introduced in Figure 7-1 as a "black box" and exhibits a high modal density in the frequency bandwidth of operation. To further increase its sensitivity to disturbances from the scan head the frame is made asymmetric.

Figure 7-10 characterizes the dynamic response of the frame for an applied load at the node that will be used for a connection with the scan head.

The response from a force in x-direction is shown in Figure 7-10a. The response in x-direction is largest for a static situation however the crosstalk creates dynamic responses in the other directions which are higher in amplitude. This is due to the fact that the frame is more compliant in y and z direction. This indicates that nodalizing will only be effective if both the x and y degree of freedom remain stationary. The current scan head is only optimized to reduce the movement in y. If a connection in x would be applied there would be a large response in y as well. Figure 7-10b Shows the response to a force applied in y-direction. Obviously the receptance in y-direction has the highest magnitude however there is also some crosstalk present creating a response in x-direction.

7-8-2 Error function

With the dynamics of the frame known the scan head and frame can be connected at their respective connection points. There are three different versions of the scan head for which the results will be discussed separately. To compare the results an error function is defined which will indicate the gain fluctuations. The error function is written as,

\[ \text{Error} = \frac{\text{FRF}_{\text{scan}}}{\text{FRF}_{\text{assembly}}} - 1. \] (7-3)

(a) FRF result from optimized scan heads with the average gain from zero Hz to the first resonance reduced by the optimization process. For each basis six modification vectors were used as design variables.

(b) Three versions of the scan head that will be connected to the frame for comparison. The original system represents a connection near a nodal point.
**Case study: designing with nodal points**

- **Figure 7-10:** Response of the frame for an applied force at the connection points that will be used by the scan head. The results indicate the high modal density in the bandwidth of operation.

**FRF**\(_{\text{scan}}\) represents the frequency response from the applied force to the payload position of the scan head before coupling. This should be a clean transfer function without gain fluctuations. **FRF\(_{\text{assembly}}\)** represents the frequency response after the coupling of the frame and scan head. The error function should clearly show the gain fluctuations that arise due to the interaction between the scan head and the frame. The assembly procedure is performed with the known techniques from dynamic sub-structuring described in [17].

The following results will only include a connection between the scan head and the frame in \(y\) direction since this is optimized.

### 7-8-3 Original scan head connected to the frame at poorly chosen location

This combination represents a system where the original scan head is connected to the frame without the intention to use nodalization as a method to improve the performance. The connection to the frame is made at the corners of the scan head at the center of the beams. The resulting transfer function from actuation force to payload position is shown in Figure 7-11a. The coupling to the frame clearly results in unwanted gain fluctuations before the first resonance frequency of the scan head. The error function defined in Eq. (7-3) is plotted in Figure 7-11b.

### 7-8-4 Original scan head connected near the nodal lines

This combination uses the original scan head however the connection is already chosen near the nodal lines. This system is referred to as "original" in Figure 7-9b. The resulting FRF and error response functions are shown in Figure 7-11. The increase in performance is substantial due to this more intelligent connection location. The result is plotted together with the results from the poorly chosen location.

### 7-8-5 Scan head with static and modal nodal line

In this paragraph the results for the two different optimization methods are compared. Figure 7-12b shows both the error response for the system with a scan head that is optimized to have the lowest average FRF amplitude as well as the response for the nodal line optimization. The results clearly
indicate that the nodal line approach results in the best performance in a large part of the frequency bandwidth. For the bandwidth near the resonance frequency of the scan head the connection with the lowest average FRF performs best. This is explained by Figure 7-9b were the gain of the average FRF is lower then the nodal line FRF in the frequency region near the resonance of the scan head. Both methods comply with the 1% gain fluctuation requirement.

7-9 Scan head sensitivity to fabrication tolerances

The previous optimization should prevent gain fluctuations in the transfer function from actuation forces to displacement at the payload position. As was already shown in Section 6-1 these fluctuations could also originate from the scan head itself by lack of symmetry. It was found that 1% parameter variations would results in similar levels of gain fluctuation. With the completion of the assembly a sensitivity analysis is performed. By selecting a set of elements which will be used as design variables an estimate can be made on the required fabrication precision.

For this investigation two design variables are defined: the thickness distribution on the lower and right side of the scan head. A change in thickness at these locations can result in two types of performance degradation. First of all the asymmetry in the scan head can initiate the actuation of asymmetric modes of the scan head in the bandwidth of operation. Secondly the symmetry will create a shift in the nodal locations. Therefore the connection points will no longer be in the optimal locations.

Figure 7-13 shows the FRF of an optimal scan head connected to the frame in combination with the sensitivity results. The sensitivities are calculated for two design variables $s_1$ and $s_2$ which represent the thickness of the lower and right beam of the scan head respectively. The sensitivity with respect to $s_1$ shows a non-zero value for the entire frequency bandwidth. This indicates the change in compliance of the payload position created by a stiffening or weakening of the beam. In the frequency region of 100 Hz, a perturbation of both design variables will create an increase in gain fluctuations. These gain fluctuations are primarily affected by the movement of the nodal points. The highest peak indicates a
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(a) FRF from applied force to displacement of the payload position for the assembly in which the scan head is optimized for the lowest FRF in the bandwidth of operation and a system where the modal and static nodal line coincide at the connection location. Due to the optimization process the static gain is slightly different.

(b) Error response function for the assembly where the scan head is optimized for the lowest FRF in the bandwidth of operation and a system where the modal and static nodal line coincide at the connection location. The latter performs best in an average sense but it shows the highest peak near resonance frequency of the scan head.

Figure 7-12: Response for optimized assembly where the scan head is optimized for the lowest FRF in the bandwidth of operation and a system where the modal and static nodal line coincide at the connection location. Both combinations are complaint with respect to the requirement.

2% gain fluctuation as a result of a 1% thickness change. In the frequency region just in front of the first actuated mode of the scan head, the sensitivity with respect to \( s_1 \) indicates a large gain fluctuation. This fluctuation triggers the actuation of an asymmetric mode due to the thickness perturbation. The expected gain fluctuation as a result of a 1% thickness change is approximated at 2%.

The sensitivity results of the assembled model have the same order of magnitude as the sensitivity results from the lumped mass model. They indicate that the fabrication accuracy would preferably be such that thickness changes of more than 0.1% are prevented. For a thickness of 20mm this results in less than 20\( \mu \text{m} \).

7-10 Summary and recommendations

This chapter describes how nodal points of both the static and modal deformation can be combined to achieve an effective suppression of gain fluctuations. The concept is based on the notion that a reduction of energy flow from the scan head to the frame will lead to minimal interference between the two.

In this section four topics will cover the summary and recommendations for further development of the demonstrator.

7-10-1 Optimizing connection for both \( x \) and \( y \) direction

The presented optimization was restricted to movement in \( y \) direction. For a practical application the movement of the connection point should be optimized for both the \( x \) and \( y \) direction. Detailed studies in ANSYS have already shown that the relative distance between the modal and static nodal line in \( x \) direction is small compared to the \( y \) direction. Moreover the sensitivity to the tuning parameters is also small. This could be both positive and negative. The positive aspect is the fact that a tuning in \( y \) direction will hardly affect the \( x \) direction. The downside is the obvious lack of tuning capability for this variable.
Summary and recommendations

7-10 Two proposals for a connection between scan head and frame

Figure 7-14 shows two conceptual layouts for the connection of the scan head with the frame. The first layout is based on the applied connection in the prototype that is described in Chapter 6. It uses small steel balls that are clamped in a socket as the fixation in all translational directions. In this layout the four nodal points are used on both sides of the scan head. This creates a stiff connection with the frame. The movement of the connection will have to be optimal in both $x$ and $y$ direction.

The second conceptual layout is derived with the knowledge of isostatic design rules. An attempt is made to constrain each degree of freedom only once yielding the connection statically determined. However, a more important constraint is the symmetric suspension of the scan head in the $x/y$ plane. This resulted in a slightly over-constrained connection. The concept is based on the idea that a folded leaf spring will constrain the scan head in $y$ direction at the optimized location. The leaf spring will show a reduced stiffness in $x$ direction however, the location of the leaf spring is close to the nodal line in the corresponding direction which reduces the amount of cross-talk even further. This concept therefore doesn’t require the optimization of the nodal lines in both $x$ and $y$ direction.

The size of the leaf springs is exaggerated to emphasize the principle layout. The sizing of the leaf springs will not be included in this report. This thesis focuses on the development of the design principle. The proof of this principle, detailed design, fabrication and testing of the demonstrator will be part of a separate project within TNO. This project has started during the writing of this thesis. Some of the lessons learned are combined in the last paragraph of this chapter.

7-10-3 Proposal for realization of the tunable parameters

The demonstrator must include tuning options to reposition the nodal lines at the connection location. Both the static and the modal nodal line have to be tuned independently. In the analysis it is shown that by lumped mass addition the modal nodal line can be manipulated without affecting the static
(a) First conceptual layout of the scan head to frame connection based on the prototype of Chapter 6. Steel balls are fixed on the scan head at the optimal nodal points. The frame will clamp the scan head on these points. Sockets on the frame will restrict movement of each point in all translational directions.

(b) Second conceptual layout of the scan head to frame connection based on isostatic design rules. The degrees of freedom are two times overdetermined. This is accepted since symmetry in the $x/y$ plane is deemed more important. Note that the size of the leaf springs is not to scale.

**Figure 7-14:** Two conceptual scan head connection layouts that could be used for the demonstrator.
response. The implementation of this mass tuning can be quite simple in the form of adding rings or nuts at a specific location.

The manipulation of the static nodal line is achieved by broadening the applied force. This will require more engineering effort to translate into a feasible design. The essence of the force tuning lies in the fact that only the width of the applied force must change. There must be no observable mass or stiffness change from a scan head viewpoint. A possible solution could be an array of reluctance or voice coil actuators. When switched off they should not affect the scan head. The width of the applied force could be changed by successively combining more actuators as the applied force. The tuning will in this case be partly incremental by the number of actuators used however, the relative forcing distribution could also be tweaked to tune within the increments provided by the actuators.

7-10-4 Insight obtained from detailed analysis

During the finishing stages of this thesis a parallel project at TNO started with the development of the demonstrator. The demonstrator must result in a proof of concept regarding the tuning of the nodal lines with the intention to suppress vibrations.

The conceptual demonstrator that was presented in this chapter was initially derived from a prototype machine described in Chapter 6. During the development of that prototype a strict requirement on the gain fluctuation led to the necessity of nodalizing. This feature was only one of many that needed to be optimized to be able to comply with the requirements. Apart from the use of nodal points the performance depended heavily on symmetry and balancing of for example the actuator forces. If the actuators were only slightly misaligned the performance was drastically decreased.

For the theoretical study into the optimal use of nodal points the importance of the other features is temporarily ignored. Yet the global concept of the scan head is directly copied from the source project and thereby its sensitivity for the previous described features.

It is essential that the performance of the demonstrator has the highest sensitivity with respect to the correct application of nodal points. If this is not the case the demonstrator can never be used for either falsification or validation of the principle. By implementing a scan head that besides nodalization relies heavily on symmetry and balancing the concept of nodalization can not be demonstrated.

This statement was confirmed during detailed analysis of the demonstrator concept that was derived from this chapter. Designing a system that has a high sensitivity with respect to nodal point placement while being less sensitive to other error sources has required a review of the concept.

In this process the development has already produced an increase in insight in the behavior of nodal lines and the current concept aims at removing all the dependence on symmetry while maintaining the nodal points principle.
During the research described in this thesis sensitivity analysis was applied for several case studies. The abstract at the third page of this document includes a summary of the most important conclusions. Section 8-1 is used for a less formal review on the lessons learned regarding the application of sensitivity analysis. Section 8-2 contains recommendations for future research after which the chapter ends with a conclusion.

8-1 Discussion on sensitivity analysis

Reviewing the application of sensitivity analysis it becomes apparent that the "what if" approach is dividable in two categories by the intent with which the analysis is performed.

The first category focuses on the change of the response with the intent to perform a tolerance analysis. Checking the stability of the response function for fabrication tolerances is a requirement in many high precision applications and sensitivity analysis provides a rigorous method to perform this. With the increasing demands on the dynamic performance such evaluations will become increasingly important. In this category both conceptual as well as detailed models are applicable. A sensitivity analysis on a detailed model should provide the analyst with an efficient method to determine the required fabrication procedure.

The second category of 'what if' analysis applies sensitivity analysis with the intent to create insight. For this category the goal is quite indistinct and it is hard to determine when it has been achieved. By studying the effect of parameter changes on the objective function the direction of optimization is retrieved. The fundamental properties of the model become more comprehensible by observation of these optimal directions.

It is found that the application of sensitivity analysis for this second category will primarily be effective on simple models or on global design parameters of complex models. In the detailed design phase the number of design variables not only increases rapidly, they are also less meaningful in their description of the 'concept'. The response to a thickness change of one single element will primarily be described by parasitic effects, such as the introduction of asymmetry. Furthermore, the sensitivities of, e.g., a detailed FE model is based on a small change in the system matrices. Such a change can not include the introduction of new elements. Due to this restriction a sensitivity analysis on such a model has a very limited meaning on a conceptual level. The definition of global parameters can reduce this problem yet it could be seen as an alternative to the simplification of the model. Furthermore, the latter is computationally more efficient.
8-2 Recommendations for future research

The focus of this thesis is the application of sensitivity analysis and not so much the computational and implementation aspects. From a user perspective the application of sensitivity analysis is mainly halted by the declaration of design variables. The selection of design variables by the user reduces the design space and thereby the possible solutions space. In addition the parametrization of models can be time consuming.

Topology optimization is a well known method that leaves the largest freedom to the optimization process and does not require design parameter definitions. However, such a method is seldom used to define a global machine concept. Parameter free shape optimization is a method that lies somewhere in the middle of classical shape and topology optimization. By first defining a structure the designer can include essential components and experience. The parameter free shape optimization is then free to move nodes at the surface of the solids. Sensitivities of such a function could be very interesting for “what if” analysis of the first category. There are crude applications of this method in commercial FE codes however, they still require a large amount of implementation effort from the user. Unreliable results from sensitivity analysis with respect to the movement of a node is currently preventing the implementation of a more general method.

The case studies have resulted in several applications of balancing. In the first case study balancing reduced the effect of floor vibrations. In the second and third case study dummy masses and additional actuators were used for the suppression of machine vibrations. Due to the strong dependency on the quality of balancing it is recommended to study this topic in more detail. A review on existing work could be combined with practical experiments with different types of actuators. Knowledge on the achievable balance in force and position is essential for future implementation of these methods.

It requires no explanation that the investigation into the nodal points principle will continue. The demonstrator project has started and should create the possibility to validate the concept. The experiments will require accurate measurements of nodal points which could be a separate topic of research. There are numerous methods for the identification of modeshapes however, seldom focusing on the locations of nodal points. Combining the accurate identification of nodal points into the production process of a component can aid the further development of the concept.

8-3 Conclusion

Sensitivity analysis is a tool that aids an analyst in the design exploration of a model. By applying the developed algorithms a formal approach results in a fair and standardized method of comparison. When aiming for insight the model complexity should be reduced to a minimum. In this way a change in design variable has a strong link with a change in concept.

Reducing the design space will lead to an additional efficiency increase. In Section 4-3 a reduction in design space by modification vectors was introduced. The method is tested in Section 7-6 for the optimization of a static nodal point. The design space reduction did not affect the obtained optimum, while the computational efficiency increased by reducing the number of design variables from 37 to 6.

Several case studies have shown how sensitivity analysis can be applied to improve the dynamic behavior of mechatronic systems. By balancing mechanical components and using nodal lines several response functions were optimized. Furthermore, the concept of tuning for optimal performance has led to the initiation of a demonstrator project. The resulting knowledge from this project could lead to the next step in mastering dynamic behavior.
Appendix A

Model Data

A-1 Model data for lumped mass model

Table A-1: Mode data for Section 6-1

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<thead>
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<th>Number</th>
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<th>damper [N s/m]</th>
<th>mass [kg]</th>
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<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1e6</td>
<td>20</td>
<td>1 (Payload)</td>
</tr>
<tr>
<td>2</td>
<td>1e6</td>
<td>92</td>
<td>20 (Frame)</td>
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</tr>
<tr>
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<td>20</td>
<td>1 (Payload)</td>
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<td>103</td>
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<td>16</td>
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<td>9</td>
<td>1e6</td>
<td>104</td>
<td>n.v.t</td>
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</table>
Appendix B

Scan head bases for design modifications
Scan head bases for design modifications

(a) Mode 10
(b) Mode 18
(c) Mode 25
(d) Mode 39
(e) Mode 55
(f) Mode 68

Figure B-1: Modal basis from six's structural modes
Figure B-2: Elastic energy distribution for six's modes
Figure B-3: Kinetic energy distribution for six’s modes
Figure B-4: Eigenvalue sensitivity basis from six's structural modes
Appendix C

Scan head optimization results
Figure C-1: Optimization results for six vectors from the modal basis
Figure C-2: Optimization results for six vectors from different bases.
Figure C-3: Optimization results for lowest average FRF for six sets of vectors from different bases.
Appendix D

L shaped cantilever beam

In this chapter a cantilever beam is used as a test case for the comparison of the Rayleigh quotient with the sensitivity analysis of the eigenfrequency. Mode shape five and six of the beam are shown in Figure D-1 and the corresponding frequencies are given in Table D-1.

<table>
<thead>
<tr>
<th>Number</th>
<th>Frequency [Hz]</th>
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<tr>
<td>5</td>
<td>25.73</td>
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<tr>
<td>6</td>
<td>34.82</td>
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D-0-1 Sensitivity and Rayleigh

The sensitivity for the eigenfrequencies is calculated with Femtools. Figure D-2 shows the derivative with respect to the Young’s modulus and Figure D-3 with respect to the density.

The distribution of the derivatives can be compared with the distribution of the potential and kinetic energy. The energy distributions are calculated with ANSYS and are plotted in Figure D-4 and Figure D-5.

![Figure D-1: Modeshape 5 and 6 of cantilever L shaped beam](image-url)
Figure D-2: Sensitivity distribution for the eigenvalues 5 and 6. The sensitivity is calculated with respect to the Young's modulus for every finite element.

Figure D-3: Sensitivity distribution for the eigenvalues 5 and 6. The sensitivity is calculated with respect to the material density for every finite element.
Figure D-4: Elastic energy distribution for the first six’s modes. The energy content is calculated for every individual finite element.

(a) $\frac{1}{2} x_5^T K x_5$

(b) $\frac{1}{2} x_6^T K x_6$

Figure D-5: Kinetic energy distribution for the first six’s modes. The energy content is calculated for every individual finite element.

(a) $\frac{1}{2} \lambda_5 x_5^T M x_5$

(b) $\frac{1}{2} \lambda_6 x_6^T M x_6$
By visual inspection it can be seen that the distributions are equal. However this can also be checked in a more rigorous way by plotting the value of each element in a graph. The graph’s of the derivatives and energies should be equal when the amplitudes are scaled. Figure D-6 shows the resulting plot and confirms the statement that the derivative of an eigenvalue with respect to the Young’s modulus or the density is equal to the potential or kinetic energy distribution.

![Comparison between Rayleigh and sensitivity](image.png)

**Figure D-6:** Comparison between eigenvalue sensitivity and elastic energy from the Rayleigh quotient
Bibliography


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Master of Science Thesis

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