CONVECTION IN A RECTANGULAR BOX OF FLUID WITH INT. HEAT SOURCES

Afstudeerverslag van: F. Korbijn
Mentor: prof. E. Palm

T.M. Daleh / Universitet i Oslo
Lab. voor Fys. Techn. / Matemåisk Instit. avd. 5 for Aktuær
oktober 1978
<table>
<thead>
<tr>
<th>Plaats</th>
<th>Er staat</th>
<th>Er moet staan</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of symbols (1)</td>
<td>$Q \ldots \ldots J/m^3 \cdot t$</td>
<td>$Q \ldots \ldots J/m^3 \cdot s$</td>
</tr>
<tr>
<td>List of symbols (1)</td>
<td>$q \ldots \text{per unit mass, on } J/kg$</td>
<td>$q \ldots \text{per unit mass and unit time, on } J/kg \cdot s$</td>
</tr>
<tr>
<td>blz. 18, na (IV.3)</td>
<td>$q \ldots \text{per unit mass}$</td>
<td>$q \ldots \text{per unit mass and unit time}$</td>
</tr>
<tr>
<td>blz. 30, (V.15)</td>
<td>$\psi = \sum_{i=1}^{\infty} \frac{b_i}{q_i} \psi_i$</td>
<td>$\psi = \sum_{i=1}^{\infty} \frac{b_i}{q_i} \psi_i$</td>
</tr>
<tr>
<td>blz. 31, (V.23)</td>
<td>$\nabla \cdot \psi = 0$</td>
<td>$\nabla \cdot \psi = 0$</td>
</tr>
<tr>
<td>blz. 52, (VI.2)</td>
<td>$\nabla \cdot \psi = 0$</td>
<td>$\nabla \cdot \psi = 0$</td>
</tr>
<tr>
<td>blz. 53, (VI.9)</td>
<td>$\nabla \cdot \psi = \ldots$</td>
<td>$\nabla \cdot \psi = \ldots$</td>
</tr>
<tr>
<td>blz. 53, (VI.11)</td>
<td>$\nabla \cdot \psi = \ldots$</td>
<td>$\nabla \cdot \psi = \ldots$</td>
</tr>
</tbody>
</table>
SUMMARY

This paper is concerned with thermal convection in a quiescent three-dimensional rectangular box of fluid heated from within by an uniform distribution of heat sources. It is assumed that the preferred mode of convective cells at the onset of convection is some number of finite roll cells (cells with two non-zero velocity components dependent on all three spatial variables). By a numerical approach the value of the critical Rayleigh number and the preferred number of finite rolls is determined for boxes with width to depth ratios $H/d$ in the range $1/4 \leq H/d \leq 6$. It is predicted that the finite rolls have their axes parallel to the short side of the box.

The critical Rayleigh number decreases rapidly to the value for the infinite horizontal layer with internal heat sources as the horizontal dimensions of the box increase.

A start is made with the study of the intensity of the motion set-up in the box for Rayleigh numbers above the critical Rayleigh number.

At the end of this paper a practical application is discussed.
CONTENTS

List of used symbols

I. Introduction 1
II. Rock-salt formations 5
III. Short literature survey on linear convection theory 11
   1. Thermal convection in a horizontal fluid layer with internal heat sources 11
   2. Thermal convection in a box heated from below 13
IV. Thermal convection in a horizontal fluid layer with internal heat sources, linear theory 18
V. Convection in a box with internal heat sources, linear theory 27
   1. Governing equations 27
   2. Finite roll approximation, Galerkin procedure 29
   3. Trial functions 35
   4. Results of the computations 38
      A. Comparison of the convergence and the results obtained with the two sets of trial functions 38
      B. Stability curves 46
VI. Convection in a box with internal heat sources, non-linear theory 52
    1. Governing equations, steady solutions 52
    2. Results 56
VII. Summary and discussion 65
References 73
Appendix I, The Galerkin method A1
Appendix II, Box with perfectly conducting side walls. Critical Rayleigh numbers for several aspect ratios A3
Appendix III, Box with perfectly insulating side walls. Critical Rayleigh numbers for several aspect ratios A5
### List of used symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>matrix defined by eq. (V.27)</td>
</tr>
<tr>
<td>$A_{mnj}$</td>
<td>coefficients in trial functions</td>
</tr>
<tr>
<td>$A_h(x)$</td>
<td>function defined by eq. (IV.22)</td>
</tr>
<tr>
<td>$a$</td>
<td>horizontal wave number</td>
</tr>
<tr>
<td>$a_c$</td>
<td>horizontal wave number corresponding with $R_c$</td>
</tr>
<tr>
<td>$a_q$</td>
<td>coefficients in trial functions</td>
</tr>
<tr>
<td>$a_{mnj}$</td>
<td>functions defined by eq. (IV.22)</td>
</tr>
<tr>
<td>$B_{qh}$</td>
<td>orthogonal functions defined by eq. (V.32)</td>
</tr>
<tr>
<td>$c(h,j)$</td>
<td>function defined by eq. (IV.26)</td>
</tr>
<tr>
<td>$c_p$</td>
<td>specific heat at constant pressure $m^2/\rho C_s^2$</td>
</tr>
<tr>
<td>$D_o$</td>
<td>integer function defined near eq. (VI.21)</td>
</tr>
<tr>
<td>$D_e$</td>
<td>integer function defined near eq. (VI.21)</td>
</tr>
<tr>
<td>$d$</td>
<td>thickness of horizontal layer, depth of box $m$</td>
</tr>
<tr>
<td>$e$</td>
<td>internal energy per unit mass $Nm/kg$</td>
</tr>
<tr>
<td>$F_h(x,z)$</td>
<td>function defined by eq. (IV.21)</td>
</tr>
<tr>
<td>$f$</td>
<td>function defined by eq. (V.33)</td>
</tr>
<tr>
<td>$f(x,y)$</td>
<td>function defined in appendix I</td>
</tr>
<tr>
<td>$g$</td>
<td>acceleration of gravity $m/s^2$</td>
</tr>
<tr>
<td>$H$</td>
<td>width of box $m$</td>
</tr>
<tr>
<td>$H_1$</td>
<td>width of box in the x-direction $m$</td>
</tr>
<tr>
<td>$H_2$</td>
<td>width of box in the y-direction $m$</td>
</tr>
<tr>
<td>$h_1$</td>
<td>dimensionless width of box in x-direction</td>
</tr>
<tr>
<td>$h_2$</td>
<td>dimensionless width of box in y-direction</td>
</tr>
<tr>
<td>$h$</td>
<td>enthalpy on page 20 only $Nm/kg$</td>
</tr>
<tr>
<td>$h$</td>
<td>summation index</td>
</tr>
<tr>
<td>$I_1$</td>
<td>integral defined by eq. (V.43)</td>
</tr>
<tr>
<td>$I_2$</td>
<td>integral defined by eq. (V.44)</td>
</tr>
<tr>
<td>$j$</td>
<td>summation index</td>
</tr>
<tr>
<td>$j_z$</td>
<td>integer defining the number of terms in the summation in the z-direction.</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>K</td>
<td>thermal conductivity</td>
</tr>
<tr>
<td>k</td>
<td>unit vector in z-direction</td>
</tr>
<tr>
<td>L</td>
<td>operator defined in appendix I</td>
</tr>
<tr>
<td>M</td>
<td>integer denoting the number of rolls</td>
</tr>
<tr>
<td>m</td>
<td>summation index</td>
</tr>
<tr>
<td>m_a</td>
<td>summation index</td>
</tr>
<tr>
<td>m_l</td>
<td>summation index</td>
</tr>
<tr>
<td>m_2</td>
<td>summation index</td>
</tr>
<tr>
<td>m_r</td>
<td>integer defining the initial value of m</td>
</tr>
<tr>
<td>m_x</td>
<td>integer defining the number of terms in the x-direction</td>
</tr>
<tr>
<td>m_x</td>
<td>integer denoting m finite x-rolls in figs. 21-28 and tables al-a8.</td>
</tr>
<tr>
<td>N</td>
<td>integer denoting the number of terms in an arbitrary summation</td>
</tr>
<tr>
<td>Nu</td>
<td>Nusselt number</td>
</tr>
<tr>
<td>Nussup</td>
<td>Nusselt number at upper boundary plane of a box</td>
</tr>
<tr>
<td>Nussdown</td>
<td>Nusselt number at lower boundary plane of a box</td>
</tr>
<tr>
<td>N_o</td>
<td>dimensionless number defined by eq. (VI.29)</td>
</tr>
<tr>
<td>N_1</td>
<td>dimensionless number defined by eq. (VI.29)</td>
</tr>
<tr>
<td>ΔN_o</td>
<td>dimensionless number defined on page 60</td>
</tr>
<tr>
<td>ΔN_1</td>
<td>dimensionless number defined on page 60</td>
</tr>
<tr>
<td>n</td>
<td>summation index</td>
</tr>
<tr>
<td>n_y</td>
<td>integer defining the number of terms in the summation in the y-direction</td>
</tr>
<tr>
<td>n_y</td>
<td>integer denoting n finite y-rolls in figs. 21-28 and tables al-a8</td>
</tr>
<tr>
<td>P</td>
<td>Prandtl number</td>
</tr>
<tr>
<td>p</td>
<td>pressure</td>
</tr>
<tr>
<td>p</td>
<td>dimensionless pressure perturbation</td>
</tr>
<tr>
<td>p'</td>
<td>pressure perturbation</td>
</tr>
<tr>
<td>p_s</td>
<td>steady state solution for the pressure if all heat transport is in the form of conduction</td>
</tr>
<tr>
<td>Q</td>
<td>generated heat per unit time and unit volume</td>
</tr>
<tr>
<td>q</td>
<td>to system supplied heat per unit mass, on page 18 only</td>
</tr>
</tbody>
</table>
List of used symbols (continued 2)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>q</td>
<td>summation index</td>
</tr>
<tr>
<td>R</td>
<td>Rayleigh number</td>
</tr>
<tr>
<td>R₀</td>
<td>Rayleigh number defining the onset of convective motion</td>
</tr>
<tr>
<td>Rₐ</td>
<td>critical Rayleigh number</td>
</tr>
<tr>
<td>r</td>
<td>summation index</td>
</tr>
<tr>
<td>S</td>
<td>two-dimensional domain (appendix I)</td>
</tr>
<tr>
<td>Sᵐ(x)</td>
<td>orthogonal functions defined by eq. (V.32)</td>
</tr>
<tr>
<td>T</td>
<td>temperature (also dimensionless) °C</td>
</tr>
<tr>
<td>T₀</td>
<td>temperature at which ρ=ρ₀ °C</td>
</tr>
<tr>
<td>Tₛ</td>
<td>steady state solution for the temperature if all heat transport is in the form of conduction °C</td>
</tr>
<tr>
<td>ΔT</td>
<td>temperature difference °C</td>
</tr>
<tr>
<td>T̅</td>
<td>dimensionless horizontally averaged temperature °C</td>
</tr>
<tr>
<td>Tₛ̅</td>
<td>dimensionless horizontally averaged steady state temperature</td>
</tr>
<tr>
<td>Tₛ'</td>
<td>dimensionless form of Tₛ</td>
</tr>
<tr>
<td>u</td>
<td>x-component of velocity m/s</td>
</tr>
<tr>
<td>u</td>
<td>dimensionless perturbation in x-component of velocity</td>
</tr>
<tr>
<td>uₐ</td>
<td>q-th. term of u in summation (V.16), (VI.14)</td>
</tr>
<tr>
<td>u'</td>
<td>perturbation in x-component of velocity m/s</td>
</tr>
<tr>
<td>u(x,y)</td>
<td>function defined in appendix I</td>
</tr>
<tr>
<td>V</td>
<td>scalar function defined on page 23</td>
</tr>
<tr>
<td>v</td>
<td>velocity m/s</td>
</tr>
<tr>
<td>v</td>
<td>dimensionless perturbation in velocity</td>
</tr>
<tr>
<td>v</td>
<td>y-component of velocity m/s</td>
</tr>
<tr>
<td>v</td>
<td>dimensionless perturbation in y-component of velocity</td>
</tr>
<tr>
<td>v'</td>
<td>perturbation in v m/s</td>
</tr>
<tr>
<td>w</td>
<td>z-component of velocity</td>
</tr>
<tr>
<td>w</td>
<td>dimensionless perturbation in z-component of velocity</td>
</tr>
<tr>
<td>wₐ</td>
<td>q-th. term of w in summation (V.17), (VI.15)</td>
</tr>
<tr>
<td>w'</td>
<td>perturbation in w m/s</td>
</tr>
<tr>
<td>x</td>
<td>position vector m</td>
</tr>
<tr>
<td>x</td>
<td>dimensionless position vector</td>
</tr>
<tr>
<td>x</td>
<td>horizontal coordinate m</td>
</tr>
<tr>
<td>x</td>
<td>dimensionless horizontal coordinate</td>
</tr>
</tbody>
</table>
List of used symbols (continued 3)

- $y$  horizontal coordinate  $\text{m}$
- $\tilde{y}$  dimensionless horizontal coordinate  $\text{m}$
- $z$  vertical coordinate  $\text{m}$
- $\tilde{z}$  dimensionless vertical coordinate

$\alpha$  coefficient of thermal expansion  $\text{oC}^{-1}$
$\beta$  arbitrary variable
$\gamma$  arbitrary variable
$\delta_{mn}$  Kronecker delta
$\delta_0$  integer function defined near eq. (V.39)
$\delta_1$  integer function defined near eq. (V.39)
$\varepsilon$  error defined by eq. (IV.23)
$\varepsilon_1$  error defined by eq. (V.18)
$\varepsilon_2$  " " eq. (V.19)
$\varepsilon_3$  " " eq. (V.20)
$\varepsilon_n(x,y)$  error function defined in appendix I
$\Theta$  perturbation in temperature  $\text{oC}$
$\Theta$  dimensionless perturbation in the temperature
$\Xi$  function variable defined as $\varphi$
$k$  function variable defined as $\varphi$
$\kappa$  thermal diffusivity  $\text{m}^2/\text{s}$
$\lambda_m$  positive roots of eqs. (V.35)
$\mu$  dynamical viscosity  $\text{kg/ms}$
$\lambda_m$  positive roots of eqs. (V.35)
$\nu$  kinematic viscosity  $\text{m}^2/\text{s}$
$\rho$  density of fluid  $\text{kg/m}^3$
$\rho_0$  standard density  $\text{kg/m}^3$
$\rho_s$  steady state solution for the density if all
$p$  heat transfer is in the form of conduction  $\text{kg/m}^3$
$\rho'$  perturbation in the density  $\text{kg/m}^3$
$\psi_i$  trial function defined in appendix I
$\psi$  stream function
I. INTRODUCTION

The development of the electricity production by means of nuclear energy in the Netherlands during the coming decades, will only be able to take place undisturbed, if and only if an adequate solution to the problem of the storage of the resulting, accumulating amounts of radioactive waste can be found in time.

Using the Dutch abbreviations the classification of solid radioactive waste is as follows (International Atomic Energy Agency, IAEA, (1)):

LAVA (Low active solid waste) < 0.2 rem/hour
MAVA (Moderate active solid waste) < 2.0 rem/hour
HAVA (High active solid waste) > 2.0 rem/hour

A separate category with its own problems, HLW (High Level Waste) is distinguished. HLW is high active solid waste which originates from the installations, in which the fissionable material of nuclear reactors is being worked up. LAVA, MAVA and HAVA originate from nuclear reactors, laboratories, hospitals etc.

We can mention the following possibilities, which have been considered for the storage of radioactive waste:
- ocean dumping (LAVA and MAVA). This possibility however is very expensive because of the necessary concrete ballast. Also environmental objections.
- storage close underneath the surface of the earth: unacceptable possibility for the Netherlands.
- storage in artificial objects on, or close underneath the surface of the earth: environmental objections.
- creation of a storage-accomodation on large depth, in one of the numerous naturally occurring rock-salt masses, which the Netherlands have to their disposal.

It is this last possibility, which now is being examined in more detail in the Netherlands. At the end of 1975 at the Technical University of Delft, a start was made with the study
of the problems that might arise from the storage of radioactive waste products in naturally occurring rock-salt masses. The main object of this research is to improve the insight in the extent of safety, with which one is able to keep the waste outside the range of the biosphere over a long period of time (Dietz, (2)). For that purpose one has to be well informed about:

1. properties of the radioactive waste
2. properties of the salt
3. possible affection of a salt formation by external groundwater
4. temperatures; the heat, produced by the HLW, is small with respect to the enormous salt masses, but locally a quite important increase of the temperature might appear. As a first limitation of the admissible temperature we can mention the melting points of some salts. Another limitation will be mentioned further on in the introduction.
5. soil movements; because of the heating the salt will expand somewhat and the soil might possibly rise a few centimeters dependent on the depth of the storage-accommodation. On the other hand a storage-accommodation might collapse (chapter II) and cause a descent of the soil of the same order of magnitude.
6. thermal mole; it has to be examined if a piece of HLW, that has broken loose, might be able to displace itself like an electric iron through ice.
7. radiolysis; as a result of radiation chemical compounds can be analysed.
8. various subterranean techniques.

It is especially subject 3, the possible affection of a salt formation by external groundwater, which is of importance for the researchgroup 'Transportphenomena' of which I am a member. Prof. J.M.M. Smith of our researchgroup has done some calculations concerning the rapidity of affection of rock-salt by groundwater at places where a protective layer (accumulated insoluble defilements, chapter II) is damaged, so that the groundwater can reach the salt (3). Smith found that this velocity is acceptable low. Smith did not account for the fact
that the insoluble defilements will accumulate and form a new protective layer.

This paper makes a start with the study of the problems that might rise because of the affection of the salt by the appearance of a convective flow in a storage-accomodation (cavity), if this cavity is filled with water and the strength of the density of the heat sources (radioactive materials) in the cavity exceeds a certain limit. Subject 4, temperatures, comes also up for discussion here, because the undesirableness of a strong convective flow in the cavity limits the acceptable strength of the density of the heat sources and thus also the acceptable temperature in the cavity.

The following simplifications have been made in this paper:

a. we do not account for the solubility of rock-salt in water.
b. the cavity is a rectangular box.
c. the upper and lower boundary plane of the box have the same temperature, i.e. we do not account for the temperature differences within the surface of the earth.
d. the heat sources (radioactive materials) are evenly distributed in the box. The other extreme is to assume that all the heat sources are placed on the bottom of the box, and to regard the problem of a box which is heated from below. It turned out however that Davis already had done a number of computations on this problem.

The aim of this paper is the following:
Given a rectangular cavity filled with water. Assume that the heat sources are evenly distributed. Find the necessary strength of the density of the heat sources, in order to get a convective motion, i.e. find the critical 'Rayleigh number'. Try to study the intensity of the motion set-up for a given 'Rayleigh number' (density of heat sources).

By means of a numerical approach the critical Rayleigh numbers are computed for boxes of several dimensions (chapter V, linear theory for a box).

A start is made with the study of the intensity of the motion set-up and the development of the temperature profile in the
box, for a given density strength of the heat sources (given Rayleigh number), which causes a convective motion to appear (non-linear theory for a box, chapter VI).

First however chapter II will give some more information about the naturally occurring rock-salt formations and how one intends to use them for the storage of radioactive waste products.
II. ROCK-SALT FORMATIONS

The use of naturally occurring rock-salt masses for the storage of radioactive waste products was first proposed in the U.S.A. during the fifties, however only for HLW, the most active category. During about 18 years the American Atomic Energy Commission (AEC) has examined the possibility of making an existing salt-mine in Kansas suitable as a storage-accommodation for radioactive waste products. For reasons which will not be discussed here, this research has not yet led to a practical application in the U.S.A.

In Western Germany however, radioactive waste is being stored in a former salt-mine near Wolfenbüttel since 1967 (LAVA, MAVA and HAVA) (Wervers(5)).

On which grounds, a special signification is attached to rock-salt as a storage medium for radioactive waste products?

Salt deposits have come into existence as a result of the evaporation of inland seas in a number of periods, situated between the Paleozoicum and Mesozoicum. The deposits in the Netherlands date from the geological period, called Zechstein, and are as a consequence more than 200 million years old. They originate from an extensive inland sea, which was situated in the eastern part of England, the North Sea, the northern part of the Netherlands, the northern part of Germany, Poland and Denmark (fig. 1).

In the geological history of the earth, these salt-layers have been covered by other deposits. Salt, being less heavy than other rock, was inclined to vault upwards to places of low resistance, in some cases close underneath the surface of the earth. At the places where this has happened we speak according to the result of salt cushions (uptil about 1000 m thick) or salt domes (uptil about 3000 m high) (fig. 2).

The bottom of the Dutch salt lies for the greater part on a depth of well over 3000 m, and the salt domes reach up to a few hundred meters underneath the surface of the earth.

The salt domes are often covered by a particular rock layer,
fig. 1. Salt areas in the Netherlands and the Dutch part of the Continental.
Fig. 4. Potassium-magnesium-salts in a salt dome

Fig. 3. Potassium-magnesium-salts in a cushion

Fig. 2. Three types of rock-salt formations
the anhydrite cap-rock. This cap-rock may have a thickness varying from a few meters till a few hundred meters. One explains the presence of the cap-rock by the penetration of a salt dome in an area with circulating water, followed by the solution of the top of the salt dome, which as a consequence becomes truncated. Rock-salt is not pure sodium chloride, and a small residue remains after solution. Over long periods of time the insoluble material becomes compacted and forms an insoluble protective layer over the top of the column, the anhydrite cap-rock.

Geological studies have shown, that the layers, cushions and domes hardly have been deformed or affected during the last 50 miljon years, which indicates, that they are isolated from subterranean water movements. We can also mention, that the Netherlands are situated in a geological very quiet area. Solids which are placed under the surface of the earth will not be driven out by geological movements. Only an object, which emits so much heat that its surroundings would melt, might be able to displace itself as a result of a deviating specific gravity.

Table 1 gives some properties of rock-salt (Wervers, (5)).

Table 1: Properties of rock-salt at 20°C.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>2.18 g/cm³</td>
</tr>
<tr>
<td>Pressure firmness</td>
<td>390 kg/cm²</td>
</tr>
<tr>
<td>Flow boundary</td>
<td>204 kg/cm²</td>
</tr>
<tr>
<td>Elasticity coefficient</td>
<td>3.5x10⁴ kg/cm²</td>
</tr>
<tr>
<td>Shear strength</td>
<td>50 kg/cm²</td>
</tr>
<tr>
<td>Solubility in water (20°C)</td>
<td>300 g/l</td>
</tr>
<tr>
<td>Melting point</td>
<td>800 °C</td>
</tr>
</tbody>
</table>

The pressure firmness of rock-salt is comparable with the pressure firmness of concrete but on the other hand rock-salt has a low flow boundary. Rock-salt is a plastic material: stresses are relieved by flow rather than by fracture, and sudden failures resulting from earth movements outside a salt mass will give cracks that heal in a relatively short space of time. The properties of rock-salt are favourable for performing mining activities, not only by means of traditional
subterranean methods, but especially because of the possibility to create cavities by lixiviation from the surface of the earth (more about the different techniques further on in this chapter).

As mentioned, sea-water not only contains sodium chloride, but also small concentrations of other salts. At the end of each evaporation-period these salts precipitate, chiefly potassium- and magnesium salts. As a consequence layers with potassium- and magnesium salts can be found between the rock-salt layers, as a rule not thicker than about 10 meters. These salts flow more easy than the rock-salt. If the salt is vaultet to a cushion, the potassium- and magnesium salts have flowed more than the rock-salt, which gave rise to an extra thickening of these salts (fig. 3).

During the formation of a salt dome, all the salt has been moving to such an extent, that the original coherence has been disturbed. The potassium- and magnesium salts can usually be found as incoherent strings with the shape of question-marks (fig. 4).

Because of the low flow boudary of rock-salt, flow might appear on a depth of about 1000 m and an atmospheric cavity might collapse completely. The potassium- and magnesium salts have lower boundary depths for flow. On larger depths a cavity can be held open for a desired length of time by the use of a counter-pressure within the cavity. In principle one can distinguish two types of storage-accomodations: shallow, rigid rooms, and deep rooms which one can admit to collapse at any moment by removing the counter-pressure (Dietz, (2)).

For the benefit of the storage of radioactive waste, several techniques will be studied:
- **Mine**, rigid rooms on a depth of about 400 m under the surface of the earth. On this depth the required space can only be found in the top of some salt domes (storage of LAVA, MAVA, HAVA and HLW).
- **Dissolved cavity**, water circulates via one or more bore-holes which dissolves the salt. The shape of the cavity can be controlled with the help of modern techniques (storage of LAVA and MAVA).
- Bore-holes, have a small capacity and can only be used for the storage of small amounts of the highly active HLW. For this purpose one can bore shallow holes in the mine galleries.

It will be clear that if one of the mentioned techniques is used, the holes have to be locked up safely, by means of iron, cement, concrete plugs. Natural locking is perhaps also possible. On sufficient depth one can allow the rock-salt to flow together over the waste. A suitable layer of the even better flowing potassium- and magnesium salts might act as a natural fluidal packing (Dietz, (2)).

The advantages and disadvantages of the different techniques under the given circumstances need to be examined together with the other items mentioned in chapter I.
III. SHORT LITERATURE SURVEY ON LINEAR CONVECTION THEORY

III.1. Thermal convection in a horizontal fluid layer with internal heat sources

Several authors have studied the problem concerning thermal convection in a horizontal fluid layer heated from within by an uniform distribution of internal heat sources. Of importance for the present paper is the situation in which the fluid is assumed to be bounded by two rigid, perfectly conducting planes, which are maintained at constant and equal temperature. As a consequence we obtain an unstably stratified fluid layer above a stably stratified fluid layer. If the strength of the density of heat sources exceeds a certain critical value, the static state of the fluid becomes unstable because the buoyancy force in the unstably stratified layer is sufficient to overcome the dissipative effects. The resulting convective motion is dependent on two dimensionless parameters, the (modified) Rayleigh number $R$ and the Prandtl number $P$ which are defined by

$$R = \frac{g \alpha |Q| d^5}{6 \beta \rho_o c_p K^2}, \quad P = \frac{\nu K}{\kappa}$$

(III.1)

where $g$ is the acceleration of gravity, $\alpha$ is the coefficient of thermal expansion of the fluid, $|Q|$ absolute value of $Q$, the constant generated heat per unit time and unit volume, $d$ is the thickness of the layer, $\rho_o$ a standard density, $c_p$ the specific heat at constant pressure, $K$ thermal diffusivity and $\nu$ the kinematic viscosity.

The linear stability problem was first solved by Sparrow, Goldstein and Jonsson (1964, (6)). They found a critical Rayleigh number $R_C$ equal to 583.2, which also is found by Tveitereid (7). Both Sparrow et al. and Tveitereid used the periodicity in the horizontal direction in temperature and velocity, and introduced a horizontal wave number $a$. They developed the solution in a power series of the vertical coordinate $z$. The Rayleigh number $R_o$, defining the onset of convective motion, then becomes a function of $a$. Fig.5 shows the qualitative behaviour of the resulting stability curve. The with the critical value of $R_o$, $R_C$, corresponding value of $a$, $a_c$, was found to be 4.0. The area under the stability curve in fig. 5 is the stable region, the area above the unstable region.
Fig. 5. Stability curve for the infinite horizontal layer with internal heat sources (qualitative)

Stability curve for a box (qualitative); $h_1$ = fixed
Tveitereid and Palm (8) have also studied the problem of a fluid layer which is bounded above by a rigid perfect conducting plane maintained at constant temperature and below by a rigid insulating plane. Using the periodicity in the horizontal direction and developing the solution in a power series of \( z \) they found

\[ R_c = 2772.27; \quad a_c = 2.63 \]

which are identical to the values found by Roberts (9).

As far as we know, no computations have been performed yet on the problem of a quiescent three-dimensional rectangular box of fluid which is heated from within by an uniform distribution of heat sources. However, several authors have studied the linear stability of a rectangular box of fluid which is heated from below. The next section of this chapter gives a survey on these studies.

III.2. Thermal convection in a box heated from below

If a quiescent three-dimensional rectangular box of fluid is heated from below, this results in a density gradient in the fluid opposite to the direction of gravity. If the temperature gradient between lower and upper plane of the box exceeds a certain critical value, the static state of the fluid becomes unstable because the buoyancy force is sufficient to overcome the dissipative effects. The resulting convective motion is again dependent on the Prandtl number \( P \) and the Rayleigh number \( R \). The Rayleigh number is now defined as

\[ R = \alpha (\Delta T)^2 \frac{\partial^3}{\partial y^3} \]

where \( \Delta T \) is the temperature difference between top and bottom of the box.

Several authors have studied the linear stability of the box of fluid heated from below. Pellew & Southwell (1940, (10)) and Zierep (1963, (11)) recognized that the presence of rigid lateral walls prevents the separation of variables in the equations that govern the system. By simplifying the boundary conditions on the walls, i.e. by assuming slip flow, Zierep calculated the influence of the walls.

Ostrach & Pnuelli (1963, (12)) obtained a sixth-order linear partial differential for the vertical velocity \( w \), with the
boundary conditions $w = \partial w / \partial z = \nabla^2 w = 0$. On vertical walls, however, the condition $\partial w / \partial z = 0$ is redundant, while no heed was taken to the fact that the horizontal velocity components must also vanish there. Separation of variables was used to obtain critical Rayleigh numbers which are insensitive to these latter conditions. Hence Ostrach & Prueili's numbers are incorrect and were later criticized by Davis (1967, (4)), who used correct boundary conditions, also for the horizontal velocity components. All boundaries were considered to be rigid and perfect heat conductors, i.e. the disturbances in velocity and temperature caused by convection are assumed to vanish at the walls. The latter assumption ensures that the temperature distribution in the lateral walls remains linear in the vertical direction even after the onset of convection. Davis assumed that the spectrum of allowable wave-numbers will be denumerable corresponding to multiples of the horizontal dimensions, since the linear stability problem is to be solved in a closed and bounded domain. To solve the linear stability problem, Davis has used trial functions which have the properties of finite rolls. With a fully enclosed geometry we can define finite rolls as cells with two non-zero velocity components dependent on all three spatial variables. This dependence guarantees that all boundary conditions can be satisfied.

Davis based his assumptions on the work of Koschmieder (1966, (13)) who performed experiments in a right circular cylinder and a rectangular parallelepiped. Koschmieder showed that the influence of lateral boundaries dominates the non-linear effects in the selection of cell shape if the width of the apparatus is an order of magnitude larger than the height. In that case roll cells of geometrical shape similar to that of the confining container seemed to appear.

Davis used a Galerkin procedure (appendix I) to obtain approximate critical Rayleigh numbers (upper bounds). The with the critical Rayleigh numbers corresponding approximate eigenfunctions satisfy all boundary conditions and continuity exactly. This last condition ensures that implicit boundary conditions such as $\partial w / \partial z = 0$ on horizontal boundaries are satisfied. Within his approximation general three-dimensional flows can be constructed by a linear superposition of finite rolls, so that he was able to predict the preferred mode at the onset of convection. The results obtained by Davis for boxes with width to depth ratios $H/d$ in the range $1/4 < H/d < 6$ are the following:
(i) The preferred mode is always some number of finite rolls (two non-zero velocity components dependent on three spatial variables) with axes parallel to the short side (square boxes excepted).

(ii) When the depth is the smallest dimension, finite rolls of near-square cross-section are predicted. Otherwise narrower finite rolls appear.

(iii) The critical Rayleigh number decreases rapidly to the value 1708, as the horizontal dimensions increase, so that most experiments, which use thin layers, would appear to have onset occur at about $R_c = 1708$.

The critical Rayleigh numbers thus increase as the side walls are brought closer together because the viscous dissipation due to the rigid side walls increases.

Without going into detail concerning the trial functions used by Davis, it can be mentioned that his results have been recalculated by Catton (1970, (14)). Catton agreed with the finite roll assumption used by Davis, but he criticized the trial functions used by Davis, because they were not a linear combination of a complete orthogonal set. This in itself is not a problem, but his trial functions were constructed out of a set of equally spaced rolls. It is inconceivable that the walls would not affect the adjacent rolls. Therefore Catton used a linear combination of a complete orthogonal set, however also representing finite rolls. Catton also used a Galerkin procedure. The upper-bound estimates of the critical Rayleigh numbers determined by Catton are lower by 15 percent than the estimates found by Davis, when one of the aspect ratios ($H/d$) is less than unity. If both horizontal aspect ratios are greater than unity, the results of Davis and Catton compare quite well.

Catton has also examined the effects of wall conduction on the stability of a fluid in a rectangular region heated from below (15), (16). He found lower critical Rayleigh numbers in the insulating case, which is a manifestation of heat loss through the sides in the conducting case.

Stork & Muller (1972, (17)) have performed an experimental study on the convective motions in rectangular boxes heated
from below. The experiments described by Stork & Müller comply largely with the assumptions of Davis (4) and their results are therefore especially suitable for comparison with his results. They determined the critical Rayleigh number, number of cells and the cell shape for boxes with various width to depth ratios $h/d$.

As an illustration fig. 6 (page 12) gives a typical stability curve for a box. In this figure, $h_1$ is the dimensionless width of the box in the horizontal x-direction and $h_2$ the dimensionless width in the horizontal y-direction. The dimensionless depth of the box is 1. During the computation of a stability curve like the one shown in fig. 6, Davis kept $h_1$ constant and varied $h_2$. Stork & Müller used the same procedure in performing the experiments. They found that the experimental values for the critical Rayleigh number generally were below the theoretical ones computed by Davis and Catton (14). Within one series of tests ($h_1$=constant) the discrepancies between theoretical and experimental results increased as the box became narrower (small $h_2$). Especially for $h_2<1$ the differences could become as large as one third of the calculated numbers. The discrepancies between the experimental and the theoretical curves are partly caused by the fact that a Galerkin method only gives upper-bounds when $R_c$ is computed using this method. Davies-Jones (1971, (18)) proved, that finite rolls, aligned perpendicular to one side wall, as was assumed by Davis, are not exact solutions of the linearized convection problem. However, they closely approximate the preferred modes of convection in a box. These statements essentially explain the discrepancies found by Stork & Müller for boxes with the ratio $h_2$ greater than about 1.5, if in addition the experimental error is considered. The large discrepancies in the region of steep slope ($h_2<1$) of the stability curves have a different cause. On practical grounds, Stork & Müller used partly insulating side walls, which influences the results increasingly if the distance between the side walls decreases. Computations of Davies-Jones (18) and Catton (15), (16) show that in the case of insulated side walls, the critical Rayleigh numbers are below those in the case of conducting side walls. The difference is considerable in the range $h_2<1.5$ (up to 40%).
The discrepancies between the theory of Davis and the experiments of Stork & Müller are therefore evident.

In general Stork & Müller observed the formation of finite rolls with the axes parallel to the short side of the box. An interesting phenomenon was that in a number of cases they observed that the formation of an even number of finite rolls was preferred to the development of an odd number. Generally their observations are in good agreement with the theoretical results of Davis.

In the next chapter we will first treat the linear stability problem of an infinite horizontal fluid layer which is heated from within by an uniform distribution of heat sources.
IV. THERMAL CONVECTION IN A HORIZONTAL FLUID LAYER WITH
INTERNAL HEAT SOURCES, LINEAR THEORY

We consider a horizontal fluid layer of constant depth \( d \), bounded above and below by two rigid perfectly conducting planes, maintained at constant and equal temperatures. The fluid is heated from within by an uniform distribution of heat sources (fig. 7). The following equations are governing the system:

Equation of motion

\[
\rho \frac{D\mathbf{v}}{Dt} = -\nabla p - \rho g k + \nabla \left\{ \mu \left( \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right) - \frac{1}{2} \mu \nabla \mathbf{v} \right\} \tag{IV.1}
\]

where \( \rho \) is the density of the fluid, \( \mathbf{v} = (u,v,w) \) the velocity, \( t \) is time, \( p \) the pressure, \( g \) the acceleration of gravity, \( k \) is the unit vector in the vertical \( z \)-direction, \( \mu \) the dynamical viscosity.

In (IV.1) we define:

\[
\nabla \mathbf{v} + (\nabla \mathbf{v})^T = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}
\]

where \( x = (x_1, x_2, x_3) = (x,y,z) \); \( v = (v_1, v_2, v_3) = (u,v,w) \); \( i=1,2,3; j=1,2,3 \).

Continuity equation

\[
-\frac{1}{\rho} \frac{D\rho}{Dt} = \nabla \cdot \mathbf{v} \tag{IV.2}
\]

Thermodynamical energy equation

\[
\left( \frac{De}{Dt} \right) = -p \frac{D(1/\rho)}{Dt} + q \tag{IV.3}
\]

where: \( (De/Dt) \) is the change of internal energy per unit mass and unit time

\[-pD(1/\rho)/Dt \] is compression work

\( q \) to the system supplied heat per unit mass

In the present case \( q \) is composed of a conductive part + heat produced by internal heat sources and is consequently defined as follows:

\[
q = \frac{1}{\rho} \nabla \cdot \mathbf{K} (\nabla T) + \frac{Q}{\rho} \tag{IV.4}
\]

where \( \mathbf{K} \) is the thermal conductivity, \( T \) the temperature and
Fig. 7. Horizontal fluid layer with internal heat sources.

Fig. 8. Shape of the temperature profile given by eq. (IV.9).
Q the constant generated heat per unit time and unit volume. Introducing eq. (IV.4) into eq. (IV.3) leads to

\[
\frac{D(e)}{Dt} = \frac{-D(P/p)}{Dt} + \frac{1}{P} \frac{DP}{Dt} + \frac{1}{P} \nabla(K \nabla T) + \frac{Q}{P}
\]

which may be written as

\[
\frac{D(e + p/p)}{Dt} = \frac{1}{P} \frac{DP}{Dt} + \frac{1}{P} \nabla(K \nabla T) + \frac{Q}{P}
\]

In this equation is \((e + p) = h=enthalpy= c_p T\), where \(c_p\) is the specific heat at constant pressure.

Introducing this into eq. (IV.5) gives:

\[
\frac{D(eP)}{Dt} = \frac{1}{P} \frac{DP}{Dt} + \frac{1}{P} \nabla(K \nabla T) + \frac{Q}{P}
\]

\(Dp/Dt\) is the mechanical work by pressure. In convection problems \((Dp/Dt)\) is negligible with respect to the other terms in eq. (IV.5)'. Thus eq. (IV.5)' can be simplified to:

\[
\frac{D(eP)}{Dt} = \frac{1}{P} \nabla(K \nabla T) + \frac{Q}{P}
\]

Equations (IV.1), (IV.2) and (IV.6) must be supplemented by an equation of state which can be written as follows

\[
\rho = \rho_0 (1 - \alpha(T - T_0))
\]

where \(\alpha\) is the coefficient of volume expansion and \(T_0\) is the temperature at which \(\rho = \rho_0\).

No assumptions have been made yet, according the constancy or otherwise of the various coefficients \((\mu, c_p, \alpha\) and \(K)\) which were introduced. Boussinesq pointed out that there are many situations of practical occurrence in which the basic equations can be simplified considerably. These situations occur when the variability in the density and in the various coefficients is due to variations in the temperature of only moderate amounts. Assuming this we can treat \(\mu, c_p, \alpha\) and \(K\) as constants.

For gases and liquids, such as we shall be mostly concerned with, the coefficient of volume expansion, \(\alpha\), is very small (in the range \(10^{-3}\) to \(10^{-4}\)). For moderate variations in the
temperature, the variation in the density will be very small and we may treat the density \( \rho \) as a constant \( \rho_0 \) in our equations. However there is one important exception: the variability of \( \rho \) in the term \( \rho g k \) in the equation of motion (IV.1) can not be ignored (Chandrasekhar, (19)). This is known as the Boussinesq approximation.

The basic equations now become:

\[
\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_0} \nabla \rho - \frac{\rho_0}{\rho} g k + \gamma \nabla^2 \mathbf{v} \tag{IV.1}'
\]

\[
\nabla \cdot \mathbf{v} = 0 \tag{IV.2}'
\]

\[
\frac{\partial T}{\partial t} = \kappa \nabla^2 T + \frac{Q}{\rho_0 c_p} \tag{IV.6}'
\]

\[
\rho = \rho_0 (1 - \alpha (T - T_0)) \tag{IV.7}
\]

where \( \gamma = \nu / \rho_0 \) is the kinematic viscosity and \( \kappa = \frac{K}{\rho_0 c_p} \) is the thermal diffusivity.

Introducing eq. (IV.7) into eq. (IV.1)' gives

\[
\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_0} \nabla \rho - (1 - \alpha (T - T_0)) g k + \gamma \nabla^2 \mathbf{v} \tag{IV.1}''
\]

We consider horizontal boundary planes which are rigid and perfectly conducting, giving the following boundary conditions:

\[
v = 0 \text{ and } T = 0 \text{ at } z = 0 \text{ and } z = d \tag{IV.8}
\]

If the generated heat \( Q \) is sufficiently small, all heat transfer will be in the form of conduction and a steady state solution of the equations (IV.1)'", (IV.2)' and (IV.6)' exists of the form:

\[
v_s = 0, \quad \rho = \rho_s(z), \quad p = p_s(z), \quad T = T_s(z) = \frac{Q(1-z)}{2K \rho_0 c_p} \tag{IV.9}
\]

Fig. 8 (page 19) shows the shape of the temperature profile given by eq. (IV.9). The temperature profile is parabolic and it will be clear that we obtain an unstably stratified fluid layer above a stably stratified fluid layer.

Now we subject the fluid layer to an arbitrary perturbation. The problem we are interested in is which values of \( Q \) cause
the fluid layer to become
a stable, i.e. the perturbations died down with the time
b neutral, i.e. the perturbations are independent of the time
c unstable, i.e. the perturbations grow with the time.

The initial state of the system is described by eq. (IV.9). The perturbed state can be described as follows

\[ \begin{align*}
  v(x,y,z,t) &= v_s + v'(x,y,z,t) = v'(x,y,z,t) \\
  p(x,y,z,t) &= p_s(z) + p'(x,y,z,t) \\
  T(x,y,z,t) &= T_s(z) + \Theta'(x,y,z,t) \\
  \rho(x,y,z,t) &= \rho_s(z) + \rho'(x,y,z,t)
\end{align*} \]

where \( v', p', \Theta' \) and \( \rho' \) denote the perturbations from the steady state (IV.9).

Introducing eq. (IV.10) into equations (IV.1)", (IV.2)' and (IV.6)' and subtracting the steady state solutions leads to:

\[ \begin{align*}
  \frac{\partial v'}{\partial t} + v' \cdot \nabla v' &= -\frac{1}{\rho_0} \nabla p' + \alpha \Theta' k + \gamma \nabla^2 v' \\
  \nabla \cdot v' &= 0 \\
  \frac{\partial \Theta}{\partial t} + v' \cdot \nabla \Theta &= \frac{\Theta}{2 K \rho_0 \sigma} + \frac{Q}{\rho_0} (\alpha z - d) - \kappa \nabla^2 \Theta'
\end{align*} \]

To get a dimensionless form of these equations we introduce \( d, d^2/K, \kappa/d, K \rho_0/d^2 \) and \( K \gamma \sigma d \) as units for length, time, velocity, pressure and temperature respectively. Omitting the primes the equations take the form

\[ \begin{align*}
  \mathcal{P}^{-1} \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) &= -\nabla p + \Theta k + \nabla^2 v \\
  \nabla \cdot v &= 0 \\
  \frac{\partial \Theta}{\partial t} + v \cdot \nabla \Theta &= \nabla^2 \Theta + \frac{Q}{|Q|} \left( 3 \kappa \mathcal{R} (\alpha z - 1) \right) \omega
\end{align*} \]

with the boundary conditions

\[ \begin{align*}
  v &= \Theta = 0 \quad ; \quad z = 0, 1
\end{align*} \]

Here \( \mathcal{P} \) is the Prandtl number and \( \mathcal{R} \) the modified Rayleigh
number defined by:

\[ P = \sqrt{\frac{\kappa}{\kappa}} \quad , \quad R = \frac{g \alpha |Q| d^5}{6 \mu \rho \kappa K^2} \quad (IV.15) \]

In the definition of \( R \), and in the last term of eq. (IV.13) we have introduced the absolute value of \( Q \), whereby the case of uniform cooling (\( Q \) negative) also is included in the equations (this case results in a stably stratified fluid layer above an unstably stratified fluid layer!).

The Rayleigh number \( R_o \), defining the onset of convection is found from the linearized, time independent version of equations (IV.11)-(IV.14), given by:

\[ \begin{align*}
0 &= -\nabla \rho + \Theta \kappa + \nabla^2 \nu \\
0 &= \nabla \nu \\
0 &= \nabla^2 \Theta + \frac{Q}{|Q|} 32R (2z-1) \nu
\end{align*} \quad (IV.11)' \quad (IV.12)' \quad (IV.13)' \]

where \( \rho \), \( \Theta \) and \( \nu \) now are time independent variables. We have solved this equations numerically, using a Galerkin procedure, which gives approximate solutions. For a general description of the Galerkin method is referred to appendix I. For this purpose it is convenient to transform the equations (IV.11)'-(IV.13)' by using the fact that the velocity is poloidal (i.e. \( \nabla \times \nu = 0 \)). The velocity may appropriately be written:

\[ \nu = (u, v, w) = (\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, -\nabla^2 V) \]

where \( \nabla^2 \) is the horizontal Laplacian, \( V \) is a scalar function. By eliminating the pressure term we obtain from (IV.11)'-(IV.13)'

\[ \begin{align*}
\nabla^4 V - \Theta &= 0 \\
\nabla^2 \Theta - \frac{Q}{|Q|} 32R(2z-1) \nabla^2 V &= 0
\end{align*} \quad (IV.16) \quad (IV.17) \]

with the boundary conditions

\[ v = \frac{\partial V}{\partial x} = 0 \quad ; \quad z = 0, 1 \quad (IV.18) \]

Since the fluid is of infinite extent in the horizontal x- and
y-direction, we expect the same periodicity in both directions. As a consequence we need only consider the x- and z-direction. Considering periodic solutions in the x-direction, we can expand θ in a complete set of Fourier modes each of them satisfying the boundary conditions.

\[
\Theta = \sum_{q,h} B_{qh} e^{iqax} \sin(h\pi z) \quad (IV.19)
\]

Here a is the wave number in the x-direction. The summation runs through all integers \(1 \leq q < \infty \) and \(1 \leq h < \infty \).

Introducing eq. (IV.19) into eq. (IV.16) and applying the boundary conditions, we obtain:

\[
V = \sum_{q,h} B_{qh} e^{iqax} F_h(\delta, z) \quad (IV.20)
\]

where \(\delta = qa\) and

\[
F_h(\delta, z) = A_h(\delta) \sin(h\pi z) + C_h^{(0)}(\delta) z \cosh(\delta z) + C_h^{(2)}(\delta) \sinh(\delta z) \quad (IV.21)
\]

In eq. (IV.21) is

\[
A_h(\delta) = \frac{1}{(h^2\pi^2 + \delta^2)^2}
\]

\[
C_h^{(2)}(\delta) = h \pi A_h(\delta) \left/ \left( (-1)^h \sinh(\delta z) - \delta \right) \right.
\]

\[
C_h^{(0)}(\delta) = -\delta C_h^{(2)}(\delta) - h \pi A_h(\delta)
\]

\[
C_h^{(1)}(\delta) = -C_h^{(2)}(\delta) - C_h^{(0)}(\delta) \cosh(\delta z) \sinh(\delta z)
\]

The unknown coefficients \(B_{qh}\) are determined from eq. (IV.17) by applying a Galerkin procedure.

We will substitute the first N terms of eqs. (IV.19) and (IV.20) in eq. (IV.17). As this is not an exact solution there will be an error:

\[
E = \sum_{q,h} B_{qh} \left( -q^2a^2 - h^2\pi^2 \right) e^{iqax} \sin(h\pi z) + \sum_{q,h} Q_{32} R(2z-1) B_{qh} Q_{10} a^2 e^{iqax} F_h(\delta, z) \quad (IV.23)
\]
According to Galerkin we require that this error is orthogonal to \( \exp(-irax)\sin(j\pi z) \) i.e.

\[
0 = \oint \left[ \sum_{n} B_{n}^{h} \left( \partial_{x}^{2} + h^{2} \partial_{z}^{2} \right) e^{i(qa-rn)z} \sin(n\pi z) \sin(j\pi z) + \right. \\
\left. \quad - \frac{Q}{2} \frac{32}{R} (2z-1) \sum_{n} B_{n}^{h} \partial_{x}^{2} e^{i(qa-rn)z} f_{h}(2z) \sin(n\pi z) dx dz \right]
\]

and we obtain

\[
\frac{1}{2} \left( j^{2} + \partial_{x}^{2} \right) B_{n}^{h} - \frac{Q}{2} \frac{32}{R} \sum_{n=1}^{N} c(h, j, j) B_{n}^{h} = 0
\]

where \( c(h, j, j) = \oint (2z-1) f_{h}(z) \sin(j\pi z) dz \)

and \( j = ra, j \) and \( r \) run from \( 1 \to N \).

It is well known that \( r=1 \) gives the lowest mode, which is the one we are looking for. Eqs. (IV.25) have a non-trivial solution for \( B_{rh} \) \( B_{rj} \) if and only if the determinant of these equations is zero. This gives \( R_{0} \) as a function of \( a \), the wave number in the \( x \)-direction. We have performed the necessary computations on the computer and found the stability curve represented by fig. 9, where we have used \( N=10 \). The minimum value of \( R_{0} \), \( R_{c} \), and the corresponding value of \( a \), \( a_{c} \), are found to be

\[
R_{c} = 583.20, \quad a_{c} = 4.00
\]

which are identical to the values given by Sparrow et al. (6) and Tveitereid (7).

Using \( N=5 \) gives us \( R_{c} = 583.41 \) and \( a_{c} = 4.00 \). The difference in \( R_{c} \) is small, less than 0.04%. The convergence is good.
Fig. 9 Stability curve for the infinite layer.

\[ R_c = 563.20 \]
\[ \alpha_c = 4.00 \]
V. CONVECTION IN A BOX WITH INTERNAL HEAT SOURCES, LINEAR THEORY

V.1. Governing equations

We consider a rectangular box (fig. 10), which is heated from within by an uniform distribution of internal heat sources. The upper and lower horizontal plane of the box are supposed to be rigid and perfectly conducting, and are maintained at constant and equal temperatures. The side walls are also supposed to be rigid, however no assumptions are made yet concerning the conductivity of these side walls.

For the convective motion in the box the equations (IV.1)', (IV.2)' and (IV.6)' are valid.

\[
\begin{align*}
\frac{\partial \psi}{\partial \xi} + \psi \nabla \psi &= -l \nabla p - (1 - \alpha (T - T_0)) \eta k + \nabla^2 \psi \\
\nabla \cdot \psi &= 0 \\
\frac{\partial T}{\partial \xi} + \psi \cdot \nabla T &= k \nabla^2 T + \frac{Q}{\rho c_p}
\end{align*}
\]

(V.1) \hspace{1cm} (V.2) \hspace{1cm} (V.3)

If the generated heat \( Q \) is sufficiently small, we are searching for a steady state solution, where all heat transfer takes place in the form of conduction, i.e.

\[
\nabla^2 T_s = -\frac{Q}{\rho c_p k}
\]

There may not exist a horizontal temperature gradient in the box, as this would cause the immediate onset of convective motion. Therefore we require that the temperature \( T_s \) is a function of the vertical coordinate \( z \) only.
Thus: \[ \frac{d^2 T_s}{dz^2} = -\frac{Q}{\rho c \kappa} \]
with the boundary conditions \( T=0 \) at \( z=0 \) and \( z=d \).
This gives us the same steady state solution as given by eq. (IV.9),
\[ T_s = \frac{Q (d-z) z}{2 \rho c \kappa} \quad (V.4) \]
As mentioned there may not exist a horizontal temperature gradient in the box. This means that on the side walls at \( x=0 \), \( x=H_1 \) and at \( y=0 \), \( y=H_2 \), the same temperature profile must exist as is given by eq. (V.4). This is attainable by supposing insulating side walls at \( x=0 \), \( H_1 \) and \( y=0 \), \( H_2 \). The boundary conditions for the temperature \( T \) in this case are:
\[ T=0 \text{ at } z=0, \ d \]
\[ \frac{\partial T}{\partial y} = 0 \text{ at } y=0, \ H_2 \]
\[ \frac{\partial T}{\partial x} = 0 \text{ at } x=0, \ H_1 \quad (V.5) \]
Another possibility is to suppose that the side walls at \( x=0, H_1 \) and \( y=0, H_2 \) are perfect conductors. In this case we must impose the temperature profile given by eq. (V.4) on these side walls. The boundary conditions for the temperature \( T \) in this case are:
\[ T=0 \text{ at } z=0, \ d \]
\[ T=0 \text{ at } y=0, \ H_2 \]
\[ T=0 \text{ at } x=0, \ H_1 \quad (V.6) \]
Actually this is an artificial case. However we will study this case too, because, in linear theory, it has certain computational advantages compared with the case of insulating side walls. Further on in this chapter (section V.3) we will introduce two different sets of trial functions, each of them able to approximate the real solution of the linear stability problem. It will be shown that it has advantages both in computation time and necessary computer space, to examine the convergence of these two sets of trial functions in the case of perfectly conducting side walls (section V.4). However, it will be clear that the case of insulating walls has more practical value.
This paper will not consider the much more complicated case of side walls which are partly conducting and partly insulating.
For the determination of the stability curves for boxes of various dimensions we can use the dimensionless, linearized and time independent perturbation equations (IV.11)\textsuperscript{'}-(IV.13)\textsuperscript{'}:

\begin{align*}
0 &= -\nabla p + \Theta \kappa + \nabla^2 \theta \\
0 &= \nabla \cdot \mathbf{v} \\
0 &= \nabla^2 \Theta + 3 \alpha \mathbf{R} (2z-1) \mathbf{v}
\end{align*}

with the boundary conditions,
\begin{align*}
z &= 0, 1; \quad \mathbf{u} = \mathbf{v} = \mathbf{w} = \Theta = 0 \\
y &= 0, h_2; \quad \mathbf{u} = \mathbf{v} = \mathbf{w} = \partial \Theta / \partial y = 0 \quad \text{(insulating)} \\
\mathbf{u} &= \mathbf{v} = \mathbf{w} = \Theta = 0 \quad \text{(conducting)} \\
x &= 0, h_1; \quad \mathbf{u} = \mathbf{v} = \mathbf{w} = \partial \Theta / \partial x = 0 \quad \text{(insulating)} \\
\mathbf{u} &= \mathbf{v} = \mathbf{w} = \Theta = 0 \quad \text{(conducting)}
\end{align*}

Here \( h_1 = H_1 / d \) and \( h_2 = H_2 / d \).

As mentioned in section III.2, the presence of rigid lateral walls excludes an analytical solution of these equations by separation of variables. Therefore we will apply a Galerkin procedure and solve the equations numerically.

V.2. Finite roll approximation, Galerkin procedure

Tveitereid and Palm (8) have shown for the infinite layer with internal heat sources, that if the linearized version of equations (IV.11)-(IV.13) is selfadjoint, the convection pattern will be two-dimensional rolls. However, in their problem and in our problem the equations are not selfadjoint and they expect hexagons for values of the Rayleigh number \( \mathbf{R} \) near the stability curve. On the other hand, Koschmieder (13) showed in his experimental work for a horizontally bounded fluid layer heated from below, that the influence of the lateral boundaries dominates the non-linear effects in the selection of cell shape if the width of the apparatus is an order of magnitude larger than the height. In that case roll cells of geometrical shape similar to that of the confining container seemed to appear. Since we in our problem also are dealing with a bounded domain, we assume that the onset of convective motion in the
box will be in the form of finite rolls.
In the case of an infinite horizontal fluid layer heated from below, the linear stability problem may be solved in the simplest case, the roll cell. A roll is a cell with only two non-zero velocity components, dependent upon two spatial variables, uniform and of infinite extent in the third direction.
In a fully enclosed geometry, Davis (4) defines finite rolls. A finite roll is a cell with only two non-zero velocity components, but by necessity dependent upon all three spatial variables. This dependence guarantees that all boundary conditions can be satisfied. We can distinguish two types of finite rolls:
(i) finite x-rolls, which have their axes parallel to the y-axis (v=0)
(ii) finite y-rolls, which have their axes parallel to the x-axis (u=0)

In section V.3 we will present appropriate trial functions which have the properties of finite rolls and satisfy all boundary conditions. For the computations of the critical Rayleigh numbers, defining the onset of convective motion, a Galerkin procedure is used. We assume the onset of convective motion to be in the form of finite x-rolls. This means that v=0. The continuity equation (V.8) then becomes:

\[ \nabla \cdot \mathbf{v} = \frac{\partial \mathbf{v}}{\partial x} + \frac{\partial \mathbf{v}}{\partial z} = 0 \]

Then we may write:
\[ u = -\frac{\partial \psi}{\partial z} \quad \omega = \frac{\partial \psi}{\partial x} \quad (V.13) \]

where \( \psi \) is a stream function. Let us for simplicity represent \( \Theta \) and \( \psi \) as follows,

\[ \Theta = \sum_{q=1}^{\infty} a_q \Theta_q \quad (V.14) \]

\[ \psi = \sum_{q=1}^{\infty} b_q \psi_q \quad (V.15) \]

Now
\[ u = -\frac{\partial \psi}{\partial z} = \sum_{q=1}^{\infty} b_q \frac{\partial \psi_q}{\partial z} = \sum_{q=1}^{\infty} b_q \psi_q \quad \text{with} \quad u_q = -\frac{\partial \psi_q}{\partial z} \quad (V.16) \]

\[ w = \frac{\partial \psi}{\partial x} = \sum_{q=1}^{\infty} b_q \frac{\partial \psi_q}{\partial x} = \sum_{q=1}^{\infty} b_q \psi_q \quad \text{with} \quad \omega_q = \frac{\partial \psi_q}{\partial x} \quad (V.17) \]
Introducing the first $N$ terms of these equations into eqs. (V.7) and (V.9) leads to

\[ \sum_{q=1}^{N} b_q \nabla^2 u_q - \frac{\partial p}{\partial x} = \varepsilon_1 \]  
(V.18)

\[ \sum_{q=1}^{N} b_q \nabla^2 w_q + \sum_{q=1}^{N} a_q \theta_q - \frac{\partial p}{\partial z} = \varepsilon_2 \]  
(V.19)

\[ \sum_{q=1}^{N} a_q \nabla^2 \theta_q + \sum_{q=1}^{N} b_q 32R (2z-1) w_q = \varepsilon_3 \]  
(V.20)

$\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ are the errors we make by only substituting $N$ terms of equations (V.14)-(V.17) into eqs. (V.7) and (V.9).

We multiply equations (V.18)-(V.20) with respectively $u_i^i$, $w_i^i$ and $\theta_i^i$ giving

\[ \sum_{q=1}^{N} b_q u_i \nabla u_q - u_i \frac{\partial p}{\partial x} = \varepsilon_1 u_i \]  
(V.21)

\[ \sum_{q=1}^{N} b_q w_i \nabla w_q + \sum_{q=1}^{N} a_q w_i \theta_q - w_i \frac{\partial p}{\partial z} = \varepsilon_2 w_i \]  
(V.22)

\[ \sum_{q=1}^{N} a_q \theta_i \nabla \theta_q + 32R (2z-1) \sum_{q=1}^{N} b_q \theta_i w_q \]  
(V.23)

We require according to Galerkin that $\varepsilon_1$ is orthogonal to $u_i$, $\varepsilon_2$ is orthogonal to $w_i$ and $\varepsilon_3$ is orthogonal with $\theta_i$.

Integrating equations (V.21)-(V.23) over the box and applying this orthogonality condition gives:

\[ \sum_{q=1}^{N} \left\{ b_q \left< u_i \nabla u_q \right> + b_q \left< w_i \nabla w_q \right> + a_q \left< w_i \theta_q \right> \right\} = 0 \]  
(V.24)

\[ \sum_{q=1}^{N} \left\{ a_q \left< \theta_i \nabla \theta_q \right> + b_q 32R (2z-1) \theta_i w_q \right\} = 0 \]  
(V.25)

where $i=1,2,\ldots,N$.

The pressure term vanishes by virtue of the fact that $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$ and $u=w=0$ on the boundaries. After integrating eqs. (V.21) and (V.22) over the box, we have added the resulting equations. $\left< \cdots \right>$ denotes integration over the entire volume of the box.

Equations (V.24) and (V.25) can also be written as follows:

\[ \sum_{q=1}^{N} \left\{ -b_q \left< \nabla u_i \cdot \nabla u_q \right> + \left< \nabla w_i \cdot \nabla w_q \right> + a_q \left< w_i \theta_q \right> \right\} = 0 \]  
(V.26)

\[ \sum_{q=1}^{N} \left\{ b_q 32R (2z-1) \theta_i w_q - a_q \left< \nabla \theta_i \cdot \nabla \theta_q \right> \right\} = 0 \]
The equations (V.26) have non-trivial solutions if and only if
\[ \det \begin{bmatrix} -\langle \nabla u_i \cdot \nabla q \rangle + \langle \nabla w_i \cdot \nabla q \rangle & \langle w_i \theta q \rangle \\ 32 R \langle \nabla z \cdot \nabla w_q \rangle & -\langle \nabla \theta \cdot \nabla q \rangle \end{bmatrix} = \det A = 0 \quad (V.27) \]

Here \( A \) is a \( 2N \times 2N \) matrix, \( i \) runs from \( 1 \rightarrow N \) and \( q \) runs from \( 1 \rightarrow N \).

The critical Rayleigh number is the smallest positive value of \( R \) for which \( \det A = 0 \) as \( N \rightarrow \infty \). The critical Rayleigh numbers are dependent upon \( h_1 \) and \( h_2 \). Considering \( M \) finite x-rolls, two types of behaviour are expected:

**Type I** We fix \( h_2 \) and allow \( h_1 \) to vary for \( M \) finite x-rolls, i.e. we vary the width of the finite rolls. The critical Rayleigh number is attained as a compromise between the conflicting needs of kinetic energy dissipation by viscosity and release of potential energy by the buoyancy force (Chandrasekhar (19)). Narrow, tall cells are inefficient because they dissipate large amounts of energy, while wide, flat cells are inefficient because a fluid particle must travel a great horizontal distance before it can release its potential energy. Hence we expect a stability curve which has a finite minimum corresponding to moderate sized cells being preferred. The shape is qualitatively similar to the stability curve for the infinite layer (fig. 11).

**Type II** We fix \( h_1 \) and vary \( h_2 \) for \( M \) finite x-rolls, i.e. we vary the length of the finite rolls. Since this variation does not materially change the length of the path of a fluid particle, the proximity of the walls at the "ends" of the cell influence \( R_c \) mostly by affecting the viscous dissipation. The dissipation due to the end walls is a small part of the total dissipation, unless the rolls are very short. We expect the stability curve to be a monotone decreasing function of the axial length of the finite roll and to approach a finite positive value of \( R_c \) rapidly (fig. 12).
Fig. 11. Qualitative behaviour of M finite x-rolls, $R_c$ versus $h_1$ with $h_2$ fixed; type I dependence.

Fig. 12. Qualitative behaviour of M finite x-rolls, $R_c$ versus $h_2$ with $h_2$ fixed; type II dependence.

Fig. 13. Qualitative behaviour of composite stability curve with $h_2$ fixed. $R_c$ versus $h_1$, $n_x$ and $m$ denote $n$ finite x-rolls and $m$ finite y-rolls respectively. The darkened portion is the minimum critical Rayleigh number.
According to Davis (4), the construction of the stability curves for the box can then be performed in the following way. We fix \( h_2 \), for example at \( h_2 = 1 \) and vary \( h_1 \). If we approximate one finite x-roll and compute \( R_c \) as a function of \( h_1 \), we obtain the stability curve denoted by \( 1x \) in figure 13. This curve is qualitatively similar to the one in fig. 11. Approximating two finite x-rolls we find the stability curve denoted by \( 2x \) in fig. 13. The minimum of this curve lies below and to the right of the curve \( 1x \). Considering three finite x-rolls we find curve \( 3x \), which has a minimum that lies below and to the right of the curve \( 2x \) etc.

Let us now construct one finite y-roll and compute its stability curve. Then we obtain the curve denoted by \( 1y \) in fig. 13. This curve is qualitatively similar to the one in fig. 12 and has for \( h_1 < h_2 \) a negative slope of smaller magnitude than the curve for one finite x-roll (\( 1x \)). For \( h_1 = h_2 \) the curves \( 1y \) and \( 1x \) have the same \( R_c \) (the same physical situation). Thus one finite y-roll has a lower \( R_c \) than one finite x-roll when \( h_1 < h_2 \). Approximating two finite y-rolls we find the curve denoted by \( 2y \) in fig. 13. This curve has a smaller negative slope but a larger asymptote (i.e. displaced upwards) for large \( h_1 \) than the curve \( 1y \) etc. Starting for example at \( h_1 = h_1' \), we see from figure 13, that if we increase \( h_1 \), curve \( 2y \) and \( 1y \) intersect, and curve \( 1y \) becomes the lowest curve for an interval of \( h_1 \).

For that interval of \( h_1 \), curve \( 1y \) forms the curve of the critical Rayleigh number.

If we increase \( h_1 \) further, curve \( 1y \) and curve \( 1x \) intersect at \( h_1 = h_2 \) and curve \( 1x \) becomes the lowest curve for an interval of \( h_1 \). For that interval of \( h_1 \) curve \( 1x \) forms the curve of the critical Rayleigh number. Increasing \( h_1 \) further, curve \( 1x \) and curve \( 2x \) intersect etc.

The curve of the critical Rayleigh number for a box (=stability curve for a box) with fixed \( h_2 \) and varying \( h_1 \), then is defined by the curve that is darkened in fig. 13. This procedure can be used to construct appropriate stability curves for various fixed \( h_2 \) values (Davis (4)).

In section V.4 some difficulties which arose with the computation and construction of the stability curves will be discussed.
V.3. Trial functions

As mentioned in section V.1, two different sets of trial functions will be introduced here. These trial functions must obey the following requirements:

(i) they must satisfy all boundary conditions and continuity exactly

(ii) the temperature field must maintain the same symmetry as the vertical velocity component

(iii) the influence of the walls on the adjacent rolls must be included

(iv) the steady state solution (V.4) results in an unstably stratified layer above a stably stratified fluid layer. Therefore the trial functions must include both symmetrical and non-symmetrical terms in the z-direction.

With regard to the conductivity of the side walls we will examine two cases,

a insulating side walls -

b conducting side walls.

During the computations of the stability curves we can use the fact that that \( R_c(\beta', \gamma') \) for \( M \) finite x-rolls equals \( R_c(\gamma', \beta) \) for \( M \) finite y-rolls (they are identical physically). Thus we only need to consider the trial functions for finite x-rolls. Finite x-rolls have a zero y-component of velocity (v=0).

Thus we may apply eq. (V.13) again:

\[
\psi = -\frac{\partial \psi}{\partial z}; \quad \omega = \frac{\partial \psi}{\partial x}
\]  

(V.13)

where \( \psi \) is a stream function.

We have examined the following two sets of trial functions.

Set 1, trial functions generated with sine and cosine:

For this type of trial functions it seemed convenient to choose the boundaries of the box at \( z=0, z=1; y=-h_2/2, y=h_2/2; x=0, x=h_1 \). The trial functions are:

\[
\psi = \sum_{m,m_1}^{\infty} \sum_{n,n_1}^{\infty} b_{mn} j \sin(n_1 \gamma) \sin(m_1 \beta) \cos((2n-1)m_2) \sin(n_2 \beta) \sin(j_2 \gamma) \]  

(V.28)

For an odd number of rolls we define: \( m_1 = (number \ of \ rolls + 1)/2 \) and \( m_2 = 2m-1 \). For an even number of rolls we define \( m_1 = (number \ of \ rolls)/2 \) and \( m_2 = 2m \).
Applying eq. (V.13) we find $u$ and $w$:

$$u = -\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{j=1}^{\infty} \left( b_{mnj} \sin \left( \frac{m\pi x}{h_1} \right) \sin \left( \frac{n\pi y}{h_2} \right) \cos \left( \frac{j\pi y}{h_2} \right) \cos \left( \frac{(m-1)\pi y}{h_2} \right) \cos \left( \frac{(n-1)\pi y}{h_2} \right) \right),$$  \hspace{1cm} (V.29)

$$w = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{j=1}^{\infty} \left( b_{mnj} \sin \left( \frac{m\pi x}{h_1} \right) \sin \left( \frac{n\pi y}{h_2} \right) \cos \left( \frac{j\pi y}{h_2} \right) \right) \right),$$  \hspace{1cm} (V.30)

a insulating side walls

$$\Theta = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{j=1}^{\infty} a_{mnj} \cos \left( \frac{m\pi x}{h_1} \right) \cos \left( \frac{(m-1)\pi y}{h_2} \right) \sin \left( \frac{j\pi y}{h_2} \right),$$  \hspace{1cm} (V.31a)

b conducting side walls

$$\Theta = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{j=1}^{\infty} a_{mnj} \left[ \cos \left( \frac{m\pi x}{h_1} \right) \sin \left( \frac{n\pi y}{h_2} \right) + m \sin \left( \frac{n\pi y}{h_2} \right) \cos \left( \frac{m\pi x}{h_1} \right) \right] \right) \right),$$  \hspace{1cm} (V.31b)

We notice that $\Theta$ in eq. (V.31a) and eq. (V.31b) has the same symmetry as $w$ in eq. (V.30).

Set 2, trial functions generated with orthogonal functions.

For this type of trial functions it seemed convenient to choose the boundaries of the box at $|z|=1/2; |y|=h_2/2; |x|=h_1/2$.

With this choice of boundaries we have to replace $(2z-1)$ in eq. (V.27) by $(2z)$ because the steady state solution (V.4) now becomes

$$\frac{T}{T_s} = \frac{Q}{2\rho c_p h_1} \left( d^2 - z^2 \right)$$  \hspace{1cm} (V.4')

The orthogonal functions are defined by:

$$C_m(x) = \frac{\cosh(\lambda_m x)}{\cosh(\lambda_m h_1)} - \frac{\cos(\lambda_m x)}{\cos(\lambda_m h_1)} \quad \text{and} \quad S_m(x) = \frac{\sinh(\mu_m x)}{\sinh(\mu_m h_2)} - \frac{\sin(\mu_m x)}{\sin(\mu_m h_2)}$$  \hspace{1cm} (V.32)

These functions are standard forms of the even and odd eigenfunctions of the characteristic value problem

$$\frac{d^2 f}{d x^2} = \lambda f$$  \hspace{1cm} (V.33)

with the boundary conditions $f=df/dx=0$ at $|x|=1/2$  \hspace{1cm} (V.34)

$\lambda_m$ and $\mu_m$ ($m=1, 2, 3, \ldots$) are the positive roots of the equations

$$\tanh(\lambda_m) + \tanh(\lambda_m) = 0 \quad \text{and} \quad \coth(\mu_m) - \cot(\mu_m) = 0$$  \hspace{1cm} (V.35)
The roots of eq. (V.35) are given in table 2 for \( m=1,2,3,4 \).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \lambda_m )</th>
<th>( \Lambda_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.7300407449</td>
<td>7.8532046241</td>
</tr>
<tr>
<td>2</td>
<td>10.9956078380</td>
<td>14.1371654913</td>
</tr>
<tr>
<td>3</td>
<td>17.2787596574</td>
<td>20.4203522456</td>
</tr>
<tr>
<td>4</td>
<td>23.5619449020</td>
<td>26.7035375555</td>
</tr>
</tbody>
</table>

For larger values of \( m \), the asymptotic formulae

\[
\lambda_m \to (2m-1)\pi \quad \text{and} \quad \Lambda_m \to (2m+1)\pi
\]

(V.36)

give the values of these roots correct to ten decimal places (Harris & Reid (21)).

When defined in this manner, the functions \( C_m(x) \) and \( S_m(x) \) have the properties

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} C_m(x) C_n(x) \, dx = \frac{1}{2} \delta_{mn} \quad \text{and} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} C_m(x) S_n(x) \, dx = 0
\]

(V.37)

(V.38)

They form therefore a complete set of orthogonal functions in the interval \((-\frac{1}{2}, \frac{1}{2})\). Fig. 14 shows the first two even and odd eigenfunctions (V.32).

The trial functions used are now as follows:

**Odd number of x-rolls**

\[
\mathcal{V} = h_1 \sum_{m=m_r}^{m_l} \sum_{n=1}^{\infty} \sum_{j=1}^{j_l} B_{mnj} \frac{C_m \left( \frac{\pi n}{h_1} \right)}{\lambda_m} \cos \left( \frac{2\pi n-1}{h_2} \right) \left[ \delta_o \cos \left( \frac{\pi j}{l} \right) + \delta_e \sin \left( \frac{\pi j}{l} \right) \right] \quad (V.39)
\]

where \( \delta_o = 1; \delta_e = 0 \) if \( j=\text{odd} \),

\( \delta_o = 0; \delta_e = 1 \) if \( j=\text{even} \),

and \( j_1 = (j+1)/2; j_2 = j/2 \).

\( B_{mnj} \) are real numbers; \( m_r = \text{(number of rolls + 1)}/2 \).

**a insulating side walls**

\[
\Theta = \sum_{m=m_r}^{m_l} \sum_{n=1}^{\infty} \sum_{j=1}^{j_l} A_{mnj} \sin \left( \frac{\pi m}{h_1} \right) \cos \left( \frac{\pi n-1}{h_2} \right) \left[ \delta_o \cos \left( \frac{\pi j}{l} \right) + \delta_e \sin \left( \frac{\pi j}{l} \right) \right] \quad (V.40a)
\]

**b conducting side walls**

\[
\Theta = \sum_{m=m_r}^{m_l} \sum_{n=1}^{\infty} \sum_{j=1}^{j_l} A_{mnj} \sin \left( \frac{\pi m}{h_1} \right) \cos \left( \frac{\pi n}{h_2} \right) \left[ \delta_o \cos \left( \frac{\pi j}{l} \right) + \delta_e \sin \left( \frac{\pi j}{l} \right) \right] \quad (V.40b)
\]
Fig. 14. The first two even and odd eigenfunctions. The curves marked 1, 2, 3 and 4 are for $C_1(x)$, $S_1(x)$, $C_2(x)$ and $S_2(x)$ respectively.
Even number of x-rolls

\[ \mathcal{V} = \sum_{m=m'}^{m''} \sum_{n=1}^{n''} B_{m,n} \sum_{j=1}^{j''} \sin(n_j x) \cos(2(n-1) \pi y) \left[ \sin \left( \frac{\pi y}{n_j} \right) \sin \left( \frac{\pi y}{n_j} \right) \right] \]  \hspace{1cm} (V.41)

where \( m = \frac{\text{number of rolls}}{2} \).

a insulating side walls

\[ \Theta = \sum_{m=m'}^{m''} \sum_{n=1}^{n''} A_{m,n,j} \cos(2(n-1) \pi y) \cos(2(n-1) \pi y) \left[ \sin \left( \frac{\pi y}{n_j} \right) \sin \left( \frac{\pi y}{n_j} \right) \right] \]  \hspace{1cm} (V.42a)

b conducting side walls

\[ \Theta = \sum_{m=m'}^{m''} \sum_{n=1}^{n''} A_{m,n,j} \cos(2(\pi n y)) \cos(2(n-1) \pi y) \left[ \sin \left( \frac{\pi y}{n_j} \right) \sin \left( \frac{\pi y}{n_j} \right) \right] \]  \hspace{1cm} (V.42b)

V.4. Results of the computations

V.4.A. Comparison of the convergence and the results obtained with the two sets of trial functions

In section V.3 we have introduced two sets of trial functions:

set 1 = trial functions generated with sine and cosine
set 2 = trial functions generated with orthogonal functions.

Looking at the trial functions for \( \mathcal{V} \) for set 1 (eq.(V.28)) and set 2 (eq.(V.39), (V.41)) we notice that for both sets

\[ \mathcal{V}_q \sim \cos \left( 2n - 1 \right) \frac{\pi y}{n_j} \]

Thus also

\[ u_q \sim \cos \left( 2n - 1 \right) \frac{\pi y}{n_j} \]

\[ w_q \sim \cos \left( 2n - 1 \right) \frac{\pi y}{n_j} \]

For the case of insulating side walls we see from the equations for \( \Theta \) for set 1 (eq.(V.31a)) and set 2 (eq.(V.40a), (V.42a)),

\[ \Theta_q \sim \cos \left( 2n - 2 \right) \frac{\pi y}{n_j} \]

For the case of perfectly conducting side walls we see from the equations (V.31b), (V.40b) and (V.42b) that

\[ \Theta_q \sim \cos \left( 2n - 1 \right) \frac{\pi y}{n_j} \]

This means that in the case of perfectly conducting side walls, \( u_q, w_q \) and \( \Theta_q \) have the same \( y \)-dependence.
Considering perfectly conducting side walls we thus see from eq. (V.27) that the following two integrals appear in all the calculations of the coefficients in the matrix $A$:

\begin{align}
I_1 &= \int_{-h_2}^{h_2} \cos(2\pi(r-1)\frac{\pi y}{h_2}) \cos(2\pi(n-1)\frac{\pi y}{h_2}) \, dy \\
I_2 &= \int_{-h_2}^{h_2} \sin((2\pi(r-1)\frac{\pi y}{h_2}) \sin((2\pi(n-1)\frac{\pi y}{h_2}) \, dy
\end{align}

Here we have defined

\[
\begin{align}
&\begin{cases}
  u_1 \\
w_1 \\
e_1
\end{cases} \sim \cos((2r-1)\pi_y/h_2) \\
&\begin{cases}
  u_q \\
w_q \\
e_q
\end{cases} \sim \cos((2n-1)\pi_y/h_2)
\end{align}
\]

Performing the integrations (V.43) and (V.44) we find

\begin{align}
I_1 &= I_2 = 0 \quad \text{if } r \neq n \\
I_1 &= \frac{h_2}{2} \quad \text{if } r = n
\end{align}

This means that $n=1$ gives the lowest mode, and $n=2, 3, \ldots$ give higher modes. Thus we can fix $n=1$ in the conducting case.

In the insulating case, however, we have to solve the integrals

\[
\langle w_q, e_1 \rangle \sim \int_{-h_2}^{h_2} \cos(2\pi(r-1)\frac{\pi y}{h_2}) \cos(2\pi(n-1)\frac{\pi y}{h_2}) \, dy
\]

and

\[
32R \langle (2z-1)\Theta; w_q \rangle \sim \int_{-h_2}^{h_2} \cos((2\pi-2)\frac{\pi y}{h_2}) \cos((2\pi-1)\frac{\pi y}{h_2}) \, dy
\]

It will be clear that we can not fix $n=1$ in this case and that the summation must run through all integers $n$ we need, to obtain a reasonable convergence.

This means that the number of unknown coefficients $b_q$ and $a_q$ in the equations (V.26) is larger ($N$ is larger) for the case of insulating side walls than for the conducting case.

For this reason we performed the examination of the convergence obtained with the set 1 and set 2 trialfunctions only for the case of conducting side walls. In this way we can save computation time as well as necessary space on the computer.
We determine the coefficients of the matrix $A$ belonging to set 1 and belonging to set 2 in the case of perfectly conducting side walls. Next we compute the critical Rayleigh numbers belonging to the respective sets. These Rayleigh numbers are dependent on both $h_1$ and $h_2$.

Fig. 15 shows two stability curves for a box with $h_2=$ fixed = 1 and varying $h_1$. The upper curve is the stability curve obtained with set 1, the lower curve is the stability curve obtained with set 2. For both sets the summation ran through 8 terms in the horizontal x-direction ($m = 8$) and 8 terms in the vertical z-direction ($j_z = 8$). With each curve the number of x-rolls is denoted by $1x$, $2x$, etc. The transition from $1x$-roll to $2x$-rolls, from $2x$-roll to $3x$-rolls, etc. is denoted by the vertical dashes which are placed respectively above and below the stability curves for set 1 and set 2.

Before we go into detail concerning the differences between the two stability curves, it seems useful to discuss the problems which were encountered in constructing these curves.

The procedure which is described in section V.2, could not be applied completely. The actual behaviour of the stability curve for $1x$-roll (type I behaviour, section V.2) is shown qualitatively in fig. 16. Both sets of trial functions showed qualitatively the same behaviour. We expected the stability curve for one $x$-roll to rise with increasing $h_1$ if $h_1 > h_{1\text{min}}$, where $h_{1\text{min}}$ is the $h_1$-value for which the stability curve for $1x$-roll was expected to reach its minimum. This is indicated by the dashed line in fig. 16. However, the curve for $1x$-roll continued to descend after the first kink in this curve. We have tried if it was possible to follow the dashed line upwards. As mentioned before, we defined the critical Rayleigh number $R_c$ as the smallest positive value of $R$ for which $\det A$ in eq. (V.27) is zero, at a certain $h_1$- and $h_2$-value. Now we also have studied the other positive values of $R$ for which $\det A=0$. We have done this for several $h_1$-values, but we could not obtain the dashed line in fig. 16.

Therefore we began to study the coefficients $b_{mlj}$, $a_{mlj}$ belonging to set 1, and the coefficients $B_{mlj}$, $A_{mlj}$ belonging to set 2. We studied their development with increasing $h_1$ and were especially interested in the coefficients belonging to $3x$-rolls,
Fig. 15a  Stability curves for set 1 (upper curve) and set 2 (lower curve)  
\[ h_2 = 1 = \text{fixed} \]  
\[ m_x = j_x = 8 \]
Fig. 16. Stability curve for in-roll (qualitative) behaviour. Deviation from the expected type I.

$h_2 = \text{Fixed}$.
which also are included in the trial functions for $1x$-roll, since the summation in the $x$-direction runs through 8 terms ($m_x=8$). It turned out that the coefficients belonging to $3x$-rolls became dominating after the first kink in the $1x$-roll curve. This is denoted by $(3x)$ in fig. 16. The $h_1$-interval where the coefficients for $1x$-roll are dominating is denoted by $1x$. Increasing $h_1$ further, we obtained a new kink in the curve and after this kink the coefficients belonging to $5x$-rolls turned out to be dominating. This is denoted by $(5x)$ in fig. 16. After the next kink the coefficients belonging to $7x$-rolls became dominating etc. The reason for the discussed behaviour is, as will be clear after the foregoing, the fact that in the performed summation not only coefficients for $1x$-roll are present, but also those for $3x$-rolls, $5x$-rolls etc. Therefore we did not obtain the type I behaviour which is discussed in section V.2.

If we do not summate in the horizontal $x$-direction, we fix $m=1$, we obtain a stability curve similar to the curve in fig. 11, thus having the type I behaviour. However, it will be clear, that we must perform a summation in the horizontal $x$-direction and vertical $z$-direction to gain reliable values for the critical Rayleigh numbers.

The same behaviour, as discussed above, appeared when performing the computations for the stability curve for $2x$-rolls. This curve also showed kinks. After the first kink the coefficients belonging to $4x$-rolls became dominating, etc.

A similar behaviour was encountered with the $y$-roll computations for $h_1<h_2$.

The discussed behaviour did not give any further difficulties with the construction of the stability curve for set 1. The construction of the stability curve for set 1, with $h_2=1$ and varying $h_1$ is shown qualitatively in figure 17. First we compute the curve for $1x$-roll and obtain a curve similar to the one in fig. 16. This curve is denoted by $1x$ in fig. 17. Next we compute the curve for $2x$-rolls, denoted by $2x$ in fig. 17. This curve intersects the curve $1x$ before the first kink in curve $1x$ and becomes the lowest curve and thus the curve
Fig. 17. Construction of the stability curve for set 1; \( h_2 = \text{fixed} \).

Fig. 18. Problems with the construction of the stability curve for set 2; \( h_2 = \text{fixed} \).
of the critical Rayleigh number for an interval of $h_1$.

Next we compute the curve for $3x$-rolls, denoted by $3x$ in fig. 17. This curve intersects the curve $2x$ before the first kink in curve $2x$ and becomes the lowest curve and consequently the curve of the critical Rayleigh number for an interval of $h_1$, etc.

Using a similar procedure if $h_1 < h_2$ (for the finite $y$-rolls), we obtain the upper stability curve in fig. 15a.

With the construction of the stability curves for set 2 however, the discussed behaviour, shown in fig. 16, gave rise to problems. What happened with the construction of the stability curve for the box with horizontal dimensions $h_2 = 1.0 =$ fixed and $h_1$ varying, is shown qualitatively in fig. 18.

First we computed the stability curve for $1x$-roll and obtained a curve similar to the curve in fig. 16. In fig. 18 this curve is denoted by $1x$. The part between the first kink and the second kink is denoted by $(3x)$.

Next we computed the stability curve for $2x$-rolls and obtained the curve denoted $2x$. The part of this curve between the first and the second kink is denoted by $(4x)$. Curve $1x$ and curve $2x$ intersect before the first kink in curve $1x$ and $2x$. The problem arose when we wanted to involve the curve for $3x$-rolls in the construction of the stability curve for the box (dashed line denoted by $3x$ in fig. 18). This curve and curve $2x$ intersected after the first kink in curve $2x$. After this kink the coefficients of $4x$-rolls are dominating and consequently this piece of the curve $2x$ represents $4x$-rolls.

With regard to the $3x$-curve we notice that the intersection point with curve $2x (=4x$) is situated after the first kink in the $3x$-curve i.e. at a place on this curve where the coefficients of $5x$-rolls have become dominating. This part of the curve, denoted by $(5x)$, lies a little lower (0.05-0.06%) than the part of curve $1x$ situated after the second kink in curve $1x$ (coefficients of $5x$-rolls are dominating).

The discussed situation is of course physically unacceptable. Changing the number of terms in the summation did not change anything about the situation. To avoid the problems discussed here, we have used the following procedure for the construction of the stability curve for the box (fig. 18).
We compute the stability curve for 1x-roll, which is shown qualitatively in fig. 16. If \( h_1 \) has reached the kink where the part of curve 1x, denoted by (2x) ends, we terminate. Next we compute the stability curve for 2x-rolls and terminate if \( h_1 \) has reached the kink where the part of curve 2x, denoted by (4x) ends. At the intersection point of curve 1x and 2x the number of rolls changes from 1x- to 2x-rolls.

We now define the intersection point of curve 2x with part (3x) of curve 1x as the point where the number of rolls changes from 2x- to 3x-rolls.

The intersection point of part (3x) of curve 1x, with part (4x) of curve 2x is defined as the point where the number of x-rolls changes from 3 to 4.

Next we compute the stability curve for 3x-rolls, which is the curve we obtain if the summation runs through all integers from 2 to 9 in the x-direction in the trial functions for an odd number of x-rolls. This is the dashed curve, denoted by 3x in fig. 18. We now define the intersection point of part (4x) of curve 2x with part (5x) of curve 3x as the point where the number of x-rolls changes from 4 to 5.

Now we compute the stability curve for 4x-rolls, which is the curve we obtain if the summation runs through all integers from 2 to 9 in the x-direction in the trial functions for an even number of x-rolls. This is the dashed curve, denoted by 4x in fig. 18. We define the intersection point of part (5x) of curve 3x, with part (6x) of curve 4x as the point where the number of x-rolls changes from 5 to 6.

An analogous procedure can be used for the determination of the stability curve for the box at values of \( h_1 < h_2 \), where we are dealing with finite y-rolls.

Following the described procedure, we obtained the lower stability curve in fig. 15a.

To assure us of the fact that we really are dealing with 1x-roll, 2x-rolls, 3x-rolls, etc., where we have denoted this under the lower stability curve (for set 2) in fig. 15a, we have determined the streamline pattern at the onset of convective motion for several points on the lower stability curve,

\[ a \ h_2 = h_1 = 1.0; \ R_c = 1508.0; \] corresponding to 1x/ly-roll (fig. 19a)

\[ b \ h_2 = 1.0, \ h_1 = 1.6; \ R_c = 1082.3; \] corresponding to 2x-rolls (fig. 19b)
Fig. 19. Streamline pattern at the onset of convective motion for a box with horizontal dimensions $h_2$ and $h_1$.

- $a$ $h_2=h_1=1.0$; $R_c=1508.0$; 1x/ly-roll
- $b$ $h_2=1.0$, $h_1=1.6$; $R_c=1082.3$; 2x-rolls
- $c$ $h_2=1.0$, $h_1=2.4$; $R_c=954.9$; 3x-rolls
- $d$ $h_2=1.0$, $h_1=3.2$; $R_c=911.7$; 4x-rolls
Fig. 19b.

Fig. 19c.

Fig. 19d.
We see from fig. 19a-19d that the streamline patterns actually correspond with the number of rolls we were expecting according to the lower stability curve in fig. 15a. Further on in this section we will discuss the streamline patterns of fig. 19a-19d in more detail. But first we will discuss the two stability curves given in fig. 15a.

As mentioned, the upper stability curve in fig. 15a is the stability curve obtained using set 1, and the lower curve is the stability curve obtained using set 2. For both sets the summation ran through 8 terms in the horizontal x-direction \((m_x=8)\) and 8 terms in the vertical z-direction \((j_z=8)\).

As we see from figure 15a, set 2 gives a lower stability curve than set 1. We also see from this figure that set 2 predicts the transition of the number of x-rolls from \(M\) to \(M+1\) for lower values of \(h^i_1\) than set 1. This can also be seen from table 3 where these \(h^i_1\)-values are presented for both set 1 and set 2. We have also included the mentioned \(h^i_1\)-values for the case that we construct the stability curve for set 1 in the same way as we have constructed the stability curve for set 2 (fig. 15b). The latter \(h^i_1\)-values lie in general closer to the \(h^i_1\)-values for set 2 than the \(h^i_1\)-values for set 1 obtained with the construction method of fig. 17. However the differences are not great, except maybe for the transition from 2 to 3 rolls.

Table 3: \(h^i_1\)-values for which the number of x-rolls changes from \(M\) to \(M+1\)

<table>
<thead>
<tr>
<th>Transition</th>
<th>(h^i_1) set 1</th>
<th>(h^i_1) set 2</th>
<th>(h^i_1) set 1 as set 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (\rightarrow) 2</td>
<td>1.35</td>
<td>1.20</td>
<td>1.35</td>
</tr>
<tr>
<td>2 (\rightarrow) 3</td>
<td>2.29</td>
<td>1.97</td>
<td>1.96</td>
</tr>
<tr>
<td>3 (\rightarrow) 4</td>
<td>2.95</td>
<td>2.71</td>
<td>2.71</td>
</tr>
<tr>
<td>4 (\rightarrow) 5</td>
<td>3.60</td>
<td>3.43</td>
<td>3.49</td>
</tr>
<tr>
<td>5 (\rightarrow) 6</td>
<td>4.27</td>
<td>4.06</td>
<td>4.17</td>
</tr>
<tr>
<td>6 (\rightarrow) 7</td>
<td>4.95</td>
<td>4.87</td>
<td>4.83</td>
</tr>
<tr>
<td>7 (\rightarrow) 8</td>
<td>5.62</td>
<td>5.39</td>
<td>5.49</td>
</tr>
</tbody>
</table>
Fig. 15b. Stability curves for set 1 (upper curve) and set 2 (lower curve), constructed using the same procedure.

\[ h_2 = 1 = \text{fixed} \]

\[ m_x = j_2 = 8 \]
Table 4 gives the $h_1$-intervals for which the preferred cell pattern consists of $M$ finite $x$-rolls.

Table 4: $h_1$-interval for which the preferred cell pattern consists of $M$ finite $x$-rolls

<table>
<thead>
<tr>
<th>M</th>
<th>$h_1$-interval</th>
<th>set 1</th>
<th>set 2</th>
<th>set 1 as set 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.94</td>
<td>0.77</td>
<td>0.61</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.66</td>
<td>0.74</td>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.65</td>
<td>0.72</td>
<td>0.78</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.67</td>
<td>0.63</td>
<td>0.68</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.68</td>
<td>0.81</td>
<td>0.66</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.67</td>
<td>0.52</td>
<td>0.66</td>
<td></td>
</tr>
</tbody>
</table>

We notice that for set 1 the $h_1$ interval is quite constant for $M=3,4,5,6,7$. This is also the case if we construct the stability curve for set 1 in the same way as we have constructed the stability curve for set 2.

Set 2 predicts considerable differences between the $h_1$-intervals for $M=4,5,6,7$. I will return to this subject in section V.4.B.

Fig. 20 shows how the critical Rayleigh number $R_c$ varies as a function of the number of terms in the horizontal $x$-direction ($m_x$) and in the vertical $z$-direction ($j_z$). The computations were performed for a box with horizontal dimensions $h_1=1.1$ and $h_2=1.0$. $R_c$ as a function of $m_x, j_z$ for set 1 and set 2 is also given in table 5.

Table 5: $R_c$ as a function of $m_x, j_z$ for set 1 and set 2

<table>
<thead>
<tr>
<th>$m_x, j_z$</th>
<th>$R_c$ set 1</th>
<th>$R_c$ set 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1828.84</td>
<td>1723.00</td>
</tr>
<tr>
<td>4</td>
<td>1501.23</td>
<td>1438.57</td>
</tr>
<tr>
<td>6</td>
<td>1443.24</td>
<td>1418.48</td>
</tr>
<tr>
<td>8</td>
<td>1427.18</td>
<td>1415.22</td>
</tr>
<tr>
<td>10</td>
<td>1421.04</td>
<td>1414.40</td>
</tr>
<tr>
<td>12</td>
<td>1418.19</td>
<td>1414.14</td>
</tr>
<tr>
<td>14</td>
<td>1416.69</td>
<td>1414.04</td>
</tr>
<tr>
<td>16</td>
<td>1415.82</td>
<td>1413.99</td>
</tr>
</tbody>
</table>
Fig. 20. $R_c$ as a function of $m_x$, $j_z$

- $x$ = set 1
- $=$ set 2

$h_1=1.1$, $h_2=1.0$; 1x-roll
It will be clear from fig. 20 and table 5 that set 2 has a better convergence than set 1. If we for example increase $m_{x',j_{z}}$ from 6 to 12, we see that $R_{c}$ for set 1 varies by 1.74%, while $R_{c}$ for set 2 varies by only 0.31%.

For $m_{x',j_{z}}=8$, $R_{c}$ for set 2 has the value 1415.22. To obtain approximately the same $R_{c}$-value for set 1 we must increase $m_{x',j_{z}}$ to 16.

Our criterion for numerical convergence is that an increase of $m_{x',j_{z}}$ by 4, may not decrease $R_{c}$ by more than 0.5%. This means for the box with horizontal dimensions $h_{1}=1.1$ and $h_{2}=1.0$, that for set 1 $m_{x',j_{z}}=10$ is sufficient. For set 2 $m_{x',j_{z}}=6$ is sufficient.

Thus, because of the better convergence, we have used set 2 during the computations of the stability curves for boxes of several dimensions. This in spite of the problems we met in constructing the stability curves with this set. As we have shown in this section, it is possible to avoid this problems and to obtain reliable stability curves. We have ensured us of the fact that we really are dealing with 1x-, 2x-, 3x-, etc. rolls where we have denoted this under the stability curve for set 2 in fig. 15a. For this purpose we studied the streamline patterns shown in fig. 19a-19d.

From fig. 19a-19d we also clearly notice the influence of the confining walls on the rolls which are adjacent to the walls. The rolls are wider at the side which is adjacent to the vertical wall. The viscous shear at the walls thus becomes apparent. We also notice that the centres of the rolls lie above $z=0$ (boundaries of the box at $z=\frac{1}{2}, -\frac{1}{2}$). This is because the internal heat sources produce an unstably stratified layer above $z=0$ and a stably stratified layer below $z=0$. As a consequence, the convective motion has its motive power in the part of the box above $z=0$. The motion in the lower part of the box is caused by the drag-force and is consequently less intense than the motion in the upper part, where the streamlines lie closer together.

V.4.B. Stability curves

1. Perfectly conducting side walls
As is discussed in section V.4.A, we have used trial functions generated with orthogonal functions (section V.3), during the rest of our computations. To obey our criterion for convergence, also in case of small boxes \((h_1 < 1, h_2 < 1)\), we have used \(m_x, n_x = 8\).

The results of our computations for boxes with perfectly conducting side walls, are presented in the figures 21-24. Fig. 21 gives the stability curve for a box with \(h_2 = 0.25\) fixed and \(h_1\) varying from 0.25 to 6.0.

Fig. 22, 23 and 24 give the stability curves for fixed \(h_2\)-values of respectively 0.5, 1.0 and 2.0.

In these figures, the number of \(x\)-rolls is denoted by \(l_x, 2x,\) etc. and the number of \(y\)-rolls by \(l_y, 2y,\) etc. The small vertical dashes, placed under the stability curves, denote a change in the number of rolls.

The results are presented in tabular form in appendix II (tables al-a4).

From the stability curves presented in the figures 21-24, and from the tables al-a4, we notice that the critical Rayleigh number \(R_c\) approaches rapidly the value for the infinite horizontal layer with internal heat sources \((R_c = 583.2,\) chapter IV\), as the horizontal dimensions of the box increase. We found \(R_c = 664.7\) for a box with horizontal dimensions \(h_2 = 2.0\) and \(h_1 = 6.0\).

We notice that the critical Rayleigh number increases considerably as the rigid lateral walls are brought closer together. The reason for this is, that the viscous dissipation due to the side walls, and the importance of the heat loss through the side walls, increase as the side walls are brought closer together. As a consequence the fluid has to do more work to overcome the additional viscous shear. This requires a larger strength of the density of the heat sources (higher Rayleigh number).

We see from the figures 21-24, that if \(h_1 < h_2\), finite \(y\)-rolls are preferred. If \(h_1 > h_2\) finite \(x\)-rolls are preferred. If \(h_1 = h_2\) there is no preference for \(M\) finite \(x\)-rolls or \(M\) finite \(y\)-rolls because they are physically identical. Thus the preferred mode at the onset of convective motion is some number of finite rolls with their axes parallel to the short side of the box.
RESULTS BOX WITH PERFECTLY CONDUCTING SIDE WALLS

Fig. 21. Stability curve for $h_2 = 0.25$, $m_x$ and $n_y$ denote finite $x$-rolls and $n_{finite}$ $y$-rolls respectively. $R_c$ versus $h_1$.

Fig. 22. Stability curve for $h_2 = 0.5$, $m_x$ and $n_y$ denote finite $x$-rolls and $n_{finite}$ $y$-rolls respectively. $R_c$ versus $h_1$. 

$R_c \times 10^{-3}$ vs $h_1$.
RESULTS BOX WITH PERFECTLY CONDUCTING SIDE WALLS

Fig. 24. Stability curve for $h_2 = 2.0$, $m_x$ and $n_y$ denote $m$ finite $x$-rolls and $n$ finite $y$-rolls respectively. $R_c$ versus $h_1$.}

Fig. 23. Stability curve for $h_2 = 1.0$, $m_x$ and $n_y$ denote $m$ finite $x$-rolls and $n$ finite $y$-rolls respectively. $R_c$ versus $h_1$.}
This is in accordance with the results of Davis (4) for a rectangular box which is confined by perfectly conducting walls and which is heated from below. We observe that the stability curves in the figures 21-24 have kinks. These kinks were also predicted by Davis for the box heated from below. Stork & Müller (17) could not exhibit the kinks predicted by Davis in their experiments, because this characteristic of the curves was on a scale within the margin of the experimental error.

Another interesting observation from the figures 21-24 is that, as $h_1$ increases, the $h_1$-intervals for an even number of rolls increase and become larger than the $h_1$-intervals for an odd number of rolls. The latter $h_1$-interval decreases with increasing $h_1$. Stork & Müller (17) observed in their experiments for a box which is heated from below, that the formation of an even number of rolls was preferred to the development of an odd number of rolls. However, to our knowledge, no experiments have been performed yet in a box which is heated from within by an uniform distribution of heat sources.

2. Insulating side walls

In this case we had to include a summation in the horizontal y-direction. It proved that a summation running through 5 terms in the y-direction ($n_y=5$) and 8 terms in the x- and z-direction ($m_x=m_z=8$) was sufficient to obey our convergence criterion. The results of our computations are presented in the figures 25-28. Fig. 25 gives the stability curve for a box with $h_2=0.5=\text{fixed}$ and $h_1$ varying from 0.25 to 6.0. The figures 26, 27 and 28 give the stability curves for fixed $h_2$-values of respectively 1.0, 2.0 and 6.0. The number of x-rolls is again denoted by $l_x$, 2x, etc. and the number of y-rolls by $l_y$, 2y, etc. The vertical dashes under the stability curves indicate a change in the number of rolls. The results are also presented in tabular form in appendix III (tables a5-a8). We notice that the critical Rayleigh number $R_c$ approaches rapidly the value for the infinite layer with internal heat sources,
Fig. 25. Stability curve for $h_2=0.5$, $m_x$ and $n_y$ denote $m$ finite $x$-rolls and $n$ finite $y$-rolls respectively. $R_C$ versus $h_1$.

Fig. 26. Stability curve for $h_2=1.0$, $m_x$ and $n_y$ denote $m$ finite $x$-rolls and $n$ finite $y$-rolls respectively. $R_C$ versus $h_1$. 
Fig. 27. Stability curve for $h_2=2.0$, $m_x$ and $n_y$ denote $m$ finite $x$-rolls and $n$ finite $y$-rolls respectively. $R_c$ versus $h_1$.

Fig. 28. Stability curve for $h_2=6.0$, $m_x$ and $n_y$ denote $m$ finite $x$-rolls and $n$ finite $y$-rolls respectively. $R_c$ versus $h_1$. 
as the horizontal dimensions of the box increase. For a box with horizontal dimensions $h_1 = h_2 = 6.0$ we found $R_C = 605.0$.

As in the box with conducting side walls, the critical Rayleigh number increases considerably as the rigid lateral walls are brought closer together.

Comparing for example fig. 23 for the conducting case, with fig. 26 for the insulating case (in both figures $h_2 = 1.0$), we notice that the kinks are much more pronounced in the case of insulating side walls.

We also notice that the stability curves for the insulating case lie below the corresponding stability curves in the conducting case. The higher critical Rayleigh numbers in the conducting case are a manifestation of heat loss through the conducting side walls. The differences become greater as the side walls are brought closer together, because the influence of this heat loss in the conducting case becomes more and more important.

Looking at fig. 28, which represents the stability curve for a box with $h_2 = 6.0$ fixed and $h_1$ varying from 0.3 to 6.0, we notice that neither $6x$-rolls nor $6y$-rolls appear. As we see from the figures 19a-19d, the cross-sections of the rolls are not 'near-square', but more 'rectangular', with the longest side in the vertical direction. The reason for this is that the centres of the rolls lie above $z = 0$ (boundaries of the box at $z = -\frac{1}{2}, \frac{1}{2}$), which causes narrower cells to appear than in the case where the centres of the rolls lie at $z = 0$.

In the case of insulating side walls we could not notice any significant difference in size between the $h_1$-intervals for an even number of rolls and the $h_1$-intervals for an odd number of rolls.

The preferred mode at the onset of convective motion is again some number of finite rolls with their axes parallel to the short side of the box.
VI. CONVECTION IN A BOX WITH INTERNAL HEAT SOURCES, NON-LINEAR THEORY

VI.1. Governing equations, steady solutions

We consider again a rectangular box, which is heated from within by an uniform distribution of internal heat sources. The dimensionless boundaries of the box are situated at \(|z| = \frac{h_1}{2}\), \(|x| = \frac{h_2}{2}\) and \(|y| = \frac{h_2}{2}\). The upper and lower horizontal planes of the box are supposed to be rigid and perfectly conducting. The vertical side walls are also supposed to be rigid. Concerning the conductivity of the side walls we consider two cases, a) insulating side walls b) conducting side walls.

The dimensionless non-linear equations that govern the system are given by the equations (IV.11)-(IV.13),

\[
\begin{align*}
\mathcal{P}^{-1} \left( \frac{\partial \mathbf{v}}{\partial \mathbf{e}} + \mathbf{v} \cdot \nabla \mathbf{v} \right) &= -\nabla p + \Theta \kappa + \nabla^2 \mathbf{v} \\
\nabla \cdot \mathbf{v} &= 0 \\
\frac{\partial \Theta}{\partial \mathbf{e}} + \mathbf{v} \cdot \nabla \Theta &= \nabla^2 \Theta + \frac{Q}{4 \pi} \mathcal{R}(2z) \mathbf{w}
\end{align*}
\]

(VI.1)

(VI.2)

(VI.3)

In eq. (IV.13) we have replaced \((2z-1)\) by \((2z)\) because we have chosen the boundaries of the box at \(|z| = \frac{1}{2}\).

In the equations (VI.1)-(VI.3) we define again,

\[
\mathcal{P} = \frac{\nu}{\kappa} \quad \text{and} \quad \mathcal{R} = \frac{g \zeta |Q| d^5}{\mathcal{b} h_0 c_p \kappa^2}
\]

(VI.4)

The boundary conditions for the system are,

\[
\begin{align*}
|z| = \frac{h_1}{2} & \quad u = v = w = \Theta = 0 \quad \text{(insulating)} \\
|y| = \frac{h_2}{2} & \quad u = v = w = \frac{\partial \Theta}{\partial y} = 0 \\
|x| = \frac{h_2}{2} & \quad u = v = w = \frac{\partial \Theta}{\partial x} = 0 \quad \text{(conducting)} \\
\end{align*}
\]

(VI.5a)

(VI.6a)

(VI.6b)

Assuming that the fluid in the box is water, we know that the Prandtl number is \(\mathcal{P} = 7.03\) at \(20^\circ\text{C}\) and 1 atm. (Smith & Stammers, (22)). In the paper by Tveitereid and Palm (8) for example,
we see from their fig. 6 (and fig. 7) that for \(P=7\) we may simplify the problem by assuming that the Prandtl number is infinite, whereby eq. (VI.1) becomes linear. It is generally believed that the solution for infinite \(P\), also is a good approximation to the solutions for moderate Prandtl numbers. We will only study steady solutions and will not present a stability analysis here. The equations (VI.1)-(VI.3) can now be simplified to (assuming positive \(Q\), for simplicity),

\[
\begin{align*}
-\nabla P + \Theta \frac{\partial}{\partial x} + \nabla^2 \nu &= 0 \\
\nabla \nu &= 0 \\
\nabla^2 \Theta + 32 \Re(\alpha z) \omega - \nabla \cdot \nabla \Theta &= 0
\end{align*}
\]

with the boundary conditions (VI.5)-(VI.7). Comparing eqs. (VI.8)-(VI.10) with eqs. (V.7)-(V.9), we notice that the only new term in eqs. (VI.8)-(VI.10) is the non-linear term \(\nabla \cdot \nabla \Theta\) in eq. (VI.10).

The solutions of eqs. (VI.8)-(VI.10) will be found by a numerical approach, using a Galerkin procedure.

Let us assume again that the convective motion takes place in the form of finite \(x\)-rolls \((v=0)\). The continuity equation (VI.9) now becomes

\[
\nabla \nu = \frac{\partial u}{\partial x} + \frac{\partial \omega}{\partial z} = 0
\]

Thus we may assume,

\[
u = -\frac{\partial \psi}{\partial x} ; \omega = \frac{\partial \psi}{\partial x}
\]

where \(\psi\) is a stream function.

Let us for simplicity represent \(\Theta\) and \(\psi\) as follows,

\[
\begin{align*}
\Theta &= \sum_q a_q \Theta_q \\
\psi &= \sum_q b_q \psi_q
\end{align*}
\]

Now

\[
u = -\frac{\partial \psi}{\partial z} = \sum_q b_q \frac{\partial \psi_q}{\partial z} = \sum_q b_q \psi_q , \quad \text{with} \quad u_q = -\frac{\partial \psi_q}{\partial z}
\]

\[
\omega = \frac{\partial \psi}{\partial x} = \sum_q b_q \frac{\partial \psi_q}{\partial x} = \sum_q b_q \psi_q , \quad \text{with} \quad w_q = \frac{\partial \psi_q}{\partial x}
\]
Introducing eqs. (VI.12), (VI.14) and (VI.15) into eq. (VI.8) and (VI.10) we obtain,

\[ \sum_q b_q \nabla^2 u_q - \frac{\partial P}{\partial x} = 0 \]  
(VI.16)

\[ \sum_q b_q \nabla^2 \omega_q + \sum_q a_q \Omega_q - \frac{\partial P}{\partial z} = 0 \]  
(VI.17)

\[ \sum_q a_q \nabla^2 \Theta_q + \sum_q b_q R(2x) \omega_q - \sum_r \sum q b_r (u_r \frac{\partial \Theta_q + w_r \frac{\partial \Theta_q}}{\partial z}) = 0 \]  
(VI.18)

The unknown coefficients \(a_q\) and \(b_q\) are determined from eqs. (VI.16)-(VI.18) by applying a Galerkin procedure. Multiplying these equations by respectively \(u_i\), \(w_i\) and \(\Theta_i\) and averaging over volume of the box we obtain,

\[ \sum_q \left\{ -b_q \langle \nabla u_i \cdot \nabla u_q \rangle + \langle \nabla \omega_i \cdot \nabla \omega_q \rangle \right\} + a_q \langle \omega_i \cdot \Theta_q \rangle = 0 \]  
(VI.19)

\[ \sum_q \left\{ b_q R(2x) \omega_q - a_q \langle \nabla \Theta_i \cdot \nabla \Theta_q \rangle \right\} + \sum_r \sum q b_r \langle \Theta_i \left( u_r \frac{\partial \Theta_q + w_r \frac{\partial \Theta_q}}{\partial z} \right) \rangle = 0 \]  
(VI.20)

In section V.4.A, where we were dealing with the linear stability problem, we have seen that the \(y\)-dependence in the case of perfectly conducting side walls disappears in eq. (V.27). Because of the non-linear term in eq. (VI.20) however, the \(y\)-dependence will not disappear in this equation. Consequently the case of perfectly conducting side walls does not give any computational advantages in the non-linear theory. We will therefore only consider the case of insulating side walls in the steady non-linear theory, because this case has most practical value. Thus we have the following boundary conditions,

\[ |z| = \frac{h_1}{2} \quad u=v=w=\Theta=0 \]  
(VI.5)

\[ |y| = \frac{h_2}{2} \quad u=v=w=\partial \Theta/\partial y=0 \]  
(VI.6a)

\[ |x| = \frac{h_1}{2} \quad u=v=w=\partial \Theta/\partial x=0 \]  
(VI.7a)

To find the solutions of eqs. (VI.19) and (VI.20) with the boundary conditions (VI.5), (VI.6a) and (VI.7a) we apply a Newton-Raphson method.

During the computations we have only considered boxes with \(h_1 > h_2\). Therefore we now only consider finite \(x\)-rolls. We can construct roll-like trial functions, using the orthogonal
functions which are defined in section V.3. In the non-linear theory we can no longer consider odd and even numbers of rolls separately, we must consider a combination of even and odd functions in the roll-like trial functions. For finite x-rolls we have achieved this as follows,

\[ \mathcal{V} = h \sum_{m_1}^{\infty} \sum_{n_1}^{\infty} \sum_{j_1}^{\infty} B_{mnj} \left( D_0 C_{mnj}^{(1)} + D_e C_{mnj}^{(2)} \right) \cos \left( \frac{(n_1-1)\pi z}{h_1} \right) \left\{ \delta_o C_j^{(1)}(x) + \delta_e C_j^{(2)}(x) \right\} \] (VI.21)

where \( D_0 = 1, D_e = 0 \) if \( m = \text{odd} \)
\( D_0 = 0, D_e = 1 \) if \( m = \text{even} \)
\( \delta_o = 1, \delta_e = 0 \) if \( j = \text{odd} \)
\( \delta_o = 0, \delta_e = 1 \) if \( j = \text{even} \)

and \( m_1 = (m+1)/2, m_2 = m/2, j_1 = (j+1)/2, j_2 = j/2. \)

\[ \Theta = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mnj} \left\{ \left[ D_0 \sin \left( \frac{(2m-1)\pi x}{h_1} \right) + D_e \cos \left( \frac{(2m)\pi x}{h_1} \right) \right] \right\} \cos \left( \frac{(2n-1)\pi z}{h_2} \right) \left\{ \delta_o \cos \left( \frac{(2j-1)\pi z}{h_2} \right) + \delta_e \sin \left( \frac{(2j)\pi z}{h_2} \right) \right\} \] (VI.22)

Introducing eqs. (VI.21), (VI.22) in eqs. (VI.19), (VI.20) and applying a Newton-Raphson method, we can determine the unknown coefficients \( B_{mnj} \) and \( A_{mnj} \) for a box of certain dimensions where the Rayleigh number \( R \) is higher than the critical value \( R_C. \)

Knowing the coefficients \( B_{mnj} \) and \( A_{mnj} \) for a certain box, we can compute the following quantities,

\( a. \) The horizontally averaged temperature

In eq. (IV.10) we have defined the perturbed temperature \( T \) as,

\[ T = T_s(z) + \Theta(x,y,z) \] (VI.23)

where \( T_s(z) \) is the steady state solution if all heat transfer is in the form of conduction. \( T_s(z) \) is defined by eq. (V.4)',

\[ T_s(z) = \frac{Q}{2\rho c_p K} (\frac{a^2}{4} - z^2) \] (VI.24)

To make it possible to compare our results with the results of Tveitereid (7) for the infinite horizontal layer with internal heat sources, we have chosen the following dimensionless form of eq. (VI.24),

\[ T_s'(z) = 4 \left( \frac{1}{4} - z^2 \right) \]
In this form the temperature $T'_s(z)$ has a maximum $T'_s = 1$ at $z = 0$. Therefore we introduce $K \sqrt{g \rho R g \alpha d^3}$ as our unit for temperature. Omitting the primes we obtain the following perturbed dimensionless temperature,

$$T = 4 \left( \frac{y}{h} - z^2 \right) + \frac{\Theta}{g R}$$  \hspace{1cm} (VI.25)

Using eq. (VI.22) we can find the horizontally averaged temperature $\overline{T}$,

$$\overline{T} = 4 \left( \frac{y}{h} - z^2 \right) + \frac{1}{8} \sum_{j \neq 0} A_{0j} \left[ \delta_0 \cos(2\pi j_1 - 1)y + \delta_0 \sin(2\pi j_2 - 1)y \right]$$  \hspace{1cm} (VI.26)

b. The Nusselt number

The Nusselt number $Nu$ is defined as the ratio between the heat transport with convection and the heat transport without convection. In our case we have to consider two different Nusselt numbers, the Nusselt number at the upper boundary plane (=Nussup) and the Nusselt number at the lower boundary plane (=Nussdown). We can determine Nussup as follows.

$$Nu_{up} = \frac{\frac{\partial T}{\partial z} \bigg|_{y=\frac{1}{2}}}{\frac{\partial T}{\partial z} \bigg|_{y=-\frac{1}{2}}} = 1 - \frac{1}{2} \sum_{j \neq 0} A_{0j} \left[ 2\pi j_1 - 1 \right] \delta_0 (2\pi j_2 - 1) \delta_0$$  \hspace{1cm} (VI.27)

Then we find

$$Nu_{up} = 1 - \frac{1}{2} \sum_{j \neq 0} A_{0j} \left[ 2\pi j_1 - 1 \right] \delta_0 (2\pi j_2 - 1) \delta_0$$  \hspace{1cm} (VI.28)

In the same way we find Nussdown,

$$Nu_{down} = \frac{\partial T}{\partial z} \bigg|_{y=-\frac{1}{2}} = 1 - \frac{1}{2} \sum_{j \neq 0} A_{0j} \left[ 2\pi j_1 - 1 \right] \delta_0 (2\pi j_2 - 1) \delta_0$$  \hspace{1cm} (VI.29)

These numbers were defined by Tveitereid (7) for the infinite horizontal layer with internal heat sources. It is therefore interesting to compare the values found by Tveitereid for the infinite layer, with the values found in this paper.

VI.2. Results

First we have computed $A_{mnj}$, $B_{mnj}$, $\overline{T}$, Nussup, Nussdown, $N_0$ and $N_1$ for a box with horizontal dimensions $h_1 = 2.6$ and $h_2 = 2.0$. 
This is corresponding with \( R_c = 683.2 \) on the stability curve given by fig. 27. The onset of convective motion in this point on the stability curve in fig. 27, will be in the form of 3\( x \)-rolls. This stability curve is computed, running the summation through 5 terms in the \( y \)-direction (\( n_y = 5 \)) and 8 terms in the \( x \)- and \( z \)-direction (\( m_x = j_z = 8 \)), using the trial functions for an odd number of rolls.

As mentioned in section VI.1, we can no longer consider odd and even numbers of rolls separately. In non-linear theory, we must consider a combination of even and odd functions in the roll-like trial functions. For \( \psi \) and \( \theta \), in steady non-linear theory, we would have to use the following summation (I), in order to fit the non-linear computations with the linear stability curve from fig. 27, where \( n_y = 5 \) and \( m_x = j_z = 8 \). Summation (I) is defined as,

\[
\psi: m=1,2,\ldots,16 \quad \theta: m=0,1,2,\ldots,16 \\
\quad n=1,2,\ldots,5 \quad \quad n=1,2,\ldots,6 \} \text{ in the case } m=0 \\
\quad j=1,2,\ldots,8 \quad \quad j=1,2,\ldots,9 \} \text{ in the case } m\neq 0
\]

This would mean that the total number of coefficients \( A_{mnj} + B_{mnj} \), which have to be computed with the Newton-Raphson method, is 240, and that the matrix which necessarily is included in the Newton-Raphson method has the dimensions \( (240,240) \). Hence the computations of \( A_{mnj} \) and \( B_{mnj} \) would become very expensive. Therefore we had to cut down the number of terms in the summation in the steady non-linear theory. We have used the following summation (II) for \( \psi \) and \( \theta \),

\[
\psi: m=1,2,\ldots,12 \quad \theta: m=0,1,2,\ldots,12 \\
\quad n=1,2,3,4 \quad \quad n=1,2,\ldots,5 \} \text{ in the case } m=0 \\
\quad j=1,2,\ldots,6 \quad \quad j=1,2,\ldots,7 \} \text{ in the case } m\neq 0
\]

Using this summation, the dimensions of the matrix in the Newton-Raphson method become \( (141,141) \) and the total number of coefficients \( A_{mnj} + B_{mnj} \), which have to be computed, becomes 141. This means a considerable decrease in necessary space on
the computer and in necessary computation time.

The critical Rayleigh number for the case \( n_y = 4, m_x = j_z = 6 \) is
\[ R_c = 694.2 \] which is only 1.6% higher than \( R_c \) for \( n_y = 5, m_x = j_z = 8 \).

Nevertheless the extension of the computations still was such, that is was quite expensive to study the behaviour of \( A_{mnj} \), \( B_{mnj} \), \( \bar{T} \), Nussup, Nussdown, \( N_o \), \( N_1 \) as a function of \( R/R_C \).

As \( R/R_C \) increased, the convergence of the Newton-Raphson method became poorer, and the necessary computation time increased. Therefore we only performed the computations for \( R/R_C \) values in the range \( 1 < R/R_C < 2.25 \).

Fig. 29 gives Nussup, Nussdown as a function of \( R/R_C \).

Fig. 30 gives \( N_o \), \( N_1 \) as a function of \( R/R_C \).

From fig. 29 we see that Nussup > 1 and Nussdown < 1. The reason for this is shown in fig. 31, where we qualitatively have displayed \( \bar{T} \) as a function of \( R/R_C \). We notice that \( \frac{\partial \bar{T}}{\partial z} \bigg|_{z=\frac{1}{2}} \) increases and that \( \frac{\partial \bar{T}}{\partial z} \bigg|_{z=\frac{1}{2}} \) decreases with increasing \( R/R_C \).

Nussup \( \sim \frac{\partial \bar{T}}{\partial z} \bigg|_{z=\frac{1}{2}} \) thus becomes greater than one and Nussdown \( \sim \frac{\partial \bar{T}}{\partial z} \bigg|_{z=\frac{1}{2}} \) becomes less than one for values of \( R/R_C > 1 \).

The values of \( N_o \), \( N_1 \) displayed in fig. 30 are much lower than the values of \( N_o \), \( N_1 \) found by Tveitereid (2) for the infinite horizontal layer with internal heat sources. This of course is not unexpected, because the vicinity of the rigid lateral side walls reduces the strength of the convective motion considerably by means of the viscous shear at these side walls.

The same reasoning is valid for \( \bar{T} \), which we did not present here because the changes in \( \bar{T} \) were very small. Tveitereid obtains a much stronger change in \( \bar{T} \) with increasing \( R/R_C \), for the infinite horizontal layer with internal heat sources. However, we expect that our results will approach the results of Tveitereid as we increase the horizontal dimensions of the box. This was confirmed by a calculation of \( N_o \), \( N_1 \) for a smaller box (\( h_1 \) and \( h_2 \) smaller), which will be presented further on in this section.

A remarkable fact which is observed from fig. 29 and fig. 30 is that the curves displayed in these figures have a kink between \( R/R_C = 2.0 \) and \( R/R_C = 2.25 \). In the same \( R/R_C \) interval the warm
Both even and odd terms in the summation.

Horizontal dimensions of the box: $h_1=2.6$, $h_2=2.0$

![Graph of Nussup and Nussdown as a function of $R/R_c$.](image)

**Fig. 29.** Nussup and Nussdown as a function of $R/R_c$.
Both even and odd terms in the summation.
Horizontal dimensions of the box: $h_1=2.6$, $h_2=2.0$

![Graph of $N_0$ and $N_1$ as a function of $R/R_c$.](image)

**Fig. 30.** $N_0$ and $N_1$ as a function of $R/R_c$.
Both even and odd terms in the summation.
Horizontal dimensions of the box: $h_1=2.6$, $h_2=2.0$
Fig. 31. The horizontally averaged temperature $\overline{T}$ for different values of $R/R_c$ (qualitative).
central core of the fluid which first was displaced upwards begins to move downwards with increasing $R/R_c$. The reason for this might be that the number of terms in the summation becomes too little for $R/R_c \geq 2.0$. It is well known from experience in non-linear theory, that one has to increase the number of terms in the summation as $R/R_c$ increases. As mentioned, we also noticed that the convergence of the Newton-Raphson method became poorer as $R/R_c$ increased. It is therefore necessary to perform a convergence investigation, to determine the reliability of the results in the non-linear theory.

To get at least some idea about the reliability of our results, we also performed some computations using the following summation (III) for $\psi$ and $\Theta$:

\[
\begin{align*}
\psi: & \quad m = 1, 2, \ldots, 8 \\
& \quad n = 1, 2, 3, 4 \\
& \quad j = 1, 2, 3, 4
\end{align*}
\]

\[
\begin{align*}
\Theta: & \quad m = 0, 1, 2, \ldots, 8 \\
& \quad n = 1, 2, \ldots, 5 \quad \text{in the case } m = 0 \\
& \quad j = 1, 2, \ldots, 5 \\
& \quad n = 1, 2, 3, 4 \\
& \quad j = 1, 2, 3, 4 \quad \text{in the case } m \neq 0
\end{align*}
\]

This is corresponding with a critical Rayleigh number $R_c = 704.6$ in linear theory, if $m_x = n_y = j_z = 4$. Using this summation (III) we computed some values for $N_o$, $N_1$, and compared these values with the values for $N_o$, $N_1$ obtained with summation (II).

We may simplify $N_o$, $N_1$ to,

\[
\begin{align*}
N_o &= 4 + \Delta N_o \\
N_1 &= 4 + \Delta N_1
\end{align*}
\]

Here $\Delta N_o$ and $\Delta N_1$ are functions of $A_{01j}$ ($j = 1, 2, \ldots$). In our computations $\Delta N_o$ and $\Delta N_1$ are small compared with four. Therefore it is reasonable not to consider $N_o$ and $N_1$ as a whole in our comparison, but to consider $\Delta N_o$ and $\Delta N_1$.

We noticed that the differences between the results obtained with summation (III) and the results obtained with summation (II) increased with increasing $R/R_c$. For example at $R/R_c = 1.25$, $\Delta N_o$ for summation (III) differs about 6% from $\Delta N_o$ for summation (II). At $R/R_c = 2.0$ this difference is about 12%.

As mentioned before, it is well known from experience in non-linear theory that one has to raise the number of terms in the summation as $R/R_c$ increases. Because of the costs however we could not do so, and hence our results become less reliable with increasing $R/R_c$. 
During the computations with summation (II) we also studied the development, with increasing $R/R_c$, of the coefficients $B_{mnj}$ in this summation, belonging to 1x-, 2x-, 3x-, 4x-, 5x-rolls. The onset of convective motion in the box with horizontal dimensions $h_1=2.6$ and $h_2=2.0$, will be in the form of 3x-rolls. As a consequence, the coefficients $B_{mnj}$ belonging to 3x-rolls were dominating for $R/R_c$ values near 1.0. However, as $R/R_c$ increased, we noticed that the coefficients belonging to 1x-roll and 4x-rolls were growing strongest. At $R/R_c=2.25$ we noticed that the largest coefficient belonging to 1x-roll had about the same value as the largest coefficient belonging to 3x-rolls. While the coefficients of 1x-roll still were increasing considerably with increasing $R/R_c$, the coefficients of 3x-rolls had become nearly constant. This might also be one of the causes of the observed kink in the figures 29 and 30. It is possibly an indication that the number of rolls tends to decrease as $R/R_c$ increases. However, we cannot be sure about this, because first a stability analysis has to be performed to determine which of the steady solutions is stable.

We have also computed $A_{mnj}$, $B_{mnj}$, $\bar{T}$, Nussup, Nussdown, $N_o$ and $N_1$ as functions of $R/R_c$ for a box with horizontal dimensions $h_1=h_2=2.0$. For $n_y=5$ and $m_x=j_z=8$ this box has a critical Rayleigh number $R_c=725.8$ and onset of convective motion in the form of 2x-rolls (2y-rolls).

We had to cut down the terms in the summation again and used summation (II) for $\psi$ and $\theta$, i.e.

$$\psi: m=1,2,\ldots,12 \quad \Theta: m=0,1,2,\ldots,12$$
$$n=1,2,3,4 \quad n=1,2,\ldots,5 \} \text{ in the case } m=0$$
$$j=1,2,\ldots,6 \quad j=1,2,\ldots,7 \} \text{ in the case } m\neq0$$

The critical Rayleigh number for $n_y=4$ and $m_x=j_z=6$ is $R_c=728.6$, which is only about 0.4% higher than $R_c$ for $n_y=5$ and $m_x=j_z=8$.

Since the onset of convective motion, according to the stability curve displayed in fig. 27, is in the form of 2x-rolls,
we started the computations with the initial conditions that
the coefficients belonging to even numbers of rolls were ≠ 0,
and that the coefficients belonging to odd numbers of rolls
were zero. As we performed some computations for \( R/R_c > 1 \), it appeared
that the coefficients \( B_{mnj} \), belonging to odd numbers of rolls,
remained zero. It can be proved from our system of equations
\((VI.19), (VI.20)\) and with the use of our trial function \((VI.21),
(VI.22)\), that the coefficients belonging to odd numbers of rolls
always must remain zero if they get initial values equal to
zero. The coefficients belonging to even numbers of rolls,
always get values ≠ 0 for \( R/R_c > 1 \), even if we give those coef-
ficients initial values equal to zero.

So actually we must give the coefficients belonging to odd
numbers of rolls initial values which do not equal zero. The
problem however is, to find a set of initial values which
lead to a rapid convergence of the Newton-Raphson method.

Therefore, for simplicity, we have cut out the coefficients
belonging to odd numbers of rolls during the computations for
the box with horizontal dimensions \( h_1=h_2=2.0 \). This probably
reduces the reliability of the results, but qualitatively the
results can give us valuable information. Since the total
number of coefficients \( A_{mnj} + B_{mnj} \) now is reduced to 79, the
computations have become a lot cheaper, and we have perfor-
med computations for \( R/R_c \) values up to 5.0.

Fig. 32 displays \( N_{\text{ussup}}, N_{\text{ussdown}} \) as a function of \( R/R_c \).
Fig. 33 displays \( N_0, N_1 \) as a function of \( R/R_c \).

We notice that the values of \( N_{\text{ussup}}, N_0 \) for this box lie above, and the va-
lues of \( N_{\text{ussdown}}, N_0 \) for this box lie below the values of respectively \( N_{\text{ussup}},
N_1 \) and \( N_{\text{ussdown}}, N_0 \) for the box with dimensions \( h_1=2.6 \) and \( h_2=2.0 \), al-
though the latter box is larger and we thus would expect the
opposite. The reason for this opposite behaviour must be the
fact that we did not account for the coefficients belonging
to odd numbers of rolls with the computations for the box
with horizontal dimensions \( h_1=h_2=2.0 \).

To make sure that our results approach the values given by
Tveitereid (7) for the infinite horizontal layer with internal
heat sources, we also performed a calculation for a smaller
box with horizontal dimensions \( h_2=1.0 \) and \( h_1=1.6 \). For \( n_y=5 \) and
Fig. 32. Nussup and Nussdown as a function of $R/R_c$. Odd terms only (=even number of rolls). Horizontal dimensions of the box: $h_1=h_2=2.0$
Fig. 33. $N_0$ and $N_1$ as a function of $R/R_c$.
Odd terms only (=even number of rolls).
Horizontal dimensions of the box: $h_1 = h_2 = 2.0$
This box has a critical Rayleigh number $R_c = 852.2$ and onset of convective motion in the form of 2x-rolls.

We use summation (II) again. For $n_y = 4$ and $m = j = 6$ we find $R_c = 855.0$. For this box we find at $R/R_c = 1.2$ that $N_o = 4.024$. For the box with horizontal dimensions $h_1 = h_2 = 2.0$ we found $N_o = 4.034$ at $R/R_c = 1.2$. Thus we see that $N_o$ increases as the horizontal dimensions of the box increase, which is what we expected in consequence of the results of Tveitereid.

Again we notice from fig. 32 and fig. 33 that the curves displayed in these figures have a kink between $R/R_c = 2.25$ and $R/R_c = 2.5$. In the same $R/R_c$ interval, the warm central core of the fluid which first was displaced upwards, begins to move downwards and continues moving downwards as $R/R_c$ is increased further. Near $R/R_c = 4.75$ the warm central core begins to move upwards again. One of the reasons for this behaviour probably is again that the number of terms in our summation is too little for values of $R/R_c \geq 2.25$. Near $R/R_c = 2.5$ we noticed however also that the coefficients belonging to 4x-rolls had become larger than the coefficients belonging to 2x-rolls, and that the latter coefficients began to decrease. This might also be a reason for the observed kink in the curves displayed in fig. 32 and fig. 33.
VII. SUMMARY AND DISCUSSION

To solve the problem of the storage of radioactive waste, the possibility of creating a storage-accommodation on large depth in one of the naturally occurring rock-salt masses, is now being studied in more detail in the Netherlands.

In this paper a start is made with the study of the problems that might rise as a result of the affection of the salt by the appearance of a convective flow in such a storage-accommodation (cavity), if this cavity is filled with water and the strength of the density of heat sources (radioactive materials) exceeds a certain critical value.

The aim of this paper was the following:
Given a rectangular cavity filled with water. Assume that the heat sources are evenly distributed. Find the necessary strength of the density of the heat sources in order to get a convective motion, i.e. find the critical 'Rayleigh number'. Try to study the intensity of the motion set-up for a given 'Rayleigh number'.

First we will summate the results obtained in this paper. After that we will discuss the simplifications made in this paper. The critical Rayleigh number for a given rectangular cavity which might appear in praxis, will be translated into a critical temperature and a critical heat production.

Summating the results obtained in this paper. In chapter IV we computed the stability curve for the infinite horizontal layer heated from within by an uniform distribution of heat sources. The layer is bounded above and below by two rigid perfectly conducting planes, maintained at constant and equal temperatures. We applied a Galerkin procedure and solved the linear stability problem numerically. The critical Rayleigh number \( R_c \) and the corresponding value of the horizontal wave number \( a, a_c \), are found to be

\[
R_c = 583.2; \quad a_c = 4.00
\]

which are identical to the values found by Sparrow et al. (6) and Tveitereid (7).
Next we considered a rectangular box heated from within by an uniform distribution of heat sources and bounded above and below by two rigid perfectly conducting planes, maintained at constant and equal temperatures. The side walls were also supposed to be rigid. Concerning the conductivity of the side walls we considered two cases,

a perfectly insulating side walls
b perfectly conducting side walls.

The case of insulating side walls has most practical value.
The case of conducting side walls is actually an artificial case, but proved to have certain computational advantages in the linear theory and was therefore also studied here.

To solve the linear stability problem for the box, we applied a Galerkin procedure and used a numerical approach.

As Davis (4) did for the rectangular box heated from below, we assumed the onset of convective motion to be in the form of finite rolls. Tveitereid & Palm (8) expected hexagons for the infinite horizontal layer with internal heat sources.

On the other hand Koschmieder (13) showed in his experimental work for a horizontal fluid layer heated from below, that the influence of lateral boundaries dominates the non-linear effects in the selection of cell shape, if the width of the apparatus is an order of magnitude larger than the height. In that case roll cells of geometrical shape similar to that of the confining container seemed to appear. Since we in our problem also are dealing with a bounded region we assumed finite rolls at the onset of convective motion.

We have constructed two sets of roll-like trial functions.

Set 1 trial functions generated with sine and cosine.
Set 2 trial functions generated with orthogonal functions.

A convergence analysis showed that set 2 had the best convergence. We used set 2 during our further computations.

First we computed stability curves for boxes with perfectly conducting side walls and with width to depth ratios in the range: x-direction $1/4 \leq H_1/d \leq 6$

$y$-direction $H_2/d=1/4$, 1/2, 1, 2 (fig. 21–fig. 24).

We noticed that the critical Rayleigh number $R_c$ approached rapidly the value for the infinite horizontal layer with internal heat sources ($R_c=583.2$) as the horizontal dimensions of the
box increased. For a box with horizontal dimensions $h_2 = H_2 / d = 2.0$, $h_1 = H_1 / d = 6.0$ we found $R_c = 664.7$.

The critical Rayleigh number increased considerably as the rigid lateral walls were brought closer together because of the increasing viscous shear due to the side walls. The preferred mode at the onset of convective motion appeared to be some number of finite rolls with axes parallel to the short side, which is in accordance with the results of Davis (4) for a box heated from below. We observed that the stability curves displayed in the figures 21-24 have kinks, which also were predicted by Davis for the box heated from below.

Next we computed the stability curves for boxes with insulating side walls and with width to depth ratios in the range: 

x-direction $1/4 \leq H_1 / d \leq 6$

y-direction $H_2 / d = 1/2, 1, 2, 6$ (fig. 25-fig. 28).

Again we noticed that the critical Rayleigh number approached rapidly the value for the infinite layer ($R_c = 583.2$) as the horizontal dimensions of the box increased. For a box with horizontal dimensions $h_1 = h_2 = 6.0$ we found $R_c = 605.0$. The kinks in the stability curves were much more pronounced in the case of insulating side walls which also was found by Catton (15) for the box heated from below. We noticed that the stability curves in the insulating case were lower than the corresponding stability curves in the conducting case which is a manifestation of the heat loss through the sides in the latter case.

In chapter VI a start is made with the non-linear theory for a box with internal heat sources. Assuming infinite Prandtl number, we have performed some computations for a box with insulating side walls. We have looked for steady solutions. To solve the equations (VI.8)-(VI.10) we used roll-like trial functions generated with orthogonal functions and we applied a Galerkin procedure. We used a Newton-Raphson method to determine the coefficients in the trial functions. We have computed the horizontally averaged temperature $T$, the Nusselt numbers at upper (=Nussup) and lower (=Nussdown) plane of the box, and the numbers $N_o, N_1$ as defined by Tveitereid (7).
Comparing the values of $N_o$ and $N_1$ found in this paper for a box with horizontal dimensions $h_1=2.6$ and $h_2=2.0$, with the values found by Tveitereid for the infinite layer, we noticed that our values were much lower. This of course is not unexpected because the vicinity of the rigid lateral walls reduces the strength of the convective motion considerably due to the viscous dissipation at these lateral walls. We expect that our results will approach the results of Tveitereid as we increase the horizontal dimensions of the box. This was confirmed by a computation of $N_o$ and $N_1$ for a smaller box.

We observed a kink in the computed curves for Nussup, Nussdown, $N_o$ and $N_1$ at Rayleigh numbers of about two times the critical Rayleigh number. One of the causes for this might be that the number of terms in the summation becomes too little. We noticed that the convergence of the Newton-Raphson procedure became poorer as the Rayleigh number increased. Because of the costs however we could not increase the number of terms in the summation. Here a disadvantage of the followed numerical approach with the use of a Galerkin procedure and a Newton-Raphson method becomes apparent. To obtain a reasonable accuracy in the results for higher supercritical Rayleigh numbers, we must increase the number of terms in the summation in x-, y- and z-direction. This however, gives a matrix of enormous dimensions in the Newton-Raphson method. Unfortunately one can only solve the non-linear problem by a numerical approach.

During the computations for a box with horizontal dimensions $h_1=2.6$ and $h_2=2.0$, having the onset of convective motion in the form of 3x-rolls, we noticed a tendency for the coefficients belonging to 1x-roll to become dominating as $R/R_c$ increased. Near $R/R_c=2.25$ the largest coefficient belonging to 1x-roll had nearly become as large as the largest coefficient belonging to 3x-rolls. This might also be a cause of the appearance of the kink in the curves for Nussup, etc. It might also be an indication that the number of rolls tends to decrease with increasing $R/R_c$. However, first a stability analysis has to be performed to determine which of the steady solutions is stable.

In the laboratory where a layer of necessity is bounded
in horizontal extent, Koschmieder (13) has found that for the layer heated from below, the number of rolls tends to decrease as $R/R_C$ increases. This is also observed by Busse & Whitehead (27). One of the causes for this could be what Busse & Whitehead called the pinching mechanism, which in its idealized form combines to roll couplets into a single couplet by joining the ends of two adjacent rolls.

Using Stuart's shape assumption (24) and a condition of maximum heat transport, Davis (23) found an indication that the preferred number of finite roll cells present in Bénard convection in a three-dimensional rectangular box tends to decrease with increasing supercritical Rayleigh number. Stuart's shape assumption is however only quantitatively correct in the interval $1 < R/R_C < 1.1$, while the transition from $M$ to $M-1$ rolls was found to take place for $R/R_C$ values greater than 3. Hence the results of Davis only give an indication.

For the infinite horizontal fluid layer with infinite Prandtl number, and heated from below, Schlüter, Lortz and Busse (1965, (25)) found that the number of rolls increased with increasing supercritical Rayleigh number. Their results have been obtained at third order in an expansion in terms of the steady-state amplitude and are therefore only approximately valid for small values of $R/R_C$.

Their work was redone by Busse (26) in 1967. Busse applied a Galerkin procedure and solved numerically the resulting system of non-linear algebraic equations for the coefficients of a complete set of functions. For disturbances with non-oscillating time dependence he found a stability region for two-dimensional convection (rolls) as is displayed in fig. 34 (fig. 5 in (26)) (infinite Prandtl number). We see from fig. 34 that the number of rolls may decrease as well as increase with increasing $R/R_C$.

It will be clear after the foregoing that a complete stability analysis for the steady solutions is necessary in order to predict the region of stability for the finite roll cells. This however will be a very expensive procedure since we have to account for all three spatial directions in the box. Before one continues the study which is started in this paper, there are several things that need to be considered seriously, concerning the practical situation which underlies this paper.
Fig. 34. The region of stability of two-dimensional convection with respect to disturbances with non-oscillatory time dependence. Infinite horizontal fluid layer bounded above and below by rigid perfectly conducting planes and heated from below (Busse (26)).
Let us return to the actual problem. We have a storage-accommodation for radioactive waste in a rock-salt formation on large depth under the surface of the earth. We assume that by some catastrophic occurrence the cavity is filled with water.

In chapter I we introduced a number of simplifications. We did not account for the temperature difference within the surface of the earth which causes the temperature at the lower plane of the cavity to be higher than the temperature at the upper plane. If we neglect the presence of heat sources in the cavity, this results in a density gradient in the fluid opposite to the direction of gravity. In cases where the temperature gradient exceeds a certain critical value, the static state of the fluid becomes unstable because the buoyancy force is sufficient to overcome the dissipative effects. In praxis we will always get a convective flow in a cavity filled with water and situated on large depth under the surface of the earth. Suppose we have a salt mine with a height $d=2\text{ m}$. Suppose that the temperature within the surface of the earth increases with $1^\circ\text{C}$ per $100\text{ m}$. This results in a temperature difference of $2\times10^{-2}\,^\circ\text{C}$ between top and bottom of the mine. The resulting density gradient is opposite to the direction of gravity.

Davis (4) defined the Rayleigh number for a box heated from below as

$$R = \alpha(\Delta T)g \rho \omega / \kappa \gamma$$

For the given situation with water at $20^\circ\text{C}$ and 1 atm. we find (x) a Rayleigh number for the cavity $R \approx 2\times10^9$. For a mine with horizontal dimensions $H_1=12\text{ m}$ ($h_1=6$) and $H_2=4\text{ m}$ ($h_2=2$) and conducting side walls, we see from fig. 8 in (4) that the critical Rayleigh number for this mine is $R_c \approx 2.3\times10^3$. In praxis the walls of the cavity will of course neither be perfectly conducting nor perfectly insulating, and we will be dealing with mixed boundary conditions as far as temperature and heat conduction are concerned. In their experiments for boxes heated from below, Stork & Müller (17) actually were dealing with mixed boundary conditions at the walls, also at the upper plane of the box. Their critical Rayleigh numbers agreed quite well with the results of Davis (4) for the box with horizontal dimensions $h_1=6$, $h_2=2$. Stork & Müller discussed the problems they met as they tried to achieve the marginal stability state. The question

(x) $\alpha \approx 2\times10^{-4}\,^\circ\text{C}^{-1}$; $\kappa \approx 1.4\times10^{-7}\,\text{m}^2/\text{s}$; $\gamma \approx 10^{-6}\,\text{m}^2/\text{s}$
therefore is if in our practical problem the marginal stability state ever can be achieved (this is even more doubtful with the presence of internal heat sources in the cavity). Anyhow we can state that the actual Rayleigh number will be much higher than the critical value and consequently a convective motion will appear in the mine, even without the presence of internal heat sources. This motion will probably not be laminar. The reason for the high Rayleigh number is the relatively large vertical dimension of the box. This dimension $d$ has the power three in the Rayleigh number. Therefore the next step should be to introduce a temperature difference between the upper and lower plane of the box into the problem, and to study the effect of this temperature difference together with the effect of internal heat sources.

Another fact is that we in praxis will not have the idealized situation of evenly distributed heat sources within the cavity. The heat sources will be spread around in the cavity. In the neighbourhood of these heat sources a convective motion will in praxis always appear. However, if we neglect the temperature field within the surface of the earth, we can describe the overall situation by the model presented in this paper.

Considering the same salt mine with dimensions $d=2\,\text{m}$, $H_1=12\,\text{m}$ ($h_1=6$), $H_2=4\,\text{m}$ ($h_2=2$), now with insulating side walls, we see from fig. 27 and table a7 in this paper that $R \approx 650$.

For a box with internal heat sources we have defined $R$ as

$$R = \frac{Q}{\rho \gamma \alpha} \left[ \frac{d}{64 \rho \gamma \alpha \kappa} \right]^5$$

Eq. (V.4)' gives the steady state solution for the temperature $T$, $T_s$, in case that all heat transport takes place by conduction. The maximum of $T_s$ lies at $z=0$ (boundaries of the mine at $|z|=d/2$) and is given by

$$T_s = \frac{Q}{8 \rho \gamma \alpha \kappa} \left( \frac{d^2}{8 \rho \gamma \alpha \kappa} \right) = \frac{R K \gamma}{d^3 \rho \gamma \alpha}$$

For water at $20^\circ\text{C}$ and 1 atm. we find that the critical value for $T_s = 0.5 \times 10^{-7}$ $^\circ\text{C}$. Thus the temperature at $z=0$ may not be higher by more than $0.5 \times 10^{-7}$ $^\circ\text{C}$ than the temperature at the upper and lower plane of the mine. The critical value for the generated heat per unit time and unit volume is $Q \approx 0.6 \times 10^{-7}$ $\text{J/m}^3\text{t}$.

In other words the critical values of $T_s$ and $Q$ are very low and the actual values of $T_s$ and $Q$ will always be much higher than the critical values in the practical situation when radioactive waste is stored in a salt mine. As mentioned on page 70
in praxis we will always be dealing with mixed boundary conditions as far as temperature and heat conduction are concerned and the question is if the marginal stability state ever can be achieved in our practical problem. If we consider this, together with the presence of the temperature field within the surface of the earth, it will be clear that we in praxis will have Rayleigh numbers far above the critical Rayleigh numbers. Thus a strong probably non-laminar convective motion will appear in the storage-accomodation and the shape of this storage-accomodation will be altered because of the solubility of the rock-salt. We did not account for the solubility of the salt yet but it is to be expected that rock-salt will be dissolved at one place and precipitated at another place. As a consequence the shape of the storage-accomodation will be altered and finally the storage-accomodation might begin to displace itself through the salt. Therefore one finally has to study the intensity of the convective motion (non-linear theory), accounting for temperature field within the surface of the earth, internal heat sources and solubility of the salt. This however will be a very difficult procedure (impossible?) since the in the practical situation occurring Rayleigh numbers will be so high that the convective motion in the cavity probably no longer can be described by a regular pattern as is done in this paper.

Considering the foregoing discussion, we have the opinion that, if one really intends to store radioactive waste in a rock-salt formation, this has to be done in such a way that external ground water never can reach or enter the storage-accomodation. This because the presence of water in the storage-accomodation will cause the immediate onset of a strong convective motion which will alter the shape of the storage-accomodation, and which finally might lead to the displacement of the cavity through the salt formation. It will be clear that this last situation is very undesirable.
REFERENCES


REFERENCES (continued)

APPENDIX I, The Galerkin method (Sokolnikoff (20))

Let it be required to solve a linear differential equation

\[ L(u) = 0 \quad \text{in } S \]  

subject to some linear homogeneous boundary conditions.

Inasmuch as the operator \( L \) is not necessarily homogeneous, the restriction on the homogeneity of the boundary conditions is not essential, since the boundary conditions can always be cast in the desired form by changing the dependent variable \( u \).

We assume, for simplicity of exposition, that the domain \( S \) is two-dimensional and seek an approximate solution of the problem in the form

\[ u_n(x,y) = \sum_{j=1}^{N} a_j \psi_j(x,y) \]  

where the \( \psi_j \) are suitable coordinate functions and the \( a_j \) are constants. \( \psi_j \) must satisfy the same boundary conditions as the exact solution \( u(x,y) \) and the set \( \{ \psi_j \} \) is supposed to be complete in the sense that every piecewise continuous function \( f(x,y) \) can be approximated in \( S \) by the sum \( \sum_{j=1}^{N} c_j \psi_j \) in such a way that

\[ \delta_n = \int_{S} \left( f - \sum_{j=1}^{N} c_j \psi_j \right)^2 dx dy \]  

(iii)

can be made as small as we wish.

The finite sum (ii) ordinarily will not satisfy eq. (i) and the substitution of \( u_n \) will yield

\[ L(u_n) = \xi_n(x,y), \quad \xi_n(x,y) \neq 0 \quad \text{in } S \]

If max. \( \xi_n(x,y) \) is small, \( u_n(x,y) \) can be considered a satisfactory approximation to \( u(x,y) \). Thus \( \xi_n(x,y) \) can be viewed as an error function, and the task is then to select the \( a_j \) so as to minimize \( \xi_n(x,y) \).

A reasonable minimization technique is suggested by the following: if on represents \( u(x,y) \) by the series \( u(x,y) = \sum_{r=1}^{\infty} a_r \psi_r \), then the orthogonality condition

\[ \int_{S} L(u_n) \psi_r(x,y) dx dy = 0 \quad \text{as } n \to \infty \]  

(iv)

is equivalent to the statement that \( L(u) = 0 \).

This led Galerkin to impose on the error function \( L(u_n) \) a set of orthogonality conditions

\[ \int_{S} L(u_n) \psi_r(x,y) dx dy = 0 \quad (r=1,2,\ldots,n) \]  

(v)
yielding the set of \( n \) equations

\[
\sum_{j=1}^{n} L \left( \sum_{j=1}^{n} a_j y_j \right) \psi_r \, dx \, dy = 0 \quad (r=1, 2, \ldots, n) \quad (vi)
\]

for the determination of the constants \( a_j \) in the approximate solution (ii).
APPENDIX II. Box with perfectly CONDUCTING side walls. Critical Rayleigh number for several aspect ratios. Number of terms in x- and z-direction: 8. $mx=number$ of x-rolls, $ny=number$ of y-rolls. $h_{lt}=h_1$value for which the number of rolls changes.

<table>
<thead>
<tr>
<th>$h_1$</th>
<th>$R_c$</th>
<th>$mx/ny$</th>
<th>$h_{lt}$</th>
<th>$h_1$</th>
<th>$R_c$</th>
<th>$mx/ny$</th>
<th>$h_{lt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>45244.3</td>
<td>1x</td>
<td></td>
<td>0.25</td>
<td>13075.6</td>
<td>1y</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>29365.6</td>
<td>1</td>
<td></td>
<td>0.3</td>
<td>9728.8</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>17222.0</td>
<td>1</td>
<td></td>
<td>0.4</td>
<td>6860.9</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>13075.6</td>
<td>1</td>
<td></td>
<td>0.5</td>
<td>5691.7</td>
<td>1x/1y</td>
<td>0.5</td>
</tr>
<tr>
<td>0.6</td>
<td>11394.3</td>
<td>1</td>
<td></td>
<td>0.6</td>
<td>4214.7</td>
<td>1x</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>10649.2</td>
<td>1</td>
<td></td>
<td>0.7</td>
<td>3488.2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>9992.9</td>
<td>2x</td>
<td>0.77</td>
<td>0.8</td>
<td>3101.0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>9243.7</td>
<td>2</td>
<td></td>
<td>0.9</td>
<td>2880.5</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>8827.3</td>
<td>2</td>
<td></td>
<td>1.0</td>
<td>2740.0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1.1</td>
<td>8605.2</td>
<td>2</td>
<td></td>
<td>1.1</td>
<td>2543.6</td>
<td>2x</td>
<td>1.04</td>
</tr>
<tr>
<td>1.2</td>
<td>8487.0</td>
<td>2</td>
<td></td>
<td>1.2</td>
<td>2381.0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.3</td>
<td>8323.3</td>
<td>3x</td>
<td>1.27</td>
<td>1.3</td>
<td>2274.4</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td>8159.3</td>
<td>3</td>
<td></td>
<td>1.4</td>
<td>2205.3</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>8060.1</td>
<td>3</td>
<td></td>
<td>1.5</td>
<td>2160.9</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.6</td>
<td>8004.8</td>
<td>3</td>
<td></td>
<td>1.6</td>
<td>2131.5</td>
<td>3x</td>
<td>1.68</td>
</tr>
<tr>
<td>1.7</td>
<td>7964.7</td>
<td>4x</td>
<td>1.69</td>
<td>1.7</td>
<td>2192.3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1.8</td>
<td>7883.8</td>
<td>4</td>
<td></td>
<td>1.8</td>
<td>2057.3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1.9</td>
<td>7829.6</td>
<td>4</td>
<td></td>
<td>1.9</td>
<td>2024.4</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>7797.5</td>
<td>4</td>
<td></td>
<td>2.0</td>
<td>2001.3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2.2</td>
<td>7736.3</td>
<td>5x</td>
<td>2.11</td>
<td>2.2</td>
<td>1974.6</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2.4</td>
<td>7688.3</td>
<td>5</td>
<td></td>
<td>2.4</td>
<td>1947.0</td>
<td>4x</td>
<td>2.29</td>
</tr>
<tr>
<td>2.6</td>
<td>7656.6</td>
<td>6x</td>
<td>2.52</td>
<td>2.6</td>
<td>1922.4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>2.8</td>
<td>7623.6</td>
<td>6</td>
<td></td>
<td>2.8</td>
<td>1910.0</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td>7603.4</td>
<td>7x</td>
<td>2.96</td>
<td>3.0</td>
<td>1896.0</td>
<td>5x</td>
<td>2.89</td>
</tr>
<tr>
<td>3.2</td>
<td>7578.3</td>
<td>7</td>
<td></td>
<td>3.2</td>
<td>1884.6</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>7565.7</td>
<td>8x</td>
<td>3.33</td>
<td>3.4</td>
<td>1878.8</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3.6</td>
<td>7552.7</td>
<td>8</td>
<td></td>
<td>3.6</td>
<td>1869.4</td>
<td>6x</td>
<td>3.43</td>
</tr>
<tr>
<td>3.8</td>
<td>7547.9</td>
<td>8</td>
<td></td>
<td>3.8</td>
<td>1862.7</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>7528.8</td>
<td>9x</td>
<td>3.81</td>
<td>4.0</td>
<td>1858.7</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>4.2</td>
<td>7520.9</td>
<td>10x</td>
<td>4.14</td>
<td>4.2</td>
<td>1853.1</td>
<td>7x</td>
<td>4.07</td>
</tr>
<tr>
<td>4.4</td>
<td>7514.3</td>
<td>10</td>
<td></td>
<td>4.4</td>
<td>1849.2</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>4.6</td>
<td>7514.0</td>
<td>10</td>
<td></td>
<td>4.6</td>
<td>1846.5</td>
<td>8x</td>
<td>4.54</td>
</tr>
<tr>
<td>4.8</td>
<td>7504.1</td>
<td>11x</td>
<td>4.68</td>
<td>4.8</td>
<td>1842.9</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>5.0</td>
<td>7497.0</td>
<td>12x</td>
<td>4.97</td>
<td>5.0</td>
<td>1840.5</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>5.2</td>
<td>7491.1</td>
<td>12</td>
<td></td>
<td>5.2</td>
<td>1839.1</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>5.4</td>
<td>7492.4</td>
<td>12</td>
<td></td>
<td>5.4</td>
<td>1836.0</td>
<td>9x</td>
<td>5.26</td>
</tr>
<tr>
<td>5.6</td>
<td>7490.9</td>
<td>13x</td>
<td>5.56</td>
<td>5.6</td>
<td>1834.2</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>5.8</td>
<td>7483.5</td>
<td>13</td>
<td></td>
<td>5.8</td>
<td>1831.9</td>
<td>10x</td>
<td>5.65</td>
</tr>
<tr>
<td>6.0</td>
<td></td>
<td></td>
<td></td>
<td>6.0</td>
<td>1830.3</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>
### APPENDIX II, Box with perfectly CONDUCTING side walls (continued)

**Table a3: $h_2 = 1.0$**

<table>
<thead>
<tr>
<th>$h_1$</th>
<th>$R_c$</th>
<th>$R_{mx/ny}$</th>
<th>$h_{lt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>8827.3</td>
<td>2</td>
<td>2y</td>
</tr>
<tr>
<td>0.3</td>
<td>6026.2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>3706.8</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>2740.0</td>
<td>1</td>
<td>0.47</td>
</tr>
<tr>
<td>0.6</td>
<td>2208.0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>1906.4</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>1719.2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>1594.9</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>1508.0</td>
<td>1x/ly 1.0</td>
<td></td>
</tr>
<tr>
<td>1.1</td>
<td>1415.2</td>
<td>1x</td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>1343.4</td>
<td>2x</td>
<td>1.20</td>
</tr>
<tr>
<td>1.3</td>
<td>1240.8</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td>1168.8</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>1118.0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.6</td>
<td>1082.3</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.7</td>
<td>1057.2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.8</td>
<td>1039.3</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.9</td>
<td>1025.9</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>1011.3</td>
<td>3x</td>
<td>1.97</td>
</tr>
<tr>
<td>2.2</td>
<td>975.7</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2.4</td>
<td>954.9</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2.6</td>
<td>943.2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2.8</td>
<td>931.2</td>
<td>4x</td>
<td>2.71</td>
</tr>
<tr>
<td>3.0</td>
<td>919.0</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.2</td>
<td>911.7</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>907.3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.6</td>
<td>900.2</td>
<td>5x</td>
<td>3.43</td>
</tr>
<tr>
<td>3.8</td>
<td>895.6</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>892.9</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4.2</td>
<td>889.0</td>
<td>6x</td>
<td>4.06</td>
</tr>
<tr>
<td>4.4</td>
<td>885.5</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>4.6</td>
<td>883.3</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>4.8</td>
<td>881.7</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>5.0</td>
<td>879.1</td>
<td>7x</td>
<td>4.87</td>
</tr>
<tr>
<td>5.2</td>
<td>877.4</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>5.4</td>
<td>876.3</td>
<td>8x</td>
<td>5.39</td>
</tr>
<tr>
<td>5.6</td>
<td>874.5</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>5.8</td>
<td>873.2</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>6.0</td>
<td>872.5</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

**Table a4: $h_2 = 2.0$**

<table>
<thead>
<tr>
<th>$h_1$</th>
<th>$R_c$</th>
<th>$R_{mx/ny}$</th>
<th>$h_{lt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>7797.5</td>
<td>4y</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>5115.2</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>2885.6</td>
<td>3y 0.35</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>2001.3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>1571.7</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>1330.4</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>1180.7</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>1081.1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>1011.3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1.1</td>
<td>960.3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>920.0</td>
<td>2y 1.12</td>
<td></td>
</tr>
<tr>
<td>1.3</td>
<td>888.6</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td>863.9</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>844.0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.6</td>
<td>827.7</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.7</td>
<td>814.3</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.8</td>
<td>803.1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.9</td>
<td>793.6</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>785.6</td>
<td>2x/2y 2.0</td>
<td></td>
</tr>
<tr>
<td>2.2</td>
<td>760.3</td>
<td>3x 2.09</td>
<td></td>
</tr>
<tr>
<td>2.4</td>
<td>738.2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2.6</td>
<td>725.1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2.8</td>
<td>717.1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td>707.6</td>
<td>4x 2.88</td>
<td></td>
</tr>
<tr>
<td>3.2</td>
<td>699.0</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>693.9</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.6</td>
<td>690.0</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.8</td>
<td>685.4</td>
<td>5x 3.66</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>681.5</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4.2</td>
<td>679.1</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4.4</td>
<td>677.0</td>
<td>6x 4.34</td>
<td></td>
</tr>
<tr>
<td>4.6</td>
<td>674.2</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>4.8</td>
<td>672.1</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>5.0</td>
<td>670.4</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>5.2</td>
<td>669.4</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>5.4</td>
<td>667.6</td>
<td>7x 5.21</td>
<td></td>
</tr>
<tr>
<td>5.6</td>
<td>666.7</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>5.8</td>
<td>665.9</td>
<td>8x 5.76</td>
<td></td>
</tr>
<tr>
<td>6.0</td>
<td>664.7</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX III, Box with perfectly INSULATING side walls. Critical Rayleigh number for several aspect ratios. Number of terms in x- and z-direction: 8. Number of terms in y-direction: 5. \( m_x \) = number of x-rolls, \( n_y \) = number of y-rolls. \( h_{lt} \) = \( h \)-value for which the number of rolls changes.

Table a5: \( h_2=0.5 \)

<table>
<thead>
<tr>
<th>( h_1 )</th>
<th>( R_c )</th>
<th>( m_x/n_y )</th>
<th>( h_{lt} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>3848.4</td>
<td>1y</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>3278.6</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>2691.0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>2403.0</td>
<td>1x/1y</td>
<td>0.5</td>
</tr>
<tr>
<td>0.6</td>
<td>1828.3</td>
<td>1x</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>1559.0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>1435.6</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>1392.4</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>1397.7</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1.1</td>
<td>1433.8</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>1408.2</td>
<td>2x</td>
<td>1.16</td>
</tr>
<tr>
<td>1.3</td>
<td>1325.0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td>1271.6</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>1239.7</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.6</td>
<td>1223.7</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.7</td>
<td>1219.8</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.8</td>
<td>1224.8</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.9</td>
<td>1235.9</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>1219.2</td>
<td>3x</td>
<td>1.93</td>
</tr>
<tr>
<td>2.2</td>
<td>1182.5</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2.4</td>
<td>1170.4</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2.6</td>
<td>1174.7</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2.8</td>
<td>1163.5</td>
<td>4x</td>
<td>2.68</td>
</tr>
<tr>
<td>3.0</td>
<td>1149.8</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.2</td>
<td>1146.7</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>1150.4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.6</td>
<td>1140.2</td>
<td>5x</td>
<td>3.42</td>
</tr>
<tr>
<td>3.8</td>
<td>1134.8</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>1134.5</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4.2</td>
<td>1134.6</td>
<td>6x</td>
<td>4.14</td>
</tr>
<tr>
<td>4.4</td>
<td>1128.6</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>4.6</td>
<td>1126.6</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>4.8</td>
<td>1127.4</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>5.0</td>
<td>1125.1</td>
<td>7x</td>
<td>4.87</td>
</tr>
<tr>
<td>5.2</td>
<td>1122.1</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>5.4</td>
<td>1121.5</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>5.6</td>
<td>1122.3</td>
<td>8x</td>
<td>5.59</td>
</tr>
<tr>
<td>5.8</td>
<td>1119.2</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>6.0</td>
<td>1118.1</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

Table a6: \( h_2=1.0 \)

<table>
<thead>
<tr>
<th>( h_1 )</th>
<th>( R_c )</th>
<th>( m_x/n_y )</th>
<th>( h_{lt} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>2858.0</td>
<td>1y</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>2278.0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>1684.0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>1397.7</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>1236.0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>1134.6</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>1066.2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>1017.5</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>981.4</td>
<td>1x/1y</td>
<td>1.0</td>
</tr>
<tr>
<td>1.1</td>
<td>993.0</td>
<td>1x</td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>1020.6</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1.3</td>
<td>958.3</td>
<td>2x</td>
<td>1.21</td>
</tr>
<tr>
<td>1.4</td>
<td>906.2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>872.5</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.6</td>
<td>852.2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.7</td>
<td>842.2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.8</td>
<td>839.7</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1.9</td>
<td>842.8</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>849.5</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2.2</td>
<td>819.3</td>
<td>3x</td>
<td>2.02</td>
</tr>
<tr>
<td>2.4</td>
<td>802.3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2.6</td>
<td>798.9</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2.8</td>
<td>802.9</td>
<td>4x</td>
<td>2.79</td>
</tr>
<tr>
<td>3.0</td>
<td>787.7</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.2</td>
<td>780.6</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>779.7</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.6</td>
<td>780.1</td>
<td>5x</td>
<td>3.56</td>
</tr>
<tr>
<td>3.8</td>
<td>773.0</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>769.6</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4.2</td>
<td>769.6</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4.4</td>
<td>768.7</td>
<td>6x</td>
<td>4.33</td>
</tr>
<tr>
<td>4.6</td>
<td>765.1</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>4.8</td>
<td>763.6</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>5.0</td>
<td>764.0</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>5.2</td>
<td>762.2</td>
<td>7x</td>
<td>5.06</td>
</tr>
<tr>
<td>5.4</td>
<td>759.8</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>5.6</td>
<td>759.3</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>5.8</td>
<td>759.7</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>6.0</td>
<td>758.1</td>
<td>8x</td>
<td>5.81</td>
</tr>
</tbody>
</table>
APPENDIX III, Box with perfectly INSULATING side walls (continued)

<table>
<thead>
<tr>
<th>$h_1$</th>
<th>$R_c$</th>
<th>mx/ny</th>
<th>$h_{lt}$</th>
<th>$h_1$</th>
<th>$R_c$</th>
<th>mx/ny</th>
<th>$h_{lt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>1983.4</td>
<td>3y</td>
<td>0.3</td>
<td>1877.9</td>
<td>8y</td>
<td>0.4</td>
<td>1365.6</td>
</tr>
<tr>
<td>0.4</td>
<td>1468.0</td>
<td>3</td>
<td>0.4</td>
<td>1118.1</td>
<td>8</td>
<td>0.5</td>
<td>978.1</td>
</tr>
<tr>
<td>0.5</td>
<td>1219.2</td>
<td>3</td>
<td>0.6</td>
<td>989.9</td>
<td>3</td>
<td>0.7</td>
<td>831.0</td>
</tr>
<tr>
<td>0.6</td>
<td>1078.3</td>
<td>3</td>
<td>0.8</td>
<td>844.1</td>
<td>2y</td>
<td>0.9</td>
<td>849.5</td>
</tr>
<tr>
<td>0.7</td>
<td>989.9</td>
<td>3</td>
<td>1.0</td>
<td>823.1</td>
<td>2</td>
<td>1.1</td>
<td>785.8</td>
</tr>
<tr>
<td>0.8</td>
<td>930.2</td>
<td>3</td>
<td>1.2</td>
<td>802.4</td>
<td>2</td>
<td>1.3</td>
<td>772.2</td>
</tr>
<tr>
<td>0.9</td>
<td>884.1</td>
<td>2y</td>
<td>1.4</td>
<td>760.9</td>
<td>2</td>
<td>1.5</td>
<td>751.6</td>
</tr>
<tr>
<td>1.0</td>
<td>849.5</td>
<td>2</td>
<td>1.6</td>
<td>743.6</td>
<td>2</td>
<td>1.7</td>
<td>736.8</td>
</tr>
<tr>
<td>1.1</td>
<td>823.1</td>
<td>2</td>
<td>1.8</td>
<td>730.8</td>
<td>2</td>
<td>1.9</td>
<td>725.8</td>
</tr>
<tr>
<td>2.0</td>
<td>725.8</td>
<td>2x/2y</td>
<td>2.0</td>
<td>707.5</td>
<td>3x</td>
<td>2.1</td>
<td>688.9</td>
</tr>
<tr>
<td>2.2</td>
<td>707.5</td>
<td>3x</td>
<td>2.2</td>
<td>683.2</td>
<td>3</td>
<td>2.3</td>
<td>685.6</td>
</tr>
<tr>
<td>2.4</td>
<td>688.9</td>
<td>3</td>
<td>2.4</td>
<td>676.4</td>
<td>4x</td>
<td>2.5</td>
<td>668.1</td>
</tr>
<tr>
<td>2.6</td>
<td>683.2</td>
<td>3</td>
<td>2.6</td>
<td>676.4</td>
<td>4x</td>
<td>2.7</td>
<td>665.8</td>
</tr>
<tr>
<td>2.8</td>
<td>685.6</td>
<td>3</td>
<td>2.8</td>
<td>676.4</td>
<td>4x</td>
<td>2.9</td>
<td>667.1</td>
</tr>
<tr>
<td>3.0</td>
<td>676.4</td>
<td>4x</td>
<td>3.0</td>
<td>668.1</td>
<td>4</td>
<td>3.1</td>
<td>667.1</td>
</tr>
<tr>
<td>3.2</td>
<td>668.1</td>
<td>4</td>
<td>3.2</td>
<td>665.8</td>
<td>4</td>
<td>3.3</td>
<td>667.1</td>
</tr>
<tr>
<td>3.4</td>
<td>665.8</td>
<td>4</td>
<td>3.4</td>
<td>667.1</td>
<td>4</td>
<td>3.5</td>
<td>667.1</td>
</tr>
<tr>
<td>3.6</td>
<td>667.1</td>
<td>4</td>
<td>3.6</td>
<td>661.7</td>
<td>5x</td>
<td>3.7</td>
<td>661.7</td>
</tr>
<tr>
<td>3.8</td>
<td>661.7</td>
<td>5x</td>
<td>3.8</td>
<td>657.6</td>
<td>5</td>
<td>3.9</td>
<td>657.6</td>
</tr>
<tr>
<td>4.0</td>
<td>657.6</td>
<td>5</td>
<td>4.0</td>
<td>656.6</td>
<td>5</td>
<td>4.1</td>
<td>656.6</td>
</tr>
<tr>
<td>4.2</td>
<td>656.6</td>
<td>5</td>
<td>4.2</td>
<td>657.5</td>
<td>5</td>
<td>4.3</td>
<td>657.5</td>
</tr>
<tr>
<td>4.4</td>
<td>657.5</td>
<td>5</td>
<td>4.4</td>
<td>653.9</td>
<td>6x</td>
<td>4.5</td>
<td>653.9</td>
</tr>
<tr>
<td>4.6</td>
<td>653.9</td>
<td>6x</td>
<td>4.6</td>
<td>651.5</td>
<td>6</td>
<td>4.7</td>
<td>651.5</td>
</tr>
<tr>
<td>4.8</td>
<td>651.5</td>
<td>6</td>
<td>4.8</td>
<td>651.0</td>
<td>6</td>
<td>4.9</td>
<td>651.0</td>
</tr>
<tr>
<td>5.0</td>
<td>651.0</td>
<td>6</td>
<td>5.0</td>
<td>651.3</td>
<td>7x</td>
<td>5.1</td>
<td>651.3</td>
</tr>
<tr>
<td>5.2</td>
<td>651.3</td>
<td>7x</td>
<td>5.2</td>
<td>648.8</td>
<td>7</td>
<td>5.3</td>
<td>648.8</td>
</tr>
<tr>
<td>5.4</td>
<td>648.8</td>
<td>7</td>
<td>5.4</td>
<td>647.8</td>
<td>7</td>
<td>5.5</td>
<td>647.8</td>
</tr>
<tr>
<td>5.6</td>
<td>647.8</td>
<td>7</td>
<td>5.6</td>
<td>647.7</td>
<td>7</td>
<td>5.7</td>
<td>647.7</td>
</tr>
<tr>
<td>5.8</td>
<td>647.7</td>
<td>7</td>
<td>5.8</td>
<td>647.4</td>
<td>8x</td>
<td>5.9</td>
<td>647.4</td>
</tr>
<tr>
<td>6.0</td>
<td>647.4</td>
<td>8x</td>
<td>6.0</td>
<td>605.0</td>
<td>7x/7y</td>
<td>6.0</td>
<td>616.0</td>
</tr>
</tbody>
</table>