SEISMIC VIBRATOR MODELLING

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ABSTRACT


The wavefield in, and at the surface of, a homogeneous, isotropic, perfectly elastic half-space, excited by a traction distribution at the surface of the medium is investigated. The emitted wavefield is a spatial convolution of the surface tractions and the spatial impulse response. The properties of the wavefield in the far-field of the medium are derived and it is shown that the far-field particle velocity is essentially equal to a weighted sum of the time derivative of the integrated surface tractions, that is, of the components of the ‘ground force’. The theory is valid for an arbitrary geometry and orientation of the surface tractions, and is independent of the boundary conditions at the surface of the medium.

The surface tractions are related to a source that consists of a mass distribution with an arbitrary force distribution imposed upon it. A boundary condition is introduced that accounts for the mass load and the forces applied to it but neglects vibrations within the mass. The boundary condition follows from the equation of motion of the surface mass load.

The theory is applied to the Vibroseis configuration, using a P-wave vibrator model with a uniformly distributed force imposed on top of the baseplate, and assuming that horizontal surface traction components are absent. The distribution of displacement and stress directly underneath the baseplate of a single vibrator and an array of vibrators is investigated. Three different boundary conditions are used: (1) assuming uniform pressure, (2) assuming uniform displacement, (3) using the equation of motion of the baseplate as a boundary condition. The calculations of the distribution of stress and displacement over the plate for different elastic media and several frequencies of operation show that only the results obtained with the mixed boundary condition agree with measurements made in the field.

The accuracy of three different phase-feedback signals is compared using synthetic data. Baseplate velocity phase-feedback leads to huge deviations in the determination of the far-field wavelet; reaction mass acceleration phase-feedback looks stable but neglects the differentiating earth filter; and phase-feedback to a weighted sum of baseplate and reaction mass accelerations becomes unstable with increasing frequency. The instability can be overcome using measurements over the whole baseplate.

The model can be extended to a lossy layered earth.

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INTRODUCTION

Beginning with the famous paper by Lamb (1904), much attention has been given to the properties of the wavefield emitted by a surface source on a homogeneous, isotropic, perfectly elastic half-space. The literature on this subject differs with respect to two model parameters: the source geometry and the boundary conditions at the surface of the medium. The source geometry in the work of Miller and Pursey (1954), Bycroft (1956), Awojobi and Grootenhuis (1965) and Robertson (1966) comprises a single disc. Lamb (1904) and de Hoop (1970) investigated the line source problem. Tan (1985) considered an array of strips and van Onselen (1980, 1982) considered a single rectangular plate. The two different boundary conditions used in these papers are (1) uniform displacement directly underneath the surface source (Awojobi and Grootenhuis 1965; Robertson 1966; Bycroft 1956; van Onselen 1982; Tan 1985), and (2) uniform stress (van Onselen 1980; Miller and Pursey 1954; de Hoop 1970).

This paper extends the theoretical work of van Onselen (1982), who investigated the wavefield emitted by a rigid, rectangular plate due to a normal surface load. The extension to an arbitrary source geometry with an arbitrary distribution of the surface load is one subject of this paper. Also, a boundary condition is considered in which the surface is mass-loaded with a distributed force applied.

First the derivation of the integral representations for the displacement components in or at the surface of the medium is presented. The displacement components are expressed in terms of the surface tractions. In this derivation, neither a description of the source that causes these surface tractions nor a specification of the boundary conditions at the surface of the medium is necessary.

The next section, in which the properties of the wavefield in the far-field of the medium are derived, is also general in that no source description or boundary conditions need to be specified. Then three different boundary conditions that can be imposed on the wavefield at the surface are discussed: uniform stress, uniform displacement and a mixed boundary condition that accounts for a mass load and additional driving forces. When this theory is applied to the Vibroseis technique, the mixed boundary condition is related to a mechanical model of the Vibroseis truck.

Numerical results are shown comparing the two conventional boundary conditions with the mixed boundary condition, and the numerical techniques employed when solving the relevant integral equations are presented.

NOTATIONS AND CONVENTIONS

A right-handed Cartesian coordinate system is used with the $x_3$-axis pointing downwards. The elastic half-space under consideration occupies the region $-\infty < x_1 < \infty$, $-\infty < x_2 < \infty$, $0 \leq x_3 < \infty$. The summation convention is used, i.e. repeated Latin subscripts imply a summation over the values 1, 2 and 3, whereas repeated Greek subscripts imply a summation over the values 1 and 2.
In the computations, a complex time factor $\exp (-i\omega t)$ is used. The partial derivative with respect to $x_i$ is denoted by $\delta_i$, the partial derivative with respect to time $t$ by $\delta_t$.

We use the symmetrical unit tensor of rank two, $\delta_{ij}$:

$$
\delta_{ij} = 1, \quad i = j
$$

$$
\delta_{ij} = 0, \quad i \neq j
$$

and the anti-symmetrical unit tensor of rank three, $\varepsilon_{ijk}$:

$$
\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1,
$$

$$
\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1,
$$

$$
\varepsilon_{ijk} = 0 \quad \text{otherwise.}
$$

Quantities in the time domain are denoted by lowercase letters, quantities in the frequency domain are denoted by the corresponding capitals, and quantities in the wavenumber domain are indicated with a circumflex.

The part of the surface covered by a surface load is denoted by $S$; the remainder of the surface, which is assumed to be stress-free, is denoted by $S'$.

**Basic Relations**

Expressions are derived for the displacement components in or at the surface of the medium (see also Aki and Richards 1980). The configuration is shown in Fig. 1. The starting point is a homogeneous, isotropic, perfectly elastic half-space on the surface of which an arbitrary distribution of traction components is present. These traction components, from which the wave motion in the medium originates, will be referred to as the surface traction components, and will later be connected to a source at the surface of the medium.

To derive the integral equations that relate surface traction components and displacement components, the frequency domain version of the linearized equation of motion is employed. In the absence of body forces, this equation follows from

$$
\delta_j \tau_{ij} + \rho \omega^2 U_i = 0, \tag{1}
$$

in which $\tau_{ij}$ denotes the stress tensor of rank two, $\rho$ the mass density of the medium, $\omega$ the angular frequency and $U_i$ the displacement vector. The components of the stress and strain are connected by means of the stress–strain relation

$$
\tau_{ij} = c_{ijpq} e_{pq}, \tag{2}
$$

For an isotropic, homogeneous medium, the stiffness tensor $c_{ijpq}$ can be written as

$$
c_{ijpq} = \lambda \delta_{ij} \delta_{pq} + \mu (\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp}), \tag{3}
$$

in which $\lambda$ and $\mu$ are Lame's constants. The strain tensor $e_{pq}$ follows from

$$
e_{pq} = \frac{1}{2} (\partial_p U_q + \partial_q U_p). \tag{4}
$$
From these equations we can derive the elastodynamic wave equation
\[(C_p^2 - C_s^2) \partial_i \partial_j U_j + C_s^2 \partial_j \partial_j U_i + \omega^2 U_i = 0,\]  
(5)
in which \(C_p\) and \(C_s\) denote the velocity of the P- and S-waves, respectively.

The elastodynamic wave equation can be separated into a curl-free part (corresponding to the propagation of P-waves) and a divergence-free part (corresponding to the propagation of S-waves). This yields
\[\begin{align*}
\partial_i \partial_i U_j^p + k_p^2 U_j^p &= 0 \quad (\epsilon_{jk} \partial_j U_k^p = 0), \\
\partial_i \partial_i U_j^s + k_s^2 U_j^s &= 0 \quad (\partial_j U_j^s = 0),
\end{align*}\]  
(6)  
(7)
in which \(k_p\) and \(k_s\) denote the P- and S-wavenumber, respectively. In the wavenumber domain, these equations transform to
\[\begin{align*}
\partial_3 \partial_3 \tilde{U}_j^p + k_p^2 \tilde{U}_j^p &= 0; \quad k_p^2 = (k_x^2 - k_y^2 - k_z^2)^{1/2}; \quad \{\text{Re, Im}\} k_p^2 \geq 0 \\
\partial_3 \partial_3 \tilde{U}_j^s + k_s^2 \tilde{U}_j^s &= 0; \quad k_s^2 = (k_x^2 - k_y^2 - k_z^2)^{1/2}; \quad \{\text{Re, Im}\} k_s^2 \geq 0
\end{align*}\]  
(8)  
(9)
Solutions to (8) and (9), which are finite for large \(x_3\), are given by
\[\tilde{U}_j^p = A^p(k_3) \exp (ik_3 x_3),\]  
(10)
\[\tilde{U}_j^s = A^s(k_3) \exp (ik_3 x_3),\]  
(11)
Using the restrictions put upon the P- and S-wave components (cf. (6) and (7)), and substituting the expressions for the particle displacement ((10) and (11)) into the constitutive relation, the amplitude factors can be obtained as a function of the surface traction components. Subsequent transformation to the space-frequency domain finally yields

\[ U_i(x_i) = \int_{-\infty}^{\infty} T_i(x_i') U_{11}^{(G)}(x_i | x_i') \, dx_i', \]  

(12)

where

\[ U_{11}^{(G)}(x_i | x_i') = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{O}_{11,ij} \exp \left( ik_d(x_i - x_i') \right) \, dk_d, \]  

(13)

and where the components of the traction at the surface of the medium are denoted by \( T_i \):

\[ T_i = \tau_{1i}(x_1 = 0). \]  

(14)

From (12) and (13) it can be seen that the displacement in or at the surface of the medium is a spatial convolution of the surface traction components \( T_i \) and the Green's state (the spatial impulse response) \( U_{11}^{(G)} \). Explicit expressions for the Green's state constituents are given in Appendix A. It must be emphasized that, in the derivation of the integral representation (12), no assumptions have been made about the nature of the surface traction distribution. Thus, the theory is valid for an arbitrary configuration of surface loads and allows for horizontal as well as vertical traction components. Equation (12) can be supplemented by the relevant boundary conditions; they need not be specified yet. The integral representation for the particle displacement can also be formulated in terms of polar coordinates. For this purpose, the substitutions

\[ k_1 = k_r \cos(\kappa), \]  

(15)

\[ k_2 = k_r \sin(\kappa), \]  

(16)

and

\[ x_1 - x_1' = r \cos(\theta), \]  

(17)

\[ x_2 - x_2' = r \sin(\theta), \]  

(18)

are made.

The integration with respect to \( \kappa \) can be performed analytically. Details of this computation are given in Appendix A. The result is that the particle displacement in or at the surface of the medium is given by

\[ U_i(x_i) = \int_{-\infty}^{\infty} T_i(x_i') U_{11}^{(G)}(x_i | x_i') \, dx_i', \]  

(19)
where now the Green's state constituents follow from

\[ U_{ij}^{\theta}(x_1'|x_2') = \int_0^\infty \mathcal{O}_{ij}^{\theta}(k, r, x_3, \theta) \, dk. \]  

(20)

Expressions for the Green's state constituents in polar coordinates are given in Appendix A.

Integral representations for the components of the traction in the medium can be obtained by substituting the relations for the particle displacement into the stress–strain relation (2).

**Far-field relations**

The far-field displacement in the medium can be obtained by taking the observation point \( x_1 \) to infinity. The steps in the computations can be summarized as follows:

1. Starting from the integral representation for the particle displacement components in polar coordinates (cf. (20)), the contour for the calculation of the Green's state constituents, corresponding to an integration over all radial wavenumbers, is deformed. The resulting integral no longer suffers from a singularity at the Rayleigh wavenumber; the contribution of the Rayleigh waves now follows from a separate, analytic expression. What remains, then, is to calculate the far-field body wave contribution.

2. The Green's state (the resulting integral over all radial wavenumbers) can be approximately by integrating over the so-called 'path of steepest descent', as the observation point tends to infinity. This contour deformation allows us to obtain an analytic expression for the far-field Green's state constituents.

3. The spatial convolution between the surface tractions and the far-field Green's state constituents can be written in a more convenient form by introducing polar and spherical coordinates for the surface tractions and displacement components, respectively.

The notation used in the following is illustrated in Fig. 2. The final result is that the surface wave contribution in the far-field, expressed in its polar displacement components, is given by

\[ U_r = A_r \frac{\exp (ik_R r)}{r^{1/2}} \int_{\mathcal{S}} T_r \exp (-ik_R r \cos \psi) \, dA \]

\[ + A_{r3} \frac{\exp (ik_R r)}{r^{1/2}} \int_{\mathcal{S}} T_3 \exp (-ik_R r \cos \psi) \, dA, \]  

(21)
\[ U_\theta = A_{30} \frac{\exp \left( i r_0 k_R \right)}{r_0^{1/2}} (i \omega)^{1/2} \iint_S T_\theta \exp \left( -i k_R r_i \cos(\psi) \right) dA, \quad (22) \]

and

\[ U_3 = A_{33} \frac{\exp \left( i r_0 k_R \right)}{r_0^{1/2}} (i \omega)^{1/2} \iint_S T_3 \exp \left( -i k_R r_i \cos(\psi) \right) dA \]

\[ + A_{33} \frac{\exp \left( i r_0 k_R \right)}{r_0^{1/2}} (i \omega)^{1/2} \iint_S T_3 \exp \left( -i k_R r_i \cos(\psi) \right) dA, \quad (23) \]

in which \( A_{ni}, A_{i0}, A_{i3}, \) and \( A_{33} \) are simple amplitude scaling factors (cf. Appendix B), and \( k_R \) represents the Rayleigh wavenumber. The body wave contribution in
the far-field, expressed in its spherical displacement components, follows from

\[
U_r = A_{p2}^R(\theta) \frac{\exp(iR_0k_r)}{R_0} \int_{S} T_r \exp(-ik_s r, \cos(\beta)) \, dA
\]

\[
+ A_{p2}^S(\theta) \frac{\exp(iR_0k_s)}{R_0} \int_{S} T_r \exp(-ik_r r, \cos(\beta)) \, dA
\]

\[
+ A_{s2}^R(\theta) \frac{\exp(iR_0k_r)}{R_0} \int_{S} T_s \exp(-ik_r r, \cos(\beta)) \, dA,
\]

(24)

\[
U_\theta = A_{p3}^R(\theta) \frac{\exp(iR_0k_\theta)}{R_0} \int_{S} T_\theta \exp(-ik_s r, \cos(\beta)) \, dA
\]

\[
+ A_{s3}^R(\theta) \frac{\exp(iR_0k_\theta)}{R_0} \int_{S} T_s \exp(-ik_r r, \cos(\beta)) \, dA,
\]

(25)

and

\[
U_z = A_{s5}^s \frac{\exp(ik_z R_0)}{R_0} \int_{S} T_z \exp(-ik_r r, \cos(\beta)) \, dA,
\]

(26)

where \(A_{p,s}^R(\theta), A_{p,s}^S(\theta), A_{s3}^R(\theta)\) and \(A_{s5}^s\) are frequency-independent directivity functions. Explicit expressions for these directivity functions can be found in Appendix B.

As can be seen from (B13)–(B17) and (21)–(23), the surface waves decay exponentially with depth, and with the reciprocal of the square root of the radial distance from the source. The amplitude increases with the square root of the frequency.

All contributions of the individual surface traction components to the respective components of the far-field particle displacement of the body waves (cf. (24)–(26)) consist of four terms: a trivial phase delay, a 1/R-scaling, a frequency-independent directivity function and a surface integral over the surface traction component, weighted with an exponential factor. As pointed out in Appendix B, this exponential factor in the surface integrals becomes unity when the dimensions of the source are small compared with a wavelength. Then the far-field particle displacement components are given by a weighted sum of the integrated surface traction components, in which the weighting factors are frequency-independent directivity functions. Note that the integrated surface traction components represent the components of the ground force. In the absence of shear stress at the surface of the medium, the radial and tangential components of the far-field particle displacement (corresponding to the propagation of P- and S-waves, respectively) are essentially in phase with the ground force. Thus, the corresponding components of the particle velocity are in phase with the time derivative of the ground force. If shear stresses are present at the surface the situation is more complicated, as can be inferred from (24)–(26).
The boundary conditions at the surface of the medium

Now the configuration where the surface tractions are caused by a mass distribution at the surface, on top of which additional forces may be acting, is considered. In the literature, two boundary conditions have been used in the derivation of the wavefield emitted by such a surface load.

The first boundary condition, assuming that the stress is uniform directly underneath the surface mass, leads to relatively simple expressions for the radiation impedance, but assumes that the surface mass is perfectly flexible.

However, the second boundary condition assumes that the surface mass is perfectly rigid and, therefore, that the displacement is uniform over the mass. This forces the vertical component of the normal stress to become infinite at the edges of the mass, which is clearly unacceptable from a physical point of view.

Field measurements show that neither the traction nor the displacement are uniform under the baseplate of a seismic vibrator (Sallas, paper presented to SEG workshop on data acquisition Monterey, 1985). A model is developed which allows for such a non-uniform distribution of displacement as well as traction. In this model, the equation of motion of the surface mass is used as the boundary condition. The only assumption is that vibrations within the plate can be neglected. Thus, the thickness of the plate is neglected but not its mass.

Consider an elemental area of such a plate of infinitesimal thickness with side lengths \( dx_1 \) and \( dx_2 \), and with a certain input force \( df_i^{\text{input}} \) imposed on it. The input force is used not only to exert a force on the ground but also to accelerate the mass present in this small elemental area. The latter contribution is given by Newton’s second law of motion. This can be expressed as

\[
df_i^{\text{input}} = df_i^{\text{ground}} + \sigma_m \, dx_1 \, dx_2 \, \frac{d^2x_i}{dt^2},
\]

(27)

in which \( df_i^{\text{ground}} \) denotes the components of the force exerted by the plate on the ground, and \( \sigma_m \) is the surface density of the mass.

Since

\[
df_i^{\text{ground}} = -t_i \, dx_1 \, dx_2,
\]

(28)

and since (27) must hold for each elemental area on the baseplate, we can write in general

\[
-t_i^{\text{input}}(x_a) + t_i(x_a) = \sigma_m(x_a) \frac{d^2u_i(x_a)}{dt^2},
\]

(29)

where

\[
df_i^{\text{input}} = -t_i^{\text{input}} \, dx_1 \, dx_2.
\]

(30)

In the frequency domain this yields

\[
-T_i^{\text{input}}(x_a) + T_i(x_a) = -\sigma_m(x_a) \omega^2 U_i(x_a).
\]

(31)
Equation (31), together with the integral representation for the displacement in or at the surface of the medium (cf. (19))

$$U_i(x_i) = \int_S T_{ij}(x_j) U_{j}'(x_i | x_j) \, dx_j,$$

(32)

completely specifies the wave motion in the medium due to a distribution of surface mass loads with an additional driving force imposed on it. The surface integration in (32) only has to be performed over the bounded area $S$ because the presence of the surface loads is restricted to $S$, and because the remainder of the surface is assumed to be stress-free.

Henceforth, the boundary condition (31) is referred to as the mixed boundary condition because both traction and displacement appear in it.

The stress and displacement distributions underneath the baseplate of a seismic vibrator

The distributions of stress and displacement over the baseplate of a seismic vibrator are investigated. This implies that the relation between the input traction in boundary condition (31) and the Vibroseis source has to be specified. Therefore, a description of a mechanical model of the Vibroseis source is given. Then this model is related to the mixed boundary condition, and the approximations made in the modelling as well as the numerical procedures employed in the calculations are explained. Finally, results are shown of the distribution of stress and displacement under the baseplate for three different boundary conditions: uniform traction, uniform displacement and the mixed boundary condition.

Source description

A model of a single vibrator is described. This model can be used to describe the performance of individual vibrators when an array is used as source configuration. In Fig. 3, the well-known mechanical model of the Vibroseis source is shown (Lerwill 1981; Sallan and Weber 1982). The forces $f$ and $f$ are equal, and represent the actuator force as would be measured at the point of application. They are the resulting forces after all effects such as pressure drops and transport delays, inherent in the hydraulic system, are taken into account. The spring between the reaction mass and the baseplate serves to support the reaction mass in its neutral position. The isolated pads by which the holddown mass and the baseplate are connected are represented by the dashpot and the spring.

The generated force $f$ is incorporated in a closed loop feedback system which adjusts the input force in such a way as to cause the selected feedback signal to be in phase with the pilot sweep. This pilot sweep is pre-selected by a sweep generator, and is used in the cross-correlation process.

Lerwill's (1981) numerical values are used for the parameters involved in the model; these are listed in Table 1.
The forces in the suspension are

\[ f_1 = s_1(u_h - u_3), \quad (33) \]

\[ f_2 = s_2(u_h - u_3), \quad (34) \]

\[ f_3 = D \frac{d}{dt}(u_h - u_3), \quad (35) \]

where \( u_h, u_r \) and \( u_3 \) denote the displacement of the holddown mass, the reaction mass and the baseplate, respectively.

**Table 1. Mechanical parameters.**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Quantity</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1/s_1 )</td>
<td>( 1.6 \times 10^{-6} )</td>
<td>Reaction mass suspension compliance</td>
<td>([m/N])</td>
</tr>
<tr>
<td>( 1/s_2 )</td>
<td>( 2.6 \times 10^{-7} )</td>
<td>Holddown isolator's total compliance</td>
<td>([m/N])</td>
</tr>
<tr>
<td>( M_p )</td>
<td>681</td>
<td>Baseplate mass</td>
<td>([kg])</td>
</tr>
<tr>
<td>( M_r )</td>
<td>1773</td>
<td>Reaction mass</td>
<td>([kg])</td>
</tr>
<tr>
<td>( M_h )</td>
<td>15700</td>
<td>Holddown mass</td>
<td>([kg])</td>
</tr>
<tr>
<td>( f )</td>
<td>56000</td>
<td>Driving force</td>
<td>([N])</td>
</tr>
<tr>
<td>( 1/D )</td>
<td>( 10^{-3} )</td>
<td>Conductance of isolator damping</td>
<td>([m/N/s])</td>
</tr>
</tbody>
</table>
The equations of motion of the holddown mass and the reaction mass are

\[-f_1 - f = M_r \frac{d^2 u_r}{dt^2}, \tag{36}\]

\[-f_2 - f_3 = M_s \frac{d^2 u_s}{dt^2}, \tag{37}\]

The equation of motion of the baseplate is usually written as (Lerwill 1981; Sallas 1984; Sallas and Webber 1982)

\[f_1 + f_2 + f_3 + f = \int_A \int_A \sigma_m(x_s) \frac{d^2 u_s(x_s)}{dt^2} \, dx_s, \tag{38}\]

where \(A\) denotes the baseplate area, i.e. the integrated form of (29).

When the mixed boundary condition is used we have

\[f_1 + f_2 + f_3 + f = -\int_A t_{3 \text{input}}(x_s) \, dx_s. \tag{39}\]

If we assume that the applied force is distributed uniformly over the baseplate, we obtain

\[t_{3 \text{input}} = -(f + f_1 + f_2 + f_3)/A_{\text{plate}}, \tag{40}\]

where \(A_{\text{plate}}\) denotes the surface area covered by the baseplate. The mixed boundary condition thus yields (cf. (29))

\[-t_{3 \text{input}} + t_3(x_s) = \sigma_m(x_s) \frac{d^2 u_s(x_s)}{dt^2}, \tag{41}\]

where the input traction is defined by (40).

Although it is not known what the distribution of the driving force over the plate would be when its thickness is neglected, the assumption of a uniformly distributed driving force over the plate is reasonable since the baseplate is stiff. Note that it is not assumed that the plate is rigid; the assumption of a stiff plate leads to a non-uniform displacement distribution over the plate, as will be shown later. The extra mass in the centre of the baseplate, due to the connection of the hydraulic piston to this centre area, was accounted for by assigning a higher value to the mass density \(\sigma_m\) of the plate of infinitesimal thickness in its centre region. For computational convenience, frictionless contact between earth and vibrator is assumed. Thus, the lateral components of the surface tractions are zero:

\[T_s = 0. \tag{42}\]

Sallas and Weber (1982) described a model of the Vibroseis truck in which hold-down mass influences are neglected. This model is shown in Fig. 4.
Since test computations with both models gave very similar results this simple model was used in the calculations. The model is governed by two equations:

\[ f_1 = s_1(u_r - u_3), \]
\[ -f_1 - f = M_r \frac{d^2u_r}{dt^2}, \]

and the equation of motion of the baseplate,

\[ f_1 + f + \int_A \int_A t_3(x_3) \, dx_3 = \int_A \sigma_m(x_3) \frac{d^2u_3(x_3)}{dt^2} \, dx_3, \]

when assuming uniform displacement or uniform stress. With the mixed boundary condition this equation is written as

\[ -t_3^{\text{input}} + t_3(x_3) = \sigma_m(x_3) \frac{d^2u_3(x_3)}{dt^2}, \]

where now

\[ t_3^{\text{input}} = -(f + f_1)/A_{\text{plate}}. \]

Since it is convenient to work in the frequency domain, a Fourier transformation is performed on (43)-(47). This yields

\[ F_1 = s_1(U_r - U_3), \]
\[ -F_1 - F = -M_r \omega^2 U_r. \]
The two versions of the equation of motion, (45) and (46), become

\[ F_1 + F + \int_A T_3(x_a) \, dA = -\omega^2 \int_A \sigma_m(x_a) U_3(x_a) \, dx_a, \]  

(50)

and

\[ -T_3^{\text{input}} + T_3(x_a) = -\sigma_m(x_a) \omega^2 U_3(x_a), \]  

(51)

respectively. Again, the driving force is assumed to be distributed uniformly and horizontal components of the surface tractions are neglected. From (47), (48) and (49) it follows that

\[ T_3^{\text{input}} = -(F + F_1)/A_{\text{plate}} = -\frac{M_r \omega^2 F}{(M_r \omega^2 - s_1)A_{\text{plate}}} + \frac{M_r \omega^2 s_3}{(M_r \omega^2 - s_1)A_{\text{plate}}} U_3(\text{centre}) \]  

(52)

Since the spring between the reaction mass and baseplate is connected to the centre part of the baseplate, the centre value of the baseplate displacement has to be used in (52).

**Numerical Considerations**

1. **Formulation of the problem**

The theory developed in the previous section is applied to a source configuration where \( K \) vibrators are acting on the half-space. The part of the surface occupied by the \( k \)th vibrator is denoted by \( S^k_1 \); displacement, stress and mechanical parameters referring to this vibrator are also indicated by the superscript \( k \). The starting point is the formulation of the integral equations pertaining to the three different boundary conditions. First, the following notations for the input force and displacement are introduced.

From the mechanical model it follows that the total input force of the \( k \)th vibrator \( F_1^{\text{input}, k} \) can be written as

\[ F_1^{\text{input}, k} = -C_1^k F^k + C_2^k U_3^k(x_a^k), \]  

(53)

where

\[ C_1^k = -\frac{M_r^k \omega^2}{M_r^k \omega^2 - s_1^k}, \]  

(54)

\[ C_2^k = C_3^1 s_1^k, \]  

(55)

and where \( x_a^k \) denotes the coordinates of the centre point of the \( k \)th vibrator.

The input traction for the \( k \)th vibrator \( T_3^{\text{input}, k} \) is given by

\[ T_3^{\text{input}, k} = \frac{C_1^k}{A^k} F^k - \frac{C_2^k}{A^k} U_3^k(x_a^k), \]  

(56)

where \( A^k \) denotes the surface area of the \( k \)th vibrator.
The relation between displacement and stress (namely (19)) is written as

\[
U^k\left(x_a\right) = \sum_{i=1}^{K} \int_{S_t} T_{ij}^{k}(x_{a}) U_{ij}^{k, 1}\left(x_{a}, x_{i}\right) \, dx_{i},
\]

(57)

where the suffixes \(k\) and \(i\) for the Green's state indicate that \(x_{a}\) and \(x_{i}\) are situated on the \(k\)th and the \(i\)th vibrator, respectively.

**Boundary conditions**

1. If the stress is uniform underneath the plate, the equation of motion of the \(k\)th baseplate is (cf. (50))

\[
A_k T_{ij}^{k} + C_k^2 U_{ij}^{k}(x_{a} - x_{i}) + \omega^2 \int_{S_t} \sigma_{ij}(x_{a}) U_{ij}^{k}(x_{a}) \, dx_{a} = C_k^4 F^k.
\]

(58)

Combining (57) and (58) yields

\[
A_k T_{ij}^{k} + C_k^2 \sum_{i=1}^{K} T_{ij}^{i} \int_{S_t} U_{ij}^{k, 1}(x_{a} - x_{i}) \, dx_{i} + \omega^2 \int_{S_t} \sigma_{ij}(x_{a}) \sum_{i=1}^{K} T_{ij}^{i} \int_{S_t} U_{ij}^{k, 1}(x_{a} - x_{i}) \, dx_{i} = C_k^4 F^k, \quad k = 1, 2, \ldots, K.
\]

(59)

Equation (59) represents a set of \(K\) equations with the \(K\) tractions as the unknowns. This set of equations can be solved for \(T_{ij}^{k}\) and the displacement distribution can be obtained by substituting these traction values in (57).

2. In the case of uniform displacement, the equation of motion of the \(k\)th baseplate is (cf. (50))

\[
\int_{S_t} T_{ij}^{k}(x_{a}) \, dx_{a} + C_k^2 U_{ij}^{k} + M_k^2 \omega^2 U_{ij}^{k} = C_k^4 F^k.
\]

(60)

Combining (57) and (60) yields

\[
\int_{S_t} T_{ij}^{k}(x_{a}) \, dx_{a} + [M_k^2 \omega^2 + C_k^2] \sum_{i=1}^{K} \int_{S_t} T_{ij}^{k}(x_{a}) U_{ij}^{k, 1}(x_{a} - x_{i}) \, dx_{i} = C_k^4 F^k
\]

(61)

for all \(x_{a}\) on \(S_t^k\), \(k = 1, 2, \ldots, K\).

This integral equation can be solved by an iterative method described later. The displacement can be calculated by substituting the traction values into (57).

3. When the mixed boundary condition is applied, the equation of motion of the \(k\)th baseplate is (cf. (51))

\[
T_{ij}^{k}(x_{a}) + \frac{C_k^2}{A_k^4} U_{ij}^{k}(x_{a} - x_{i}) + \sigma_{ij}(x_{a}) \omega^2 U_{ij}^{k}(x_{a}) = \frac{C_k^4}{A_k^4} F^k
\]

(62)
Combining (57) and (62) results in
\[ T_3^k(x_a) + \frac{C_3^k}{A^k} \sum_{i=1}^{K} \int_{S_i} T_3^k(x_a^i) U_{3;3}^{G_i k_i}(x_{a^i}^i | x_a^i) \, dx_a^i 
+ \sigma_{a^i}^k(x_{a^i}) \omega^2 \int_{S_i} T_3^k(x_a^i) U_{3;3}^{G_i k_i}(x_{a^i}^i | x_a^i) \, dx_a^i = \frac{C_3^k}{A^k} F^k \] (63)

for all \( x_a \) on \( S_k^k, k = 1, 2, \ldots, K. \)

This integral equation can be solved by an iterative technique and the displacement distribution can be found by substituting the stress distribution into (57).

Note that all cases can be given in a general way in operator form:
\[ L[T_3^k](x_a) = C_3^k F^k. \] (64)

**Analysis**

The analysis can be divided into three distinct problems: (A) the evaluation of the Green's state; (B) the evaluation of the surface integral; and (C) the solution of an equation that has the general form of (64).

A. Calculation of the Green's state

The Green's state follows from
\[ U_{3;3}^{G}(x_a | x_a^i) = \frac{k_s^2}{2\pi i \mu} \int_{0}^{\infty} k_s \gamma_3^s J_0(k_s r_a) / \Delta(k_s) \, dk_s. \] (65)

If the substitutions
\[ k_s = s \omega \] (66)
and
\[ \Omega = \omega r \] (67)
are made, (65) becomes
\[ U_{3;3}^{G}(x_a | x_a^i) = -\frac{\omega}{2\pi} \frac{1}{C_3^s} \int_{0}^{\infty} \gamma_3^s J_0(\Omega s) / \Delta_\omega(s) \, ds, \] (68)

where the scaled vertical P- and S-wavenumbers, and the scaled Rayleigh denominator are introduced:
\[ \gamma_3^s = (1/C_3^s - s^2)^{1/2}, \] (69)
\[ \Delta_\omega(s) = (1/C_3^s - 2s^2)^2 + 4s^2 \gamma_3^s \gamma_3^s \] (70)

In the following, the integral in (68) is denoted as the Green's integral.
It follows from (68) that there is no difference in calculating the Green's integral for different values of \( r \) and calculating it for different frequencies. Hence, if the Green's integral is calculated for a whole range of \( \Omega \)-values, the solution to the integral equation can be obtained for all desired frequencies and for any desired source geometry by choosing the appropriate \( \Omega \)-values in the subsequent integration over the surface area.

We decide on the maximum frequency of interest and obtain the maximum distance between two points in the vibrator array (i.e. the maximum value of \( r \)) from the chosen source geometry. This gives the maximum value of \( \Omega \) for which the Green's integral has to be calculated. The \( \Omega \)-sampling is determined by the desired spatial sampling and the lowest frequency of interest. The problem of calculating the Green's function for all values of \( r \) and \( \omega \) is thus reduced to the calculation of the Green's function for a finite number of \( \Omega \)-values. By interpolation in this table, the Green's function can be obtained in an efficient way for every frequency and \( r \)-value of interest. The computation of the Green's integral involves three major steps: (1) extraction of the Rayleigh-pole contribution; (2) elimination of square-root singularities in the derivative of the integrand by goniometric substitutions; and (3) acceleration of the convergence of the integral by subtracting the asymptotic expansion of the integrand.

For details on the numerical procedure see Appendix C.

B. The surface integral

The only problem encountered in the evaluation of the surface integral was the non-convergence of the Green's state when \( r \) is equal to zero. This is due to a \( 1/\Omega \) term that emerges from the evaluation of the Green's state. The solution to this problem is given by Herman (1981) (Appendix C).

C. Solving the integral equation

The integral equation is written in operator form, as already described (cf. (64)):

\[
L[T^3_\omega](x_\omega) = C^3_\omega F^\omega.
\]

The error function after \( n \) iterations \( F^{n,k}(x_\omega) \) is defined as

\[
F^{n,k}(x_\omega) = L[T^{n,k}_3](x_\omega) - C^3_\omega F^\omega,
\]

where \( T^{n,k}_3 \) is the estimate of \( T^3_3 \) after the \( n \)th iteration, and the integrated square error after \( n \) iterations \( E^n \) as

\[
E^n = \sum_{k=1}^{K} \int_{S_\omega} |F^{n,k}(x_\omega)|^2 \, dx_\omega.
\]

The integrated square error is minimized in a least-squares sense by an iterative procedure in which subsequent estimates for the stress distribution are obtained by application of the conjugate gradient method. The method is taken from van den
Berg (1982) who discusses the method in detail. We stopped the iteration process when the integrated square error was less than $10^{-6}$.

**RESULTS**

We give the results obtained with the earth–vibrator model described. Three distinct issues are investigated:

a. First, the distribution of stress and displacement directly underneath the baseplate of the seismic vibrator is calculated using three different boundary conditions: uniform stress, uniform displacement and the mixed boundary condition. Subsequently, only the mixed boundary condition is used in the model to calculate:

b. the performance of a single vibrator compared with an array of vibrators, and

c. the performance of different phase-feedback signals presently in use compared with the true far-field wavelet in the model; i.e. the time derivative of the integrated surface traction (the time derivative of the ground force).

a. The distributions of stress and displacement

The distributions of stress and displacement over a rectangular baseplate of length 1.5 m and width 1 m are calculated. The baseplate is divided into $15 \times 10$ equal samples. Because of the presence of the hydraulic piston in the centre area, the nine centre samples were given a mass density of 850 kg/m$^2$, whereas the remainder of the plate was given a mass density of 450 kg/m$^2$. The earth model used in the calculations has a P-wave velocity of 400 m/s, an S-wave velocity of 150 m/s and a density of 1700 kg/m$^3$. These velocities are quite low, and lead to a Rayleigh wavelength that is comparable to the baseplate dimensions at 80 Hz. The effect of an increase in velocity is shown later. The results obtained with this model are given in Fig. 5, where data are shown for a line parallel to the plate along its centre line. The frequency of operation is 80 Hz. Figure 5a shows the modulus of the stress. As expected, the stress goes up at the edges of the plate when we assume uniform displacement. Using the mixed boundary condition results in a non-uniform stress distribution. From Fig. 5b, it can be seen that the phase of the stress differs by as much as 80° over the baseplate using the mixed boundary condition. From Figs 5a and 5b it is clear that the results using uniform displacement and the mixed boundary condition show little resemblance. The increase in modulus and phase of the traction in the centre region of the baseplate when using the mixed boundary condition is caused by the higher mass density assigned to this area.

Figures 5c and 5d show the modulus and phase of the displacement underneath the baseplate. Again, the mixed boundary condition leads to a non-uniform distribution. The assumption of uniform stress resembles the results with the mixed boundary condition more closely than the assumption of uniform displacement.

Figures 6a–d show distributions of stress and displacement along a line at the edge of, and parallel to, the plate. Again, the mixed boundary condition leads to non-uniform behaviour of both stress and displacement, although the variations are less pronounced than in Fig. 5. The resemblance between the results obtained with the uniform traction assumption and the mixed boundary condition is better than in
Fig. 5. The distribution of traction and displacement underneath the single vibrator, as a function of plate position and for three different boundary conditions. The frequency of operation is 80 Hz, data are shown along the centreline of the vibrator. The earth model has a P-wave velocity of 400 m/s, an S-wave velocity of 150 m/s and a density of 1700 kg/m$^3$. 

(-----) uniform stress; (-----) mixed boundary condition; (---) uniform displacement.

Fig. 5b. Phase of the traction.
Fig. 5c. Modulus of the displacement.

Fig. 5. If the results shown in Figs 5 and 6 using uniform displacement as a boundary condition are compared, it can be seen that along the entire edge of the plate the stress is greater than at the centre line of the plate.

b. Interaction

The response of an array of two identical vibrators, vibrating simultaneously on the
earth model used in section a, is calculated. Also, the geometry and mechanical parameters of each individual vibrator are the same as used in previous plots. The distributions of stress and displacement on the adjacent vibrator, situated on the right of the single vibrator (seen from the front), are not shown but can easily be deduced from symmetry considerations. In all calculations the mixed boundary condition is used.
The response of the vibrator array is compared with the performance of a single vibrator by showing the percentage difference between them. For example, if the modulus of the displacement is shown, $\Delta U_3(x_d)$ is plotted, where

$$\Delta U_3(x_d) = \frac{|U_3^{\text{array}}(x_d)| - |U_3^{\text{single}}(x_d)|}{|U_3^{\text{single}}(x_d)|} \times 100\%.$$ \hspace{1cm} (73)

The superscripts 'array' and 'single' refer to the array response and the response of a single vibrator, respectively.
FIG. 7a. Modulus of the traction.

FIG. 7. Plot of the percentage difference (cf. (73)) between the array response and the response of a single vibrator. Data are shown for one vibrator only (along its centre line), for different values of $L/\lambda_R$. The spacing between the two vibrators is 2 m. (----) $L/\lambda_R = 0.25$; (-----) $L/\lambda_R = 0.50$; (---) $L/\lambda_R = 0.75$; (-----) $L/\lambda_R = 1$.

Figure 7a shows the percentage difference in the modulus of the traction as a function of the position of the plate. $x_x$ is varying along the centre line of the baseplate. Results are shown for four different frequencies. The frequencies are chosen such that $L/\lambda_R$ (the ratio of centre line to centre line distance between the two plates.
and the Rayleigh wavelength) increases from 0.25 to 1. The spacing between the two vibrators was kept constant (2 m), which corresponds to a centre line to centre line distance \( L \) of 3.5 m. The highest percentage difference occurs when \( L \) equals \( \lambda_R \) but does not exceed 10%. The difference in the phase of the traction, shown in Fig. 7b, is even less significant: it lies between \(-1\%\) and \(+1\%\). Only the modulus of the displacement (Fig. 7c) shows differences that exceed 10\%: when \( L \) equals \( \lambda_R \) the difference varies between 20\% and 30\%.
When the frequency of operation is increased, the interaction becomes more important at certain values of $L/\lambda_R$. This is illustrated in Fig. 8, where the ratio $L/\lambda_R$ varies from 1.25 to 2. Again, the percentage difference in the modulus and phase of the traction (Figs 8a and 8b) is modest, but the modulus and phase of the displacement (Figs 8c and 8d) are strongly affected by the presence of a second vibrator, especially at the edges of the plate.
c. Phase-feedback signals

The performance of three phase-feedback signals presently in use is investigated using the mixed boundary condition: baseplate velocity, reaction mass acceleration, and the weighted sum method. The shape of the amplitude spectrum used for the various phase-feedback signals is flat. The phase-feedback signals are compared to the true far-field wavelet. In our model, the true far-field wavelet is equal to the time
derivative of the integrated surface stress (the ground force). In practice, it may be more appropriate to control the ground force itself instead of its time derivative, for the following reasons. First, vibrators are operated as close to the maximum hold-down weight as possible without decoupling. Second, the time derivative can be handled in processing. Third, controlling the time derivative leads to a reduced output at higher frequencies. Because of absorption, we need all the high frequency energy we can get. In the model, however, the time derivative is included in the true far-field wavelet, as well as in the weighted sum method.

Figure 9 shows the phase difference between the three phase-feedback signals and the true far-field signal (i.e., the time derivative of the integrated surface stress), as a function of frequency. The earth model is the same as in the previous sections. All figures refer to a single rectangular baseplate 1.5 × 1 m. The range in which the phase error lies when using the baseplate velocity is well over 360°, the error with the weighted sum method is negligible at low frequencies but increases towards the higher frequencies, and the results with the acceleration of the reaction mass as a phase feedback signal show a fairly stable phase error of 90°. The influence of these phase errors on the cross-correlation functions of the three different phase-feedback signals with the far-field wavelet is calculated and compared with the autocorrelation of the sweep. A linear 5 s upsweep ranging from 10 to 90 Hz was used. The results are shown in Fig. 10. Figure 10a shows the far-field cross-correlation when

![Figure 9](image_url)

**Fig. 9.** The phase difference between three phase-feedback signals and the true time derivative of the ground force, as a function of frequency. The earth model is the same as in Fig. 5. (— — —) Time derivative of the weighted sum; (— — —) baseplate velocity; (— — —) reaction mass acceleration.
Fig. 10a. The cross-correlation using baseplate velocity as a phase-feedback signal.

Fig. 10. Far-field particle velocity cross-correlation functions using three different phase-feedback signals. A 5 s, 10-90 Hz upsweep was used in the calculations.

Fig. 10b. The cross-correlation using the time derivative of the weighted sum as a phase-feedback signal.
the baseplate velocity is used as a phase-feedback signal. If the result is compared with the autocorrelation of the sweep, Fig. 10d, it is obvious that using the velocity of the baseplate as a phase-feedback signal leads to unsatisfactory results. The main peak is not at zero time and the side lobes are too strong. Figure 10b shows the far-field cross-correlation using the weighted sum method as a phase-feedback signal. It can be seen that, although the result is not perfect, the cross-correlation
function matches the autocorrelation of the sweep (Fig. 10d) quite closely. The cross-correlation shown in Fig. 10c, where the acceleration of the reaction mass is used as a phase-feedback signal, is approximately a differentiated version of the autocorrelation function.

The results can be explained as follows. All three methods neglect the non-uniformity of both traction and displacement. All the methods use the concept of the baseplate velocity and the pressure underneath the plate. It has been shown that both these quantities are not constant but vary over the baseplate area. The errors specific for the individual phase-feedback signals will now be discussed.

Using baseplate velocity as a phase-feedback signal assumes that the particle velocity directly underneath the plate is in phase with the velocity of the baseplate. Then it is implicitly assumed that the far-field particle velocity is in phase with the near-field particle velocity. However, the far-field particle velocity is in phase with the time derivative of the near-field stress. As seen in Figs 5 and 6, stress and displacement are not in phase with each other directly underneath the plate. In conclusion, using baseplate velocity as a phase-feedback signal gives rise to the following errors: (1) it neglects the phase difference between stress and particle velocity in the near-field; (2) it neglects the differentiating earth filter; and (3) it neglects the non-uniformity of stress and displacement directly underneath the baseplate.

Using reaction mass acceleration as a phase-feedback signal assumes that the acceleration of the reaction mass is a measure of the force applied to the baseplate. It is then assumed that this driving force is in phase with the stress directly underneath the plate, and that the far-field particle velocity is in phase with the near-field stress. However, the driving force is used not only to exert a force on the ground but also to accelerate the baseplate. Furthermore, the method fails to consider that the near-field stress is differentiated when travelling to the far-field. In conclusion, using the acceleration of the reaction mass gives rise to the following errors: (1) it neglects the inertia of the baseplate; (2) it neglects the differentiating earth filter; and (3) it neglects the non-uniformity of stress directly underneath the baseplate.

The signal looks reasonably stable, but this is probably because vibrations in the plate are neglected in the model. We think that including these vibrations will degrade the performance of the reaction mass phase-feedback signal, because then the reaction mass will no longer give a stable indication of the force applied to the plate. The issue deserves further attention but has not yet been included in this project.

The weighted sum method for the phase-feedback signal is based on the mechanical model of the Vibroseis truck, as described by (48)-(50). If (49) and (50) are added, it follows that the integrated surface traction equals a weighted sum of the baseplate and the reaction mass accelerations, in which the weighting factors are the mass of the baseplate and the reaction mass, respectively. This method, however, neglects the non-uniformity of the displacement over the baseplate.

It must be realized that the cross-correlation functions shown in Fig. 10 are for one particular vibrator model and source geometry, for one specific earth model and for a certain frequency bandwidth. In fact, the result depends on all these quantities. The discussion of different vibrator models is beyond the scope of this paper but the influence of the earth model and the frequency bandwidth was investigated.
The influence of the earth model is illustrated in Fig. 11, where the cross-correlation function is shown using baseplate velocity as a phase-feedback signal and with an earth model with a P-wave velocity of 1000 m/s and an S-wave velocity of 400 m/s. All other parameters were kept constant. It was found that the radiation impedance was mainly affected by the shear-wave velocity of the medium. Using a higher shear-wave velocity results in a much less fluctuating radiation impedance, thus minimizing the phase error. This explains why the new cross-correlation function in Fig. 11 looks better than the one in Fig. 10a.

In Fig. 12, the influence of the frequency bandwidth is shown. A sweep ranging from 10 to 200 Hz was used in the calculations. All other parameters are the same as used in the calculation of Fig. 10. Figure 12a shows the cross-correlation function using the weighted sum method as a phase-feedback signal. The autocorrelation function is shown in Fig. 12b. Comparing Fig. 12a with Fig. 10b, it follows that the performance of the weighted sum method gets worse as the frequency increases because the non-uniformity of the displacement becomes more severe at higher frequencies.

**Extensions**

The earth model can be improved by simulating absorption in the medium by the introduction of complex velocities (Aki and Richards 1980, equation 5.88), and calculating the response of a layered elastic medium.
Fig. 12a. The cross-correlation function.

Fig. 12. The far-field particle velocity cross-correlation function using the time derivative of the weighted sum as a phase-feedback signal, with the same earth model as in Fig. 10, but with different sweep parameters: a 5 s, 10–200 Hz up-sweep was used.

Fig. 12b. The autocorrelation of the sweep.
The vibrator model can be improved by considering the baseplate as an elastic body, with the same dimensions and elastic constants as the baseplate, on which a driving force is applied. The plate vibrations could, alternatively, be included using finite-difference techniques or advanced mechanical models of the baseplate.

Conclusions

A theory has been developed to describe the wavefield emitted by a distribution of surface tractions on a homogeneous, isotropic, perfectly elastic half-space.

The theory is used to describe the wavefield in the far-field of the medium. The components of the far-field particle displacement are a weighted sum of the components of the ground force. Therefore, the far-field particle velocity components are in phase with the time derivative of this weighted sum of ground force components. The weighting factors are independent of frequency. The far-field analysis is independent of the boundary conditions that are applied at the surface of the medium. The source that causes the surface tractions consists of a mass load with an additional driving force imposed on it. The source and the earth model are connected by a new ('mixed') boundary condition, which accounts for the inertia of the baseplate and the presence of an additional driving force distribution.

The theory is applied to the Vibroseis configuration. The mixed boundary condition is compared with two conventional boundary conditions, i.e. uniform stress and uniform displacement. Only the mixed boundary condition shows the non-uniform distribution of both stress and displacement over the baseplate, which is present in field measurements. Application of the mixed boundary condition to a two-plate configuration shows that interaction between vibrators depends on the Rayleigh wavelength, and influences baseplate velocity more than the ground force. In the loss-free medium we investigated, interaction is not a typical low-frequency phenomenon. The mixed boundary condition is used to simulate the far-field cross-correlation functions using three different phase-feedback signals presently in use: baseplate velocity, reaction mass acceleration, and the weighted sum method. All three methods neglect the non-uniform behaviour of both stress and displacement. This implies that the performance of all the methods depends on the earth model as well as on the frequency of operation.

Apart from the non-uniform behaviour of stress and displacement, there are other reasons for the dependence of the far-field cross-correlation functions on the medium parameters and the frequency. Using baseplate velocity as a phase-feedback signal neglects the differentiating earth filter and the phase difference between stress and displacement in the near-field. Using reaction mass acceleration as a phase-feedback signal, however, neglects the differentiating earth filter and the inertia of the baseplate. In our model, the true far-field wavelet can only be obtained by differentiating the total force exerted by the plate on the ground. For practical considerations it is desirable to account for the differentiation at the processing stage rather than during the data acquisition. In practice, the measurement of the exact ground force is a non-trivial matter that requires measurements over the whole baseplate.
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Appendix A. Derivation of the Integral Representation for the Displacement

Using the restrictions put upon the P- and S-wave components, (6) and (7), and defining

$$A^s_1(k_s) = k_s^s \Lambda, \quad k_s^s = k_s,$$

we can express the amplitude factors in terms of the surface tractions, and obtain

$$\Lambda = \frac{2k_1(k_1^2 - k_2^2)}{k_3^s i \mu \Delta} \hat{T}_1 + \frac{2k_2(k_2^2 - k_3^2)}{k_3^s i \mu \Delta} \hat{T}_2 + \frac{(k_3^2 - 2k_2^2)}{i \mu \Delta} \hat{T}_3,$$  \quad (A2)

$$A_1^s = \frac{(k_1^2 - 2k_2^2)(k_1^2 - k_2^2 - 2k_3^2) + 4k_2^2 k_3^2 k_3^s}{k_3^s i \mu \Delta} \hat{T}_1 + \frac{k_1 k_2 (k_3^2 - 2k_2^2) - 4k_2^2 k_3^s}{k_3^s i \mu \Delta} \hat{T}_2 - \frac{2k_1 k_3^s k_3^s}{i \mu \Delta} \hat{T}_3,$$  \quad (A3)

$$A_2^s = \frac{k_1 k_2 ((k_3^2 - 2k_2^2) - 4k_3^2 k_3^s)}{k_3^s i \mu \Delta} \hat{T}_1 + \frac{(k_1^2 - 2k_2^2)(k_1^2 - k_2^2 - k_3^2) + 4k_2^2 k_3^2 k_3^s}{k_3^s i \mu \Delta} \hat{T}_2 - \frac{2k_2 k_3^s k_3^s}{i \mu \Delta} \hat{T}_3,$$  \quad (A4)

and

$$A_3^s = \frac{-k_1 (k_1^2 - 2k_2^2)}{i \mu \Delta} \hat{T}_1 + \frac{-k_2 (k_2^2 - 2k_3^2)}{i \mu \Delta} \hat{T}_2 + \frac{2k_2 k_3^s}{i \mu \Delta} \hat{T}_3,$$  \quad (A5)

in which the Rayleigh denominator $\Delta(k_s)$ is given by

$$\Delta(k_s) = (k_3^2 - 2k_2^2)^2 + 4k_2^2 k_3^2 k_3^s.$$  \quad (A6)

From (A2)–(A5), we can write

$$\Lambda = F \hat{T}_1$$  \quad (A7)

and

$$A^s_i = G_{ij} \hat{T}_j.$$  \quad (A8)

Substitution of these relations in the expression for the total particle displacement

$$\hat{U} = \hat{U}^P + \hat{U}^s,$$

where $\hat{U}^P$ and $\hat{U}^s$ follow from (10) and (11), yields

$$\hat{U} = \{k_1^s F_j \exp (ik_3^s x_j) + G_{ij} \exp (ik_3 x_j) \hat{T}_j,$$  \quad (A10)
or, alternatively,
\[ \mathcal{U}_i = \mathcal{U}_{ij} \mathcal{T}_j. \]  
(A11)

From this equation, it follows that the displacement is a multiplication in the frequency–wavenumber domain of the Green’s state and the surface traction components. This results in a spatial convolution in the space–frequency domain.

Using the transformation
\[ \mathcal{T}_j(k_a) = \int_{\infty}^{0} T_j(x_a) \exp(-ik_a x_a) \, dx_a, \]  
(A12)
the following expression is obtained for the displacement components in or at the surface of the medium:
\[ U_i(x_0) = \int_{\infty}^{0} T_i(x_0) U_{ij}^{\Omega} U_j(x_0) \, dx_0', \]  
(A13)

where
\[ U_{ij}^{\Omega}(x_a | x_0') = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} U_{ij}^{\Omega} \exp(i k_a (x_a - x_0')) \, dk_a. \]  
(A14)

Analytic expressions for \( U_{ij}^{\Omega} \) can easily be inferred from (A2)–(A11).

In our calculations (A14) was not used, but the substitutions
\[ k_1 = k_r \cos(\kappa), \]  
(A15)
\[ k_2 = k_r \sin(\kappa), \]  
(A16)
and
\[ x_1 - x_1' = r \cos(\theta), \]  
(A17)
\[ x_2 - x_2' = r \sin(\theta), \]  
(A18)
were made.

The Green’s state constituents follow from an integration over \( k_r \) and over \( \kappa \).

The integration with respect to \( \kappa \) can be performed analytically, so that the Green’s state can be obtained by a single integration over the radial wavenumber \( k_r \). This simplifies the numerical evaluation of the Green’s state.

Introducing the functions \( F_i \):
\[ F_1 = k_r^3 k_\Omega^3/(i\Delta), \]  
(A19)
\[ F_2 = k_r^3 (k_r^2 - 2k_\Omega^2)/(i\Delta), \]  
(A20)
\[ F_3 = -\frac{1}{2} k_r^3 [(k_r^2 - 2k_\Omega^2) - 4k_r \kappa k_\Omega^3]/(k_r^3 i\Delta), \]  
(A21)
\[ F_4 = k_r^3 (k_r^2 - 2k_\Omega^2)/\Delta, \]  
(A22)
\[ F_5 = 2k_r^3 k_\Omega^3/\Delta, \]  
(A23)
\[ F_6 = k_r^3 (k_r^2 - 2k_\Omega^2)/(i\Delta) \]  
(A24)
and
\[ F_0 = 2k_0^3 k_0^3 (i \Delta), \]  
(A25)

and writing
\[ J^+ (r_k, \theta) = J_0 (r_k) \pm \cos (2\theta) J_2 (r_k), \]  
(A26)

the final result after performing the integration over \( \xi \) is that the displacement components are given by
\[ U_i(x_i) = \int_{-\infty}^{\infty} T_i (x'_i) U^{(j)}_{i, j} (x_i, x'_i) \, dx'_i, \]  
(A27)
\[ U^{(g)}_{i, j} (x_i, x'_i) = \frac{1}{2\pi i \mu} \int_{0}^{\infty} \tilde{U}^{(g)}_{i, j} (k, r, x_i, x'_i, \theta) \, dk, \]  
(A28)

and the components of the Green's state \( U^{(g)}_{i, j}, \tilde{U}^{(g)}_{i, j} \), are given by
\[ \tilde{U}^{(g)}_{1, 1} (k, r, x_i, \theta) = F_1 J^+ (r_k, \theta) \exp (i k_0^3 x_i), \]  
(A29)
\[ + [F_2 J_0 (r_k) + F_3 J^- (r_k, \theta)] \exp (i k_0^3 x_i), \]
\[ \tilde{U}^{(g)}_{1, 2} (k, r, x_i, \theta) = [-F_1 \exp (i k_0^3 x_i) + F_3 \exp (i k_0^3 x_i)] \sin (2\theta) J_2 (r_k), \]  
(A30)
\[ \tilde{U}^{(g)}_{1, 3} (k, r, x_i, \theta) = [F_4 \exp (i k_0^3 x_i) - F_5 \exp (i k_0^3 x_i)] \cos (\theta) J_1 (r_k), \]  
(A31)
\[ \tilde{U}^{(g)}_{2, 1} (k, r, x_i, \theta) = [-F_1 \exp (i k_0^3 x_i) + F_3 \exp (i k_0^3 x_i)] \sin (2\theta) J_2 (r_k), \]  
(A32)
\[ \tilde{U}^{(g)}_{2, 2} (k, r, x_i, \theta) = F_1 J^+ (r_k, x_i, \theta) \exp (i k_0^3 x_i), \]  
(A33)
\[ + [F_2 J_0 (r_k) + F_3 J^- (r_k, \theta)] \exp (i k_0^3 x_i), \]
\[ \tilde{U}^{(g)}_{2, 3} (k, r, x_i, \theta) = [F_4 \exp (i k_0^3 x_i) - F_5 \exp (i k_0^3 x_i)] \sin (\theta) J_1 (r_k), \]  
(A34)
\[ \tilde{U}^{(g)}_{3, 1} (k, r, x_i, \theta) = [F_4 \exp (i k_0^3 x_i) - F_5 \exp (i k_0^3 x_i)] \cos (\theta) J_1 (r_k), \]  
(A35)
\[ \tilde{U}^{(g)}_{3, 2} (k, r, x_i, \theta) = [F_5 \exp (i k_0^3 x_i) - F_4 \exp (i k_0^3 x_i)] \sin (\theta) J_1 (r_k), \]  
(A36)
and
\[ \tilde{U}^{(g)}_{3, 3} (k, r, x_i, \theta) = [F_6 \exp (i k_0^3 x_i) + F_7 \exp (i k_0^3 x_i)] J_0 (r_k). \]  
(A37)

**Appendix B. Derivation of Far-field Relations**

The starting point for the derivation of far-field properties is the integral representation for the particle displacement components in polar coordinates. The Green's state is separated into a Rayleigh wave contribution and a body wave contribution. The Green's state of the Rayleigh wave contribution is given by a simple analytical expression, whereas the Green's state pertaining to the body wave contribution is given by an integral whose integrand contains a first-order Hankel function. The latter integral is approximated by performing the integration over the so-called path of steepest descent. Finally, the traction and displacement components are expressed in their polar or spherical coordinates. The notation used in this section is shown in Fig. 2.
The coordinate transformations are performed according to the formulae:

\[
\begin{bmatrix}
    f_R \\
    f_\theta \\
    f_\xi
\end{bmatrix}
= \begin{bmatrix}
    \sin (\theta) \cos (\zeta) & \sin (\theta) \sin (\zeta) & \cos (\theta) \\
    \cos (\theta) \cos (\zeta) & \cos (\theta) \sin (\zeta) & -\sin (\theta) \\
    -\sin (\zeta) & \cos (\zeta) & 0
\end{bmatrix}
\begin{bmatrix}
    f_1 \\
    f_2 \\
    f_3
\end{bmatrix}
\] (B1)

and

\[
\begin{bmatrix}
    f_R \\
    f_\theta \\
    f_\xi
\end{bmatrix}
= \begin{bmatrix}
    \cos (\theta) & \sin (\theta) & 0 \\
    -\sin (\theta) & \cos (\theta) & 0 \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    f_1 \\
    f_2 \\
    f_3
\end{bmatrix}
\] (B2)

The introduction of spherical and polar displacement components is only meaningful when they refer to a fixed direction. It can be inferred from Fig. 2 that the variation in the angles \( \theta \) and \( \xi \) are of order \( D/r \) and \( D/R \), respectively, where \( D \) denotes the source dimension. Therefore, we require that

\[ r \gg D \] (B3)

and

\[ R \gg D. \] (B4)

The distances \( R \) and \( r \) are approximated as follows. Using

\[
R^2 = R_0^2 + r_0^2 - 2R_0 \ r \ \cos (\beta), \quad (B5)
\]

\[
r^2 = r_0^2 + r_1^2 - 2r_0 \ r_1 \ \cos (\psi), \quad (B6)
\]
yields

\[
R = R_0 - r, \ \cos (\beta) + O(R_0^{-1}), \quad (B7)
\]

\[
r = r_0 - r, \ \cos (\psi) + O(r_0^{-1}). \quad (B8)
\]

Until now, the position of the origin of our coordinate system was not specified. Choosing the origin in the centre of the source leads to the smallest error, since then the maximum value of \( r \) is minimized.

Furthermore, the error in the far-field approximation for the Vibroseis configuration can be minimized by treating each vibrator separately; then the total far-field wavefield can be obtained by vector summation of all individual contributions, and the far-field relations are independent of the array dimensions.

Now the evaluation of the far-field Green's state will be discussed. The starting point is a contour deformation that separates the surface wave and the body wave contributions. This is achieved by deforming the path of integration in the \( k_z \)-plane (van Onselen 1982). The imaginary part of the vertical wavenumbers is kept positive in the entire plane. This procedure, combined with the application of the asymptotic expansion of the Hankel function (Abramowitz and Stegun 1970, equation 9.2.2),

\[
H_{\nu}^{(1)}(z) = \left[2/(\pi z) \right]^{1/2} \exp \left[i(z - m\pi/2 - \pi/4) \right] + O(z^{-3/2}), \quad (B9)
\]
yields the far-field surface wave contribution expressed in its polar displacement components:

\[
U_r = A_{r} \frac{\exp \left( \frac{ir_0 k_R}{r_0^{1/2}} \right)}{r_0^{1/2}} \int_{S} T_r \exp \left( -ik_r r_r \cos \left( \psi \right) \right) dA \\
+ A_{3r} \frac{\exp \left( \frac{ir_0 k_R}{r_0^{1/2}} \right)}{r_0^{1/2}} \int_{S} T_3 \exp \left( -ik_r r_r \cos \left( \psi \right) \right) dA, \tag{B10}
\]

\[
U_\theta = A_{\theta} \frac{\exp \left( \frac{ir_0 k_R}{r_0^{1/2}} \right)}{r_0^{1/2}} \int_{S} T_\theta \exp \left( -ik_r r_r \cos \left( \psi \right) \right) dA, \tag{B11}
\]

\[
U_3 = A_{3\theta} \frac{\exp \left( \frac{ir_0 k_R}{r_0^{1/2}} \right)}{r_0^{1/2}} \int_{S} T_3 \exp \left( -ik_r r_r \cos \left( \psi \right) \right) dA \\
+ A_{33} \frac{\exp \left( \frac{ir_0 k_R}{r_0^{1/2}} \right)}{r_0^{1/2}} \int_{S} T_3 \exp \left( -ik_r r_r \cos \left( \psi \right) \right) dA. \tag{B12}
\]

\(k_R\) denotes the Rayleigh wavenumber and \(A_r, A_{3r}, A_{\theta}, A_{3\theta}\) and \(A_{33}\) are amplitude scaling factors, whose magnitude is given by

\[
A_{r} = \left( C_R/(2\pi) \right)^{1/2} \Delta_{nr}^{-1}(1/C_R)\left[1/C_R^2 \left(1/C_R^2 - 1/C_p^2\right)^{1/2}\right] \\
\times \exp \left[-\omega(1/C_R^2 - 1/C_p^2)^{1/2} x_3\right] + 2/C_R(1/C_R^2 - 1/C_p^2)^{1/2} \left(1/C_R^2 - 2/C_p^2\right) \\
\times \exp \left[-\omega(1/C_R^2 - 1/C_p^2)^{1/2} x_3\right] / \mu, \tag{B13}
\]

\[
A_{3r} = \left( C_R/(2\pi) \right)^{1/2} \Delta_{nr}^{-1}(1/C_R)\left[1/C_R^2 \left(1/C_R^2 - 2/C_p^2\right)^{1/2}\right] \\
\times \exp \left[-\omega(1/C_R^2 - 1/C_p^2)^{1/2} x_3\right] + 2/C_R(1/C_R^2 - 1/C_p^2)^{1/2} \left(1/C_R^2 - 1/C_p^2\right)^{1/2} \\
\times \exp \left[-\omega(1/C_R^2 - 1/C_p^2)^{1/2} x_3\right] / \mu, \tag{B14}
\]

\[
A_{\theta} = -\left( C_R/(2\pi) \right)^{1/2} \Delta_{nr}^{-1}(1/C_R)\left[1/C_R \Delta_{nr}(1/C_R)\right] \\
\times \exp \left[-\omega(1/C_R^2 - 1/C_p^2)^{1/2} x_3\right] / \left((1/C_R^2 - 1/C_p^2)^{1/2} \mu\right), \tag{B15}
\]

\[
A_{3\theta} = \left( C_R/(2\pi) \right)^{1/2} \Delta_{nr}^{-1}(1/C_R)\left[2/C_R^2(1/C_R^2 - 1/C_p^2)^{1/2} \left(1/C_R^2 - 2/C_p^2\right)\right] \\
\times \exp \left[-\omega(1/C_R^2 - 1/C_p^2)^{1/2} x_3\right] + 2/C_R(1/C_R^2 - 1/C_p^2)^{1/2} \\
\times \exp \left[-\omega(1/C_R^2 - 1/C_p^2)^{1/2} x_3\right] / \mu, \tag{B16}
\]

and

\[
A_{33} = \left( C_R/(2\pi) \right)^{1/2} \Delta_{nr}^{-1}(1/C_R)\left[1/C_R(1/C_R^2 - 1/C_p^2)^{1/2} \left(1/C_R^2 - 2/C_p^2\right)\right] \\
\times \exp \left[-\omega(1/C_R^2 - 1/C_p^2)^{1/2} x_3\right] + 2/C_R(1/C_R^2 - 1/C_p^2)^{1/2} \\
\times \exp \left[-\omega(1/C_R^2 - 1/C_p^2)^{1/2} x_3\right] / \mu. \tag{B17}
\]

The scaled Rayleigh denominator \(\Delta_{nr}(s)\) is defined in (70) and its derivative with respect to \(s\) is denoted by \(\Delta'_{nr}(s)\).
This concludes the discussion of the far-field surface wave contribution. To obtain the contribution of the body waves in the far-field region, the substitutions

$$r = R \sin (\zeta),$$
(B18)

$$x_3 = R \cos (\zeta),$$
(B19)

are made and the relevant integrals are evaluated by using the method of steepest descent. For details on this procedure see Båth and Berkhout (1984), van Onselen (1982) and Miller and Pursey (1954). The body wave contribution is given by

$$U_R = A_{Rd}^S(\theta) \frac{\exp (iR_R k_R)}{R_o} \int \int_S T_r \exp (-ik_r r \cos (\beta)) dA$$

$$+ A_{Rd}^P(\theta) \frac{\exp (iR_R k_R)}{R_o} \int \int_S T_r \exp (-ik_r r \cos (\beta)) dA$$

$$+ A_{Rd}^S(\theta) \frac{\exp (iR_R k_R)}{R_o} \int \int_S T_3 \exp (-ik_r r \cos (\beta)) dA,$$
(B20)

$$U_\phi = A_{\phi d}^S(\theta) \frac{\exp (iR_\phi k_\phi)}{R_o} \int \int_S T_\phi \exp (-ik_\phi r \cos (\beta)) dA$$

$$+ A_{\phi d}^P(\theta) \frac{\exp (iR_\phi k_\phi)}{R_o} \int \int_S T_3 \exp (-ik_\phi r \cos (\beta)) dA,$$
(B21)

and

$$U_\zeta = A_{\zeta d}^S(\theta) \frac{\exp (iR_\zeta k_\zeta)}{R_o} \int \int_S T_\zeta \exp (-ik_\zeta r \cos (\beta)) dA.$$  
(B22)

in which

$$A_{Rd}^S(\theta) = \frac{-1/C_\phi^2 \sin^2 (\theta)(1 - 2 \sin^2 (\theta))}{2\pi \mu \Delta_m (1/C_\phi \sin (\theta))},$$
(B23)

$$A_{Rd}^P(\theta) = \frac{1/C_\phi^2 [1/C_\phi^2 - 1/C_\phi^2 \sin^2 (\theta)]^{1/2}}{2\pi \mu \Delta_m (1/C_\phi \sin (\theta))},$$
(B24)

$$A_{Rd}^P(\theta) = \frac{1/C_\phi^2 \cos (\theta)(1/C_\phi^2 - 2/C_\phi^2 \sin^2 (\theta))}{2\pi \mu \Delta_m (1/C_\phi \sin (\theta))},$$
(B25)

$$A_{Rd}^S(\theta) = \frac{1/C_\phi^4 \cos (\theta)(1 - 2 \sin^2 (\theta))}{2\pi \mu \Delta_m (1/C_\phi \sin (\theta))},$$
(B26)

$$A_{Rd}^P(\theta) = \frac{-1/C_\phi^2 [1/C_\phi^2 - 1/(C_\phi^2 \sin^2 (\theta))]^{1/2}}{2\pi \mu \Delta_m (1/C_\phi \sin (\theta))},$$
(B27)
and
\[ A'^2_{ii} = 1/(2\pi \mu). \]  
(B28)

The exponential factors in the integrals (B20)–(B22) can be neglected provided that
\[ k_{p,s} r, \cos (\beta) \ll 1 \]  
(B29)
or
\[ 2\pi r, \ll \lambda_{p,s}, \]  
(B30)
where \( \lambda_{p,s} \) denotes the P- and S-wavelengths respectively. This condition holds if the far-field evaluation is performed for each vibrator separately, since then the maximum value of \( r_{v} \) for present industrial applications is about 2 m, whereas the wavelengths of seismic interest are in the range of 10–100 m. If the vibrator array is considered as a whole, an additional phase shift is no longer negligible when the array dimensions become large compared to the wavelength. If the influence of the exponential factor can be neglected, it can be seen from (B20)–(B22) that the far-field displacement is essentially a weighted sum of the components of the ground force, where the weighting factors are frequency-independent directivity functions.

**Appendix C. Numerical Computations**

The evaluation of the integral
\[ I = \int_{0}^{\infty} s y_{p} J_{0}(\Omega s) / (A_{w}(s)) \, ds = \int_{0}^{\infty} G(s) J_{0}(\Omega s) \, ds, \]  
(C1)
is discussed. Use has been made of the numerical scheme van Onselen (1982) developed.

Since the integrand in (C1) is of order \( O(s^{-1/2}) \) we have to accelerate the convergence of the integral. Taylor expansion of the \( G \)-function results in
\[ G(s) = \sum_{k=0}^{\infty} C_{2k} s^{-2k}. \]  
(C2)

Subtracting the first term of the asymptotic expansion yields
\[ I = \int_{0}^{\infty} [G(s) - C_{0}] J_{0}(\Omega s) \, ds + C_{0}/\Omega, \]  
(C3)
in which we used Abramowitz and Stegun's (1970) equation 11.4.17. Due to the singularity at \( s = 0 \), the second term of the expansion cannot be subtracted over the whole integration interval, but only from a certain limit \( a \) up to infinity. This yields
\[ I = \int_{a}^{\infty} [G(s) - C_{0}] J_{0}(\Omega s) \, ds + \int_{a}^{\infty} [G(s) - C_{0} - C_{2} s^{-2}] J_{0}(\Omega s) \, ds \]  
\[ + C_{0}/\Omega + C_{2} \int_{a}^{\infty} s^{-2} J_{0}(\Omega s) \, ds. \]  
(C4)
The second integral on the right-hand side of (C4) can be evaluated by realizing that
\[ G(s) - C_0 - C_2 s^{-2} = C_4 s^{-4} + O(s^{-6}). \]  
(C5)

Therefore, neglecting orders \( s^{-6} \) and higher, we have
\[ \int_\alpha^\infty [G(s) - C_0 - C_2 s^{-2}] J_0(\Omega s) \, ds = \int_\alpha^\infty C_4 s^{-4} J_0(\Omega s) \, ds. \]  
(C6)

We can determine \( \alpha \) such that the contribution of the integral (C6) is negligible, i.e.
\[ \int_\alpha^\infty C_4 s^{-4} J_0(\Omega s) \, ds < \epsilon, \]  
(C7)

where \( \epsilon \) is small. A rough estimate for \( \alpha \) can be obtained using the inequality
\[ \left| \int_\alpha^\infty C_4 s^{-4} J_0(\Omega s) \, ds \right| \leq \int_\alpha^\infty |C_4 s^{-4}| |J_0(\Omega s)| \, ds < \int_\alpha^\infty C_4 s^{-4} \, ds = |C_4|/(3\pi^3) < \epsilon, \]  
(C8)

where we have used the knowledge that the value of the Bessel function is always smaller than one, so that
\[ \alpha > \left| \frac{|C_4|}{(3\pi^3)} \right|^{1/3}. \]  
(C9)

The third integral on the right-hand side of (C4) can be evaluated by performing an integration by parts, and by subsequent application of Abramowitz and Stegun's (1970) equation 11.4.16 and 9.1.27. The final result is that (\( \Omega \neq 0 \))
\[ I = \int_0^\alpha G(s) J_0(\Omega s) \, ds + \left[ C_4 \Omega - C_0 / \Omega \right] \int_0^\Omega J_0(t) \, dt + C_0 / \Omega \]
\[ + \frac{C_2 J_0(\Omega \alpha)}{\alpha} - C_2 \alpha [1 + J_1(\Omega \alpha)]. \]  
(C10)

where \( \alpha \) follows from (C9).

The second integral on the right-hand side of (C10) was tabulated using Abramowitz and Stegun's (1970) equation 11.1.16, thus allowing for a very fast evaluation of this integral.

If \( \Omega = 0 \), the result is
\[ I(\Omega = 0) = \int_0^\infty G(s) \, ds + C_0 / \Omega + C_2 / \alpha. \]  
(C11)

The singularity in (C11) is removed in the subsequent surface integration. For this purpose, we recall that we substituted
\[ \Omega = \omega r. \]  
(C12)
Herman (1981) showed that

\[
\int_A \frac{dx}{r} = x_1^+ \ln \left[ \frac{r_2 + x_2^+}{r_1 + x_1^+} \right] + x_2^+ \ln \left[ \frac{r_3 - x_1^+}{r_2 - x_2^+} \right] - x_1^+ \ln \left[ \frac{r_4 - x_2^+}{r_3 - x_3^+} \right] - x_2^+ \ln \left[ \frac{r_1 + x_1^+}{r_4 + x_3^+} \right]
\] (C13)

The notations used in this expression are explained in Fig. 13. The first integral on the right-hand side of (C10) was evaluated using Simpson’s rule. Singularities in the derivative of the integrand were removed by substituting

\[
s^2 = a^2 \cos^2 (\theta) + b^2 \sin^2 (\theta),
\] (C14)

in the integration intervals

\[
[a, b] = [0, 1/C_p],
\]

\[
[a, b] = [1/C_p, 1/C_s]
\]
and

\([a, b] = [1/C_*, x]\),

respectively.

The contribution of the Rayleigh pole was extracted by subtracting from the \(G\)-function its Taylor expansion around the singular point; i.e.

\[
\int_p^q G(s)J_0(2s) \, ds = \int_p^q \frac{N(s)}{\Delta_n(s)} \, ds = \int_p^q \frac{N(s)}{\Delta_n(s)} \left( -\frac{N(1/C_R)}{\Delta_n'(1/C_R)(s - 1/C_R)} \right) \, ds + \frac{N(1/C_R)}{\Delta_n'(1/C_R)} \int_p^q \frac{ds}{s - 1/C_R}, \tag{C15}
\]

where the interval \([p, q]\) contains the singular point \(1/C_R\).

The first integral on the right-hand side of (C15) no longer suffers from a singularity; the second integral can be solved using Cauchy's theorem. This yields

\[
\int_p^q \frac{ds}{s - 1/C_R} = i\pi + \ln \left[ \frac{q - 1/C_R}{1/C_R - p} \right], \tag{C16}
\]

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