Client:
DG Rijkswaterstaat, RIZA

Wave growth limit in shallow water

Analytical study of wave growth curves and SWAN source terms

July 2003
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ABSTRACT:

In this report insight has been gained in the physical mechanisms that are important for the wave growth in shallow water in general and for the limit of that growth in particular. The wave growth in shallow water is only known within a broad margin. Wave models predict relatively mild wave conditions within this margin. The large extent of uncertainty in the predicted wave conditions, that lead to ‘unsafe’ computational data, are of great concern for the determination of the design conditions of sea defences along the IJsselmeer and Markermeer.

To obtain insight in the uncertainties of physical processes modelled in Swan in wave growth limit situation in shallow water, first an overview has been given of the formulations of physical processes, that have been modelled in Swan and their performance in general. By combining this to the relative importance of source terms under wave growth limit situations in shallow water, conclusions have been drawn which physical processes cause the largest amount of uncertainty in wave growth limit situations. Quadruplets and wind contribute significantly to incorrect predictions of wave growth limit situations.

The expected wave height over depth ratio predicted by Swan has been investigated both qualitatively and quantitatively. Firstly, the possibility of determining an equilibrium spectrum from the action balance equation in Swan has been investigated briefly. Secondly, the opposite path was followed in which for a number of spectral shapes has been investigated whether they approach the equilibrium spectrum. From the ‘approximate’ equilibrium spectrum the wave height over depth ratio has been determined, without using the numerics in Swan. The relative wave height is significantly larger than obtained with Swan computations. In Swan probably the presence of the limiter on the action density acts as a dissipative source term and limits the wave growth. If the effect of the limiter were relaxed, higher values for the relative wave height would be obtained.

An attempt has been made to scale the present formulations of the source terms, such that it is applicable for wave growth limit situations and can be applied for various wind speeds, wave conditions and depths. For this purpose the integrated balance equation has been considered.

For a number of available formulations of wave growth curves, estimates for the wave height over depth ratio have been determined in the wave growth limit situation in shallow water.

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Contract no. RI-3806, d.d. 5 November 2002
Revised contract no. RI-3806A, d.d. 21 January 2003
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\[ \begin{align*}
A & \quad \text{coefficient in (linear) wind input source term} \quad \text{m}^2 \\
A_{br} & \quad \text{near-bottom orbital excursion amplitude} \quad \text{m} \\
a & \quad \text{wave amplitude} \quad \text{m} \\
a_0 & \quad \text{typical wave amplitude} \quad \text{m} \\
B & \quad \text{coefficient in (exponential) wind input source term} \quad \text{Hz} \\
b & \quad \text{ratio between rms and maximum wave height} \quad \text{b} = \frac{H_{\text{rms}}}{H_b} \quad - \\
C & \quad \text{factor in D1A approach for quadruplets} \quad - \\
C_0 & \quad \text{equivalent drag coefficient} \quad - \\
C_{\text{bot}} & \quad \text{bottom friction coefficient} \quad \text{m}^2 \text{s}^{-3} \\
C_d & \quad \text{wind drag coefficient} \quad - \\
C_{\text{ds}} & \quad \text{tunable coefficient in white-capping source term} \quad - \\
C_f & \quad \text{bottom friction coefficient in Collins' model} \quad - \\
C_{\text{uc}} & \quad \text{tunable coefficient in white-capping source term} \quad - \\
C_{\text{uc}}^{\text{ss}} & \quad \text{tunable coefficient in cumulative steepness method} \quad - \\
C_z & \quad \text{drag coefficient at height z above water surface} \quad - \\
C_o & \quad \text{cospectra between wind velocity components} \quad \text{m}^2 \text{s}^{-2} \\
c & \quad \text{phase velocity} \quad \text{m} \text{s}^{-1} \\
c_{\text{bore}} & \quad \text{propagation velocity of turbulent bore} \quad \text{m} \text{s}^{-1} \\
c_p & \quad \text{group velocity} \quad \text{m} \text{s}^{-1} \\
D_b & \quad \text{dissipation function due to depth-induced wave breaking} \quad \text{J m}^{-2} \text{s}^{-1} \\
D'_{b} & \quad \text{dissipation function in Baldock et al.'s formulation} \quad \text{J m}^{-2} \text{s}^{-1} \\
D''_{b} & \quad \text{dissipation function using modified Glukhovskiy distribution} \quad \text{J m}^{-2} \text{s}^{-1} \\
D' & \quad \text{dissipation of a turbulent bore} \quad \text{J m}^{-1} \text{s}^{-1} \\
D_f & \quad \text{dissipation function due to bottom friction} \quad \text{J m}^{-2} \text{Hz} \\
E & \quad \text{wave energy density or variance density} \quad \text{J m}^{-2} \text{Hz}^{-1} \quad (\text{m}^2 \text{Hz}^{-1}) \\
\hat{E} & \quad \text{dimensionless wave energy density} \quad - \\
\hat{E} & \quad \text{total wave energy} \quad \text{J m}^{-2} \quad (\text{m}^2) \\
F_c & \quad \text{nonlinearity parameter} \quad - \\
F_r & \quad \text{Fröude number} \quad - \\
f & \quad \text{cyclic frequency} \quad \text{Hz} \\
f_\text{mean} & \quad \text{mean frequency in Battjes-Janssen breaker formulation} \quad \text{Hz} \\
f_m & \quad \text{peak frequency} \quad \text{Hz} \\
f_{m-10} & \quad \text{mean spectral wave frequency} \quad (m_{0/10}) \quad \text{Hz} \\
f_{\text{PM}} & \quad \text{Pierson-Moskowitz frequency} \quad \text{Hz} \\
f_{\text{PM}}^{\text{eq}} & \quad \text{equilibrium peak frequency according to Pierson-Moskowitz} \quad \text{Hz} \\
f_w & \quad \text{friction coefficient} \quad - \\
g & \quad \text{gravity acceleration} \quad \text{m s}^{-2} \\
H & \quad \text{wave height} \quad \text{m} \\
H_b & \quad \text{breaker or maximum wave height} \quad \text{m} \\
H_b & \quad \text{dimensionless maximum wave height} \quad - \\
H_{m0} & \quad \text{significant wave height} \quad (H_{m0} = 4\sqrt{m_0}) \quad \text{m} \\
H_N & \quad \text{wave height with exceedance probability } 1/N \quad \text{m}
\end{align*} \]
\( H_{rms} \)  rms-value of wave height \((H_{rms} = \sqrt{2}H_{m0})\)  
\( H_s \)  significant wave height  
\( h \)  water depth  
\( h_0 \)  typical water depth  
\( \hat{h} \)  dimensionless water depth  
\( h_{wc} \)  height of whitecap  
\( J \)  interaction coefficient in LTA formulation  
\( k \)  wave number  
\( k \)  length of wave number vector  
\( k_0 \)  typical wave number  
\( k \)  dimensionless wave number  
\( k_b \)  equivalent bottom roughness \((k_b = 30\alpha_0)\)  
\( \overline{k} \)  spectrally-averaged wave number  
\( \overline{k}^{(n)} \)  spectrally-averaged wave number  
\( l_2 \)  Euclidian norm for sum of source terms  
\( l_{\infty} \)  maximum norm for sum of source terms  
\( m \)  local bottom slope  
\( m_0 \)  zero-th order spectral moment (or total wave energy)  
\( m_0 \)  dimensionless zero-th order spectral moment  
\( m_n \)  \(n\)-th moment of surface elevation  
\( m_n \)  \(n\)-th moment of frequency spectrum  
\( M_n \)  \(n\)-th moment of wave height probability distribution  
\( N \)  wave action density  
\( p_w \)  pressure exerted on water surface by whitecap  
\( Q_b \)  fraction of breaking or broken waves  
\( Q_b' \)  fraction of breaking waves for method of Baldock et al.  
\( Q_b'' \)  fraction of breaking waves using modifi. Glukhovskyi distr.  
\( R_F \)  stability correction factor to adjust wind speed  
\( S \)  source function, change in wave energy  
\( S_{in} \)  source function for wind input  
\( S' \)  spectrally-integrated source function  
\( S_{bot} \)  source function for bottom friction  
\( S_{tri} \)  source function for triad wave interaction  
\( S_{qua} \)  source function for quadruplet wave interaction  
\( S_{src} \)  source function for white-capping  
\( s_0 \)  deep-water steepness  
\( T_m \)  peak period  
\( T_s \)  significant wave period  
\( t \)  time  
\( t_\chi \)  dimensionless duration belonging to certain fetch \(\chi\)  
\( U \)  depth-averaged current velocity  
\( U_A \)  adjusted windspeed in SPM84 formulation  
\( u^b \)  velocity just outside bottom boundary layer  
\( U_r \)  Ursell number  
\( U_z \)  wind velocity at height \(z\) above water surface  
\( u_* \)  friction velocity at sea surface

---

**Notes:**
- **m** : meter
- **rad** : radian
- **Hz** : Hertz
- **J m^-2 Hz^-2** : Joule per meter square per Hertz square
- **N m** : Newton meter
- **J m^-2 m^2** : Joule per meter squared
- **J m^-2 s** : Joule per second
- **J m^-2 m^2** : Joule per meter squared
- **m s^-1** : meter per second
\( \hat{u}_s \)  
Dimensionless friction velocity at sea surface

\( x \)  
Spatial coordinate

\( z_0 \)  
Roughness height

\( \alpha \)  
Scale factor in Battjes-Janssen formulation

\( \varepsilon \)  
Wave steepness

\( \delta \)  
Deep-water wave steepness

\( \delta_{PM} \)  
Deep-water wave steepness for Pierson-Moskowitz spectrum

\( \beta \)  
Miles' parameter

\( \hat{\beta} \)  
Approximation for the bi-phase in triad formulation

\( \beta_b \)  
Fraction of turbulent bore

\( \beta_Q \)  
Correction factor for the fraction of breaking waves

\( \gamma \)  
Growth rate of waves due to wind in Miles' formulation

\( \gamma_b \)  
Breaker parameter

\( \gamma_r \)  
Ratio between significant wave height and water depth

\( \gamma_1 \)  
Skewness

\( \gamma_d \)  
Deep-water steepness parameter

\( \delta \)  
Dimensionless depth

\( \delta_{ij} \)  
Kronecker delta

\( \delta \)  
Tunable coefficient in white-capping source term

\( \epsilon \)  
Dimensionless wave energy

\( \chi \)  
Dimensionless fetch length

\( \chi_{A} \)  
Dimensionless fetch length w.r.t. \( U_A \)

\( \Phi \)  
First order moment of energy flux

\( \kappa \)  
Von Kármán constant

\( \lambda \)  
Wave length

\( \lambda \)  
Factor in quadruplet formulations

\( \lambda_0 \)  
Deep-water wave length

\( \lambda_w \)  
Length of whitecap

\( \mu \)  
Parameter in Miles' formulation

\( \mu \)  
Factor in quadruplet formulations

\( \nu \)  
Dispersion parameter \((\mu = k h)\)

Dimensional frequency

\( \zeta \)  
Free surface elevation

\( \theta \)  
Wave direction

\( \theta_w \)  
Mean wind direction

\( \rho \)  
Density of water

\( \rho \)  
Covariance of surface displacement

\( \rho_a \)  
Density of air

\( \rho_w \)  
Density of water

\( \Sigma \)  
Source function, change in action

\( \Sigma' \)  
Spectrally-integrated source function

\( \Sigma_{sd} \)  
Steepness of wave spectrum below certain frequency

\( \tau_b \)  
Bottom shear stress

\( \tau_w \)  
Shear stress at water surface

\( \omega \)  
Absolute angular frequency
<table>
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<td>( \tilde{\omega}_r )</td>
<td>dimensionless angular frequency</td>
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<tr>
<td>( \omega_r )</td>
<td>relative angular frequency</td>
<td>rad s(^{-1})</td>
</tr>
<tr>
<td>( \bar{\omega}_r )</td>
<td>spectrally-averaged relative angular frequency</td>
<td>rad s(^{-1})</td>
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<td>( \bar{\omega}_r^{(n)} )</td>
<td>spectrally-averaged relative angular frequency</td>
<td>rad s(^{-1})</td>
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<td>( \Omega )</td>
<td>profile parameter in Miles' formulation dispersion relation</td>
<td>s(^{-1})</td>
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1 Introduction

1.1 Background of the study

The department Institute for Inland Water Management and Waste Water Treatment RIZA of Rijkswaterstaat (the Directorate-General for Public Works and Water Management) has commissioned WL | Delft Hydraulics by means of the contract RI-3806A (d.d. 31 October 2002, revised 21 January 2003) to investigate the wave-growth limit in shallow water.

The wave-growth limit in shallow water is an important subject. Not only in lakes as IJsselmeer and Markermeer, but also on tidal flats in tidal inlet systems as the Wadden Sea, it is important to have knowledge of the maximum wave height for a given depth for determining design conditions for sea defences. These are examples in The Netherlands, but in fact any water system satisfying the following criterion are subject to the wave growth limit:

- The water depth is almost constant and relatively small;
- The fetch length is relatively large;
- The wind velocity is relatively large.

Most of the wave prediction models have been developed for oceans, where the wave growth is hardly limited by the water depth, or for coastal areas, where in general waves propagate from deep to shallow water over a sloping bottom. The wave-growth limit in shallow water is underexposed in wave modelling.

The wave-growth limit can roughly be expressed in an equilibrium value for the ratio between significant wave height \( H_s = H_{ms} \) and water depth \( h : H_s / h \). The ratio is roughly in between 0.3 and 0.5, indicating the uncertainty. HISWA predicts relatively small wave height over depth ratios for the wave-growth limit over a horizontal bed, varying in between 0.33 and 0.35, whereas a value in the order of 0.4 is to be expected. This implies that the expected wave height is 15 – 20% higher than computed. For the determination of the hydraulic boundary conditions the performance of HISWA is relatively poor (and the results are even unsafe).

The wave model SWAN is more advanced than HISWA. On the other hand, the implementation of the dominant processes for the situation described above hardly deviates in SWAN and HISWA, apart from the fact that in HISWA a single frequency is used, and in SWAN the full spectrum is used. The results of SWAN and HISWA are therefore expected to be more or less similar.

This report describes the analysis of the wave-growth limit at shallow water, using wave growth curves from literature and the theory behind SWAN. The study is split into two major parts:

- Analysis of wave growth curves;
- Analysis of source terms of SWAN in limit situations.
This study has been performed by M.W. Dingemans and J. Groeneweg, the latter being project leader.

1.2 Purpose of the study

The aim of the study is to gain insight in the physical mechanisms that are important for the wave growth in shallow water in general and for the limit of that growth in particular. The wave growth in shallow water is only known within a broad margin. Wave models predict relatively mild wave conditions within this margin. The large extent of uncertainty in the predicted wave conditions, that lead to 'unsafe' computational data, are of great concern for the determination of the design conditions of sea defences along the IJsselmeer and Markermeer.

To reach the goal the following questions will be answered:

- Which physical processes cause the largest uncertainties in predicting the wave field in shallow water?
- Does an equilibrium spectrum exist that is based on the present formulations of the source terms in SWAN in the limit situation?
- What is the wave-growth limit in terms of the wave height over depth ratio?
- Are the wave-growth limit conditions scalable for the present formulations of the source terms in the action balance equation in SWAN? If so, which scale rules should be applied?
- Do the presently available formulations of wave-growth curves provide reliable estimates for the wave conditions in limit situations?

1.3 Approach of the study

To get insight in the uncertainties of physical processes modelled in SWAN in wave-growth limit situation in shallow water, first an overview is given of the formulations of physical processes, that have been modelled in SWAN and their performance in general. By combining this to the relative importance of source terms under wave-growth limit situations on shallow water, conclusions can be drawn which physical processes cause the largest amount of uncertainty in wave-growth limit situations.

The expected wave height over depth ratio predicted by SWAN has been investigated both qualitatively and quantitatively. Firstly, the possibility of determining an equilibrium spectrum from the action balance equation in SWAN is briefly investigated. Secondly, the opposite path is followed in which for a number of spectral shapes has been investigated whether they approach the equilibrium spectrum. From the 'approximate' equilibrium spectrum the wave height over depth ratio is determined.

An attempt has been made to scale the present formulations of the source terms, such that it is applicable for wave-growth limit situations and can be applied for various wind speeds, wave
conditions and depths. For this purpose the integrated balance equation has been considered. Furthermore, for a number of available formulations for wave-growth curves, estimates for the wave height over depth ratio have been determined in the wave-growth limit situation in shallow water.

1.4 Outline

The analysis of wave growth curves and the theory behind SWAN has been described in the Chapters 2–4. In Chapter 2 an inventory is given of relevant source terms for the modelling of wind-induced waves in shallow water. In Chapter 3 a qualitative analysis of the limit situation has been carried out. This has been quantified in Chapter 4. The analysis of existing growth curves has been worked out in Chapter 5. Finally, conclusions and recommendations are given in Chapter 6.
2 Inventory of source terms

2.1 Introduction

In this Chapter we address the action-balance equation and, especially, its source terms. We start with a discussion of the action-balance equation and its validity (Section 2.2); this discussion rests heavily on the one given in Dingemans (1998). Then we discuss the other source terms and their usual mathematical descriptions in Sections 2.3 - 2.8. Finally, in Section 2.9 the uncertainties in the presently-used source terms in SWAN will be discussed. In this chapter the balance equation and the source terms are discussed for general situations. In Chapter 3 we focus on the particular situation of growth limit conditions with strong wind, infinite fetch, horizontal bottom and shallow water.

In the review of Hasselmann (1968) the source term $S(k, x, t)$ was taken to be the sum of seven contributions, representing:

- A constant energy transfer to the wave field through turbulent atmospheric pressure fluctuations;
- Miles’ instability mechanism;
- a non-linear correction to Miles’ theory;
- energy transfer due to interactions with the atmosphere;
- energy transfer due to non-linear wave-wave interactions (only four-wave interactions);
- dissipation in shallow water due to turbulent bottom friction;
- a then still unknown dissipation mechanism due to breaking waves, later to be identified with whitecapping, see Hasselmann (1974).

The first four contributions are important for the wave generation due to wind. In SWAN the first two of these four have been taken into account. Besides the terms identified by Hasselmann (1968) in SWAN two more contributions are incorporated:

- energy dissipation in shallow water due to depth-induced wave breaking;
- energy transfer due to three-wave interactions.

Apart from the source terms mentioned above, more physical effects can be mentioned, which are not accounted for in SWAN:

- a source term due to scattering of waves against the (random) bottom;
- a source term describing the loss of energy due to the interaction of waves with a mud bottom. When mud is present, the attenuation of waves may be very large. A start for studying this phenomenon may be the paper of Liu, Davis and Downing (1996);
- effects of wave diffraction;

For a definition of the source term see Eqs. (2.2b) and (2.4)
• effects of wave trapping, especially in trenches such as entrance channels to harbours (e.g., the Euro-Meuse shipping channel).

In this chapter we discuss the various source terms.

2.2 The action–balance equation

The action–balance equation is described here as used in SWAN (see Booij et al., 1999). We first define the dispersion relation in the general form\(^2\) for the absolute frequency \(\omega\),

\[
\omega = \Omega(k, \mathbf{x}, t).
\]

(2.1)

For a discussion of the kinematics of wave propagation see, e.g., Dingemans (1978) or Dingemans (1997, §2.2), or the references mentioned therein.

The SWAN model is a so-called third-generation wave-evolution model and has especially been designed for application in coastal areas. With the wave action density given as \(N(k, \mathbf{x}, t)\) the spectral action balance equation reads\(^3\)

\[
\frac{dN}{dt} = \Sigma, \quad (2.2a)
\]

or, writing out the total derivative \(d/dt\),

\[
\frac{\partial N}{\partial t} + \dot{x}_i \frac{\partial N}{\partial x_i} + \dot{k}_i \frac{\partial N}{\partial k_i} = \Sigma, \quad (2.2b)
\]

where \(k\) is the wavenumber, \(\dot{x}_i \equiv dx_i/dt = \partial \Omega/\partial k_i = c_{gi} + U_i\) with \(c_g\) the group velocity and \(\dot{k}_i \equiv dk_i/dt = -\partial \Omega/\partial x_i\) is the rate of change of the wave number due to refraction\(^4\). The left-hand side of (2.2) describes, in addition to horizontal transport and refraction of energy, the interaction with non-homogeneous or time-varying mean currents and water depths. The source function \(\Sigma(k, \mathbf{x}, t)\) is the rate of change of action density and is due to several physical processes to be discussed later in §§2.3-2.8 of the present Chapter. When the action density is not considered as a function of \(k\), but as a function of relative frequency \(\omega_r\) and direction \(\theta\). The relative frequency is defined by \(\omega = \omega_r + k \cdot U\) where \(U\) is the depth-averaged current; for a derivation, see Dingemans (1997, pp. 79-83). The balance equation is written in the form

\[
\frac{\partial N}{\partial t} + \frac{\partial(c_{gi}N)}{\partial x_i} + \frac{\partial(c_{\omega r}N)}{\partial \omega_r} + \frac{\partial(c_{\theta}N)}{\partial \theta} = \Sigma, \quad (2.2c)
\]

\(^2\)\(\omega\) is the radian frequency in rad/s and \(\mathbf{x} = (x_1, x_2)^T = (x, y)^T\) is the horizontal spatial vector measured in m. Later we also use the cyclic frequency \(f\) measured in Hz and defined by \(\omega = 2\pi f\).

\(^3\)The wave action \(N\) has dimension of energy/frequency. Time \(t\) is measured in s.

\(^4\)We consider vectors in the horizontal plane and therefore \(i\) runs from 1 to 2.
which is the form used in SWAN\textsuperscript{5}.

2.2.1 Definition of wave action

For the definition of the action one usually has to take recourse to the averaged Lagrangian $L$. Only for linear wave propagation one may state that action equals energy over relative frequency:

$$ N(\omega, \theta; x, t) = \frac{E(\omega, \theta; x, t)}{\omega} \quad \text{or} \quad N(k; x, t) = \frac{E(k; x, t)}{\omega}, $$

(2.3)

which form has been given originally by Bretherton and Garret (1968).

It has to be stressed that equipartition of kinetic and potential energy is the basis for this action definition. It can be demonstrated easily that for non-linear wave propagation the kinetic energy is always larger than the potential energy and therefore another definition of wave action should be used, e.g. see Lighthill (1978, p. 457) or Dingemans (1997, p. 874). For slowly-varying waves the usual procedure is via the averaged Lagrangian $L$ or the averaged Hamiltonian $H$ (e.g., see Dingemans (1997, §8.2.2 and 8.2.4)\textsuperscript{6}. In this way also wave-action definitions for non-linear waves are possible, although only for slowly-varying wave systems (which is an assumption also needed for the wave-propagation part because that is based on the refraction approximation, in absence of diffraction effects). The wave-propagation part in SWAN is linear, but the four-wave interaction term is definitely non-linear. That the four-wave interaction formulation is nominally only valid for weak interactions is well known, but this source of interaction is almost the single source of non-linearity in wave-evolution models and therefore is taken to be responsible for all non-linear effects.

2.2.2 Definition of the source term

The source term $\Sigma$ describes the change in action density due to several physical processes such as wave growth due to wind, wave reduction due to dissipation processes and also changes due to non-linearity due to four and three-wave interaction processes. A problem with the definition of these terms is that some of them are derived for energy, not action. The unavoidable so-called modelling constants are determined using the spectral energy density, not the action density. A direct formulation in terms of wave action is therefore not always possible and the relation between the change in action ($\Sigma$) and that in energy ($S$) has to be

\textsuperscript{5}The potential energy of a regular linear wave is $E_p = \frac{1}{2} \rho g a^2$ with $a$ the wave amplitude. The dimension of the total energy $E$ then is $J/m^2$. The energy density $E(\omega)$ then has the dimension $J/m^2$ and the action density $N(\omega) = E(\omega)/\omega$ then has dimension $J s^2/m^2$. The source terms $\Sigma$ then have the dimension $J s^2/m^2$ (notice that the source terms are then in the form of densities). In this report also $m_0$ is used, where $m_0$ has the dimension of $m^2$; we thus have $m_0 = \frac{1}{2} \int_0^\infty E(\omega) d\omega$.

\textsuperscript{6}An extension to non-slowly-varying wave fields seems to be possible only by means of the generalised Lagrangian mean (GLM) formulation of Andrews and McIntyre (1978a, b)
found. In SWAN it is simply assumed:

$$\Sigma = \frac{S}{\omega_r}.$$  \hspace{1cm} (2.4)

In Appendix A this concern is underpinned. Nevertheless, it is very likely that relation (2.4) is correct. However, it has not been proven yet.

2.2.3 The wave-propagation part

The wave propagation part as defined in the left-hand side of the action balance permits in principle only linear wave propagation; only linear refraction without diffraction is permitted. This is the case in almost all wave evolution models. For a description of some effects of diffraction in random wave models the discussion in Dingemans (1978) is relevant. Here we focus on the linearity in the wave propagation part. As (2.2) is in essence a refraction model (that is, a model in which the geometric-optics approximation has been imposed), transport of wave action proceeds along linear wave rays, the paths of which do not depend on the local amplitude. In essence this means that the transport of action along a wave ray is independent of the transport of action along neighbouring wave rays. As shown first by Longuet-Higgins and Phillips (1962) there is a definite effect on the phase velocity due to tertiary wave interactions (interactions among three wave components). This analysis has recently been reconsidered by Hogan, Grumman and Stiassnie (1988). To our knowledge in only two, albeit very important papers the effect of tertiary wave interactions on the wave propagation has been accounted for, namely in Willebrand (1975) and in Watson and West (1975). The effect has usually been ignored because attention was focused on wave generation in deep water, where three-wave interaction effects are insignificant. However, SWAN is meant to perform well for shallow water and it is here that this effect may enlarge the wave heights by up to 25%, see Willebrand (1975), which is a significant amount.

Willebrand’s approach

The radiative transfer equation, as derived by Willebrand (1975) is found to be:

$$\frac{d}{dt} \left[ \frac{E(k,x,t)}{(gk \tanh kh)^{1/3}} (1 + J(k,x,t)) \right] = \Sigma(k,x,t),$$ \hspace{1cm} (2.5a)

with

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial \Omega}{\partial k_i} \frac{\partial}{\partial x_i} - \frac{\partial \Omega}{\partial \xi_i} \frac{\partial}{\partial k_i}$$ \hspace{1cm} (2.5b)
and the frequency is given by

\[ \Omega(k, x, t) = k \cdot U + (gk \tanh kh)^{1/2} \left[ 1 + \int S(k, k') E(k') \, dk' \right]. \tag{2.5c} \]

For a detailed description of the non-linear contribution we refer to Willebrand (1975). Let us consider the effect of the non-linear term \( J \) in (2.5a). This discussion is taken from Willebrand.

- Because of the dependence of \( \Omega \) on the wave spectrum, also the group velocity \( \partial \Omega / \partial k \)
  depends on the wave spectrum. This means in particular that the direction of \( \partial \Omega / \partial k \)
  is generally different from the wave number direction, even in the case of absence of a mean current.
- The refraction term, proportional to \( \partial \Omega / \partial x \), may be non-vanishing, even in the case of a horizontal bottom and a homogeneous current. This means that in a fetch area where \( \partial E / \partial x \neq 0 \) also \( \partial \Omega / \partial x \neq 0 \).
- The change in action density due to the presence of the term \( J(k, x, t) \) in (2.5a) leads to an additional flow of energy between the waves and the mean current and therefore represents a higher-order correction to the radiation stress effects.

**The approach of Watson and West**

In Watson and West (1975) new terms to the wave-action balance equation are found which are of second order in the wave spectrum and non-vanishing only if the wave spectrum has a spatial variation. Also here the effect of non-linearity is the coupling across the wave spectrum. The analysis of Watson and West is not specifically focused on shallow water and therefore we do not consider their approach in this report. However, when it is decided to account for non-linear effects on the wave propagation and the wave action, the results of this paper should also be taken into account.

### 2.3 Wind input

The wind input in SWAN is modelled in the usual way via linear and exponential growth of waves:

\[ S_{in} = A + BE(\omega_r, \theta). \tag{2.6} \]

This way of accounting for wave growth due to wind has been used for a long time already and much effort has been given to estimating the coefficients\(^7\) \( A \) and \( B \), which depend on the local wind speed and wind direction. In §2.3.1 linear wave growth in SWAN is discussed, followed by exponential wave growth in §2.3.2.

\(^7\)Notice that the dimension of \( S_{in} \) and thus also of \( A \) is in m\(^2\). For use in the action balance equations a multiplication by \( \rho g \) is necessary. The dimension of \( B \) is Hz.
2.3.1 Linear growth

For $A$ the formulation of Cavaleri and Malanotte-Rizzoli (1981) has been implemented in SWAN, in which case the growth for frequencies lower than the so-called equilibrium frequency or Pierson-Moskovich frequency\(^8\) $f_{PM}$ is exponentially reduced because growth due to wind happens at the high-frequency range. The formulation taken for the linear growth $A$ is given in Ris (1997, p. 137) as

$$A = \frac{1.5 \times 10^{-3}}{g^2 2\pi} \left[ u_* \cdot \max \{0, \cos(\theta - \theta_o)\} \right]^4 \cdot \exp \left[ - \left( \frac{f}{f_{PM}^*} \right)^{-4} \right]$$

with $f_{PM}^* = \frac{0.13 g}{28 u_*}$,

with $u_*$ the friction velocity (in m/s), $\theta_o$ the mean wind direction and $f_{PM}^*$ the equilibrium peak frequency (in Hz) for a fully-developed sea according to Pierson and Moskowitz (1964).

**Comment.** It has to be remarked that linear wave growth as modelled in SWAN is only correct in homogeneous conditions. It can be argued quite simply that this is usually the case because for these short small waves the water is effectively deep, but that is not the whole of the story. Current refraction and the influence felt by short waves riding on longer waves (and on wave groups) are not accounted for.

2.3.2 Exponential growth

For exponential growth several possibilities exist in SWAN.

**First choice.**

The expression for $B$ used in SWAN (see Eq. (2.6)) is the one used in WAM cycle 3 and reads\(^8\)

$$f_{PM} = \frac{1}{2\pi} \frac{g}{28 u_*} ,$$

see Tolman (1992a, Eq. (7)). Notice that in Ris (1997, Eq. (A-2)), the PM frequency is $0.13 \ast (2\pi)^2 \approx 5.13$ times the value given by Tolman:

$$f_{PM} = 2\pi \frac{0.13 g}{28 u_*} ,$$

Which definition is used, is not clear when comparing Booij et al. (1998) and Ris (1997). In Komen et al. (1994) this frequency is also used, but not defined, as is the same for WAMDI (1988). A further different definition is used by Graber and Madsen (1988, Eq. (27)):

$$f_{PM} = \frac{0.13 g \tanh k_{PM} h}{U_{10} \cos(\theta_0 - \theta)} .$$
(Ris, 1997, p. 138):

\[
B = \max [0, \frac{1}{4} \frac{\rho_a}{\rho_w} \left( \frac{28 u_r}{c} \cos (\theta - \theta_w) - 1 \right) \omega_r],
\]

(2.8)

where \( \rho_a \) and \( \rho_w \) are the densities of air and water respectively and \( \theta_w \) is the wind direction.

Booij et al. (1999) relate the friction velocity \( u_* \) to the wind velocity at 10 m height by the well-known relation of Wu (1980, 1982), (see also Dingemans, 1997, p. 418). With \( u_*^2 = C_{10} U_{10}^3 \) and \( U_* \) the wind velocity at a height \( z \) above the water surface, the (dimensionless) drag coefficient \( C_{10} \) follows, according to Wu (1980, Eq. 11) by

\[
C_{10} = [0.8 + 0.065 U_{10}] \times 10^{-3}, \quad \text{with} \quad U_{10} > 1 \text{m/s}.
\]

(2.9)

Notice that Charnock’s relation plays no role in (2.9). Only when relating \( u_* \) with \( z_0 \), Charnock’s constant \( a \) plays a role:

\[
z_0 = a u_*^2 / g \quad \text{with} \quad a = 0.0185.
\]

(2.10)

We notice that Ris (1997, p. 138) uses (2.9) only for \( U_{10} \geq 7.5 \text{ m/s} \), and takes a constant value of \( 1.2875 \times 10^{-3} \) (which is obtained for \( U_{10} = 7.5 \text{ m/s} \)) below that wind velocity. This is also implemented in SWAN, but it is in contradiction with Wu (1980, 1982) to which Booij et al. (1999) refer.

**Second choice.** In the first choice the effect of the wind has been modelled, but the reverse effect of the waves on the wind has been ignored. Because of the large scatter between the observed aerodynamic drag over waves as a function of \( U_{10} \), as reported by Donelan (1982), it was argued by Janssen (1989) that this variation can only be explained by sufficient wave-age dependency of the drag.

Janssen (1989, 1991b) investigated a model for the coupling of wind and waves. This model is solved in an iterative way, see Mastenbroek et al. (1993), in which the unknown friction velocity \( u_* \) is determined. This method is also contained in WAM, cycle 4. The formulation of the model is also given in Ris (1997, Eqs. (A-6)-(A-10)). The expression for \( B \) is based on a quasi-linear wind-wave theory and is given by

\[
B = \beta \frac{\rho_a}{\rho_w} \left( \frac{u_*}{c} \right)^2 \max [0, \cos (\theta - \theta_w)]^2 \omega_r,
\]

(2.11)

where \( \beta \) is the Miles constant.

An important simplification is that from the outset it is assumed that the wave stress is in the wind direction so that only one component of the momentum equation for air has to be considered. It has to be noted that this simplification is only a good approximation for deep water. Near the coast the approximation breaks down, so for SWAN this condition should be relaxed. Some properties of this description, given in Janssen (1991b), are:
The aerodynamic drag depends on the sea state.

- A proper description of the momentum transfer from air to waves can only be given by coupling an aerodynamic boundary layer model with a wave prediction model.
- Especially in the initial stages of growth, a strong interaction between wind and waves is found, resulting in a substantial increase in the stress in the surface layer.
- A realistic scatter around the mean is found from the coupled model results, similar to the in-situ observations of Donelan (1982).
- Growth of both high-frequency and low-frequency waves is found from the model. This is in contrast with the Miles mechanism in which only dependence on the curvature of the wind field is accounted for.

- With a Miles parameter \( \beta \) defined by
  \[
  \beta = \frac{1.2}{\kappa^2} \mu \log \mu \quad , \quad \mu \ll 1
  \]  
  (2.12)

the growth rate of waves due to wind is found to be
  \[
  \frac{\gamma}{\omega} = \varepsilon \beta \left( \frac{u_s}{K_C} \right)^2 \cos^2 \theta ,
  \]  
  (2.13)

where \( \mu \) is given by:
  \[
  \mu = \left( \frac{u_s}{K_C} \right)^2 \Omega \exp \left[ \frac{K_C}{u_s} \cos \theta \right].
  \]  
  (2.14a)

Here \( \kappa \) denotes Von Kármán's constant and \( \Omega \) is the profile parameter, given by
  \[
  \Omega = g (z_0 + z_1) \left( \frac{K_C}{u_s} \right)^2.
  \]  
  (2.14b)

Here \( z_0 \) is the roughness length of the gravity-capillary waves and \( z_1 \) is the one of the short gravity waves.

**Comment 1.** As reported in Ris (1997, p. 48), the cycle-4 formulation did not work well in SWAN. It could not generate the growth curves of, amongst others, Kahma and Calkoen (1992) and therefore this formulation has not been used in further verification studies.

**Comment 2.** One of the problems is the relation between the roughness of the free surface and the drag coefficient. In the second choice this has partly been solved via the wave age dependence. However, wave age is a well-defined concept and may be too coarse for the problem at hand. A solution may be to get some formulation for the drag in which the effect of long waves is eliminated. Such a solution may be the method of Rieder and Smith (1998), which will be discussed in the next section.

**Comment 3.** According to Ris (1998, personal communication) the problems with the present method of wind input formulation come from interference with the used method for wave dissipation due to white-capping and it also has to do with the chosen cut-off frequency. If this would be the true reason why the second choice for wave generation does not work,
then it is time to consider more fundamentally wave breaking due to white-capping. However, the explanation given in Comment 4 seems more probable.

Comment 4. Discussions at the WISE meeting in May 1998 led us to believe that the discrete interaction approximation (DIA), together with the fixed cut-off frequency in SWAN is responsible for this behaviour. A major improvement can be achieved by letting the cut-off frequency breathe in the same way as done in WAM cycle 4. This would also lead to a solution of the difference in mean frequency between SWAN and WAM.

Comment 5. As discussed by Phillips (1987) the effect of rain on wind generation has also not been considered. This effect may be twofold: 1) the rapid disappearance of the short waves due to different effects on the forward and backward side of the wave and 2) the effect of the rain on the mean wind profile.

Comment 6. As stated by Jones and Toba (1995), in a discussion of the paper of Donelan et al. (1993), the relation between surface roughness and wave age is only a function between two variables for the case that the wind and waves are in equilibrium. In deep water this may well be the case, but near the coast effects of shoaling and refraction, amongst others, play a role and the resulting wave field is not in equilibrium any more with the wind field.

Jones and Toba argue that, while it is true that at a constant wave height old waves have a lower aerodynamic roughness than young waves, waves in equilibrium with the wind have an increasing wave height with wave age and wind stress. Jones and Toba present a Figure in which the drag coefficient is seen to be increasing with wave age (and therefore also wave height) for a constant wind velocity.

Donelan et al. (1995) argue that the reason of the difference in the analysis of Jones and Toba (1995) and that of Donelan et al. (1993) lies in the fact that Donelan et al. (1993) considered both laboratory and field data in one data set.

Comment 7. The influence of swell on wave growth due to wind has not been considered in SWAN. Swell has a diminishing effect on the capillary waves generated by wind and consequently, swell hinders the effectiveness of generation of wind waves. A further effect of swell is that it also has effect on the wind-wave part through four-wave interactions (Balk, 1996 and Watson and McBride, 1993).

Comment 8. The effect of wind-gustiness has not been accounted for in SWAN. The effect of wind speed gustiness is relevant to growth of significant wave height because it causes an increase in the surface stress. Notice that the growth is roughly proportional to the wind velocity squared, see e.g. (2.13), and that variations around the mean have thus effect on the growth of waves. Experiments by Ponce and Ocampo-Torres (1998) showed that for an increase of 7 and 24 % of the white noise in a synthetic signal, the increase in surface stress was 2 and 7 % respectively. This was obtained under the condition that the friction velocity would be a function of only the wind speed at 10 m height. In reality, the increase in friction velocity would be more.

It is our opinion that it is sufficient to account for the gross effect of wind-gustiness, so
that also for stationary SWAN computations its effects may be accounted for. In a later stage the instationarity of the gustiness may be used more accurately; this would mean a highly instationary SWAN version and therefore it is questionable whether to account for the instationarity of the gustiness would lead to a practical solution. We note that a description of the effect of wind-gustiness has also been given by Janssen in Komen et al. (1994, pp. 88-90).

Comment 9. We note that differences between air and sea temperatures can lead to stable or unstable stratified water with effect on the atmospheric boundary layer. These temperature effects are not accounted for in the used atmospheric boundary-layer models. We do not think this to be a very pressing subject, but some estimate of its effect on wave generation due to wind seems advisable.

2.3.3 Alternative methods to model the effect of wind input

A different boundary layer model

A different boundary layer model is formulated by Burgers and Makin (1993). This boundary layer model is used to obtain results for the energy flux to monochromatic waves. The results are formulated in terms of the Miles parameter $\beta$:

$$\beta = \frac{1}{\omega_r E} \frac{\partial E}{\partial t}. \quad (2.15)$$

Note that this means that the source term $S_{in}(f, \theta)$ is

$$S_{in}(f, \theta) = \beta \omega_r E(f, \theta). \quad (2.16)$$

For a value $\rho_a/\rho_w = 1.3 \times 10^{-3}$ they fitted $\beta$ by means of piecewise linear and quadratic functions of $U(z)/c$ where $c$ is the phase velocity, the results of which still depend on an equivalent drag coefficient$^9 C_0$. For $C_0$ is taken $10^{-3}$ to $3 \times 10^{-3}$. Notice that Tolman and Chalikov (1996) proposed a similar function for wind input. In their case they started with wind velocity at some level together with the phase velocity and frequency of the waves and estimated the other parameters. Instead of non-dimensionalising $U(z)$ with the phase velocity $c$ as such, they used the deep-water value for $c$, $g/\omega$, for the non-dimensionalisation. It is worth noticing that Burgers and Makin used a formulation for $S_{in}$ which also has a negative component.

It should be noted that Stewart (1974) gave the following estimate for the Miles parameter:

$$\beta = 1.2 \frac{\rho_a}{\rho_w} \sqrt{C_0} \left| \frac{U(z)}{c} \left( \frac{U(z)}{c} - 1 \right) \right| \quad (2.17)$$

$^9$In Burgers and Makin the drag coefficient is denoted as $c_q$. 
which is shown to give almost the same result as obtained from Burgers and Makin (1993), see Figure 2.1. From Batchelor (1967, Appendix 1) we find for $\rho_a$ at 0 m height for the standard atmosphere a value of 1.226 kg/m$^3$. With $\rho_w = 1025$ kg/m$^3$ for sea water, we have $\rho_a/\rho_w = 1.20 \times 10^{-3}$. To comply with the value used in Burgers and Makin (1993), we also use $\rho_a/\rho_w = 1.3 \times 10^{-3}$ in (2.17).

![Graphs showing parameter against wind velocity over wave celerity](image)

Figure 2.1: Miles parameter against wind velocity over wave celerity, $C_0 = 10^{-3}$ (left) and $C_0 = 2 \times 10^{-3}$ (right).

Burgers and Makin (1993) notice that a new retuning for the wave dissipation is necessary because their wind parameterisation is different from that of WAM. Tuning is a necessity, according to Burgers and Makin, because no direct measurements or theory is available for the dissipation in the energy-containing range of the wave spectrum. Presumably they meant that in deep water wave dissipation never can be measured in absence of wave growth by wind. As is well known, wind growth curves are usually determined while wave breaking of the higher-frequency waves is occurring so as to maintain a saturation form in the spectrum. However, in shallow water the situation is not that bad. The wind growth in shallow water is insignificant locally in high seas and the wave attenuation in shallow water can be attributed almost totally to the dissipation (after a simple correction for shoaling). In shallow water, the wave breaking is the dominant feature. Burgers and Makin finally settled for a dissipation source function that equals the standard one of WAM, divided by 5.

The wind stress vector

The primary agent for wave growth due to wind is through the wind stress acting on the free surface. Usually the shear stress $\tau^w$ (here in m$^2$/s$^2$) is related to the wind speed by a quadratic law:

$$\tau^w = C_d |U_{10}| U_{10}$$

where the drag coefficient $C_d$ is still a function of the roughness of the free surface. In method 2 above (Janssen, 1989, 1991b) the roughness was related to the wave age. In the presence
of swell both the magnitude and direction of the stress were found to deviate considerably from the stress obtained for a situation without swell (see e.g. Rieder and Smith, 1998). Wave-age scaling is useful only in the absence of swell. As SWAN is meant to be applied in coastal areas, wave generation due to wind takes almost nowhere place in absence of larger gravity waves, except perhaps very close behind an island. However, computations of Yeh et al. (1994) and Liu et al. (1995) have shown that in the case of earthquakes also wave run-up behind an island occurs (in fact most damage occurs at the lee-side of the island). It has to be remarked that for normal random waves this effect will be not so strong, but effects due to diffraction and wave-directional effects do occur closely behind an island. In many cases effects of swell-type waves on the wave generation have to be accounted for.

Rieder and Smith (1998) present a method to remove the wave effect from the wind stress vector. The wind stress vector is calculated via the cospectra $C_{\nu}$ between the downwind $u'$, the cross-wind $\varphi$ and the vertical $w'$ fluctuating wind velocities

$$\tau'(f) = C_{\nu u'}(f)i + C_{\nu \varphi w'}(f)j,$$  

(2.19)

where $f$ is the frequency in Hz and $(i, j)$ are the unit vectors in the $(u', \varphi)$ directions.

1. First they notice that significant wave energy occurs in a frequency band much smaller than for which the wind stress is of importance. The frequency band for water waves is typically between 0.05 Hz and 0.5 Hz for sea conditions\(^{10}\), whereas turbulent wind fluctuations occur between 0.002 Hz and 1 Hz. The wind stress is considered in three frequency bands of similar magnitude: a low-, a middle- and a high-frequency band. The wind stress in the high-frequency band is linked to the short waves and the waves are roughly in equilibrium with the wind. The stress in the low-frequency band has not much correlation with the waves because only negligible wave energy occurs in this band. It is mainly in the middle band that the waves can change the stress significantly.

2. Secondly, the complex cross-spectra between the fluctuations in the wind and the seasurface elevation are computed. In the experiments which Rieder and Smith describe, the wind velocity was measured at 8 m height. In the higher-frequency band not much correlation between the waves and wind at 8 m height is expected. In the low-frequency band negligible wave energy is present. Consequently, only in the middle-frequency band appreciable correlation between the waves and the wind might be expected. The stress vector which accounted for the correlations between waves and wind is computed and subtracted from the total stress.

3. Rieder and Smith (1998) subsequently show that the removal of the correlated stress removes the anomalous magnitudes and directions from the estimated stress. The three frequency bands used were: a low band for $f < 0.06$ Hz, a middle band for $0.06 \leq f < 0.16$ Hz and the high-frequency one for $f \geq 0.16$ Hz. The frequency 0.16 Hz has been chosen corresponding to $k = 1/8$ m. The correlated stress is estimated using the complex cross-spectra $C$ between the horizontal velocities $u', \varphi$, the vertical

\(^{10}\)Both frequencies may be larger for IJsselmeer and Markermeer conditions.
velocity \( w' \) and the free surface elevation \( \zeta \):

\[
\tau_{\text{corr}}(f) = \frac{\text{Re} \left[ C_{w'\zeta}(f)C_{w'\zeta}^*(f) \right]}{C_{\zeta\zeta}(f)} i + \frac{\text{Re} \left[ C_{\phi\zeta}(f)C_{w'\zeta}^*(f) \right]}{C_{\zeta\zeta}(f)} j,
\]

with \( i \) and \( j \) the unit vectors in \( x \) and \( y \) direction, respectively.

The result is that the subtraction of the correlated part of the stress results in much less scatter in \( \tau_w/\rho \) for the middle band. Not much difference occurs for the high-frequency band.

Notice that the finding of Rieder and Smith is in line with the result found by Dobson et al. (1994)\(^{11}\) that after deleting the swell part from the spectrum the existing drag coefficient versus wave-age relationships would hold.

The modelling of the reduced stress is therefore much easier than the modelling of the full stress. For the reduced stress the usual wave-age dependence may be used. The idea now is to base the part of the stress without the wave part (the radiation-stresses part) on the usual quadratic law and afterwards to add the wave contribution to \( \tau \).

For prognostic use one has to determine the wave contribution to the surface stress (i.e., the radiation stress) by an iterative way, similar to the method of Janssen (1991b).

**Finite-depth effects**

Growth curves for wave-growth due to wind are usually obtained for rather deep water. Typical shallow-water effects are mostly not accounted for in the wind-wave growth curves used in the third-generation wave evolution models. A recent experiment in a lake with still-water depth of 2 m by Young and Verhagen (1996a) showed that the growth curve for the dimensional energy as function of the non-dimensional fetch depends strongly on the dimensionless depth \( \delta = gh/U^2 \) with \( h \) being the water depth, averaged over the fetch.

Results of this study can be summarised as follows:

1. In contradiction with the case for deep water where the fetch-limited evolution of the energy is defined by a single power law, for finite depth a number of curves exist, one for each non-dimensional depth.
2. With increasing fetch, the effects of finite depth become more pronounced. The total energy is smaller and the peak frequency is higher than would be attained in deep water.

Young and Verhagen (1996a) show that a possible intuitive estimate of the various processes is at fault with the results. This reasoning goes as follows, see Young and Verhagen (1996a):

\(^{11}\) Mentioned in Donelan, Drennan and Katsaros (1997).
• The atmospheric input for deep water is of the form

\[ S_{in} \sim \alpha \left( \frac{U_{10}}{c} - 1 \right) , \]

with \( c \) the phase velocity. Because \( c \) decreases for decreasing \( kh \), this relation would suggest increased atmospheric input and therefore higher waves.

• The magnitudes of the discrete interaction approximation yield stronger interactions in shallower water. However, as we will discuss later in §3.4, this is due to diverging Stokes-type approximations in shallow water, which invalidates the use of the commonly used four-wave interaction Boltzmann integral in shallow water. Other more suitable approximations are available for shallow water, see §2.6.

• Because the measurements show an increase of non-dimensional peak frequency with either increasing fetch or decreasing depth instead of a decrease as would be obtained from the above two reasons, Young and Verhagen conclude that therefore necessarily wave breaking should explain the difference.

Measurements of Young and Verhagen (1996a) clearly show that the relation between non-dimensional energy \( \epsilon \) and non-dimensional fetch \( \chi \) is strongly dependent on a shallowness parameter \( \delta \).

Young (1997, Eqs. (12) and (2)) devised an analytic solution for the relation between \( \epsilon \) and \( \chi \):

\[ \chi = 46675g \int \frac{c_p}{\omega_p U_{10}^3} \left( \frac{U_{10}}{c_p} \right) \left( \frac{U_{10}}{c_p} - 0.83 \right)^{-1} \left[ \tanh \left( \frac{U_{10}}{c_p} - B \right) \right]^{-0.45} dc_p , \quad (2.21) \]

and

\[ \epsilon = \frac{Eg^2}{U_{10}^4} = 0.0023 \left( \frac{U_{10}}{c_p} \right)^{-3.2} . \quad (2.22) \]

A relation of the dimensionless energy \( \epsilon \) in terms of the dimensionless fetch \( \chi \) may be obtained from these two equations. This solution compared well\(^\text{12}\) with the result of Young and Verhagen (1966a), as is shown in Figure 5 of Young (1997), which is reproduced here in Figure 2.2.

Effects of finite depth on the evolution of spectra and directional spectra are studied in Young and Verhagen (1996b) and Young et al. (1996).

\(^{12}\)One might think that an error of 25 % is large; for a growth curve due to wind this is a small error, therefore it gives a good comparison, not merely reasonable.
Figure 2.2: Growth curves for $\mathcal{E}(\chi)$ for shallowness parameters 0.5 and 0.1. Analytic result (drawn line) and measured one are compared.

2.4 Effects of drift currents

As is well known and well documented, wind flowing over water not only produces waves but also produces drift currents in the upper layer of the water. These drift currents are, in equilibrium, typically of magnitude $0.033U_{10}$ (Tsahalis, 1979, Table 1). Some effect of wave-current interaction can thus be expected. For wave growth due to wind, the upper layer is of most importance and it is in this layer that Doppler-shift effects occur. Usually these effects are small and therefore neglected, compared to the other wave-propagation effects.

It has been shown by Shrir (1998) that, although the wave-current interaction effect in itself may be small, the effect on the wave generation by wind may be large because the effects of the non-linear interactions are large.

Shrira discussed the problem that, while the short waves have a wide directional distribution, the energy-carrying larger waves have a narrow distribution in models. This is an artefact of the non-linear interaction model, together with the neglect of the influence of the larger waves on the wind. A new mechanism for wind generation is proposed by Shrira, which is based on induced scattering by shear flow, which phenomenon is well known in plasma turbulence.

Taking the drift current into account leads to the appearance of new eigenmodes of fluid motion. These motions form resonant triads with the spectral components of the water waves. The phenomenon induced scattering can be understood as the evolution of the wave field due to an interaction between the difference harmonics of the field which appear due to the non-linearity of the hydrodynamic equations and critical layers in the fluid.
One of the components of the mechanism considered by Shira is the generation by fixed difference harmonics of the perturbations of a given shear flow, which has been studied before as Langmuir circulations. Shira shows that the induced scattering mechanism is the most effective non-linear wave interaction mechanism, both for stochastic wave field evolution and for dynamics of isolated wave groups.

We are of the opinion that this work deserves further attention to be applied in a (fourth?)-generation wave model.

2.5 Models for wave breaking

In this section both wave breaking on steepness (Section 2.5.1) and depth-induced wave breaking (Sections 2.5.2 - 2.5.10) are discussed. In Section 2.5.11 the modelling of wave breaking for very mildly-sloping bottoms, as suggested by De Waal et al. (1999, Appendix B), is discussed.

2.5.1 Wave breaking in deep water

Wave breaking due to steepness is especially of importance for the waves with small wave lengths, which are generated directly by the wind. White-capping sets a limit to further growth of these waves and gives a transfer of momentum to the mean flow. In Komen et al. (1994, pp. 143-155) white-capping approaches have been described by Donelan and Yuan. Under the condition that the whitecap and the wave are geometrically similar (i.e., \(\lambda_w/\lambda\) and \(h_w/a\) a constant, where \(a\) is the wave amplitude and \(\lambda_w\) and \(h_w\) are the length and the height of the whitecap), the pressure (in N/m\(^2\) exerted by the whitecap on the water surface is given by

\[
p_w \sim \rho_w g h_w .
\]

With \(\zeta\) denoting the free surface elevation, the drag, or negative momentum transfer to the wave, then is

\[
-\frac{\rho_w g}{2c} \frac{dE}{dt} = \left\langle p_w \frac{\partial \zeta}{\partial x} \right\rangle ,
\]

which becomes, upon applying the following approximations: \(h_w \approx a\) and \(\left\langle p_w \partial \zeta / \partial x \right\rangle \approx \rho_w g a \times k a = 2\rho_w g k E\),

\[
\frac{dE}{dt} = -4\omega E ,
\]

where the deep-water dispersion relation has been used. This is a source function linear in both the energy density and the frequency.
Because the breaking of large-scale waves also causes rapid attenuation of short waves in its wake, this attenuation has to be accounted for also. This may be represented by requiring the dissipation function to depend on a relative frequency \( \omega / \bar{\omega} \), relative to the peak. The dependence is taken to be of the form \( \omega / \bar{\omega} \). Furthermore, the pressure-induced decay and the attenuation of the short waves are sensitive to whitecap coverage and thus to a measure of wave slope. As wave slope measure is taken, the wave slope compared with the one for a Pierson-Moskowitz spectrum: \( \hat{\alpha} / \hat{\alpha}_{PM} \). The resulting source function for dissipation due to white-capping is

\[
S_{wc} = -C_{wc} \left( \frac{\hat{\alpha}}{\hat{\alpha}_{PM}} \right)^m \left( \frac{\omega}{\bar{\omega}} \right)^n \omega N(k) ,
\]  

(2.26)

where the dimensionless parameter \( \hat{\alpha} = m_0 \omega^4 / g^2 \) is an integral wave steepness parameter, and \( C_{wc} \), \( m \) and \( n \) are the unavoidable adjustable dimensionless parameters. The model (2.26) is implemented in WAM and the same form has been used in SWAN. In SWAN it is used in the form as adapted by Komen et al. (1984),

\[
S_{wc} = -C_{wc} \frac{\bar{w}_r k}{k} E(\omega_r, \theta) ,
\]  

(2.27)

where the coefficient \( n \) is chosen to be equal to one. Furthermore, \( \langle \cdots \rangle \) denotes an average value of frequency or wave number, weighted with respect to the energy wave spectrum. The dependence on the steepness parameters \( \hat{\alpha} \) is taken into the coefficient \( C_{wc} \) which reads (Ris, 1997, Eq. A.13):

\[
C_{wc} = C_{ds} \left[ (1 - \delta) + \delta \frac{k}{\bar{k}} \right] \left( \frac{\overline{\alpha}}{\overline{\alpha}_{PM}} \right)^m
\]  

(2.28)

with \( \overline{\alpha} = m_0 \bar{k}^2 \) (\( \overline{\alpha} \to \hat{\alpha} \) for \( k \to \infty \)). Furthermore, \( C_{ds} \) and \( \delta \) are other dimensionless tunable coefficients.

The widely used formulations (2.26) and (2.27) for white-capping dissipation produce unrealistic results for multi-peaked wave spectra since it depends on wave steepness that is based on a mean wave period. Consequently, it underpredicts the dissipation of the wind-sea when a small amount of low frequency energy is added, and it overpredicts the dissipation of swell waves when a wind sea is present. Both of these effects lead to an underprediction of wave period measures. Since integral quantities of the spectrum depend on the shape of the spectral tail, it is important that the spectral decay with frequency is properly described. The 'Komen' formulation (2.27) depends on integral quantities of the spectrum and relates the dissipation at a certain frequency with the occurrence of wave energy at higher frequencies. This implies that the dissipation predicted by the 'Komen' method is sensitive to the description of the spectral tail.

Because of the problems for multi-peaked spectra, three alternatives for the 'Komen' method (2.27) are briefly described below (taken from Groeneweg et al., 2002, §7.3).
The cumulative steepness method

The cumulative steepness method was first implemented by Cecchi (Ris et al., 1999). The dissipation by white-capping at a particular frequency depends on the steepness of the wave spectrum at and below that frequency,

$$\Sigma_{st}(\omega) = \int_{0}^{\omega} k^2 E(\omega') d\omega'.$$

(2.29)

The white-capping source term is given by

$$S_{wc}(\omega, \theta) = -C_{wc}^{st} \Sigma_{st}(\omega) E(\omega, \theta),$$

(2.30)

with $C_{wc}^{st}$ another tunable coefficient. The relative dissipation increases with frequency because the cumulative steepness also increases with frequency. This is physically plausible. It seems unlikely that the dissipation of wave energy at a certain frequency depends on what happens at higher frequencies. One of the physical mechanisms behind the cumulative steepness method is that as surface straining, in which shorter waves riding on top of longer waves are modulated causing enhanced dissipation where the shorter waves are steepened.

The cumulative steepness method also has the considerable practical advantage that the shape of the tail of the spectrum need not be considered - it will result from the use of the formulation without being required by it. This behaviour follows directly from the basic concept of this method, only energy at lower frequencies is accounted for.

The extended Komen method

Ris et al. (1999) developed the extended Komen method, which has been extended and tested by Holthuijsen and Booij (2000). In this method, the dissipation by white-capping for a particular frequency depends only on parameters computed from the energy spectrum at frequencies above that frequency. The formulation behaves as expected and desired in the presence of swell (white-capping in the wind sea part of the spectrum) and therefore may be considered a considerable improvement. It is unlikely to require extensive re-calibration of the model, since it should give very similar results to the previous formulation for situations with single peaked spectra. In fact, Holthuijsen and Booij (2000) require that their extended method reduces to the basic formulation of Komen et al. (1984) for single peaked spectra. It is therefore expected that their method still suffers from the problems associated using mean period measures to scale the white-capping dissipation. The extended Komen method has the disadvantage that the dissipation at a particular frequency depends on the steepness at frequencies higher than that being considered. The steepness still has to be computed based on an average wave number and the corresponding spectral energy, including the energy in the spectral tail.
The Tolman and Chalikov method

Tolman and Chalikov (1996) developed a method in which the dissipation mechanism above the spectral peak frequency is different from the one below the peak frequency. Below the peak frequency the dissipation is described using an analogy with dissipation of wave energy due to oceanic turbulence. A diagnostic parameterisation for the high frequency dissipation has been obtained by assuming a quasi-steady balance of source terms in the corresponding regime. An intrinsic problem with the Tolman-Chalikov method is the fact that for many spectra the position of the peak frequency is not unique, especially in mixed seas this poses a problem.

2.5.2 Wave breaking due to depth limitations

A review of methods to account for dissipation due to wave breaking in shallow water has been given in Southgate (1995), which is an updated version of Southgate (1993). Some discussion is also given in Dingenmans (1997, Section 3.3). A number of dissipation formulations for accounting wave breaking are available nowadays. Many are based on a total-energy concept, only the decrease in the variance of the spectrum is described, usually for one characteristic wave period. Because these models depend on only a few parameters, they are quite successful in locally predicting the total energy. Other parameters such as skewness and kurtosis are predicted not so well, see also the discussion in Chen, Guza and Elgar (1997). Some of the more well-known methods are the ones of Battjes and Janssen (1978), a variation on this by Thornton and Guza (1983), Dally, Dean and Dalrymple (1985) and Roelvink (1993). Especially Battjes and Janssen's model has been applied in many models. However, this wave breaking model has been applied in an incorrect way in a number of circumstances, including SWAN. The SWAN user manual is wrong in this respect. Therefore we discuss the application of this model first.

Furthermore, we discuss the Rayleigh distribution and the modified Glukhovskyi distribution to obtain probabilities of wave breaking, $Q_b$. We also discuss a method of Baldock et al. (1998), which, according to Vink (2001) gave somewhat better results on steep slopes than the approach of Battjes and Janssen (1978). Lastly we investigate the formulation of the breaking wave height $H_b$.

Battjes and Janssen approach

In a number of wave propagation models the method of Battjes and Janssen (1978) has been applied. This method is based on a periodic bore formulation and it has two or three free parameters. The setting of these parameters is essential for obtaining a good result. Early examples of this were provided by Dingenmans (1983) and Dingenmans (1987).

\footnote{Usually the parameter for wave breaking on steepness is taken to fixed so that two parameters remain: the parameter for depth-limited wave breaking, $\gamma$ and the intensity of wave dissipation parameter, $\alpha$.}
A very succinct description of the method of Battjes and Janssen (1978) can be given as follows (see also Dingemans, 1983 and Dingemans, 1997, §3.3.6). Based on a maximum wave height adapted from Miche:

\[ H_b = \frac{0.88}{k} \tanh \left( \frac{\gamma}{0.88} kh \right) \]  
\( (2.31) \)

the dissipation of a periodic bore is computed as

\[ D_b = \frac{\alpha}{4} \rho g f \frac{H^3}{h}. \]  
\( (2.32) \)

For application to random waves the maximum wave height \( H_b \) is used for \( H \) and the fraction of breaking and broken waves, \( Q_b = P(H \geq H_b) \) is used. Also is the formula simplified somewhat by substituting the fact that \( H_b/h = \mathcal{O}(1) \); it is needed anyhow to introduce an unknown scale factor \( \alpha \); this coefficient has to be of order one and had been chosen simply as \( \alpha = 1 \). The dissipation function \( D_b \) then becomes\(^{14}\):

\[ D_b = \frac{\alpha}{4} \rho g f Q_b H_b. \]  
\( (2.33) \)

Based on calculations with the one-dimensional program ENDEC an optimal curve for the setting of the parameter \( \gamma \) was found, see Stive and Dingemans (1984) and Battjes and Stive (1985). The result was:

\[ \gamma = 0.5 + 0.4 \tanh(33s_0), \]  
\( (2.34a) \)

where \( s_0 \) is the deep-water steepness, based on the peak-frequency in the spectrum. Later Nairn (1990) found, partly based on the same measurements, extended with data with more swell present, a slightly different best fit:

\[ \gamma = 0.39 + 0.57 \tanh(33s_0). \]  
\( (2.34b) \)

These two fits are shown in Figure 2.3. It is seen that a considerable difference between the two fits exists. For the steeper waves the curve of Nairn yields higher \( \gamma \) values, meaning less breaking and thus less dissipation due to wave breaking.

Besides the optimal setting of the break-parameter \( \gamma \), Beyer (1994) showed that for \( \alpha = 0.5 \) instead of 1, together with an adapted value for \( \gamma \), better results were obtained between computation and measurements. As explained by Dingemans (1997, pp. 397-398), an equivalent wave dissipation was obtained when was used

\[ \alpha \gamma^{-5} = \text{constant}. \]  
\( (2.35) \)

With \( (\alpha_0, \gamma_0) = (1, 0.80) \) then is obtained \( (\alpha_1, \gamma_1) = (0.5, 0.70) \). The setting of Beyer instead of the one of Battjes and Stive then gives earlier wave breaking, but not so intense.

\(^{14}\)The dimension of \( D_b \) then is \( J/(m^2s) \). Notice that this \( D_b \) is not a density as is used in the action balance equation.
Proposition 1

We suggest as a first step to use Nairn’s optimal value for $\gamma$, and also apply Beyers method subsequently for $\alpha = 0.5$, thus:

$$\gamma_n = 0.39 + 0.57 \tanh(33s_0) ,$$

(2.36a)

and, with

$$\alpha_1 \gamma_1^{-5} = \alpha_2 \gamma_2^{-5} \quad \text{we obtain} \quad \gamma_2 = \left( \frac{\alpha_2}{\alpha_1} \right)^{1/5} \gamma_1 ,$$

(2.36b)

With $\alpha_1 = 1$ and $\alpha_2 = 1/2$ we then obtain

$$\gamma_o = \left( \frac{1}{2} \right)^{1/5} \gamma_n \approx 0.87 \gamma_n ,$$

(2.36c)

where $\gamma_o$ is the optimal value to be used. In this connection we notice that SWAN is usually performed with $\gamma = 0.73$ and a maximum wave height $H_b = \gamma h$, which choice has already been discussed and criticised at length in Dinge- mans (1998). Notice that with (2.36c) the optimum value of $\gamma$ can be determined immediately from

$$\gamma_o = 0.44 + 0.35 \tanh(33s_0) , \quad \alpha = \frac{1}{2} \quad \text{(Battjes & Janssen)} \quad \text{(2.36d)}$$

and

$$\gamma_o = 0.34 + 0.50 \tanh(33s_0) , \quad \alpha = \frac{1}{2} \quad \text{(Nairn)} .$$

(2.36e)

Alternatively, a value of the wave-breaking parameter proposed by Nelson (1994), dependent on also bottom slope, is used in conjunction with the BJ approach:

$$\gamma = 0.55 + 0.88 \exp \left( -\frac{0.012}{m} \right) ,$$

(2.37)

with $m$ the local bottom slope. In contrast to the previously mentioned values for the breaker parameter the parameter proposed by Nelson does not depend on the wave condition (local or at deep water). Furthermore, unrealistically high values for $\gamma$ are obtained for steeper slopes. For $m > 0.018$ we obtain $\gamma > 1$. Therefore, the wave-dependent formulations by Battjes and Stive (1985) or Nairn (1990) are preferred.

2.5.3 The maximum wave height and the probability of wave breaking

In the dissipation function $D_b$ also the parameters $Q_b$ and $H_b$ appear. In the original formulation of Battjes and Janssen (1978) the maximum wave height was chosen as an adaption of one given by Miche, see Eq. (2.31). It appeared that this value did not give an accurate
Figure 2.3: The optimal $\gamma$'s of Battjes and Stive (1985) (circles) and Nairn (1990) (full line) for $\alpha = 1/2$ and $\alpha = 1$.

description of the maximum wave heights observed in the field; this $H_b$ is to be regarded only as a suitable computational parameter.

We now identify the following problems to be addressed:

- The determination of the probability function of the wave heights.
- The determination of the probability of wave breaking (i.e. the determination of $Q_b = P(H > H_b)$).
- The determination of the maximum wave height $H_b$ (i.e. the wave height above which all waves are breaking or broken).

We consider two types of distributions for the wave heights, the much-used Rayleigh distribution and the modified Glukhovskiy distribution.

It has to be reminded that the Rayleigh distribution is only valid for linear waves with a narrow spectral width. Also it is assumed that the phases of the wave components are uncorrelated and uniformly distributed. The case of non-uniformly distributed phases is discussed by Beckman (1964). Tayfun (1981) investigated the difference of using zero-crossing wave height to the wave-envelope wave heights used to derive the Rayleigh distribution. He found that the the density of the zero-crossing wave heights displays an excess of waves with heights about the mean wave height, and deficiencies at both ends away of the mean. A principal effect of a deficiency toward the high wave tail is an underestimate in the exceedance probability of larger waves. The modified Glukhovskiy distribution remedies this deficiency, see Figure 2.5.
2.5.4 The probability function for the wave heights

The Rayleigh distribution

The probability $Q_b$ that the waves are breaking or have been broken is expressed as $Q_b = P(H \geq H_b)$. The probability distribution of the wave heights has then still to be chosen. The simplest one is provided by the Rayleigh distribution (see Rayleigh, 1880, or Vanmarcke, 1984, p. 54). Vanmarcke states that when two independent random variables $x_1$ and $x_2$ are normally distributed with zero mean and common variance $\sigma$, the derived random variable $r = (x_1^2 + x_2^2)^{1/2}$ will follow a Rayleigh distribution with probability density function:

$$f(r) = \frac{r}{\sigma^2} \exp \left[ -\frac{1}{2} \left( \frac{r}{\sigma} \right)^2 \right], \quad r \geq 0 . \quad (2.38a)$$

Translated to water waves, $r$ is the amplitude of the wave envelope and $\sigma^2$ is the variance $m_0$. Introducing the wave height $H = 2r$, and imposing $f(r)dr = f(H)dH$, so that $\int_0^\infty f(H)dH = 1$, the Rayleigh probability density function can be written as:

$$f(H) = \frac{H}{4m_0} \exp \left[ -\frac{H^2}{8m_0} \right] = \frac{2H}{H_{\text{rms}}^2} \exp \left[ -\left( \frac{H}{H_{\text{rms}}} \right)^2 \right]. \quad (2.38b)$$

The probability distribution $F(H)$ follows from the definition $F(H) = \int_0^H f(H')dH'$ as

$$F(H) = 1 - \exp \left[ -\left( \frac{H}{H_{\text{rms}}} \right)^2 \right]. \quad (2.38c)$$

For later use also the normalised distribution is useful. Introducing the normalised wave height $\rho$ by $\rho = H/H_{\text{rms}}$, we have

$$F(\rho) = 1 - e^{-\rho^2} \quad \text{and} \quad f(\rho) = 2\rho e^{-\rho^2} , \quad (2.38d)$$

which have been plotted in Figure 2.4. The relative mean wave height $\bar{\rho} = \bar{H}/H_{\text{rms}}$ follows as\textsuperscript{15}:

$$\frac{\bar{H}}{H_{\text{rms}}} = \bar{\rho} = \frac{M_1}{\sqrt{M_2}} = \frac{\int_0^\infty 2\rho^2 e^{-\rho^2} d\rho}{\int_0^\infty 2\rho^4 e^{-\rho^2} d\rho} = \frac{1}{2} \sqrt{\pi} \approx 0.886 . \quad (2.38e)$$

The wave height $H_N$ with exceedance probability $Q(H_N) = P(H > H_N) = 1/N$ follows then from (2.38c) as\textsuperscript{16}:

$$\frac{H_N}{H_{\text{rms}}} = \rho_N = \sqrt{\log N} . \quad (2.38f)$$

\textsuperscript{15}In this case $\bar{H}$ also follows directly from $\bar{H} = M_1$; notice also that $\int_0^\infty 2\rho^3 e^{-\rho^2} d\rho = 1$.

\textsuperscript{16}$\log$ is the natural logarithm.
Figure 2.4: The Rayleigh distribution $F(\rho)$ and its density $f(\rho)$.

The mean of the waves higher than $H_N$ is given by\textsuperscript{17}:

\[
\frac{\bar{H}_{1/N}}{H_{\text{rms}}} = \bar{\rho}_{1/N} = \frac{\int_{\rho_N}^{\infty} 2\rho^2 e^{-\rho^2} d\rho}{\int_{\rho_N}^{\infty} 2\rho e^{-\rho^2} d\rho} = \sqrt{\log N} + \frac{\sqrt{\pi}}{2} N \operatorname{erfc}\left(\sqrt{\log N}\right) .
\]

(2.38g)

Notice that the numerator in (2.38g) is needed as a scaling of the probability.

For some values of $N$ the relative wave heights $H_N/H_{\text{rms}}$ and $\bar{H}_{1/N}/H_{\text{rms}}$ are given in Table 2.1.

<table>
<thead>
<tr>
<th>$N$</th>
<th>3</th>
<th>10</th>
<th>50</th>
<th>100</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_N = \frac{H_N}{H_{\text{rms}}}$</td>
<td>1.04814</td>
<td>1.51743</td>
<td>1.97788</td>
<td>2.14597</td>
<td>2.49291</td>
<td>2.62826</td>
</tr>
<tr>
<td>$\bar{\rho}<em>{1/N} = \frac{\bar{H}</em>{1/N}}{H_{\text{rms}}}$</td>
<td>1.41573</td>
<td>1.79991</td>
<td>2.20633</td>
<td>2.35924</td>
<td>2.68020</td>
<td>2.80698</td>
</tr>
</tbody>
</table>

Table 2.1: The relative wave heights $H_N/H_{\text{rms}}$ and $\bar{H}_{1/N}/H_{\text{rms}}$ as function of $N$.

The modified Gluhkovskiy distribution

The Gluhkovskiy distribution is in effect a two-parameter Weibull distribution, originally

\textsuperscript{17}We again introduce $\rho = H/H_{\text{rms}}$. The error function $\operatorname{erf}(x)$ and its complement $\operatorname{erfc}(x)$ are defined by (Abramowitz and Stegun 1964, Eqs. 7.1.1 and 7.1.2):

\[
\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad \text{and} \quad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt = 1 - \operatorname{erf}(x) .
\]
formulated in the mean wave height and the relative wave height parameter. For wave models like SWAN it is easier to have a formulation in the root-mean-square wave height, which is directly related to the variance in the wave train. We therefore follow the formulation of Klopman (1996) and we write the probability distribution of the wave heights as

\[ F(H) = P(H \leq H) = 1 - \exp \left[ -B \left( \frac{H}{H_{\text{rms}}} \right)^\kappa \right], \]  

(2.39a)

and thus is the probability density function \( f(H) = dF(H)/dH \) is given by:

\[ f(H) = \frac{\kappa B}{H_{\text{rms}}} \left( \frac{H}{H_{\text{rms}}} \right)^{\kappa - 1} \exp \left[ -B \left( \frac{H}{H_{\text{rms}}} \right)^\kappa \right]. \]  

(2.39b)

The exponent \( \kappa \) is a function of the relative wave height \( d = H_{\text{rms}}/h \). For \( \kappa \) is chosen

\[ \kappa = \frac{2}{1 - \beta d}, \]  

(2.39c)

with \( \beta > 0 \). Notice that \( \kappa > 2 \) for \( d > 0 \).

The shape parameter \( B \) is determined by imposing the condition that the second moment equals \( H_{\text{rms}}^2 \). We first determine the \( n \)-th moment \( M_n \) as\(^{18}\)

\[ M_n = \int_0^\infty H^n f(H) dH = \kappa B H_{\text{rms}}^n \int_0^\infty \rho^{n+\kappa-1} \exp \left( -B \rho^\kappa \right) d\rho = \]

\[ = H_{\text{rms}}^n B^{-n/\kappa} \int_0^\infty t^{n/\kappa} e^{-t} dt = H_{\text{rms}}^n B^{-n/\kappa} \frac{n}{\kappa + 1}. \]  

(2.39d)

Imposing \( M_2 = H_{\text{rms}}^2 \) then yields \( B \) as:

\[ B = \left\{ \Gamma \left( \frac{2}{\kappa} + 1 \right) \right\}^{\kappa/2}. \]  

(2.39e)

The value of \( \beta \) in the choice for \( \kappa \) was chosen as \( \beta = 0.7 \), which gave a good fit with the laboratory data also used in Klopman and Stive (1989), see Figure 2.5.

In terms of the relative wave height \( \rho = H/H_{\text{rms}} \) the probability function \( F(\rho) \) and its probability density function \( f(\rho) \) are given by:

\[ F(\rho) = 1 - e^{-B\rho^\kappa} \quad \text{and} \quad f(\rho) = \kappa B \rho^{\kappa-1} e^{-B\rho^\kappa}. \]  

(2.39f)

These functions are plotted in Figure 2.6 for \( d = 0, d = 0.5 \) and \( d = 1 \). Notice that for \( d = 0 \) the Rayleigh distribution is recovered.

---

\(^{18}\)We first set \( \rho = H/H_{\text{rms}} \) and, subsequently, \( B \rho^\kappa = t \), and thus \( \kappa B t^{\kappa-1} dt = dt \). Notice also that \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \), see Abramowitz and Stegun (1964, Eq. (6.1.1)).
We now have, for example, the mean wave height given by

\[
\frac{\overline{H}}{H_{\text{rms}}} = \overline{\rho} = \frac{M_1}{\sqrt{M_2}} = \frac{\Gamma\left(\frac{1}{\kappa} + 1\right)}{\sqrt{\Gamma\left(\frac{2}{\kappa} + 1\right)}}.
\]  

(2.39g)

Notice that for \( d = 0 \) and thus, \( \kappa = 2 \) and \( \Gamma(2) = 1 \), we again have the result (2.38g) from the Rayleigh distribution, \( \overline{\rho} = \sqrt{\pi}/2 \).

The wave height \( H_N \) with the exceedance probability \( Q(H_N) = P(H > H_N) = 1/N \) follows then from (2.39a) as:

\[
\frac{H_N}{H_{\text{rms}}} = \rho_N = \left\{ \frac{1}{B} \log N \right\}^{1/\kappa} = \frac{\left(\log N\right)^{1/\kappa}}{\sqrt{\Gamma\left(\frac{2}{\kappa} + 1\right)}}.
\]  

(2.39h)
(a) $F(\rho)$, $d = 0$ (left), $d = 0.5$ (middle) and $d = 1$ (right) graph in lower half.

(b) $f(\rho)$, $d = 0$ (lowest), $d = 0.5$ (middle) and $d = 1$ (upper) graph.

Figure 2.6: The modified Glukhovsky distribution $F(\rho)$ and its density $f(\rho)$ for $d = 0$, 0.5 and 1.

Some examples of $H_N/H_{rms}$ have been plotted in Figures 2.7. Notice that for $d = 0$ the results of the Rayleigh distribution are recovered. It follows from the Figures that the $H_N$ of the modified Glukhovsky distribution are generally lower than the ones following from the Rayleigh distribution; only for $H_3$ and in lesser extent, for $H_4$ the wave heights increase initially for increasing values of $d$.

The relative wave height $\rho_N$ is plotted as function of $N$ for three values of $d$ in Figure 2.8.

The relative wave height $\bar{\rho}_{1/N} = \bar{H}_{1/N} / H_{rms}$ is defined as the mean of the wave heights larger than $H_N$:

\[
\frac{\bar{H}_{1/N}}{H_{rms}} = \bar{\rho}_{1/N} = \frac{\int_0^\infty \rho f(\rho) d\rho}{\int_0^\infty f(\rho) d\rho} = \frac{\int_0^\infty \rho B \rho^\kappa \exp(-\rho^\kappa) d\rho}{\int_0^\infty \rho^\kappa \exp(-\rho^\kappa) d\rho} = B^{-1/\kappa} \int_0^\infty t^{1/\kappa} e^{-t} dt / \beta_N t^{-\kappa} e^{-t} dt \quad \text{with} \quad t_N = B \rho_N^\kappa = \log N. \quad (2.39i)
\]

Therefore we have\(^{19}\):

\[
\frac{\bar{H}_{1/N}}{H_{rms}} = \bar{\rho}_{1/N} = B^{-1/\kappa} N \frac{1}{\kappa} \left( 1 + \log N \right), \quad (2.40a)
\]

and thus, with the value (2.39e) substituted for $B$:

\[
\frac{\bar{H}_{1/N}}{H_{rms}} = \bar{\rho}_{1/N} = N \left( \frac{1}{\kappa} + 1, \log N \right) \frac{1}{\sqrt{\Gamma \left( \frac{2}{\kappa} + 1 \right)}}. \quad (2.40b)
\]

\(^{19}\text{The incomplete Gamma function is defined as } \Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt \text{ and } \gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt, \text{ and thus, } \Gamma(a, x) = \Gamma(a) - \gamma(a, x).\)
Notice that for $\kappa = 2$ and $N = 3$ we should get the significant wave according to the Rayleigh distribution. From (2.40b) we indeed obtain $\bar{H}_{1/3} = 1.41573$. The dependence on $d$ and $\kappa$ of $\bar{H}_{1/3}$ has been shown in Figure 2.9. It becomes clear that for increasing $d$ the wave height $\bar{H}_{1/3}$ is decreasing.

The composite Weibull distribution

In deep water the wave height distribution is accurately described by the Rayleigh distribution. However, in shallow water wave breaking can cause the distribution to differ significantly from the Rayleigh distribution, in a manner that at present is not well known.

Recently, Battjes and Groenendijk (2000) proposed a model distribution based on a Rayleigh distribution, or a Weibull distribution with exponent equal to 2, for the lower wave heights.
and a Weibull distribution with a higher exponent for the higher wave heights. The composite Weibull distribution is given by

\[
F(H) = P(H \leq H) = \begin{cases} F_1(H) = 1 - \exp \left[ - \left( \frac{H}{H_1} \right)^{k_1} \right] & H \leq H_{tr} \\ F_2(H) = 1 - \exp \left[ - \left( \frac{H}{H_2} \right)^{k_2} \right] & H \geq H_{tr} \end{cases}
\]

(2.41)

The exponents \( k_1 \) and \( k_2 \) are shape parameters of the distribution. \( H_1 \) and \( H_2 \) are scale parameters. At the transition wave height \( H_{tr} \) continuity is required. The parameters in the assumed model distribution have been calibrated and validated by using experimental wave flume data obtained from experiments on shallow foreshores, ranging from 1:20 to 1:250. After validation \( k_1 = 2, k_2 = 3.6 \). The transition wave height is assumed to be proportional to the local water depth \( h \) and a slope dependent coefficient \( \gamma_{tr}(m) \) with \( m \) the slope of the foreshore:

\[
H_{tr} = \gamma_{tr} h.
\]

(2.42)

Assuming a linear variation of \( \gamma_{tr} \) Battjes and Groenendijk found that \( \gamma_{tr} = 0.35 + 5.8m \). Finally, the scaling parameters \( H_1 \) and \( H_2 \) can be determined from an implicit relation (Battjes and Groenendijk, 2000, Eq. (7)). The scaling parameters are a function of \( H_{tr} \). For a given value of \( H_{tr} \) these values are listed in an extensive table (see Battjes and Groenendijk, 2000, Table 2).

The proposed distribution model based on the composite Weibull distribution depends on total wave energy, local water depth and bed slope. For sloping bottoms this distribution is a promising alternative for the conventional Rayleigh and Glukhovskiy distributions. However, it is unclear how the performance of the model will be on horizontal bottoms. Since the model calibration includes 1:250 slopes the model distribution of Battjes and Groenendijk (2000) may be successful for horizontal bottoms as well. This needs to be investigated in the future.
2.5.5 The probability of wave breaking, $Q_b$

The clipped Rayleigh distribution

Battjes and Janssen (1978) define the probability that at a certain location a wave height is associated with a broken or breaking wave ($H \geq H_b$) by means of

$$Q_b = P(H \geq H_b) = \exp \left\{ -\frac{1}{2} \left( \frac{H_b}{H} \right)^2 \right\}, \quad (2.43)$$

with $\bar{H}$ an unknown wave height. They thus use a clipped Rayleigh distribution for the wave heights, which is given by:

$$F(H) \equiv P(H \leq H) = 1 - \exp \left[ -\frac{1}{2} \left( \frac{H}{\bar{H}} \right)^2 \right] \quad ; \quad 0 \leq H < H_b$$

$$= 1 \quad ; \quad H \geq H_b. \quad (2.44)$$

The root-mean-square wave height $H_{rms}$ then follows as:

$$H_{rms}^2 = \int_0^\infty H^2 dF(H) = \int_0^{H_b} H^2 dF(H) + Q_b \bar{H}^2.$$

Carrying out the integration yields

$$H_{rms}^2 = 2\bar{H}^2 (1 - Q_b). \quad (2.45)$$
Elimination of the unknown \( \hat{H} \) then gives the implicit relationship for \( Q_b \):

\[
Q_b = \exp \left[ -\frac{(1 - Q_b)}{b^2} \right] \quad \text{with} \quad b = \frac{H_{\text{rms}}}{H_b}.
\] (2.46)

As shown in Dingemans (1983), a sufficient approximation of this implicit relation is given as follows. First an initial value \( q_0 \) is chosen as:

\[
q_0 = (2b - 1)^2 \quad \text{for} \quad 0.5 < b < 1
\]
\[
q_0 = 0 \quad \text{for} \quad b \leq 0.5.
\] (2.47a)

(2.47b)

Performing one iteration of a Newton-Raphson procedure yields the next approximation \( q_1 \) as

\[
q_1 = q_0 - b^2 \frac{q_0 - p}{b^2 - p} \quad \text{with} \quad p = \exp \left[ \frac{(q_0 - 1)}{b^2} \right].
\] (2.47c)

The value \( q_1 \) is accurately enough to serve as an approximation for \( Q_b \).

The clipped modified Glukhovskiy distribution

The clipped modified Glukhovskiy distribution is given by

\[
F(H) = P(H \leq H) = \begin{cases} 
1 - \exp \left[ -B \left( \frac{H}{\hat{H}} \right)^\kappa \right] & \text{for} \quad H \leq H_b \\
1 & \text{for} \quad H > H_b.
\end{cases}
\] (2.48)

In the same way as before we introduce \( \rho = H/\hat{H} \) and obtain

\[
Q_b = \exp \left[ -B \left( \frac{H_b}{\hat{H}} \right)^\kappa \right],
\] (2.49a)

and with \( t = B \rho^\kappa \) we get

\[
H_{\text{rms}}^2 = \int_0^{H_b} H^2 dF(H) + Q_b H_b^2 = \kappa B \hat{H}^2 \int_0^{q_0} \rho^{\kappa+1} \exp \left( -B \rho^\kappa \right) d\rho + Q_b H_b^2
\]

\[
= \hat{H}^2 B^{-2/\kappa} \int_0^{q_0} t^{2\kappa} e^{-t} dt + Q_b H_b^2
\]

\[
= \hat{H}^2 B^{-2/\kappa} \left\{ \frac{2}{\kappa} + 1, B \left( \frac{H_b}{\hat{H}} \right)^\kappa \right\} + Q_b H_b^2.
\] (2.49b)

From (2.49a) we have

\[
\log Q_b = -B \left( \frac{H_b}{\hat{H}} \right)^\kappa \quad \text{and thus} \quad \hat{H}^2 = H_b^2 B^{2/\kappa} \left( -\log Q_b \right)^{-2/\kappa}.
\] (2.50)

Substitution of these expressions into (2.49b) then yields the implicit relation for \( Q_b \):

\[
Q_b = b^2 - \gamma \left( \frac{2}{\kappa} \right) - \log Q_b \quad \text{with} \quad b = \frac{H_{\text{rms}}}{H_b}.
\] (2.51a)
To see whether (2.51a) has solutions, we introduce the function

$$f(q) = q - b^2 + \gamma \left( \frac{2}{\kappa} + 1, - \log q \right).$$  \hfill (2.51b)

We now draw $f(q)$ for a number of values $b$. From Figures 2.10 it is seen that only within the range $0.9 < b < 1$ solutions for $q$ are obtained. For $b = 0.90, 0.98$ and $1.0$ we look in more detail. For $\kappa = 2, 6$ and $8$, we plot $f$ as function of $q$ in Figure 2.11. We see that the solutions for $q$ which can be obtained are mostly very close to 1 and therefore unrealistic. We therefore conclude that in this way the modified Glukhovskiy distribution cannot be used. It has to be remembered that this distribution is based upon measurements, in which wave breaking has already been accounted for.

(a) $f(q)$ for $b = 0.95$

(b) $f(q)$ for $b = 1.0$

Figure 2.10: The function $f(q)$.

(a) $b = 0.90$

(b) $b = 0.98$

(c) $b = 1.0$

Figure 2.11: $f(q)$ for three values of $b$. Per subplot $\kappa = 2$ (lowest line), $\kappa = 6$ (middle line) and $\kappa = 8$ (upper line) have been considered.
The method of Baldock et al. (1998)

Baldock et al. (1998) use the Rayleigh distribution, which they assume valid for both broken and unbroken waves:

\[ F(\rho) = 1 - e^{-\rho^2} \quad \text{and} \quad f(\rho) = 2\rho e^{-\rho^2} \quad \text{with} \quad \rho = \frac{H}{H_{\text{rms}}} . \] (2.52)

We then have:

\[ Q_b = P (\rho > \rho_b) = \int_{\rho_b}^{\infty} f(\rho) d\rho = \int_{\rho_b}^{\infty} 2\rho \exp\left(-\rho^2\right) d\rho = \exp\left(-\rho_b^2\right) . \] (2.53)

We thus obtain an explicit expression for \( Q_b \). This \( Q_b \) is compared with the one of Battjes and Janssen (1978) in Figure 2.12.

![Figure 2.12: \( Q_b \) as a function of \( b = H_{\text{rms}}/H_0 \) for Battjes and Janssen (1978) and Baldock et al. (1998).](a)\( Q_b \) according to Battjes and Janssen (circles) and Baldock et al. (line)

(b)\( Q_b \) for the range \( 0.4 \leq b \leq 0.8 \).

It is seen that Baldock et al. (1998) obtain \( Q_b = 0.4 \) for \( b = 1 \) whereas Battjes and Janssen (1978) obtain \( Q_b = 1 \) for \( b = 1 \). This difference is due to different assumptions. While Battjes and Janssen suppose the surf zone to be saturated (i.e. fully determined by depth-limited wave breaking), Baldock et al. take the surf zone as unsaturated. This is in accordance with findings of Thornton and Guza (1983, Figure 11), i.e. \( Q_b = 0.5 \) for \( b = 1 \).

The dissipation function is obtained from (2.32), with using the approximation \( H/h \approx 1 \) by multiplying each wave \( H \) with the probability that it is broken. Writing \( A = \frac{1}{4} \alpha \rho_w g f_p \), this yields\textsuperscript{20}:

\[ D_b = A \int_{H_b}^{\infty} H^2 f \left( \frac{H}{H_{\text{rms}}} \right) d \left( \frac{H}{H_{\text{rms}}} \right) = AQ_b \left( H_b^2 + H_{\text{rms}}^2 \right) . \] (2.54)

\textsuperscript{20}To avoid confusion, we now write \( \rho_w \) for the density of the water.
In the saturated surf zone (for which $H_{rms} = H_b$) then $D_b = (2/e)A H_b^2 \cong 0.74 A H_b^2$, whereas from Battjes and Janssen follows $D_b = A H_b^2$. We now investigate the difference in dissipation function for the two cases in some detail. The dissipation function and the probability of wave breaking are denoted with a prime for Baldock et al.’s formulation. We then have, using $b = H_{rms}/H_b$:

$$Q_b = \exp\left[-\frac{(1 - Q_b)}{b^2}\right] \quad \text{and} \quad Q_b' = \exp\left(-\frac{1}{b^2}\right) \quad (2.55a)$$

$$D_b = A H_{rms}^2 \frac{Q_b}{b^2} \quad \text{and} \quad D_b' = A H_{rms}^2 Q_b' \left(\frac{1}{b^2} + 1\right) \quad (2.55b)$$

We now plot $D_b/(A H_{rms}^2)$ and $D_b'/(A H_{rms}^2)$ as function of $b$, see Figure 2.13. We also plot $D_b'/D_b$. We see that the dissipation function according to Baldock et al. is less than the one of Battjes and Janssen for $b > 0.71$ and is higher for lower values of $b$. For practical applications the dissipation function of Baldock et al. leads to higher waves heights.

Figure 2.13: Comparison of dissipation functions according to Battjes and Janssen (1978) and Baldock et al. (1998).

The use of the modified Glukhovskiy distribution

We now investigate the difference between $Q_b$ obtained with the Rayleigh distribution and the modified Glukhovskiy distribution. With the density $f(\rho)$ given in (2.39f) we have

$$Q_b'' = \kappa B \int_{\rho_b}^{\infty} \rho^{\kappa-1} \exp\left[-B \rho^\alpha\right] \, d\rho = \exp\left[-B \rho_b^\alpha\right] = \exp\left[-\frac{B}{b^\alpha}\right]. \quad (2.56)$$

In Figure 2.14 we plot $Q_b''$ for the values $\kappa = 2, 4, 6, 8$ (thus, from relation (2.39c) with $\beta = 0.7$, we obtain $d = H_{rms}/h = 0, 5/7, 20/21$ and 15/14). Notice that for $\kappa = 2$ (and thus, $d = 0$) we recover the case of Baldock et al. It should be noted that while the Figures are
given up to a value $\kappa = 8$ (for which $d = 1.07$), the applicability region is in fact only for values $d \leq 0.6$, and thus, well below $\kappa = 4$.

The dissipation function is obtained as follows:

$$D_b'' = AH_{rms}^2 \int_{\rho_b}^{\infty} \rho^2 \kappa B \rho^{-1} \exp \left[-B \rho^\kappa\right] d\rho = AH_{rms}^2 B^{-2/\kappa} \int_{t_b}^{\infty} t^{2/\kappa} e^{-t} dt = AH_{rms}^2 B^{-2/\kappa} \Gamma \left(\frac{2}{\kappa} + 1, \frac{B}{B_{b'}}\right) \Gamma \left(\frac{2}{\kappa} + 1, \frac{B}{B_{b'}}\right). \tag{2.57}$$

It follows that the dissipation is less for more non-linear waves (higher values of $d$ and thus $\kappa$). Appreciable dissipation starts later (for higher values of $b = H_{rms}/H_b$). When this method is used in SWAN, then at each computational step the local value of $d = H_{rms}/h$ has to be determined, from which the local value for $\kappa$ can be determined. With these values for $\kappa$ and the break parameter $b$ the local dissipation function can be determined. It is suggested to try this procedure first in a 1D model, such as ENDEC. It is obvious that this dissipation function cannot be used immediately in SWAN without performing an extensive study, comparing computations with measurements to provide the correct setting of the parameters $\alpha$ and $\gamma$ and find a prescription for the curve of Battjes and Stive (1985).

2.5.6 The maximum wave height

According to the SWAN papers (e.g. Ris, 1997, p. 16 and Booij et al, 1999) the wave breaking in shallow water is accounted for by using the formulation of Battjes and Janssen (1978). Inspection of the source code shows that the maximum wave height $H_b$, which is estimated
in BJ from the adapted Miche criterion (2.31), is replaced by:

\[ H'_b = \gamma h \, , \]  

(2.58)

where a prime has been added to ease the distinction with (2.31).

In SWAN the wave breaking is more sudden than in the original Battjes and Janssen formulation. This can be seen in Figure A.6 in Appendix A where the dissipation function has been plotted as obtained for the original Battjes and Janssen formulation and the SWAN formulation. Care has been exercised that only wave breaking due to depth limitation occurred in these examples.

The examples in Dingemans (1998, Appendix A) show that not much difference is found in very shallow water for the different definitions of the maximum wave height, which was to be expected; in intermediate depth the difference increases and it is questionable whether this difference can be attributed wholly to the neglect of dissipation due to white-capping. This neglect is due to the fact that \( H_b \) is defined as a smooth transition between 0.88/\( k \) for large \( kh \) for which breaking on steepness occurs and \( \gamma h \) for small \( kh \) where wave breaking due to restricted depth occurs. It is not clear whether in the intermediate region both dissipation mechanisms of breaking on steepness and on depth occur.

An objection to the maximum wave height \( H'_b = \gamma h \) is that the dependence of the maximum wave height on the relative depth \( kh \) is lost. Hamm (1995, pp. 178-179) shows that a good approximation for some measurements (Danel, 1951 and Yamada and Shiotani, 1968) is given by the formula:

\[ \frac{H_b}{h} = 0.806 - 1.15 \frac{h}{\lambda_0} \quad \text{for} \quad \frac{h}{\lambda_0} \leq \frac{1}{4} \, , \]  

(2.59)

where \( \lambda_0 \) is the deep-water wave length. Using the deep-water dispersion relation \( \omega^2 = gk \), we rewrite this expression in terms of \( kh \) as

\[ \frac{H_b}{h} = 0.806 - 1.15 \frac{kh}{2\pi \tanh kh} = 0.806 - 0.183kh \tanh kh \, . \]  

(2.60)

This expression is compared with the adapted Miche expression (2.31) for \( \gamma = 0.806 \) so as to start for \( kh = 0 \) with the same value. The two curves are given in Figure 2.15. Also included are the graphs for \( \gamma = 0.7 \) and 0.6. What is clear from these graphs is that the maximum wave height is strongly dependent on the relative depth \( kh \). Notice that \( h/\lambda_0 = 1/4 \) translates to \( kh = 1.683 \).

Recently, Rattanapitikon and Shibayama (2000) compared a large number of breaker height formula's and checked them against measured breaker heights from 24 sources. It was found that the root-mean-square error, \( \varepsilon_{\text{rms}} \), increased for increasing bottom slopes \( m \). Therefore a new form of bottom slope effect was derived to decrease the error in the breaker height formulae. This was done for a number of formulae which performed reasonably well originally. These were the formulae of Komar and Gaughan (1972), Goda (1970, 1974) and Ostendorf
Figure 2.15: A few curves for the maximum wave height, Eq. (2.60) from Hamm (1995) and Eq. (2.31) for $\gamma = 0.806$, 0.7 and 0.6.

and Madsen (1979). For completeness, these formulae are given here$^{21}$:

Komar and Gaughin (1972):

$$H_b = 0.56H_0 \left( \frac{H_0}{\lambda_0} \right)^{-1/5}$$

(2.61)

Goda (1974) expressed the diagramme given by Goda (1970) in formula form:

$$H_b = 0.17\lambda_0 \left\{ 1 - \exp \left[ -1.5 \frac{\pi h}{\lambda_0} \left( 1 + 15m^{4/3} \right) \right] \right\}$$

(2.62)

Ostendorf and Madsen (1979) Modified Miche’s formulation by including effects of bottom slope. After comparison with laboratory data they found:

$$H_b = 0.14\lambda_0 \tanh \left[ (0.8 + 5m) \, kh \right] \quad \text{for} \quad m \leq 0.1$$

(2.63a)

$$H_b = 0.14\lambda_0 \tanh \left[ 1.3kh \right] \quad \text{for} \quad m > 0.1$$

(2.63b)

Rattanapitikon and Shibayama (2000) now modified these formulæ by replacing the constant in these formulæ by an unknown coefficient. By performing a regression analysis between

$^{21}$We write $kh$ for $k_0h_0$ for simplicity; $\lambda$ is the wave length and $\lambda_0$ is the deep-water wave length.
measured breaker height as function of the bottom slope, this unknown coefficient was determined as function of the bottom slope. In this way they found the following modified formulae:

\[ H_b = \left(10.02m^3 - 7.46m^2 + 1.32m + 0.55\right) H_0 \left(\frac{H_0}{\lambda_0}\right)^{-1/5} \]  
\[ H_b = 0.17\lambda_0 \left\{1 - \exp\left[\frac{\pi h}{\lambda_0} \left(16.21m^2 - 7.07m - 1.55\right)\right]\right\} \]  
\[ H_b = 0.14\lambda_0 \tanh \left\{\left(-11.21m^2 + 5.01m + 0.91\right) kh\right\}. \]  

(2.64a)  
(2.64b)  
(2.64c)

The conclusion of Rattanapitikon and Shibayama (2000) is that the modified Goda formula (2.64b) gives the best results in the general case. The modified Ostendorf and Madsen expression (2.64c), the modified Komar and Gaughan expression (2.64a) and the original Komar and Gaughan (2.61) expression are second, third and fourth choice. We therefore concentrate on the modified Goda formulation (2.64b). Using \( \omega^2 = gk_0 = gk \tanh kh \), we rewrite (2.64b) as:

\[ \frac{H_b}{h} = \frac{1.07}{kh \tanh kh} \left\{1 - \exp\left[\frac{1}{2} kh \tanh(kh) \left(16.21m^2 - 7.07m - 1.55\right)\right]\right\}. \]  

(2.65)

We plot this expression for \( m = 0 \), \( m = 0.01 \) and \( m = 0.1 \) in Figure 2.16. For \( m = 0.001 \) no visible difference with \( m = 0 \) could be noted. We also compare this expression for \( m = 0 \) with Hamm’s breaking-wave expression (2.60).

![Figure 2.16: The breaking wave heights according to the modified Goda formula, \( m = 0 \) (lowest), \( m = 0.01 \) (middle) and \( m = 0.1 \) (upper) (left) and Goda for \( m = 0 \) (upper, at \( kh = 0 \)) and Hamm (lower) (right).](image)

A further objection to the implementation in SWAN is that with the maximum wave height \( H_b' \) the previous studies (Battjes and Stive, 1985 and Nairn, 1990) for the parameter setting
of the wave breaking parameter $\gamma$ cannot be used anymore. These studies are not valid for the setting (2.58). Some implications of this setting have been studied in Dingemans (1998, Appendix A).

A third objection to the chosen formulation in SWAN is that for SWAN a constant wave-breaking parameter is advised, constant irrespectively of the wave conditions at hand. As discussed before, this gives very poor results in wave-height predictions. With the wave-breaking formulation according to Battjes and Janssen (1978), it is impossible to take a standard value for $\gamma$, simply because such a value does not exist. It is therefore antiquated to advise a value 0.73 (being the mean over the computations from Battjes and Stive (1985), see Booij et al., 1999) or 0.8 as is taken in the code of SWAN.

Conclusion

Considering the various formulae for breaking wave heights, it is seen that they are not very different, especially when noting that Goda’s formula (for $m = 0$) and Hamm’s formula both start at $\gamma \approx 0.8$ for $kh = 0$ and are almost the same as the adapted Miche formula for $\gamma = 0.8$. We should note that with Goda’s formulation variation with the bottom slope is obtained, while, via the setting of Battjes and Stive (1985) for the adapted Miche formula variation for the deep-water wave condition is possible. Considering the various possibilities, we still have preference for the adapted Miche maximum wave height as we have the impression that then the best prediction of wave height in the coastal zone is possible. For the special case of horizontal bottom, see next paragraph.

2.5.7 Wave breaking on horizontal bottoms

Fenton (1990) gives the following maximum wave height, based on very accurate computations of Williams (1981) (see also Williams, 1985):

$$
\frac{H_b}{h} = \frac{0.141063 \left( \frac{\lambda}{h} \right) + 0.0095721 \left( \frac{\lambda}{h} \right)^2 + 0.0077829 \left( \frac{\lambda}{h} \right)^3}{1 + 0.0788340 \left( \frac{\lambda}{h} \right) + 0.0317567 \left( \frac{\lambda}{h} \right)^2 + 0.0093407 \left( \frac{\lambda}{h} \right)^3}.
$$

(2.66)

Notice that the limit for short waves gives the correct behaviour $H_b/\lambda = 0.141063$ and the shallow-water limit $\lambda/h \to \infty$ leads to $H_b/h \to 0.0077829/0.0093407 = 0.8332$.

The maximum wave height according to (2.66) has been plotted in Figure 2.17.

As is also mentioned by Fenton (1990, p. 20), experiments on mild slopes have shown that observed maximum wave heights remain below $H/h = 0.55$, see Nelson (1994) and also Massel (1996). This is ascribed to the fact that the waves become unstable before breaking when they propagate on a horizontal bottom. As discussed in Dingemans (1997, p. 320) also Van
der Meer (1990) and Gourlay (1994) found these results. Nelson gave for the breaking wave height the following expression:

\[ \frac{H_b}{h} = 0.55 + 0.88 \exp \left( -\frac{0.012}{m} \right) \],

(2.67)

with \( m \) the bottom slope. This relation is plotted as function of bottom slope \( m \) in Figure 2.18a.

Nelson also used a non-linearity parameter, originally introduced by Swart and Loubser (1978,1979). It is a function of wave period \( T \), water depth \( h \) and wave height \( H \) in the following way:

\[ F_c = \frac{g^{5/4}H^{1/2}T^{5/2}}{h^{7/4}} = \left( \frac{H}{h} \right)^{1/2} \left( T \sqrt{\frac{g}{h}} \right)^{5/2}, \]

(2.68)

where the second form has been given by Massel (1996). We now have shallow-water waves for \( F_c > 500 \), intermediate water waves for \( 10 < F_c < 500 \) and deep-water waves for \( F_c < 10 \). Nelson (1994) now gives a curve which acts as an envelope indicating the maximum of \( H/h \), which is valid for both shallow and intermediate water waves as:

\[ \frac{H}{h} = \frac{F_c}{22 + 1.82F_c}. \]

(2.69)

This curve is plotted in 2.18b.

Massel (1996) investigated the meaning of the parameter \( F_c \) further. Nelson (1994) found that waves with the same \( F_c \) had also approximately the same wave profile \( \zeta(t)/H \). The
parameter introduced by Goda (1983) reads:

\[ \Pi = \frac{H}{\lambda} \coth^3(kh) , \]  

(2.70)

with \( k = 2\pi/\lambda \). For deep water \((kh \to \infty)\) we have \( \Pi \to H/\lambda \) and for shallow water \((kh \to 0)\) we have

\[ \Pi \to \left( \frac{1}{2\pi} \right)^3 \left( \frac{H}{\lambda} \right) / \left( \frac{h}{\lambda} \right) = \left( \frac{1}{2\pi} \right)^3 U_r , \]  

(2.71)

where the Ursell number is given by \( U_r = (H/h)/(h/\lambda) \) By using the linear dispersion relation Massel expresses \( \Pi \) as function of \( F_c \) his equation (8)).

Substitution of the definition of \( F_c \), i.e. Eq. (2.68) into relation (2.67) leads to

\[
\frac{H_b}{h} = \left( \frac{\sqrt{1 + 0.01504h_* - 5/2} - 1}{0.1654h_* - 5/4} \right)^2 \quad \text{with} \quad h_* = \frac{h}{gT^2} . \]  

(2.72a)

or

\[
\frac{H_b}{gT^2} = \left( \frac{\sqrt{1 + 0.01504h_* - 5/2} - 1}{0.1654h_* - 5/4} \right)^2 h_* . \]  

(2.72b)

Notice that small values for \( h_* \) correspond to shallow water, while large values for \( h_* \) correspond to deep water. Massel notes that the relation between \( H/(gT^2) \) and \( h_* \) is almost linear
on logarithmic scale and, for values of $F_c$ between 100 and 1000 there is:

$$\frac{H_b}{gT^2} \cong 0.481 \frac{h}{gT^2} \quad \text{and thus,} \quad \frac{H_b}{h} \cong 0.481.$$  \hspace{1cm} (2.73)

This can be interpreted as a further indication of rather low maximum wave heights on horizontal bottoms.

After looking into the skewness $\gamma_1 = m_3 \cdot (m_2)^{-3/2}$ (with $m_n = \int_0^1 (\zeta/H)^n \, d(x/\lambda)$) of the waves, Masel gives a relation between $\gamma_1$ and $F_c$ in the range $100 \leq F_c \leq 1000$:

$$\gamma_1 = -1.2120 + 0.0165 F_c - 3.49 \cdot 10^{-5} F_c^2 + 3.6460 \cdot 10^{-8} F_c^3 - 1.4223 \cdot 10^{-11} F_c^4.$$  \hspace{1cm} (2.74)

Masel concludes that $F_c$ cannot be presented as a combination of wave non-linearity and wave dispersion, but should be recognised as being a quantity proportional to the skewness of the wave profile. Waves with the same $F_c$ approximately have the same skewness.

Masel (1996) remarks that both in the laboratory and in the field, the ratio of 0.55 seems to be the largest possible for stable, shallow water oscillatory waves propagating in water of constant depth.

For non-linear waves, for example calculated with high-order Stokes-type approximations and also by higher-order cnoidal-type approximations, higher limiting values of $H/h$ have been found, for example, Williams found 0.833, as mentioned earlier. He also states that the highest values of $H/h$ in a naturally occurring random wave train are located below the empirical envelope (2.69) of Nelson (1994).

2.5.8 Spectral depth-limited wave breaking

The Battjes and Janssen formulation for wave dissipation due to wave breaking has been proven to be an accurate and robust method for prediction of total energy through a wave-breaking region such as a surf zone. The most restrictive feature is that it only gives a (good) estimate for the total energy, but no information for the distribution of the dissipation over the variance spectrum is available from it. From experiments of wave propagation over an underwater barrier in a wave flume, it appeared that the resulting wave spectrum behind the bar was of a similar shape for non-breaking waves, for spilling breakers on the bar and for plunging breakers, see e.g. Beji and Battjes (1993). This led to the assumption that a good approximation might be to apply a reduction factor over the spectrum which was constant. Thus no dependence on the frequency for wave breaking was assumed. The amount of wave breaking in total energy was determined in the usual way, estimating first a wave breaking parameter $\gamma$.

Other investigators used frequency-dependent dissipation functions, see the discussion in Chen et al. (1997). These authors conclude that the resulting spectra are quite insensitive to the precise form of dependency on $f$ which is taken, $f$, $f^2$ or even $f^4$. However, it is very impor-
tant that the non-linear wave interaction is modelled adequately. Chen et al. (1997) used a Boussinesq-type model for their numerical experiments. SWAN is a spectral model. Similar non-linearities in SWAN as in a Boussinesq-type model consist of three-wave interaction. It is therefore essential that three-wave interaction is modelled in SWAN so that the wave-breaking formulation might lead to a correct prediction of the resulting spectrum. Depth-induced wave breaking and three-wave interaction formulation should be used in conjunction. Using only one of them might lead to erroneous results. In the words of Chen et al. (1997): Consistent with the suggested importance of non-linear, cross-spectral energy transfers in maintaining energy levels at high frequency, additional simulations indicate that non-linear energy transfers rapidly restore high-frequency levels to approximately their original values even when artificially reduced to nearly zero in the initial conditions.

Chen et al. (1997) also notice that whereas the spectral shape is roughly the same with frequency-dependent and frequency-independent methods of dissipation, the wave shapes are not. For frequency-independent methods the skewness and kurtosis of the waves is poorly predicted, whereas frequency-dependent methods give a rather good agreement.

We conclude therefore that further investigations in the form of the dissipation function over the spectrum remain necessary, but, as long as one is satisfied with the spectrum itself, it is not very important to find another formulation. When, however, one wants to use the results of SWAN for morphological studies, one should make haste with another formulation because all kinds of sand-transport processes critically depend on the skewness of the waves. A useful parameter which can be determined from the spectrum itself is the groupiness parameter $\kappa$ (see e.g. Dingsmans (1987, Appendix G). The determination of the skewness and kurtosis from the spectrum is not clear. This aspect should be investigated.

2.5.9 Alternative application of wave breaking formulations in shallow water

One of the reasons for the modellers of SWAN to change the Battjes-Janssen formulation is presumably the fact that the authors of SWAN argue that wave breaking on steepness is modelled separately via dissipation due to white-capping. Since wind-input formulations are usually only tested with wave breaking present, some effect of wave breaking on steepness is already hidden in the wind formulation. Consequently, the separate physical processes are not modelled independently, which may lead to modeling of other physical processes in a somewhat incorrect way in order to repair the incorrect modelling of the process at hand.

For wave breaking at restricted depth, the following procedure may give a way out.

A correct way would be to account only for depth-breaking in the BJ process. That means that the steepness-breaking should be turned off in some way. The way to do that in the Battjes-Janssen approach is to first apply the white-capping; then the resulting spectrum is tested for wave breaking in the Battjes-Janssen approach. A characteristic frequency (the one for which BJ is tested) is then tested whether its slope surpasses the maximum wave slope 0.88$/k$. If this is true, the wave breaking process typically will be spilling wave breaking,
and BJ is not applied further. Notice that this situation should usually not be met, because that means that the white-capping dissipation process did not do its work properly. When the slope is less than the maximum slope, the BJ dissipation is applied in the correct form, so with (2.31) as the maximum wave height and with chosen parameters $\alpha$ and $\gamma$, selected specifically for that situation. In this situation, all wave breaking is considered to be due to breaking on depth.

In this way nothing is doubly accounted for and the wave breaking in restricted depth is done as accurately as possible, without hindering the computation time much. Whether this is really the case, should be investigated further.

2.5.10 Wave breaking and downshifting

Some indications exist that wave breaking acts differently on the different side bands of a non-linearly modulated wave (so-called Benjamin-Feir effect). This is also related to wave breaking on wave groups. Such effects are not accounted for in SWAN, but it may be useful to consider these effects in some detail. Already quite a few papers exist on this subject. We mention Su (1986), Trulsen and Dysthe (1989), and Tulin (1996).

2.5.11 Discussion of Appendix B of De Waal et al. (1999): Wave breaking parameters for very mildly-sloping bottoms

In this section we discuss Appendix B of de Waal et al. (1999). In this Appendix the parameters of the wave-breaking model are considered, especially for the case of very-mildly sloping bottoms.

The authors state the conjecture that wave height calculated with HISWA in case of very-mildly sloping bottoms, large wind and long fetches gives rise to wave heights being too low. They base their conjecture on the following general statements (but without giving any proof for these statements):

1. HISWA gives appreciably lower wave heights than common design wave heights. Several reasons might be due to this:
   a. The common design wave heights may be wrong. If the rules of thumb were always correct, further computations were not necessary anymore.
   b. The standard parameter set is chosen (with a fixed constant break-parameter $\gamma$);
   c. Too much bottom friction has been used. It is not usual to choose the bottom friction coefficient $f_w$ according to the local grain size of the sediment in the region; also no sheet-flow is modelled (giving much lower dissipation due to bottom friction). Moreover, for low waves, bottom ripples may be generated, given a higher dissipation due to bottom friction. See e.g. the discussion in §3.3.3 of Dingemans (1997).
d. The authors remark that in the manual of ENDEC, where also wave breaking according to Battjes and Janssen (1978) has been modelled, a warning is given that wave heights are too low for small bottom slopes (say, 1 : 250). In contrast to HISWA, where the frequency $f_{m-10}$ has been used, the peak frequency $f_m$ has been used in ENDEC, which has also effect on the choice of $\gamma$, see Dingemans (1987, pp. 20-21 and Appendix U). Based on $f_{m-10}$ the parameter $\gamma$ may be considerable higher than the one based on the peak frequency $f_m$. The statement in the manual of ENDEC that the waves may be too low for mildly-sloping bottoms, is based on incorrectly performed measurements, see the discussion in Dingemans (1997, pp. 323-324). We do not state that the statement of too low wave heights is not true, only that it cannot be based on these measurements and thus, that there is no evidence that the statement is true.

e. No wave growth due to wind has been taken into consideration in the computation. Even with ENDEC it is possible to take into account some effect of wind, by means of a GONO formulation. Usually ENDEC is performed without wind input.

2. In their Figure 2 de Waal et al. (1999) compare measurements with HISWA computations performed with the standard (default) values. As discussed many times, there are no default values such as a default value for $\gamma$, and therefore this Figure 2 has no value for this discussion. Furthermore, the fact that for $U > 25$ m/s the computed wave heights increase only slightly, does not prove that they are wrong because measurements are lacking in this region; one may not automatically assume that measured wave heights will increase in the same way as those below 25 m/s wind velocity. This is especially true for the shallow-water regions we are studying here. The conjecture of de Waal et al. may be true, but the information given in their Figure 2 is insufficient to come to their conclusion.

De Waal et al. (1999) conclude that the problem lies in the modelling of the wave-breaking formulation in HISWA and leads, under normal use to too low wave heights. As discussed above, the evidence given by the authors does not necessarily lead to such a conclusion. To reach such a conclusion the computations should be performed consistently, and all effects should be modelled. The results have to be compared to the measurements and only then any conclusion can be drawn.

In order to achieve lower wave heights, the authors decided to change the input parameters $\alpha$ and $\gamma$, if need be outside their validity range to obtain higher waves. We strongly oppose such a solution because the wave breaking process has been checked against measurements with specific parameter settings, from which also a prediction of setting of the pair ($\alpha, \gamma$) was found. With $\alpha = 1$, $\gamma$ is set by either the curve of Battjes and Stive (1985), Eq. (2.34a), or the one by Nairn (1990), Eq. (2.34b). When decreasing the scaling parameter $\alpha$ to 0.5, the parameter $\gamma$ has to be increased using the formula $\alpha \gamma^{-5} = \text{constant}$ in which way Beyer’s (1994) setting is recovered. It has to be remarked that Beyer’s setting has the additional advantage that the fraction of breaking waves, $Q_b$ much more closely resembles the fraction of breaking waves found in nature than with the scaling $\alpha = 1$.

De Waal et al. (1999) discuss the Battjes and Janssen breaking-wave dissipation method and
start, as also used in HisWA, with Miche’s maximum wave height written as\(^{22}\):

\[
H_b = \frac{\gamma d}{k} \tanh \left( \frac{\gamma}{\gamma_d} kh \right).
\]  

(2.75)

Now de Waal et al. (1999) state that a lead value for both \(\gamma\) and \(\gamma_d\) is 0.88. For \(\gamma_d\) this is correct, but we do not understand why \(\gamma\) is taken equal to 0.88. It is known that for solitary wave breaking \(\gamma = 0.78\), already found in 1894 by McCowan. Based on recent work of non-linear wave motion a theoretical value of 0.83 would be appropriate (see Southgate, 1995). As reported by Southgate (1995), Wiegel and Mash (1961) gave

\[
\gamma = 0.73 - 1.12 \left( \frac{H_0}{gT^2} \right)^{1/2} \left( \frac{h_b}{H_0} \right)^{1/2}.
\]  

(2.76)

The choice of the intensity parameter \(\alpha\)

De Waal et al. recapitulate the derivation of the dissipation function as given by Battjes and Janssen (1978). A detailed account is given by Battjes (1986), and a shorter derivation is also given in Dingemans (1983) and in Dingemans (1997, pp. 321-322). Starting point is a turbulent bore as given by Rayleigh (1914):

\[
D' = \frac{1}{4} \rho g (d_2 - d_1)^3 \sqrt{g (d_1 + d_2)} \frac{2 d_1 d_2}{2d_1 d_2},
\]  

(2.77)

where \(d_1\) and \(d_2\) are the depths at both sides of the bore. For a periodic bore this transforms into (e.g., see Svendsen et al. (1978) and Stive, 1984):

\[
D' = \rho g c_{bore} \frac{h (d_2 - d_1)^3}{4d_1 d_2} \quad \text{and} \quad c_{bore} = \left[ \frac{1}{2} \frac{1}{2} g \frac{d_1 d_2 (d_1 + d_2)}{h^2} \right]^{1/2}
\]  

(2.78)

where \(h\) is the mean level.

For shallow water the quantities are interpreted as

\[
d_2 - d_1 = H, \quad d_1 d_2 \gg h^2 \quad \text{and} \quad c_{bore} = c_{\text{wave}}.
\]  

(2.79)

Then relation (2.78) becomes

\[
D'_{\text{wave}} = \frac{\alpha}{4} \rho g c_{\text{wave}} \frac{H^3}{h}.
\]  

(2.80)

For periodic waves an average rate of dissipation per unit area, \(D\), is considered and therefore

\(^{22}\gamma_d\) is the deep-water steepness parameter, equal to 0.88 as the maximum wave steepness. We keep writing \(\gamma\) instead of \(\gamma_d\) as used by de Waal et al. for consistency of notation.
\[ D = D' / \lambda \]. We then have

\[ D_b = \frac{\alpha}{4} \rho g f^2 H^3 / h \]  

(2.81)

where \( f \) is the frequency in Hertz (\( \omega = 2\pi f \)).

De Waal et al. (1999, Eq. (B.17)) now suppose that \( d_2 - d_1 \) represents only a fraction of the wave height of the periodic wave. They argue that this would be the case because there would only be \textit{foam} over part of the turbulent bore. This is a faulty reasoning. It is not the foam which is of importance, but the \textit{turbulence}. However, turbulence is not always visible as foam. Moreover, the starting point (Eq. (2.77)) is only valid for turbulent bores, \textbf{not for partial-turbulent bores}. That means that, when using Eq. (B.17) of De Waal et al. (1999), their starting point (B.13) should be changed, because the suppositions which led Rayleigh to this formula are not valid anymore. Our conclusion is that the author's \( \beta_0 \) should be chosen equal to one.

We are aware of the fact that a turbulent bore only can be sustained on a sloping bottom. For a bottom which becomes of constant depth, or of increasing depth, a turbulent bore may become an undular bore, see e.g. Dingemans (1989) and Sobey and Dingemans (1992).

The parameter \( \beta_Q \)

De Waal et al. (1999) introduce a parameter \( \beta_Q \), to correct for the fraction of breaking waves occurring in practice. The parameter is arbitrarily set equal to 0.5. The effect is that the probability of wave breaking halves. It has to be stressed that \( Q_b \) as appears in the wave dissipation model of Battjes and Janssen only can be interpreted as a \textit{computational parameter}, it does not give an accurate description of visible encountered breaking waves, see e.g. Southgate and Wallace (1993). See also Beyer (1994), where with the setting \( \alpha = 0.5 \) and an adapted \( \gamma \) so that equivalent dissipation occurs, a computational value of \( Q_b \) occurs which is much closer to the fraction of breaking waves in practice. It should be stressed that the value of \( Q_b \) results of a computation, which in itself is exact and therefore the value may not be changed arbitrarily. Our conclusion is that \( \beta_Q = 1 \) should be set.

The parameter \( \gamma \)

The choice of the break parameter \( \gamma \) poses a problem for De Waal et al. (1999). To obtain less wave breaking, and therefore a higher wave height, a large value of \( \gamma \) is wanted. However, there are many indications that for horizontal bottom and for very-mildly sloping bottoms, a much smaller value for \( \gamma \) would be appropriate. In SWAN there is the possibility to choose \( \gamma \) according to Nelson's relation (2.67), yielding a much smaller value than when the relation of Battjes and Stive is used. It should be remembered that the relation of Battjes and Stive (1985) is based on comparisons with measurements where always has been used \( \alpha = 1 \) and the computations for one situation have been repeated until satisfactory agreement was obtained.
In Dingemans (1983, pp. 50-55) the relation between the model of Battjes and Janssen with the one of Thornton and Guza (1983) (see also Thornton and Guza, 1989) has been investigated. Thornton and Guza adopted a saturation range $\gamma_T = 0.40$ of wave breaking by $H_{\text{rms}} = \gamma_T h$. In terms of the parameters of Battjes and Janssen this was calculated as $\alpha = 1$ and $\gamma = 0.59$ for the field. It should be noted that the results of Thornton and Guza were obtained for Torrey Pines beach which has a very mildly-sloping beach. This finding is thus in accordance with a lower value for $\gamma$ for nearly-horizontal bottoms.

**The approach suggested by De Waal et al. (1999)**

In view of the above discussions, it cannot come as a surprise that we do not think the setting of parameters outside their physical validity domain is a good suggestion. It should be remembered that parameters of the models are chosen in such a way that comparison with measurements gives best results. When these parameters are changed, again some check with measurements is required to check on the validity of the adjustments. It is bad engineering judgement to adjust the parameters without such a check. As already remarked in the beginning of this section, the too-low waves could also be caused by an incorrect modelling of wave growth by wind. A number of parameters have been set, which might not be the optimal ones for high winds. Also the parameters of bottom friction might not be chosen correctly when only default values have been used. It is possible to obtain less dissipation due to wave breaking in a physically acceptable way by following one of our suggestions in the previous subsections.

**The value $H/h$ which may be expected**

For an estimate of the possible wave heights due to wind in shallow water we turn to Young and Verhagen (1996). The limit of the non-dimensional energy $\varepsilon = g^2 m_0 / U_{10}^2$ for small dimensionless depth $\delta = gh / U_{10}^2$ is given by (Young and Verhagen (1996, Eq. (17)):

$$\varepsilon = 1.06 \times 10^{-3} \delta^{1.3}.$$  \hspace{1cm} (2.82)

We rewrite this expression to obtain $H/h$ and get

$$\frac{H_s}{h} = 0.13 \left( \frac{gh}{U_{10}^2} \right)^{-0.35},$$  \hspace{1cm} (2.83)

where $H_s = H_{\text{rms}}$ is the significant wave height defined by $H_s = 4\sqrt{m_0}$. We now plot (2.83) for the depths $h = 2, 5, 6$ and 8 m as a function of the windspeed $U_{10}$ from 0 to 60 m/s in Figure 2.19.

We also plot (2.83) as function of the depth for a wind speed of 30 and 60 m/s in Figure 2.20. It is clear from Figures 2.19 and 2.20 that in order to obtain a relative wave height $H/h = 0.6$ excessive wind speeds are necessary for the depths occurring in the IJsselmeer.
and Markersnet. We also can pose the question which wind speed is necessary to obtain a prescribed value of $H/h$? A few examples are given in Table 2.2.

\[
\begin{array}{c|cc}
H/h & h = 5 & h = 8 \\
\hline
0.4 & 34.9 & 44.1 \\
0.6 & 62.3 & 78.9 \\
\end{array}
\]

Table 2.2: Necessary wind velocity $U_{10}$ in m/s to obtain the $(h, H/h)$ combination.

From Table 2.2 we draw the conclusion that for realistic winds values for $H/h$ are of the order 0.4 rather than 0.6.

2.6 Four-wave interaction

In this Section properties of the four-wave interaction models are discussed. For water waves, Hasselmann (1962) gave a perturbation approach which is valid for weakly non-linear waves obeying the random-phase approximation. Relevant discussions for this approach are given in Hasselmann (1968), Hasselmann (1977) and in Webb (1978a,b). The approach is not very different from some approaches in turbulence and in plasma physics. For the latter approach good sources are Davidson (1972), Tsytovich (1970) and Galeev and Sagdeev (1979). For the relation with turbulence see Monin and Yaglom (1975) and Lesieur (1987).

We first give a short discussion of the four-wave interaction processes. We follow here the discussion given in Dingemans (1980, Chapter 9) closely.

\footnote{Notice that $H/h = 0.6$ is the maximum value which is possible to be stable in the case of a horizontal bottom.}
In the present Section we will, in Section 2.6.1, first give a short description of the four-wave interaction process as widely accepted. Then, in §2.6.2, we discuss the fourth-cumulant discard hypothesis, which is basic for the usual description. In §2.6.3 we discuss the invalidity of the fourth-cumulant discard hypothesis approach, meaning that the usual Boltzmann-integral formulation as used in WAM and SWAN is an inconsistent approach.

For the case that one still wants to use the inconsistent approach, the way in which the integral is computed is done in different ways. These ways are discussed in Sections 2.6.4 and 2.6.5. It has to be ascertained that the four-wave resonance interaction integral has primarily been derived for deep water. As our interest with SWAN is primarily the not-so-deep water, the properties of the integral for shallow water become important. It then appears that the integral becomes large in shallow water because the perturbation approach which was the basis for the derivation rests upon Stokes-type of expansions, which diverge in shallow water.

A sketch of the derivation of the Zakharov equations and the comparison of the result with Hasselmann's approach is given in Appendix B.

### 2.6.1 Short description of four-wave interaction

Studies of evolution of random non-linear surface waves have mostly been restricted to the study of wave-wave energy transfer within a broad spectrum due to non-linear coupling in a nearly random ocean (e.g., Hasselmann, 1968). In terms of the spectral action density $N(k)$ the net rate of change of action density at wave number $k_1$ is due to resonance between quartets for which

$$k_1 + k_2 = k_3 + k_4, \quad \omega_1 + \omega_2 = \omega_3 + \omega_4,$$  \hspace{1cm} (2.84)
is given by

\[
\frac{\partial N(k_1)}{\partial t} = \iint G(k_1, k_2, k_3, k_4) [(N_1 + N_2) N_3 N_4 - (N_3 + N_4) N_1 N_2] \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \delta(k_1 + k_2 - k_3 - k_4) \, dk_2 \, dk_3 \, dk_4
\]  
(2.85)

where \( N_i \equiv N(k_i) \), \( \delta(\cdot) \) is the delta function, \( \omega_i \equiv \omega(k_i) = \{ g |k_i| \tanh(|k_i| h) \}^{1/2} \) and \( G \) is a complicated function of its arguments. Note that \( \partial N_1 / \partial t = 0 \) when all \( N_i \) are equal; furthermore, it follows that, with \( N_2 = N_3 = N_4 = N \), \( N_1 \neq N \), the term in square brackets becomes \( N^2 (N - N_1) \) and \( N_1 \to N \) as \( t \) evolves. In other words, Eq. (2.85) predicts that the net result of all interactions is to redistribute the energy of the spectrum more uniformly over all wave numbers. For a clear discussion of the behaviour of (2.85) the reader is referred to Webb (1978a,b). Insight can be gained by writing the term in square brackets of Eq. (2.85) as

\[
N_1 N_3 (N_4 - N_2) + N_2 N_4 (N_3 - N_1)
\]

Because Eq. (2.85) describes the transfer of action between the pairs of wave numbers \( k_1, k_2 \) and \( k_3, k_4 \), the term \( N_2 N_4 (N_3 - N_1) \) can be considered as describing a diffusive transfer between \( k_1 \) and \( k_3 \). This diffusive transfer now pumps the transfer between \( k_2 \) and \( k_4 \) in order that conservation of energy and momentum is retained.

The entropy of the wave spectrum is given by \( H = \int \log N(k) dk \) and the total rate of entropy production \( dH/dt \geq 0 \). By considering the entropy production separately for the diffusive transfer and the pumped transfer, \( dH/dt \) and \( dH/dt \) respectively, Webb (1978a,b) showed that \( dH/dt \geq 0 \) whereas \( dH/dt \) is of the same sign as the term \( (1/N_1 - 1/N_3) - (1/N_2 - 1/N_4) \). This gives an explanation of the growth of the spectral peak and its shifting to lower frequencies during the wave-generation stage.

For one-dimensional wave propagation in deep water it can be shown that the interaction coefficient \( G \) is zero (Rasmussen, 1995, pp. 30-31). However, for 1D propagation in water of restricted depth, the interaction coefficient is unequal to zero.

2.6.2 The zero-fourth-order cumulant hypothesis

To explain the difficulties in closing the system of equations, we follow Holloway (1978). We consider the general system of first-order equations with second-order non-linearity in the amplitudes \( a_\ell \) with \( \ell \) a large, but finite, number:

\[
\frac{da_\ell}{dt} + (i\omega_\ell + \nu_\ell) a_\ell + \sum_{m,n} c_{\ell mn} a_m a_n + f_\ell = 0
\]  
(2.86)

where \( \omega_\ell \) and \( \nu_\ell \) are the real frequency and dissipation rates in mode \( \ell \). When the flow field is treated statistically and we ignore the external force \( f_\ell \) for this discussion, we take the
ensemble average of (2.86) and obtain:

$$\frac{d \langle a_\ell \rangle}{dt} + (i\omega_\ell + \nu_j) \langle a_\ell \rangle + \sum_{m,n} c_{\ell mn} \langle a_m a_n \rangle = 0.$$  \hspace{1cm} (2.87)

It is now not possible to compute the motion of $\langle a_\ell \rangle$ because the quantity $\langle a_m a_n \rangle$ is not known. By multiplying the equation (2.86) by $a_j$ and taking the average, we obtain an equation for the second moment $\langle a_\ell a_j \rangle$:

$$\frac{d}{dt} \langle a_\ell a_j \rangle + [i(\omega_\ell + \omega_j) + (\nu_\ell + \nu_j)] \langle a_\ell a_j \rangle + \sum_{m,n} [c_{\ell mn} \langle a_m a_n a_j \rangle + c_{j mn} \langle a_m a_n a_\ell \rangle] = 0.$$  \hspace{1cm} (2.88)

In turn, the second moment cannot be computed because of the dependence of the third moments. Some closure is thus needed.

In water wave problems we have third-order non-linearity and the differential equation for the second-order moments then depends on fourth-order moments. An $n$-th order moment of random quantities $a_1 \cdots a_n$ can be reduced to a sum of products of lower-order moments, plus a irreducible term, the $n$-th order cumulant:

$$\langle a_1 \cdots a_n \rangle = \sum_{j=1}^{n-1} \langle a_1 \cdots a_j \rangle \langle a_{j+1} \cdots a_n \rangle + \langle a_1 \cdots a_n \rangle^C.$$  \hspace{1cm} (2.89)

For a discussion of cumulants\textsuperscript{24} for both random variables and random fields is referred to Monin and Yaglom (1975, pp. 223 ff.) or to Kendall and Stuart (1977, Chapter 3).

With four random variables with zero mean (2.89) reduces to

$$\langle a_1 a_2 a_3 a_4 \rangle = \langle a_1 a_2 \rangle \langle a_3 a_4 \rangle + \langle a_1 a_3 \rangle \langle a_2 a_4 \rangle + \langle a_1 a_4 \rangle \langle a_2 a_3 \rangle + \langle a_1 a_2 a_3 a_4 \rangle^C.$$  \hspace{1cm} (2.90)

Usually the fourth-order cumulant in (2.90) is chosen to be zero. This approximation is also called the quasi-normal or the random-phase approximation. When the phase of each mode $a_\ell$ is statistically independent of all other modes, then $\langle a_1 a_2 a_3 a_4 \rangle^C = 0$.

Instead of taking the $n$-th order cumulant to be zero, $\langle a_1 \cdots a_n \rangle^C = 0$, Holloway (1978, 1979) substitutes for the $n$-th order cumulant a term linear in the $(n-1)$-th cumulant $\langle a_1 \cdots a_{n-1} \rangle^C$ with a forefactor which is an unknown function of lower-order cumulants. Effectively what is happening is that instead of supposing the fourth-order cumulant to be zero, the difference between the fourth-order cumulant and some linear functional of the triple correlation is supposed to be zero.

\textsuperscript{24}In the Russian literature cumulant is also termed semi-invariant.
2.6.3 The validity of the Boltzmann-integral approach

Eq. (2.85) has been derived under the condition that the wave field is homogeneous. Furthermore, it has been assumed that the distribution of the wave field is initially normal (Gaussian) and that it remains Gaussian; that is, the fourth-order cumulant is taken to be zero. In weak turbulence theory this condition is known as the Millionschikov zero-fourth-cumulant hypothesis (see, e.g., Monin and Yaglom, 1975, p. 241 and section 19.3). It is known that this hypothesis ultimately leads to negative energy densities (see Monin and Yaglom (1975, Fig. 43 on page 285).

![Figure 2.21](image)

Figure 2.21: Computation showing that negative energy densities evolve; from Monin and Yaglom (1975, Fig. 43, p. 285).

It is furthermore noted that in these turbulence applications some dissipation due to viscosity is also present; this acts as a check on the build-up of the cumulants, see also Holloway (1978). For water-wave problems, without such a dissipation model added in the interaction equations, the Boltzmann integral remains positive, according to Komen et al. (1994, p. 136). We note in this respect that the ultimate dissipation of the shortest waves is not included in the model for the wave-wave-interaction formulation, as it is in the turbulence models. The cut-off frequency above which no waves are considered anymore also plays some role here. In wave-interaction problems the zero-fourth-cumulant hypothesis is often called the random-phase approximation, see, e.g., Davidson (1972, p. 136) or Tsytovich (1970, p. 51).

Hasselmann (1962) used the random-phase approximation together with an updating scheme and Benney and Saffman (1966) showed by using a multiple-scale expansion that, for a simplified equation, a closure assumption is not needed on time scales of $O(\varepsilon^{-2})$, where $\varepsilon$ is the root-mean-square wave-slope parameter. Because the time scale of the action flux is $O(\varepsilon^{-4})$, this result is not enough, but Newell (1968) showed that, for water waves, even at this order no closure hypothesis is needed.
A general discussion of weak-coupling theory is given in Olbers (1979) and Hasselmann (1977) discusses coupling of systems with widely different intrinsic time scales. We note that for water waves exchange of wave action takes place due to resonance between four wave components, as is expressed by Eq. (2.85); this description is obtained as lowest-order correction to the linear description. At higher order also resonances between more than four components take place, but the lowest-order spectral-transport equations are already quite involved.

Because the energy-density spectrum is the Fourier transform of the covariance, and thus of second-order moments of the free-surface elevation \( \zeta(x, t) \) which is regarded as a stochastic variable, higher-order moments also occur in the evolution equations for the covariances. The closure hypothesis usually considered is to apply the random-phase approximation. The random-phase approximation means that the phases of the various components are taken to be statistically uncorrelated. Application of the random-phase approximation thus means that the initially Gaussian distribution remains Gaussian upon evolution of the wave field and the phases remain uncorrelated; this follows from application of the law of large numbers. This is typically not true because one of the results of non-linear interaction is the coupling of the various phases as can be ascertained by a bi- or tri-spectrum analysis. Moreover, already in lower-order (i.e., second-order non-linearity) bound waves are formed as exemplified by second-order Stokes theory. These bound second-order components make uncorrelated phases impossible to remain uncorrelated so the assumed Gaussian distribution fails. Due to the non-linearity, the Gaussian probability function of the free-surface elevation evolves into a Gram-Charlier distribution, of which the Gaussian distribution is its first term when expanding it into a series with respect to the non-linearity parameter.

**A different view**

In this respect it should be mentioned that Janssen (1991a) notices that for a homogeneous wave field the zero-fourth-order cumulant hypothesis leads to the absence of any transfer of action. It is a necessity that the fourth-order cumulant is different from zero to obtain the possibility of transfer of action over the components. The analysis of Janssen is performed for deep water; it is not immediately clear whether this result also remains valid for water of restricted depth.

Janssen now applies the random-phase approximation on the sixth-order moment (that is, the sixth-order cumulant is taken to be zero), and solving the equation for the fourth moment then leads to the Hasselmann formulation for four-wave interaction. Janssen (1998, personal communication) stresses the fact that the papers in which the fourth-order-cumulant discard hypothesis is applied, are totally wrong.

A property of Hasselmann’s equation which we want to stress here is the fact that **Hasselmann’s equations are only valid for homogeneous wave conditions**. For homogeneous conditions the time scale of the interactions is \( O\left(\varepsilon^{-4} \omega_0^{-1}\right) \). For non-homogeneous situations the time scale is \( O\left(\varepsilon^{-2} \omega_0^{-1}\right) \) according to Janssen (1991a).

Janssen (1991a) concludes that Hasselmann’s equations are only valid for weakly non-linear
waves with a spectrum which is wide enough and for homogeneous waves. For narrow spectra 
and inhomogeneous waves the modulational instability will occur, giving interaction on a much 
faster time scale. He argues that on a grid of 10 by 10 km the wave field may be considered 
to be homogeneous, and Hasselmann's equations are adequate, but on a 5 by 5 km grid the 
inharmonic non-linear action transfer should be taken into account. We therefore conclude 
that for SWAN, which is meant to be applied in the coastal zone, Hasselmann's equations are 
not accurate enough. The effects of an inhomogeneous wave field have also to be taken into 
account. See for example the models of Willebrand (1975) and Watson and West as described 
in §2.3 of the present review.

2.6.4 The accuracy of the approximation

The perturbation scheme as used by Hasselmann (1962) has a major drawback: the pertur-
bation for all modes starts from the mean free surface, \( z = 0 \). For short waves riding on 
longer waves the correct level from which the approximation should proceed is the surface of 
the long waves. See the discussion in Brueckner and West (1988).

A different view on the accuracy of the resulting equations was given by Crawford et al. 
(1981), as also reported by Lin and Perrie (1997). These authors considered the computed 
non-linear action transfer still reliable when the discrepancy with respect to the true action 
transfer was less than 10%. As a measure of the accuracy the dimensionless perturbation wave 
number for the most unstable mode was considered. Comparison with the exact equations 
as used by Longuet-Higgins (1978) showed that the approximation of Zakharov (1968) is still 
viable for \( \varepsilon = 0.3 \), whereas the one of Hasselmann (1962) is viable up to \( \varepsilon = 0.06 \). The 
non-linearity parameter \( \varepsilon \) is defined as the ratio of the amplitude of the second harmonic to 
the amplitude of the first harmonic, \( \varepsilon = a_2/a \). What this means in a realistic situation can be 
ascertained in the following way. From a second-order Stokes expansion it is found that (i.e., 
see Dingemans (1997, Eq. (2.422a)) \( a_2/a = k a/2 \). The amplitude \( a \) can be estimated from 
\( k a/2 = 0.06 \) as \( a = 0.06 \lambda / \pi \). Using for \( \lambda \) the deep-water estimate \( \lambda = gT^2/(2\pi) \) and using 
\( T = 7 \) s as typical for the coastal zone in the Southern North Sea, we get \( a = 0.03T^2 = 1.46 \) 
m. As the deep-water wave length is an overestimate for the wave length at restricted depth, 
we conclude that Hasselmann then is valid for wave heights of less than 2.5 m.

2.6.5 Various parameterisations of the Boltzmann integral

To visualise the solutions of the wave number vectors, we use a method presented in, amongst 
others, Young and Van Vledder (1993), in which is taken \( k_1 + k_2 = k_3 + k_4 = k \). For deep 
water there is obtained for the frequencies \( \sqrt{k_1 + \sqrt{k_2}} = \sqrt{k_3 + \sqrt{k_4}} = \gamma \sqrt{k} \). For \( k = 1 \) and \( \theta \) 
the angle between \( k_1 \) and \( k \), the locus of all values of \( k_1 \) which obey the resonance conditions 
is given by Young and Van Vledder as:

\[
\cos \vartheta = \frac{1}{2k_1} \left[ 1 + k_1^2 - (\gamma - \sqrt{k_1})^4 \right] \tag{2.91}
\]
For various values of $\gamma$ from 1.8 (the outermost) to 1.25, the loci are plotted in Figure 2.22.

Figure 2.22: Resonance curves $k_1 + k_2 = k_3 + k_4 = k$ for various values $\gamma$; after Young and Van Vledder (1993)

A solution of the Boltzmann integral in which no further approximations are made is possible; this numerical solution is called EXACT-NL. EXACT-NL is a computer program in which initially the full grid of possible wave interactions is determined. In a second step the relevant part of the number of possible interactions is determined; this depends on the input spectrum, and, if the evolved spectrum changes too much, a new set of interacting components has to be determined. Point is that the interaction is computed as exact as possible, only not all possible interaction sets are considered, only those with enough energy content. A full description of the program can be found in Van Vledder and Weber (1988). The numerical effort is quite large and therefore several approximation schemes have been put forward. The most used one is called the discrete interaction approximation (DIA). The essence of this method is as follows. Instead of using the full set of quadruplets which obey the resonance surface, only a small number quadruplets is considered, e.g., see Komen et al. (1994, pp. 226-227) or Young and Van Vledder (1993) which we follow here. The quadruplets to be chosen for the DIA computation satisfy $k_1 = k_2$, whereas $k_3$ and $k_4$ are of different magnitude and lie at an angle to the first two wave numbers, in such a way that the resonance conditions are fulfilled. The corresponding four frequencies are related by:

$$
\begin{align*}
\omega_1 = \omega_2 &= \omega \\
\omega_3 &= \omega (1 + \lambda) = \omega^+ \\
\omega_4 &= \omega (1 - \lambda) = \omega^- .
\end{align*}
$$

(2.92)

In the discrete interaction approximation usually $\lambda = 1/4$ is chosen, on experimental grounds. As is clear from Komen (1994, Fig. 3.6), reproduced here as Figure 2.23, the DIA is not particularly accurate. The effect of the choice of cut-off frequency may be made clear from the computed changes of the energy densities within one quadruplet (Young and Van Vledder, 1993, Eq. (3.6)):

$$
\begin{align*}
\begin{bmatrix}
\delta S_{nl} \\
\delta S_{nl}^+ \\
\delta S_{nl}^-
\end{bmatrix} &= C \cdot g^{-4} f^{11} \left[ E^2 \left( \frac{E^+}{(1 + \lambda)^4} + \frac{E^-}{(1 - \lambda)^4} \right) - 2 \frac{E E^+ E^-}{(1 - \lambda^2)^4} \right] .
\end{align*}
$$
where \( C = 3 \times 10^7 \) is a constant, \( f = \omega/(2\pi) \) and \( E = E(f, \theta) \). It is clear that the choice of the cut-off frequency is very important because the dependence on \( f \) is of the eleventh power.

Figure 2.23: Comparison of discrete interaction approximation with Exact-NL; from Komen et al. (1994), p. 228.

Moreover, DIA typically produces spectra with a broader directional spreading than do models which use a full solution of the non-linear interaction term.

Another approximation is the diffusion approximation, e.g., discussed in Komen et al. (1994, p. 224). According to Figures 3.4 of Komen et al. (1994, p. 225), the diffusive approximation is much more accurate.

Landrini et al. (1998) studied the long-time evolution of gravity-wave systems, both in a flume and numerically. They note that it is advantageous to use the kinetic equations in the form as they are derived by Krasitskii (1994), because in this form the Hamiltonian character of the water waves is preserved. It seems that the kinetic equations are not often solved in this form. We do think it to be wise to look at the possibilities of rewriting the kinetic equations in Krasitskii’s form.
The xDIA approach

Recently Van Vledder et al. (2000) and Van Vledder (2001) described an extension of the DIA approach, called xDIA 25. The essential difference from DIA is to deviate from the choice $k_1 = k_2$. Now the following relationship between the four wave number vectors is proposed:

\[
\begin{align*}
\omega_1 & = \omega \\
\omega_2 & = (1 + \mu) \omega = \omega^* \\
\omega_3 & = (1 + \lambda) \omega = \omega^+ \\
\omega_4 & = (1 - \lambda - \mu) \omega = \omega^-, \\
\text{and} & \\
\theta_2 & = \theta_1 + \Delta \theta,
\end{align*}
\]

(2.94a)  (2.94b)  (2.94c)  (2.94d)  (2.94e)

in which $\Delta \theta$ is the angle between the wave number vectors $k_1$ and $k_2$.

As shown in Figure 2.24, the new approach is a considerable improvement over the DIA approximation. The xDIA approximation comes close to the exact formulation.

![Comparison of DIA and xDIA approximation with EXACT-NL](image)

Figure 2.24: Comparison of DIA and xDIA approximation with EXACT-NL. (Figure from Van Vledder et al. (2000)).

\[25\text{In effect different approaches are followed, called the shallow-water approximation, or sDIA, as described in Van Vledder and Bottema (2002) and the multiple sum configuration (mDIA) and the general form (gDIA). The total of different approximation is denoted as xDIA.}\]
2.7 Three-wave interaction

Nonlinear wave interactions are taken into account by modelling three and four wave interactions, triads and quadruplets respectively. The modelling of the latter has been discussed in Section 2.6. Eldeberky and Battjes (1995) proposed the Discrete Triad Approximation for the modelling of triads. In SWAN the triads are modelled by applying the slightly modified Lumped Triad Interaction (LTA) method of Eldeberky (1996) in each spectral direction:

\[ S_{n3} = S_{n3}^- + S_{n3}^+ , \]  

with

\[ S_{n3}^+(\omega_r, \theta) = \max \left\{ 0, 2\pi \alpha c c_g J^2 \sin \hat{\beta} \left[ E^2 \left( \frac{\omega_r}{2}, \theta \right) - 2E \left( \frac{\omega_r}{2}, \theta \right) E \left( \omega_r, \theta \right) \right] \right\} , \]  

in which \( c \) and \( c_g \) denote the phase velocity and the group velocity. Furthermore,

\[ S_{n3}^-(\omega_r, \theta) = -2S_{n3}^+(2\omega_r, \theta) , \]  

in which \( \alpha \) is a tunable proportionality coefficient, \( \hat{\beta} \) is an approximation for the biphase and \( J \) is an interaction coefficient (in \( \text{rad}^2/\text{m}^2 \)):

\[ \hat{\beta} = -\frac{\pi}{2} + \frac{\pi}{2} \tanh \left( \frac{0.2}{U_r} \right) , \quad U_r = \frac{gH_s}{2\sqrt{2}C_\rho^2 h^2} , \]  

\[ J = \frac{k_{\omega_r/2}^2 \left( gh + 2c_{\omega_r/2}^2 \right)}{k_{\omega_r} h \left( gh + \frac{3}{15} gh^3 k_{\omega_r}^2 - \frac{3}{2} \omega_r^2 h^2 \right)} . \]  

The parameter \( U_r \) denotes the Ursell number. Furthermore, \( k_{\omega_r} \) is the wave number corresponding to the frequency \( \omega_r \). Both quantities are related through the linear dispersion relation. The phase velocity is defined in the usual way: \( c_{\omega_r} = \omega_r/k_{\omega_r} \).

The LTA method is a very approximate one. Part of the bi-spectrum, in casu the bi-phase, is not predicted at all. From a limited set of measurements a simple prescription is found. Consequently, the formulation may not be general enough to be suitable for all kinds of conditions.

The simplified one-equation stochastic model represents an average effect of triad wave interaction, transferring energy from lower to higher frequencies through self-self interaction. The model does neither take into account the energy transfer to subharmonics nor the non-resonant wave interaction. Consequently, the approach is in general unsuitable for frequency spectra that are not uni-modal (and unidirectional). In cases of sea states involving swells and wind waves, the LTA model is not expected to model accurately the nonlinear energy exchange between the two frequency regimes.

Results of simulations on a horizontal bottom, performed by Rasmussen (1998, sect. 11), indicate that the LTA model and other one-equation models are not applicable for prediction.
of long-term evolution on a (nearly) horizontal bottom in shallow water. As a consequence of the introduced simplifications, the LTA model is mainly appropriate for relatively short evolution distances on sloping bathymetries, where the generation of bound super-harmonics is substantial. However, Dingemans (1998) noticed that the LTA method is only valid for horizontal bottom, because, in course of the derivation, the shoaling term in the amplitude equations was temporarily neglected (Eldeberky, 1996, p.164). However, the term was never put back in the equations. This contradiction in the model setup might give rise to serious problems. Furthermore we notice that the LTA method works best for a shoaling bottom.

The triad wave interactions considered by Eldeberky (1996) are only valid in 1D and can be applied to two-dimensional problems accounting only for colinear interactions. In SWAN the LTA approach is used in each spectral direction. Interactions between waves with different directions have not been considered.

Alternatives have been recommended by Dingemans (2000). A further recommendation was given by Klopman and Dingemans (2001).

### 2.8 Dissipation due to bottom friction

Dissipation due to bottom friction is generated in the bottom-boundary layer. By considering a boundary-layer approximation (in fact, Prandtl's equations), it can be shown (e.g., Dingemans, 1997, Note 2.5) that the dissipation function (in J m$^{-2}$ Hz) in the energy balance equation is given by

$$
\langle D_f \rangle = \left\langle \tau^b \cdot \mathbf{u}^b \right\rangle,
$$

where $\tau^b$ (in N m$^{-2}$) is the shear stress at the bottom and $\mathbf{u}^b$ is the velocity just outside the bottom boundary layer. The real problem is the determination of the shear stress. For regular waves some approaches have been discussed in Dingemans (1997, §3.3). Most of the approaches are based on a quadratic friction law:

$$
\frac{1}{\rho} \tau^b = \frac{1}{2} f_w | \mathbf{u}^b | \mathbf{u}^b.
$$

In this way the problem is moved to the determination of the friction coefficient $f_w$. In the case that ambient currents are present, the situation is much more complicated for two reasons:

1. One has to decide how to use the quadratic model, on the current and wave velocity separately, or on the sum of them. For a discussion of this aspect, see Tolman (1992). A good overview for random waves in the presence of ambient currents is given by Mathisen and Madsen (1993).
2. One has to consider the interaction in the fluid between the current and the wave part of the velocities. The turbulence model used in the wave and the current boundary layers is essential. From experiments it followed that a single roughness length
scale can be used to characterize pure currents, pure waves and combined waves and currents, when a fixed bottom can be assumed, see Mathisen and Madsen (1993). In reality a single roughness length scale cannot be used in the case of ripple formation (low bottom velocity) and in the case of sheet flow (large bottom velocity). These two situations need much different roughness parameters.

2.8.1 Bottom friction in absence of ambient currents

For random water waves two models are often used:

1. a model based on shear stress which is supposed to vary linearly with the bottom velocity:

   \[ \tau^b = \rho C u^b \quad \text{and} \quad (2.101) \]

2. a model based on the quadratic law (2.100).

A model in the first class was proposed by Hasselmann et al. (1973). The empirical model was used in the JONSWAP experiment, in which tidal currents were much more important than the wave part (Hasselmann et al., 1973, p. 73), and, with an always fixed and constant coefficient in the whole region this may be the reason of its success for that JONSWAP situation.

Collins (1972) developed a drag-law model, fitting in the second class. In fact, this model is a simplification of the model of Hasselmann and Collins (1968) in that the dependence over the wave rays is neglected. We remark that Graber (1984, p. 154) notices that Eq. (13) of Collins (1972) is incorrect, the relation should be replaced by Eq. (3.3.53) of Graber (1984).

The model of Collins now is (see also Padilla-Hernández and Monbaliu, 2001):

\[ S_{bot} = -\frac{C_{bot}}{g^2} \cdot \frac{\omega_r^2}{\sinh^2 kh} E(\omega_r, \theta) = -\frac{C_{bot}}{g} \cdot \frac{2k}{\sinh(2kh)} E(\omega_r, \theta) \quad (2.102a) \]

with

\[ C_{bot} = 2C_f g \left( \left\langle u^b \right\rangle^2 \right)^{1/2}, \quad C_f = 0.015 \quad (2.102b) \]

where the total wave-induced bottom velocity follows from

\[ \left\langle \left( u^b \right)^2 \right\rangle = \int \int \frac{2gk}{\sinh 2kh} E(\omega_r, \theta) \, d\omega_r \, d\theta . \quad (2.102c) \]

An intercomparison of the models of Hasselmann et al. (1973) and Collins (1972) has been given by Cavalieri and Lionello (1990). Their main conclusion was that the JONSWAP method overestimated the actual energy loss below a certain critical point and it underestimated it above that critical point.
As a third possibility the model of Madsen et al. (1988) is mentioned, which is also based on the drag friction law. In this model the friction factor $f_w$ is chosen according to a formula of Jonsson:

$$\frac{1}{4\sqrt{f_w}} \log \frac{1}{4\sqrt{f_w}} = 10 \log \frac{A_{br}}{k_b} - 0.17$$  (2.103)

where $k_b = 30z_b$ is used for the equivalent bottom roughness and $A_{br}$ is the near-bottom orbital excursion amplitude of the root-mean-square equivalent wave. Notice that Madsen et al. (1988) took (2.103) from the 1966 paper of Jonsson, whereas Jonsson (1980) changed this relation in that the constant 0.08 is subtracted instead of 0.17, see also Dingemans (1997, Eq. (3.167b)).

The bottom friction models of Hasselmann et al. (1973), Collins (1972) and Madsen et al. (1988) have been implemented in SWAN. These models are relatively simple, compared to a more advanced method of Weber (1989, 1991a,b). Recently Padilla-Hernández and Monbaliu (2001) have compared the three available models in SWAN and the eddy-viscosity formulation of Weber for shallow water situations (Lake George). They found that Weber’s model showed the best performance in the cases of depth- and fetch-limited wave growth.

Padilla-Hernández and Monbaliu (2001) correctly state that formulations for dissipation by bottom friction, such as the model by Madsen et al. (1988) and Weber (1989, 1991a,b), which take explicitly physical parameters for bottom roughness into account, should be preferred in wave modelling in shallow water. These parameters offer the possibility to adapt the changing roughness under different wave or wave-current conditions.

Furthermore, we stress the importance to account for various situations which may occur within a single computational area. In coastal regions one encounters on the one hand ripple formation and on the other hand sheet flow. Dissipation in these regions is quite different. For information concerning bottom friction on movable-bed models, one might consult Grant and Madsen (1982), Nielsen (1992) and Tolman (1994).

2.8.2 Bottom friction for waves on currents

In SWAN no provision has been made for the situation that bottom dissipation is to be determined for the case that both waves and currents are present. For application as part of a series of models for morphological changes, such a provision should be present in SWAN.

The bottom boundary-layer flow under spectral waves plus current was investigated by Madsen et al. (1988). The wave-friction factor and wave-energy dissipation were predicted based on the concept of an equivalent sinusoidal wave having the same near-bed wave orbital velocity amplitude $u_b^{rms}$ and excursion amplitude $\tilde{a}_b^{rms}$ as the wave spectrum. This paper was extended by Madsen (1994). The equivalent sinusoidal-wave concept has been extended to spectral wave-current bottom boundary-layer flow. Explicit formulae were given for the wave friction factor under random waves which also apply to friction factors for spectral waves and
current, when the ratio of mean to oscillatory friction velocity for the combined wave-current flow is known.

A parametric model predicting the shear-stress amplitudes under random waves alone was proposed by Myrhaug (1995). This approach has recently been extended by Myrhaug et al. (2001) to weak wave-current interactions. Shear-stress amplitudes under random waves plus current have been calculated by Holmedal et al. (2000), using Monte Carlo simulations of Soulsby’s parameterised wave-current models that are valid for sinusoidal waves plus current. Mathisen and Madsen (1999) investigated the bottom roughness for spectral waves and current. Their experiments show that sinusoidal and spectral wave-current bottom boundary layer flow over a fixed rippled bed can both be characterised by a single bottom roughness when used in conjunction with a representative equivalent wave.

Holmedal (2002) investigated the rough turbulent bottom boundary layer under random waves with and without a superposed mean current. The boundary layer structure and the statistical properties of the individual bottom shear stress maxima have been examined. The mean bottom shear stress has been calculated for a range of wave-current conditions. Furthermore, an equivalent sinusoidal wave approach was suggested.

2.9 Accuracy and reliability of source-term formulations

In the previous sections the source terms that are presently used in SWAN have been described. Besides, a number of alternative approaches have been given. Some of them are believed to lead to more reliable results, also for the exceptional equilibrium situation, which is characterised by strong wind, wave growth with a very long fetch, shallow water and a rather horizontal bottom. This situation may occur e.g. at the IJsselmeer in the Netherlands.

Here we will discuss the accuracy and reliability of the present formulations of the source terms in SWAN and compare them with the alternatives that are given as well. Note that the comparison is purely qualitative. For a quantitative comparison, an extensive set of computations should be carried out. This is beyond the scope of this project.

2.9.1 Action balance equation

The action balance equation consists of a wave propagating part (the left-hand side) and a number of source terms (the right-hand side). Here we are concerned with the wave propagating part. We notice the following:

1. The wave action is defined only linearly by means of $E/\omega_r$. Furthermore, the wave propagation part is linear. The absence of non-linearity in the description is felt in two ways:
   a. Along each wave ray only a linear amplitude description is used. Non-linear descriptions, as, e.g. a type of Korteweg de Vries (KdV) equation along a wave
ray is not used.

b. More important is the fact that there is no interaction between neighbouring wave rays.

2. Only refraction is modelled (the geometric optics approximation is used); diffraction is not included in the formulation.

As discussed earlier, inclusion of non-linearity in the wave-propagation description is available, see Willebrand (1975) and Watson and West (1975). Willebrand (1975) shows that inclusion of non-linearity may result in wave heights to be 25% higher in shallow water than without non-linearity.

2.9.2 Wind input

The model for wind input as is used in SWAN now is a rather simple one. We commented upon this model with nine comments in Section 2.3.2. As commented there one of the possible reasons for not so good behaviour (the growth curve of Kajma and Calkoen could not be obtained) might be the poor representation of the DIA approximation for the four-wave interactions, combined with the imposition of the cut-off frequency.

Furthermore it has to be stressed that many effects, important for the effect of wind upon waves have not been modelled. We mention 1) the effect of wave age on the wind stress, 2) the effect of long waves on the wind stress, 3) the lack of the effect of wind gustiness on the wave growth, 4) the effect of drift currents on the wave interactions and 5) the effect of shallow water on the atmospheric input.

We also think that the setting of the parameters for the wind-input terms is that of a not too large wind because for that case comparison with measurements can be made. It might be that the setting of the parameters is not suitable for large wind speeds. This has to be investigated.

For a better formulation of the wind input, it is necessary to apply a boundary-layer model of the form of the one discussed in Section 2.3.3.

2.9.3 Wave breaking

For the accuracy of the wave-breaking formulations we make distinction in wave breaking due to white-capping and wave breaking due to depth limitation.

White-capping

The Komen formulation (2.27)-(2.28) for white-capping has been implemented in SWAN. For narrow-banded spectra the method leads to reasonable results. However, for broad spectra
the white-capping problem exists, as mentioned in Section 2.5.1. The dissipation of the high-frequency waves is underpredicted, whereas the dissipation of the low-frequency waves is overestimated. A number of alternatives have been given in Section 2.5.1: the cumulative steepness method, the extended Komen method and the Tolman and Chalikov method. The latter two approaches still suffer from inaccuracies in mixed seas, due to the use of an average wave number in the extended Komen method. For the Tolman-Chalikov method the position of the peak frequency is not unique. Although the cumulative steepness method is not widely validated yet, the approach looks promising for spectra including both significant low and high frequent wave energy.

**Depth-limited wave breaking**

As already discussed at length in Dingemans (1998, pp. 27-30), a number of objections to the treatment of depth-induced wave breaking in SWAN can be formulated. These objections are:

1. **The definition of the breaker height:**

   In SWAN is used for the maximum wave height \( H'_b = \gamma h \) instead of the adapted Miche formulation (2.31), which is used in the Battjes and Janssen (1978) formulation. As can be expected, the difference between the two formulations is not much in extremely shallow water (measured in terms of \( kh \)), but for intermediate depth the difference can be appreciably, as shown, for example, in Figure A.6 in Dingemans (1998).

2. **The fixed value \( \gamma = 0.73 \) as used in SWAN.**

   Instead of relating the value of \( \gamma \) to the wave conditions at hand (for example using the formulae (2.34a) or (2.34b), see also Fig. 2.3, a so-called mean value for \( \gamma \) is used as standard value. It has already been shown long ago (Dingemans, 1983) that the use of one standard value for different wave conditions leads to computational results which are way off from measured values for the same conditions. Ergo, such computational results have no value.

3. **Known parameter settings cannot be used anymore when \( H'_b \) is used.**

   Even when \( \gamma \) is chosen by, e.g., Battjes and Stive (1985) for the wave conditions at hand, the difficulty remains that the known parameter settings cannot be used because the Battjes and Stive setting has been based on the maximum wave height (2.31).

2.9.4 **Four-wave interaction**

Because presently the DIA approximation is used in SWAN, results from four-wave interaction may be totally wrong. As becomes clear from Fig. 2.23, local errors of 400% can be obtained. Recently, alternatives for the DIA approximation have been developed, which are very close.
to the exact results of the Boltzmann integral. These new approximations (xDIA approximations) should be implemented as soon as possible, to obtain a trustworthy result of four-wave interactions.

2.9.5 Three-wave interaction

As remarked earlier (Dingemans, 1998), the present formulation of the three-wave interaction, modelled by the LTA approach is very inaccurate. In the first place it is based on a form of the Zakharov formulation which is dissipative and should be replaced by Krasitskii’s (1994) approach. Secondly, it contains a parameter of which the value remains unknown and its choice has large effects on the computational result, see e.g. Figures 5.1 and 5.2 in Dingemans (1998). In our opinion the results of present three-wave interaction formulation are so bad that it may be better not to use the present formulations. Alternatives have been recommended by Dingemans (2000). A further recommendation was given by Klopman and Dingemans (2001).

2.9.6 Bottom friction

In SWAN bottom friction models of Hasselmann et al. (1973), Collins (1972) and Madsen et al. (1988) have been implemented. Padilla-Hernández and Monbaliu (2001) note that for shallow areas methods that explicitly take into account physical parameters for bottom roughness, should be preferred. These models offer the possibility to adapt the dissipation rate according to the changing roughness under different wave or wave-current conditions. The model of Madsen et al. fits in this class. A very good alternative is the eddy-viscosity formulation of Weber (1989, 1991). The latter showed the best performance in the cases of depth- and fetch-limited wave growth tests, carried out by Padilla-Hernández and Monbaliu (2001). For example, the models of Hasselmann et al. (1973) and Madsen et al. (1988) lead to an overestimation of the total amount of wave energy in shallow water, compared to the formula of Young and Verhagen (1996a).

For strong winds sheet flow may become important. Effects of sheet flow cannot yet be taken into account in the present modelling of bottom friction in SWAN. By adjusting the bottom roughness a first step towards including current effects can be taken.

2.9.7 Discussion

By dividing the physical processes into wave generation, dissipation and nonlinear wave interaction, we can conclude that the latter class is modelled worst in SWAN. The standard DIA formulation that is used for modelling four-wave interactions may lead to large errors. Especially the scaling to shallow water is questionable. Fortunately, trustworthy alternatives have been developed and are ready to be implemented in SWAN. In shallow water also triad
wave interactions become important. Also their modelling by means of the LTA approach is erroneous. Also for this type of wave interactions alternative formulations exist. However, implementation of these formulations in SWAN requires a major effort.

The wind input source term presently implemented in SWAN is rather simple, lacking a number of effects which are important for the effect of wind upon waves. Furthermore, the wind input source term is sensitive for the interaction with the DIA formulation for the quadruplets in combination with the imposition of the cut-off frequency. The modelling of the dissipative source terms is probably more accurate than the presently implemented wind input source term. An extensive sensitivity analysis is required to give accurate estimates of the accuracy level of the source terms involved.
3 A qualitative analysis of the equilibrium situation

In §3.1 the possible mutual dependence of the various source terms is investigated. It should be remarked that the terms are modelled quite independently of each other. In §3.2 the possibility of simplification of source terms, in limit situations, is investigated. In §3.3, the aspects that are relevant in finding equilibrium spectra are mentioned. Furthermore, a qualitative reasoning is given for the existence of an equilibrium spectrum for the limit conditions. Finally the scalability of the balance of the source terms for the situation of limit of wave growth is studied here in §3.4.

3.1 Correlation between source terms

In this section we will discuss some interrelations between various source terms, as well as physical processes that are not modelled within SWAN. The omissions may introduce uncertainties in the prediction of waves in general, and in extreme situations in particular. Especially, the case considered here is the one of strong winds, long fetches, a nearly-horizontal bed and shallow water.

First we will summarize a number of interrelations between physical processes. In fact these are interrelations between classes. Of the following interrelations, the first two are wave generation (by wind) versus dissipation (by white-capping), the third and fourth are examples of wave generation versus nonlinear wave interactions (quadruplets). The last one fits in the class of wave-current interaction.

1. One can imagine that for high winds much foam due to white-capping will result, which in its turn has influence on the shear stress at the surface and therefore influences the wave growth due to the wind. Similarly, the presence of turbulence has effects on the wave growth due to wind.

2. Janssen (1989, 1991b) investigated a model for the coupling of wind and waves. Unfortunately, the implemented wave generation method did not work in SWAN. At first, one thought the interference of the wind formulation with the used method for white-capping (2.27)-(2.28) was responsible, but nowadays we believe that the DIA approach for the quadruplets, together with the fixed cut-off frequency in SWAN, is responsible for this behaviour. A major improvement can be obtained by dynamically adapting the cut-off frequency (see also WAM model Cycle 4).

3. As discussed by Phillips (1987), the effect of rain on wave generation by wind has not been considered. This effect may be twofold. Firstly, there is the rapid disappearance of the short waves due to the different effects on the forward and backward side of the wave. Secondly, the mean wind profile is affected by the rain.

4. The influence of swell on wave growth due to wind has not been considered in SWAN. Swell has a diminishing effect on the capillary waves generated by wind and consequently, swell hinders the effectivity of generation of wind waves. A further effect of
swell is that it also has effect on the wind-wave part through four-wave interactions (Balk, 1996 and Watson and McBride, 1993).

5. Shrir (1998) noticed that drift currents may have large effect on the generation of waves due to wind.

Out of these five points, the first two have major effect on the results in situations of strong wind. For very strong wind the wave form is not such that wave generation by wind and dissipation by white-capping act similarly as for mild conditions. Nevertheless, the same physics is applied in SWAN, irrespective of the kind of wave and wind conditions. Secondly, the model of Janssen (1989, 1991b) might lead to improved wave generation. Most certainly, releasing the fixed cut-off frequency when using the DIA approach for the quadruplet wave interactions will lead to significantly better results. Without implementing these changes in SWAN and carrying out a sensitivity study, we are not able to determine how much better the results will be in storm and extreme conditions.

Among others, the following physical processes are not modelled in SWAN:

1. As already mentioned in Section 2.8, the effect of an ambient current is not taken into account in the modelling of bottom friction in SWAN. For very strong winds sheet flow will be present and should be accounted for in the bottom friction models.

2. The effect of wind-gustiness has not been accounted for in SWAN. The effect of wind speed gustiness is relevant to growth of significant wave height because it causes an increase in the surface stress (see e.g. Komen et al., 1994). The growth is roughly proportional to the wind velocity squared, see e.g. (2.13). See also Abdalla and Cavaleri (2002).

3. The usual formulation for dissipation due to white-capping (Komen et al., 1984) has been used in SWAN. As noted by Banner et al. (1989), wave breaking of larger waves also causes rapid attenuation of shorter waves in its wake. Such effects are not accounted for in SWAN.

4. For spectral depth-limited wave breaking we advise to consider also different methods than uniformly distributing the dissipation over the spectrum. Some indications exist that a frequency-dependent redistribution might lead to better properties of kurtosis, which is of importance for morphological studies requiring SWAN input.

5. Effects of wave breaking as presently modelled in SWAN are based on a spectral description, in which the spectral components are independent. However, wave breaking is a strongly nonlinear phenomenon. Consequently, the pure spectral wave-breaking description is unable to capture the essentials due to wave breaking (skewness, asymmetry).

6. The effect of long waves, either free or bounded, is not taken into account in SWAN. Waves of the order of 10 wave lengths not only cause an increase of the water level, but also play an essential role in the interaction of waves with dunes and dikes (wave run-up and wave overtopping). Long waves also influence the wind stress at the surface because long waves tend to diminish the roughness of the short waves, see, e.g., Balk (1996). Also the wave age plays a role in this respect, see Drennan et al. (2003).

7. The LTA approach proposed by Eldekerky (1996) is in essence only valid in 1D. Interactions between waves with different directions have not been considered, leading to unrealistic results for short-crested wave fields.
Omission of long-wave effects in SWAN and omission of a proper description of triad wave interactions will lead to wrongly predicted spectral shapes by SWAN in very shallow areas where depth-induced breaking and nonlinear wave interactions play a dominant role. Especially the different measures for the wave period will be underpredicted. The effect on the total amount of wave energy is less pronounced, although the amount of discarded long-wave energy may be up to 20%.

The omission of the processes mentioned under items 2, 4 and 5 will probably have less effect on the predicted significant wave height and wave period. For more quantitative conclusions an extensive investigation is required, which is beyond the scope of the present study.

3.2 Simplification of source terms in limit situations

In this section we briefly repeat the formulation of the source terms, as implemented in SWAN. An attempt is made to simplify these source terms for application in further analysis by considering an analytical situation which is characterized by strong wind, infinitely long fetch length, a horizontal bed at shallow water and no spatial and temporal variations of active physical processes. Due to the fact that the source terms in SWAN are simplifications, which model real-life processes as well as possible, major simplifications of the source terms are not to be expected.

The dominant physical processes for situations of strong wind and shallow water, as far as they are modelled in SWAN, are wind forcing, depth-induced wave breaking, white-capping, bottom friction and three and four wave interactions. In the following we will consider each of these source terms.

In SWAN spectrally-averaged values for wave number and wave frequency are used. In the formulations for the source terms given below, the following notation will be used:

\[ \phi^{(n)} = \frac{1}{m_0} \left( \int_0^\infty \int_0^{2\pi} \phi^n E(\omega_r, \theta) d\theta d\omega_r \right)^{1/n}, \] (3.1)

with \( \phi \) denoting wave number or frequency and \( m_0 \) the total wave energy:

\[ m_0 = \int_0^\infty \int_0^{2\pi} E(\omega_r, \theta) d\theta d\omega_r. \] (3.2)

The wave number and the wave frequency are coupled by the linear dispersion equation. In shallow water this relation yields

\[ \omega_r^{(n)} = k^{(n)} \sqrt{gh}. \] (3.3)

Note that relation (3.3) is approximate since for higher frequencies, and thus wave numbers, the deep-water approximation \( \omega_r^2 = gk \) is more appropriate. Therefore, equation (3.3) will be
applied for mean values only and is reliable especially for spectra with small amount of high-frequency energy. The shallow water assumption will not be applied on single frequencies.

The wind input is modelled via linear and exponential growth of waves by means of (2.6), using (2.9) and (2.8) respectively. The modelling of the wind input is simple and in the analytical situation a simpler expression for $S_{in}$ is not obtained. If exponential wind growth is assumed, then $S_{in} \sim \omega_r E$. The main emphasis is on the higher frequencies.

The white-capping formulation in SWAN is given by (2.27) and (2.28). With $\delta = 0$ and $m = 2$ the source term for white-capping reduces to:

$$S_{wc} = -C_{ds} \left( \frac{\overline{\alpha}}{\alpha_{PM}} \right)^2 \frac{\omega_r^{(-1)}}{k^{(-\frac{1}{2})}} kE(\omega_r, \theta), \quad (3.4)$$

Usually, $\frac{C_{ds}}{16\alpha_{PM}^2} = 2.36 \cdot 10^{-5}$ and $\overline{\alpha}_{PM} = 3.02 \cdot 10^{-3}$. Relation (3.3) does not apply to the ratio between the average values of wave frequency and wave number, since the deep-water approximation for the dispersion relation has been used. Nevertheless, using $\overline{\alpha} = m_0 \left( \frac{k^{(-\frac{1}{2})}}{k^{(-\frac{1}{2})}} \right)^2$ (3.4) can be approximated by

$$S_{wc} = -\frac{C_{ds}}{(16\alpha_{PM}^2)} \sqrt{gh} \left( \frac{H_{m0}}{h} \right)^a \left( \frac{k^{(-\frac{1}{2})} h}{k^{(-\frac{1}{2})}} \right)^{\frac{1}{4}} kE(\omega_r, \theta), \quad (3.5)$$

The significant wave height is defined as $H_{m0} = 4\sqrt{m0} = \sqrt{2} H_{rms}$. For narrow-banded spectra a further reduction is obtained, since then $k^{(-\frac{1}{2})} \approx k^{(-\frac{1}{2})}$. As expected, $S_{wc} \sim kE$, which is equivalent to the exponential behaviour of the wind input term at the high frequencies.

The dissipation term (2.33) of Battjes-Janssen, together with with $H_b = \gamma h$, is used to describe depth-induced wave breaking. The breaker fraction $Q_b$ is determined by relation (2.46). In the particular situation of a horizontal bed, the breaker fraction will be small. Relation (2.46) then reduces to

$$Q_b = \exp \left[ -\left( \frac{H_b}{H_{rms}} \right)^2 \right] = \exp \left[ -b^{-2} \right] \quad \text{with} \quad b = \frac{H_{rms}}{H_b} = \frac{H_{m0}}{\sqrt{2} \gamma h}. \quad (3.6)$$

Equation (2.33) can now be reduced, resulting in the following source term,

$$S_b = -\frac{\alpha \omega_r^{(-1)}}{4} Q_b h_b^2 E(\omega_r, \theta)$$

$$= -\frac{\alpha \omega_r^{(-1)}}{\frac{2}{k}} \sqrt{gh} b^{-2} \exp \left( -b^{-2} \right) E(\omega_r, \theta). \quad (3.7)$$

The breaker source term acts uniformly on the entire spectrum, $S_b \sim E$.

The source term for bottom friction is given by (2.102):

$$S_{bot}(\omega_r, \theta) = -C_{bot} \frac{\omega_r^2}{g^2 \sinh^2(2kh)} E(\omega_r, \theta) = -C_{bot} \frac{2k}{gh} \sinh(2kh) E(\omega_r, \theta). \quad (3.8)$$
The latter equality is obtained using the dispersion relation. Hasselmann et al. (1973) assumed the bottom-friction coefficient to be constant \( C_{bot} = 0.038 \text{ m}^2 \text{s}^{-3} \) for swell conditions. For fully-developed wave conditions in shallow water Bouws and Komen (1983) selected a bottom-friction coefficient of \( C_{bot} = 0.067 \text{ m}^2 \text{s}^{-3} \). More advanced models are available (see Section 2.8). Due to the exponential decay in (3.8) for high frequencies, \( S_{bot} \) has its impact on the lower frequencies at which \( S_0 \approx \{C_{bot}/(gh)\}E \).

It is well known that the LTA approach of Eldekerby (1996), given by (2.95)-(2.98) is the most basic approach to describe three-wave interactions. As already mentioned in §2.7, the shoaling term has been neglected in the derivation of the LTA approach. For the special situation of a horizontal bed, the expression for the three-wave interaction term will therefore not simplify.

A parameterization of the Boltzmann integral leads to the DIA approximation, basically given by (2.93), for four-wave interactions. As for the triads, the parameterization is such a simplification that also in an analytical situation significant simplifications will not be obtained.

### 3.2.1 Quantitative comparison of source terms in limit situation

The dissipative source terms for white-capping (3.5), depth-induced wave breaking (3.7) and bottom friction (3.8) using the Jonswap formulation: \( C_{bot} = 0.067 \text{ m}^2 \text{s}^{-3} \) have been compared in Figure 3.1 for the characteristic situation of a horizontal bed \( (h = 5 \text{ m}) \), strong wind \( (U_{10} = 30 \text{ m/s}) \) and an infinite fetch. To estimate the values for wave frequency and wave number in the expressions for \( S_{we} \) and \( S_b \), a Jonswap spectrum is assumed. Furthermore, the growth curves of Young and Verhagen (1996, p.62) for the dimensionless wave energy and frequency have been used to determine a characteristic value for the significant wave height \( H_{m0} \) and the wave peak period \( T_m \). The expressions

\[
\varepsilon = 1.06 \cdot 10^{-3} \delta^{1.3}, \quad \varepsilon = \frac{g^2 m_0}{U_{10}^4}, \quad \delta = \frac{gh}{U_{10}^2}
\]

(3.9)

\[
\nu = 0.20 \delta^{-3/8}, \quad \nu = f_m \frac{U_{10}}{g}
\]

(3.10)

lead to the following estimate:

\[
H_{m0} = 1.8 \text{ m}, \quad T_m = 5.14 \text{ s}
\]

(3.11)

The breaker parameter \( \gamma \) is related to the deep-water steepness according to the relation of Battjes and Stive (1985), given by (2.34a). The deep-water steepness is given by \( S_0 = H_{m0}/L_0 \) with \( L_0 = gT_m^2/2\pi \) and results in a value for the breaker parameter \( \gamma = 0.86 \). For the situation under consideration bottom friction and white-capping clearly dominate the process of depth-induced wave breaking. For this situation the bottom friction is even twice as large as the white-capping at the peak frequency. Notice that this is not an equilibrium situation.
Figure 3.1: Dissipative source terms in SWAN scaled with the energy wave spectrum for a situation with $h = 5 \text{ m}$, $U_{10} = 30 \text{ m/s}$ and infinite fetch.

Secondly, SWAN version 40.16 with the full non-simplified source term formulations was used on a horizontal bottom ($h = 5 \text{ m}$) in 1D and stationary mode. At the boundary $x = 0$ small waves were imposed ($H_{m0} = 0.20 \text{ m}$, $T_m = 3.0 \text{ s}$). The strong wind ($U_{10} = 30 \text{ m/s}$) caused a spectrum that did not change spatially after 40 km. For the same situation as sketched above, the spectrum and individual source terms at $x = 80 \text{ km}$ have been plotted in Figure 3.2. The source terms have been obtained using the 'test'-option SWAN. The SWAN inputfile for this run has been given in Appendix C.

Whereas in the first exercise the magnitude of the source term for the depth-induced wave breaking was small (see Figure 3.1), its magnitude in the balance is comparable to bottom friction and white-capping. At $x = 80 \text{ km}$ SWAN determines a significant wave height to depth ratio of 0.40. Consequently, the value for $Q_b$ and thus the magnitude of $S_b$ is larger. The magnitude of $S_b$ seems to be sensitive for the total amount of wave energy. A small change is $H_{m0}$ causes relatively large changes in $S_b$, according to (3.6)-(3.7).

The three dissipative source terms mentioned here are all important in the limit situation under consideration. For the high frequencies the white-capping is dominant over the other two dissipative processes. At the lower frequencies both bottom friction and depth-induced wave breaking are important. Which source term has the largest impact on the spectrum, cannot be predicted beforehand. This depends on the settings of the model parameters and of the situation under consideration. They should all be taken into account.
Figure 3.2: Wave energy spectrum (up), source terms (middle), and sum of the source terms (low) computed with SWAN at $x = 80\,\text{km}$ for a situation with $h = 5\,\text{m}$, $U_{10} = 30\,\text{m/s}$.

In the limit situation both source terms for modelling nonlinear wave interactions are important. An equilibrium situation cannot be obtained without transferring energy from one part of the spectrum to the other. Therefore, the source terms for non-linear wave-wave interactions are relatively important compared to the input and dissipative source terms. Since
the triad formulation is very approximate and limited (only transfer two double frequency through self-self interaction) the relative importance that SWAN gives to triads, is not correct. Three-wave interaction modelling by the LITA approach should better be omitted.

3.2.2 Dominant inaccuracy for wave-growth limit situation

Based on the reasoning in the previous paragraph, the relative importance of the source terms can be determined for wave-growth limit situations. Since the wind input is the only wave energy generation mechanism in SWAN, this term is extremely important in equilibrium situations. The dissipative terms are equally important as sketched above. Without non-linear interactions the transfer of wave energy across the spectrum cannot be modelled. This mechanism is required to obtain realistic predictions of the wave spectrum in equilibrium situations. Therefore, the source terms for non-linear wave-wave interactions are relatively important compared to the input and dissipative source terms. Especially quadruplets play an important role in the evolution of the spectrum towards an equilibrium. Triads affect the full-grown spectrum in very shallow areas.

The dominant uncertainties for the determination of the wave conditions in the wave-growth limit with SWAN can be obtained by combining the dominant uncertainties in the formulation of the source terms in general situations (see Section 2.9.7) with the relative importance of the source terms for wave growth limit situations. Therefore, the conclusion is drawn that the present implementation of the quadruplet source terms is responsible for the largest uncertainties in the modelling of equilibrium situations at moderate to shallow water. Since triads will be less important, their effect on the final SWAN results will be less.

The modelling of the dissipative source terms is relatively good, compared to the modelling of the wind input source term, provided that the correct calibration parameters are applied. Although not as pronounced as the quadruplets the presently implemented wind input source term contributes significantly to incorrect predictions of wave growth limit conditions. In the order specified below the source terms have a decreasing negative effect on the accuracy of the SWAN predictions in wave-growth limit situations:

1. quadruplets
2. wind input
3. triads
4. white-capping, depth-induced wave breaking, bottom friction
3.3 Spectrum in equilibrium situation

3.3.1 Determination of equilibrium spectrum

Among others, Phillips (1977, p.140) states that, under given wind forcing, the spectra of ocean gravity waves attain an equilibrium state at frequencies much higher than the peak frequency. At shallow water also aspects as depth-induced wave breaking and bottom friction play a role, which complicate the analysis of equilibrium states significantly.

Following Phillips (1977, §4.1) and Banner (1990) we first define the various forms of wave spectra. The wave spectrum for a homogeneous, stationary wave field is defined by:

$$X(k, \omega) = (2\pi)^{-3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\rho dt \rho(r, t) \exp[-i(k \cdot r - \omega t)] ,$$  \hspace{1cm} (3.12)

where $\rho(r, t) = \langle \zeta(x, t)\zeta(x + r, t_0 + t) \rangle$ is the covariance of the surface displacement function $\zeta(x, t)$, $r$ is the spatial displacement vector and $t_0$ is the time separation. $X(k, \omega)$ has the property that

$$\int_{-\infty}^{\infty} dk d\omega X(k, \omega) = \zeta^2 .$$  \hspace{1cm} (3.13)

Various reduced spectra can be defined:

1. The directional wavenumber spectrum:

$$\Phi(k) = 2 \int_{0}^{\infty} d\omega X(k, \omega) ;$$  \hspace{1cm} (3.14a)

2. The wavenumber spectrum:

$$\Phi_s(k) = \int_{-\infty}^{\infty} d\omega X(k, \omega) ,$$  \hspace{1cm} (3.14b)

representing the folded wavenumber distribution, with $\Phi_s(k) = \frac{1}{2} [\Phi(k) + \Phi(-k)]$. The $\Phi_s(k)$ is obtained from aerial photographs and does not contain actual wave propagation information. There is a $180^\circ$ ambiguity of direction.

3. One-dimensional (transverse) spectra:

$$\phi (k_1) = \int_{-\infty}^{\infty} dk_2 \Phi(k_1, k_2)$$  \hspace{1cm} (3.14c)

and

$$\phi (k_2) = \int_{-\infty}^{\infty} dk_1 \Phi(k_1, k_2) .$$  \hspace{1cm} (3.14d)

4. Omni-directional wavenumber spectrum:

$$\bar{\phi}(k) = \int_{-\pi}^{\pi} d\theta k \Phi(k, \theta) .$$  \hspace{1cm} (3.14e)

5. Directional-averaged wavenumber spectrum:
\[ f(k) = \int_{-\pi}^{\pi} d\theta \, \Phi(k, \theta) . \] (3.14f)

6. Directional frequency spectrum:
\[ G(\omega, \theta) = 2 \int_{0}^{\infty} dk \, k X(k, \omega) . \] (3.14g)

7. Frequency spectrum:
\[ F(\omega) = \int_{-\pi}^{\pi} d\theta \, G(\omega, \theta) . \] (3.14h)

The equilibrium spectrum for deep water was postulated by Phillips (1958) as
\[ F(\omega) = \alpha g^2 \omega^{-5} , \quad f(k) = B \Theta(\theta) k^{-4} , \] (3.15)

where \( \theta \) is the angle between the wind and the wave number \( k \) and \( \alpha \) and \( B \) are equilibrium range constants. Phillips (1985) re-examined the equilibrium spectrum. Given new evidence, he concludes that it has become evident that the idea of a hard, saturated upper limit to the spectrum is no longer tenable. It was recognised soon that the proportionality constant implicated in (3.15) was not really a constant. Toba (1973) found both experimentally and theoretically that a spectrum of the form
\[ \Phi(\omega_r) \sim u_* g \omega_r^{-4} \] (3.16)

fitted the data better than a power -5 tail. Zitman (1985) (see also Battjes et al., 1987) re-examined the JONSWAP data set and found that with a power -4 tail in the frequency spectrum the proportionality constant varied much less than in the original JONSWAP spectral form. This spectrum is written by Dingemans (1987, Appendix G, p.203) as:

\[ E(f) = A_4 S(\nu) \quad \text{with} \quad S(\nu) = \nu^{-4} \exp \left( -\nu^{-4} \right) \gamma(\nu) \] (3.17a)

\[ \gamma(\nu) = \gamma_0 \exp \left[ -\frac{1}{2\sigma^2} (\nu - 1)^2 \right] \] (3.17b)

\[ A_4 = \alpha_4 u_* (2\pi)^{-3} f_m^{-4} \] (3.17c)

\[ \nu = f/f_m ; \quad \sigma = \sigma_a \quad , \quad \nu \leq 1 \quad \text{and} \quad \sigma = \sigma_b \quad , \quad \nu > 1 , \] (3.17d)

where \( f_m \) is the modal frequency (and thus, the peak frequency). The mean parameters are given by:
\[ \langle \gamma_0 \rangle = 3.64 \quad , \quad \langle \alpha_4 \rangle = 0.13 \quad , \quad \langle \sigma_a \rangle = 0.12 \quad , \quad \langle \sigma_b \rangle = 0.11 . \] (3.17e)

The question of an equilibrium spectrum for shallow water is taken up by Resio et al. (2001) and also by Huang et al. (2001). For shallow water the matter is taken up by Katsaras (2001).

The question of equilibrium spectra is discussed also by Banner (1990) and by Hara and Belcher (2002). Banner (1990) notices that Phillips (1985) predicted an equilibrium spectrum of the form
\[ \Phi_s(k) \sim u_* g^{-1/2} k^{-7/2} g(\theta) , \] (3.18)
where $g(\theta)$ is some directional spreading function. The one-dimensional spectral forms corresponding to (3.16) are

$$ \phi(k_i) \sim u_+ g^{-1/2} k_i^{-5/2} , \quad i = 1, 2 $$

and

$$ F(\omega_r) \sim u_+ \omega_r^{-4} . $$

(3.19a)

(3.19b)

Phillips stated that the equilibrium spectral range contained gravity wave components with wave lengths much shorter than that of the spectral peak. Kitaigorodskii (1983) obtained (3.18) and (3.19) for $\omega > \omega_g$ where $\omega_g \sim 4g/U$ is an empirical transition frequency.

3.3.2 Equilibrium state in SWAN

In the previous subsection the analysis is restricted to the spectral tail. If the wave height over depth ratio in limit situations should be determined from wave-energy spectra, knowledge about the entire spectrum in limit situation is required. Here we will give a qualitative explanation why SWAN in its present state will not predict an equilibrium spectrum.

An equilibrium state is defined as a state in which the rate of change in action, or wave-energy density, vanishes for all frequencies. Mathematically, this means that the sum of the source terms is zero for all frequencies.

As stated by Weber (1988), it is assumed that a deep-water expression for the dissipation by white-capping, written in wavenumber space, also holds for shallow water. The general idea of wave growth is that the non-linear interactions preserve the spectral shape and cause the spectrum to shift to the lower frequencies, so that waves become larger, until an equilibrium is reached between the source terms. Weber states that it is not known whether this applies also for shallow water or which of the source terms dominates wave growth.

Weber performed four computations, for 15, 30, 60 and 120 m depth, all with a friction velocity of 0.71 m/s (or, $U_{10} = 16.5$ m/s). The 120 m depth is effectively deep water. For the 15 m run equilibrium was reached at 500 km and for the 30 m depth equilibrium was reached at 900 km. The 60 m run oscillates slightly around equilibrium values at 2000 km. The deep-water run (120 m depth) was continued up to 2300 km; from former deep-water runs it is expected that equilibrium (a Pierson-Moskowitz spectrum) is reached after 6000 km. These computations show that a long distance is required to obtain an equilibrium state, even for shallow regions. Based upon these computations the conclusion might be drawn that Dutch lakes as the IJsselmeer and Markermeer are too small to reach an equilibrium state.

Weber (1988) solves the energy balance equation for different water depths, including bottom dissipation and explicit calculation of the resonant four-wave interactions. The equilibrium which is reached between bottom friction and the resonant interactions on the forward face of the spectrum (that is, the part for which $f < f_m$ where $f_m$ is the peak frequency) turns out to be important. In SWAN for the higher frequencies wind input, white-capping and quadruplets are the dominant source terms. As already shown in the previous section the
asymptotic behaviour of wind input and white-capping is similar. For the higher frequencies an equilibrium state might be reached in principle.

At shallow water the three-wave interactions may also play an important role. The present triad formulation (LTA) prevents the occurrence of an equilibrium state. In shallow water \((kh < 1)\) the triad source term transfers the bulk of energy around the peak frequency \(f = f_m\) towards the first harmonic at \(f = 2f_m\). A phenomenological study by Groeneweg et al. (2002) underpinned the strong overestimation of the energy transfer towards higher frequencies. Moreover, the region in which SWAN predicts triads to be important is overpredicted as well, enhancing the overestimation of high-frequent wave energy. Since the white-capping formulation is tuned to the wind input formulation at higher frequencies, the white-capping source term as such is not capable of decreasing the wave energy transferred by triad wave interactions. The depth-induced wave breaking source term acts uniformly over all frequencies. This source term could only dissipate the energy at the higher frequencies without affecting the wave energy at lower frequencies, if the dissipation rate would have been frequency-dependent. However, this is not the case in the present implementation of the Battjes-Janssen (1978) formulation or derivatives thereof given in the present study, or in any of the suggested alternatives given in Groeneweg et al. (2002, Ch.2).

The presently implemented quadruplet formulation in SWAN, i.e. the DIA formulation, is not accurate enough to transfer the energy generated by the triads towards the frequencies around the peak frequency. Figure 2.24 shows that energy is taken from frequencies higher than the peak frequency, but the amount of energy taken from these frequencies is strongly overpredicted. This also holds for the energy generated by the wind. For the really short waves, an equilibrium between wind input and white-capping is obtained. On the backward face of the spectrum the DIA approximation is so inaccurate (see also Figure 2.24) that an equilibrium between wind, white-capping and quadruplets (and triads) cannot be obtained.

A further discrepancy is to be expected at the forward face of the spectrum. There is some wind input, but for low frequencies the contribution is small compared to the bottom friction source term, since \(S_{in} \sim \omega f E\) and \(S_{bot} \sim \sqrt{gh} E\). Therefore the dissipation due to bottom friction should be balanced by the quadruplet source term. Here, once again the inaccuracy of the DIA approximation prevents a balance between the two source terms.

An equilibrium cannot be obtained with SWAN in the sense that the sum of the source terms vanishes for all frequencies. On the other hand, Figure 3.2 shows that for a situation of wave growth the magnitude of the sum of the source terms (computed with SWAN) is smaller than the magnitude of each individual source term in the region around the peak frequency. Apparently, SWAN determines wave conditions which the program itself considers as more or less an equilibrium state. The obtained spectrum shown in Figure 3.2 does not change spatially. De Waal (2001) considered a spectrum as an equilibrium, if it is spatially non-varying. Using this weaker definition of an equilibrium state, he also showed that the sum of the dissipative source terms is smaller than the wind input source term in an absolute sense (Figure 6 in De Waal, 2001). This might be due to the fact that limiters act as source terms in SWAN. These limiters are necessary to avoid numerical instabilities and should as such be a numerical tool, that should not influence the final (i.e. after convergence) solution. As long as limiters are required in SWAN, which affect the action balance itself, the numerical model
can certainly not predict an equilibrium state in the sense that the sum of the source terms vanishes for all frequencies.

3.4 Scaling of source terms in equilibrium situation

In this section we will address the question whether the wave conditions in wave-growth limit situations, as obtained by SWAN, are scalable. This question can only be answered by making the assumption that an equilibrium condition exists, despite the fact that an equilibrium will probably not be obtained with SWAN (see previous section). The right-hand side of the balance equation, consisting of the sum of the source terms, will be set to zero. Obviously, if the source terms in the balance equation scale differently, the wave conditions in the limit situation do not scale, since they are the result of the obtained balance of the source terms. This topic should however not be confused with the problem how to scale the physics related to wind-wave interactions. These interactions do not scale according to Froude scaling law due to the influence of surface tension. The latter becomes especially important for small scales (e.g. wave heights of a few decimeters).

The scaling rules that are applied are based on the constant ratio of inertia and gravity forces. This implies that horizontal and vertical length scales are the same. As a consequence the wave length over depth ratio is constant and the dispersion relation remains valid. A typical consequence is that the Froude number remains constant (see e.g. Section 2.4 in Chakrabarti, 1994). The Froude number is given by

\[ F_r = \frac{U_0}{\sqrt{g h}} \]  

(3.20)

with \( U_0 \) a typical velocity.

First of all, the source term for the wind input, i.e. Eq. (2.6), with \( A = 0 \) and \( B \) given by Eq. (2.8), is considered. From the expression for \( B \) it is clear that one should take \( U = u_* \). Because \( U_{10} \) does not depend linearly on \( u_* \), due to the \( U_{10} \)-dependent constant \( C_{10} \) in relation (2.9), scaling with \( U_{10} \) is not possible.

The following dimensionless parameters are introduced in which water depth \( h_0 \) is a characteristic length scale \( h_0 \) and wind stress velocity \( U_0 = u_* \) a characteristic velocity. Since scaling with Froude applies, the dimensionless parameters can be given either in terms of \( g \) and \( U_0 \), or \( g \) and \( h_0 \). We chose for the latter.

- \( \hat{h} = h / h_0 \)
- \( \hat{k} = k h_0 \)
- \( \hat{H}_b = H_b / h_0 \)
- \( \hat{\omega}_r = \omega_r / \sqrt{g h_0} \)
- \( \hat{E} = E \sqrt{g / h_0^3} / h_0^2 \)
\[ \hat{m}_0 = \frac{m_0}{h_0^2} \]
\[ \hat{u}_* = \frac{u_*}{\sqrt{gh_0}} \]

Using these scaling rules, the wave generating source term (2.6) and wave dissipating source terms for white-capping (2.27), for depth-induced wave breaking (3.7) and for bottom friction (2.102) can be written in dimensionless quantities as:

\[ S_{\text{th}} = \frac{1}{4} \frac{\rho_a}{\rho_w} \max \left[ 0, \, 28F_c \cos (\theta - \theta_w) \frac{k \hat{u}_*}{\hat{\omega}_r} \hat{E} h_0^2 \right] \]
\[ S_{\text{wc}} = -C_{\text{wc}} \left( \frac{\tau^2}{k} \frac{\hat{\omega}_r}{\hat{\omega}_*} \right) \frac{1}{k} \hat{E} h_0^2 \]
\[ S_b = -\frac{\alpha}{4} \frac{\hat{H}^2}{Q_b} \frac{\hat{E} h_0^2}{\hat{m}_0} \]
\[ S_{\text{bot}} = -C_{\text{bot}} \frac{2k}{g \sinh \left( \frac{2k \hat{h}}{k} \right)} \hat{E} \frac{h_0^2}{\sqrt{gh_0}} \]

Clearly, the bottom friction source term scales differently than others if \( C_{\text{bot}} \) is taken as a constant, as in the formulations suggested by Hasselman et al. (1973) or Bouws and Komen (1983). On the other hand, Collins’ formulation (2.102b) for \( C_{\text{bot}} \) scales with \( \sqrt{gh} \). So either Collins’ (or Madsen et al.’s, 1988) approach should be used if the wave generating and wave dissipating source terms are required to be scalable in their balance.

As for the source terms above, one can show that the triad source term (2.95)-(2.97) scales with \( h_0^2 \), since

\[ S_{\text{tri}}(\omega_r, \theta) = \max \left\{ 0, \, 2\pi \alpha \hat{\varepsilon}_g \hat{J}^2 \sin \hat{\beta} \left[ \hat{E} \left( \frac{\hat{\omega}_r}{2}, \theta \right) - 2 \hat{E} \left( \frac{\hat{\omega}_r}{2}, \theta \right) \hat{E} (\hat{\omega}_r, \theta) \right] \right\} h_0^2, \]

with \( \hat{\varepsilon} = c/\sqrt{gh} \), \( \hat{\varepsilon}_g = c_g/\sqrt{gh} \) and \( \hat{J} = J h_0^2 \).

The exact formulation of the quadruplet source term is given by the right-hand side of (2.85). The DIA and XDIA formulations are derivatives of the exact formulation. These formulations are obtained by considering different numerical techniques to solve the Boltzmann integral. The complicated function \( G \) in the integral can be approximated by \( 4\pi |K_1| \). By applying the scaling rules mentioned above to (2.85), it is unclear how the quadruplet source term given by the right-hand side of (2.85) scales.

Under the assumption that the quadruplet source term scales with \( h_0^2 \), as the other source terms do, the balance of the source terms also scales and so will the wave conditions in the 'approximate' equilibrium. We will return to this issue in Section 4.3.
4 A quantitative analysis of the equilibrium situation

In Section 3.3 we concluded that an equilibrium spectrum cannot be obtained with the present formulations of the source terms in SWAN. For the situation of a constant wind speed and horizontal bed SWAN predicts the waves to grow to a certain limit for which the energy density does not change in time and space, but the sum of the source terms is not equal to zero for all frequencies.

Here we quantitatively determine the 'approximate' equilibrium spectrum that SWAN predicts for a given wind speed and water depth. For a number of spectral shapes, the wave height and wave steepness have been varied. For each combination the SWAN source terms have been determined. The wave conditions corresponding to the combination for which the sum of source terms is minimal (in a certain norm over all frequencies) is considered as equilibrium wave condition. The resulting wave height over depth is of special interest.

4.1 Determination of SWAN source terms

For a given spectrum it is possible to determine the source terms within SWAN using the TEST option. The spectrum under consideration is imposed as a boundary condition. Only two grid points are required, one of which is the boundary location. In SWAN the option NOUPDATE exists in the MODE environment. Applying this functionality, the action density is not updated when the source terms have been calculated. In the first iteration step initialisations are carried out by running in second generation mode. After the first iteration step the action density is left unchanged. In the second iteration step SWAN runs in third-generation mode. Firstly, the source terms are calculated, given the spectrum in the previous iteration step. Secondly, the resulting action density would have been determined. However, this step is not carried out. After the two iterations the test output is written to an output file. In the second grid points we then obtain the (originally imposed) wave energy density and the source terms for wind input, white-capping, bottom friction, depth-induced wave breaking, triads and quadruplets for all frequencies.

The SWAN computations have been carried out with SWAN version 40.16. The advantage of this version is that triad and quadruplet source terms can be determined simultaneously, using the following commands

- TRIAD [urSELL]
- LIMITER [urSELL]

The first command indicates the value of the Ursell number below which the triad wave interactions are deactivated and hence not calculated (default: 0.1) The Ursell parameter in the second command is the value of the Ursell number above which quadruplets and the limiter on the action density are both switched off (default: 0.1). Notice that with the default values of these parameters either triads or quadruplets are activated. Actually, by using the
defaults values version 40.11 is applied. In our computations we used values of 10 and 0.001 for the Ursell parameters for the triads and quadruplets respectively. Effectively, triads and quadruplets are always calculated.

The bottom friction is modelled by means of Hasselmann et al.'s (1973) formulation, i.e. a constant bottom friction coefficient \( C_{bot} = 0.038 \, \text{m}^2 \, \text{s}^{-3} \). The breaker parameter \( \gamma \) in the Battjes-Janssen formulation is not fixed for all conditions, but depends on the wave condition valid for the test under consideration, using the expression of Battjes and Stive for \( \gamma \), i.e. (2.34a). The deep-water steepness is estimated by

\[
s_0 = \frac{2\pi H_{m0}}{gT_m^2}. \tag{4.1}
\]

For the modelling of the quadruplet interaction source term the default values for the constants in formulation (2.93) have been used, i.e. \( C = 3 \times 10^7 \) and \( \lambda = 0.25 \). For shallow water the result of the deep-water non-linear interaction integral is simply multiplied with a scale factor which is a function of \( kh \). The form is (see WAMDI, 1988, Eq. 3.8):

\[
R(kh) = 1 + \frac{C_{sh1}}{kh} (1 - C_{sh2} \, kh) \exp (C_{sh3} \, kh). \tag{4.2}
\]

Note that it is recommended by WAMDI (1988) not to apply this function for \( kh < 1 \). In all tests the following default values are used: \( C_{sh1} = 5.5 \), \( C_{sh2} = 0.8333 \), \( C_{sh3} = -1.25 \).

4.2 Approximate equilibrium spectra in SWAN

Applying SWAN in its setting described in the previous subsection, an approximate equilibrium spectrum and corresponding wave conditions have been derived for a given water depth \( h \) and wind speed \( U_{10} \). By considering 21 values of wave height relative to the water depth \( \gamma \) in the range of \((0.35, 0.45)\), 10 values for \( T_m \) such that the wave steepness \( (s_0, \text{according to } (4.1)) \) is in the range of \((0.04, 0.07)\), and three spectral shapes (Jonswap, Pierson-Moskowitz and TMA) in total \(3 \cdot 21 \cdot 7 = 441\) spectra have been considered. The wave spectrum for which the sum of the source terms is minimal is considered as the approximate equilibrium spectrum.

The SWAN computations have been carried out in 1D mode. For all tests the directional spreading has been taken as 5 degrees. The directional space is divided in a sector of 80 degrees around the mean direction \( \theta_w = 0 \). The directional space consists of 20 bins of 4 degrees. Frequency space is ranging from 0.1 Hz to 1.2 Hz and is divided into 60 logarithmically distributed frequency bins.

First of all, we considered \( h = 5 \, \text{m} \) and \( U_{10} = 35 \, \text{m/s} \) which is characteristic for design conditions at e.g. IJsselmeer. The following measures for the deviation from zero of the sum
of the directionally-averaged source terms are defined:

\[
l_2 = \frac{1}{m_0} \left( \sum |S_{in}(f_i) + S_{we}(f_i) + S_{bot}(f_i) + S_b(f_i) + S_{nt3}(f_i) + S_{nt4}(f_i)|^2 \right)^{1/2} \quad (4.3a)
\]

\[
l_\infty = \frac{1}{m_0} \max_i |S_{in}(f_i) + S_{we}(f_i) + S_{bot}(f_i) + S_b(f_i) + S_{nt3}(f_i) + S_{nt4}(f_i)| \quad (4.3b)
\]

where the sum or maximum is taken over all frequencies \( f_i \). Notice that the norms are scaled with the total variance, because all source terms scale with wave energy, excluding nonlinear source terms.

These norms have been determined and for those conditions around the expected minimum listed in Table (4.1) for the three considered wave energy spectra. For each spectrum and each norm the minimum value for each column has been marked.

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<th>( T_m )</th>
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Table 4.1: Measures for \( l_2 \) and \( l_\infty \) for \( h = 5 \text{ m} \) and \( U_{10} = 35 \text{ m/s} \).
In Figures 4.1-4.3 the contour plots of the norms for the three spectral shapes have been given. A circle indicates the combination for relative wave height and wave steepness where the minimum of the norms occur.

Figure 4.1: Contour plots of $l_2$ (left) and $l_\infty$ (right) for $h = 5 \text{ m}$ and $U_{10} = 35 \text{ m/s}$ with Jonswap spectrum. Circle indicates location of minimum.

Figure 4.2: Contour plots of $l_2$ (left) and $l_\infty$ (right) for $h = 5 \text{ m}$ and $U_{10} = 35 \text{ m/s}$ with Pierson-Moskowitz spectrum. Circle indicates location of minimum.

For a Jonswap spectrum (Figure 4.1) and a TMA spectrum (Figure 4.3) the wave steepness in the 'approximate' equilibrium situation is approximately $s_0 = 0.05$. The relative wave height equals $H_{\text{r0}}/h = 0.40$ for a TMA spectrum and significantly lower for a Jonswap spectrum ($H_{\text{r0}}/h \approx 0.38$). The Pierson-Moskowitz spectrum yields slightly higher optimal values, viz. $H_{\text{r0}}/h = 0.42$ and $s_0 = 0.06$ (Figure 4.2). In an absolute sense the deviation of the sum of the source terms from zero is smallest for the latter spectral form.

In Figures 4.4-4.6 the wave energy spectra, the source terms for the considered spectrum
Figure 4.3: Contour plots of \( l_2 \) (left) and \( l_{\infty} \) (right) for \( h = 5 \) m and \( U_{10} = 35 \) m/s with TMA spectrum. Circle indicates location of minimum

and their sum are plotted for the conditions at which the 'approximate' equilibriums occur. The sum of the source terms is clearly smallest for the Pierson-Moskowitz spectrum. For the optimal condition wind input and white-capping are the dominant terms in the balance. The quadruplet source term is significant as well. Moreover, the four-wave interactions are responsible for the fact that an equilibrium is not reached. The deviation from zero of the sum of the source terms more or less follows the shape of the quadruplet source term. In other words, there seems to be a balance between wave generation and wave dissipation. Only the quadruplet source term does not cooperate. This is least pronounced for the Pierson-Moskowitz spectrum.

4.3 Scalability of source terms in balance equation

The scalability of wave conditions and balancing source terms in a limit situation have been studied here briefly. The same steps as in the previous section have been carried out for a water depth of \( h = 2 \) m. The wind speed that has been considered is chosen such that the Froude number \( F_n \), based on the wind stress velocity \( u_* \), is the same as in the previous setting. A wind speed \( U_{10} = 35 \) m/s corresponds to \( u_* = 1.94 \) m/s (relation (2.9)). Reduction of water depth by a factor 2.5 yields a reduction of \( u_* \) by a factor \( \sqrt{2.5} \) and thus \( u_* = 1.23 \) m/s. By solving relation (2.9) we find that for the new situation \( U_{10} = 25 \) m/s. The resulting norms of the deviations from zero of the sum of the source terms are listed in Table 4.2 for the new situation.

The absolute minimum is again obtained with a Pierson-Moskowitz spectrum. The difference between the norms obtained with the other spectra is a factor 2. The wave conditions in the 'approximate' equilibrium situation are the same as obtained with the stronger wind. Again the wave height over depth ratio is \( H_{m0}/h = 0.42 \) and the wave steepness equals \( \eta_0 = 0.06 \). For this condition the individual source terms and their sum are plotted in Figure 4.7.
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Table 4.2: Measures for $l_2$ and $l_\infty$ for $h = 2\text{ m}$ and $U_{10} = 25.0 \text{ m/s}$

Since the wave conditions in the 'approximate' equilibrium are the same for $h = 5\text{ m}$ and $h = 2\text{ m}$ the limit conditions appear to be scalable. For the condition at which for the Pierson-Moskowitz spectrum the 'approximate' equilibrium is obtained, i.e. $H_{m0}/h = 0.42$ and $s_0 = 0.06$ the source terms are plotted in Figure 4.8. The source terms at $h = 2\text{ m}$ are scaled to $h = 5\text{ m}$ by multiplying them with a factor $(h_1/h_2)^2 = 6.25$. The frequency is scaled with a factor $\sqrt{h_2/h_1} = 0.63$. Clearly, all source terms in SWAN are scalable in a similar way. Even the quadruplets appear to be scalable. Consequently, the balance of source terms is scalable and so is the 'approximate' equilibrium solution obtained with SWAN.

A similar exercise was also carried out with $U_{10}$-scaling. For $h = 2\text{ m}$ and $U_{10} = 35/\sqrt{2.5} = 22.1 \text{ m/s}$ lower values for the wave height over depth ratio of approximately 0.38 were obtained in the 'approximate' equilibrium situation than for $h = 5\text{ m}$. This indicated that $U_{10}$-scaling does not apply in the balance of source terms of SWAN.
4.4 Discussion

First of all, it has become clear that SWAN cannot reach an equilibrium spectrum. From the theoretical spectra considered here, the Pierson-Moskowitz spectrum is closest to being an equilibrium spectrum. The wave conditions corresponding to the equilibrium condition are surprisingly high. The wave height over depth ratio equals $H_{m0}/h \approx 0.42$ and the wave steepness $s_0 \approx 0.06$.

De Waal (2001) noticed that SWAN predicted significantly smaller values for $H_{m0}/h$. Here only the source terms have been determined, given a wave condition and spectral shape. Consequently, the numerical scheme of SWAN has not been applied. As already mentioned in Section 3.3.2 the limiter in SWAN acts as a source term in the SWAN computations. In the analysis in this section the extra undesired source term has not been taken into account. From De Waal (2001, Figure 6) is observed that the limiter can be up to 10% of the wind input and acts as a dissipative source term. Effectively 10% of the wind input term is balanced by a non-physical term in the total balance. This means that the calculated wave height in the equilibrium is too small. In other words, without 'limiter source term' the wave height would have been higher. In Section 3.2.1 we already noticed that SWAN (version 40.16) predicts 'approximate' equilibrium solutions with significantly higher values for the relative wave height ($H_{m0}/h \approx 0.40$) if a non-default value for the breaker parameter is used according to Battjes-Stive (1985).

The main conclusion drawn here is that SWAN version 40.16, as used e.g. by De Waal (2001), predicts values for the wave height over depth ratio that are too small. Besides an improper setting of the breaker parameter $\gamma$, it is likely that the underprediction of the wave height is due to numerical deficiencies. However, if the limiter does not act as a source term, an equilibrium situation will still not be obtained due to poor modelling of the quadruplets. Nevertheless, the wave height over depth ratio in the 'approximate' equilibrium situation would have been significantly larger, i.e. approximately 0.40 instead of 0.34 (De Waal, 2001, Figure 3, for dimensionless depth $\delta \approx 0.04$).
Figure 4.4: JONSWAP energy spectrum (up), corresponding SWAN source terms (middle), and sum of the source terms (low) for a situation with $h = 5$ m, $U_{10} = 35$ m/s, $H_{m0}/h = 0.38$ and $s_0 = 0.05$
Figure 4.5: Pierson-Moscowitz energy spectrum (up), corresponding SWAN source terms (middle), and sum of the source terms (low) for a situation with $h = 5\, \text{m}$, $U_{10} = 35\, \text{m/s}$, $H_{max}/h = 0.42$ and $\zeta_0 = 0.06$
Figure 4.6: TMA energy spectrum (up), corresponding SWAN source terms (middle), and sum of the source terms (low) for a situation with $h = 5 \, \text{m}$, $U_{10} = 25 \, \text{m/s}$, $H_{m0}/h = 0.40$ and $\sigma_0 = 0.05$.
Figure 4.7: Pierson-Moskowitz energy spectrum (up), corresponding SWAN source terms (middle), and sum of the source terms (low) for a situation with $h = 2 \text{ m}$, $U_{10} = 25 \text{ m/s}$, $H_{m0}/h = 0.42$ and $s_0 = 0.06$. 
Figure 4.8: The SWAN source terms for a Pierson-Moskowitz energy spectrum with $H_m0/h = 0.42$ and $s_0 = 0.08$ for both $h = 5$ m and $h = 2$ m. The latter is scaled to 5 m.
5 Analysis of wave growth curves

Trends in experimental data have led to wave growth curves. By extrapolating the wave growth curves, the trends are obtained in limit situations, as predicted by the underlying formulation of the growth curve. In this way the lack of experimental data in limit situations may be filled up partly.

In this chapter a number of well-known growth curves have been analysed. We restrict ourselves in this study to finite water depth formulations and arrive at the formulations given by Young and Verhagen (1996) and derivatives of the deep-water formula of Brettschneider (1970). First the expressions of the curves are given in general form in terms of dimensionless depth and fetch. Secondly, the limit situation of infinite fetch and very shallow water is considered. The applicability of the growth curves in this situation is investigated. Partly, such an analysis has also been carried out by De Waal (2002). His analysis is briefly discussed here as well.

5.1 Inventory of wave growth curves

5.1.1 The SMB method

Brettschneider (1958, 1970) revised prediction curves of Sverdrup and Munk (1947) using empirical data. Henceforth the method is called the SMB method. The method has been revised several times. For deep water the SMB method was given by Brettschneider (1970), see also Massel (1996, p. 241) and the Shore Protection Manual (CERC, 1977, p. 3-35).

Deep-water SMB formulae

In the following the significant wave height $H_s$ at the end of the fetch is equal to $H_m$, $X$ is the fetch, $U = U_{10}$ the wind velocity at 10 m height, $T_s$ the significant wave period, $f_m$ the peak frequency and $t_X$ the minimum duration required for fully developed wave conditions at that fetch. The non-dimensional quantities for energy ($\varepsilon$), frequency ($\nu$), fetch ($\chi$), duration ($t_X$) and depth ($\delta$) are given by:

$$
\varepsilon = \frac{m \nu g^2}{U^4}, \quad \nu = \frac{f_m U}{g}, \quad \chi = \frac{g X}{U^2}, \quad t_X = \frac{g t}{U} \quad \text{and} \quad \delta = \frac{g h}{U^2}.
$$

The SMB formulae for nondimensional wave height, wave period and duration are given by

$$
\frac{g H_s}{U^2} = 0.283 \tanh \left[ 0.0125 \chi^{0.42} \right]
$$

\[(5.2a)\]
\[
\frac{gT_a}{U} = 7.54 \tanh \left[ 0.077 \chi^{0.25} \right] \tag{5.2b}
\]

and
\[
\frac{gT_s}{U} = K \exp \left\{ \left( A (\log \chi)^2 - B \log \chi + C \right)^{1/2} + D \log \chi \right\} \tag{5.2c}
\]

where \( K = 6.5882, A = 0.0161, B = 0.3692, C = 2.2024 \) and \( D = 0.8798 \).

In terms of energy, Young (1999) reformulated these deep-water formulae in:
\[
\varepsilon = 5.0 \times 10^{-3} \tanh^2 \left( 0.0125 \chi^{0.42} \right) \tag{5.3a}
\]
\[
\nu = 0.133 \left\{ \tanh \left( 0.077 \chi^{0.25} \right) \right\}^{-1} \tag{5.3b}
\]

In the following only fully developed wave conditions are considered. As a consequence the duration \( t_X \) will not be studied here.

**Shallow-water SMB formulations**

For shallow water several formulations are available. The Shore Protection Manual (SPM) 1977 formulation has been revised in the 1984 manual. Hurdle and Stive (1989) found inconsistencies and reformulated the method. We give all three formulations here, together with the formulation of Bouws (1986) which is also based on the SPM77 formulation.

**The shallow-water formulation according to SPM77**

In CERC (1977, p. 3-46) the significant wave height and period are given as:
\[
\frac{gH_s}{U^2} = 0.283 \tanh \left[ 0.530 \delta^{0.75} \right] \cdot \tanh \left\{ \frac{0.0125 \chi^{0.42}}{\tanh \left[ 0.530 \delta^{0.75} \right]} \right\} \tag{5.4a}
\]
\[
\frac{gT_s}{2\pi U} = 1.20 \tanh \left[ 0.833 \delta^{0.375} \right] \cdot \tanh \left\{ \frac{0.077 \chi^{0.25}}{\tanh \left[ 0.833 \delta^{0.375} \right]} \right\} \tag{5.4b}
\]

**The shallow-water formulation according to SPM84**

The wave-growth curve given in SPM77 was adjusted in 1984 in order to be consistent with the Jonswap result. In CERC (1984, p. 3-55) the significant wave height and period are given as:
\[
\frac{gH_s}{U^2} = 0.283 \tanh \left[ 0.530 \delta^{3/4} \right] \cdot \tanh \left\{ \frac{0.00565 \chi_A^{1/2}}{\tanh \left[ 0.530 \delta_A^{3/4} \right]} \right\} \tag{5.5a}
\]
\[
\frac{gT_s}{U_A} = 7.54 \tanh \left[ 0.833 \delta_A^{3/8} \right] \cdot \tanh \left\{ \frac{0.0379 \chi_A^{1/3}}{\tanh \left[ 0.833 \delta_A^{3/8} \right]} \right\} \tag{5.5b}
\]
with
\[ \delta_A = \frac{gh}{U_A^2}, \quad \chi_A = \frac{gX}{U_A^2}. \] (5.6)

The wind-stress factor \( U_A \) (in m/s) is an adjusted wind velocity and is determined as follows. First a stability correction \( R_T \) is introduced to account for temperature differences (SPM84, p. 3-30). An effective wind speed \( U \) is obtained by
\[ U = R_T U_{10}, \] (5.7a)

where, in absence of temperature information, \( R_T = 1.1 \) may be assumed (otherwise, see Figure 3-14 in SPM84). The adjusted windspeed then follows as
\[ U_A = 0.71 U^{1.23} = 0.80 U_{10}^{1.23} \quad \text{when} \quad R_T = 1.1. \] (5.7b)

Relation (5.7b) is shown in Figure 5.1.

![Figure 5.1: \( U_A \) as function of \( U_{10} \) according to relation (5.7b) with \( R_T = 1.1 \).](image)

The revision of Hurdle and Stive (1989)

The revised formulation of Hurdle and Stive (1989) is given by their equations (4.1) and (4.2):
\[ \frac{gH_s}{U_A^2} = 0.25 \tanh \left( 0.6 \delta_A^{3/4} \right) \left( \tanh \left[ \frac{4.3 \times 10^{-5} \chi_A}{\left( \tanh \left( 0.6 \delta_A^{3/4} \right) \right)^2} \right] \right)^{1/2} \] (5.8a)

---

1 One might be inclined to note that the dimension of \( U_A \) cannot be correct. The equation (5.7b) has been proposed in this form in SPM84. Presumably the constant has also dimension of (m/s)\(^{0.23}\). In SPM84, Eq. (3-28b), the corresponding equation for the case that the wind velocity \( U \) is given in miles per hour (mph), has been given as \( U_A = 0.589 U^{1.23} \) mph.

2 The co-author, D.P. Hurdle, confirmed us in June 2003 that the exponents are indeed 3/8 and 3/4 because these exponents were used in the curve fitting procedures.

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\[
\frac{gT_s}{U_A} = 8.3 \tanh \left(0.76 \delta_A^{3/8}\right) \left\{ \tanh \left[ \frac{4.1 \times 10^{-5} \chi_A}{\tanh \left(0.76 \delta_A^{3/8}\right)}\right]^3 \right\}^{1/3}
\]

(5.8b)

**Adaptation of SPM curve by Bouws (1986)**

When analysing observations in Lake Marken, Bouws (1986) rescaled the SPM77 growth curve in terms of \( u_* \) simply by substituting \( u_*/\sqrt{C_D} \) for \( U_{10} \), taking a value \( C_D = 1.775 \times 10^{-3} \). This value corresponds to a wind speed \( U_{10} = 15 \text{ m/s} \), according to (2.9). This gives the following result:

\[
\frac{gH_s}{u_*^2} = 159 \tanh \left[ 4.6 \times 10^{-3} \delta_*^{0.75} \right] \cdot \tanh \left\{ \frac{0.14 \chi_*^{0.42}}{159 \tanh \left[ 4.6 \times 10^{-3} \delta_*^{0.75} \right]} \right\},
\]

(5.9a)

\[
\frac{gT_s}{2\pi u_*} = 28.5 \tanh \left[ 0.077 \delta_*^{0.375} \right] \cdot \tanh \left\{ \frac{0.0158 \chi_*^{0.25}}{\tanh \left[ 0.077 \delta_*^{0.375} \right]} \right\},
\]

(5.9b)

with the non-dimensional parameters \( \delta_* \) and \( \chi_* \) given by (5.1) with \( U = u_* \). Since the scaling is based on a wind speed of 15 m/s, the expressions (5.9) are not likely to be generally applicable for situations of much stronger wind. Therefore, the growth curve formulation of Bouws will not be considered further in this study.

Since the observed scatter was reduced in the results, Bouws (1986) adopted a nondimensional scaling in terms of the friction velocity rather than the \( U_{10} \). In general, application of the scaling given in (5.1) appears to lead to inconsistencies between various measured data sets and scatter within individual data sets. Attempts to explain these discrepancies (Battjes et al., 1987; Kahma and Calkoen, 1992) have been insightful, but questions about the appropriate scaling of such experimental data still exist. The question whether within the widely-used scaling technique given by (5.1) scaling with either \( U_{10} \) or \( u_* \) applies cannot be answered a priori. Kahma and Calkoen (1992) show that experimental results concerning the effect of stability on the momentum retained by waves (Donelan, 1979) contradict the view that the use of \( u_* \) instead of \( U_{10} \) could alone bring the data sets into line.

### 5.1.2 The growth curves of Young and Verhagen

Young and Verhagen (1996a) gave the following growth curves for the dimensionless wave energy \( \varepsilon = g^2E/U^4 \) and wave frequency \( \nu = f_mU/g \) with \( E = m_0 \) the total wave energy, \( f_m \) the peak (or modal) frequency and \( U \) a characteristic wind velocity, for which has been used \( U_{10} \) averaged over the down-wind fetch in their study (see Young and Verhagen, 1996a, §6 and Eq. (15)). For the water depth \( h \) likewise an average over the down-wind water depths is taken.
For fetch-limited growth they then propose (Young and Verhagen, 1996a, Eqs. (25)-(30)):

\[
\varepsilon = 3.64 \times 10^{-3} \left\{ \tanh \left( 0.493 \delta^{0.75} \right) \tanh \left[ \frac{3.13 \times 10^{-3} \chi^{0.57}}{\tanh \left( 0.493 \delta^{0.75} \right)} \right] \right\}^{1.74},
\]

\[
\nu = 0.133 \left\{ \tanh \left( 0.331 \delta^{1.01} \right) \tanh \left[ \frac{5.215 \times 10^{-4} \chi^{0.73}}{\tanh \left( 0.331 \delta^{1.01} \right)} \right] \right\}^{-0.37},
\]

where the non-dimensional depth \( \delta \) and fetch \( \chi \) are given as in (5.1) with \( U = U_{10} \). Notice that for infinite fetch and small dimensionless depth \( \delta \) we obtain from (5.10a) the shallow-water limit:

\[
\varepsilon = 1.06 \times 10^{-3} \delta^{1.3},
\]

which expression has been used in section 2.5.11.

Because the other growth curves are formulated in wave height instead of energy, relation (5.10a) is reworked. We first notice that \( \sqrt{\varepsilon} = \frac{g}{U_{10}^2} \sqrt{E} \) and with \( E = m_0 \) we then have:

\[
\sqrt{\varepsilon} = \frac{1}{4 \ U_{10}^2} \quad \text{with} \quad H_s = H_{m0}. \quad (5.12)
\]

Relation (5.10a) then becomes

\[
\frac{g H_s}{U_{10}^2} = 0.24 \left\{ \tanh \left( 0.493 \delta^{0.75} \right) \tanh \left[ \frac{3.13 \times 10^{-3} \chi^{0.57}}{\tanh \left( 0.493 \delta^{0.75} \right)} \right] \right\}^{0.87}. \quad (5.13)
\]

5.1.3 Comparison between wave growth curves in general situations

Before considering limit situations such as infinite fetch and small water depth, the wave growth curves of Young and Verhagen and the modified SMP84 formulation of Hurdle and Stive have been compared here. In Figure 5.2 the dimensionless wave height (5.13) has been plotted as a function of dimensionless fetch \( \chi \) for dimensionless waterdepths \( \delta = 0.1 \) and \( 0.2 \). We also plot the dimensionless wave height given by Hurdle and Stive, Eq. (5.8a). Because in the latter formulation the wind velocity \( U_A \) is used, we rewrite this equation in terms of \( U_{10} \) which has been used by Young and Verhagen, by using relation (5.7b) between \( U_A \) and \( U_{10} \). With \( \delta_A = gh/U_A^2 \) and \( \chi_A = gX/U_A^2 \) we then obtain

\[
\delta_A = \frac{gh}{U_{10}^2 \ U_A^2} = \frac{\delta}{0.64 \ U_{10}^{0.46}} \quad \text{and} \quad \chi_A = \frac{\chi}{0.64 \ U_{10}^{0.46}}. \quad (5.14)
\]
Substituting these expressions in (5.8a), we obtain for the dimensionless height $gH_s/U_{10}^2$ of Hurdle and Stive the following expression:

$$
\frac{gH_s}{U_{10}^2} = \frac{1}{4} \left( \frac{U_A}{U_{10}} \right)^2 \tanh \left( \frac{0.6\delta_A^{3/4}}{\tanh \left( \frac{0.6\delta_A^{3/4}}{4.3 \times 10^{-5}\chi_A} \right)} \right)^{1/2}.
$$  \hspace{1cm} (5.15)

The difficulty now is that it is not possible to plot (5.15) for some choices of $\delta$ as function of the fetch $\chi$ without specifying the wind velocity $U_{10}$. Therefore we first plot (5.8a) as function of $\chi_A$ to get a feeling for the expression of Hurdle and Stive.

![Graphs](image)

(a) $gH_s/U_{10}^2$ as function of the fetch $\chi = gX/U_{10}^2$ for $\delta = 0.01$ (lowest), 0.1 (middle) and 0.2 (upper), according to Eq. (5.13).

(b) $gH_s/U_{10}^2$ as function of the fetch $\chi_A = gX/U_A^2$ for $\delta_A = 0.01$, 0.1 and 0.2, according to (5.8a).

Figure 5.2: Relations (5.13) and (5.8a).

For fixed values $h = 5$ m and $\chi = 4000$ we compute $U_{10} = \sqrt{gh/\delta}$, $U_A$, $\delta_A = gh/U_A^2$ and $\chi_A/\chi$ (see Eq. (5.14)) for a number of values of the dimensionless water depth $\delta$. Using these values of $\delta$ we compute the dimensionless wave height (5.13) according to Young and Verhagen. The dimensionless wave height of Hurdle and Stive, Eq. (5.15) is computed using the values of $U_{10}$, $U_A$, $\delta_A$ and $\chi_A$ corresponding to $\delta$ from the Table 5.1.

It is clear from Table 5.1 that the dimensionless wave height of Young and Verhagen is somewhat smaller than the adapted SPM84 one of Hurdle and Stive. Looking in detail for $\delta = 0.04$ for which the wind velocity $U_{10} = 35 \text{ m/s}$, we have the dimensionless wave heights 0.0179 and 0.0159 for respectivelt SPM84 and Young and Verhagen. The dimensional wave heights $H_s$ then follow by multiplication of $U_{10}^2/g$ as respectively 2.24 m and 1.99 m. We then have the ratios $H/h = 0.447$ and 0.397. In this case the difference is $(.447 - .397)/.397 = 12.6\%$ between these two wave growth curves (at this value of dimensionless depth).

The differences between the growth curves of Young and Verhagen, SPM77 and Hurdle and Stive's variant of SPM84 are further illustrated by Figure 5.3. For a fixed depth $h = 5$ m
and wind speed $U_{10} = 35 \text{ m/s}$, a clear difference in the predicted wave height over depth ratio $H_s/h$ is obtained with the different formulations. For infinite fetch ($X > 150 \text{ km}$) the differences are over 10%. If the fetch is fixed at $X = 500 \text{ km}$ and the wind speed $U_{10}$ is considered between 10 m/s and 70 m/s, the SPM77 growth curve seems to give small values for $H_s/h$. As already seen in Table 5.1 the values for the significant wave height predicted by the growth curve of Young and Verhagen are smaller than those of SPM84 as revised by Hurdle and Stive. Based on the value at $U_{10} = 35 \text{ m/s}$ Young and Verhagen’s curve seems to be more realistic for the situations of strong wind forcing. In the next section this will be discussed in more detail.

![Figure 5.3](image)

(a) $H_s/h$ as function of the wind speed $U_{10}$ for $h = 5 \text{ m}$ and $X = 500 \text{ km}$.

(b) $H_s/h$ as function of the fetch wind speed $X$ for $h = 5 \text{ m}$ and $U_{10} = 35 \text{ m/s}$.

Figure 5.3: Wave growth formulations of Young and Verhagen, SPM77 and Hurdle and Stive’s variant of SPM84
Notice that for the situation considered here the fetch required to reach a limit situation is significantly larger for Young and Verhagen’s curve than for the other two curves. In the next section infinite fetch is assumed. Before applying the expressions derived in Section 5.2, one should always check whether the situation at hand is a limit situation, or more specifically, if the fetch is such that it can be assumed as infinite for the considered growth curve.

5.2 Applicability of growth curves in limit situations at shallow water

In a note De Waal (2002) gave a preliminary analysis of two wave growth curves, according to Bretschneider (shallow-water version SPM77) and Young and Verhagen. From their general formulations the wave-growth limit for shallow water has been derived. Some preliminary conclusions have been drawn from that exercise. These will be discussed here, after having given the main results in the analysis by De Waal (2002).

De Waal took the general expressions for the wave growth curves of SPM77 and Young and Verhagen, given by (5.4) and (5.10) respectively, as a starting point for his analysis. The following limit situations were considered:

1. infinite fetch, arbitrary depth
2. arbitrary fetch, infinite depth
3. small fetch, infinite depth
4. infinite fetch, small depth

Here we restrict ourselves to the situation considered in the present study, i.e. infinite fetch and small depth.

For small dimensionless depth (which is obtained for either small depth $h$ or large $U_A$, or both) we obtain the limit by approximating $\tanh x$ by $x$. If also the fetch is infinite, the expressions for the SPM77 curves (5.4) reduce to

$$\frac{gH_s}{U_{10}^2} = 0.15 \delta^{0.75},$$

$$\frac{gT_s}{U_{10}} = 6.28 \delta^{0.375}. \quad (5.16a)$$

Obviously the wave height over depth ratio $\frac{H_s}{h} = \frac{gH_s}{\delta U_{10}^2}$ is given by

$$\tilde{\gamma}_{SPM77} = 0.15 \delta^{-0.25} . \quad (5.17)$$

The limit expressions for Young and Verhagen’s growth curve reads

$$\frac{gH_s}{U_{10}^2} = 0.241 \left( 0.493 \delta^{0.75} \right)^{0.87} = 0.13 \delta^{0.65}, \quad (5.18a)$$

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\[
\frac{gT_m}{U_{10}} = 5.0 \delta^{0.375}.
\]  
(5.18b)

whereas the expression for the dimensionless wave peak period is determined as the inverse of the dimensionless peak frequency, given by (5.10b). Notice that in (5.16) the significant wave period \( T_s \) has been used. From (5.18) the following wave height over depth ratio is obtained:

\[
\tilde{\gamma}_{YV} = 0.13 \delta^{-0.35}.
\]  
(5.19)

A similar analysis can also be applied to the growth curve of Hurdle and Stive (1989). For infinite fetch and shallow water expression (5.15) with \( \delta_A \) given by (5.14) reduces to

\[
\frac{gH_s}{U_{10}^2} = 0.25 \frac{U_{10}^2}{U_{10}^2} 0.6 \delta^{0.75} = 0.134 U_{10}^{0.115} \delta^{0.75}.
\]  
(5.20)

The wave height over depth ratio then equals:

\[
\tilde{\gamma}_{HS} = 0.134 U_{10}^{0.115} \delta^{-0.25}.
\]  
(5.21)

For completeness, the dimensionless wave period (5.8b) reduces for infinite fetch and at shallow water to

\[
\frac{gT_s}{U_{10}} = 5.97 U_{10}^{0.0575} \delta^{0.375}.
\]  
(5.22)

From their expressions it is already clear that the variation in the curve of Young and Verhagen is larger than in the other two curves. In Figure 5.4 the wave height over depth ratio for the situation of infinite fetch and small depth is plotted as a function of dimensionless depth for the three considered wave growth curves, i.e. SPM77, the adapted SPM84 and Young and Verhagen. In order to plot the SPM84 curve we considered a wind speed \( U_{10} = 35 \) m/s. Clearly for small values of \( \delta \) both the Young and Verhagen-curve and the SPM84-curve predict unrealistic large values for \( H/h \). For a strong wind of \( U_{10} = 40 \) m/s and a water depth \( h = 3.5 \) m the dimensionless depth equals \( \delta = 0.021 \). For this value \( \tilde{\gamma}_{SPM77} = 0.39 \), \( \tilde{\gamma}_{YV} = 0.50 \) and \( \tilde{\gamma}_{HS} = 0.53 \). For even stronger winds over shallower water the differences between the growth curves become even larger. It looks as if the SPM77 curve predicts values for the wave height over depth ratio that are small, whereas the other two formulations lead to rather high values.

It is well-known that for a less strong wind and larger water depth, and thus larger \( \delta \), the growth curves are closer to each other and closer to measured data. The growth curves have been determined by fitting data through measurements. Extreme situations are not included in these measured data sets. Therefore, applying the growth curves for conditions for which measurements are not available is dangerous.

De Waal (2002) also derived expressions for the wave steepness as a function of dimensionless water depth in the limit situation from both Bretschneider’s and Young and Verhagen’s expressions for the wave height and wave period (i.e. (5.16) and (5.18)). Both curves do not vary much for increasing wind speed, but Bretschneider’s curve yields decreasing wave
steepness values with increasing wind speed, whereas surprisingly Young and Verhagen’s curve shows the opposite behaviour. Furthermore, expressions for the required fetch to obtain a certain percentage of the limit wave height have been derived. Since the focus is on the wave height to depth ratio, these topics will not be discussed here.

5.3 Concluding remarks about wave growth curves

In the analysis of wave growth curves three formulations have been considered: the shallow-water approximations of the original SBM curves, namely SPM77 and the revision of SPM84 given by Hurdle and Stive (1989). Thirdly, the curve of Young and Verhagen (1996) has been considered. The wave growth under normal and extreme circumstances have been considered. Based upon the analysis in the previous sections and the work of De Waal (2002) the following conclusions can be drawn:

1. The present growth curves provide reasonable estimates for the wave height over depth ratio in area of measurement data. Application in areas for which they have not been derived, such as the limit situation of infinite fetch and very shallow water, makes them questionable to apply. As long as they are not tuned to what is considered to be a limit condition, they should only be applied for these conditions taking into account significantly large error bands.

2. Under limit conditions the growth curve of Young and Verhagen and the adapted SPM84 curve of Hurdle and Stive predict values for the wave height over depth ratio that are probably too large.

3. Fetches needed to reach a limit situation are significantly larger for the Young and Ver-
hagen curve than for the other two formulations. Application of Young and Verhagen’s expressions derived for limit conditions might therefore not be valid for fetch-limited regions as the Dutch lakes.
6 Discussion and recommendations

6.1 Discussion

In this report insight has been gained in the physical mechanisms that are important for the wave growth in shallow water in general and for the limit of that growth in particular. The wave growth in shallow water is only known within a broad margin. Wave models such as SWAN predict relatively mild wave conditions within this margin. The large extent of uncertainty in the predicted wave conditions, that lead to 'unsafe' computational data, are of great concern for the determination of the design conditions of sea defences along the IJsselmeer and Markermeer.

In this study special attention was paid to the following subjects:

Uncertainties in SWAN source terms

By dividing the physical processes into wave generation, dissipation and nonlinear wave interaction, we can conclude that the latter class is modelled worst in SWAN. The standard DIA formulation that is used for modelling four-wave interactions may lead to large errors. Especially the scaling to shallow water is questionable. Fortunately, trustworthy alternatives such as the XDIA approximation have been developed and are ready to be implemented in SWAN. In shallow water also triad wave interactions become important. Also their modelling by means of the LTA approach is erroneous. For this type of wave interactions alternative formulations exist as well.

The wind input source term presently implemented in SWAN is rather simple, lacking effects of wind on waves and sensitive for the interaction with the DIA formulation for the quadruplets in combination with the imposition of the cut-off frequency. The modelling of the dissipative source terms is probably more accurate than the presently implemented wind input source term.

Furthermore, omission of long-wave effects in SWAN and a good description of triad wave interactions will lead to wrongly predicted spectral shapes by SWAN in very shallow areas where depth-induced breaking and nonlinear wave interactions play a dominant role. Especially the different measures for the wave period will be underpredicted. The effect on the total amount of wave energy is less pronounced, although the amount of discarded long-wave energy may reach levels of for instance 20% of the total amount of wave energy.

Dominant uncertainties of SWAN source terms in wave-growth limit situations

Conclusions about the dominant uncertainties for the determination of the wave conditions in
the wave-growth limit with SWAN have been obtained by combining the dominant uncertainties in the formulation of the source terms in general situations with the relative importance of the source terms for wave-growth limit situations. Therefore, the conclusion is drawn that the present implementation of the quadruplet source terms is responsible for the largest uncertainties in the modelling of equilibrium situations at moderate to shallow water. Since triads are only important in really shallow water, their effect on the final SWAN results will be less in computations for the Dutch lakes (moderate to shallow relative water depth).

The modelling of the dissipative source terms is relatively good, compared to the modelling of the wind input source term, provided that the correct calibration parameters are applied. Although not as pronounced as the quadruplets the presently implemented wind input source term contributes significantly to incorrect predictions of wave growth limit conditions. For very strong wind the wave form is not such that wave generation by wind and dissipation by white-capping act similarly as for mild conditions. Nevertheless, the same physics is applied in SWAN, irrespective of the kind of wave and wind conditions.

For the source terms implemented in SWAN we conclude the following. Ranging from large to small the quadruplets, wind input, triads and dissipative source terms have a negative effect on the accuracy of the SWAN predictions in wave-growth limit situations.

**Equilibrium conditions in SWAN**

An equilibrium cannot be obtained with SWAN in the sense that the sum of the source terms vanishes for all frequencies. This is mostly caused by the poor representation of the four-wave interactions by means of the DIA approach. On the other hand, for a situation of very strong wind with infinite fetch at shallow water the magnitude of the sum of the source terms is smaller than the magnitude of the individual source terms computed with SWAN. Since the corresponding spectra did not change spatially, they could be considered as an approximate equilibrium spectrum. The wave height over depth ratio was approximately 0.4, which is significantly larger than obtained in earlier computations. In our computations we used values for the breaker parameter \( \gamma \), based on deep-water wave steepness, that were significantly larger than the default value, which contributes to higher relative wave heights.

In a more quantitative study the SWAN source terms have been determined for a number of spectral shapes and wave conditions. SWAN was only used to compute the source terms for given spectra. Consequently, the numerical techniques in the propagation part of SWAN did not have to be applied. The minimum sum of the source terms (in some norm) is considered as 'approximate' equilibrium spectrum. The relative wave height corresponding to the equilibrium spectrum, having the shape of a Pierson-Moskowitz spectrum, appeared to be significantly larger than obtained in earlier computations.

By studying the earlier computations (e.g. De Waal, 2001) we conclude that the underestimation of the relative wave height by SWAN might not only be caused by the settings of physical parameters applied in the study of De Waal (2001), but might also be due to the fact that stability-ensuring limiters act as a dissipative source terms in SWAN. As long as limiters are
required in SWAN, which affect the action balance itself, the numerical model can certainly not predict a 'true' equilibrium state in the sense that the sum of the source terms vanishes for all frequencies.

**Scalability of SWAN source terms in the action balance**

By considering the WAM cycle 3 formulation for wind input, which is implemented in SWAN, it becomes clear that Froude scaling applies with the wind stress velocity $u_*$ and not $U_{10}$, due to the nonlinear relation between $U_{10}$ and $u_*$. The source terms in SWAN all scale in the same way, unless the bottom friction is modelled by means of a constant bottom friction coefficient. Therefore, Collins' (1972) or Madsen et al.'s (1988) formulation should be used. The scalability of the quadruplet source term was not proven analytically but experimentally. Since the balance of all source terms is scalable, the wave conditions in the wave-growth limit situation are scalable as well.

**Estimates of existing wave growth curves for limit situation**

For a number of available formulations for wave growth curves, estimates for the wave height over depth ratio have been determined in the wave-growth limit situation for the shallow-water approximations of the original SBM curves, namely SPM77 and the revision of SPM84 given by Hurdle and Stive (1989). Thirdly, the curve of Young and Verhagen (1996) has been considered. These growth curves provide reasonable estimates for the wave height relative to water depth in the areas where measurement data are available. Application in areas for which they have not been calibrated against measurements, such as the limit situation of infinite fetch and very shallow water, makes them questionable to apply.

Under limit conditions the growth curve of Young and Verhagen and the adapted SPM84 curve of Hurdle and Stive predict values for the wave height over depth ratio that are probably too large. Fetches needed to reach a limit situation are significantly larger for the Young and Verhagen curve than for the other two formulations. Application of Young and Verhagen's expressions derived for limit conditions might therefore not be valid for fetch-limited regions as the Dutch lakes.

**Applicability for practical situations**

By using SWAN with default settings for the physical parameters, realistic predictions for the wave conditions in the wave-growth limit will not be obtained. The most obvious improvements and suggestions for obtaining better results are given here.

In previous computations of wave conditions at IJsselmeer, Markermeer and Slotemeer mostly default values for physical parameters have been used, for both storm and extreme conditions.
Application of a default value for the breaker parameter $\gamma$ leads to underprediction of the wave height over depth ratio for wave limit conditions. In this study the dependence of $\gamma$ on the deep water steepness, by means of the relations of Nairn (1990) or Battjes and Stive, 1985), has been emphasised. The deep-water steepness of the waves is obtained by translating the local wave conditions to deep water. In the areas mentioned above a larger value for the breaker parameter than the default value will be obtained, resulting in less dissipation and consequently in a higher value for the wave height relative to the water depth.

Also the modelling of physical processes in SWAN could be improved. Based on the discussion about dominant uncertainties in the results obtained with SWAN in wave-growth limit situations, both the modelling of four-wave interactions and wind input should be improved. Setting up and carrying out a study to develop new formulations for those processes will probably cost a lot of effort. However, for the DIA formulation a good alternative has already been obtained in terms of the XDIA approach, especially for shallow water conditions.

Furthermore, the wind input model of Janssen (1989, 1991b) might lead to improved wave generation. Most certainly, releasing the fixed cut-off frequency when using the DIA approach for the quadruplet wave interactions will lead to significantly better results.

### 6.2 Recommendations

Based on the study presented in this report and the drawn conclusions, the following recommendations are given:

- **An extensive sensitivity analysis is required to give accurate estimates of the accuracy level of the source terms involved.**
- **Alternative formulations for the DIA approach exist and should be implemented in SWAN as soon as possible to obtain more accurate predictions in shallow water. The XDIA approach is very promising in this respect. Also for triad wave interactions alternative, more accurate formulations exist and should be implemented in SWAN. However, implementation of alternative triad formulations requires a major effort.**
- **The default setting for the breaker parameter $\gamma$ is too small to provide reliable estimates for wave conditions in limit conditions. A more realistic value is obtained by applying the relations given by either Nairn (1990) or Battjes and Stive (1985). These are based on deep-water steepness of the waves, which is obtained by translating the local wave conditions to deep water. At the Dutch lakes a larger value for the breaker parameter than the default value will be obtained, which yields less dissipation and a higher value for the wave height relative to the water depth.**
- **Avoid the use of limiters in SWAN that act as a dissipative source term in the action balance equation. A step forward has been taken in the new release of SWAN by implementing a frequency-dependent underrelaxation technique, which reduces the effect of the limiter on the bulk of the action density.**
- **As long as the growth curves have not been tuned to what is considered to be a limit condition, they should only be applied for these conditions taking into account significantly large error bands.**
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A  Definition of source terms in action balance equation

It is not immediately clear why the relation between the change in action and change in energy can be given by (2.4). Ris (1997, p. 12) refers to Hasselmann et al. (1973), Mei (1983), Phillips (1977) and Whitham (1974), without giving any detail where to find it in those sources. In fact, in the latter three references the wave action equation inclusive the source term does not appear. Only in Hasselmann et al. (1973, p. 71) it is given, albeit without any derivation. The wave action equation is normally given as a conservation equation, that is, in absence of source terms. And it is the form of the source term which concerns us here.

In Komen et al. (1994, p. 47) the action balance equation (2.2) is obtained from a combination of

\[
\frac{\partial E}{\partial t} = S \quad \text{and} \quad \frac{d}{dt}\left(\frac{E}{\omega_r}\right) = 0, \tag{A.1}
\]

(their equations (1.173) and (1.185)). Notice that the energy equation they consider is in absence of any inhomogeneity, such as current or unevenness of the bottom. It is then stated that the final equation

\[
\frac{d}{dt}\left(\frac{E}{\omega_r}\right) = \frac{S}{\omega_r}, \tag{A.2}
\]

simply follows by a combination of the two equations (A.1). This does not help much.

Suppose that the linear wave action definition \( N = E/\omega_r \) is to be used. Then the wave action equation (2.2a) can be written in the following form:

\[
\frac{dN}{dt} - \Sigma = \frac{d}{dt}\left(\frac{E}{\omega_r}\right) - \Sigma = \frac{1}{\omega_r} \frac{dE}{dt} + E \frac{d}{dt}\left(\frac{1}{\omega_r}\right) - \Sigma = 0. \tag{A.3}
\]

To bring this result in the form of a balance equation for the energy density \( E \) we multiply with the relative frequency \( \omega_r \) and obtain:

\[
\frac{dE}{dt} + \omega_r E \frac{d}{dt}\left(\frac{1}{\omega_r}\right) = \omega_r \Sigma. \tag{A.4}
\]

Notice that also in absence of the source term \( \Sigma \) this equation shows that the energy is not conserved in a moving medium. The principle of conservation of wave action means that wave energy increases wherever the wave paths (rays) move into regions of larger \( \omega_r \); this increase is at the expense of the mean flow (see also Lighthill (1978, p. 331) or Dingemans (1978, p. 7)).
For the special case of the energy density $E$ being integrated over $k$ space (or over $(\omega, \theta)$ space) giving the total energy $\mathcal{E}$ we obtain

$$\frac{D\mathcal{E}}{Dt} + \omega_r \mathcal{E} \frac{D}{Dt} \left( \frac{1}{\omega_r} \right) = \omega_r \Sigma' \quad \text{or} \quad \frac{D\mathcal{E}}{Dt} - \frac{E}{\omega_r} \frac{D\omega_r}{Dt} = \omega_r \Sigma', \quad (A.5a)$$

with the following definitions:

$$\mathcal{E}(x, t) = \iiint dk \, E(k; x, t) , \quad \Sigma'(x, t) = \iiint dk \, \Sigma(k; x, t) \quad (A.5b)$$

and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \dot{x}_i \frac{\partial}{\partial x_i} . \quad (A.5c)$$

Similarly, with $\mathcal{N} = \iint dk \, N$, Eq. (2.2) becomes

$$\frac{D\mathcal{N}}{Dt} = \Sigma'. \quad (A.6)$$

It has been shown in Bretherton and Garret (1968) and Phillips (1977, p. 69) that the term

$$\omega_r \mathcal{E} \frac{D}{Dt} \left( \frac{1}{\omega_r} \right)$$

can be rewritten as\textsuperscript{1}

$$\mathcal{E} \frac{\partial}{\partial x_i} (U_i + c_{g_i}) + S_{ij} \frac{\partial U_j}{\partial x_i} , \quad (A.7)$$

with the radiation-stress tensor $S_{ij}$ given as

$$S_{ij} = \mathcal{E} \left[ 2 \left( \frac{c_g}{c} \right) \ell_i \ell_j + \left( \frac{2c_g}{c} - 1 \right) \delta_{ij} \right] , \quad (A.8)$$

with $c_g = |c_g|$, $k = |k|$, $c = \omega_r/k$ the celerity, $\ell_i = k_i/k$ and $\delta_{ij}$ the Kronecker delta. This expression for the radiation stress is only valid for propagating linear waves. The resulting energy balance equation can then be written as

$$\frac{D\mathcal{E}}{Dt} + \mathcal{E} \frac{\partial}{\partial x_i} (U_i + c_{g_i}) + S_{ij} \frac{\partial U_j}{\partial x_i} = \omega_r \Sigma' \quad (A.9a)$$

or, equivalently, in the form as one usually encounters,

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial}{\partial x_i} \left[ (U_i + c_{g_i}) \mathcal{E} \right] + S_{ij} \frac{\partial U_j}{\partial x_i} = \omega_r \Sigma' . \quad (A.9b)$$

It is seen now that $S' = \iint d\mathcal{K} \, S$ equals $\omega_r \Sigma'$ from the right-hand member in (A.9b). For the linear definition for the wave action as is considered here (see Eq. (2.3)), the expression

\textsuperscript{1}The dispersion relation (2.1) then is $\omega = \Omega + k \cdot U$ with $\omega^2 = g k \tanh (kh)$.
\( \Sigma' = S'/\omega_r \) is correct. It is therefore very likely (but not yet proven) that the relation (2.4) for the densities is also correct.

**Comment 1.** The present derivation rests heavily on the special form for the wave action, \( N = E/\omega_r \), valid strictly only for linear waves. The relation between (A.9) and (A.6) is furthermore only valid for travelling waves, that is, diffraction effects are ignored. This is consistent in that the wave propagation part is also based on the geometric-optics approximation. However, extending SWAN to include diffraction effects has a number of consequences. Not only the eikonal equation should be adapted, but also the definition of the radiation-stress tensor has to be extended to account for diffraction effects. This may be done in the way used in Dingemans et al. (1987) (see also the derivation in Dingemans (1997, §2.9.6)). We do expect that such a new definition of radiation stress (with its effects on morphology via different wave-driven current fields) has not only its effect on \( \omega_r \), but also necessitates a more involved definition of wave action, rather than simply \( N = E/\omega_r \).

**Comment 2.** One can also follow the similar derivation of Christoffersen (1982, Chapter 4). He uses the commonly used averaged equations for mass, momentum and energy, also given in Dingemans (1997), Eqs. (2.436), (2.470) and (2.500). It is also possible to use the exact averaged equations, given in Dingemans (1997) by Eqs. (2.436), (2.462) and (2.496). When using these exact equations it might be necessary to use a more involved definition of the action density, to comply with the added accuracy of the equations.
B Zakharov's equations

B.1 Sketch of the derivation of Zakharov's equations

The present sketch of the derivation of Zakharov's equation is taken from Dingemans and Otta (2001).

That the NLS equation is a special case of the Zakharov equation, has been proven by Stiassnie (1984) for the case of deep water. Originally, Zakharov (1968) derived a deep-water evolution equation for the amplitude of a wave field. A little later, the equations for restricted depth were given by Zakharov and Kharitanov (1970), still for horizontal bottom.

We first sketch the steps along which the Zakharov equation can be obtained. Here we follow Stiassnie and Shemer (1984), see also Shemer and Stiassnie (1991), where the bottom has been assumed to be horizontal. A detailed account can also be found in Rasmussen (1998).

The kinematic and dynamic free-surface conditions are written in terms of the free-surface potential $\varphi(x,t) = \Phi\{x, z = \zeta(x,t), t\}$, the vertical velocity $w^s = (\partial \Phi / \partial z)_{z=\zeta}$ and have been given in Eqs. (B.1):

\[
\frac{\partial \varphi}{\partial t} + \frac{1}{2} (\nabla \varphi)^2 - \frac{1}{2} (w^s)^2 \left[1 + (\nabla \zeta)^2 \right] + \frac{p}{\rho} + g\zeta = 0 \quad (B.1a)
\]

\[
\frac{\partial \zeta}{\partial t} + \nabla \varphi \cdot \nabla \zeta - w^s \left[1 + (\nabla \zeta)^2 \right] = 0 . \quad (B.1b)
\]

Together with the Laplace equation $\nabla^2 \Phi = 0$ and the kinematic bottom condition $\partial \Phi / \partial z = 0$ at $z = -h$ these equations constitute the description of the physical problem. We stress the fact that in this derivation the bottom is taken to be horizontal. The derivation of the Zakharov equation proceeds in the following steps.

1. The horizontal Fourier transform of Eqs. (B.1) yields two integro-differential equations for the Fourier transforms $\hat{\zeta}$ and $\hat{\varphi}$, where the Fourier transform of a function $f(x)$ is defined by:

\[
\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \, f(x) \, e^{-ikx} , \quad (B.2a)
\]

and the delta function is defined as

\[
\delta(k) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dx \, e^{ikx} . \quad (B.2b)
\]

In addition to the variables $\hat{\zeta}$ and $\hat{\varphi}$, $\hat{w}^s$ also features in the transformed form of Eqs.
(B.1). An expression for \( \hat{w}^a \) has to be found. This is achieved in the following steps.

2. Taking the horizontal Fourier transform of the Laplace equation and satisfying subsequently the bottom condition, yields a separation of the vertical structure in the following way:

\[
\hat{\varphi}(k, z, t) = \hat{\phi}(k, t) \cosh \left[ |k| (z + h) \right] ;
\]

this makes it possible to express the free-surface variables \( \varphi \) and \( w^a \) in terms of \( \hat{\phi}(k, t) \) and \( \zeta(x, t) \) in the following way:

\[
\begin{align*}
\varphi(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \left[ \cosh \left[ |k| h \right] \cosh \left[ |k| \zeta(x, t) \right] + \\
&\quad + \sinh \left[ |k| h \right] \sinh \left[ |k| \zeta(x, t) \right] \right] e^{ikx} \\
\end{align*}
\] (B.4a)

\[
\begin{align*}
w^a(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \left[ |k| \hat{\phi} \cosh \left[ |k| h \right] \sinh \left[ |k| \zeta(x, t) \right] + \\
&\quad + \sinh \left[ |k| h \right] \cosh \left[ |k| \zeta(x, t) \right] \right] e^{ikx} \\
\end{align*}
\] (B.4b)

3. The next step is to express \( w^a \) in terms of \( \zeta \) and \( \hat{\varphi} \). It is here that the first approximations have to be made. The expressions \( \sinh(|k| \zeta) \) and \( \cosh(|k| \zeta) \) are replaced by their Taylor expansions up to \( O \left( \left( |k| \zeta \right)^3 \right) \); \( \zeta \) is expressed by its Fourier transform \( \hat{\varphi} \). Finally, the Fourier transform of (B.4) is considered. This yields two equations in which \( \hat{\varphi} \) and \( \hat{w}^a \) are expressed in terms of \( \hat{\phi} \) and \( \hat{\zeta} \). An iterative solution of the equation for \( \hat{\varphi} \) is applied to obtain \( \hat{\varphi} \) as a function of \( \hat{\phi} \) and the subsequent use of \( \hat{\phi} \) in the equation for \( \hat{w}^a \) yields an expression for \( \hat{w}^a \) in terms of \( \hat{\zeta} \) and \( \hat{\varphi} \). This expression for \( \hat{w}^a \) is now used in the Fourier transform of the free-surface equations (B.1). Multiplying the equation for \( \zeta \) by \( \sqrt{g/(2\omega(k))} \) and multiplying the equation for \( \phi \) by \( \omega(k)/(2g) \) and adding these two equations together, the result is an evolution equation for the complex variable

\[
b(k, t) = \left[ \frac{g}{2\omega(k)} \right]^{1/2} \zeta(k, t) + i \left[ \frac{\omega(k)}{2g} \right]^{1/2} \hat{\varphi}(k, t) ,
\]

where the dispersion relation is

\[
\omega = \left[ g \left| k \right| \tanh \left( \left| k \right| h \right) \right]^{1/2}.
\] (B.6)

The evolution equation for \( b(k, t) \) then is:

\[
\frac{\partial b}{\partial t}(k, t) + i\omega(k)b(k, t) \\
+ i \sum_{n=1}^{3} \int_{-\infty}^{\infty} dk_1 dk_2 V^{(n)}(k, k_1, k_2) C_{2n} \\
+ i \sum_{n=1}^{4} \int_{-\infty}^{\infty} dk_1 dk_2 dk_3 W^{(n)}(k, k_1, k_2, k_3) C_{3n} \\
+ i \sum_{n=1}^{5} \int_{-\infty}^{\infty} dk_1 dk_2 dk_3 dk_4 X^{(n)}(k, k_1, k_2, k_3, k_4) C_{4n}
\]
\[ C_{\ell n} = \left( \prod_{m=1}^{n-1} b^* (k_m, t) \right) \left( \prod_{m=n}^{\ell} b (k_m, t) \right) \delta \left( k + \sum_{m=1}^{n-1} k_m - \sum_{m=n}^{\ell} k_m \right) \] (B.8)

with * denoting the complex conjugate and \( \sum_{m=n}^{\ell} () = 0 \) and \( \prod_{m=n}^{\ell} () = 1 \) whenever \( \ell < n \).

Notice that \( \zeta \) and \( \varphi \) can be expressed in terms of \( b \) as

\[ \zeta (k, t) = \frac{\omega (k)}{2g} \left[ b(k, t) + b^* (-k, t) \right] \] (B.9a)
\[ \varphi (k, t) = -i \frac{g}{2 \omega (k)} \left[ b(k, t) - b^* (-k, t) \right] \] (B.9b)

4. We now use the transformation

\[ B(k, t) = b(k, t) e^{-i \omega (k) t} \] (B.10)

The term \( i \omega b \) then disappears from (B.7) while for \( b \) can be read \( B \) in (B.7). The next assumption now is that we suppose that \( B \) is composed of a (in time) slowly-varying part, \( \bar{B} \), and faster varying parts \( B', B'' \) and \( B''' \):

\[ B(k, t) = \bar{B}(k, t_2, t_3) + \epsilon B'(k, t, t_2, t_3) + \epsilon^2 B''(k, t, t_2, t_3) + \epsilon^3 B'''(k, t, t_2, t_3) \] (B.11)

where \( t_j = \epsilon^j t, j = 2, 3 \). The slow time \( t_1 \) is omitted because no exact resonance between three waves is possible. When surface tension is included exact resonance is possible for three-wave interaction, but these waves are very short and not of interest for us here. It is assumed also that most of the wave energy is contained in \( \bar{B} \). The representation (B.11) is substituted into the evolution equation (B.7); separating terms of equal power in \( \epsilon \) leads to evolution equations for \( \bar{B} \) and \( B', B'' \) and \( B''' \). For \( \epsilon^1 \) no information is obtained. Terms with \( \epsilon^2 \) yield an evolution equation for \( \partial B' / \partial t \) which depends on \( \bar{B} \) and therefore can be integrated to \( t \), while keeping \( t_2 \) and \( t_3 \) fixed.

At \( O (\epsilon^3) \) is obtained an equation for \( i \partial_{t_2} \bar{B} + i \partial_t B'' \) in which the right-hand member depends on terms with \( \bar{B} \), where both slow and fast-varying terms are present. Separating this equation into a slow and fast-varying part, and also writing \( B = \epsilon \bar{B} \), we obtain the evolution equation:

\[ i \frac{\partial \bar{B}}{\partial t} = \iint_{-\infty}^{\infty} dk_1 dk_2 dk_3 \beta_{0,1,2,3}^{(2)} \bar{B}_{k_1} B_{k_2} B_{k_3} \delta (k + k_1 - k_2 - k_3) \times \exp \left[ i (\omega + \omega_1 - \omega_2 - \omega_3) t \right] \] (B.12)
where $B_j$ stands for $B(k_j, t)$. For $B''$ an evolution equation is obtained in which the right-hand side does not depend on the slow time-scale so that an integration to $t$ is possible.

Equation (B.12) is the so-called Zakharov equation which is also valid for restricted depth when the dispersion relation $\omega(k)$ and the definition of $T^{(2)}$ are adapted for finite depth.

Once $B$ has been determined, the free-surface elevation $\zeta(x, t)$ follows by

$$
\zeta(x, t) = \int_{-\infty}^{\infty} dk \left( \frac{\omega(k)}{2g} \right)^{1/2} \left[ B(k, t) e^{i(k \cdot x - \omega t)} + CC \right].
$$

(B.13)

A problem with the Zakharov equation is that many different forms for the interaction coefficient $T^{(2)}$ exist. The reason for this is that there is some freedom in the definition of $T^{(2)}$ without changing the value of the integral in (B.12). The interaction coefficients can be symmetrised, as noted by Stiassnie and Shemer (1984).

The Zakharov equation has been reconsidered by Krasitskii (1994) who showed that previously used forms did not give a truly Hamiltonian system of equations; this had to do with the definition of the interaction coefficients. It appears that in the older form of the Zakharov equation the coefficients were not sufficiently symmetric. For an extensive discussion of these matters is referred to Krasitskii (1994) and Badulin et al. (1995). According to Krasitskii (1994), the symmetry conditions are not clear without considering the Hamiltonian formulation.

Notice that Rasmussen (1998, Eq. 2.61) writes the Zakharov equation (B.12) in the form

$$
\frac{\partial B}{\partial t}(k, t) = -i \iint d^3k_1 d^3k_2 d^3k_3 X^{(2)}_{0,1,2,3} C^\prime_{3,2}
$$

(B.14a)

with for $C^\prime$ the expression

$$
C^\prime_{3n} = \left( \prod_{m=1}^{n-1} b^* (k_m, t) \right) \left( \prod_{m=n}^{l} b (k_m, t) \right) \cdot \delta \left( k + \sum_{m=1}^{n-1} k_m - \sum_{m=n}^{l} k_m \right) \times
$$

$$
\times \exp \left[ i \left( \sum_{m=1}^{n-1} \omega_m - \sum_{m=n}^{l} \omega_m \right) t \right].
$$

(B.14b)

Taking $T^{(2)} = -X^{(2)}$, the same equation as given in (B.12) is obtained. For $X^{(2)}_{0,1,2,3} = -T^{(2)}_{0,1,2,3}$, Rasmussen (1998, Eq. (2.60)) gives the expression

$$
X^{(2)}_{0,1,2,3} = \frac{1}{2} \left( Y^{(2)}_{0,1,2,3} + Y^{(2)}_{0,1,3,2} \right)
$$

(B.15)

whenever both $k + k_1 - k_2 - k_3 = 0$ and $|\omega + \omega(k_1) - \omega(k_2) - \omega(k_3)| \leq O(\varepsilon^2)$; otherwise $X^{(2)} = 0$. The coefficient $Y^{(2)}_{0,1,2,3}$ has been given in Rasmussen (1998, Eq. (A.18)).
Taking the symmetric form of Krasitskii (1994) (in his notation \( T^{(2)} \) is called \( \tilde{V}^{(2)} \)), the underlying system is a Hamiltonian system and we have the property that (see also Badulin et al. 1995):

\[
T^{(2)}(k, k_1, k_2, k_3) = T^{(2)}(k_1, k, k_2, k_3) = T^{(2)}(k_1, k_3, k_2, k) = T^{(2)}(k_2, k_3, k, k_1). \tag{B.16}
\]

### B.2 Comparison between Hasselmann’s and Zakharov’s approach

Dyachenko and Lvov (1995) show that the formulations of Hasselmann (1962) and Zakharov (1968) are the same at the resonant surface, at least for the case of deep water. The formulations are written in the form

\[
\frac{\partial N(k_1)}{\partial t} = \alpha^2 \iiint T_2^2 [(N_1 + N_2) N_3 N_4 - (N_3 + N_4) N_1 N_2] \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \delta(k_1 + k_2 - k_3 - k_4) \, dk_2 dk_3 dk_4, \tag{B.17}
\]

with

\[
N(k)\delta(k - k_1) = \frac{\langle \zeta_k \zeta_{k_1} \rangle}{\alpha \omega_k}. \tag{B.18}
\]

and for Zakharov the following equation is derived:

\[
\frac{\partial N(k_1)}{\partial t} = 4\pi \iiint T_2^2 [(N_1 + N_2) N_3 N_4 - (N_3 + N_4) N_1 N_2] \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \delta(k_1 + k_2 - k_3 - k_4) \, dk_2 dk_3 dk_4, \tag{B.19}
\]

with the action density defined by

\[
N(k)\delta(k - k_1) = \langle a_k a_{k_1} \rangle, \tag{B.20}
\]

and the quantities \( a \) are defined in terms of the Fourier transform of the free surface elevation \( \zeta(x, t) \) and the wave potential at the free surface \( \varphi(x, t) \), \( \zeta_k \) and \( \varphi_k \) as

\[
\zeta_k = \left( \frac{\omega_k}{2g} \right)^{1/2} (a_k + a_{-k}^*) \quad \text{and} \quad \varphi_k = -i \left( \frac{2g}{\omega_k} \right)^{1/2} (a_k - a_{-k}^*). \tag{B.21}
\]

For the one-dimensional case two different solutions for the resonant surface are obtained: 1) the trivial one,

\[
k_3 = k_2, k_4 = k_1 \text{ or } k_3 = k_1, k_4 = k_2.
\]
and 2) a non-trivial one:

\[
\begin{align*}
    k_1 &= a (1 + \eta)^2, \\
    k_2 &= a (1 + \eta)^2 \eta^2, \\
    k_3 &= -a \eta^2, \\
    k_4 &= a (1 + \eta + \eta^2)^2; \quad \text{with} \quad 0 < \eta < 1.
\end{align*}
\]  

On the resonant surface (B.22) \( T_Z \) and \( T_H \) differ only in sign (which is irrelevant since the squares are needed) and are

\[
T_Z = -T_H = \frac{1}{4} \pi^2 (k_1 k_2) \min(|k_1|, |k_2|).
\]  

On the resonant surface (B.23) both \( T_H \) and \( T_Z \) are zero, as also remarked by Rasmussen (1995) for the 1D situation.

It should be remarked that off the resonant surface \(|T_Z|\) and \(|T_H|\) do not coincide. Notice that for the one-dimensional trivial case the kernel is different from zero, whereas with the discrete interaction approximation it is zero, as also is the case for the non-trivial solution.

For the two-dimensional case also non-trivial solutions are given, see the paper.
C SWAN input file

For the SWAN 40.16 computation carried out in Section 3.2.1 the following inputfile has been used:

```
PROJECT 'Source terms' '4173' 'Wave growth at shallow water'
MODE STATIONARY ONED
SET LEVEL=0.0 DEPMIN=0.01 MAXMES=999
     MAXERR=3 pwtail=4
CGRID REGULAR 0. 0. 0. 100000. 0 1000 0 &
     CIRCLE 72 flow=.050 fhigh=1.000 50
INPGRID BOTTOM REGULAR 0. 0. 0 1 100000. 1.
READ BOTTOM 1 'swanio.bot' 1 0 FREE
BOUNDARY SIDE WEST UNIFORM PAR 0.2 3.0 0 20

WIND 30.0 0.
GEN3 KOMEN
QUAD iquad=2 lambda=.250 cnl4=.300E+08
     csh1=5.5 csh2=0.8333 csh3=-1.25
FRIC JONSWAP
BREAK alpha=1 gamma=0.86
TRIAD URSELL=0.01
LIM URSELL=10 qb=1 iter=100
NUM ACCUR 0.01 0.01 0.01 99. 100

POINTS 'point' 0. 0. 10000. 0. 20000. 0. 30000. 0. &
     40000. 0. 50000. 0. 60000. 0. 70000. 0. &
     80000. 0. 90000. 0. 100000. 0.
SPEC 'point' SPEC1D ABS 'fetch100km.sp1'
TABLE 'point' .HEAD 'fetch100km.tab' XP YP HSIGN &
     RTP TMO1 TMO2 DIR DSPR FSPR QB
TEST 30 0 POINTS XY
     0. 0. 10000. 0. 20000. 0. 30000. 0. &
     40000. 0. 50000. 0. 60000. 0. 70000. 0. &
     80000. 0. 90000. 0. 100000. 0.
S1D 'fetch100km.srl'

POOL
COMPUTE
STOP
```
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