Technical Notes and Correspondence

Discrete Event Systems with Stochastic Processing Times

G. J. OLSDER, J. A. C. RESING, R. E. DE VRIES,
M. S. KEANE, AND G. HOOGHMSTRA

Abstract—Discrete event dynamic systems are studied in which the underlying algebra is the max-algebra and the coefficients in the system, referring to processing times in practice, are stochastic.

I. INTRODUCTION

The phrase "discrete event dynamic system," or "discrete event system" has been coined in at least three research directions viz. 1) control theory approach to automata, 2) perturbation analysis of trajectories, and 3) max-algebra approach. Recent publications in each of these directions are [1]–[3], respectively. In this note we shall concentrate on the last direction.

The discrete event dynamic systems (DEDS) we consider are linear in the max-algebra (also referred to as path-algebra or dioid). Linearity is a most tractable property and many phenomena are suitably modeled as linear systems. Many problems, however, for instance with respect to material flow in production processes, flows in communication networks and job flows in multiprogrammed computer systems, are nonlinear in the conventional arithmetic but are linear in the max-algebra. The elements of the max-algebra are the real numbers (and minus infinity) and the (only) admissible operations are maximization and addition.

A systematic theory, analogous to conventional linear algebra, exists for the max-algebra. Reference [4] is the basic work in this direction. In [3], [5], and [6] an analogy between the conventional linear system theory [7] and the theory of DEDS has been developed. The emphasis in [3] was on the periodic behavior of closed-loop DEDS, where the output is fed back to the input. The notions of eigenvalue and eigenvectors in the max-algebra sense played an important role. In [5] the concepts of stability, controllability, and observability for DEDS have been introduced and in [6] realization theory for DEDS has been studied.

Up until now the study of DEDS in the context of max-algebra has been purely deterministic. In this note we shall discuss certain stochastic extensions. Our motivation is that in practice processing times and/or transportation times are quite often stochastic quantities. Such stochastic fluctuations can, for instance, be caused by machine failure or depreciation.

The content of this note is as follows. Section II provides a motivation for the specific models used and sets the notation. Section III gives a general introduction to stochastic DEDS. It is assumed here that the probability distributions of the uncertain factors in the system are continuous. The asymptotic behavior of the system is investigated, and in particular the average duration of one cycle of the process is calculated. Section IV deals with a specific example of the theory provided in Section III. The state space is two-dimensional and the probability distributions are exponential. It is shown here that the process approaches a stationary limit as time proceeds. Sections V and VI deal with the case when the probability distributions are discrete and the theory of Section III does not directly apply. The problem is transformed into a Markov chain and results comparable to those of Sections III and IV are obtained. Several examples are given. Section V treats two-dimensional systems, and Sec-

tion VI deals more generally with finite-dimensional systems. The note ends with a Conclusion in Section VII.

II. DETERMINISTIC DISCRETE EVENT DYNAMIC SYSTEMS

In this section we will review some results related to deterministic DEDS, which have been described more extensively in [3], [4], and [8]. In conventional linear system theory [7] the standard model is

$$x(k+1) = Ax(k) + Bu(k); \quad y(k) = Cx(k) \quad (2.1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$. In scalar notation these equations read

$$\left. \begin{aligned} x_i(k+1) &= \sum_{j=1}^n a_{ij}x_j(k) + \sum_{j=1}^m b_{ij}u_j(k), & i &= 1, \dots, n; \\ y_i(k) &= \sum_{j=1}^n c_{ij}x_j(k), & i &= 1, \dots, p. \end{aligned} \right\} \quad (2.2)$$

In these equations there are two operations: addition and multiplication. These two operations will be changed into maximization and addition, respectively, such that the new way of calculation of (2.1) becomes

$$\left. \begin{aligned} x_i(k+1) &= \max(a_{i1} + x_1(k), \dots, a_{in} + x_n(k), \\ &\quad b_{i1} + u_1(k), \dots, b_{im} + u_m(k)), & i &= 1, \dots, n; \\ y_i(k) &= \max(c_{i1} + x_1(k), \dots, c_{in} + x_n(k)), & i &= 1, \dots, p. \end{aligned} \right\} \quad (2.3)$$

In order to show many analogies between the conventional linear system theory and the new one more clearly, the operation max will be written as $\oplus$ and the addition in (2.3) will be written as $\otimes$. Now (2.3) can be written as

$$\left. \begin{aligned} x_i(k+1) &= \sum_{j=1 \oplus}^n a_{ij} \otimes x_j(k) \\ &\quad \oplus \sum_{j=1 \oplus}^m b_{ij} \otimes u_j(k), & i &= 1, \dots, n; \\ y_i(k) &= \sum_{j=1 \oplus}^n c_{ij} \otimes x_j(k), & i &= 1, \dots, p. \end{aligned} \right\} \quad (2.4)$$

The summation symbols $\sum_{\oplus}$ used here are to be read in the maximization sense; in order to distinguish them from the conventional summation symbol, the subscript $\oplus$ has been added. In matrix notation (2.4) can be written as

$$x(k+1) = A \otimes x(k) \oplus B \otimes u(k); \quad y(k) = C \otimes x(k). \quad (2.5)$$

This model can describe flows in networks. The network (or directed graph) which corresponds to the $n \times n$ matrix A has n nodes and the entry a_{ij} corresponds to the arc going from node j to node i . Parts of the network corresponding to the input u and the output y can be similarly defined. The interpretation of the quantities used is as follows.

- $x_i(k)$ Earliest time instant at which node i of the network becomes active for the k th time.
- $u_j(k)$ Time instant at which resource j becomes available for the k th time.
- a_{ij} Transportation time from node j to node i , including the duration of the activity at node j .
- b_{ij} Transportation time between resource j and node i .

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The authors are with the Department of Technical Mathematics and Informatics, Delft University of Technology, Delft, The Netherlands.

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c_{ij} Transportation time between node j and sink i .

$y_i(k)$ Time instant at which product i is delivered for the k th time.

In this note, we concentrate on the equation $x(k+1) = A \oplus x(k)$, which for instance can be obtained from (2.5) if all resources are immediately available at the beginning of the process. Also, if a feedback $u(k) = F \oplus x(k)$ is used, a homogeneous equation of the form $x(k+1) = (A \oplus B \oplus F) \oplus x(k)$ results. For the behavior of $x(k)$, as k goes to infinity, the eigenvalue(s) of the matrix A play(s) a crucial role. If A is irreducible, then this eigenvalue is directly related to the cycle time of the network [3].

III. STOCHASTIC DISCRETE EVENT DYNAMIC SYSTEMS

We consider the equation

$$x(k+1) = A(k) \oplus x(k) \quad (3.1)$$

where the entries of the matrix $A(k)$ are stochastic. It is assumed that the sequence of matrices $A(k)$, $k = 0, 1, \dots$, is an independent sequence, and moreover that for each k the entries $a_{ij}(k)$, $i = 1, \dots, n$; $j = 1, \dots, n$ are mutually independent, although many of the derived results will hold through for dependent entries. For reasons of simplicity it is also assumed that all $a_{ij}(k)$ have the same distribution which will be denoted by F .

Define

$$\begin{aligned} z_1(k) &= x_1(k) \\ z_i(k) &= x_i(k) - x_1(k), \quad i = 2, \dots, n. \end{aligned} \quad (3.2)$$

The variables $z_i(k)$, $2 \leq i \leq n$, denote the differences in starting times of the k th activities between node i and node 1. Then

$$z_i(k+1) - z_i(k) = \left[a_{i1}(k) \oplus \sum_{j=2}^n (a_{ij}(k) \oplus z_j(k)) \right] \quad (3.3)$$

and

$$\begin{aligned} z_i(k+1) &= \left[a_{i1}(k) \oplus \sum_{j=2}^n a_{ij}(k) \oplus z_j(k) \right] \\ &- \left[a_{i1}(k) \oplus \sum_{j=2}^n a_{ij}(k) \oplus z_j(k) \right], \quad i = 2, \dots, n. \end{aligned} \quad (3.4)$$

It is clear from (3.4) that $(z_2(k), \dots, z_n(k))$ is a Markov chain with an $(n-1)$ -dimensional state space.

Among other things we want to study

$$\lim_{k \rightarrow \infty} x_m(k+1) - x_m(k), \quad 1 \leq m \leq n \quad (3.5)$$

and for irreducible A -matrices, the limit behavior is independent of the subscript m . This is trivially true in our case, in which we assumed that all a_{ij} have identical distributions. Therefore, we take $m = 1$ in (3.5). The joint distribution of $(z_2(k+1), \dots, z_n(k+1))$ given $(z_2(k), \dots, z_n(k))$ is obtained from (3.4)

$$\begin{aligned} P\{z_2(k+1) \leq y_2, \dots, z_n(k+1) \leq y_n | z_2(k) = x_2, \dots, z_n(k) = x_n\} \\ = P\{X_2 - X_1 \leq y_2, \dots, X_n - X_1 \leq y_n\} \end{aligned} \quad (3.6)$$

where $X_1, \dots, X_n$ are independent and

$$X_i \stackrel{d}{=} a_{i1}(k) \oplus \sum_{j=2}^n a_{ij}(k) \oplus x_j \quad (3.7)$$

(here $\stackrel{d}{=}$ denotes equality in distribution).

From (3.6), (3.7), and the fact that all $a_{ij}(k)$ possess distribution F ,

it follows that

$$\begin{aligned} K(x_2, \dots, x_n, y_2, \dots, y_n) \\ := P\{z_2(k+1) \leq y_2, \dots, z_n(k+1) \leq y_n | z_2(k) = x_2, \dots, z_n(k) = x_n\} \\ = \int_{-\infty}^{\infty} \left[\prod_{j=2}^n P\{X_j \leq y_j + t\} \right] P\{X_1 \in dt\} \\ = \int_{-\infty}^{\infty} \left[\prod_{j=2}^n H(t + y_j, t + y_j - x_2, \dots, t + y_j - x_n) \right] \\ \cdot \left(\frac{d}{dt} H(t, t - x_2, \dots, t - x_n) \right) dt \end{aligned} \quad (3.8)$$

where

$$H(x_1, x_2, \dots, x_n) = \prod_{i=1}^n F(x_i).$$

The time-dependent probability

$$\pi_k(x_2, \dots, x_n) := P\{z_2(k) \leq x_2, \dots, z_n(k) \leq x_n\}$$

satisfies

$$\begin{aligned} \pi_{k+1}(y_2, \dots, y_n) &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} K(x_2, \dots, x_n, y_2, \dots, y_n) \\ &\cdot \pi_k(dx_2, \dots, dx_n) \end{aligned} \quad (3.9)$$

where the kernel K is given in (3.8). If $\lim_{k \rightarrow \infty} \pi_k = \pi$ exists in the distributional sense (precise conditions are given in [11], [9]) then this limit is a solution of

$$\begin{aligned} \pi(y_2, \dots, y_n) &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} K(x_2, \dots, x_n, y_2, \dots, y_n) \\ &\cdot \pi(dx_2, \dots, dx_n). \end{aligned} \quad (3.10)$$

Furthermore, by (3.3),

$$\begin{aligned} \lim_{k \rightarrow \infty} P\{x_1(k+1) - x_1(k) \leq x\} \\ = \lim_{k \rightarrow \infty} P\{z_1(k+1) - z_1(k) \leq x\} \\ = \lim_{k \rightarrow \infty} P\{a_{11}(k) \leq x, a_{12}(k) + z_2(k) \leq x_2, \dots, a_{1n}(k) \\ + z_n(k) \leq x_n\} = \lim_{k \rightarrow \infty} P\{a_{11}(k) \leq x, a_{12}(k) \\ \leq x_2 - z_2(k), \dots, a_{1n}(k) \leq x_n - z_n(k)\} \\ = F(x) \lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \prod_{i=2}^n F(x - y_i) \pi_k(dy_2, \dots, dy_n) \\ = F(x) \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \prod_{i=2}^n F(x - y_i) \pi(dy_2, \dots, dy_n). \end{aligned} \quad (3.11)$$

The limit and the integral in (3.11) can be interchanged by the theory of weak convergence of probability measures if F is continuous (cf. [12]).

IV. AN EXAMPLE

Consider the case $n = 2$. The transition kernel $K(x, y)$ of the Markov chain $\{z_2(k)\}$ is given by

$$\begin{aligned} K(x, y) &:= P\{z_2(k+1) \leq y | z_2(k) = x\} \\ &= \int_{-\infty}^{\infty} H(t + y, t + y - x) \left(\frac{d}{dt} H(t, t - x) \right) dt \\ &= \int_{-\infty}^{\infty} F(t + y) F(t + y - x) (F'(t) F(t - x) \\ &\quad + F(t) F'(t - x)) dt. \end{aligned}$$

Explicit calculations will be made for this case $n = 2$ and

$$F(x) = (1 - e^{-x})1_{[0, \infty)}(x). \quad (4.1)$$

If $\phi(x)$ denotes the density of the stationary distribution π we obtain from (3.10),

$$\phi(y) = \int_{-\infty}^{\infty} \left(\frac{d}{dy} K(x, y) \right) \phi(x) dx$$

or equivalently after some calculations,

$$\phi(y) = \frac{1}{2} \int_{-\infty}^{\infty} \left[-\frac{1}{3} e^{-|y|-2|x|} + \frac{2}{3} e^{-2|y|-2|x|} + \frac{2}{3} e^{-|y|-|x|} - \frac{4}{3} e^{-2|y|-|x|} + e^{-|y|} \right] \phi(x) dx. \quad (4.2)$$

This is an integral equation of which the kernel is degenerate, see [10, sect. 1.4]. The corresponding theory tells us that the solution $\phi(y)$ must be of the form

$$\phi(y) = c_1 e^{-|y|} + c_2 e^{-2|y|}. \quad (4.3)$$

Substitution into (4.2) leads to

$$8c_1 + 23c_2 = 0.$$

Together with $2c_1 + c_2 = 1$ which results from $\int_{-\infty}^{\infty} \phi(y) dy = 1$, the constants c_1 and c_2 are uniquely determined and $\phi(y)$ becomes

$$\phi(y) = \frac{23}{38} e^{-|y|} - \frac{4}{19} e^{-2|y|}, \quad y \in \mathbb{R}.$$

It is easy to show that $\phi(y) \geq 0$ for all $y \in \mathbb{R}$, and hence $\phi(\cdot)$ is indeed a probability density function

$$\lim_{k \rightarrow \infty} P(x_2(k) - x_1(k) \in dy) = \phi(y) dy.$$

Remark: The coefficients (c_1, c_2) in (4.3) can also be obtained by iterating the equation

$$\phi_{k+1}(x) = \int_{-\infty}^{\infty} K(x, y) \phi_k(y) dy. \quad \square$$

Once $\phi(y)$ has been obtained, it is straightforward from (3.11) to calculate $P(x_1(k+1) - x_1(k) \in dy)$ in the stationary situation (or when $k \rightarrow \infty$);

$$P(x_1(k+1) - x_1(k) \in dy) = \left(\frac{233}{228} e^{-|y|} + \frac{23}{38} y e^{-|y|} - \frac{100}{228} e^{-2|y|} - \frac{23}{19} y e^{-2|y|} - \frac{6}{19} e^{-3|y|} \right) dy$$

and hence

$$\lim_{k \rightarrow \infty} E(x_1(k+1) - x_1(k)) = \frac{407}{228}.$$

On the average it takes 407/228 time units between two subsequent activities of either node of the network. If $x_1(0) = 0$, then also

$$\lim_{k \rightarrow \infty} \frac{1}{k} E(x_i(k)) = \frac{407}{228}.$$

V. STOCHASTIC DISCRETE EVENT DYNAMIC SYSTEMS IN $\mathbb{R}^2$ WITH POINT DISTRIBUTIONS

In the previous two sections we assumed that the processing and transportation times are stochastic variables with continuous distribution functions. In this section we consider discrete distributions. The analysis of Section III cannot be used anymore but it will be seen that the same train of thought can be used. This train of thought is the following: first calculate the stochastic dependence (or correlation) between the elements of

the state vector $x(k)$ for a fixed k ; next consider the stochastic evolution of the system by considering the limit as $k \rightarrow \infty$; finally use this stationary limit behavior in order to find the average growth of the system related to $\lim (x(k+1) - x(k))$ as $k \rightarrow \infty$. Related results, derived from the central limit theorem, will be given in [9].

The starting point is again

$$x(k+1) = A(k) \oplus x(k) \quad (5.1)$$

where the sequence $A(k)$, $k = 0, 1, 2, \dots$, is an independent sequence. Also, we assume again that the entries of $A(k)$ are mutually independent and (for simplicity) that all $a_{ij}(k)$, $i, j = 1, \dots, n$; $k = 0, 1, \dots$ have the same underlying distribution function. Furthermore, we will restrict ourselves to uniform distributions of the following kind:

$$P\left(a_{ij}(k) = \frac{l}{m}\right) = \frac{1}{m+1}, \quad l = 0, 1, \dots, m \quad (5.2)$$

where m is an integer ≥ 1 .

Example 2: Suppose $n = 2$, $m = 1$. Hence, the following system is considered:

$$\left. \begin{aligned} x_1(k+1) &= a_{11}(k) \oplus x_1(k) \oplus a_{12}(k) \oplus x_2(k), \\ x_2(k+1) &= a_{21}(k) \oplus x_1(k) \oplus a_{22}(k) \oplus x_2(k) \end{aligned} \right\} \quad (5.3)$$

where

$$P(a_{ij}(k) = 0) = P(a_{ij}(k) = 1) = \frac{1}{2}. \quad (5.4)$$

It is still true that $z_2(k) = x_2(k) - x_1(k)$ forms a Markov chain, and for any initial vector $(x_1(0), x_2(0))$, the Markov chain $z(k)$ takes values in the set $\{-1, 0, 1\}$ in the stationary situation. The transition probabilities on $\{-1, 0, 1\}$ are given by the matrix

$$P = \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 3/16 & 5/8 & 3/16 \\ 1/4 & 1/2 & 1/4 \end{pmatrix} \quad (5.5)$$

and the stationary probabilities turn out to be

$$\pi_{-1} = 3/14, \pi_0 = 8/14 \text{ and } \pi_1 = 3/14. \quad (5.6)$$

The quantity $x_1(k+1) - x_1(k)$ in the stationary situation (or when $k \rightarrow \infty$) is hence obtained from

$$x_1(k+1) - x_1(k) = a_{11}(k) \oplus (a_{12}(k) \oplus z)$$

where the quantity z has distribution (5.6) and is independent of $a_{11}(k)$ and $a_{12}(k)$. It follows that

$$\lim_{k \rightarrow \infty} E(x_1(k+1) - x_1(k)) = \frac{6}{7}.$$

On the average it takes 6/7 time units between two subsequent activities of either node of the network. $\square$

This example can be generalized to higher values of n and m . We will consider $n = 2$ and general m in some detail. The Markov chain $z_2(k) = x_2(k) - x_1(k)$ will now take values in the set

$$S = \left\{ -\frac{m}{m}, \dots, -\frac{1}{m}, 0, \frac{1}{m}, \dots, \frac{m}{m} \right\}.$$

For $l/m \in S$ and $j = 0, \dots, m$, the transition probabilities are

$$\begin{aligned} P\left\{z_2(k+1) = \frac{j}{m} \mid z_2(k) = -\frac{l}{m}\right\} \\ = \sum_{i \geq j} P\left\{a_{21}(k) \oplus \left(a_{22}(k) - \frac{l}{m}\right) = \frac{i}{m}\right\} \\ \cdot P\left\{a_{11}(k) \oplus \left(a_{12}(k) - \frac{l}{m}\right) = \frac{i-j}{m}\right\} \\ = \sum_{i \geq j} c_{l,i+j} \cdot c_{l,i-j} \end{aligned} \quad (5.7)$$

where

$$c_{l,j} := \begin{cases} \frac{2j+l+1}{(m+1)^2}, & \text{if } j+l \leq m, \\ \frac{1}{m+1}, & \text{if } j+l > m. \end{cases} \quad (5.8)$$

The remaining transition probabilities ($j < 0$) follow by symmetry. The stationary distribution π is obtained from the equation ${}^t\pi P = {}^t\pi$, where P is the square matrix of order $(2m+1)$ with entries given in (5.7). For instance, when $m = 2$ the quantities π_i become

$$\pi_1 = \pi_{-1} = 0.075; \pi_{1/2} = \pi_{-1/2} = 0.234; \pi_0 = 0.382.$$

The mean cycle time becomes 0.803. If we compare this result to the answer for $m = 1$, which equals $6/7 \approx 0.857$, we note that the mean cycle time decreases. Intuitively this is clear: as m grows the distribution of $a_{ij}(k)$ approaches the uniform distribution on $(0, 1)$ and so the extreme outcomes like 0 and 1 decrease in probability. From numerical work we conclude that for $m \rightarrow \infty$ the mean cycle time converges to 0.719.

VI. SOME REMARKS ON STOCHASTIC DEDS IN R^n WITH POINT DISTRIBUTIONS

We consider again $x(k+1) = A(k) \oplus x(k)$, where now $x \in R^n$ and where all quantities $\{a_{ij}(k), i, j = 1, \dots, n; k = 0, 1, \dots\}$ are independent random variables, all with the same distribution

$$P\left(a_{ij}(k) = \frac{l}{m}\right) = \frac{1}{m+1}, \quad l = 0, 1, \dots, m. \quad (6.1)$$

In order to keep the calculation as simple as possible we only need to consider

$$x_i(k) \geq x_{i+1}(k); \quad i = 1, \dots, n-1. \quad (6.2)$$

The other cases can, by means of reordering of the elements of x , be brought back to this situation. Suppose $x_1(k) = v - l_1/m$, $x_2(k) = v - l_2/m, \dots, x_n(k) = v - l_n/m$, with $0 = l_1 \leq l_2 \leq \dots \leq l_n \leq m$. The conditional probability of $x_\alpha(k+1) = v + j/m$, given that $x_i(k) = v - l_i/m$, $1 \leq i \leq n$ does not depend on α , is

$$\begin{aligned} & c_{l_2, \dots, l_m, j} \\ & \triangleq P\left(x_\alpha(k+1) = v + \frac{j}{m} \mid x_i(k) = v - \frac{l_i}{m}, i = 1, \dots, n\right) \\ & = P\left(a_{\alpha 1} = \frac{j}{m}, a_{\alpha i} \leq \frac{j+l_i}{m}, i = 2, \dots, n\right) \\ & + P\left(a_{\alpha 1} < \frac{j}{m}, a_{\alpha 2} = \frac{j+l_2}{m}, a_{\alpha i} \leq \frac{j+l_i}{m}, i = 3, \dots, n\right) \\ & + \dots = \sum_{r=1}^n \left[\left(\frac{(j+l_r+1)^*}{m+1} \right) \left(\prod_{i=1}^{r-1} \frac{(j+l_i)^*}{m+1} \right) \right. \\ & \quad \left. \left(\prod_{i=r+1}^n \frac{(j+l_i+1)^*}{m+1} \right) \right], j = 0, \dots, m \end{aligned} \quad (6.3)$$

where the following notation has been used:

$$(\gamma)^* = \begin{cases} 1 & \text{if } \gamma \leq m+1 \\ 0 & \text{if } \gamma > m+1 \end{cases}, \quad (\gamma)^* = \begin{cases} \gamma & \text{if } \gamma \leq m+1 \\ m+1 & \text{if } \gamma > m+1 \end{cases}.$$

In general the transition probabilities of the Markov chain $(z_2(k), \dots, z_n(k))$ can be reconstructed from (6.3). These calculations

are not difficult, but can be very tedious. We will show the case $n = 3$ and $m = 1$. Denote by $p_j(i_2, i_3) := P\{z_2(k+1) = j | z_2(k) = i_2, z_3(k) = i_3\}$, then it follows from (6.3) or directly from the definition of $(z_2(k), \dots, z_n(k))$ that

$$p_0(-1, -1) = p_0(-1, 1) = p_0(1, -1) = p_0(0, 1) = p_0(1, 0) = \frac{1}{2},$$

$$p_0(-1, 0) = p_0(0, -1) = p_0(1, 1) = \left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2,$$

$$p_0(0, 0) = \left(\frac{1}{8}\right)^2 + \left(\frac{7}{8}\right)^2. \quad (6.4)$$

From (6.4) the transition probabilities follow immediately, for instance

$$\begin{aligned} & P\{z_2(k+1) = 1, z_3(k+1) = 0 | z_2(k) = -1, z_3(k) = -1\} \\ & = c_1(-1, -1) \cdot c_0(-1, -1) \\ & = \frac{1}{2}(1 - c_0(-1, -1)) \cdot c_0(-1, -1) = \left(\frac{1}{2}\right)^3. \end{aligned}$$

The mean cycle time $E(x_1(k+1) - x_1(k))$ in the stationary situation equals 0.979. Observe that this mean cycle time is larger than the value obtained for $n = 2$ and $m = 1$ (namely 0.857). This is to be expected since cycle times will increase as the number of nodes increases. Of course, for the case considered here, the upperbound will be one.

VII. CONCLUSION

During the last couple of years the deterministic theory of discrete event dynamic systems (DEDS) has received much attention and important contributions have been made. The practical motivation for such systems stems from concurrent activities in processes. This note provides a window to stochastic extensions. The theory given deals with the analysis of stochastic DEDS in which the processing times and/or the transportation times within a network show stochastic fluctuations. Restrictions in the current note are that the stochastic processing times of the nodes in the network are independent and identically distributed. These, however, are no fundamental restrictions since the same approach can essentially be used for dependent processing times and/or different distribution functions. In [9] a specific form of the central limit theorem will be used in order to study the asymptotic behavior more precisely.

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