Stellingen

behorende bij het proefschrift

"Numerical Simulation of Crack Growth in Pressurized Fuselages"

door H.A.J. Knops

1. Het sterk niet-lineaire gedrag van scheuren in vlietgigrompen maakt een analytische aanpak van dit probleem vrijwel onmogelijk (Hoofdstuk 2 van dit proefschrift)

2. Wanneer gebruik wordt gemaakt van een plaattheorie met afschuifvorming door de dikte moet rond de scheurtip een zeer fijn elementennetwerk worden toegepast om de buigtermen van de K-factoren te bepalen. Maakt men echter gebruik van een Kirchhoff plaattheorie (zonder afschuifvormingen door de dikte) dan kunnen deze buigtermen bepaald worden met een grover elementennetwerk t.p.v. de scheurtip. (Hoofdstuk 2 van dit proefschrift)

3. Het automatisch genereren en verfijnen van een elementennetwerk bestaande uit louter vierhoekige elementen is vele malen moeilijker dan het genereren en verfijnen van een elementenverdeling bestaande uit driehoekige elementen, getuige het aantal publicaties in de literatuur. (Hoofdstuk 4 van dit proefschrift)

4. De elastische T-spanning lijkt voor dunwandige constructies een gunstige invloed te hebben op de nauwkeurigheid van het numeriek bepaalde scheurgroeipad. (Hoofdstuk 5 van dit proefschrift)

5. Tot op de dag van vandaag is het onduidelijk welke rol de buigtermen van de K-factoren spelen bij het scheurgroeiprocès van scheuren in dunwandige vlietgigrompen. Afgaande op de voorbeelden in dit proefschrift is de invloed van deze buigtermen op het scheurgroeigedrag klein. (Hoofdstuk 5 van dit proefschrift)


7. Bij een medische behandeling van ernstig zieke patiënten is de vraag niet langer of een bepaalde ingreep technisch haalbaar is maar ethisch verantwoord.

8. Door het eigen huizen bezit te stimuleren en financieel aantrekkelijker te maken moet het mogelijk zijn om het aanzien van bepaalde stadswijken drastisch te verbeteren.

9. Een begeleider van een jeugdteam bij een sportvereniging moet vaak meer tijd besteden aan het in toom houden van de ouders dan aan het werkelijk begeleiden en coachen van de jeugdspelers.

10. Het ingrijpen van de westerse wereld in militaire conflicten en burgeroorlogen om humanitaire redenen is in werkelijkheid vaak niets anders dan het veilig stellen van economische belangen.

11. In het kader van 'de verbruiker betaalt' moet de wegenbelasting worden vervangen door een toeslag op de brandstofprijzen.
Numerical Simulation of Crack Growth in Pressurized Fuselages

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Notation

Most of the symbols used in this dissertation are listed below. However, there are symbols that have different meanings in different chapters in order to keep them close to the international conventions. In that case these symbols are explained as they appear.

- $a$: half crack length
- $A_i$: base vectors for the undeformed state
- $A_{i(cr)}$: base vectors at the crack tip for the undeformed state
- $b_i$: base vectors for the intermediate state
- $b_{i(cr)}$: base vectors at the crack tip for the intermediate state
- $D$: deformation gradient
- $C$: material stiffness matrix
- $\frac{d}{dx}(\ )$: derivative with respect to $x$
- $\frac{\partial}{\partial x}(\ )$: partial derivative with respect to $x$
- $E$: Young's modulus
- $\| e \|$: energy norm
- $(e_n, e_t, n)$: base vectors of an orthogonal triad at the shell edge
- $F_i$: vector of external forces
- $f_i$: vector of residual forces
- $G_{ij}$: Green-Lagrange strain tensor
- $G_c$: energy release rate
- $h$: plate height
- $h_i$, $h_r$: element size
- $h_\alpha$: weight functions
- $k$: vector of stress intensity factors
- $K_I$, $K_{II}$: membrane mode I and mode II stress intensity factors
- $K_{III}$: stress intensity factor for the tearing mode
- $K_{BI}$, $K_{BII}$: bending mode I and mode II stress intensity factors for a transverse shear deformable plate theory
- $k_1$, $k_2$: bending mode I and mode II stress intensity factors for the classical plate theory
\( M \)  
- bending moment per unit distance

\( M_n, M_t \)  
- normal and in-plane moments at the edge of a shell

\( N, N^i \)  
- finite element shape functions

\( g_i \)  
- vector of nodal displacements for the underlying displacement field

\( g_i \)  
- vector of total nodal displacements

\( Q \)  
- transverse shear load per unit distance

\( Q \)  
- transverse shear resultant in the classical plate theory directed along the normal of the shell mid-surface

\( r, \theta \)  
- polar coordinate system with respect to the crack tip

\( R \)  
- rotation matrix

\( R \)  
- radius of a fuselage

\( S_{ij} \)  
- stiffness matrix

\( t \)  
- plate thickness

\( T \)  
- elastic T-term

\( T_n, T_t \)  
- normal and in-plane stress resultants in a shell

\((\cdot)^T\)  
- transposed

\( u, v, w \)  
- global displacements in \( x-, y-, \) and \( z- \) direction

\( U \)  
- stretch matrix

\( u_n, u_t \)  
- normal and in-plane displacement at the edge of a shell

\( u \)  
- approximate displacement field

\( U_p \)  
- potential energy

\( U_s \)  
- strain energy density

\( V \)  
- volume

\( v_i \)  
- analytical displacement solutions corresponding to the stress intensity factors

\( y_i \)  
- nodal values of the analytical displacement solutions

\( W \)  
- plate width

\( X \)  
- vector denoting the undeformed state

\( x \)  
- vector denoting the final deformed state

\( x' \)  
- vector denoting the intermediate state

\( x, y, z \)  
- cartesian coordinate system

\( \beta \)  
- angle between the crack faces and the loading direction

\( \beta_{\text{bulge}} \)  
- bulge factor

\( \varepsilon_e \)  
- engineering strain

\( \Theta \)  
- analytical shape functions corresponding to the stress intensity factors

\( \lambda \)  
- load parameter

\( \mu \)  
- shear modulus

\( \nu \)  
- Poisson's ratio
\( \sigma \)  
Cauchy stress

\[ \sum_{i=i_1}^{i_2} \]  
summation from \( i=i_1 \) to \( i=i_2 \)

\( \varphi_n, \varphi_t \)  
rotation about the normal and in-plane vector at the edge of a shell

\( \Omega \)  
surface
Chapter 1

Introduction

1.1 Importance of fracture mechanics

Fatigue is a failure mechanism that occurs as a result of a large number of repeated load applications where the applied load is (far) below the static failure load. Therefore fatigue has to be considered for structures and vehicles that are subjected to many load cycles during their economical life. As an example of such a structure we can mention an aircraft fuselage where the cyclic load is represented by the cabin pressure.

Nowadays fatigue has become a dominant aspect of structural airworthiness for different reasons. First of all, a considerable part of the present jet fleet is operated beyond their original design life goal. This means that a lot of airplanes that are still in service have been subjected to a large number of load cycles already and thus the incidences of fatigue may become widespread.

Furthermore, as far as the aircraft cabin is concerned, the design load in the fuselage skin of modern airplanes has more than doubled compared to the first jet transport aircraft. This increase, which is mainly due to the larger fuselage diameter and the higher cabin pressures, will result in a higher load intensity for each load cycle.

A final reason for the growing interest in the fatigue problem is that aircraft operators are pressuring for longer intervals between inspections. Therefore it is important to gain more experience and knowledge on the fatigue problem. One has to be sure that cracks which develop between inspections will not cause complete failure of the structure before the next inspection.

In practice it is impossible to maintain pressurized aircraft fuselages without cracks. Therefore, one of the most important aspects in the design process is to consider skin cracks resulting from fatigue. The material and stress level have to be chosen in such a way that these cracks can be detected before they reach a critical length. In this way structural integrity of the fuselage can be assured, provided that damaged components are replaced before cracks grow too large.

The most obvious way to create realistic inspection intervals and quality assurance standards is by performing full scale tests. In combination with these tests it is important to develop
analytical and numerical techniques for simulating the behavior of skin cracks. With these techniques, which are relatively inexpensive as compared to full scale tests, one can obtain some insight in the behavior of all kind of crack configurations for a certain fuselage model before performing a costly full scale experiment.

However, a problem with these numerical techniques is that the influence of all the structural details of the actual fuselage on the crack behavior is difficult to account for. This means that a numerical simulation of the crack behavior can only be established after making some approximations with respect to the structural layout and the applied loading. These approximations have to be chosen in such a way that conformity between the numerical model and the actual fuselage is violated as little as possible. Hence, one may conclude that numerical crack analysis techniques may only be used in combination with, and not instead of, full scale tests.

The main part of the loading in the fuselage skin is caused by the internal pressure. Due to these so-called boiler stresses, skin cracks can be initiated at material discontinuities (such as rivet holes, thickness jumps and structural cutouts). Experience has shown that cracks are most likely to initiate in longitudinal direction due to the larger (circumferential) hoop stress (as compared to the longitudinal stress). However, cracks which initially propagate in longitudinal direction can change direction and can turn to a circumferential line when exceeding a certain critical crack length. This so-called crack flapping phenomenon allows a rapid decompression of the fuselage prior to complete failure of the structure and thus is preferred over a crack which continues to propagate in longitudinal direction.

The behavior of longitudinal through cracks in pressurized fuselages is dominated by a number of influence factors and is more difficult to handle than the behavior of circumferential cracks. First of all, there is a geometrically nonlinear effect caused by the curvature of the fuselage and by the longitudinal tensile membrane stresses in the direction parallel to the crack faces. These longitudinal stresses will restrain the so-called bulging (i.e. out of plane displacement) of the crack and therefore will cause a reduction of the intensity of the stress singularity at the crack tip compared to the (incorrect) linear results.

Another important aspect concerns the geometry of the structure, i.e. the presence of longerons and bulkheads. These stiffening components will have a significant influence on the stress distribution in the shell and this will lead to a change of loading of the crack.

Finally, we should mention the plasticity effects at the crack tip, which will cause a local redistribution of the stresses around the crack tip.

It is clear that the analysis of longitudinal through cracks in pressurized fuselages is not straightforward. In literature it was shown that the finite element method is an excellent means for dealing with a problem of this complexity. In this way the problem can be tackled for all kinds of different crack geometries, including the reinforcing structural members such as longerons and bulkheads. An additional advantage of the finite element method is that it offers
the opportunity to model crack growth in a discrete way so that it is possible to simulate crack propagation numerically.

1.2 Research objectives

The main objective of the present research is to develop a number of tools for the numerical simulation of crack behavior in pressurized fuselages. In these calculations the geometrically nonlinear effects described above and the influence of stiffening components on the crack behavior will be taken into account.

Since the flapping phenomenon that is observed in experiments on pressurized aircraft fuselages with initially longitudinal cracks is not yet well understood, a major goal of the present research is to simulate crack flapping numerically and to predict whether a crack will flap and at what crack length.

In real structures there will always be a plastic zone around the crack tip. However, the behavior of such 'real' cracks is still believed to be dictated by a number of linear elastic crack tip parameters. Therefore in our research we will assume that linear-elastic fracture mechanics conditions hold and that the state of stress around the crack tip is characterized by these linear-elastic crack tip parameters.

1.3 Crack analysis tools

Numerical simulation of crack behavior can be divided into two equally essential aspects. The first aspect concerns the representation of the physical crack problem in terms of a finite element model. This includes, for instance, the geometry of the cracked structure, the boundary conditions and routines that will automatically update a finite element mesh as the crack evolves. This first aspect can be seen as a mechanism for simulating crack propagation. The second aspect concerns the principals of fracture mechanics that are used to drive the crack growth simulation. These principals are governed by some characteristic parameters at the crack tip, and they provide a strategy for the crack propagation mechanism.

Crack growth simulation can be seen as a combination of a crack propagation mechanism and a crack propagation strategy. It is an iterative process of determining the parameters that govern the fracture process, evaluating the new crack direction and increment on the basis of these parameters and modeling the new crack increment. To perform these tasks some special numerical tools have to be developed.

First of all, the characteristic crack tip parameters must be calculated. Over the years several methods were proposed to determine these parameters for linear crack problems. However, for
our purpose a technique has to be developed that is also applicable to geometrically nonlinear problems.

A next step is the determination of the crack propagation angle. In the literature one can find a number of criteria for calculating this angle and all these criteria are governed by the crack tip parameters calculated at the current crack tip. In our research we will not pursue the development of a new crack propagation criterion. Some of the existing crack propagation criteria will be tested and the calculated crack trajectories will be compared with some experimental results found in literature.

Finally, a finite element mesh generator has to be available that produces the mesh for the problem at hand automatically. This mesh generator must be able not only to generate a finite element mesh for the initial crack problem but also to remesh the crack tip region for an arbitrary propagation angle and increment size.

1.4 Organization of the thesis

In chapter 2 the calculation of the stress intensity factors which govern the singular stress field at the crack tip is discussed. The proposed technique is applied to classical (linear) as well as geometrically nonlinear crack problems, and the calculated stress intensity factors are shown to be in close agreement with results found in literature.

In chapter 3 a method is described for calculating the so-called elastic T-term. This elastic T-term actually represents a constant stress parallel to the crack faces and will be used in one of the criteria for predicting the crack growth direction for a propagating crack.

The details of the routine for automatically generating a finite element mesh consisting of quadrilateral elements only is discussed in chapter 4. Along with this mesh generator, an error estimation routine and a solution data mapping routine have been developed.

The error estimation routine calculates the global and local (element) errors in one of the solution data over the total finite element model. Moreover, it determines the local degrees of mesh refinement that are needed to obtain a certain (user supplied) global accuracy. The output of this routine is used by the finite element mesh generator to produce a kind of optimum grid layout that will give accurate results with a minimum number of degrees of freedom.

The solution data mapping routine is used to map the solution data from an initial finite element mesh onto a refined one. After regaining equilibrium again for this new mesh at the current load level, the nonlinear calculation can be proceeded at this load level without having to impose the external load again for each new finite element model. This feature is especially interesting when we simulate crack growth for geometrically nonlinear problems. The resulting displacements and rotations are mapped from one mesh onto another after modeling the new crack increment, and the crack growth simulation can be continued by opening this new crack increment.
In chapter 5 the actual crack propagation mechanism and strategy are discussed in more detail. First some of the crack propagation criteria that can be found in literature will be briefly summarized. As already mentioned, it is not our goal to pursue the development of a new crack propagation criterion, but to analyse the results obtained with some of the existing criteria.

Next the crack propagation routine is discussed. The implementation of such a procedure is not straightforward because the resulting finite element mesh has to meet some special conditions. All the elements around the new crack increment have to be quadrilateral and the local mesh around the crack tip has to be symmetrical with respect to the crack tip.

Finally all these tools are integrated in one routine for simulating crack propagation. This routine is applied to some linear and geometrically nonlinear crack problems and the results will be shown to be in good agreement with some of the experimental findings reported in the literature.
Chapter 2
Mode enrichment

2.1 Introduction

The so-called mode enrichment (or singular mode superposition) technique [9] is a powerful method to compute the stress intensity factors of through cracks in thin-walled structures under service loads. It has been applied for some time now, apparently with great success. So far, however, its usefulness has been limited to the field of linear elasticity. The particular application we have in mind is the longitudinal crack in a pressurized fuselage which is to a great extent governed by geometrically nonlinear effects [1,19,76,78,81]. Therefore we want to explore the possibility to extend the range of applications of this method to geometrically nonlinear problems.

The idea of extracting the stress intensity factors in finite element computations by using singular modes in addition to a regular interpolation field goes back to Wilson [110], Mote [58], Strang and Fix [92], Foschi and Barrett [32], Hepler and Hansen [36], Heyliger and Kriz [37], Tracey [104] and Benzley [9]. The particular way in which this idea is used here was probably first described by Benzley [9]. The mode enrichment technique can be implemented as an additional computational feature in an existing nonlinear finite element code and it does not require any interference with the finite element drivers of the code.

In comparison with other methods that aim for the same objective, such as the J-integral [73] or the nodal release technique [83], it has some important advantages. First of all, the mode enrichment technique offers the opportunity to distinguish easily between the various modes of cracking (i.e. the two membrane modes, the two bending modes and the tearing mode) which is not the case for the path independent J-integral. Furthermore, the technique is very appealing because it is fully integrated in a finite element code. The stress intensity factors are part of the set of computational degrees of freedom and are directly produced, in contrast to the other methods mentioned where the stress intensity factors are determined indirectly via the energy release rate. Finally we should mention that via mode enrichment the agreement between the numerically obtained displacement field around the crack tip and the exact analytical displacement solution will become much closer as compared to the other techniques. This advantage will become especially interesting if one wants to calculate one (or
more) of the higher order terms of the analytical displacement solution around the crack tip such as the elastic T-term (see chapter 3).

However, in contrast to the J-integral or the nodal release technique, application of the mode enrichment method to geometrically nonlinear problems necessitates a couple of measures on the algorithmic level which are not entirely obvious at first sight. One of them is the formulation of the superposition model that takes into account the finite rotation of the crack tip during the loading sequence. Another concerns the coupling of the in-plane strains around the crack tip because of the use of Green-Lagrange strains which are not truncated as in linear elasticity theory. Finally the impact that the superposition process has on the solution of the governing equations has to be considered.

2.2 Description of the computational procedure

2.2.1 Basic principle

For a shell or a plate the general form of the linear elasticity solution for the stresses in the immediate surrounding of a sharply edged crack does not depend on the loads and boundary conditions applied to the structure at a finite distance of the crack tip. Only the stress intensity factors, which are characteristic parameters in this solution, depend on these conditions [12,63]. It is this property that is exploited by the singular mode superposition technique.

The general solution for a crack in a plate or a shell in terms of the displacements \( \mathbf{u} = (u,v)^T \) and stresses \( \boldsymbol{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{xy})^T \) tangential to the midsurface can be decomposed in two parts namely

![Figure 2.1](image)

*Figure 2.1* Straight crack in an infinite plate
\[
\mathbf{u} = u_1(r, \theta, z) + u_2(r, \theta, z) \tag{2.1a}
\]
\[
\sigma = \sigma_1(r, \theta, z) + \sigma_2(r, \theta, z) \tag{2.1b}
\]

where \(r\) and \(\theta\) are the components of a polar coordinate system at the tip of the crack (as is depicted in figure 2.1) and \(z\) is the distance measured along the normal \(n\) to the mid-surface of the plate. The subscript (1) refers to the regular part of the solution and the subscript (2) refers to the singular crack tip solution that is dominant in a close neighborhood of the crack tip. When using a plate theory that includes transverse shear deformation there is also a tearing mode that is governed by an out of plane displacement field \(w\) and by transverse shear stresses \(\sigma = (\sigma_{xz}, \sigma_{yz})^T\).

For a shell in a general state of deformation, the in-plane crack tip solutions can be cast in the form (more detailed expressions will be given later)

\[
u_2(r, \theta) = \sum_{i=1}^{2} \left\{ K_i \Phi_i(\theta) + z k_i X_i(\theta) \right\} \tag{2.2a}
\]
\[
\sigma_2(r, \theta) = \sum_{i=1}^{2} \frac{1}{\sqrt{r}} \left\{ K_i \Psi_i(\theta) + z k_i \Gamma_i(\theta) \right\} \tag{2.2b}
\]

where

\[
K_i (i=1,2) = (K_1, K_2)
\]
\[
k_i (i=1,2) = (k_1, k_2)
\]

and the particular functions \(\Phi_i, X_i, \Psi_i\) and \(\Gamma_i\) depend only on the specific shell theory that is used. If the transverse shear deformation is taken into account one also has

\[
u_{2\text{III}}(r, \theta) = w = \sqrt{r} \left\{ K_{\text{III}} \Xi(\theta) \right\} \tag{2.3a}
\]
\[
\sigma_{2\text{III}}(r, \theta) = \frac{1}{\sqrt{r}} K_{\text{III}} Y(\theta) \tag{2.3b}
\]

The stress intensity factors in these expressions are determined by the load intensity, the boundary conditions, the geometry and the physical properties of the structure surrounding the crack. The solutions (2.2a-b) and (2.3a-b) are generally applicable as long as the problem remains within the realm of the theory of linear elasticity.

It is now noted that the function \(u_2\) varies with \(\sqrt{r}\) while the lowest order terms of \(u_1\) are proportional to \(r\). For the stresses a similar qualification holds. The crack tip solutions \(\sigma_2\) are
proportional to $1/\sqrt{r}$ while the leading term in $\sigma_1$ is a constant. Due to these singular $1/\sqrt{r}$ terms the crack tip solution is dominated by $u_2$ and $\sigma_2$, while away from the crack the solution field is represented by the composition (2.1a-b). At the crack tip $r = 0$ all derivatives of $u_2$ with respect to $r$ are singular, while away from the crack tip the solution is smooth with derivatives that are continuous in the entire remaining domain of the structure.

As noted [9,32,36,37,58,92,104,110] the separation of the general solution into two parts, i.e. into a local singular field plus a regular global field can be carried over into a finite element solution scheme in an almost natural way. The first step in this approach is to construct a representation of the global field $u_1$ in terms of a regular finite element distribution. The second step is to add to this field (locally) the functions $u_2$ so that the completion of the solution is obtained. The result of such a discretization is that the solution is represented in terms of a nodal displacement set $q$ belonging to the finite element distribution and a set $k$ of stress intensity factors that are coupled with the local singular fields (2.2a-b) and (2.3a-b). In this way the stress intensity factors appear as an additional set of computational freedoms. In other words, this technique produces the stress intensity factors as an integrated part of the set of computational freedoms.

The extension of the range of applications of the method to geometrically nonlinear problems requires some careful consideration with regard to the formulation of the superposition process. In the following sections we will discuss three major aspects of this approach that need clarification. The first concerns the choice of the local singular mode for the part of the deformation field corresponding to the stress intensity factors, i.e. the actual specification of the displacement functions $u_2$ and the requirements that we have to impose on the continuity of the modes $u_1$ and $u_2$ combined. Another issue which needs attention is the implementation of the local crack tip solution in a geometrically nonlinear environment with large rotations and translations. Finally, because the governing equations are nonlinear and because it is advantageous to keep the required changes in the solution algorithm at a minimum, it is necessary to discuss the way the governing equations are solved in the altered situation.

2.2.2 The singular crack tip solution

The cracks that we will consider in our research are (relatively long) through cracks that occur in thin-walled structures, such as the fuselage of an airliner. 'Long' here means that the ratio $\rho = a/t$ is typically: $\rho > 10-1000$ where $a$ is the half crack length and $t$ is the thickness of the shell in which the crack resides. Further the state of stress at the crack tip is assumed to be approximately a state of plane stress.

The state of deformation at the crack tip can be seen to be composed of a set of distinct fracture modes (fig. 2.2). These fracture modes must be defined in a way that is consistent with the plate theory used.
In the general case it is possible to distinguish between five separate modes, two extensional fracture modes, a transverse shear mode and two bending modes, as can be seen in figure 2.2. In the context of classical plate theory it is not possible to define these five modes independently. This is due to the simplifying assumptions on which classical plate theory is based [30,45,62]. We will discuss this point in more detail below.

Consider the edge of a plate or shell (fig. 2.3). The edge forces and moments that are defined in the classical plate theory consist of five separate components. These components are the in-plane resultants $T_n$ and $T_t$ normal and tangential to the shell edge respectively, the in-plane edge moments $M_n$ and $M_t$ which are aligned with $T_n$ and $T_t$ and the transverse shear resultant $Q$ directed along the normal of the shell mid-surface. This transverse shear resultant $Q$ is only introduced to be able to fulfill the moment equilibrium equations. In vector form we can lump these resultants together as

$$
T = T_n e_n + T_t e_t + Q n
$$

$$
M = M_n e_n + M_t e_t
$$

(2.4)

where $e_n$, $e_t$ and $n$ are the base vectors of an orthogonal triad which is aligned with the shell boundary. If a through crack in a shell is considered we can view the crack faces as boundary surfaces without loading. This implies that $T$ and $M$ should vanish along the centre lines of the free edges of the crack leading to 5 independent conditions according to (2.4).

When using a plate theory that includes transverse shear deformation the 5 degrees of freedom
\( (u_n, u_t, w, \phi_t, \phi_n) \) along the crack faces are independent. Consequently, it is possible to fulfill the 5 boundary conditions along the centre line of the crack, i.e. \( (T_t, T_n, Q, M_t, M_n) = 0 \).

However, in classical plate theory \( w \) and \( (\phi_t, \phi_n) \) are coupled and thus not independent. The rotations \( (\phi_t, \phi_n) \) are defined by the rotation of the normal to the midsurface of the plate (see fig 2.3) as

\[
\phi_t = -\frac{\partial w}{\partial x_1}
\]

\[
\phi_n = \frac{\partial w}{\partial x_2}
\]  

(2.5)

The consequences that this coupling of \( w \) and \( (\phi_t, \phi_n) \) has on the boundary conditions along the crack faces is worked out in more detail in [30,45,62]. It is shown that now only 4 conditions can be formulated, namely

\[
T_t = 0
\]

\[
T_n = 0
\]

\[
M_t = 0
\]

\[
Q^* = Q - \frac{\partial M_n}{\partial t} = 0
\]  

(2.6)

where \( Q^* \) is the so-called effective shear force.
Thus, within the context of classical plate theory there can only be 4 independent stress intensity factors \((K_1,K_{II},k_1,k_2)\) while for a plate theory that includes transverse shear deformation the number of independent modes is 5 \((K_1,K_{II},K_{III},K_{BI},K_{BII})\). This will have some impact on the shape of the crack tip solutions for the two different plate theories as we will see below.

For the two membrane fracture modes the crack tip solutions are identical for both theories. They can be written as follows [44].

**membrane mode I:**

\[
\begin{align*}
    u &= \frac{K_1}{\mu} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left( \frac{1 - v}{1 + v} + \sin^2 \left( \frac{\theta}{2} \right) \right) + O(r) \\
    v &= \frac{K_1}{\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left( \frac{2}{1 + v} - \cos^2 \left( \frac{\theta}{2} \right) \right) + O(r) \\
    w &= 0
\end{align*}
\]

(2.7)

**membrane mode II:**

\[
\begin{align*}
    u &= \frac{K_{II}}{\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left( \frac{2}{1 + v} + \cos^2 \left( \frac{\theta}{2} \right) \right) + O(r) \\
    v &= \frac{K_{II}}{\mu} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left( -\frac{1 - v}{1 + v} + \sin^2 \left( \frac{\theta}{2} \right) \right) + O(r) \\
    w &= 0
\end{align*}
\]

(2.8)

In these equations \(\mu\) represents the shear modulus and \(v\) is Poisson’s ratio.

Williams [108] was the first one to obtain the analytical bending displacement solutions for the classical plate theory. These solutions can be written as below.

**bending mode I:**

\[
\begin{align*}
    u &= -zk_1 \frac{\partial w_1}{\partial x} \\
    v &= -zk_1 \frac{\partial w_1}{\partial y} \\
    w &= w_1 = \frac{\sqrt{2} (1 + v) r^{3/2}}{3 + v} \left( \frac{1}{3} (7 + v) \cos \frac{3\theta}{2} - (1 - v) \cos \frac{\theta}{2} \right) + O(r^2)
\end{align*}
\]

(2.9)
bending mode II:

\[
\begin{align*}
    u &= -zk_2 \frac{\partial w_2}{\partial x} \\
    v &= -zk_2 \frac{\partial w_2}{\partial y} \\
    w &= w_2 = \frac{\sqrt{2} (1 + v) r^{3/2}}{(3 + v) E_t} \left( \frac{1}{3} (5 + 3v) \sin \frac{3\theta}{2} - (1 - v) \sin \frac{\theta}{2} \right) + O(r^2)
\end{align*}
\] (2.10)

For the transverse shear deformable plate theory one has in addition to the two bending modes also a tearing mode III. The bending and tearing displacement solutions for this plate theory can be written as follows [34,35].

bending mode I:

\[
\begin{align*}
    u &= \frac{K_{Bl}}{\mu} 2z \sqrt{2} \frac{r}{t} \cos \frac{\theta}{2} \left( \frac{1 - v}{1 + v} + \sin^2 \left( \frac{\theta}{2} \right) \right) + O(r) \\
    v &= \frac{K_{Bl}}{\mu} 2z \sqrt{2} \frac{r}{t} \sin \frac{\theta}{2} \left( \frac{2}{1 + v} - \cos^2 \left( \frac{\theta}{2} \right) \right) + O(r) \\
    w &= K_{Bl} f_{Bl} (r^{3/2}, \theta)
\end{align*}
\] (2.11)

bending mode II:

\[
\begin{align*}
    u &= \frac{K_{BII}}{\mu} 2z \sqrt{2} \frac{r}{t} \sin \frac{\theta}{2} \left( -\frac{2}{1 + v} + \cos^2 \left( \frac{\theta}{2} \right) \right) + O(r) \\
    v &= \frac{K_{BII}}{\mu} 2z \sqrt{2} \frac{r}{t} \cos \frac{\theta}{2} \left( \frac{1 - v}{1 + v} + \sin^2 \left( \frac{\theta}{2} \right) \right) + O(r) \\
    w &= K_{BII} f_{BII} (r^{3/2}, \theta)
\end{align*}
\] (2.12)

tearing mode III:

\[
\begin{align*}
    u &= v = 0 \\
    w &= \frac{K_{III}}{\mu} \sqrt{\frac{2r}{\pi}} \sin \frac{\theta}{2} + O(r)
\end{align*}
\] (2.13)
The angular variation of the bending displacement fields for the plate theory that includes transverse shear deformation is identical to that of the corresponding membrane modes. This correspondence, which does not exist in the classical theory, is the result of the inclusion of the transverse shear deformation in the formulation.

### 2.2.3 Relation between the Kirchhoff and Reissner stress intensity factors

Hui and Zehnder [38] showed that for a plate theory that includes transverse shear deformation the analytical displacement solutions for the two bending modes and the tearing mode are valid only within a boundary layer of thickness \( t/10 \). For thin-walled structures the bending displacement field away from the crack tip is characterized by the classical plate theory. This means that the stress intensity factors \( k_1 \) and \( k_2 \) corresponding to the classical plate theory uniquely determine the inner crack tip field for a plate theory that includes transverse shear deformation. Therefore the two sets of stress intensity factors \( (K_{B1}, K_{BII}, K_{III}) \) and \( (k_1, k_2) \) must be uniquely related to each other and this relation is universal in the sense that it is independent of the specimen geometry and loading configurations. Therefore we have

\[
\begin{align*}
k_1 &= f(K_{B1}) \\
k_2 &= f(K_{BII}, K_{III})
\end{align*}
\tag{2.14}
\]

Hui and Zehnder used both the J-integral [73] and the crack closure integral [83] to determine the relation between the two sets of stress intensity factors and they showed the results to be identical. The shear deformable plate theory that they used, the Reissner theory [34,35], assumes a parabolic through the thickness distribution of the transverse shear stresses and thus a stress free plate surface. The corresponding energy release rate for the tearing mode is calculated as

\[
G_c(K_3) = K_3^2 \frac{1 + \nu}{E} \frac{8}{15}
\tag{2.15}
\]

where \( K_3 \) is the tearing stress intensity factor for the Reissner theory.

In our research we used shell elements with a linear distribution of the transverse shear stresses through the thickness of the plate [34,35] thus violating the condition that the plate surface is stress free. The energy release rate that we obtain for the tearing mode is (appendix A)
\[ G_e(K_{III}) = K_{III}^2 \frac{1 + v}{E} \]  

(2.16)

It can be shown that the energy release rates for the two different assumptions for the transverse shear stress have to be identical. This yields for the mode III stress intensity factors

\[ K_3 = \sqrt{\frac{15}{8}} K_{III} \]  

(2.17)

The Kirchhoff stress intensity factors are calculated from the stress intensity factors for a shear deformable plate theory by equating the energy release rates. Consequently, these different assumptions for the transverse shear stress through the thickness of the plate will not have any influence on the Kirchhoff stress intensity factor \( k_2 \) as expected. 

The relations between the two sets of stress intensity factors are calculated in appendix A as

\[ k_1^2 = \frac{3 + v}{1 + v} K_{II}^2 \]  

(2.18a)

\[ k_2^2 = \frac{3 + v}{1 + v} K_{II}^2 + \frac{3(3 + v)}{\pi} K_{III}^2 \]  

(2.18b)

In our finite element calculations we can distinguish between 5 different modes of cracking because of the transverse shear deformable plate theory that we used. In section 2.4.3-4 it will be shown that our results for the bending and tearing stress intensity factors compare well with the analytical and numerical results for the classical Kirchhoff plate theory.

### 2.2.4 The enriched element domain

Another aspect which needs attention is the size of the domain in which mode enrichment is applied. As mentioned above the crack tip solutions for the two bending modes and the tearing mode as stated in equations (2.11-13) are only valid within a small boundary layer. Therefore mode enrichment also has to be restricted to this small boundary layer in order to obtain accurate results for the bending and tearing stress intensity factors. This will be discussed in more detail in section 2.3.3.

We will restrict the mode enrichment process to some (small) area around the crack tip. In a strict sense one should require that the summation of \( u_1 \) and \( u_2 \) remains \( C_0 \)-continuous everywhere in the domain. In practice this means that one should enforce this condition at the boundary of the enriched area. A simple way to accomplish this is to use a transition function. This transition function is equal to 1.0 for the enriched elements next to the crack tip. For the
transition elements though this function is equal to 1.0 next to the enriched elements and equal to 0.0 next to the conventional elements (fig. 2.4) while the descent of this transition function is assumed to be bilinear.

As already mentioned in the introduction, we want to apply the mode enrichment technique to the problem of a longitudinal crack in a pressurized fuselage. This implies that we are dealing with a curved surface. However, this difficulty is circumvented by restricting mode enrichment to a small area around the crack tip which is approximately flat. Therefore the possible curvature of a structure plays no role as far as the mode enrichment process is concerned.

### 2.2.5 Corotational formulation

The set of singular functions discussed in section 2.2.3 is taken from the linear elasticity solution for an infinite plate with a crack under general loading conditions. It is defined with respect to the orientation of the undeformed crack geometry, which, in the linear theory, does not appreciably change during the loading phase. When the problem is geometrically nonlinear, as is in our case, the crack front may undergo a finite displacement and rotation under the given load. In that case the set of shape functions $\mathbf{u}_2$ should be defined with respect to the rotated geometry. Consequently, in the nonlinear case it is not correct to simply add the analytical crack tip solution to the global solution. This can only be done after these special solutions are modified appropriately. The following development is included to make this issue clear.

Suppose that the shell midsurface in the undeformed state $B_0$ is described by the mapping (fig 2.5)

$$X = X(\xi) = y(\xi, 0)$$

(2.19)

After deformation, in state B, the shell midsurface is described by
Figure 2.5  Definition of the undeformed, the intermediate and the final deformed state

\[ x = x (\xi, \zeta) = y (\xi, \zeta) \]  \hfill (2.20)

where \( \zeta \) is the parameter that represents the intensity of the deformation or the time, whichever is appropriate. The inclusion of the evolution parameter \( \zeta \) in this discussion is purely for the sake of clarity of exposition.

The difference between the two states \( B_0 \) and \( B \) is the displacement field

\[ u (\xi, \zeta) = x (\xi, \zeta) - X (\xi) \]  \hfill (2.21)

The displacement \( u \) at the crack tip comprises the global field \( u_1 \) furnished by the finite element discretization

\[ u_1 = N^T (\xi) q (\zeta) \]  \hfill (2.22)

and the singular crack tip solution

\[ u_2 = \Theta (\xi) k (\zeta) \]  \hfill (2.23)
While the global field \( \mathbf{u}_1 \) extends itself over the whole domain, the singular field \( \mathbf{u}_2 \) is nonzero only in a small domain around the crack tip.

In equation (2.23) \( \Theta (\bar{\xi}) \) contains the shape functions corresponding to the stress intensity factors. These shape functions can be obtained from the analytical displacement solutions in equations (2.7-13). However, these solutions are not directly applicable to the nonlinear case because we have to take into account the rotation of the crack tip in space during the deformation process. If the analytical displacement solutions in the undeformed state are given by \( v (\bar{\xi}, \bar{\zeta}) \), the function \( \mathbf{u}_2 \) is given by

\[
\mathbf{u}_2 = \Theta (\bar{\xi}) \mathbf{k} (\bar{\zeta}) = \mathbf{R} (\bar{\xi}_T) v (\bar{\xi}, \bar{\zeta}) = \mathbf{R} (\bar{\xi}_T) v_0 (\bar{\xi}) \mathbf{k} (\bar{\zeta})
\]

(2.24)

where \( \mathbf{R} (\bar{\xi}_T) \) denotes the rotation at the crack tip (\( \bar{\xi}_T \) = the location of the crack tip) and \( v_0 (\bar{\xi}) \) are the analytical shape functions with respect to the undeformed crack tip zone. Note that the rotation matrix is dependent on the solution \( \mathbf{u}_1 (\bar{\zeta}) \) so that one could write

\[
\mathbf{u}_2 = \mathbf{R} (\bar{\xi}_T, \bar{\zeta}) v (\bar{\xi}, \bar{\zeta}) = \mathbf{R} (\bar{\xi}_T, \bar{\zeta}) v_0 (\bar{\xi}) \mathbf{k} (\bar{\zeta})
\]

(2.25)

The configuration in the deformed state can be seen to be attained in two stages, the first stage by the displacement \( \mathbf{u}_1 \), the second by the addition of \( \mathbf{u}_2 \). In this way we identify an intermediate state B' described by

\[
x' = \mathbf{X} + \mathbf{u}_1 = y (\bar{\xi}, 0) + \mathbf{u}_1 (\bar{\xi}, \bar{\zeta})
\]

(2.26)

The base vectors that belong to the undeformed state are given by

\[
A_i = \frac{\partial}{\partial \bar{\xi}_i} v (\bar{\xi}, 0)
\]

(2.27)

In the intermediate state the base vectors are

\[
b_i = \frac{\partial}{\partial \bar{\xi}_i} x' (\bar{\xi}, \bar{\zeta}) = A_i + \frac{\partial}{\partial \bar{\xi}_i} (\mathbf{u}_1 (\bar{\xi}, \bar{\zeta})) = A_i + \mathbf{u}_{1,i}
\]

(2.28)

Note that a comma followed by an index (i) denotes partial differentiation with respect to \( \bar{\xi}_i \).

In our formulation we use the Green-Lagrange strains, which can be written as

\[
G_{ij} = \frac{1}{2} \{ (A_i)^T \mathbf{u}_{1,j} + (\mathbf{u}_{1,i})^T A_j + (\mathbf{u}_{1,j})^T \mathbf{u}_{1,i} \}
\]

(2.29)
Substitution of the composite field \( u_1 + u_2 \) gives

\[
G_{ij}(u_1, Rv) = \frac{1}{2} \left\{ (A_i)^T u_{1,i} + (u_{1,i})^T A_i + (u_{1,i})^T v_{1,j} + (A_i + u_{1,i})^T Rv_{i,j} \right. \\
\left. + (Rv_{i,j})^T (A_j + u_{1,j}) + (Rv_{i,j})^T Rv_{i,j} \right\} \tag{2.30}
\]

or

\[
G_{ij}(u_1, Rv) = G_{ij}(u_1) + \frac{1}{2} \left\{ (A_i + u_{1,i})^T Rv_{i,j} + (Rv_{i,j})^T (A_j + u_{1,j}) \right. \\
\left. + (v_{i,j})^T R^T Rv_{i,j} \right\} \tag{2.31}
\]

Substituting equation (2.28) into (2.31) and making use of the orthogonality of the rotation matrix \( R \) (thus \( R^T R = I_3 \)) yields

\[
G_{ij}(u_1, Rv) = G_{ij}(u_1) + \frac{1}{2} \left\{ (b_i)^T Rv_{i,j} + (Rv_{i,j})^T b_j + (v_{i,j})^T v_{i,j} \right\} \tag{2.32}
\]

Equation (2.32) can be rewritten as

\[
G_{ij}(u_1, Rv) = G_{ij}(u_1) + G^\ast_{ij}(u_1, Rv) \tag{2.33}
\]

where

\[
G^\ast_{ij}(u_1, Rv) = \frac{1}{2} \left\{ (b_i)^T Rv_{i,j} + (Rv_{i,j})^T b_j + (v_{i,j})^T v_{i,j} \right\} \tag{2.34}
\]

\( G_{ij}^\ast \) describes the increase of the deformation in going from \( B' \) to \( B \). It follows from the foregoing that \( G_{ij}^\ast \) is only locally defined, i.e. in a small area \( \Omega^\ast \) around the crack tip. Everywhere outside this area \( G_{ij}^\ast \) is equal to zero. As long as the initial curvature and the deformation \( G_{ij}(u_1) \) remain small in \( \Omega^\ast \), which is a reasonable assumption, the variation of \( b_i \) over this area will also be small. Therefore, we can replace the base vectors \( b_i(\xi) \) belonging to state \( B' \) in equation (2.34) by base vectors \( b_i(\xi_T) = b_{i(\text{cr})} \) that are attached to a point determined by \( \xi_T \) in the region \( \Omega^\ast \) (for which the node at the crack tip presents itself as a natural candidate).

Let \( v \) be specified by
\[ v = v_k(\xi) A(\xi) \xi \kappa = v_k(\xi) A_{k(cr)} \] (2.35)

where we assume that the base vectors \( A_{k(cr)} \) at the crack tip for the undeformed state are unit vectors and mutually orthogonal. Substitution in equation (2.34) yields

\[
G_{ij}^*(u_1, Rv) = \frac{1}{2} \left\{ (b_{i(\text{cr})})^T RA_{k(cr)} v_{k,j} + v_{k,i} (RA_{k(cr)})^T b_{j(\text{cr})} + v_{k,i} v_{k,j} \right\} \] (2.36)

The base vectors \( b_{i(\text{cr})} \) are obtained from the base vectors \( A_{k(cr)} \) by a rotation, which we identify with \( R \), and a small deformation caused by the field \( u_1 \). It is now natural and consistent with our previous assumptions to neglect the deformation of the triad \( A_{k(cr)} \) and replace \( b_{i(\text{cr})} \) by \( RA_{k(cr)} \). The additional strains \( G_{ij}^* \) that arise can then be simplified to the expressions

\[
G_{ij}^*(u_1, u_2) = G_{ij}^*(u_1, Rv) = \frac{1}{2} (v_{i,j} + v_{j,i} + v_{k,i} v_{k,j}) \] (2.37)

Comparing this additional Green-Lagrange strain \( G_{ij}^* \) with the (truncated) strain field derived on the basis of linearized elasticity theory one can see that the only difference lies in the nonlinear term \( v_{k,i} v_{k,j} \). The effect of this nonlinear term on the total additional strain \( G_{ij}^* \) remains small as long as the displacement gradients are small. However, when approaching the crack tip these displacement gradients will increase, and so will the nonlinear term \( v_{k,i} v_{k,j} \). This increase of \( v_{k,i} v_{k,j} \) will cause a discrepancy between the truncated strain field and the Green-Lagrange strain field.

The concept of singular stress and strain field around the crack tip is derived on the basis of linearized elasticity theory which is formulated in terms of small displacement gradients. Actually this presents a contradiction in terms because the stresses and strains will go to infinity when closing in to the crack tip. This contradiction is generally accepted as a modeling error that occurs only in a very small neighborhood of the crack tip. However, to be consistent with the singular stress and strain field for the linearized theory the additional Green-Lagrange strain in \( G_{ij}^* \) also has to be truncated. This linearization, which is easy to apply to equation (2.37), actually represents a linearization of the in-plane strain field carried out in the rotated orientation of the crack tip. The additional (truncated) Green-Lagrange strain \( G_{ij}^* \) that has to be implemented therefore is

\[
G_{ij}^*(u_1, u_2) = \frac{1}{2} (v_{i,j} + v_{j,i}) = \frac{1}{2} (v_{0,i,j} + v_{0,j,i}) k(\xi) = \Theta_{0ij}(\xi) k(\xi) \] (2.38)
It is obvious that $\Theta_{0ij}(\xi)$ denotes the derivatives of the analytical shape functions (equations (2.7-13)) with respect to the undeformed crack tip zone.

The additional strains caused by insertion of the classical solutions for the crack tip behavior in the global representation of the displacements are thus obtained by a direct evaluation of the Green-Lagrange functional expressions in terms of the local components of this displacement field. It is further noted that the two fields $\mathbf{u}_1$ and $\mathbf{u}_2$ are not coupled in the strain measures, but only in the specific elastic energy $E = 0.5 (G_{ij} C_{ijkl} G_{kl})$, at least to an acceptable degree of approximation.

A further justification of the assumptions of small initial curvature and small deformation inside $\Omega^*$ for the intermediate state is given in appendix B.

2.2.6 The resulting nonlinear equations

The nonlinear equations that result from the foregoing definitions can be written as

$$F(q, k, \lambda) = 0$$

(2.39)

where $\lambda$ represents the intensity of the loading, $k$ is the vector of stress intensity factors and $q$ is the vector of total nodal displacements.

These total nodal displacements for the enriched elements are obtained by summation of the analytical displacement solution corresponding to the stress intensity factors and the underlying conventional finite element displacement field (fig. 2.6). Thus

![Figure 2.6](image)

**Figure 2.6** Separation of the analytical displacement solution corresponding to the stress intensity factors and the underlying finite element displacement field
\[ q_i = q_i + R y_i k \]  
(2.40a)

or

\[ q_i = q_i - R y_i k \]  
(2.40b)

where

\( y_i \) are the nodal values of the analytical displacement solution corresponding to the stress intensity factors,

\( q_i \) are the nodal displacements determined by the underlying finite element displacement formulation,

\( q_i \) are the total nodal displacements,

\( R \) represents the rotation of the crack tip with respect to the undeformed state and

\( k \) is the vector of stress intensity factors.

The total displacement field can be written now as

\[
\mathbf{u} = N^T(\xi) q_i + R y k \\
= N^T(\xi) q_i + R (v - N^T(\xi) y_i) k \\
= N^T(\xi) q_i + \Theta_m(\xi) k
\]  
(2.41)

where \( N^T(\xi) \) and \( \Theta_m(\xi) \) denote the finite element shape functions and a modified version of the analytical shape functions respectively.

Returning to equation (2.39) we decompose this set of nonlinear equations in the form

\[
f(q_i, \lambda) + f^*(q_i, k, \lambda) = 0 \]  
(2.42a)

\[
h^*(k, \lambda) = 0 \]  
(2.42b)

where

\[ f, f^* \in R_N \quad ; \quad h^* \in R_k \]

The first term in equation (2.42a) corresponds to the contributions of the finite element discretization scheme. The two other terms in these equations arise when we apply the enrichment scheme described in the previous sections. Equations (2.42a-b) can be seen to be derived from the requirement that the total potential energy of the structure with its loads must
be stationary. The contributions $f, f^*$ and $h^*$ are then computed from

$$f = \frac{\partial P}{\partial q_i} ; \quad f^* = \frac{\partial P^*}{\partial q_i} ; \quad h^* = \frac{\partial P^*}{\partial k}$$

(2.43)

where the potential energy is given in two parts.

$$P_{total} = P(q_i, \lambda) + P^*(q_i, k, \lambda)$$

(2.44)

The first term in equation (2.44) refers to the global part of the formulation while the second term $P^*$ is due to the mode enrichment process.

### 2.2.7 Solution of the corrector equations

The nonlinear equations (2.42a-b) are solved for a range of values of the loading depending on the particular requirements for the analysis of the problem at hand. This solution process is carried out step by step by variation of the load parameter $\lambda$ or, more generally, by variation of a generalized path parameter $\eta$. It is characteristic for these type of procedures that each new step along the loading path is carried out by a predictor corrector scheme [77].

The addition of the modes (2.7-8) and (2.11-13) to the regular discretization scheme, however, has an impact on the solution algorithm that needs some clarification. The changes can best be described at the level of the corrector equations.

The solution of the corrector equations in each new step of the step by step solution algorithm furnishes sequentially a set of corrections $\Delta \tau^i : i = 1, 2, 3 \ldots$ to the predicted solution $\tau^0$ of the equation set (2.39). This system of equations is usually a linearization of (2.39), and for the sake of simplicity we will assume that this linearization corresponds to Newton’s method.

The governing equations on which the operations are carried out are first written as

$$H(t, \eta) = 0$$

(2.45)

where

$$t = \begin{bmatrix} q \\ k \\ \lambda \end{bmatrix} \in \mathbb{R}_N + \mathbb{R}_k + 1 \quad H = \begin{bmatrix} F(t) \\ g(t, \eta) \end{bmatrix}$$

(2.46)
The extra equation that is introduced here is a device to introduce the path parameter $\eta$ that measures the distance between the solution points along the loading path.

Newton's method is based on the linearization

$$H_\lambda (t, \eta) \Delta \tau = -H(t, \eta)$$

(2.47)

which we call the Newton form of equation (2.39). The configuration $\tau$ is here some approximation to the actual solution, while $\Delta \tau$ stands for the correction to this approximation at the prescribed value of $\eta$ that determines a particular point on the loading path.

In matrix form equation (2.47) can be written as

$$\begin{bmatrix} S_{11} & S_{12} & F_\lambda \\ S_{12}^T & S_{22} & F_{k\lambda} \\ n_q & n_k & \eta_0 \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta k \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} f_i \\ f_{k\lambda} \\ 0 \end{bmatrix}$$

(2.48)

where

$S_{11}$ is the global stiffness matrix corresponding to the conventional finite element displacement formulation,

$S_{12}$ is the global (sparse) stiffness matrix containing the coupling terms between the conventional displacement formulation and the analytical crack tip displacement solution,

$S_{22}$ is the global stiffness matrix containing terms due to the analytical crack tip displacement solutions only,

$F_\lambda$ is the vector of nodal forces corresponding to the conventional displacement field,

$F_{k\lambda}$ is the load vector corresponding to the analytical crack tip displacement solutions,

$n_q$ is the tangential vector to the solution field for the current load,

$n_k$ is the 'stress intensity'-part of the total tangential vector,

$n_0$ is a constant that defines the auxiliary surface needed in the continuation procedure,

$\Delta k$ is the correction of the stress intensity factor(s),

$\Delta q$ is the correction of the nodal displacements,

$\Delta \lambda$ is the correction of the load step size,

$f_i$ is the vector of residual forces corresponding to the conventional displacement field and

$f_{k\lambda}$ is the vector of residual forces corresponding to the analytical crack tip displacement solutions.

The stiffness terms $S_{11}, S_{12}$ and $S_{22}$ in equation (2.48) can be derived from the total potential energy as described in section 2.2.6. Together with the kinematic relations described in
section 2.2.5 this yields

\[ S_{11} = \int_V \mathbf{B}_L^T \mathbf{C} \mathbf{B}_L \, dV + \int_V \mathbf{B}_{NL}^T \Sigma_t \mathbf{B}_{NL} \, dV \]

\[ S_{12} = \int_V \mathbf{B}_L^T \mathbf{C} \Theta_{mij} \, dV \]

\[ S_{22} = \int_V \Theta_{mij}^T \mathbf{C} \Theta_{mij} \, dV \]  

(2.49)

In these equations \( \mathbf{B}_L \) and \( \mathbf{B}_{NL} \) denote the strain displacement relations that are used for constructing the linear and geometric stiffness matrix \([6]\). Furthermore, \( \mathbf{C} \) represents the material stiffness matrix while \( \Sigma_t \) denotes the second Piola Kirchhoff stresses \([21]\). \( \Theta_{mij} \) contains the derivatives of the modified analytical shape functions corresponding to the stress intensity factors (equation (2.41)). As already explained in section 2.2.5 (see equation (2.38)) \( \Theta_{mij} \) is calculated with respect to the undeformed crack tip zone.

It is obvious that \( S_{12} \), which incorporates the coupling terms between the conventional displacement formulation and the analytical displacement solutions, will contain a considerable number of nonzero values because a number of elements around the crack tip are involved in the enrichment process.

Entering this sparse matrix together with the conventional global stiffness matrix in an equation solver would be inconvenient because it would destroy the bandwidth. Therefore the corrector equations (2.48) will be solved with a Block-Gauss elimination scheme as follows.

First equation (2.48) is written as

\[ S_A \Delta a + S_B \Delta b = r_a \]  

(2.50a)

\[ S_C \Delta a + S_D \Delta b = r_b \]  

(2.50b)

where

\[ S_A = \begin{bmatrix} S_{11} \end{bmatrix} \]

\[ S_B = \begin{bmatrix} S_{12} F_{\lambda} \end{bmatrix} \]

\[ S_C = \begin{bmatrix} S_{12}^T \\ n_q \end{bmatrix} \]

\[ S_D = \begin{bmatrix} S_{22} F_{k\lambda} \\ n_k \\ n_0 \end{bmatrix} \]

\[ \Delta a = \begin{bmatrix} \Delta q \end{bmatrix} \]

\[ \Delta b = \begin{bmatrix} \Delta k \\ \Delta \lambda \end{bmatrix} \]
\[ r_a = \begin{bmatrix} f_1 \\ f_{k\lambda} \\ 0 \end{bmatrix} \quad r_b = \begin{bmatrix} f_{k\lambda} \\ 0 \end{bmatrix} \]  

(2.51)

Working out equation (2.50a) yields

\[ \Delta a = S^{-1}_A r_a - S^{-1}_A S_B \Delta b \]  

(2.52)

and substitution of equation (2.52) into equation (2.50b) will result in

\[ S_C S^{-1}_A r_a - S_C S^{-1}_A S_B \Delta b + S_D \Delta b = r_b \]  

(2.53)

or

\[ (S_D - S_C S^{-1}_A S_B) \Delta b = r_b - S_C S^{-1}_A r_a \]  

(2.54)

From this equation \( \Delta b \) can be calculated and \( \Delta a \) is obtained from equation (2.52) by substitution of \( \Delta b \).

In the mode enrichment process, the stress intensity factors are calculated by extracting the corresponding displacement solutions from the total displacement field around the crack tip. The accuracy of this extraction can be checked by requiring orthogonality between the analytical displacement solutions and the underlying finite element displacement field. It can be shown that in this way a (small) correction of the calculated stress intensity factors can be obtained (appendix C) and some tests showed that the magnitude of this correction was always of the order of O(2%) (at most).

### 2.2.8 A posteriori mode enrichment

In section (2.2.5) it was shown that the difference between the intermediate state \( B' \) and the final deformed state \( B \) is furnished by the inclusion of the analytical displacement solution \( u_2 \) in the total displacement field. This analytical displacement solution is only locally defined and therefore its influence on the total displacement solution is restricted to a small area around the crack tip. Actually, the enrichment process represents a linear technique applied to a small part of a geometrically nonlinear deformed structure which gives rise only to small changes in the general solution. Therefore one can apply mode enrichment as an additional step after the nonlinear state of deformation of the structure described by the global field \( u_1 \) is completed. It turns out that because of the linear character of the mode enrichment process, the stress intensity factors and thus the final deformed state is obtained within a few iterations.
only. This technique is particularly useful for cases where the stress intensity factors do not need to be calculated at each point of the load-deformation path, but only for the final load value.

There is yet another reason why this a posteriori technique is attractive to use. As will be described in section 2.3.2, a higher order Gaussian integration scheme has to be applied for the enriched elements to get accurate solutions for the stress intensity factors. This means that the formation of the stiffness matrices for the enriched elements is rather time consuming due to the high number of integration points and the evaluation of the additional stiffness terms in these integration points. A posteriori application of the mode enrichment technique is thus an effective means to reduce the volume of the computation because the higher order integration scheme has to be applied only once (namely after the nonlinear deformation of the structure is completed).

2.3 Implementation

2.3.1 The B2000 finite element code

The mode enrichment technique described in the previous chapter was implemented in the B2000 finite element code that is operative at the Department of Structures and Materials of the Faculty of Aerospace Engineering of the Technical University of Delft. B2000 is a database oriented software package [56,57] that consists of a number of separate modules (called processors) which communicate with a common database. Some of these modules are e.g. an element processor, a matrix assembler and an equation solver. This means that a complete analysis of a certain problem can be performed by calling the required processors one after the other.

We implemented the mode enrichment technique in the quadrilateral nonlinear Bathe element [6]. This isoparametric plate element has 4, 8 or 9 nodes and is based on a theory that includes transverse shear deformation. It has 5 degrees of freedom per node namely three translations and two rotations about two in-plane axes, and it is suitable to describe small deformations under large displacements and rotations.

2.3.2 Numerical integration

In finite element analysis the integrals involved in the equilibrium equations are evaluated numerically. The type and order of this numerical integration procedure has to be chosen in such a way that an acceptable level of accuracy is achieved for the finite element results (including the stress intensity factors).
A widely used numerical integration technique is the so-called Gaussian quadrature procedure [6]. In this procedure the position and the weight of the integration points is optimized in such a way that the integrations involved in the finite element calculation are performed with a high degree of accuracy. These accurate integrations can be achieved with a low order (2x2 or 3x3) integration scheme depending on the type of element used (4 or 8/9 nodes). The integration will be exact if the concerning element is rectangular.

The enriched elements, however, also contain shape functions corresponding to the analytical crack tip solutions for the different modes of cracking. Due to these additional shape functions a low order integration scheme is not accurate enough for these elements. Benzley [9], for instance, used a 7x7 Gaussian integration scheme for the enriched elements, while Hepler and Hansen [36] applied 9x9 integration. We performed a number of calculations for the centre cracked plate (see section 2.4.1) with an increasing order of integration and we found that converged solutions for the stress intensity factors can be obtained for an integration order of 7 (or higher).

As already mentioned in section 2.2.4 enrichment is restricted to a small domain around the crack tip and a logical choice is to let the boundary of this domain coincide with element boundaries. It was found that the accuracy of the calculated stress intensity factors is determined by the enriched crack tip elements only. The shape functions of the patch of elements immediately surrounding the crack tip elements are capable of describing the analytical solution away from the crack tip quite accurately. Therefore, enrichment of these elements will not have a noticeable influence on the calculated stress intensity factors. This finding is in agreement with the results found in literature [9,36,37]. It also became clear that the influence of the transition zone on the accuracy of the calculated stress intensity factors is very small (within the order of 1.0 percent).

This made us decide only to enrich the elements that have one node in common with the crack tip for the examples in section 2.4 and 2.5 and to use the surrounding layer of elements as transition zone between the enriched and conventional element patches. Therefore the higher order integration scheme can be restricted to a small number of elements in the enriched and transition element patches. Further, we decided to use a 10x10 Gaussian integration scheme for the enriched and transition elements in all the examples in sections 2.4-5 which guarantees that we will obtain converged solutions for the stress intensity factors.

2.3.3 The size of the enriched element domain

Membrane stress intensity factors calculated for different element meshes showed that the size of the crack tip elements did not have much influence on the accuracy of the results. This finding is in accord with the conclusion reached by Heyliger and Kriz [37].

As far as bending is concerned, we can only calculate the classical bending stress intensity
factors \( k_1 \) and \( k_2 \) indirectly after determining the bending and tearing stress intensity factors \( K_{BI}, K_{BII} \) and \( K_{III} \). The reason for this must be sought in the properties of the elements that we used in the present development. The stress intensity factors according to the Kirchhoff representation can be calculated from the stress intensity factors for a shear deformable plate theory via equations (2.18a-b).

Extensive testing with the formulation based on a shear deformable plate theory showed that the size of the crack tip elements is of great importance for the accuracy of the results. As it turned out, accurate results can only be obtained if the size \( l \) is taken of the order of magnitude of \( l \leq O(t/2) \), where \( l \) is the length of an element side. In a rough way this finding is in agreement with the observations of Hui and Zehnder [38] who predicted that the integration zone should be of the order of magnitude of the zone in which the Reissner boundary layer of the solution differs from the Kirchhoff solution. This is a layer with a depth of the order of magnitude of \( l \leq O(t/10) \). With the choice of \( l \leq O(t/2) \) the error in the computed \( K_{BI}, K_{BII} \) and \( K_{III} \) factors was no greater than 2%, in comparison with well known benchmark solutions reported in the literature.

The reader will have noted from the previous discussion that for the bending terms, it is necessary to shrink the area of each crack tip element to approximately one half -times- one half of the wall thickness. The total area occupied by the crack tip elements is then approximately the square of the wall thickness. This is a very small patch, but in our experience it does not give rise to difficulties in the development of the remaining element distribution. It turned out that it is very easy to blend the small crack tip elements in a surrounding of larger elements without impairing the accuracy of the results. The possibility to do this must be due to the accuracy of the higher order (8- or 9-node) elements that we used throughout our research on crack behavior. An illustration of a mesh with these small elements is given in figure 2.13.

2.4 Calibration tests on some classical problems

2.4.1 Mode I stretching

The first classical problem concerns a centre cracked plate loaded in the direction perpendicular to this crack (fig. 2.7). The out of plane displacements and the rotations about the two in-plane axes are restrained. The applied finite element mesh is plotted in figure 2.8. The theoretical solution for \( K_1 \) for an infinite plate is

\[
K_1 = \sigma \sqrt{\pi a}
\]  

(2.55)
Figure 2.7 One quarter of a centre cracked plate subjected to a tensile load (crack length from $a_1 = 50$ mm to $a_2 = 150$ mm with steps of 25 mm)

Figure 2.8 Finite element mesh of a quarter of the centre cracked plate (see fig. 2.7)
Figure 2.9  Membrane mode I stress intensity factor versus the crack length for the centre cracked plate (see fig. 2.7)

where

$\sigma$ is the tensile stress at infinity and

$a$ is the half crack length.

Because of the finite width of the plate a correction has to be applied to this value of $K_I$. In literature one can find a number of correction formulas [25] of which we will use the one defined by Feddersen.

$$K_I = C_F \sigma \sqrt{\pi a}$$  \hspace{1cm} (2.56)

where

$$C_F = \sqrt{\sec \left( \frac{\pi a}{W} \right)}$$  \hspace{1cm} (2.57)

and $W$ is the total plate width.

$K_I / K_{I(\text{inf})}$ is calculated for several values of the crack length, varying from $a_1 = 50$ mm to $a_2 = 150$ mm with steps of 25 mm. As can be seen in figure 2.9 the results are in close agreement with the expected values obtained with Feddersen's correction formula.
2.4.2 Combined mode I and mode II stretching

Next the crack is placed under an angle of 45° to the loading direction. Note that, because of the absence of symmetry or antisymmetry, the total plate has to be considered. The dimensions and material properties of the plate are given in figure 2.10.

![Diagram of a plate with a central crack under an angle β subjected to a tensile load]

**Figure 2.10** Plate with a central crack under an angle β subjected to a tensile load

For an infinite plate the variation of $K_I$ and $K_{II}$ with $\beta$, where $\beta$ is the angle between the crack faces and the loading direction, can be written as [88]

$$K_I(\beta) = (\sin^2\beta) \sigma \sqrt{\pi a} \quad (2.58a)$$

$$K_{II}(\beta) = (\sin \beta \cos \beta) \sigma \sqrt{\pi a} \quad (2.58b)$$

Because of the finite width of the plate, correction factors have to be applied to both $K_I$ and $K_{II}$. These correction factors can be found in a handbook [59] and their accuracy is claimed to be better than 0.5%. Applying these factors to equations (2.58a-b) yields

$$K_I(45^\circ) = 8.31 \sigma \quad (2.59a)$$

$$K_{II}(45^\circ) = 7.80 \sigma \quad (2.59b)$$
Figure 2.11  Finite element mesh of the plate with a central crack under an angle $\beta$ (see fig 2.10)

We calculated the stress intensity factors with the mesh plotted in figure 2.11 and found

$$K_I (45^\circ) = 8.33 \sigma$$ \hspace{1cm} (2.60a)

$$K_{II} (45^\circ) = 7.89 \sigma$$ \hspace{1cm} (2.60b)

which is a very satisfactory result.

2.4.3 Mode I bending

For the validation of the mode I bending stress intensity factor we analyzed the same problem as Hui and Zehnder [38], namely a square centre cracked plate loaded by a uniform bending moment $M$ (fig. 2.12). The finite element mesh that was used for this problem is pictured in figure 2.13. The length of the sides of the 4 crack tip elements is approximately 0.25 mm.

The analytical solution for this problem was first obtained by Sih et al. [88]. They found for the classical bending mode I stress intensity factor

$$k_I = \frac{6M}{l^2 \sqrt{a}}$$ \hspace{1cm} (2.61)
Figure 2.12 Plate with a central crack subjected to a bending moment \( M \) or a transverse shear load \( Q \)

Figure 2.13 Finite element mesh of the plate defined in figure 2.12. Due to symmetry only half of the plate has to be analyzed.
For the current plate dimensions this yields

\[ k_1 = 42.43 \text{ M} \]  \hspace{1cm} (2.62)

where M is expressed in N (moment per unit width). Hui and Zehnder used the virtual crack extension procedure [64], a standard postprocessing tool in the STAGS computer code [69], and found

\[ k_1 = 42.20 \text{ M} \]  \hspace{1cm} (2.63)

This value is \(-0.5 \%\) in error as compared to equation (2.62). The result that we obtained indirectly by means of the implementation based on a shear deformable plate theory is

\[ k_1 = 42.86 \text{ M} \]  \hspace{1cm} (2.64)

The agreement between the result in equation (2.62) and ours is thus excellent (deviation of \(-1.0 \%\)!)!

### 2.4.4 Mode II bending and tearing

The final test problem concerns combined mode II bending and tearing of a centre cracked plate. The dimensions are the same as in the previous section but the plate is now subjected to a transverse shear load Q instead of a bending moment M (fig. 2.12). The analytical solution for this problem was first determined by Sih et al. [88] and later on corrected by Hui and Zehnder [38]. This analytical solution looks as follows

\[ k_2 = \frac{3Q}{t^2} a \sqrt{a} \]  \hspace{1cm} (2.65)

For the current example this yields

\[ k_2 = 1060.66 \text{ Q} \]  \hspace{1cm} (2.66)

where Q is expressed in N/mm (force per unit width). The result that Hui and Zehnder computed is
\[ k_2 = 1040.40 \, Q \]  \hspace{1cm} (2.67)

which implies an error of -1.9\% compared to the analytical result. We used the same mesh as in the previous section and obtained

\[ k_2 = 1076.07 \, Q \]  \hspace{1cm} (2.68)

The deviation of our result from that of equation (2.66) is 1.5 \%.

### 2.5 Geometrically nonlinear problems

#### 2.5.1 Rotation of a plate with a central crack

As discussed in section 2.2.5, the results for the stress intensity factors should be unaffected by the finite rotations that the crack tip may experience. To test whether the present formulation obeys this requirement we conducted a numerical experiment whereby the centrally cracked plate under load (section 2.4.1) was rotated (by a finite angle) in space. The stress intensity factors were calculated before and after the rotation had taken place. No influence of the rotation on the computed K-factors could be detected.

#### 2.5.2 Buckling of a stretched plate with a central crack

The problem of buckling of a stretched plate with a central crack was already considered in a previous research by Riks et al. [80]. They showed that due to buckling the intensity of the membrane mode I crack tip singularity increases in comparison with the situation where the plate is forced to remain flat. Because buckling is accompanied by out of plane displacements a mode I bending crack tip singularity will develop. Riks et al. [80] argued that the stress intensity due to bending would be negligible compared to the membrane stress intensity.

The dimensions and boundary conditions of the plate considered here are the same as for the plate in section 2.4.1 (fig. 2.7) except that now the out of plane displacements and the rotations about the two in-plane axes are free. The crack length is set equal to 120.0 mm. To facilitate the transition from the primary path onto the postbuckled state, we introduced a small imperfection. This was accomplished by a point force of 0.05 N perpendicular to the plate at the midpoint P of the crack (see fig. 2.7). The tensile load perpendicular to the crack was applied as a uniform prescribed displacement v in y-direction.

First a calculation of \( K_t \) is performed with the finite element mesh given in figure 2.14. For the midpoint P of the crack (fig. 2.7), the in-plane displacement v and the out of plane
Figure 2.14  Finite element mesh of the centre cracked plate subjected to a tensile load (see fig.2.7). The out of plane displacements and rotations about the two in plane axes are free.

Figure 2.15  In plane displacement $v$ perpendicular to the crack for the midpoint $P$ of the crack versus the applied stress $\sigma$ (see fig. 2.7 and fig. 2.14)
**Figure 2.16** Out of plane displacement $w$ perpendicular to the crack for the midpoint $P$ of the crack versus the applied stress $\sigma$ (see fig. 2.7 and fig. 2.14)

**Figure 2.17** $K_I$ versus $\sigma$ for the centre cracked plate subjected to a tensile load perpendicular to the crack (see fig. 2.7 and fig. 2.14)
**Figure 2.18** Finite element mesh of the centre cracked plate used for the calculation of \( K_1 \) and \( k_1 \) (see fig. 2.7)

**Figure 2.19** \( K_1 \) and \( k_1 \) versus \( \sigma \) for the centre cracked plate subjected to a tensile load perpendicular to the crack (see fig. 2.7 and fig. 2.18)
Figure 2.20  \( k_1 \) versus \( \sigma \) for the centre cracked plate subjected to a tensile load perpendicular to the crack (see fig. 2.7 and fig. 2.18)

displacement \( w \) perpendicular to the crack are plotted versus the stress \( \sigma \) at the loaded edge (fig. 2.15-16). A similar plot for \( K_I \) is made in figure 2.17. Next, \( K_I \) and \( k_1 \) are calculated using the refined finite element mesh plotted in figure 2.18. We can see from the data in figure 2.19 that \( k_1 \) starts to grow once the buckling load is passed, but it remains small as compared to \( K_I \) for all load levels considered.

The agreement between our calculations and those reported in [80] is satisfactory but pertains only to the membrane part of the solution because bending stress intensity factors were not considered in [80]. To get an idea about the quality of the bending stress intensity factor \( k_1 \), we compared our solutions with the results obtained with the crack closure integral (see appendix A). As one can see in figure 2.20 the agreement between the results for \( k_1 \) produced by these two methods is close.
2.5.3 Bulging of a longitudinal crack in a pressurized fuselage

2.5.3.1 Historical review

Peters and Kuhn [66] observed that the residual strength of a pressurized fuselage with a longitudinal crack was considerably below the residual strength of a comparative cracked flat sheet. They described this effect as a result of the deflection and rotation of the crack edges due to the internal pressure. This reduction of the residual strength for cylindrical shells as compared to a flat sheet is called the 'bulge effect'. The ratio of the $K_I$-factors for the two configurations is expressed as the bulge factor $\beta_{\text{bulge}}$:

$$\beta_{\text{bulge}} = \frac{K_{I_{\text{curved}}}}{K_{I_{\text{flat}}}}$$ (2.69)

where

$K_{I_{\text{curved}}}$ is the stress intensity factor of a curved panel with a crack and

$K_{I_{\text{flat}}}$ is the stress intensity factor of an infinite flat panel made of the same material and with the same thickness, crack length and in-plane remote tensile stress perpendicular to the crack.

Pioneering contributions to the analytical solution of the problem of crack bulging were made by Folias [31], Erdogan and Kibler [23] and Barrois [5]. The formulas for the bulge factor $\beta_{\text{bulge}}$ that they obtained are

Folias [31]:

$$\beta_{\text{bulge}} = \sqrt{1 + 0.317 \lambda_f^2}$$ (2.70)

where

$$\lambda_f = \frac{4\sqrt{12} (1 - v^2)}{\sqrt{R_t}} a$$ (2.71)

Erdogan and Kibler [23] and Barrois [5]:

...
\[ \beta_{\text{bulge}} = 0.64 + 0.49 \lambda _{r} \]  

(2.72)

These analytical bulge factors are not affected by the internal pressure \( p \) and thus do not incorporate the nonlinear dependence of the bulge factor on the internal pressure. The first nonlinear approach of the problem of a cracked pressurized fuselage was made by Lemaitre et al. [48]. They recognized that the bulge factor \( \beta_{\text{bulge}} \) was dependent on the internal pressure \( p \). This dependence can be explained as follows.

For small load values, the out of plane displacement of the crack edges will be relatively large because the bending stiffness does not contribute much to the crack opening resistance in this direction. When the load increases the (nonlinear) coupling between out of plane and in-plane displacements will cause the resistance against further opening to increase. The radial displacement can only continue to grow if the fibers along the crack edges become longer and/or the crack tips move towards each other. However, there is a considerable resistance against fibre stretching due to the in-plane material stiffness, and this resistance is further amplified by the tensile component in axial direction in the far field loading.

This reasoning makes it obvious that the bulge factor will decrease for increasing load. Further it underscores the choice for a code with appropriate nonlinear capabilities for studying the behavior of longitudinal cracks in pressurized fuselages.

### 2.5.3.2 Application of the mode enrichment technique

To present a particular example we will now focus on a model of a cracked cylindrical shell segment previously treated by Riks and den Reijer [81] and by Riks et al. [78]. The dimensions and boundary conditions of the configuration are shown in figure 2.21. This model is a pressurized thin-walled cylindrical shell with a periodic distribution of longitudinal and circumferential cracks. Due to symmetry, it is only necessary to analyze a small segment of the total shell (fig. 2.21). Along boundary 3 the shell segment is forced to keep its cylindrical shape. This can be seen as a first approximation to model the influence of the ring stiffeners that are normally present in a fuselage.

First, a calculation of \( K_1 \) alone is performed with the finite element mesh shown in figure 2.22. In figure 2.23 the radial displacement of the midpoint \( P \) of the crack (see fig. 2.21) is plotted versus the load factor \( \lambda \). As one can see, the current results compare well with the results obtained by Riks and den Reijer [81]. It is interesting to see the influence of the tensile axial stresses on the radial displacements. They tend to reduce the radial displacement of the midpoint of the crack, and thus it is reasonable to assume that they will also reduce the bulge factor for an increasing load factor. Indeed in figure 2.24 one can see that the bulge factor will decrease for an increasing pressure because of the growing resistance against crack opening.
**Figure 2.21** Part of a fuselage with a longitudinal crack subjected to internal pressure
**Figure 2.22** Finite element mesh of a segment of the fuselage loaded by internal pressure (see fig. 2.21)

**Figure 2.23** Radial displacement at the midpoint P of the crack versus the load factor $\lambda$ for the cracked fuselage segment (see fig. 2.21)
Figure 2.24  Bulge factor $\beta_b$ versus the load factor $\lambda$ for the cracked fuselage segment (see fig. 2.21)

Figure 2.25  Finite element mesh for the cracked fuselage segment used for the calculation of the bending stress intensity factor $k_1$ (see fig 2.21)
Figure 2.26 Mode I bending stress intensity factor $k_1$ versus the load factor $\lambda$ for the cracked fuselage segment (see fig. 2.21)

Riks and den Reijer [81] only reported the bulge factor for a load factor of 1.0 and as can be observed this value agrees well with the present results.

The linear solutions obtained by Folas [31] and Erdogan and Kibler [23] and Barrois [5] are plotted in the same figure. According to the previous section linear conditions are valid only for a vanishing load. Indeed it can be seen that the analytical (linear) solutions found in the literature compare well with our results for the bulge factor corresponding to a very small load factor.

Finally the mode I bending stress intensity factor is calculated after refining the crack tip elements (fig. 2.25). The results obtained with the mode enrichment technique are in close agreement with the results calculated with the crack closure integral (fig. 2.26).

2.6 Conclusion

The mode enrichment technique, that was first described by Benzley [9] for linear problems, is implemented for the calculation of stress intensity factors for geometrically nonlinear crack problems. A number of linear and geometrically nonlinear problems were analyzed to test the method, and very good results were obtained.
For the membrane mode I and mode II stress intensity factors, one can do with a rather coarse element mesh (fig. 2.8, 2.11, 2.14 and 2.22). For plate bending problems, the size of the enriched area has to be adapted to the plate thickness (fig. 2.13, 2.18 and 2.25). As it turned out, the size $l$ of the crack tip elements has to be taken of the order $l \leq O(t/2)$ to obtain accurate results for the bending stress intensity factors.

In the examples dealt with in sections 2.4-5, we have shown that the results obtained with the mode enrichment technique are in close agreement with the results obtained with the crack closure integral (appendix A). Therefore one could argue why to use a relatively complex technique (the mode enrichment method) if the same accuracy for the stress intensity factors can be obtained with a very simple method (the crack closure integral). The answer to this question consists of two parts.

First of all, the accuracy of the crack closure integral in the examples that we considered is partly due to the application of mode enrichment to the solution used in the crack closure integral. It can be shown that if the finite element mesh is relatively coarse, the accuracy of the crack closure results increases considerably by enriching the crack tip elements.

Another argument in favor of the mode enrichment technique has to do with the calculation of the elastic T-term, which represents the first higher order term in the membrane mode I crack tip solution. This elastic T-term is believed to be an important parameter to predict the direction of crack propagation. In chapter 3 it will be shown that mode enrichment offers the possibility to extract this elastic T-term from the underlying finite element displacement field in the crack tip elements with a reasonable accuracy.

All in all, one may conclude that the mode enrichment technique is a powerful technique for calculating stress intensity factors. It can be applied to linear as well as geometrically nonlinear problems, and it permits one to distinguish easily between the various modes of cracking.
Chapter 3
The elastic T-term

3.1 Introduction

In conjunction with the stress intensity factors the so-called elastic T-term is a useful parameter for the prediction of the fracture process for thin-walled structures. This elastic T-term is represented by the first higher order term in the membrane mode I crack tip solution [44] and denotes a constant stress tangential to the crack, thus

$$\sigma_x = K_1 f(r^{1/2}, \theta) + T + \sum_{n=3}^{\infty} H_n f(r^{n/2}, \theta)$$  \hspace{1cm} (3.1)

where $r$ and $\theta$ again are the polar coordinates with respect to the crack tip.

Larsson and Carlsson [47] and Rice [74] observed that inclusion of the elastic T-term in a small scale yielding procedure of elastic-plastic fracture increases the accuracy of the predicted plastic zone shape and size. Finite element calculations showed that for different specimens, initiation of yielding occurred at different $K_I$-values. This means that the shape and size of the plastic zone can not be dictated by the mode I stress intensity factor alone. However, including the elastic T-term in the boundary layer formulation gave results that were in close agreement with the finite element calculations. Later on Bilby et al. [10] also showed that inclusion of the elastic T-term extends the validity of the small scale yielding conditions at finite strains.

The elastic T-term is also believed to be an important parameter for the prediction of instability of a straight crack path under mode I loading conditions. Cotterell and Rice [20] argued that the path of a crack propagating under pure mode I conditions will become unstable for a positive elastic T-term. Leevers et al. [51] did some experiments for the centre cracked plate subjected to a biaxial load (fig. 3.1), and they showed that the stability of the crack path depends on the load biaxiality ratio $\chi$ (which determines the sign of the elastic T-term). Increasing the load tangential to the crack faces beyond a critical value will cause the crack path to become unstable. These experimental results were in reasonable agreement with the
analytical results of Cotterell and Rice [20].
From literature [100] it is known that for a pressurized cracked fuselage (which is governed by geometrically nonlinear effects), a longitudinal mode I crack path can become unstable for a certain crack length. As observed the crack will change direction, and crack propagation will continue along an almost circumferential line. In order to predict this crack flapping phenomenon numerically, it is necessary to develop a tool for calculating the elastic T-term for geometrically nonlinear problems.

3.2 Calculation of the elastic T-term

3.2.1 Historical review

3.2.1.1 Larsson and Carlsson

The method that Larsson and Carlsson [47] applied is rather straightforward. They extracted the elastic T-term from the stress field that remains after subtracting the analytical stresses corresponding to the stress intensity factor $K_I$ from the total finite element stress field. Larsson and Carlsson used constant strain triangular elements for their calculations. They considered the stress field in the elements behind the crack tip for determining the elastic T-term because the contribution of the singular stresses corresponding to $K_I$ is small there. In
this way possible errors in the calculated elastic T-term due to the singular stress field are minimized.
To obtain accurate results for the elastic T-term Larsson and Carlsson were forced to apply rather dense meshes around the crack tip. This difficulty is linked to the inability of the constant strain triangular elements to accurately approximate the singular part of the stress field.

3.2.1.2 Swedlow et al.

Swedlow et al. [97] directly incorporated the membrane mode I crack tip solution (or Williams' eigenfunction expansion) in a variational formulation. The resulting linear system of equations was used to solve for the vector of Williams' series coefficients $H_n$ (see equation (3.1)). This vector represents a high order approximation of the elastic stress/strain field. Different implementations of this technique were used by Urriolagoitia [105] in studies on crack path instability, by Ewing et al. [26] in work on mixed-mode crack extension and, elsewhere, by Swedlow himself [96].

Leevers [49] and Leevers and Radon [50] tested the method for a number of different plate configurations. They calculated $K_T$-factors with an accuracy up to about 1.0%. It was found that only a few terms of the Williams' series coefficients $H_n$ had to be considered to obtain this accuracy for $K_T$.

On the other hand, the elastic T-term converged much slower than the stress intensity factor $K_T$. The number of terms considered in the calculation had to be much larger in order to obtain a converged solution for the elastic T-term. Leevers and Radon estimated the maximum error in most of their solutions for the elastic T-term to be less than 3.0%.

3.2.1.3 Sham

The method that Sham [86] applied for calculating the elastic T-term at the tip of a crack is based on elastic reciprocity, which is essential to the theory of the so-called higher order weight functions. These weight functions can be determined through a work conjugate integral. This method is closely related to the use of the Bueckner-Rice weight functions [13] for calculating the stress intensity factor.

$$K_\alpha = \int_{area} F_\Omega h_\alpha d\Omega + \int_{volume} F_\nu h_\alpha dV \quad (3.2)$$
In this expression \( K_\alpha \) is the stress intensity factor for a cracked body under arbitrary combinations of mode I and mode II plane strain or plane stress deformation at the crack tip. \( F_\Omega \) and \( F_V \) represent the forces per unit area and per unit volume respectively, and \( h_\alpha \) are weight functions where \( \alpha \) takes on values 1 or 2 with each value denoting mode I or mode II deformation respectively. The weight functions corresponding to the stress intensity factors are of order \( r^{-1/2} \) while the weight functions for calculating the elastic T-term are of order \( r^{-1} \) [86].

A unique feature of the weight functions is that they are universal functions for the given crack configuration and body geometry. The stress intensity factor and the elastic T-term for a specific crack tip in a given body can be calculated for any surface and body loading once the weight functions associated with this configuration are known. They can be identified as displacements which obey all the equations of linear elasticity except that the weight functions corresponding to the elastic T-term, the so-called second order weight functions, give rise to unbounded energy in any finite area surrounding the crack tip. Furthermore, the stress fields corresponding to the weight functions are self-equilibrating.

Analytical expressions for the second order weight functions are only available for simple crack geometries. For more complex geometries numerical procedures have to be used to determine them [86].

Sham applied this technique to the problem of an edge cracked plate and showed his results to be in close agreement with analytical results found in literature. Sham's results will be used in section 3.3 as a reference to validate the working of the currently proposed technique for calculating the elastic T-term.

### 3.2.1.4 Eshelby

Another technique, due to Eshelby and worked out by Cardew, Goldthorpe, Howard and Kfouri [15] and Kfouri [42], makes use of the fact that the J-integral is nonzero only if the strain energy density singularity at an internal point is of type \( r^{-1} \) [75].

Suppose that the external loading \( F \) on a body produces the stress, strain and displacement fields \( \sigma_{ij}, \epsilon_{ij} \) and \( u_i \) in this body and that another self equilibrated force system produces the fields \( \sigma'_{ij}, \epsilon'_{ij} \) and \( u'_i \) (fig. 3.2). This additional force system consists of a point force \( f_p \) at the crack tip tangential to the crack faces equilibrated by tractions \( t_s \) along a contour around the crack tip. One way to express these tractions is

\[
  t_s = \frac{-f_p \cos \theta}{\pi r} \quad (3.3)
\]
Figure 3.2  Cracked body subjected to external forces $F$. A point force $f_p$ applied at the crack tip is resisted by tractions $t_s$.

As one can see these tractions are purely radial and they result in a stress singularity of order $r^{-1}$ at the crack tip. It is true that the crack tip force can be equilibrated in more than one way. However, the particular choice made above (equation (3.3)) has an advantage that becomes clear in what follows.

Combining the two fields in the J-integral yields

$$ J(F + f_p) = \int_C \left\{ \frac{1}{2} (\sigma_{ik} + \sigma'_{ik}) (\epsilon_{ik} + \epsilon'_{ik}) \delta_{ij} - (\sigma_{ij} + \sigma'_{ij}) (u_{i,1} + u'_{i,1}) \right\} \mathrm{d}s \quad (3.4) $$

This equation can be written as

$$ J(F + f_p) = J_2(F) + J_{11}(F, f_p) + J_2(f_p) \quad (3.5) $$

where $J_2(.)$ is a quadratic term while $J_{11}(.)$ represents a bilinear form that can be written as

$$ J_{11}(F, f_p) = \int_C \left\{ \frac{1}{2} (\sigma_{ik} \epsilon'_{ik} + \sigma'_{ik} \epsilon_{ik}) \delta_{ij} - \sigma_{ij} u_{i,1} - \sigma'_{ij} u'_{i,1} \right\} \mathrm{d}s \quad (3.6) $$
$J_{11}(F, f_p)$ is the integral associated with the cross-terms, and $J_2(F)$ and $J_2(f_p)$ are the values of the $J$-integral for each of the applied loads alone. Note that the stress field due to the point force $f_p$ corresponds to a strain energy density of order $r^{-2}$ and this renders $J_2(f_p)$ to be zero. It follows from the structure of the stress field near the crack tip

$$\sigma_{ij} = K_1 r^{-1/2} f_{ij}^{(1/2)}(\theta) + T f_{ij}^{(0)}(\theta) + O(r^{1/2})$$

(3.7a)

$$\sigma'_{ij} = f_p r^{-1} f_{ij}^{(1)}(\theta)$$

(3.7b)

that only the cross-term between $T$ and $f_p$ produces nonzero terms in $J_{11}(F, f_p)$. It turns out that for plane stress conditions $J_{11}(F, f_p)$ yields

$$J_{11}(F, f_p) = \frac{T f_p}{E}$$

(3.8)

The final result is therefore

$$\frac{T f_p}{E} = J(F + f_p) - J_2(F)$$

(3.9)

The procedure is now to calculate the $J$-integral for the problem at hand with and without inclusion of the force system as in equation (3.3). The elastic T-term is then obtained from equation (3.9) by substitution.

The method can also be used with a reacting force system different from that in equation (3.3). If this is the case this force system will also render a stress intensity factor $K_p$ and an elastic T-term $T_p$. In other words

$$\sigma'_{ij} = f_p r^{-1} f_{ij}^{(1)}(\theta) + K_p r^{-1/2} f_{ij}^{(1/2)}(\theta) + T_p f_{ij}^{(0)}(\theta) + O(r^{1/2})$$

(3.10)

In this case $J_{11}(F, f_p)$ yields

$$J_{11}(F, f_p) = \frac{T f_p}{E} + \frac{2K_p K_1}{E}$$

(3.11)

where $K_I$ is the stress intensity factor for the case when the load $F$ is applied on its own.

The expression for the elastic T-term is then given by
\[ \frac{T_{fp}}{E} = J(F + f_p) - J_2(F) - J_2(f_p) - \frac{2K_pK_t}{E} \] (3.12)

Note that \(J_2(f_p)\) is not equal to zero now because the stress field due to the point force \(f_p\) contains a strain energy density singularity of order \(r^{-1}\). The calculation of the elastic T-term now involves more effort as compared to the calculation by equation (3.9).

The results obtained with this technique were shown to be in close agreement with the results obtained by Leivers and Radon [50]. However, the method loses a lot of its appeal if one deals with geometrically nonlinear crack problems. This is so because the principle of superposition of stress and strain fields, which is essential for this technique, no longer holds for these kinds of problems.

### 3.2.2 The elastic T-term via mode enrichment

#### 3.2.2.1 Introduction

As stated in the introduction the main purpose of our research is to develop a tool for calculating the elastic T-term for linear as well as geometrically nonlinear crack problems. The nonlinear calculations are usually performed with a so-called path following technique [77]. This means that the volume of operations is most of the time quite substantial. Against this background, it is clear that a straightforward computation of the elastic T-term is to be preferred.

Therefore, our approach is to extract the elastic T-term directly from the finite element displacement field around the crack tip. The analytical plane stress displacement solution around the crack tip corresponding to the membrane mode I fracture mode can be written as

\[ u = \frac{K_I}{\mu} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left( \frac{1-v}{1+v} + \sin^2 \left( \frac{\theta}{2} \right) \right) + \frac{T}{E} r \cos \theta + \sum_{n=3}^{\infty} \frac{H_{n,1}r^{n/2}}{2\mu} \left( \frac{3-v}{1+v} \cos \frac{n\theta}{2} - \frac{n}{2} \cos \left( \frac{n}{2} - 2 \right) \theta + \left( \frac{n}{2} + (-1)^n \right) \cos \frac{n\theta}{2} \right) \] (3.13a)

\[ v = \frac{K_I}{\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left( \frac{2}{1+v} - \cos^2 \left( \frac{\theta}{2} \right) \right) - \frac{T}{E} r \sin \theta + \sum_{n=3}^{\infty} \frac{H_{n,1}r^{n/2}}{2\mu} \left( \frac{3-v}{1+v} \sin \frac{n\theta}{2} + \frac{n}{2} \sin \left( \frac{n}{2} - 2 \right) \theta - \left( \frac{n}{2} + (-1)^n \right) \sin \frac{n\theta}{2} \right) \] (3.13b)

where \(\mu\) is the shear modulus, \(v\) is Poisson’s ratio and \(E\) is Young’s modulus.
The conventional finite element displacement formulation is not able to describe the first part of the displacement field corresponding to the stress intensity factor. In our method this 'singular' part of the displacement field is taken care of via the mode enrichment technique. In that case the finite element formulation has to approximate the displacements for the higher order terms starting with the elastic T-term.

The displacement field for the elastic T-term varies in a linear way in x- and y-direction and therefore can be described exactly by the finite element displacement field. However, the conventional finite element formulation is not able to describe the analytical displacement fields for the other higher order terms exactly. This poses a difficulty, because the interpolation functions of the finite element mesh try to approximate the lower and higher order terms at the same time. Consequently, the linear part of the finite element displacement field, which should correspond to the elastic T-term, is affected by the higher order terms, thus impairing the accuracy of the calculated elastic T-term. How this difficulty can be at least partially circumvented will be discussed in the next section.

3.2.2.2 Solution strategy

According to equation (3.13a) the analytical displacement solution u ahead of the crack tip (θ = 0°) contains, besides the elastic T-term, a contribution not only of the stress intensity factor K₁ but also of all the other higher order terms. The linear part of the finite element displacement field in this region will be disturbed significantly by these higher order terms. Consequently it is difficult to compute the elastic T-term from the displacement field ahead of the crack tip with a reasonable accuracy.

On the other hand, behind the crack tip (θ = 180°) the analytical displacement solution only contains contributions from the elastic T-term and from the higher order terms with even index n (H₄, H₆, H₈ etc.). Furthermore, there is no contribution of the displacement solution corresponding to K₁. This implies that behind the crack tip the disturbance of the linear displacement field is reduced to a minimum and that the elastic T-term can be computed rather accurately along the crack faces (which are situated at θ = 180°). This accuracy can even be increased by the use of quadratic (8- or 9-node) elements which are able to describe the analytical displacement fields corresponding to T and H₄ exactly.

One of the problems is that the nodal displacements along the crack face (which will be used for calculating the elastic T-term) also have to describe the displacement field within the element they are part of. Such an element covers an area for which θ < 180°. Consequently, the mode I stress intensity factor and all the higher order terms are still able to affect this linear displacement field along the crack face to some extent.

It is obvious that the major part of this disturbance is due to the displacement field corresponding to K₁, because this field is dominant around the crack tip. Larsson and Carlsson
[47] solved this problem by applying very dense element meshes around the crack tip. The elements that they used for calculating the elastic T-term were situated in a boundary zone along the crack face and thus one may expect only a slight disturbance of the linear part of the finite element displacement field in this region. In our approach this problem does not exist because the crack tip elements are enriched. However, there still is a contribution of the higher order terms with even index n (H_4, H_6, H_8 etc.) to the total displacement field along the crack faces. This means that for a general case the stress along the crack face is not constant. Therefore it was decided to use the following technique for calculating the elastic T-term.

![Figure 3.3](image.png)

**Figure 3.3** Refined finite element mesh around the crack tip

First of all, the finite element mesh around the crack tip is refined by introducing a number of new element layers inside the original crack tip elements (fig. 3.3). Next, the mode enrichment technique is applied to all the elements in this crack tip zone, with the outermost layer of elements serving as transition zone between the enriched and the conventional elements. The enriched and transition elements are integrated with a higher order 10x10 Gauss integration scheme (see section 2.3.2) while the conventional elements are integrated with a low order 2x2 or 3x3 integration scheme.

After completing the finite element calculation, the element stresses corresponding to the linear finite element displacement field tangential to the crack face are determined along the element sides that coincide with this crack face. The final step is then to determine the elastic T-term from these element stresses. This will be accomplished by making a least squares fit over a number of element sides behind the crack tip. The details of this technique will be described in the next section.

### 3.2.2.3 Numerical procedure

The finite element displacement solution for an element side (of a quadratic quadrilateral element) that coincides with the crack face can be written as (fig. 3.4)
\[ u = -\frac{1}{2} \xi_1 (1 - \xi_1) u_1 + (1 - \xi_1^2) u_3 + \frac{1}{2} \xi_1 (1 + \xi_1) u_2 \]  

(3.14)

where \( \xi_1 \) is one of the two parameters that are used to define the isoparametric quadrilateral element (\( \xi_1 \in [-1, 1] \)).

Because the elastic T-term represents a constant stress parallel to the crack faces, we will only use the 'constant stress part' to calculate this elastic T-term. Therefore we are interested only in the linear part of the finite element displacement field along these crack faces, thus

\[ u_{\text{lin}} = \frac{1}{2} \xi_1 (u_2 - u_1) \]  

(3.15)

Differentiating with respect to \( x \) yields

\[ \varepsilon_{\text{lin}} = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi_1} \frac{\partial \xi_1}{\partial x} = \frac{1}{2} \frac{\partial \xi_1}{\partial x} (u_2 - u_1) = \frac{1}{l} (u_2 - u_1) \]  

(3.16)

For geometrically nonlinear problems one has to note that the crack tip region will rotate in space during deformation. Therefore the in plane displacement field after rotation of the crack tip has to be considered for the calculation of the elastic T-term. For the strain tangential to the crack face this yields (fig. 3.5)

\[ \tilde{\varepsilon}_{\text{lin}} = \frac{\partial \tilde{u}}{\partial x} = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right) \]  

(3.17)

\[ \text{Figure 3.4} \quad \text{Part of the refined finite element mesh behind the crack tip} \]
Figure 3.5 Crack tip before and after rotation

or

\[ \bar{\varepsilon}_{\text{lin}} = \frac{1}{l} (u_2 - u_1) + \frac{1}{2l^2} \left( (u_2 - u_1)^2 + (v_2 - v_1)^2 + (w_2 - w_1)^2 \right) \]  \hspace{1cm} (3.18)

The element stresses corresponding to the elastic T-term can be obtained by simply multiplying these strains with the elasticity constants (see equation (3.13a)).

Now, in order to establish a least squares fit over these element stresses, it is first necessary to choose a continuous function for the stress field behind the crack tip. As already mentioned in the previous section, this stress field is governed by the elastic T-term and the higher order terms with even index \( n \). It follows from the previous remarks that this continuous stress function should look as follows:

\[ \sigma(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots. \] \hspace{1cm} (3.19)

In this equation \( C_0 \) represents the elastic T-term. Furthermore, \( C_1, C_2 \) etc. correspond to the higher order terms while \( x \) denotes the distance to the crack tip.

According to the least squares fit technique, the error between this continuous function for the stress field and the discrete stresses calculated in the elements behind the crack tip has to be minimized, where this error is defined as

\[ E = \sum_{j=1}^{n_e} (\sigma(x_j) - \sigma_j)^2 \] \hspace{1cm} (3.20)
In this equation $x_i$ denotes the distance of the midpoint of the element sides along the crack face to the crack tip while $\sigma(x_i)$ and $\sigma_i$ are the values of the continuous stress field $\sigma(x_c)$ and the discrete finite element stress at point $x_i$ respectively (fig. 3.6). Moreover, $n_e$ represents the number of elements over which this least squares fit is performed.

**Figure 3.6** Extrapolation of the element stresses to the crack tip via the least squares fit procedure

Minimizing the error $E$ error with respect to the coefficients $C_n$ in the polynomial function yields

$$\frac{\partial E}{\partial C_n} = 0 \quad (3.21)$$

for all coefficients $C_n$. This results in a matrix equation the order of which is determined by the number of coefficients in the continuous stress function $\sigma$. The elastic T-term ($C_0$ in equation (3.19)) is obtained by solving this matrix equation.

The question remains how to choose the number of terms that have to be considered in the stress function behind the crack tip (equation (3.19)) in order to render the elastic T-term with a reasonable accuracy. Moreover, it must be determined over which elements behind the crack tip this least squares fit must be established. These two questions will be answered in the next section on the basis of some classical examples.
3.3 Calibration tests on some classical problems

3.3.1 Edge cracked plate

The first classical example consists of an edge cracked plate loaded in tension (fig. 3.7). Sham [86] showed that for values of $a/W$ up to about 0.6, the elastic T-term is negative while for values of $a/W$ larger than 0.6 the elastic T-term becomes positive. He calculated the elastic T-term with the weight function technique discussed in section 3.2.1.3 and claims his results to be very accurate.

![Figure 3.7: Edge cracked plate subjected to a tensile load](image)

The overall finite element mesh that we used in our calculations is shown in figure 3.8. The two overall crack tip elements are refined as in figure 3.3 by introducing 8 element layers inside these two crack tip elements and mode enrichment is applied as described in section 3.2.2.2.

Before discussing the results we will first introduce a nondimensional expression for the elastic T-term that is often used in literature [47,50]. This so-called stress biaxiality ratio $B$ is written as

$$B = \frac{T\sqrt{\pi a}}{K_1} \quad \text{(3.22)}$$

and in what follows the stresses in the elements behind the crack tip are nondimensionalized in the same way as in equation (3.22).
Figure 3.8  Overall finite element mesh for the edge cracked plate (see fig. 3.7)

In figure 3.9 these nondimensional stress components along the element sides that coincide with the crack face are plotted for two different crack lengths. It is clear that the stress behind the crack tip is not constant but varies in an approximately linear way. In the crack tip element itself, the constant stress component tangential to the crack face is far off this linear approximation. This crack tip element has to describe the rapidly varying displacement field for $90^\circ < \theta < 180^\circ$ and thus the stresses along the element side that coincides with the crack face suffer under the influence of the stress intensity factor and all the other higher order terms.

Some tests with the examples dealt with in this chapter showed that $C_0$ and $C_1$ are the most dominant terms in the polynomial expression for the stress tangential to the crack face. In fact, inclusion of $C_2$, $C_3$, $C_4$ etc. had only a slight influence on the value of $C_0$ (within 2.0 %) and this observation made us decide to use only two terms ($C_0$ and $C_1$) in this polynomial expression. Furthermore, it is advantageous to skip the tangential stress in the crack tip element from the least squares fit procedure. This removes the negative influence that this crack tip element will have on the final solution for the elastic T-term. It turns out that this method produces very satisfactory results. A comparison with Sham's results [86] is shown in figure 3.10.

As mentioned in section (3.2.2.1), mode enrichment of the crack tip zone improves the accuracy of the obtained B-factor (representing the nondimensional stress component tangential to the crack face). This can be illustrated by performing a calculation of the nondimensional stress components along the crack face with the same finite element mesh but without enriching the crack tip zone (fig. 3.11). The beneficial influence of mode enrichment on the accuracy of the obtained results is clear.
Figure 3.9  B-factor versus the distance behind the crack tip for the edge cracked plate. The crack tip elements are enriched.

Figure 3.10  B-factor versus a/W for the edge cracked plate
3.3.2 Centre cracked plate

The second classical example concerns a centre cracked plate subjected to a uniaxial or biaxial load (fig. 3.1). This problem was also dealt with by Kfouri [42] and Leevers and Radon [50], and they showed that the load biaxiality ratio affects the elastic T-term in a pronounced way. We calculated the elastic T-term for a square centre cracked plate \((h/W = 1.0)\) for two load cases: pure uniaxial tension, \(\sigma_Q = 0\), and biaxial tension with \(\sigma_Q = \sigma_P\). The finite element mesh is pictured in figure 3.12 and the two overall crack tip elements are refined and enriched as described in the previous section.

From literature it is known that for the centre cracked plate the elastic T-term is constant over a considerable distance behind the crack tip [47,50]. In our research we made the same observations, as can be seen in figure 3.13. Again the tangential stress component in the crack tip element has a negative influence on the accuracy of the elastic T-term and therefore is not taken into account in the least squares fit procedure. The results obtained in this way are in close agreement with the results reported by Kfouri [42].

As in the previous example we performed another calculation without enriching the crack tip zone. It can be seen in figure 3.14 that this will give poor results for the B-factor.
**Figure 3.12** Overall finite element mesh for the centre cracked plate subjected to a tensile load perpendicular to the crack (see fig. 3.1)

**Figure 3.13** B-factor versus the distance behind the crack tip for the centre cracked plate. The crack tip elements are enriched.
**Figure 3.14** B-factor versus the distance behind the crack tip for the centre cracked plate obtained with and without mode enrichment.

**Figure 3.15** B-factor versus $a/W$ for the centre cracked plate.
In figure 3.15 the B-factors for different crack lengths are plotted for the two load cases. The agreement between the current results and the ones obtained by Kfouri [42] and by Leevers and Radon [50] is close for both load cases. Kfouri and Leevers and Radon analyzed the centre cracked plate up to \( a/W = 0.6 \) and for the uniaxial load case they found an almost linear relation between B and \( a/W \) namely [50]

\[
B = - (1 + 0.085 \left( \frac{2a}{W} \right))
\]  

(3.23)

However, a further increase of the crack length shows that this relation does not hold beyond \( a/W = 0.6 \) (fig. 3.15). The elastic T-term will increase faster than the stress intensity factor \( K_1 \) for a growing crack length and this will cause the absolute value of B also to increase. The same trend can be seen for the biaxial load case.

### 3.4 Geometrically nonlinear problems

#### 3.4.1 Buckling of a stretched plate with a central crack

The first geometrically nonlinear crack problem that we solved is buckling of a centre cracked plate. This problem was already dealt with in section 2.5.2 and it was found that the crack faces tangential to the crack will buckle due to the compressive stresses along these crack faces. In figure 2.17 it can be seen that this buckling phenomenon will cause a discontinuous slope for the curve of \( K_1 \) versus \( \sigma \), and that it tends to amplify the severity of the stress singularity at the crack tip after buckling has occurred.

The overall finite element mesh that we used for the calculation of the elastic T-term is the same as in section 2.5.2 (fig. 2.14), while the crack tip elements are refined and enriched again as in the previous sections. A similar slope discontinuity as for the \( K_1-\sigma \) curve is also found for the T-\( \sigma \) curve (fig. 3.16). In contrast to the stress intensity factor \( K_1 \) buckling has a softening effect on the elastic T-term. This means that in the postbuckling range the elastic T-term will not increase as fast as before buckling. Consequently the B-factor will show a sharp decrease right after buckling has occurred (fig. 3.17).

Note that the smooth pre- and post-buckling branches of the T-\( \sigma \) curve are almost linear. This observation is not surprising because the same trend was found for \( K_1 \) (figure 2.17).
Figure 3.16  Elastic T-term versus the applied stress $\sigma$ for the centre cracked plate (see fig. 2.7 and fig. 2.14)

Figure 3.17  B-factor versus the applied stress $\sigma$ for the centre cracked plate (see fig. 2.7 and fig. 2.14)
3.4.2 Bulging of a longitudinal crack in a pressurized fuselage

For our second geometrically nonlinear problem we used the same model of a cracked pressurized fuselage that was introduced in section 2.5.3 (fig. 2.21). It was shown that the longitudinal boiler stresses tend to close the crack for an increasing pressure causing a reduction of the bulge factor (fig. 2.24). The overall finite element mesh that we used for the calculation of the elastic T-term is plotted in figure 2.22.

It can be seen that the calculated elastic T-term first increases until the load factor \( \lambda \) is about 1.2 (fig. 3.19). The maximum value is approximately 31.0 N/mm\(^2\). A further increase of the load factor will cause the elastic T-term to decrease. Comparing figures 3.18 and 3.19 it is obvious that the internal pressure has an even more pronounced influence on the elastic T-term than on the stress intensity factor \( K_I \). Consequently the B-factor first increases rapidly for small load factors and then starts to decrease monotonically (fig. 3.20).

3.5 Conclusion

The linear examples dealt with in section 3.3 show that the elastic T-term can be calculated rather accurately with the technique proposed here. The results obtained for the edge cracked

![Figure 3.18](image)

**Figure 3.18** Stress intensity factor \( K_I \) versus the load factor \( \lambda \) for the cracked fuselage segment (see fig. 2.21 and fig. 2.22)
Figure 3.19  Elastic T-term versus the load factor $\lambda$ for the cracked fuselage segment (see fig. 2.21 and fig. 2.22)

Figure 3.20  B-factor versus the load factor $\lambda$ for the cracked fuselage segment (see fig. 2.21 and fig. 2.22)
plate and the centre cracked plate are in close agreement with the results found in literature. Compared to Sham's weight function method [86] and Eshelby's method [15] the present technique is very simple. For geometrically nonlinear problems it is especially advantageous that the elastic T-term is obtained directly from the finite element solution for the problem at hand.

In contrast, the determination of the weight function for Sham's method can become rather complex for a geometrically nonlinear problem. On the other hand, Eshelby's method [15] requires a separate additional finite element calculation for a special force system applied around the crack tip (see section 3.2.1.4). Moreover, application of this special force system can become quite complicated if one wants to deal with geometrically nonlinear problems.

All in all one may conclude that the present technique is a simple but effective tool for calculating the elastic T-term for linear as well as geometrically nonlinear crack problems. The main advantage as compared to other techniques is that the elastic T-term is extracted directly from the finite element displacement solution around the crack tip. This eliminates the need for any cumbersome pre- or post-processing tools.
Chapter 4
Adaptive mesh refinement

4.1 Introduction

The accuracy of finite element calculations mainly depends on the quality of the finite element model. Besides the type and shape of the elements a major part of this accuracy is dictated by the size of the separate elements. This so-called mesh density has to be appropriate to describe the stress and strain gradients over the complete model with a reasonable accuracy. The question is how to determine the mesh density that is needed to achieve this goal.

One of the techniques for estimating the accuracy of a given finite element solution is to recompute the solution using a uniformly refined mesh. However, a major drawback of this technique is that the measure of refinement needed to obtain accurate results over the complete finite element model is difficult to determine. This means that refining by hand is not only very time consuming, but it can also result in a finite element mesh that is still not properly refined at all places. A solution to this problem is to automate this process by developing a tool that generates an optimum finite element mesh rendering accurate results for a minimum number of computational freedoms. This so-called adaptive mesh refinement technique involves some special routines that will be discussed in what follows.

In a previous work on adaptive mesh refinement Stanley et al. [90] argued that for the analysis of plate and shell structures quadrilateral elements are superior as compared to triangular

![Diagram](image)

**Figure 4.1** Finite element mesh with and without hanging nodes
elements. Indeed one can see that most finite element users (including the author) prefer the use of quadrilateral plate and shell elements over triangular ones. Therefore the current research is focussed on the development of a mesh generator for quadrilateral elements. One of the requirements that this mesh generator has to meet is that it has to produce meshes without so-called 'hanging nodes' (fig. 4.1) in order to avoid the need to introduce multipoint constraints to preserve interelement compatibility. Furthermore, it is also beneficial if the mesh generator produces only quadrilateral elements in the transition zone between the fine and coarse quadrilateral element regions. It is rather easy to generate such a transition zone using triangular elements (fig. 4.2). However, the absence of good triangular $C^0$ shell elements rules out the use of these triangular based transition zones.

![Transition zone of triangular elements](image)

**Figure 4.2** Transition zone of triangular elements

Another numerical tool that is needed is a routine for calculating an element-by-element estimate of the error in the solution over the entire finite element model. Mesh refinement is performed on the basis of this error where the degree of refinement is proportional to the size of the error in each element as compared to the total error. Moreover, adaptive mesh refinement also needs a routine for mapping the solution data from one finite element mesh onto another. This routine is especially important for nonlinear calculations.

Suppose that the nonlinear equilibrium path I for a certain problem is calculated (fig. 4.3). After refinement of the initial mesh a second calculation can be performed yielding equilibrium path II. In the second calculation, it takes, just as in the first run, a number of load steps to get to equilibrium point $P'$, that corresponds to $P$ in the first calculation. It is clear that this time consuming (re)calculation of the first part of equilibrium path II can be avoided by directly determining the equilibrium state at point $P'$ from the solution data obtained at point $P$ of equilibrium path I. The equilibrium state at point $P'$ is then obtained by first mapping the solution data at equilibrium point $P$ for the initial finite element mesh onto the refined one and
regaining equilibrium after a certain number of load steps. The remaining part of equilibrium path II can be calculated now by starting from point $P'$. For materially elastic geometrically nonlinear problems application of the mapping technique does not influence the accuracy of the final solution because for these kind of problems, the final equilibrium state is path independent. This means that the error at equilibrium point $P$ due to a coarse mesh does not influence the final equilibrium state for equilibrium path II. However, when material nonlinearities are included, the final equilibrium state is path dependent, and thus errors made previously do influence the final result. This implies that, for such problems, the error has to be minimized along the total equilibrium path.

4.2 Adaptive mesh refinement tools

4.2.1 Mesh generation and mesh refinement

Many papers in the literature deal with mesh generators for triangular elements [16,17,53,65,112]. However, the automatic generation of a mesh consisting of quadrilateral elements is much more difficult and most of today's quadrilateral mesh generators start from a triangular mesh [4,41,103,114]. They simply join the triangular elements two by two to produce quadrilateral elements (fig. 4.4). Mesh refinement can be performed in two ways. One can either refine an existing finite element mesh [90] or generate a complete new mesh that fulfills the required mesh density conditions [4,41,103,114].
The main advantage of generating a complete new finite element mesh is that the quality of this new mesh does not depend on the shape of the elements in the initial mesh. When refining an existing finite element mesh by element subdivision the initial mesh has to consist of well-proportioned elements in order to guarantee a good quality for the refined elements. However, a major drawback of generating a new mesh is that the geometry of the model has to be reproduced again during each mesh refinement step and this can become quite complicated when dealing with complex shell structures. On the other hand, the geometry of the model has to be defined only once when refining an existing finite element mesh. This made us decide to develop a technique that is based on refining the current mesh by element subdivision instead of generating a complete new one in each refinement step.

The refinement technique that we propose can be divided in four steps.

(i) An initial quadrilateral finite element mesh is introduced manually. This initial mesh is needed to define the geometry of the model, the boundaries where constraints are applied and the nodes that are to be fixed during the mesh smoothing process (see section 4.2.1.4).

(ii) Mesh refinement is performed in such a way that the required mesh density conditions are met.

(iii) The so-called hanging nodes are removed by adding elements in the transition zone between different levels of refinement. As already mentioned in the introduction these transition elements are also quadrilateral.

(iv) The average quality of the elements is increased by a node shifting technique. Sharp angles are enlarged and obtuse angles are decreased in order to improve the overall quality of the elements.
4.2.1.1 Generating an initial mesh

In the initial stage of the modeling process the overall geometry of the model is described. This is accomplished by assigning each element to an element group and defining a certain geometrical shape for each element group. For the example of a quarter of a circular plate with a circular hole (fig. 4.5), for instance, all elements are part of one and the same element group that describes a plane surface.

![Diagram of physical problem and finite element model]

**Figure 4.5** Quarter of a circular plate with a hole subjected to an external load F

Next, the nodal lines are defined. This provides information that the mesh generator must have to be able to maintain the geometry of a boundary during each mesh refinement step. For the example, in figure 4.5, the nodes on boundary 2 and 4 are part of two different node groups describing a circular segment. If, for example, a new node $n_n$ is generated half way between nodes $n_6$ and $n_7$, this node is shifted automatically to $n'_n$, a position that corresponds to the prescribed geometry for this nodal line.

It is obvious that nodal lines can also be used to simplify the application of boundary conditions. For the current example, boundary 1 and boundary 3 are defined as two different nodal lines with prescribed boundary conditions, while boundary 5 is subjected to a distributed load.
4.2.1.2 Mesh refinement

The mesh refinement process is started on the basis of mesh density conditions that are either calculated by the error estimation procedure or specified by the user.

![Mesh representation](image)

*Figure 4.6 Definition of parent and child elements*

The elements that are generated during this refinement process all belong to a certain element level. The initial hand-made elements are all of level 1. Each of these elements can be divided into four new elements of level 2 (fig. 4.1). If the required element size is still not satisfied the refinement routine will generate 4 elements of level 3 for each level 2 element and so on. An element that is divided into 4 new ones is called a parent element while the 4 new elements are called children. This means that a level 2 element has a parent among the level 1 elements while it may also have 4 child elements of level 3 (fig. 4.6). All this information about parents and children of all the elements is stored and one can always determine the parent of an arbitrary element of a certain element level. In this manner the coarsening of a previously refined element mesh is accomplished by skipping 4 neighboring child elements and introducing their parent again in the mesh.

4.2.1.3 Removing the hanging nodes

Hanging nodes can be removed either two by two by generating a number of new elements and nodes in the transition zone, or, if close to a boundary, by introducing a new boundary node (fig. 4.7).

This technique can be applied between element zones that differ by only one level. In figure 4.8, for instance, one can see that node P is part of element e₁ and e₂ of level 4, element e₃ of level 3 and element e₄ of level 2. Therefore this last element e₄ has to be refined one more step in order to fulfill the condition described previously.

There are some cases which need special attention. If an element of level n is surrounded by 3 (or 4) elements of level n+1 and the above described technique would be applied, one is still left with 1 (or 2) hanging nodes (fig. 4.9). This also occurs if an element of level n is
two by two removal of hanging nodes

removal of hanging nodes near a boundary

Figure 4.7 Removing hanging nodes in a refined finite element mesh

$e_1, e_2 : \text{level 4}$  
$e_3 : \text{level 3}$  
$e_4 : \text{level 2}$

Figure 4.8 Node P is connected to elements of 3 different levels

one hanging node remains  
all hanging nodes removed

Figure 4.9 A level n element surrounded by 3 elements of level $n+1$
A level \( n \) element with 2 neighboring elements of level \( n+1 \) at opposite sides (fig. 4.10). The solution to these problems is to refine the concerning element one more step before removing the hanging nodes.

If, during successive steps of mesh refinement, an element is repeatedly used as a transition element (fig. 4.11), its quality (or shape) can become quite poor. This problem is circumvented by first removing the transition elements before starting a new mesh refinement step.

4.2.1.4 Mesh smoothing

In the final stage of the refinement process a node shifting technique is applied to improve the overall quality of the quadrilateral elements. Each node is shifted to a position that is defined by the average of the coordinates of the nodes that the considered node is connected to (fig. 4.12).

During the mesh smoothing procedure, boundary nodes have to remain at the physical boundary of the model. Therefore they are averaged only between the two neighboring
boundary nodes that they are connected to. This also implies that the corner nodes of the model have to be defined as fixed nodes in order to maintain the geometry of the model. For the example in figure 4.5, for instance, nodes $n_1$ to $n_4$ are corner nodes and thus must be fixed. Moreover, node $n_5$ must be fixed in order to apply the external load $F$ correctly.

4.2.2 Error calculation

The mesh refinement process is guided by an error estimator that calculates both local and global errors. The local errors determine where to refine or coarsen a finite element mesh, while the global errors are used to provide termination conditions for the mesh refinement process. Most of today's error estimators are based on a projection estimate \cite{115}, a residual estimate \cite{28,2} or an extrapolation estimate \cite{33}.

Projection estimators project the finite element results for one of the solution data onto a continuous field of higher order that can be obtained via a least squares fit \cite{115}. The difference between the finite element results and the continuous higher order field, which is assumed to be a good approximation of the exact solution, is taken then as a measure for the error.

Residual error estimators approximate the error by performing independent local analyses on each element with higher order shape functions, or, alternatively, by increasing the number of elements employed in the interior of the considered element. In these analyses the coarse mesh displacements are imposed as local boundary conditions \cite{28}. These residual error estimators thus calculate the change in some solution quantity that would be experienced if the element approximation would be improved locally without actually solving the more expensive global problem with such refinements. These category of error estimators are well suited for parallel processing because all element local analyses can be performed independently. Babuska et al. \cite{2} proved that these error estimators are more accurate than the projection error estimators. However, the major drawback is that they can become quite
complicated for nonlinear or dynamic problems where the local analyses also become nonlinear or dynamic.

Extrapolation error estimators calculate an error from a sequence of mesh refinement steps [33]. In this way a higher order approximation of the error is obtained as compared to an estimator that determines the error from querying the solution for a single mesh. Mesh refinement must be uniform for this method in order to be effective. The extrapolation error estimators have an accuracy advantage over the projection and residual error estimators. However, this accuracy comes at a higher cost and with greater implementational complexity. For our purposes the applicability of the residual and extrapolation error estimators is limited because we want to deal with nonlinear problems. All the local analyses for the residual error estimator will be nonlinear too and this can become quite complicated as mentioned above. For the extrapolation error estimator nonlinear calculations have to be performed for a number of meshes of increasing density. Because the volume of these succeeding calculations will become quite substantial this error estimator also loses a lot of its attraction.

Therefore in the present research the so-called \( Z^2 \) projection error estimator, proposed by Zienkiewicz and Zhu [115], is employed. This error estimator is relatively easy to apply and has proved its applicability to all kinds of problems in literature [54,115]. First the theoretical background of the \( Z^2 \) error estimator will be discussed. Next a description of a modified version of this error estimator which we implemented will be given.

### 4.2.2.1 The energy norm

The \( Z^2 \) error estimator (or energy norm) is defined as [115]

\[
\| e \| = \left( \int_V e_\sigma^T C^{-1} e_\sigma dV \right)^{1/2} \tag{4.1}
\]

where

\[
e_\sigma = \sigma - \bar{\sigma} \tag{4.2}
\]

In equation (4.1) \( C^{-1} \) is the inverted material stiffness matrix and \( V \) denotes the volume of the finite element model. The finite element stresses are represented by \( \sigma \) (equation (4.2)) while \( \bar{\sigma} \) is the still unknown (close) approximation of the exact continuous stress field.

One way to approximate the exact stress field is by assuming the same interpolation functions for this stress field \( \sigma \) as for the finite element displacement field \( \mathbf{u} \). This implies that the interpolation polynomials used for the approximate stress field \( \sigma \) are one order higher than the ones used for the finite element stresses \( \bar{\sigma} \) (which are described by the derivatives of the
interpolation functions for \( y \). The nodal values of the stress field \( \sigma \) are determined via a least squares fit with respect to the finite element stress field \( \sigma \). So if

\[
y = N^i u_i
\]

(4.3)

we now assume that

\[
\sigma = N^i \sigma_i
\]

(4.4)

where \( \sigma_i \) represents the discrete nodal values of the still unknown continuous stress field \( \sigma \) [115]. Next an error \( E_\sigma \) in the stresses is introduced as [11]

\[
E_\sigma = \frac{1}{2} \int_V (\sigma - \bar{\sigma})^2 dV
\]

(4.5)

Working out equation (4.5) yields

\[
E_\sigma = \frac{1}{2} \int_V \sigma^2 dV - \int_V \sigma \sigma dV + \frac{1}{2} \int_V \bar{\sigma}^2 dV
\]

(4.6)

Substitution of equation (4.4) into equation (4.6) yields for each element

\[
E_{\sigma_e} = \frac{1}{2} \int_V \sigma_i^T N^i \sigma_i dV - \int_V \sigma_i^T N^i \bar{\sigma} dV + \frac{1}{2} \int_V \bar{\sigma}_i^2 dV
\]

(4.7)

or

\[
E_{\sigma_e} = \frac{1}{2} \sigma_i^T B_e \sigma_i - \sigma_i^T f_e + A_e
\]

(4.8)

where

\[
B_e = \int_V N^i^T N^i dV
\]

(4.9a)

\[
f_e = \int_V N^i \sigma dV
\]

(4.9b)
\[ A_e = \frac{1}{2} \int_V \sigma^2 dV \]  

(4.9c)

Summation of the contribution \( E_\sigma \) over all the elements yields for the total error

\[ E_\sigma = \frac{1}{2} \sigma_{it}^T B \sigma_{it} - \sigma_{it}^T f + A \]  

(4.10)

where \( \sigma_{it} \) represents the total nodal stress vector.

This error \( E_\sigma \) has to be minimized with respect to \( \sigma_{it} \), thus

\[ \frac{\partial E_\sigma}{\partial \sigma_{it}} = 0 \]  

(4.11)

or

\[ B \sigma_{it} = f \]  

(4.12)

where \( B \) and \( f \) were already defined in equations (4.9a-b). The continuous stress field \( \sigma \) as described by equation (4.4) is now obtained by solving equation (4.12). This way of working can be applied for all stress components.

Next the energy norm per element can be calculated as

\[ \| e_e \| = ( \int_V e_{\sigma e}^T C^{-1} e_{\sigma e} dV )^{1/2} \]  

(4.13)

For the total energy norm this yields

\[ \| e \|^2 = \sum_e \| e_e \|^2 \]  

(4.14)

Although the absolute value of the energy norm has little physical meaning the relative percentage error is more easily interpreted.

\[ \eta = \frac{\| e \|}{\| U \|} \times 100\% \]  

(4.15)

where
\[ \|U\| = \left( \int_{\Omega} \sigma^T \mathbf{C}^{-1} \sigma \, dV \right)^{1/2} \] (4.16)

and represents the square root of two times the total strain energy. Equation (4.15) is the so-called weighted RMS (root mean square) percentage error in the stresses. The percentage error \( \eta \) is calculated for the whole domain as well as for all elements separately. The energy norm for the separate elements gives the contribution of each element to the total error and local mesh refinement is performed on the basis of these element errors.

A very common requirement is to specify a maximum percentage error for the total energy norm. One thus requires that after the final analysis is completed the condition

\[ \eta \leq \bar{\eta} \] (4.17)

is satisfied for the whole domain, where \( \bar{\eta} \) is the maximum permissible percentage error. Equation (4.17) can be translated into placing a limit on the error in each element. One can require, for instance, that the total error is equally distributed over all the elements. This yields for the permissible error for each element

\[ \|\varepsilon_e\| \leq \frac{\bar{\eta}\|U\|}{100\sqrt{m}} = \varepsilon_m \] (4.18)

where \( m \) denotes the total number of elements. According to equation (4.18) the ratio

\[ \xi_i = \frac{\|\varepsilon_e\|}{\varepsilon_m} > 1 \] (4.19)

defines the elements to be refined and its value can decide the degree of subdivision or the element size needed. Assuming, for instance, the rate of convergence to be \( O(h^p) \) the required element size \( h_r \) should be

\[ h_r = \frac{h_i}{\xi_i^{1/p}} \] (4.20)

where \( h_i \) is the current element size. Generally \( p \) is assumed to be equal to the polynomial order of the shape functions used [115]. Equation (4.19) also implies that elements with an error \( \|\varepsilon_e\| \) much smaller than the required error \( \varepsilon_m \) can be enlarged.
In contrast to equation (4.18) one can also require the error to be equally distributed over the total volume of the model. This will yield for the maximum permissible error per element

\[
\| e_c \| \leq \frac{\bar{V}_e}{100 \sqrt{V_t}} \cdot \sqrt{\text{V}_e} = e_m
\]  

(4.21)

where

\[ V_e \] denotes the volume of element \( e \) and
\[ V_t \] represents the total volume of all the elements.

Compared to equation (4.18) the maximum permissible error is proportional now to the volume of the elements [14].

\subsection{4.2.2.2 Modified version of the energy norm}

A potential disadvantage of the error estimator described in the previous section is that it is difficult to apply to complex structures involving discontinuities. These discontinuities can exist of shell intersections, discrete stiffeners, material- or thickness jumps and/or concentrated forces. For such problems the model has to be partitioned at such discontinuity boundaries because of the stress and strain discontinuities that exist at these boundaries.

Another way to deal with problems involving physical discontinuities is to use an error estimator based on the strain energy (rather than stress or strain). The advantage of this approach is that the strain energy will be continuous over these discontinuous boundaries. Furthermore, one only has to deal with a scalar quantity now (the strain energy density) instead of a tensor quantity (stress or strain).

The continuous stress field \( \sigma \) and strain field \( \epsilon \) are obtained from the nodal stresses and strains via the shape functions \( N^i \) (equation (4.4)). Because the strain energy density \( U_s \) is proportional to the product of \( \sigma \) and \( \epsilon \) it is obviously inconsistent to use the same shape functions for \( U_s \) as for \( \sigma \) and \( \epsilon \). To correct this Stanley et al. [90] suggested to interpolate \( \sqrt{U_s} \) instead of \( U_s \) thus

\[
\sqrt{U_{s_c}} = N^i \sqrt{U_{s_i}}
\]

(4.22)

and this approach seems reasonable.

The energy norm defined in the previous section is
\[ \| e \| = \left( \int_V (\sigma - \bar{\sigma})^T C^{-1} (\sigma - \bar{\sigma}) \, dV \right)^{1/2} \]  

(4.23)

Continuing as above one can substitute \( \sqrt{U_s} \) for \( \sigma \) yielding

\[ \| e_s \| = \left( \int_V (\sqrt{U_s} - \sqrt{U_s})^2 \, dV \right)^{1/2} \]

\[ = \left( \int_V (\sqrt{U_s}^2 - \sqrt{U_s}^2) \, dV \right)^{1/2} \]

(4.24)

It is obvious that in equation (4.24) the material stiffness matrix \( C \) is incorporated in the strain energy densities.

Stanley et al. [90] proved this error estimator (eq. 4.24) to converge to the exact error for a decreasing element size

\[ \lim_{h \to 0} \frac{\| e_s \|}{\| e_{ex} \|} = 1 \]  

(4.25)

Further, the error estimator of equation (4.24) is a lower bound for the \( Z^2 \) error estimator (eq. 4.23).

### 4.2.3 Mapping of the solution data

Mapping of the solution data from one finite element mesh onto another can be performed in several ways. Dyduch et al. [22], for instance, determine the solution data in a node of the new mesh from the corresponding data in a number of old nodes which are closest to this new node. They suggested a formula that looks as follows

\[ S_j = \frac{\left( \sum_{k=1}^{N} \frac{S_k}{R_{kj}^2} \right) + \frac{W_p S_p}{R_{pj}^2}}{\left( \sum_{k=1}^{N} \frac{1}{R_{kj}^2} \right) + \frac{W_p}{R_{pj}^2}} \]  

(4.26)

where
$S_j$ are the solution data in the new node $j$,
$S_p$ are the solution data in the old node $p$ that is closest to the new node $j$,
$S_k$ are the solution data in the old node $k$,
$W_p$ is the weight of the contribution of $S_p$ to $S_j$ and
$R_{pj}, R_{kj}$ denotes the distance between the nodes $p$ (or $k$) of the old mesh and node $j$ of
the new mesh.

Further there are some additional requirements namely

if $R_{kj} > R_{\text{max}}$ : point $k$ is not taken into account
if $R_{kj} < R_{\text{min}}$ : $S_j = S_k$

Equation (4.26) implies that the weight of the contribution of the solution data in the old nodes
to the mapped solution data in the considered new node decreases as the distance to this new
node increases and this approach seems reasonable. However, this formula is rather empirical
and it takes some experience to choose appropriate values for $W_p$, $R_{\text{max}}$ and $R_{\text{min}}$ in equation
(4.26).

In the present research we applied a (in our opinion) more rational mapping technique. The
position of the new nodes in the old element mesh is determined by a projection procedure
and the solution data are mapped from the old element mesh onto the new one via the finite
element shape functions.

### 4.2.3.1 Node projection

In the projection process the nodes of the new mesh are projected by dealing with the new
elements in a sequential manner. The advantage of this approach is that one can always check
whether the projection points of the nodes of a new element are situated in and around one and
the same element of the old mesh. This way of working will prove to be very helpful in the
case of a structure with two or more boundaries that are almost coincident, as, for example, in
the case of a cracked plate.

First a so-called internal node, i.e a node that is not coincident with the boundary of the model,
is located for each new element. This procedure is followed to ensure that the projection of the
first node of each new element does not fall outside the old element mesh (fig. 4.13).

Subsequently, one has to find the node $n_0$ of the old mesh which is closest to this internal node
$n_n$ and also to identify an arbitrary element $e$ of the old mesh that node $n_0$ belongs to (fig.
4.14). Node $n_n$ is projected onto the surface described by element $e$ and the parametric $\xi_1$- and
$\xi_2$-coordinate of this projection point $P$ are checked to see whether this point is situated inside
element $e$. If, for instance, it turns out that $\xi_1 < -1.0$ the projection process must be repeated
Figure 4.13 The projection of node $n_b$ of new element $e_n$ is situated outside the old element mesh.

with the element in the old mesh that has side 1-4 in common with the previous element $e$ (fig. 4.15). This technique of projecting node $n_n$ and checking the size of the parameters $\xi_1$ and $\xi_2$ must be continued until $\xi_1$ and $\xi_2$ lie between -1.0 and 1.0.

The positioning of the remaining nodes of the new element is carried out with the understanding that the projection process is always started from the old element $e$ that contains the projection of the first new (internal) node. The reason for this can best be explained by means of an example.

Suppose that one has two finite element meshes for a cracked plate consisting of 4-node elements (fig. 4.16). $P$, the projection of the new node $n_n$ which is part of the lower crack face, is geometrically situated in element $e_3$ at the upper crack face of the old mesh (fig. 4.16a). As a consequence, the mapped displacements and orthogonal triads for point $P$ on the lower crack face will be dictated by the corresponding solution data of some old nodes at the upper crack face. It is clear that this will yield incorrect results.

Figure 4.14 Projection of new node $n_n$ onto element $e$ of the old mesh
Figure 4.15  Values of the parameters $\xi_1$ and $\xi_2$ in and around element e

Figure 4.16  Node projection for a model with a curved crack

Starting the projection process for node $n_n$ from element $e_1$, the element that contains the projection of the first (internal) node $n_n$, one will end up with element $e_2$ to be the best choice for projecting node $n_n$ on. Due to the curvature of the crack faces however the projection point $P$ is not situated inside element $e_2$. Therefore $P$ is shifted to $P'$ by setting the value of $\xi_1$ and/or $\xi_2$ that is/are outside the interval $[-1.0,1.0]$ equal to the closest boundary of this interval. If, for instance, the final values of $\xi_1$ and $\xi_2$ are 0.1 and 1.05 respectively than $\xi_2$ is set equal to 1.0 (fig. 4.16b).
4.2.3.2 The mapping process

The mapping process for the deflections and the base vectors of the orthogonal triads (describing the rotations in the element nodes) from the old finite element mesh onto the new one is straightforward once the position of the new nodes is determined with respect to the old mesh. The coordinates of a projection point \( P \) were obtained as

\[
(x, y, z)_P = N^i (\xi_{1_p}, \xi_{2_p}) (x, y, z)_i
\]

(4.27)

where

\((x, y, z)_P\) are the coordinates of the projection point \( P \),

\(\xi_{1_p}, \xi_{2_p}\) are the parametric coordinates of \( P \),

\(N^i\) denotes the corresponding shape functions and

\((x, y, z)_i\) are the nodal coordinates of the element of the old mesh in which \( P \) is situated.

The deflections and orthogonal triads in this projection point \( P \) (and thus in node \( n_i \) of the new mesh) can be determined now in the same way yielding

\[
(u, n)_P = N^i (\xi_{1_p}, \xi_{2_p}) (u, n)_i
\]

(4.28)

where

\((u, n)_P\) are the deflections and triads corresponding to the projection point \( P \) and

\((u, n)_i\) are the deflections and triads of the nodes of the old element that contains \( P \).

As already mentioned in the introduction, the solution data that are mapped do not correspond to an equilibrium state for the new finite element mesh. However, when the difference between the new and the old equilibrium state is small enough, this new equilibrium state can be obtained within a few iterations. If this difference is 'too large' a load relaxation procedure has to be used whereby the residual force vector resulting from the mapped solution data is used as an external load (see [77]).
4.3 Calibration tests on some classical problems

4.3.1 L-shaped domain

The first test problem that is considered concerns the plane stress analysis of a square plate with a square cutout (fig. 4.17) loaded by uniform tension on all four outer edges (load case A) or on two opposing outer edges (load case B). Due to symmetry only one quarter of the plate is modeled. This example has been used extensively in the literature as a benchmark for adaptive mesh refinement techniques [90,115] and therefore offers the opportunity to make comparisons.

![Figure 4.17: Square plate with a cutout](image)

\[ E = 70000.0 \text{ N/mm}^2 \]
\[ t = 1.0 \text{ mm} \]
\[ \nu = 0.3 \]
\[ L = 100.0 \text{ mm} \]
\[ \text{Load A} = 1.0 \text{ N/mm} \]
\[ \text{Load B} = 1.0 \text{ N/mm} \]

Figure 4.17: Square plate with a cutout

Preliminary tests showed that the meshes that are determined with the 2 different error distribution techniques, discussed in section 4.2.2.1, were almost identical. Therefore, only the results that were obtained via equal distribution of the global error over the total volume of the model will be discussed.

In figure 4.18 (load case A and 4-node Bathe elements [6]) one can see that the mesh density around the corner of the cutout increases for the succeeding meshes. Despite the considerable increase of the number of degrees of freedom (up to about 5000) the global error will remain about 6-7%. For the higher order 8-node Bathe elements the errors will be much smaller for a comparable number of degrees of freedom (fig. 4.19). Furthermore, one can observe that our results for the energy norm and for the strain energy for the 8-node elements compare well with previous results obtained by Stanley et al. for the 9-node Lagrange elements [90].
Figure 4.18  Mesh refinement results for the L-shaped region (load case A, 4-node elements)
Figure 4.19  Mesh refinement results for the L-shaped region (load case A, 8-node elements)
Figure 4.20  Mesh refinement results for the L-shaped region (load case B, 4-node elements)
Figure 4.21  Mesh refinement results for the L-shaped region (load case B, 8-node elements)
In figures 4.20 and 4.21, the results for load case B are plotted. Again these results compare well with previous results found in literature [90,115]. It is noted that, in contrast to load case A (a symmetrical load system), the meshes for load case B are no longer symmetrical.

### 4.3.2 Circular plate with a hole

The next problem concerns a circular plate with a circular hole (fig. 4.22) subjected to a uniform radial load at the boundary of the hole [14,115]. Due to symmetry again only one quarter of the structure has to be modeled.

![Circular plate with a circular hole](image)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>70000.0 N/mm²</td>
</tr>
<tr>
<td>t</td>
<td>1.0 mm</td>
</tr>
<tr>
<td>ν</td>
<td>0.3</td>
</tr>
<tr>
<td>r₁</td>
<td>5.0 mm</td>
</tr>
<tr>
<td>r₂</td>
<td>20.0 mm</td>
</tr>
<tr>
<td>p</td>
<td>1.0 N/mm</td>
</tr>
</tbody>
</table>

**Figure 4.22** Circular plate with a circular hole

Adaptive mesh refinement steps were performed with the 4-node elements while both error distribution options, described in section 4.2.2.1, were applied.

Distributing the error equally over the total volume results in a finite element mesh with a dense element region around the hole as expected (fig. 4.23). However, using the other error distribution option (equal distribution over all the elements) will yield a more overall refinement of the model (fig. 4.24). It can be observed that the final energy norm and strain energy obtained with the two different error distribution techniques are almost identical.
Figure 4.23  Mesh refinement results for the circular plate with a circular hole (4-node elements, required error distributed over the total volume)
Figure 4.24  Mesh refinement results for the circular plate with a circular hole (4-node elements, required error distributed over the elements)
4.3.3 Zienkiewicz and Zhu's problem

Another classical example, Zienkiewicz and Zhu's problem [116], concerns a plate with two circular holes and some sharp and rounded corners along the outside boundary (fig. 4.25). Due to the rather complex geometry this problem is an excellent benchmark for testing the mesh generator.

The coarse mesh plotted in figure 4.26 is produced by hand. This initial mesh is used to define the geometry of the total model and some nodal lines with special constraints. In the first

![Diagram of Zienkiewicz and Zhu's problem with dimensions and forces]

\[ E = 70000.0 \text{ N/mm}^2 \]
\[ t = 1.0 \text{ mm} \]
\[ v = 0.3 \]
\[ r_1 = 12.0 \text{ mm} \]
\[ r_2 = 10.0 \text{ mm} \]
\[ p = 1.0 \text{ N/mm} \]

Figure 4.25  Zienkiewicz and Zhu's problem [116]

refinement step each element of the initial mesh is divided into 16 new ones (fig. 4.27). This new finite element mesh is used as the initial mesh in the adaptive refinement process for both the 4-node and the 8-node elements. Only one refinement step is performed using the option of equal distribution of the required error over the total surface of the model.

The most severe stress concentrations (and thus the smallest elements after mesh refinement) are located around the hole which is closest to the applied load and around the two sharp corners. Furthermore, one can see that for a comparable number of degrees of freedom, the error for the final mesh consisting of the higher order 8-node elements is again far below the error for the mesh consisting of the 4-node elements.
Figure 4.26 Initial hand-made coarse mesh for Zienkiewicz and Zhu's problem

Figure 4.27 Refined mesh for Zienkiewicz and Zhu's problem. Each element is divided into 16 new ones.
4-node elements:
- number of elements: 1806
- degrees of freedom: 3822
- required error: 6.0%
- achieved error: 5.16%
- strain energy: 0.11041 Nmm

Figure 4.28 Adaptively refined mesh for Zienkiewicz and Zhu's problem (4-node elements)

8-node elements:
- number of elements: 545
- degrees of freedom: 3482
- required error: 1.0%
- achieved error: 1.07%
- strain energy: 0.11074 Nmm

Figure 4.29 Adaptively refined mesh for Zienkiewicz and Zhu's problem (8-node elements)
4.4 Geometrically nonlinear problems

4.4.1 Pinched cylinder

We will now consider some problems that are geometrically nonlinear. The first is a pinched cylinder (fig. 4.30). This problem was also analyzed by Mathisen et al. [55] and by Stander et al. [89].

![Pinched Cylinder Diagram]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>$L = 77.419\ mm$</td>
</tr>
<tr>
<td>Radius</td>
<td>$r = 25.806\ mm$</td>
</tr>
<tr>
<td>Thickness</td>
<td>$t = 0.762\ mm$</td>
</tr>
<tr>
<td>Young's modulus</td>
<td>$E = 32062.0\ N/mm^2$</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>$\nu = 0.3$</td>
</tr>
</tbody>
</table>

**Figure 4.30** Pinched cylinder

First, a calculation is performed with a uniform mesh consisting of 8-node elements (fig. 4.31) applying a reduced 2x2x2 Gauss integration scheme [7]. In figure 4.32 the radial displacement of point A is plotted versus the applied load in this point, and these results agree well with the results of Mathisen et al. [55].

Next, the final equilibrium state is recalculated with a full 3x3x2 Gauss integration scheme [7]. In this process the displacements and orthogonal triads for the final equilibrium state are mapped from the initial finite element mesh onto the new one, which in this case, is a trivial process because the meshes are identical. The new equilibrium state is obtained within a few load steps, starting from the final equilibrium state corresponding to the reduced integration scheme. As one can see the full integration scheme will cause the model to behave too stiff [7]; it renders a radial displacement in point A that is too small as compared to the results found in literature.

Finally, a refinement step was performed for the final equilibrium state resulting in a mesh plotted in figure 4.33. Again the displacements and deflections from the initial finite element calculation were mapped onto the refined one and the new equilibrium state was calculated with the reduced 2x2x2 integration scheme. It turned out that the influence of this mesh refinement step on the initial results (which are believed to be rather accurate) is negligible.
Figure 4.31  Uniform 8x16 finite element mesh for the pinched cylinder problem

Figure 4.32  Radial displacement of point A versus the applied load F
4.4.2 Buckling of a stretched plate with a central crack

In chapter 2, it was shown that for the problem of buckling of a centre cracked plate in tension, accurate results for the membrane stress intensity factors can be obtained with a rather coarse finite element mesh. However, for calculating the bending stress intensity factors, mode enrichment has to be restricted to a small area around the crack tip. Consequently, the finite element mesh must be dense around this crack tip causing a considerable increase in the number of degrees of freedom as well as in the time involved in computing the complete equilibrium path.

However, if one is interested in the final stress intensity factors only one can also make use of the mesh refinement tools discussed in section 4.2. First a calculation of the final equilibrium state is performed with the coarse finite element mesh (fig. 2.14) without enriching the crack tip elements. Next the solution data for the final equilibrium state at $\sigma = 20.0$ N/mm$^2$ are mapped onto a finite element mesh which is extremely dense around the crack tip (fig. 4.34). The equilibrium state for this refined finite element mesh is calculated within a few load steps with application of the mode enrichment technique to the 2 crack tip elements and the 6 surrounding elements in the transition zone. The conventional elements are integrated with a low order 2x2 Gaussian integration scheme while for the enriched and transition elements the integration order is 10x10. The resulting stress intensity factors are
Figure 4.34 Dense finite element mesh for calculating the membrane and bending mode I stress intensity factors for the centre cracked plate (see fig. 2.7).

\[
\begin{align*}
K_1 &= 646.0 \text{ N/mm}^{3/2} \\
K_1 &= 27.2 \text{ N/mm}^{3/2}
\end{align*}
\]  

which agree well with the results obtained in chapter 2.

4.4.3 Bulging of a longitudinal crack in a pressurized fuselage

The same methods as discussed above were applied to the problem of a bulging crack in a pressurized fuselage (fig. 2.21). The coarse and dense finite element meshes are plotted in figures 2.22 and 4.35 respectively.

First the coarse finite element mesh was loaded up to a load factor \( \lambda = 1.0 \). Next the solution data were mapped onto the dense finite element mesh and mode enrichment was applied in the same way as in the previous example. For the stress intensity factors we obtained

\[
K_1 = 2607.7 \text{ N/mm}^{3/2}
\]
\[ k_1 = 125.6 \text{ N/mm}^{3/2} \]  \quad (4.30)

which again is in close agreement with the results in chapter 2.

### 4.5 Conclusion

Adaptive mesh refinement for geometrically nonlinear problems involves three different tools: a mesh generator, an error estimator, and a solution data mapping routine. The theoretical background and implementation of these tools is discussed in section 4.2.

In sections 4.3 and 4.4 some linear and geometrically nonlinear problems are dealt with and the solutions are shown to be in close agreement with the results found in literature and in chapter 2 of the present work.

The solution data mapping routine, discussed in section 4.2.3, is very appealing when dealing with geometrically nonlinear crack problems. The two examples in section 4.4.2-3 showed that the calculation of the bending stress intensity factors, which requires a very dense mesh

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**Figure 4.35** Dense finite element mesh for calculating the membrane and bending mode I stress intensity factors for the pressurized fuselage (see fig. 2.21)
around the crack tip, can be performed without using a dense finite element mesh (with a large number of computational freedoms) throughout the complete analysis. In chapter 5 this mapping routine will prove to be very useful for the simulation of cracks that are propagating through a structure.
Chapter 5
Crack growth simulation

5.1 Introduction

Crucial elements in the simulation of crack propagation are the prediction of the crack growth direction and a technique for updating the finite element mesh. Simulations of this type should be able to predict whether a longitudinal crack will 'flap' when it reaches a certain length, i.e. when and how a crack will turn to a circumferential line (fig. 5.1). Crack flapping (followed by crack arrest) allows a decompression of the fuselage, thereby diminishing the loading of the crack. The ability to predict whether a crack will flap or will continue to grow in a self similar fashion is of great practical importance, which is why there is so much recent interest in simulating the crack growing process.

![Diagram of crack growth](image)

**Figure 5.1** Flapping of a longitudinal crack in a pressurized fuselage

The problem of crack growth simulation was already dealt with by Saouma [84], Saouma and Ingraffea [85], Swenson [98], Wawrzynek and Ingraffea [106,107], Lin and Abel [52], Remzi, Blackburn and Hellen [72], Kim [43], Fish, Nath and Pandheeradi [29] and Potyondy [67]. Except for Potyondy, they all considered linear crack problems and applied crack propagation criteria that only incorporate the stress intensity factors and thus are defined with respect to the crack tip. Further, most of them (except for Fish et al. [29] and Potyondy [67]) make use of triangular elements because it is relatively easy to generate a triangular grid layout around the
new crack tip after crack extension.
In our research, however, we want to deal with geometrically nonlinear crack problems such as cracked pressurized fuselages. Further, we will make use of quadrilateral elements only because for shell structures they are generally accepted as being superior compared to triangular elements (see also section 4.1). Finally, in addition to a crack growth criterion that is governed by the stress intensity factors only, we will also apply another criterion which includes the elastic T-term as one of the parameters for calculating the crack propagation angle.

The crack growth criteria described in section 5.2 incorporate only the two membrane stress intensity factors ($K_I$ and $K_{II}$) and the elastic T-term. However, in a pressurized fuselage the state of stress around the crack tip can never be purely membrane. Since there will always be bending components at the crack tip ($k_1$ and $k_2$) we are faced with the question how to construct a valid crack growth criterion that includes the effect of these bending components. It is clear that the theory on crack growth criteria in shell structures is still incomplete, and that there is still a lot of work to do in this field.

The present routine for simulating crack propagation can be divided into several steps.
(i) The crack tip coefficients $K_I$, $K_{II}$ and $T$ are computed with the techniques described in chapter 2 and 3.
(ii) The crack propagation angle $\theta_p$ is calculated by substitution of the crack tip parameters in a crack growth criterion.
(iii) The finite element mesh is updated by modeling the new crack increment. The shape

![Diagram](image-url)

**Figure 5.2** Crack growth simulation for a geometrically nonlinear crack problem
of the new crack is not only defined by the crack propagation angle but also by the size of the crack increment. How to choose a proper value for this crack growth increment will be discussed in more detail in section 5.2.3.

The crack growth simulation is proceeded by calculating the crack tip coefficients $K_I$, $K_{II}$ and $T$ again (step(i)). For geometrically nonlinear problems the calculation of the new crack tip coefficients is preceded by a step in which the solution data are mapped from the previous mesh onto the new one (see section 4.2.3). The simulation is then continued by opening the newly modeled crack increment (path II in figure 5.2) without having to ramp up the external load $\lambda_c$ again since this was already done for the initial mesh (path I in figure 5.2).

5.2 Calculation of the crack propagation angle

5.2.1 Crack propagation criteria

For some crack problems the direction of crack propagation seems obvious. For example, for a centre cracked plate subjected to a tensile load perpendicular to the initial crack, the crack path will coincide with the line of symmetry perpendicular to the applied tensile load (fig. 5.3). For pressurized cylindrical shells with longitudinal pure mode I cracks, the same can be said as long as these cracks remain short.

When dealing with mixed mode crack propagation the situation is different. It turns out from experiments that a mixed mode crack propagates in such a way that the pure mode I state of

![Figure 5.3](image)

*Figure 5.3* Observed crack path in a centre cracked plate subjected to a tensile load
stress is approximated along the total crack path. This means that, compared to the centre cracked plate subjected to a tensile load only (fig. 5.3), the crack trajectory for a mixed mode crack is not known a priori.

In the literature one can find several theories for predicting the crack propagation direction, for instance, the maximum principal stress theory [24], the maximum energy release rate theory [39] and the minimum strain energy density theory [87]. All these theories have in common that they are defined with respect to the crack tip itself and only incorporate the two membrane stress intensity factors.

The maximum principal stress theory, originally presented by Erdogan and Sih [24], states that a crack will propagate in a direction perpendicular to the direction of greatest tension, i.e. in the direction where $\sigma_\theta$ reaches a maximum (fig. 5.4). According to the maximum energy release rate theory [39] a crack will propagate in the direction of greatest potential energy release rate, i.e. in the direction where $G(\theta)$ is maximum. The minimum strain energy density criterion [87] finally states that a crack will propagate in the direction in which the strain energy density $S(\theta)$ is minimum while the tensile hoop stress $\sigma_\theta$ has to be positive [96]. For an increasing ratio of $K_{II}/K_I$ the crack propagation angles predicted by these three techniques will start to differ. For most realistic crack configurations, however, the size of $K_{II}$ is a small fraction of $K_I$ and the difference between the predicted crack propagation angles is negligible [106].

In the present research, we only applied the maximum principal stress criterion because it is the most popular of the 3 theories described above. The circumferential stress $\sigma_\theta$ can be written as

$$\sigma_\theta = \frac{K_I}{\sqrt{2\pi r}} \left\{ \frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{3\theta}{2} \right\} + \frac{K_{II}}{\sqrt{2\pi r}} \left\{ - \sin \frac{\theta}{2} - \frac{3}{4} \sin \frac{3\theta}{2} \right\}$$

(5.1)

![Figure 5.4](image)

**Figure 5.4** Definition of the circumferential stress $\sigma_\theta$
The crack propagation angle $\theta_p$ on the basis of the maximum principal stress hypothesis can be calculated now by setting the derivative of $\sigma_0$ with respect to $\theta$ equal to zero. This yields

$$\cos \theta_p = \frac{3s_e^2 + \sqrt{1 + 8s_e^2}}{1 + 9s_e^2}$$  \hfill (5.2)

where

$$s_e = \frac{K_{II}}{K_I}$$  \hfill (5.3)

### 5.2.2 Crack path stability criteria

Swift observed that above a certain crack length longitudinal cracks in unstiffened pressurized fuselages flapped and started to grow in circumferential direction [100,101]. However, due to symmetry, these longitudinal cracks are pure mode I. This means that the observed flapping phenomenon can not be predicted by one of the criteria discussed in section 5.2.1, because the crack propagation angle according to these criteria will only differ from zero for a mixed mode crack.

This implies that there has to be another criterion for predicting when a pure mode I crack path becomes unstable and when it starts to deviate from its initially stable path. Three of these so-called crack path stability criteria, in which the elastic $\mathbf{T}$-term plays an important role, will be summarized below. For further details the interested reader is referred to the work of Zaal [113].

#### 5.2.2.1 Cotterell and Rice

The crack path stability criterion of Cotterell and Rice [20] is obtained from a perturbation analysis of a kinked and slightly curved crack. Starting from the state of stress ahead of the crack tip

$$\sigma_x = \frac{k_I}{\sqrt{2\pi x}} + T + O(\sqrt{x})$$  \hfill (5.4a)

$$\sigma_y = \frac{k_I}{\sqrt{2\pi x}} + O(\sqrt{x})$$  \hfill (5.4b)
Figure 5.5  Small extension of a crack according to Cotterell and Rice

\[ \sigma_{xy} = \frac{k_{II}}{\sqrt{2\pi x}} + O(\sqrt{x}) \]  \hfill (5.4c)

they derived an expression for the mode II stress intensity factor at the tip of a, so far unknown, crack extension (fig. 5.5).

\[ K_{II} = k_{II} + \frac{1}{2} k' (l) k_1 - \frac{2}{\sqrt{\pi}} T \int_0^l \frac{k'(x)}{\sqrt{l-x}} \, dx \]  \hfill (5.5)

In the above expressions \( k_1 \) and \( k_{II} \) are the mode I and mode II stress intensity factors at the tip of the original crack, \( K_{II} \) is the mode II stress intensity factor at the tip of the new crack extension and \( l \) represents the total amount of crack extension (fig. 5.5). Assuming that the mode II stress intensity factor remains zero while the crack extends, they derived an approximate solution for the shape \( \kappa(x) \) of the crack extension.

\[ \kappa(x) = \theta_0 x \left[ 1 + \frac{8}{3} \frac{T}{k_1 \sqrt{\pi}} \frac{2x}{\sqrt{l-x}} \right] \]  \hfill (5.6)

where

Figure 5.6  Influence of the elastic T-term on the stability of an extending crack
\[ \theta_0 = -2 \frac{k_{II}}{k_1} \] (5.7)

From equation (5.6) it can be seen that a negative elastic T-term results in a decreasing slope for \( \kappa(x) \) and thus a stable crack path (fig. 5.6). On the other hand if \( T \) is positive \( \kappa(x) \) will show an increasing slope and thus the crack path is unstable.

### 5.2.2.2 Sumi et al.

Sumi et al. [94,95] extended the work of Cotterell and Rice by including one more term in the perturbation analysis and starting from a state of stress ahead of the crack tip which also incorporates the higher order terms corresponding to \( \sqrt{x} \). Thus

\[
\sigma_x = \frac{k_1}{\sqrt{2\pi x}} + T + b_1 \frac{x}{2\pi} + O(x) \tag{5.8a}
\]

\[
\sigma_y = \frac{k_1}{\sqrt{2\pi x}} + b_1 \frac{x}{2\pi} + O(x) \tag{5.8b}
\]

\[
\sigma_{xy} = \frac{k_{II}}{\sqrt{2\pi x}} + b_{II} \frac{x}{2\pi} + O(x) \tag{5.8c}
\]

To describe the crack path extension Sumi et al. introduced three variables

\[
\kappa(x) = \alpha_1 x + \alpha_2 x^{3/2} + \alpha_3 x^2 + O(x^{5/2}) \tag{5.9}
\]

where \( x \) denotes the amount of crack extension measured in the direction of the original straight crack path and \( \kappa \) again represents the deviation of the crack path from this line (fig. 5.5). The expression that Sumi et al. derived for the mode II stress intensity factor \( K_{II} \) at the tip of the crack extension looks as follows

\[
K_{II} = (k_{II} + \frac{1}{2} \alpha_1 k_1) + (\frac{3}{4} \alpha_2 k_1 - 2 \frac{\sqrt{2}}{\sqrt{\pi}} \alpha_1 T) \frac{l}{\sqrt{2}}
\]

\[
+ \left[ b_{II} \frac{\alpha_1}{4} b_1 - \frac{3}{4} \sqrt{2\pi} \alpha_2 T + \alpha_3 k_1 + \alpha_1 k_1 \left( \frac{1}{2} k_{11} - k_{22} \right) + k_{II} k_{22} \right] l \tag{5.10}
\]

In this expression \( k_{11} \) and \( k_{22} \) introduce the influence of the far field stress and displacement boundary conditions on the stress field at the tip of the extending crack. Assuming that \( k_1 \) and
\( k_{II} \) are the stress intensity factors of the original straight crack path, and that \( K_{II} \) remains zero while the crack extends. Sumi et al. obtained the following equations for the crack extension coefficients \( \alpha_1, \alpha_2 \) and \( \alpha_3 \).

\[
\alpha_1 = -2 \frac{k_{II}}{k_1} \quad (5.11a)
\]

\[
\frac{\alpha_2}{\alpha_1} = \frac{8}{3} \sqrt[4]{\frac{2T}{\pi k_1}} \numL^* \quad (5.11b)
\]

\[
\frac{\alpha_3}{\alpha_1} = \left[ \frac{k_1(2k_{22} - k_{11}) + b_1}{2} \right] + \frac{1}{2k_1} \left( \frac{T}{k_1} \right)^2 \numL^* \quad (5.11c)
\]

where

\[
\alpha_2^* = \alpha_2 \numL^* \quad (5.12a)
\]

\[
\alpha_3^* = \alpha_3 \numL^* \quad (5.12b)
\]

and \( \numL^* \) is a characteristic length (e.g. the crack length).

For their crack path stability criterion Sumi et al. give the following expression

\[
C = \frac{\alpha_2^*}{\alpha_1} + \frac{\alpha_3^*}{\numL^*} \quad (5.13)
\]

where

- \( C < 0 \) predicts a stable crack path and
- \( C > 0 \) predicts an unstable crack path.

This criterion provides four different situations. If both \( \alpha_2^*/\alpha_1 \) and \( \alpha_3^*/\alpha_1 \) are less than zero the crack path is always stable (fig. 5.7). If \( \alpha_2^*/\alpha_1 \) is less than zero and \( \alpha_3^*/\alpha_1 \) is greater than zero an initially stable crack path will become unstable after some crack extension if

\[
\left( \frac{\alpha_2^*}{\alpha_3^*} \right)^2 < \frac{l}{\numL^*} \quad (5.14)
\]

On the other hand if \( \alpha_2^*/\alpha_1 \) is greater than zero and \( \alpha_3^*/\alpha_1 \) is less than zero, an initially
unstable crack path is stabilized after some distance under the condition in equation (5.14). Finally if both $\alpha_2^*/\alpha_1$ and $\alpha_3^*/\alpha_1$ are greater than zero the crack path is unstable for any extension $l$.

In addition to the crack path stability criterion of Cotterell and Rice [20] the criterion of Sumi et al. [94,95] not only predicts whether the original crack is stable or not, but also considers the increase or decrease of crack path stability while the crack deflects.
5.2.2.3 Finnie and Saith

The stability criterion of Finnie and Saith [27] is based on the hypothesis of Erdogan and Sih [24] that a crack will grow in a direction where the tensile stress component in circumferential direction reaches a maximum. Erdogan and Sih applied this theory in the crack tip itself and found $K_1$ and $K_{II}$ to be the only coefficients that determine the crack propagation angle. Williams and Ewing [109] suggested to apply the above theory at a certain material dependent distance $r_c$ away from the crack tip. This means that, in contrast to Erdogan and Sih’s criterion, the elastic T-term can now affect the size of the crack propagation angle.

For $\sigma_\theta$ at the crack tip one can write

$$\sigma_\theta = \frac{K_1}{\sqrt{2\pi r}} \left\{ \frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{3\theta}{2} \right\} + \frac{K_{II}}{\sqrt{2\pi r}} \left\{ -\frac{3}{4} \sin \frac{\theta}{2} - \frac{3}{4} \sin \frac{3\theta}{2} \right\} + T \sin^2 \theta + O(\sqrt{r}) \quad (5.15)$$

Taking the derivative with respect to $\theta$ yields

$$\frac{\partial \sigma_\theta}{\partial \theta} = \frac{K_1}{\sqrt{2\pi r}} \left\{ -\frac{3}{8} \sin \frac{\theta}{2} - \frac{3}{8} \sin \frac{3\theta}{2} \right\}$$

$$+ \frac{K_{II}}{\sqrt{2\pi r}} \left\{ -\frac{3}{8} \cos \frac{\theta}{2} - \frac{9}{8} \cos \frac{3\theta}{2} \right\} + 2T \sin \theta \cos \theta \quad (5.16)$$

As already mentioned in section 5.2.1 the stress $\sigma_\theta$ at some distance away from the crack tip can have a maximum only if its derivative with respect to $\theta$ is zero. Zaal [113] solved this problem for $K_{II} = 0$ and he found that the crack propagation angle for a pure mode I crack can only be different from zero if

$$T > T_c = \frac{3}{8} \frac{K_1}{\sqrt{2\pi r_c}} \quad (5.17)$$

The corresponding crack propagation angle $\theta_p$ can be determined as

$$\cos \frac{\theta_p}{2} = s_F + \sqrt{\frac{1}{2} + s_F^2} \quad (5.18)$$

where
\[ s_F = \frac{3}{32} \frac{K_I}{\sqrt{2\pi r_c}} \]  

(5.19)

This means that, according to the crack path stability criterion of Finnie and Saith (eq. 5.17), for \( T < T_c \) the stress \( \sigma_\theta \) has a maximum only at the line of local symmetry ahead of the crack tip (fig. 5.8). However for \( T > T_c \) there will be a maximum at either side of this line of symmetry. The exact position of these maxima depends on the size of \( T \) with respect to \( K_I \).

\[ T = 0.9 \ T_c \quad \sigma_\theta_{\text{max}} \quad T = 1.0 \ T_c \quad \sigma_\theta_{\text{max}} \quad T = 1.1 \ T_c \quad \sigma_\theta_{\text{max}} \]

\[ 28^\circ \]

**Figure 5.8**  The maximum circumferential stress \( \sigma_\theta \) for different values of \( T \)

In contrast to Erdogan and Sih's criterion [24] the criterion of Finnie and Saith can predict a nonzero crack propagation angle also for a pure mode I crack. The question remains how the crack will propagate, because there will be two equally sized maxima for the circumferential stress \( \sigma_\theta \) at both sides of the line of symmetry (fig. 5.8). Ramula and Kobayashi [68] observed that cracks, that propagated with a very high speed through brittle material often branched into two cracks. However, Kosai et al. [46] showed that for aluminum the crack front speed is so small that crack branching is not likely to place. Therefore we will not consider this phenomenon in our research.

If \( K_{II} \) is not equal to zero there will be maxima for \( \sigma_\theta \) at both sides of the line of symmetry ahead of the crack tip which are not of the same size and at the same position. This means that for this case there is only one absolute maximum for the circumferential stress \( \sigma_\theta \) and thus only one option for crack propagation.

### 5.2.3 Size of the crack growth increment

Numerical simulation of crack propagation with a finite element code involves some approximations with respect to the physical crack problem. One of these approximations stems from the fact that the crack growth process of a crack, which actually propagates in a continuous fashion, must be discretized. Modeling considerations force us to simulate the behavior of a continuously propagating crack by making use of discrete crack growth steps, which in turn means that the calculated crack path can differ from the physical path. This can
be explained on the basis of one of the crack growth criteria discussed in the previous sections. We will use Finnie and Saith's criterion because this crack growth criterion will also be applied in sections 5.4-5 in the examples. However, note that this explanation can be based on any of the crack growth criteria discussed in sections 5.2.1-2.

In our simulation the physical crack path is described by a number of straight crack extension $d\alpha$ (fig. 5.9). After calculating the crack growth direction at the current crack tip ($x = 0$), the crack is extended over a distance $d\alpha_n$ in the calculated direction. However, during the extension of the crack, the state of stress (and thus the crack growth direction) at the intermediate crack tip position may change. This means that the membrane part of the circumferential stress along the crack extension $d\alpha_n$ can be written as

$$\sigma_\theta (x, \theta) = (K_I + g_I(x)) f_I(\theta) + (K_{II} + g_{II}(x)) f_{II}(\theta) + (T + g_T(x)) f_T(\theta)$$

(5.20)

In this equation, $K_I$, $K_{II}$ and $T$ are the membrane mode I and mode II stress intensity factors and the elastic T-term respectively at the initial crack tip ($x = 0$), while the functions $g_I(x)$, $g_{II}(x)$ and $g_T(x)$ account for the change of these crack tip parameters along the crack extension $d\alpha$. Note that

$$g_I(0) = g_{II}(0) = g_T(0) = 0$$

(5.21)

The crack propagation angle according to Finnie and Saith's criterion is calculated by requiring that

$$\frac{\partial \sigma_\theta}{\partial \theta} = 0$$

(5.22)

![Figure 5.9](image)  
*Figure 5.9*  
Discrete modeling of a continuously propagating crack
For equation (5.20) this yields
\[
\frac{\partial \sigma_\theta}{\partial \theta} = (K_I + g_I(x)) \frac{df_I(\theta)}{d\theta} + (K_{II} + g_{II}(x)) \frac{df_{II}(\theta)}{d\theta} + (T + g_T(x)) \frac{df_T(\theta)}{d\theta}
\] (5.23)

The crack propagation angle \( \theta_{p(n)} \) for the crack extension \( d_{a_n} \) is obtained by solving this equation for \( x = 0 \). However, because the crack tip parameters are dependent on the position along the crack extension, equation (5.23) will not be zero along the total crack growth increment. Actually the residual (nonzero) part of \( \partial \sigma_\theta / \partial \theta \) along the crack extension \( d_{a_n} \) can be written as
\[
\left( \frac{\partial \sigma_\theta}{\partial \theta} \right)_T = g_I(x) \frac{df_I(\theta)}{d\theta} + g_{II}(x) \frac{df_{II}(\theta)}{d\theta} + g_T(x) \frac{df_T(\theta)}{d\theta}
\] (5.24)

and this residue can be seen as a measure for the error in crack growth direction along the crack extension \( d_{a_n} \). In a qualitative sense one can state that this error depends on the variation of the functions \( g_I(x), g_{II}(x) \) and \( g_T(x) \) with respect to \( x \); if the gradient of these functions with respect to \( x \) increases, one may expect the error in crack growth direction along the crack extension \( d_{a_n} \) also to increase.

The most straightforward method to deal with this problem is to decrease the size of the crack increment if the gradient of the functions \( g_I(x), g_{II}(x) \) and \( g_T(x) \) (i.e. the gradient of the crack tip parameters) with respect to the crack growth direction \( x \) increases. However, the question is how to determine the variation of \( K_I, K_{II} \) and \( T \) along the crack extension. One way to do this is by considering the size of the crack propagation angle \( \theta_{p(n+1)} \) for the next following crack extension \( d_{a_{n+1}} \). This new crack propagation angle is calculated by setting equation (5.24) equal to zero and thus is proportional to the error in crack growth direction along the crack extension \( d_{a_n} \). In this way the error along the crack extension \( d_{a_n} \) is determined a posteriori, i.e. after the crack has been extended over a distance \( d_{a_n} \).

However, the crack propagation angle calculated at the current crack tip (\( x = 0 \)) also gives some information about the dependence of the crack tip parameters on the crack length. If this crack propagation angle is 'rather large' (for instance, above \( 10^6 \)) one may expect a relatively large error for the crack growth direction along the crack extension \( d_{a_n} \) and therefore the applied crack increment must be small. On the other hand, if the crack propagation angle is 'small' (for instance, below \( 2^6 \)) the error along the crack extension will also be small. For this case one can take a relatively large crack growth step.

In our research we did not automate this process of choosing a proper size for the crack
growth increment. In most of the examples, dealt with in the next section, we used a constant (relatively small) crack growth increment. Comparing our results with experimental data gives us confidence that our choice for the size of the crack growth increment gives accurate results for the crack path.

In order to get an idea about the influence that the size of the crack growth increment has on the calculated crack path, we determined one of the crack trajectories for the DCB specimen (section 5.4.2) with two different crack growth increments. Although the considered crack branches off over an angle of almost 90°, the influence that the size of the crack increment has on the calculated crack path is shown to be of minor nature.

5.2.4 Discussion

As described above, there are several theories for predicting the crack propagation direction. Some of them consider the state of stress in the crack tip itself and therefore are governed by the stress intensity factors only. Others also incorporate one [20,27] or more [94,95] of the higher order terms in the William's eigenfunction expansion for the state of stress around the crack tip.

The first group of theories can only predict a nonzero crack propagation angle if $K_{II}$ at the current crack tip is not equal to zero. However, from experiments [51,100,101] it became clear that a pure mode I crack can also deviate from its initially straight path. This crack path instability can only be predicted by criteria which incorporate one or more of the higher order terms for the state of stress around the crack tip.

Zaal [113] applied some of the crack path stability criteria described above to investigate the problem of crack flapping in a pressurized fuselage. He concluded that the results obtained with the theory of Finnie and Saith [27] showed the closest agreement with the experimental results of Swift [100,101].

In our research we will make use of the maximum principal stress criterion [24] and the crack path stability criterion of Finnie and Saith [27]. These two techniques, which show much resemblance, will be applied to both linear as well as geometrically nonlinear problems, and the predicted crack trajectories are compared with experimental results found in literature.
5.3 Discrete modeling of crack propagation

5.3.1 Historical review

The simplest technique for simulating crack propagation involves symmetrical cracks which remain at the line of symmetry [18]. For these cases, one only has to change the boundary conditions to model the extending crack (fig. 5.10). The shortcoming of this approach is that crack propagation along a plane of symmetry is the exception, not the rule.

![Diagram of crack propagation](image)

*Figure 5.10* Crack propagation by changing the boundary conditions

![Diagram of crack propagation](image)

*Figure 5.11* Crack propagation using the nodal release technique

The first application of crack propagation for non-trivial problems was the nodal release technique [60,61]. The crack is assumed to propagate along existing element sides by releasing the element nodes one by one (fig. 5.11). With this technique the crack trajectories are not constrained to straight lines. The major drawback of this technique is that considerable
effort may be necessary to generate a finite element mesh where the expected crack trajectory (which is usually not known a priori) is situated along element sides.

Another technique, introduced by Rashid [71] in 1968, is the so-called smeared crack method. The crack is modeled implicitly through modifications of the material constitutive relations for the elements along the crack path. In short, when the tensile stress in an element reaches a critical value, the constitutive relations for that element are changed so that the element has little or no stiffness in the direction of this tensile stress. This technique also has some shortcomings. First of all the results are highly sensitive to the finite element mesh used [8]. Furthermore, the stress along the crack faces will become inaccurate when the direction of crack propagation is not parallel to the element sides [82].

The first attempt to model crack propagation without regards to an existing finite element mesh was done by Saouma [84] and Saouma and Ingraffea [85]. With this technique elements which are crossed by the crack path are simply split into a number of smaller elements (fig. 5.12). The total mesh is only locally modified around the crack region. Meshing templates were developed for all possible topologies of a crack entering or crossing an element. Therefore the remeshing process becomes one of identifying the particular topology present
and using the appropriate template to remesh the problem. For the first time, arbitrary
curvilinear crack propagation was modeled and the remeshing process was automated.
However, the problem of this approach is that the resulting meshes are frequently of poor
quality.
Swenson [98] introduced the so-called delete and fill technique to model crack propagation.
Here the strategy is to delete a group of elements in a region around the crack tip, extend the
crack tip into this region and fill the region around the new crack tip again with well-shaped
elements (fig. 5.13). In this approach the number, size and shape of the crack tip elements can
be controlled.
Most of the remeshing techniques described above make use of triangular elements to model a
new crack increment. However, as already mentioned in the introduction, quadrilateral
elements are generally accepted as being superior compared to triangular ones when dealing
with shell structures. Therefore our concern is to develop a crack propagation technique that
uses quadrilateral elements only.

5.3.2 Present remeshing strategy

Chapter 4 discussed the development of the three basic tools to perform adaptive mesh
refinement for geometrically nonlinear problems. An important aspect of this refinement
policy is the hierarchical structure of the generated elements. This hierarchical structure not
only facilitates the mesh refinement process but also offers the opportunity to coarsen a
previously refined mesh where necessary.
Therefore one of the main requirements for the routine used for modeling crack propagation is
that it has to affect the hierarchical element structure as little as possible. Moreover, it has to
be able to generate a new mesh of quadrilateral elements around a crack increment for an
arbitrary crack propagation angle and size of this increment. Finally, the mesh around the new
crack tip has to be symmetrical with respect to this crack tip in order to obtain accurate results
for the stress intensity factors (see section 5.3.2.4).
The developed remeshing routine that meets these requirements is dealt with step by step in
the next following sections. These successive steps will be explained on the basis of the
example of a square plate with an initial edge crack of 100.0 mm (fig. 5.14). A fictitious crack
increment of 35.0 mm will be modeled under an angle of 40°. For each of the steps discussed
below the intermediate finite element meshes will be plotted.
5.3.2.1 Level equalization

The meshing routine that we propose starts from a mesh configuration where all the elements around the new crack increment are of the same element level. Therefore the first step is to refine all these elements up to a level which is chosen in such a way that the length of the sides of these refined elements is smaller than the size of the new crack increment.

This level equalization step is performed by first calculating the position of the new crack tip and next refining all elements which have at least one node within a distance of 2 times the new crack increment to the old or new crack tip position.

The initial mesh for the example in figure 5.14 consists of 4 level 1 elements of 100x100 mm. By requiring the elements around the new crack increment to be of level 5 we obtained an intermediate finite element mesh as plotted in figure 5.15.

5.3.2.2 Crack propagation

Before starting the actual modeling of the new crack increment, a search is made for the node in the intermediate finite element mesh which is closest to the new crack tip location. This node is shifted to this position and will serve as the new crack tip.

![Diagram showing geometry of a cracked square plate with dimensions and angles labeled.](image)

*Figure 5.14* Geometry of a cracked square plate
Figure 5.15  Intermediate finite element mesh of the square plate after level equalization

Figure 5.16  Crack propagation along element sides

Figure 5.17  Crack propagation along element diagonals
Figure 5.18 Remeshing element $e_1$ and $e_2$ in order to obtain valid elements with the crack propagation routine.

Next the crack faces between the old and the new crack tip are modeled. It is obvious that these crack faces have to be constructed of element sides. The most straightforward method to achieve this is to align one of the element sides with these new crack faces. However, this technique will only give satisfying results for a crack propagation angle $\theta_p$ close to 0° or 90° (fig. 5.16). For a propagation angle somewhere in between 0° and 90° a remeshing technique is developed that generates new quadrilateral elements along the crack faces (fig. 5.17).

Crack propagation modeling is started at the old crack tip and is performed step by step with one of the two above described techniques until the new crack tip is reached. This means that for each step a choice has to be made which of the two techniques (fig. 5.16 or fig. 5.17) will give the best results.

There are a few cases in this modeling process which need special care. Element corners, for instance, may become 180° when applying the element side alignment technique after a remeshing step (fig. 5.18). If this situation shows up, the previously generated elements are modified in order to prevent the appearance of bad-shaped elements.

Another exceptional case occurs if the 'free' node $p_1$ of an element $e_1$ which has to be remeshed is connected to only 2 elements ($e_2$ and $e_3$) at one side of the new crack face (fig.

Figure 5.19 Crack propagation strategy for an element which is connected to only 2 elements at one side of the crack face.
Figure 5.20 Intermediate finite element mesh of the cracked square plate after crack propagation

5.191). When performing a remeshing step for element $e_1$ one will end up with 2 elements $e_1'$ and $e_2'$ that have 3 nodes in common (fig. 5.192) and thus are invalid. However, this problem can simply be overcome by shifting node $p_1$ to the crack face (fig. 5.193) instead of remeshing element $e_1$.

The new crack increment may grow 'into' a parent element which will cause the involved child elements to be separated from each other. In addition, there will be a local loss of the hierarchical mesh structure around the new crack increment due to the remeshing technique as described in figure 5.17. Therefore the elements which are directly involved in the remeshing process around the new crack increment are 'locked' at their current level. This means that, in order to preserve the shape of the new crack increment, it is not possible to coarsen these elements again.

It is obvious that the nodes along the new crack faces have to be duplicated in order to define the upper and lower crack faces. Moreover, the old and new crack tip nodes must be fixed in order to preserve the shape of the crack during the mesh smoothing step.

Applying the above discussed techniques to the considered example of an edge cracked plate will yield an intermediate mesh as plotted in figure 5.20.
Figure 5.21 Intermediate finite element mesh of the cracked square plate after level restoring

5.3.2.3 Level restoring

The next step is to coarsen the finite element mesh again at locations which are not involved in the crack propagation process. This is done by releasing a group of connected child elements and introducing their parent element again in the finite element mesh (see section 4.2.1.2). Applying this level restoring routine to the current example will result in a mesh as pictured in figure 5.21. As one can see most of the refined elements generated by the level equalization step are removed again.

5.3.2.4 Making the crack tip mesh symmetrical

A number of tests showed that the most accurate results for the stress intensity factors can be obtained with crack tip meshes which are symmetrical with respect to the crack tip. It was found that the shape of the enriched element zone has a more pronounced influence on the accuracy of the mode II stress intensity factor than on the accuracy of the mode I stress intensity factor, while the best results were obtained with a symmetrical shape of the crack tip mesh. Therefore a routine is developed to generate such a symmetrical element zone around the new crack tip.
Figure 5.22 Generating a symmetrical crack tip mesh suitable for application of the mode enrichment technique

Figure 5.23 Remeshing one of the crack tip elements and one of its neighbors in order to obtain 4 crack tip elements

First, the nodes of the 4 elements which are directly connected to the crack tip are repositioned in order to make this element region symmetrical with respect to the crack tip (fig. 5.22). Next a layer of new elements is introduced in this element region. Moreover, mode enrichment is applied as already described in section 3.2.2.2. It is obvious that all the nodes involved in this symmetrical element region have to be fixed in order to prevent them from being shifted due to mesh smoothing.

If the number of crack tip elements is not equal to 4 one has to add or release crack tip elements before making the crack tip mesh symmetrical. If, for instance, the number of crack tip elements is equal to 3, a new element can be added by remeshing one of the old crack tip elements together with one of its neighbors (fig. 5.23). This technique can also be used for releasing a crack tip element.

It is possible that one may obtain low quality elements around the symmetrical crack tip mesh due to fixing the nodes belonging to this crack tip mesh. This undesirable effect can be remedied by application of the same remeshing technique as is used for increasing or decreasing the number of crack tip elements (fig. 5.24).
**Figure 5.24** Improvement of the element quality around the crack tip mesh

1. initial mesh
2. adding a crack tip element
3. making the crack tip mesh symmetrical
4. remeshing low quality elements
5. smoothing

**Figure 5.25** Succeeding steps involved in the construction of a symmetrical smoothed finite element mesh around the crack tip

To show how the routine described above works, an example is chosen for which the crack tip mesh as obtained from the crack propagation routine consists of 3 elements only (fig. 5.25). As can be seen, the quality of the final mesh obtained with the techniques described above is quite good.

For the example of the edge cracked square plate dealt with in this chapter, the final mesh is pictured in figure 5.26.
5.4 Calibration tests on some classical problems

5.4.1 Biaxially loaded centre cracked plate

The first example concerns crack propagation in a centre cracked plate subjected to a biaxial load (fig. 5.27). Leevers et al. [51] performed some experiments for such a specimen where they applied a number of different load biaxiality ratios. They used a PMMA plate (polymethyl methacrylate) for which Ramula and Kobayashi [68] calculated the material property $r_c$, needed for applying Finnie and Saith’s crack propagation criterion, as $r_c = 1.3$ mm.

In their experiments Leevers et al. [51] observed that the direction of crack propagation depends on the load biaxiality ratio $\chi$ (fig. 5.27). For small values of $\chi$ (less than 1.0) the propagating crack remains at the line of symmetry. However, increasing the stress biaxiality ratio causes the crack to deviate from this line of symmetry while the amount of path deviation depends on the size of $\chi$.

Later on Cotterell and Rice [20] and Sumi et al. [94,95] applied their crack path stability criteria to this problem, and they both predicted the crack path to become unstable for a load biaxiality ratio that exceeds 1.0. According to figure 5.27 these predictions agree well with the experimental results of Leevers et al. [51].
Figure 5.27 Crack trajectories in the biaxially loaded centre cracked plate as observed by Leewers and Radon

We computed the crack path of a centre cracked square plate \((h = W = 1000.0 \text{ mm}, a = 75.0 \text{ mm} \text{ and } r_c = 1.3 \text{ mm})\) subjected to a biaxial load for three different values of the biaxiality ratio \(\chi\). The crack is assumed to propagate in an anti-symmetrical way with respect to the y-axis. Leewers et al. [51] observed that for all but 1 specimen (of a total of about 30) the experimental crack path turned out to be symmetric with respect to the y-axis, and thus the above assumption seems reasonable.

First the crack path is calculated with the maximum principal stress criterion of Erdogan and Sih [24]. As stated in section 5.2.2 this criterion can not predict a nonzero crack propagation angle for a pure mode I crack. This implies that with this criterion the behavior of a (initially pure mode I) crack propagating in a biaxially loaded plate (see fig. 5.27) can not be described without artificial help. Therefore a trial imperfection of \(2.0^\circ\) is introduced for the initial crack in order to induce a nonzero value for \(K_P\) at the crack tip. It will be shown that with this initial imperfection the crack behavior as pictured in figure 5.27 can be described reasonably well with Erdogan and Sih's criterion.

Another crack growth simulation is performed with the crack path stability criterion of Finnie and Saith [27]. This criterion can also predict crack path deviation for a pure mode I crack and thus the need for the initial path imperfection has vanished.

The elastic T-term is calculated as explained in chapter 3 by introducing 8 new element layers within the 4 crack tip elements of the overall mesh. However, in contrast to the examples in chapter 3, the tangential displacement field along the crack faces now also contains a contribution due to membrane mode II crack opening. These mode II displacements have to be eliminated before calculating the elastic T-term. Because the mode II displacement solution is anti-symmetric with respect to the crack faces, this can be achieved by averaging the nodal displacements along the upper crack face with the corresponding nodal displacements along the lower crack face before calculating the elastic T-term.
**Figure 5.28** Crack trajectories in the biaxially loaded plate obtained with the maximum principal stress criterion of Erdogan and Sih

**Erdogan and Sih's criterion**

The crack growth data obtained with Erdogan and Sih's criterion are printed in table 5.1. For $\chi = 1.0$ the obtained crack trajectory almost coincides with the line of symmetry. Increasing the load biaxiality ratio to 2.0 will cause a considerable deviation of the crack path from the line of symmetry. For a load biaxiality ratio of 3.0 the crack will branch off over an angle of about $75^\circ$-$80^\circ$ and crack propagation will proceed almost perpendicular to the more intense tensile stress field in x-direction. As one can see these numerically obtained crack trajectories for $\chi = 1.0$, 2.0 and 3.0 (fig. 5.28) are in reasonable agreement with the experimental results of Leevers et al. [51] (fig. 5.27).

In figure 5.29 $K_I$ and $K_{II}$ are plotted versus the crack length for the three load cases. As expected $K_{II}$ is always small compared to $K_I$. For $\chi = 2.0$ the crack deviates considerably from the line of symmetry (fig. 5.28). However, $K_I$ is still dictated by the load applied in y-direction. For $\chi = 3.0$ the change of direction of the propagating crack causes a sharp increase of $K_I$ because of the more severe stress field in x-direction.

To get an idea about the finite element meshes involved in these crack growth simulations, three of these meshes (for $\chi = 3.0$) are plotted in figures 5.30a-c.
Finnie and Saith's criterion

The agreement between the crack trajectories obtained with Finnie and Saith's criterion [27] (fig. 5.31) and the ones obtained with the criterion of Erdogan and Sih (fig. 5.28) is close. For $\chi = 1.0$ the 2 results are almost identical (within 5.0 %) because the elastic T-term is less than zero along the total crack path. Therefore these results are not pictured again in figures 5.31-33. For $\chi = 2.0$ the elastic T-term is positive along the main part of the crack path (fig. 5.32), resulting in a deviation from the line of symmetry which is a little bit greater as compared to the corresponding crack path in figure 5.28.

**Table 5.1** Crack growth data for the biaxially loaded plate using Erdogan and Sih's criterion

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<th>$y$</th>
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<th>$K_{II}/\sigma$</th>
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**Table 5.1a : $\chi = 1.0$**

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**Table 5.1b : $\chi = 2.0$**

**Table 5.1c : $\chi = 3.0$**

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</table>
Figure 5.29 K_I and K_II for the extending cracks using Erdogan and Sih's criterion

Figure 5.30a Finite element mesh for the extending crack: $\chi = 3.0$, $a = 125.0$ mm
Figure 5.30b  Finite element mesh for the extending crack: $\chi = 3.0$, $a = 225.0$ mm

Figure 5.30c  Finite element mesh for the extending crack: $\chi = 3.0$, $a = 350.0$ mm
Figure 5.31 Crack trajectories in the biaxially loaded plate obtained with the criterion of Finnie and Saith

However, for $\chi = 3.0$ the crack path changes direction more rigorously as compared to Erdogan and Sih's criterion (fig. 5.28). This is due to the large positive elastic T-term for the initial cracked model under these loading conditions. The size of this elastic T-term (as compared to the stress intensity factors) is a measure for the difference between the crack

Table 5.2 Crack growth data for the biaxially loaded plate using Finnie and Saith’s criterion

<table>
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<tr>
<th>x</th>
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<th>$K_I/\sigma$</th>
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<table>
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Figure 5.32  Elastic T-term for the propagating cracks in the biaxially loaded plate using the criterion of Finnie and Saith

Figure 5.33  $K_I$ and $K_{II}$ for the extending cracks using Finnie and Saith's criterion
propagation angles predicted by Erdogan and Sih's criterion and by Finnie and Saith's criterion. The larger the elastic T-term, the greater the difference between the predicted crack propagation angles.

After the crack has branched off over an angle of about 60° the elastic T-term changes sign and becomes negative. Consequently the influence of the elastic T-term on the crack propagation angle for the remaining part of the crack path will be negligible, and the shape of this part of the crack trajectory will be almost identical for the 2 different crack growth criteria.

Comparing these results with the experimental work of Leevers et al. [51] it seems that now the resemblance is even better than for the results obtained with Erdogan and Sih's criterion (fig. 5.28). Especially for $\chi = 3.0$ the sharp change of direction that was found by Leevers et al. (fig. 5.27) is approximated even closer now (fig. 5.31) as compared to Erdogan and Sih's criterion. Therefore one may conclude that the crack trajectories according to Finnie and Saith's criterion seem to match the experimental findings better than the results obtained with the criterion of Erdogan and Sih.

### 5.4.2 Double cantilever beam

The second (linear) example concerns crack propagation in a so-called double cantilever beam specimen (DCB). Sumi [93] did some experiments for this specimen and investigated the influence of some of the specimen dimensions on the observed crack path. He found that for $W = 180.0$ mm (figure 5.34) the crack deviated only slightly from the line of symmetry while for $W = 240.0$ mm the crack branched off over almost 90° after some crack extension.

![Figure 5.34](image)

**Figure 5.34** Double cantilever beam specimen (DCB) according to Sumi [93]

The results that we obtained for these two cases ($W = 180.0$ mm and $W = 240.0$ mm) are tabulated in table 5.3 and 5.4 respectively. Note that now a trial path imperfection of 1.0° is introduced for Erdogan and Sih's criterion.
**Erdogan and Sih's criterion**

For case 1 (W = 180.0 mm) the calculated crack path according to Erdogan and Sih's criterion coincides with the line of symmetry (fig. 5.35). This agrees well with the experimental results of Sumi who observed only a slight deviation of the crack path for this case. However, for case 2 (W = 240.0 mm) the sharp change of direction of the crack path that was found in the experiments can only be approximated roughly. The calculated crack path seems to drift away from the experimental results as the crack extends.

**Finnie and Saith's criterion**

The results obtained with the criterion of Finnie and Saith are much closer to the experimental findings. For case 1 the crack first seems to be unstable till x is about 140.0 mm, but then becomes stable again and continues to propagate almost parallel to the line of symmetry. For case 2 the experimental and numerical results agree rather well now. If we look at the experimental crack trajectories one can see some scatter: only 1 experimental result is plotted here.

The mode I stress intensity factor $K_I$ for case 2 increases much faster now than the one obtained with Erdogan and Sih's criterion (fig. 5.36). This is due to the fact that with Finnie

![Graph](image)

*Figure 5.35* Experimental and numerical crack trajectories for the DCB specimens
### Table 5.3: Crack growth data for the DCB-specimen (W = 180.0 mm)

<table>
<thead>
<tr>
<th>x</th>
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### Table 5.4: Crack growth data for the DCB-specimen (W = 240.0 mm)

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<table>
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</table>

and Saith's criterion the crack branches off more sharply and through that reaches the boundary of the specimen at an earlier stage. This same edge effect can be seen for the elastic $T$-term.

In order to get an idea about the influence of the increment size on the obtained crack path another calculation is performed for case 2 where now the crack increment is halved (7.5 mm instead of 15 mm). Although the crack path changes direction rather rapidly, it can be seen that there is only a slight difference between the 2 crack trajectories. This seems to indicate that even for this extreme case (crack deviation over almost 90°) the influence of the increment size on the numerically obtained crack trajectory is not important.
Figure 5.36  $K_I$ and $K_{II}$ for the DCB specimen ($W = 240.0$ mm)

Figure 5.37  The elastic T-term for the DCB specimen ($W = 240.0$ mm)
5.5 Geometrically nonlinear problems

5.5.1 Swift's cylinder

We consider next crack propagation in a longitudinally cracked unstiffened cylinder with closed ends under internal pressure (fig. 5.38). The dimensions and initial crack length for this cylinder are taken from the experiments of Swift [100,101]. He pressurized this cracked cylinder until the crack started to propagate in a dynamic way. After some longitudinal crack extension the crack suddenly branched off and started to propagate in circumferential direction. The observed crack flapping phenomenon caused a rapid decompression of the cylinder and crack growth stopped. A rough sketch of Swift's findings is made in figure 5.38.

![Diagram of cylinder with crack flapping](image_url)

**Figure 5.38** Dimensions and material properties for Swift's cylinder [100,101]

The model that we used in our analysis occupies only half of the cylinder in circumferential direction. The boundary conditions are assumed to be symmetrical except for boundary 4. Due to the presence of a pressure bulkhead this boundary is free to deform in axial direction while the radial displacement has to be uniform. These assumptions imply that the total cylinder has 2 cracks in circumferential direction with a pitch of 180°. Due to the large distance between the two cracks the interaction between these cracks is believed to be small. Moreover, the longitudinal crack is assumed to propagate in a fashion which is symmetrical with respect to a circumferential line through the midpoint of the crack. The idea for this
assumption is inspired by some experiments and accidents [66,100,101] and it implies that crack flapping will occur as soon as the crack starts to deviate from the straight crack path and turns onto a circumferential line. Note that, due to our assumption, only a quarter of the cylinder need be considered (see fig. 5.38).

In our calculations the pressure remains constant during the complete crack growth simulation. However, in reality the pressure will decrease as the crack propagates. Due to this decompression the stress intensity factors for the evolving crack will also decrease and propagation of the crack will stop.

Modeling the influence of decompression on the actual loading of the cracked fuselage is a problem in itself. For a crack that propagates in longitudinal direction, this decompression will occur smoothly because the size of the gap in the fuselage skin increases rather slowly. However, as soon as the crack flaps, the area of the opening will grow much faster and a rapid decompression will occur. In our calculations we will only simulate crack propagation before, during and right after crack flapping. Consequently, as a first approximation we will assume that the influence of decompression on the external loading of the cracked fuselage model is negligible.

Finally, we should mention that for dynamic crack propagation, the stress distribution at the crack tip actually depends on the crack front speed because the material around the extending crack needs some time to adjust to the static singular solution. Instead of the quasi-static crack tip coefficients, one should use the dynamic ones which depend on the crack growth speed [68]. However, Kosai et al. [46] reported that the measured crack front speed in aluminum is so small that the difference between the dynamic and quasi-static crack tip parameters is negligible. Therefore we assume that we can use the quasi-static crack tip parameters for simulating dynamic crack growth in the pressurized fuselage model.

As mentioned in section 5.2.4, Zaal [113] calculated the elastic T-term for Swift’s cylinder (with the routine discussed in chapter 3) to compare the crack path stability results obtained from different criteria with the experimental findings of Swift. He analyzed a number of longitudinally cracked models with different crack lengths and found that, as expected, the elastic T-term at the crack tip (and thus the instability) of a longitudinal crack increases with increasing crack length. However, according to the T-stress criterion of Cotterell and Rice, crack path instability (i.e. change of sign of the elastic T-term from negative to positive) would already occur for cracks of about 50 mm, while the critical crack length in Swift’s experiments seems to be about 300-350 mm. The crack path stability criterion of Finny and Saith [27], which states that the elastic T-term should be sufficiently larger than zero, predicted the crack path to become unstable for cracks of about 150-200 mm which is much closer to the experimental results.
Erdogan and Sih's criterion

The crack trajectory obtained with the criterion of Erdogan and Sih [24] is pictured in figure 5.39 together with the experimental results of Swift. In accordance with the previous example on the cantilever beam, this calculated crack trajectory also seems to drift away from the experimental crack path observed by Swift.

In figure 5.40 the final mesh is pictured in the deformed state. The influence of crack flapping on the local deformation of the model is obvious.

Finnie and Saith's criterion

Now the agreement between the numerically obtained crack trajectory and Swift's experimental results becomes much closer (fig. 5.39). This is due to the positive elastic T-term along the total crack path (fig. 5.42) which has an amplifying effect on the change of direction of the crack path.

In figure 5.41 it can be seen that the values for KI along the two crack trajectories are almost identical. On the other hand the elastic T-term for the crack trajectory obtained with Finnie and Saith's criterion does not increase as fast as for the crack path obtained with the criterion of Erdogan and Sih (fig. 5.42). This implies that the current crack trajectory is 'more stable'.

Table 5.5

Crack growth data for Swift's cylinder

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Table 5.5a : Erdogan and Sih

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Table 5.5b : Finnie and Saith
Figure 5.39  Experimental and calculated crack trajectories for Swift's cylinder

Figure 5.40  Final deformed state for Swift's cylinder using the criterion of Erdogan and Sih
Figure 5.41  
$K_I$ and $K_{II}$ along the crack trajectories for Swift's cylinder

Figure 5.42  
The elastic T-term along the crack trajectories for Swift's cylinder
Figure 5.43  Final deformed state for Swift's cylinder using the criterion of Finnie and Saith

than the one corresponding to Erdogan and Sih's criterion.
Just as the linear problems that were considered in section 5.4 this example also suggests that
the elastic T-term plays an essential role in the growth of a crack and therefore must be
incorporated in the crack growth criterion.

5.5.2 Stiffened fuselage segment

In this section we consider a segment of a fuselage that is representative for a similar segment
taken from a present day airliner [67,70,111]. For this example we will only apply the crack
growth criterion of Finnie and Saith. The total fuselage model contains a periodic distribution
of longitudinal and circumferential cracks which enables us to assume symmetrical boundary
conditions for the chosen segment. The dimensions and material properties of the considered
panel are pictured in figure 5.44.
The analyzed cracks are small compared to the size of the considered shell segment and
therefore the interaction between the different cracks is believed to be negligible. This
statement will be supported by comparing our results with the ones obtained by Rankin et al.
[70] and Potyondy [67].
Rankin et al. [70] dealt with a comparable cracked shell segment. However, they first analyzed the complete fuselage model subjected to a combined load that can occur during flight in order to obtain the kinematic boundary conditions for the cracked shell segment. Next the stress intensity factors were calculated by applying these kinematic boundary conditions (together with the internal pressure) to a refined model of this cracked shell segment. They argued that the problem has to be tackled in this way to account for the influence that the overall (rather complex) load can have on the behavior of a crack in one of the bays. Moreover, there will be some interaction between the behavior of the extending crack in the considered (small) shell segment and the response of the surrounding structure. Although the quality of the calculated stress intensity factors is believed to be very good (due to the realistic overall loads) the computational effort involved in these calculations can become quite substantial because the boundary conditions for the refined shell segment have to be reformulated for each new crack geometry. This reformulating of the kinematic boundary conditions involves a new analysis of the complete fuselage model for all considered crack geometries.

As in the previous example (Swift’s cylinder) the crack is assumed to propagate symmetrically in both directions. In accordance with this assumption, which agrees with the numerical and experimental results reported by Potyondy [67] and Worden [111] respectively, the model in figure 5.44 represents only half of the cracked shell segment. We performed a crack growth simulation for 2 different initial crack configurations (fig. 5.44). In the first case the initial crack with a half crack length of 143.0 mm is positioned halfway between the two longerons. In order to compare our results with the ones obtained by Rankin et al. [70], the tear straps are removed for this first case.

Another crack growth simulation is performed for the case where a crack with a half crack length of 63.5 mm is positioned at a distance of 0.0625 times the longerons pitch to one of the longerons. Now the shell segment is used as pictured in figure 5.44 (with the tear straps) while the crack cuts completely through both the tear strap and the skin.

This second case resembles an experiment performed by Worden [111] who determined the crack path and fatigue life of a fuselage panel with an initial crack along one of the longerons. Potyondy [67] tried to simulate this experiment numerically by using a finite element model in which the stringers are modeled as hat sections and the frames as Z-sections just like in the experiment. The agreement between these numerical results and the experimental findings is close. However, this agreement is not self-evident because there are some (to the author's opinion) important differences between Worden's test set-up and Potyondy's numerical simulation. The discussion of this matter is postponed to the section where our numerical results will be described.

Note that the internal pressure for case 1 and case 2 is not the same. For case 1 the internal pressure as used by Rankin et al. is applied \( (p = 0.05516 \text{ N/mm}^2 = 8.0 \text{ psi}) \) while for case 2 the pressure load is taken from Worden's experiment \( (p = 0.05378 \text{ N/mm}^2 = 7.8 \text{ psi}) \).
Figure 5.44  Segment of a pressurized fuselage

Symmetrically positioned crack

The elastic T-term is small compared to the membrane mode I stress intensity factor along the major part of the crack path. Consequently the crack will remain at the line of symmetry until it reaches one of the frames. Due to the alleviating effect that the frame has on the loading perpendicular to the crack, $K_I$ starts to decrease rapidly in contrast to $T$, which still increases as the crack approaches the frame. Consequently, the influence of the elastic T-term on the crack propagation direction will also increase, and the crack may flap instead of growing under the frame. Although in figure 5.45 this effect is difficult to see, the angle between the final crack direction and the line of symmetry is approximately 20° (table 5.6).

The mode I stress intensity factors obtained by Rankin et al. [70] are always higher than ours. This is caused by the different external loads in longitudinal direction. According to the overall load (including bending) that Rankin et al. applied to the fuselage model, their shell segment is compressed in longitudinal direction. This axial compression will cause an
Table 5.6  Crack growth data for the case of a symmetrically positioned initial crack

<table>
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<th>θ_p</th>
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increase of the bulge out displacements, as compared to our results, and thus higher values for the stress intensity factors. To prove this we made another calculation of the mode 1 stress intensity factor for some of the crack geometries, where now the shell segment is compressed by a load of 50.0 N/mm. This load is believed to approximate the compressive load for Rankin’s case and it is clear that the agreement is now excellent (fig. 5.46).

![Figure 5.45](image-url)  Crack path for the case of a symmetrically positioned initial crack
Figure 5.46  Stress intensity factors for the case of a symmetrically positioned initial crack

Figure 5.47  Elastic T-term for the case of a symmetrically positioned initial crack
Crack close to a longeron

In contrast to Potyondy, who calculated the crack path with the maximum principal stress criterion of Erdogan and Sih, we used Finnie and Saith's criterion thus including the elastic T-term in the calculation of the crack propagation angle. However, because the elastic T-term for this problem (fig. 5.52) remains small compared to the membrane mode I stress intensity factor (fig. 5.51), the inclusion of the elastic T-term in the crack propagation criterion will not have much influence on the crack growth direction. Note that, although the distance of the initial crack to the nearest longeron is not exactly the same for the three cases, the numerically obtained crack trajectories are shifted a little bit in circumferential direction in such a way that the initial cracks coincide.

Despite the difference in stringer and frame modeling and the difference in the size of the tear strap, our crack path is in close agreement with the one obtained by Potyondy. Both the numerical crack trajectories differ a little from the experimental result. These (small) differences may be caused by the effect of neglecting fastener flexibility in both the numerical analyses. The importance of these fastener flexibilities will increase as the distance of the crack to the longerons or frames decreases.

Moreover, one should investigate whether the experimental results are reproducible. Due to
Table 5.7

Crack growth data for the case of an initial crack along one of the longerons

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the complexity of the test model, it is hard to construct a number of specimens that are exactly identical. This implies that there will be a slight variation in some of the properties of the test specimens that might cause some scatter in the experimental crack behavior.

Finally, the direction of the crack path will be affected by the slight difference in material

Figure 5.49

Crack trajectories for the case of an initial crack along one of the longerons
Figure 5.50  Antisymmetrical crack tip loads due to a difference in bulge out displacement

properties in and perpendicular to the rolling direction of the plate material. Worden [111] already noticed that the critical stress intensity factors for these 2 perpendicular directions are not the same (see appendix D). Therefore it is reasonable to assume that rolling will cause some kind of fibre structure in the material and that cracks are inclined to grow parallel to this fibre structure.

It can be seen that the propagating crack will start with a kink, which seems odd but can be

Figure 5.51  Stress intensity factors for the case of an initial crack along one of the longerons
Figure 5.52 Elastic T-term for the case of an initial crack along one of the longerons

explained as follows. Due to the position of the longerons with respect to the crack (fig. 5.50), the bulge out displacement along the lower crack face closest to the longeron will be smaller than the bulge out displacement along the upper crack face. Consequently, the elongation along the lower crack face will be less than the elongation along the upper crack face. Due to this difference in tangential strains along the crack faces, there will be a set of loads at the crack tip similar to those pictured in figure 5.50. These anti-symmetrical loads will induce a membrane mode II stress intensity factor that will cause the crack to propagate with a kink.

As far as the stress intensity factors are concerned, there is a slight difference between the two numerical results. The slope of our $K_I$-curve is larger as compared to Potyondy's results while the initial value is a little bit lower. Further, our values for $k_2$ are smaller than the ones obtained by Potyondy over the whole range of the crack. Part of this discrepancy may be caused by the difference in modeling the longerons and frames and by the fact that the initial crack in Potyondy's model is a little bit closer to the longeron ($d = 11.43$ mm) as compared to our model ($d = 14.65$). However, there are two other reasons which might be a cause for this discrepancy.

First of all, Potyondy analyzed the small fuselage panel as if it is part of a cylinder with open ends. In the experiment, however, a real fuselage model (with closed ends) is used. This means that in the test set-up there will be a significant load in longitudinal direction that is not taken into account in his numerical calculations. These longitudinal stresses will not only
influence the crack path but they will also have an alleviating effect on the stress intensity factors.

Furthermore, Potyondy used the kinematic boundary conditions for the initial crack configuration throughout the complete crack growth simulation for the shell segment. Only the initial crack is analyzed with the 'correct' kinematic boundary conditions, and the 'error' in the results will increase as the crack evolves. These 'errors' can be significant because of the small size of the panel that Potyondy analyzed. This means that the membrane mode I stress intensity factor that is calculated for the propagating crack, will be too low as compared to the results that would be obtained with the reformulated kinematic boundary conditions. In our calculations this problem is of minor importance because we used a larger shell segment and we applied the external loading directly to this segment.

To support our explanation for the discrepancy between the two numerical results we calculated the stress intensity factors $K_1$ and $K_2$ for the initial crack without applying a longitudinal load and thus simulating (in correspondence with Potyondy) an open cylinder under internal pressure. For the initial crack configuration Potyondy used the 'correct' kinematic boundary conditions and thus the agreement between the two finite element models is rather close now. In figure 5.51 one can see that this adjustment indeed makes our results (especially for $K_2$) to approach the values obtained by Potyondy even closer. Furthermore, the

Figure 5.53 Final deformed state for the cracked shell segment
'error' because of applying the 'incorrect' kinematic boundary conditions should increase as the crack propagates and this increase can be seen in figure 5.51.

Finally, the fatigue life of the cracked panel is predicted using an appropriate fatigue crack growth model (appendix D). This so-called Paris law uses the sequence of calculated stress intensity factors to predict the fatigue life of a cracked structure. Actually it only incorporates $K_I$, thus ruling out the influence of the other stress intensity factors on the fatigue life of a damaged structure. In reality, however, one can imagine that the presence of the other stress intensity factors should increase the crack growth rates. Therefore Potyondy proposed to replace $K_I$ in the fatigue crack growth model by the so-called effective stress intensity factor $K_{eff}$ which is defined as

$$K_{eff} = \sqrt{EG_{eff}}$$  \hspace{1cm} (5.25)

where

$$G_{eff} = G_{K_I} + \alpha (G_{K_{II}} + G_{k_1} + G_{k_2})$$  \hspace{1cm} (5.26)
In figure 5.54 the predicted fatigue life is plotted for $\alpha = 1.0$ and the agreement with Potyondy's results as well as with the experimental data is close. Although the initial crack growth rate calculated by Potyondy agrees better with the experimental findings, our total life prediction is in closer agreement with the experiment. It is noted that, because of the high crack growth rates throughout the test (and thus the extreme sensitivity of the predicted fatigue life to small changes in the stress intensity factors) life predictions within a factor of 2 are considered good for this case [67].

5.6 Conclusion

In addition to the tools discussed in chapters 2, 3 and 4, numerical simulation of crack propagation requires a routine for calculating the crack propagation angle and a remeshing technique for modeling the new crack geometry. The different crack propagation criteria that can be found in literature are summarized in section 5.2. Two of them, the principal stress criterion of Erdogan and Sih [24], which only involves the membrane stress intensity factors, and Finnie and Saith's criterion [27], which also incorporates the elastic T-term, are applied to some linear as well as geometrically nonlinear crack problems.

In section 5.3 the routine for modeling crack propagation is discussed. This routine can generate a mesh of all quadrilateral elements around a new crack extension for an arbitrary crack propagation angle and size of the crack growth increment. Crack propagation for linear as well as geometrically nonlinear problems is simulated by applying all the developed routines one after the other and the obtained results are satisfactory. The problem of crack propagation in a centre cracked plate subjected to a biaxial load (section 5.4.1) seems to indicate that the elastic T-term should be incorporated in the crack growth criterion. The second linear example (the DCB specimen in section 5.4.2) corroborates the above statement.

Crack propagation in an unstiffened pressurized cylinder is simulated in section 5.5.1. Again the crack path obtained with Finnie and Saith's criterion seems to match the experimental findings better than the results obtained with the criterion of Erdogan and Sih.

For the stiffened pressurized fuselage with a symmetrically positioned crack (section 5.5.2), the calculated stress intensity factors are in close agreement with the ones calculated by Rankin et al. [70]. The elastic T-term is small compared to the membrane mode I stress intensity factor along the major part of the crack path. Therefore the elastic T-term will not have much influence on the direction of crack propagation. When approaching the frame, however, $K_1$ decreases rapidly in contrast to T which still increases. Consequently the influence of T on the crack propagation direction will also increase and this may cause the crack to flap instead of growing under the frame.
The same holds for the case where the initial crack is positioned close to one of the longerons. The results (crack path, stress intensity factors and fatigue life) for this second case are shown to be in reasonable agreement with the numerical results of Potyondy [67] and the experimental findings of Worden [111].
Chapter 6
Conclusions

6.1 General crack growth simulation

This thesis discusses the development of an algorithm for simulating general crack propagation in a pressurized fuselage. Compared to the classical crack problems, such as, for instance, a centre cracked plate in tension, the behavior of a crack in a pressurized fuselage is much more difficult to handle because of the geometrically nonlinear effects. These nonlinearities arise from the curvature of the fuselage and from the crack opening resistance due to the stresses tangential to the crack faces. Moreover, there will be a significant influence of the stiffening components (frames and longerons) on the crack growth process.

In order to deal with the problem of a cracked pressurized fuselage some of the techniques that already exist for linear problems (e.g. the mode enrichment technique) had to be extended to the geometrically nonlinear range. Furthermore, we had to develop some new numerical tools that are especially important when dealing with geometrically nonlinear problems (e.g. the displacement mapping technique).

The final algorithm, that we developed for simulating crack propagation in a cracked aircraft fuselage under internal pressure can be divided in a number of steps.

step 1 Define the geometry and the initial finite element mesh of the considered crack problem.

The geometry of the model (including boundary conditions and external loads) has to be defined only for the initial finite element mesh because the mesh generator automatically maintains this information during the successive crack extensions.

step 2 Calculate the crack tip parameters.

The stress intensity factors are calculated with the mode enrichment technique while the elastic T-term is extracted from the underlying finite element displacement field.

step 3 Determine the direction of crack propagation.

In our research we applied two existing criteria that can be found in the literature, and we did not pursue the development of a new criterion for the calculation of the
crack propagation angle.

step 4 Update the finite element mesh for the extended crack.

The finite element mesh is updated automatically on the basis of the crack propagation angle calculated in step 3 and the geometrical information introduced in step 1.

step 5 Map the solution data from the final equilibrium state for the previous mesh onto the new grid layout and continue the crack growth simulation at step 2.

This step will cause a considerable improvement of computational efficiency, because the crack growth simulation can be continued by opening the newly modeled crack increment without having to start the analysis from the beginning load.

By comparing our results to some experimental and numerical data that can be found in literature we may conclude that the agreement is quite good. Although this agreement underscores the correctness of our approach to the crack problem, it does not prove that this is the one and only way to simulate crack propagation. Therefore our work must be seen as a first step in getting a better understanding of how to simulate crack propagation within a finite element environment.

All in all, one may conclude that we succeeded in combining the numerical tools discussed in the previous chapters into a software framework which provides a powerful tool for the simulation of crack growth in thin-walled structures. This software framework can be seen as a first attempt to model crack propagation in pressurized aircraft fuselages and can serve as a testbed for future developments. It offers the opportunity to implement and test new ideas in this field of computational fracture mechanics with a minimum of computational effort.

6.2 Recommendations for further research

Nowadays numerical fracture mechanics has become an important topic in the design and maintenance of aircraft structures due to the increasing fatigue sensitivity of aging aircraft that are still in service, the increased design load in the fuselage skin of modern airplanes and the pressure that aircraft operators are putting on larger intervals between inspections. Enormous progress in this field has been made very recently, brought on by the development of extremely powerful computer facilities that are needed to deal with problems of this size.

As mentioned above, the crack growth simulation algorithm that we developed is already a powerful tool for dealing with problems of this complexity. However, the range of applications of this routine can still be extended and some of the underlying methodologies should be verified more extensively on the basis of experimental results. This implies that
there is still a lot of work to do in this field. Some of the subjects that need further investigation are discussed below.

Implementation of the mode enrichment technique in a Kirchhoff type of element

In chapter 2 the mode enrichment technique, which is already available for linear problems, is extended to the geometrically nonlinear range and the implementation of this technique in a shear deformable plate element is discussed. For such an element the state of stress at the crack tip due to bending is governed by two bending stress intensity factors ($K_{BI}, K_{BII}$) and a tearing stress intensity factor ($K_{III}$). However, the modes corresponding to these stress intensity factors are valid only within a boundary layer of thickness $O(t/10)$, where $t$ refers to the thickness of the plate. Away from the crack tip the bending displacement field is characterized by two bending stress intensity factors ($k_1,k_2$) that correspond to the classical plate theory. The region of dominance of these classical stress intensity factors is $O(A/10)$, where $A$ is a relevant in-plane dimension, such as the crack length. In section 2.2.3 it was shown that these 2 sets of stress intensity factors ($K_{BI}, K_{BII}, K_{III}$) and ($k_1,k_2$) are uniquely related to each other.

The question now is which of these 2 sets of stress intensity factors should be used to correlate experimental crack growth data. Only recently, Hui and Zehnder [38] presented an argument in favor of the classical stress intensity factors. This argument is based on a requirement (which is quintessential to the theory of linear elastic fracture mechanics) which states that the inelastic deformation zone at the crack tip should be contained in the region of dominance of the crack tip field. This requirement, which is known as the concept of small scale yielding, renders the crack tip fields corresponding to a shear deformable plate theory to be invalid even for a small amount of crack tip plasticity.

In our approach we make use of a shear deformable plate element (see section 2.3.3). As a consequence, we have to calculate the classical bending stress intensity factors in an indirect way via the bending and tearing stress intensity factors for a shear deformable plate theory. Because mode enrichment must be restricted to the region of dominance of the stress intensity factors in order to render accurate results, we are forced to apply an extremely dense mesh around the crack tip. Therefore a subject of further research would be to implement the mode enrichment technique in a plate element based on the Kirchhoff plate theory. In this way it should be possible not only to calculate the classical stress intensity factors in a direct way but also to apply a much coarser element mesh around the crack tip.

Influence of the bending singularities on the crack propagation process

Most of the crack growth simulations that were performed until now are restricted to problems where the crack tip is subjected to membrane stresses only. This means that one only has to
deal with the two membrane stress intensity factors $K_I$ and $K_{II}$ and the elastic T-term as far as
the state of stress at the crack tip is considered. This may be one of the reasons why all the
-crack propagation criteria that can be found in literature are restricted to the membrane crack
tip parameters and do not incorporate, for instance, the bending stress intensity factors.
When dealing with a crack in a pressurized fuselage, the state of stress at the tip of the crack is
no longer purely membrane. The question now is whether the crack propagation process can
still be described adequately without taking into account the bending stress intensity factors.
The examples dealt with in section 5.5 seem to confirm that the influence of the bending stress
intensity factors on the crack propagation angle is of minor importance. However, this does
not prove that the influence of the bending stress intensity factors on the direction of crack
propagation is always negligible. Therefore this matter should be further investigated in future
work.

**Numerical crack growth simulations for fiber reinforced materials**

Nowadays fiber reinforced materials are becoming more and more popular in the aircraft
industry because they have some important advantages as compared to the classical materials,
such as aluminum. The most important advantage is that the fatigue resistance of these new
materials is much greater than for aluminum. This superiority, as far as fatigue is concerned, is
mainly due to an effect called 'crack bridging'; if a crack has been initiated the (unbroken)

![Diagram of crack bridging fibers](image)

**Figure 6.1** Schematic picture of crack bridging fibers
fibers will bridge the crack and thus restrain crack opening (fig. 6.1). This crack bridging reduces the stress intensity factors at the crack tip significantly. Because of the high fatigue resistance, the use of these fiber reinforced materials is especially interesting for the aircraft components that are highly sensitive to fatigue, such as, the aircraft fuselage. This implies that along with the developments of new aircraft materials the theoretical and numerical techniques for simulating crack growth in a pressurized fuselage should be extended to cope with these fiber reinforced (anisotropic) materials. Compared with the classical (isotropic) materials, this extension brings to the surface some new difficulties with the material and crack modeling. Some of these problems involve, for instance, the strong variation of the stress field at the crack tip through the thickness of the material, the modeling of crack bridging and partly through cracks, and the influence that the fibers will have on the crack growth direction. It is clear that there is still a lot of work to do in the field of crack propagation modeling for fiber reinforced materials.
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Appendix A

Relation between $k_1$, $k_2$ and $K_{BI}$, $K_{BII}$, $K_{III}$

The relation between the two bending stress intensity factors $k_1$ and $k_2$ for the classical plate theory and the stress intensity factors $K_{BI}$, $K_{BII}$ and $K_{III}$ for a shear deformable plate theory can be determined with the so-called crack closure integral [83]. This integral calculates the energy release rate $G_c$, i.e. the negative of the work done by the tractions $\sigma_y$ per unit new crack surface (fig. A.1). For mode I crack opening this yields

$$G_c = \frac{2}{tda} \int_{0}^{\frac{\nu}{2}} \int_{-\frac{\nu}{2}}^{\frac{\nu}{2}} \frac{1}{2} \sigma_y(x) \cdot v(da - x) \, dz \, dx$$

(A.1)

where

$\sigma_y$ is the stress component in $y$-direction in front of the crack tip,
$v$ is the displacement in $y$-direction behind the crack tip,
is the amount of crack tip extension and 

is the plate thickness.

For membrane mode I crack opening one has

\[
v = \frac{K_I}{\mu \sqrt{2\pi}} \sin \frac{\theta}{2} \left( \frac{2}{1 + \nu} - \cos^2 \left( \frac{\theta}{2} \right) \right)
\]

\[
v (\theta = 180^\circ) = \frac{K_I}{\mu \sqrt{2\pi}} \frac{2}{1 + \nu}
\]  

(A.2)

and

\[
\sigma_y = K_I \frac{1}{2\pi r} \cos \frac{\theta}{2} \left( 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)
\]

\[
\sigma_y (\theta = 0^\circ) = K_I \frac{1}{2\pi r}
\]  

(A.3)

For the energy release rate this yields the well known expression

\[
G_e (K_I) = \frac{K_I^2}{E}
\]  

(A.4)

For mode I bending a comparative relation between the energy release rate and the bending stress intensity factor can be obtained. The displacement \(v\) behind the crack tip and the stress \(\sigma_y\) in front of the crack tip are

\[
v (\theta = 180^\circ) = \frac{K_{BI} z}{\mu} \frac{2}{t} \sqrt{\frac{r}{2}} \frac{2}{1 + \nu}
\]  

(A.5)

\[
\sigma_y (\theta = 0^\circ) = K_{BI} \frac{2z}{t} \sqrt{\frac{1}{2}}
\]  

(A.6)

Substitution of equations (A.5) and (A.6) in equation (A.1) yields

\[
G_e (K_{BI}) = \frac{2}{t \partial a} \int_0^{u_2} \int_0^{v_2} \sigma_y (x) \cdot v (da - x) \, dz \, dx
\]
\[ \frac{2}{\pi} \int_0^{u/2} \frac{1}{2} K_{BII} \frac{2z}{t} \sqrt{\frac{1}{2x}} \frac{K_{BII}}{\mu} \sqrt{\frac{da-x}{2}} \frac{2}{1+v} \frac{dz}{dx} \]

\[ = \frac{K_{BII}^2}{\mu t^3 (1+v)} \frac{4}{u^2} \int_0^{u/2} \frac{da}{\sqrt{x}} \frac{dz}{dx} \]

\[ = \frac{1}{3} \frac{K_{BII}^2}{\mu (1+v)} = \frac{1}{3} \frac{K_{BII}^2}{E} \]

(A.7)

For mode II bending a similar calculation can be performed where \( \sigma_y \) and \( v \) in equation (A.1) have to be replaced by \( \sigma_{xy} \) and \( u \) respectively. The displacement \( u \) is calculated as

\[ u = \frac{K_{BII} 2z}{\mu t} \sqrt{\frac{r}{2}} \sin \left( \frac{\theta}{2} \right) \left( 2 \frac{2}{1+v} + \cos^2 \left( \frac{\theta}{2} \right) \right) \]

\[ u (\theta=180^\circ) = \frac{K_{BII} 2z}{\mu t} \sqrt{\frac{2}{1+v}} \]

(A.8)

and for \( \sigma_{xy} \) one will find

\[ \sigma_{xy} = \frac{K_{BII} 2z}{\mu t} \sqrt{\frac{1}{2r}} \cos \left( \frac{\theta}{2} \right) \left( 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \]

\[ \sigma_{xy} (\theta=0^\circ) = \frac{K_{BII} 2z}{\mu t} \sqrt{\frac{1}{2r}} \]

(A.9)

Comparing equations (A.8) and (A.9) with (A.5) and (A.6) respectively it is obvious that, analogous to equation (A.7), the final result for the energy release rate for mode II bending will be

\[ G_e (K_{BII}) = \frac{1}{3} \frac{K_{BII}^2}{E} \]

(A.10)

Finally the energy release rate for the tearing mode has to be determined. For \( w \) due to tearing we have

\[ w = \frac{K_{III} \sqrt{\frac{2r}{\pi}} \sin \frac{\theta}{2}}{\mu \sqrt{\pi}} \]

\[ w (\theta=180^\circ) = \frac{K_{III} \sqrt{2r}}{\mu \sqrt{\pi}} \]

(A.11)
and for $\sigma_{yz}$

$$\sigma_{yz} = K_{III} \sqrt{\frac{1}{2\pi r}} \cos \frac{\theta}{2} (1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2})$$

$$\sigma_{yz} (\theta=0^o) = K_{III} \sqrt{\frac{1}{2\pi r}}$$  \hspace{1cm} (A.12)

Substitution of these equations into (A.1) now yields after replacing $\sigma_y$ and $v$ by $\sigma_{yz}$ and $w$ respectively

$$G_e(K_{III}) = \frac{2}{tda} \int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{2} \sigma_{yz} (x) \cdot w (da - x) \, dz \, dx$$

$$= \frac{2}{tda} \int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{2} K_{III} \sqrt{\frac{1}{2\pi x}} \frac{1}{\mu} \sqrt{\frac{2}{\pi}} \frac{da}{x} (da - x) \, dz \, dx$$

$$= K_{III}^2 \frac{1}{\mu \pi t da} \int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\frac{da - x}{x}} \, dz \, dx = K_{III}^2 \frac{(1 + v)}{E}$$  \hspace{1cm} (A.13)

Hui and Zehnder [38] calculated the energy release rate for mode I and mode II bending according to the classical plate theory as

$$G_e(k_1) = \frac{(1 + v)}{3(3 + v)} k_1^2 \pi$$ \hspace{1cm} (A.14a)

$$G_e(k_2) = \frac{(1 + v)}{3(3 + v)} k_2^2 \pi$$ \hspace{1cm} (A.14b)

Equating the energy release rates for the two plate theories now yields

$$G_e(k_1) = G_e(K_{B_1})$$ \hspace{1cm} (A.15a)

$$G_e(k_2) = G_e(K_{B_II}) + G_e(K_{III})$$ \hspace{1cm} (A.15b)

or

$$k_1^2 = \frac{3 + v}{1 + v} k_{B_1}^2$$ \hspace{1cm} (A.16a)

$$k_2^2 = \frac{3 + v}{1 + v} k_{B_{II}}^2 + \frac{3(3 + v)}{\pi} k_{III}^2$$ \hspace{1cm} (A.16b)
Appendix B

Error analysis for $G^*$

The base vectors $b_{i(cr)}$ at the crack tip for the intermediate state $B'$ (fig. B.1) can be expressed in the base vectors $A_{k(cr)}$ for the undeformed state $B$ as

$$b_{i(cr)} = D A_{i(cr)} = R U A_{i(cr)} \quad \text{(B.1)}$$

where $D$ is the deformation gradient, $R$ represents the rotation matrix and $U$ is the stretch matrix. $U$ can be written as

$$U = I_3 + |O(\varepsilon_e)|_3 \quad \text{(B.2)}$$

where $|.|_3$ denotes a $3x3$ identity matrix. In this equation $\varepsilon_e$ is the so-called engineering strain [21]. Substitution of equation (B.2) into (B.1) yields

$$b_{i(cr)} = R (I_3 + |O(\varepsilon_e)|_3) A_{i(cr)} \quad \text{(B.3)}$$

Next the base vectors $b_i$ in an arbitrary point $P$ inside $\Omega^*$ for the intermediate state $B'$ are obtained via a rotation of the base vectors $b_{i(cr)}$ (fig. B.1). Denoting the distance between point $P$ and the crack tip in the intermediate state as $d_\Omega$ and the curvature of the crack tip region $\Omega^*$ as $r_\Omega$ one can write

$$b_i = R_\Omega \ b_{i(cr)} \quad \text{(B.4)}$$

where

$$R_\Omega = I_3 + |O(d_\Omega/r_\Omega)|_3 \quad \text{(B.5)}$$

This finally yields for the base vectors $b_i$
Figure B.1  Definition of the base vectors for the undeformed and the intermediate state

\[ b_i = R_{\Omega} \cdot b_{i\,(cr)} \]
\[ = (I_3 + |O\,(d_{\Omega}/r_{\Omega})|_3) \cdot R \cdot (I_3 + |O\,(\varepsilon_c)|_3) \cdot A_{i\,(cr)} \]
\[ = (R + R|O\,(\varepsilon_c)|_3 + |O\,(d_{\Omega}/r_{\Omega})|_3 \cdot R + |O\,(d_{\Omega}/r_{\Omega})|_3 \cdot R|O\,(\varepsilon_c)|_3) \cdot A_{i\,(cr)} \]  \hspace{1cm} (B.6)

The engineering strains \( \varepsilon_c \) are small compared to 1. Further \( d_{\Omega}/r_{\Omega} \) can be made arbitrary small by choosing an appropriate size of the enriched area \( \Omega^* \). Therefore it is reasonable to write equation (B.6) as

\[ b_i = R \cdot A_{i\,(cr)} \]  \hspace{1cm} (B.7)

and to express the additional Green-Lagrange strains \( G_{ij}^* \) as in equation (2.37).
Appendix C
Correction of the stress intensity factors

The total displacement field can be written as

$$\mathbf{u} = \mathbf{u}_g + \mathbf{R} \mathbf{v}_s \mathbf{k}$$  \hspace{1cm} (C.1)

where $\mathbf{u}_g$ is the underlying finite element displacement field, $\mathbf{k}$ is the vector of stress intensity factors, $\mathbf{v}_s$ is the vector of corresponding shape functions and $\mathbf{R}$ represents the rotation of the crack tip with respect to the undeformed state. Furthermore, according to equation (2.41) we can write

$$\mathbf{u}_g = \mathbf{N}^T (\mathbf{q} - \mathbf{R} \mathbf{v}_s \mathbf{k})$$  \hspace{1cm} (C.2)

where $\mathbf{q}$ are the total nodal displacements and $\mathbf{v}_s$ are the nodal values of the analytical displacement solution corresponding to the stress intensity factors. Now suppose that the underlying finite element displacement field $\mathbf{u}_g$ contains a small part of the analytical displacement solutions for the stress intensity factors. This means that a correction $\mathbf{c}$ has to be applied to the calculated stress intensity factors $\mathbf{k}$ in order to increase the accuracy of our results. Thus

$$\mathbf{u} = \mathbf{u}_g + \mathbf{R} \mathbf{v}_s (\mathbf{k} - \mathbf{c})$$  \hspace{1cm} (C.3)

where now

$$\mathbf{u}_g = \mathbf{N}^T (\mathbf{q} - \mathbf{R} \mathbf{v}_s (\mathbf{k} - \mathbf{c}))$$  \hspace{1cm} (C.4)

By requiring orthogonality between the analytical displacement solutions and the underlying finite element displacement field one obtains
\[ \langle u_{ge}, Rv_s \rangle = 0 \quad \text{(C.5)} \]

where

\[ \langle a, b \rangle = \frac{1}{V} \int_V a \cdot b \, dV \quad \text{(C.6)} \]

is an integral over the volume of the considered elements.

Substitution now yields

\[ \langle N^T q_1, Rv_s \rangle - \langle N^T R y_s k, Rv_s \rangle + \langle N^T R y_s c, Rv_s \rangle = 0 \quad \text{(C.7)} \]

Working out this equation one finally obtains

\[ c = k - A^{-1} V \quad \text{(C.8)} \]

where

\[ V = \frac{1}{V} \int_V \begin{bmatrix} N^T q \cdot Rv_1 \\ N^T q \cdot Rv_2 \\ N^T q \cdot Rv_3 \\ N^T q \cdot Rv_4 \\ N^T q \cdot Rv_5 \end{bmatrix} dV \quad \text{(C.9a)} \]

\[ A = \frac{1}{V} \int_V \begin{bmatrix} N^T R y_1 \cdot Rv_1 & \cdots & \cdots & N^T R y_5 \cdot Rv_1 \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ N^T R y_1 \cdot Rv_5 & \cdots & \cdots & N^T R y_5 \cdot Rv_5 \end{bmatrix} dV \quad \text{(C.9b)} \]

The corrected stress intensity factors can be calculated now as

\[ k_{corr} = k_{mr} - c \quad \text{(C.10)} \]
Appendix D

Paris' fatigue crack growth law

In experiments it can be observed that an important part of the crack growth rate curve is characterized by a linear log-log relation between the crack growth rate \( \frac{da}{dN} \) and \( \Delta K \) (the difference between the maximum and minimum stress intensity factor in 1 load cycle).

![Diagram of Paris' fatigue crack growth law]

**Figure D.1** Characteristics of the fatigue crack growth rate curve \( \frac{da}{dN} \cdot \Delta K \)

In correspondence with these observations Paris formulated his crack growth law as \([25]\)

\[
\frac{da}{dN} = C \cdot (\Delta K)^n
\]

(D.1)

In this equation \( C \) and \( n \) are material dependent variables which are determined by experiment. For 2024-T3 aluminum, the material that was used in Worden's experiment, these variables are
\[ n = 4.9 \quad \text{(D.2a)} \]
\[ C = 5.7853 \cdot 10^{18} \text{ mm}^{1+\frac{3}{2}n}/\text{N}^n \quad \text{(D.2b)} \]

where \( \Delta K \) is expressed in N/mm\(^{3/2} \) and \( da \) in mm. In our case the minimum stress intensity factor for each load cycle is zero and as a consequence \( \Delta K \) can be replaced by \( K \).

Further Worden determined the critical stress intensity factor \( K_c \) for aluminum 2024-T3 as

\[ K_{C_{T-L}} = 3476.2 \text{ N/mm}^{3/2} \quad \text{(D.3a)} \]
\[ K_{C_{L-T}} = 3650.0 - 4171.5 \text{ N/mm}^{3/2} \quad \text{(D.3b)} \]

where \( T-L \) refers to cracks aligned with the rolling direction, while \( L-T \) refers to cracks perpendicular to the rolling direction. For the curved panel that Worden used in his experiment the rolling direction coincides with the longitudinal direction. This means that the initial crack is aligned with the rolling direction and thus experiences \( K_{C_{T-L}} \).

The crack growth law can be integrated by assuming that the stress intensity factor is piecewise linear with respect to the crack length. This implies that

\[ \int dN = N_1 - N_0 = \int_{a_0}^{a_1} \frac{da}{CK^n} \quad \text{(D.4)} \]

where

\[ K = K_0 + \frac{a - a_0}{a_1 - a_0} (K_1 - K_0) = K_0 \frac{a_1 - a}{a_1 - a_0} + K_1 \frac{a - a_0}{a_1 - a_0} \quad \text{(D.5)} \]

In this equation the indices 0 and 1 refer to two succeeding crack tip locations. Substituting equation (D.5) into (D.4) and integrating with respect to the crack length \( a \) yields

\[ N_1 - N_0 = \int_{a_0}^{a_1} \frac{1}{C} \left( K_0 \frac{a_1 - a}{a_1 - a_0} + K_1 \frac{a - a_0}{a_1 - a_0} \right)^{-n} da \]
\[
\frac{1}{C} \frac{1}{1 - n} \frac{a_1 - a_0}{K_1 - K_0} \left( K_0 \frac{a_1 - a}{a_1 - a_0} + K_1 \frac{a - a_0}{a_1 - a_0} \right)^{1-n} a_1
\]

\[
= \frac{1}{C (1 - n)} \frac{a_1 - a_0}{K_1 - K_0} (K_1^{1-n} - K_0^{1-n})
\]

(D.6)

and this final equation is used to calculate the number of load cycles that is needed to extend the crack from \(a_0\) to \(a_1\).
Summary

The behavior of cracks in thin-walled pressurized fuselages is dominated by a number of influence factors that are difficult to handle in an analytical way. First of all there is a geometrically nonlinear effect that is caused by the curvature of the fuselage and by the boiler stresses. Another aspect which makes this problem rather complex is the influence that the stiffening components (longerons and bulkheads) will have on the crack behavior.

In the literature it was shown that the finite element method is an excellent means for dealing with problems of this complexity. However, in order to simulate crack propagation in thin-walled structures, a number of additional numerical techniques have to be developed. First, there has to be a tool for calculating the stress intensity factors at the tip of a crack. For this purpose we will use the so-called mode enrichment (or mode superposition) technique, a method that is already available for linear crack problems. Because we want to deal with cracks in fuselage models, this technique has to be extended to the geometrically nonlinear range.

Moreover, one has to be able to calculate the elastic T-term, a crack tip parameter that is needed for the prediction of the crack propagation angle. All the techniques that are available in the literature are difficult to apply to geometrically nonlinear crack problems. Therefore we decided to develop a method that is used in combination with the mode enrichment technique and that extracts the elastic T-term directly from the underlying finite element displacement field around the crack tip.

As far as the crack propagation angle is concerned one can use several criteria that can be found in literature. All these criteria are governed by the crack tip parameters (i.e. the stress intensity factors and the elastic T-term) calculated at the current crack tip. Two of them are selected to be tested on some linear and geometrically nonlinear crack problems, and the results are compared with experimental findings.

Another indispensable tool for simulating crack propagation with a finite element method is an automatic mesh generator. The mesh generator that we developed makes use of quadrilateral elements and is used not only to generate an initial grid layout, but also to update the local mesh around the crack tip after crack propagation.

A final technique that is especially important for geometrically nonlinear crack problems is the solution data mapping routine. This routine maps the solution data from one mesh onto
another after refining or updating a finite element mesh. In this way one has the opportunity to continue a crack growth simulation by opening the newly modeled crack increment instead of having to start the external load sequence all over again using the updated finite element mesh. The routines mentioned above were implemented in the B2000 finite element code and crack growth simulations were performed for some linear and geometrically nonlinear crack problems. The calculations seem to confirm that the numerical tools that we developed are quite efficient. Moreover, the agreement between our results and some numerical and experimental work found in literature is shown to be excellent.
Samenvatting

Het gedrag van scheuren in dunwandige vliegtuigrompen onder inwendige druk wordt sterk bepaald door een aantal factoren die op analytische wijze nauwelijks te beschrijven zijn. Allereerst is er een geometrisch niet-lineair effect dat wordt veroorzaakt door de kromming van de romp en door de ketelspanningen ten gevolge van de inwendige overdruk. Verder draagt ook de invloed van verstijvers en spanten op het scheurgedrag bij aan de complexiteit van dit probleem.

In de literatuur is aangetoond dat de eindige elementen methode bij voorbaat geschikt is om een dergelijke complex probleem te analyseren. Voor het simuleren van scheurgroei in dunwandige constructies zullen echter nog een aantal numerieke werktuigen ontwikkeld moeten worden.

Allereerst moeten we over een methode beschikken voor het bepalen van de spanningsintensiteits faktoren. Wij zullen hiervoor gebruik maken van de zogenaamde 'mode enrichment' (of 'mode superposition') techniek, een methode die al wordt toegepast voor lineaire scheurproblemen. Omdat wij echter het gedrag van scheuren in vliegtuigrompen willen bestuderen moet het toepassingsgebied van de 'mode enrichment' techniek worden uitgebreid naar geometrisch niet-lineaire problemen.

Verder moet de zogenaamde elastische T-spanning, een grootheid die benodigd is voor het bepalen van de scheurgroeirichting, berekend kunnen worden. De technieken die beschikbaar zijn in de literatuur zijn nauwelijks toepasbaar op geometrisch niet-lineaire scheurproblemen. Daarom hebben wij een methode ontwikkeld die toegepast wordt in combinatie met de 'mode enrichment' techniek en die de elastische T-spanning berekent aan de hand van het onderliggende eindige elementen verplaatsingsveld rond de scheurtip.

Uit de literatuur zijn meerdere criteria bekend voor het berekenen van de scheurgroeirichting. In al deze criteria wordt de scheurgroeirichting bepaald door de scheurtip parameters (de spanningsintensiteits faktoren en de elastische T-spanning) ter plaatse van de huidige scheurtip. Twee van deze criteria zullen worden toegepast op lineaire en geometrisch niet-lineaire problemen en de resultaten worden vergeleken met experimenten.

Een ander onmisbaar numeriek werktuig voor het simuleren van scheurgroei met behulp van een eindige elementen methode is een routine voor het automatisch genereren van een elementenverdeling. De elementen generator, die wij hebben ontwikkeld, maakt uitsluitend
gebruik van vierhoekige elementen en kan niet alleen een initiële elementenverdeling genereren maar is ook in staat om het elementennetwerk ter plaatse van de scheurtip opnieuw aan te passen nadat de scheur is gegroeid.

Een laatste techniek die vooral belangrijk is voor geometrisch niet-lineaire scheurproblemen is een routine voor het projekteren van de oplossingsdata van het ene eindige elementen model op het andere nadat het elementennetwerk is verfijnd of aangepast. Op deze wijze kan een scheurgroeisimulatie worden voortgezet door het nieuw gemodelleerde scheurincrement te openen en hoeft de externe belasting niet steeds opnieuw te worden aangebracht op het aangepaste eindige elementen model.

De numerieke werktuigen, die hierboven beschreven staan, zijn geïmplementeerd in het eindige element programma B2000 en er zijn scheurgroeisimulaties uitgevoerd voor enkele lineaire en geometrisch niet-lineaire scheurproblemen. De uitgevoerde berekeningen lijken de efficiënte werking van de ontwikkelde routines te bevestigen. Verder blijken onze resultaten goed overeen te komen met vergelijkbare numerieke en experimentele resultaten uit de literatuur.
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Curriculum Vitae


Vanaf 1 maart 1989 was hij als assistent in opleiding werkzaam bij de Technische Universiteit Delft in de vakgroep Constructie, Sterkte en Trillingen van de Faculteit der Luchtvaart- en Ruimtevaarttechniek. Hier heeft hij onder leiding van Prof. Dr. J. Arbocz en Dr. Ir. E. Riks onderzoek verricht naar de mogelijkheden om het gedrag van scheuren in vliegtuigrompen numeriek te simuleren.

Momenteel is de schrijver van dit proefschrift werkzaam bij MARC Analysis Research Corporation - Europe te Zoetermeer als 'support engineer' op het gebied van niet-lineaire eindige elementen berekeningen.