ANILISA - Computational Module for Koiter's Imperfection Sensitivity Theory

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J. Arbocz / J.M.A.M. Hol

TU Delft
Faculty of Aerospace Engineering
Delft University of Technology
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ANILISA - Computational Module for Kolter's Imperfection Sensitivity Theory

On p. 7
Eq. (15) ... + C_2 \ddot{x} + C_2 ... should be ... + \tilde{C}_1 \ddot{x} + \tilde{C}_2 ...

On p. 7
3rd line from below ... constants of integration C_1 and C_2 ... should be ...
... constants of integration \tilde{C}_1 and \tilde{C}_2 ...

On p. 8
Eq. (21) ... W'_1 ... should be ... w'_1 ...

On p. 10
Eq. (29) ... n^2 \{w'_1 w_{1,xx} + ...} should be ... n^2 \{w'_1 w_{1,xx} + ...

On p. 13
Eq. (40) ... + D_6 (...) should be ... - D_6 (...)  

On p. 13
Eq. (42) ... -(D_{16} + D_{19})w'_\gamma should be ... -(D_{16} + D_{19})w'_\beta

On p. 39
Line 7 from below ...Eq. (34),... should be ...Eq. (39)

On p. 41
Eq. (C11) ... w'_0 = \Sigma ... should be ... w'_0 = \Sigma ...

On p. 45
Eq. (D2) ... + f'_o (x)]... should be ... + R^2 f'_o (x)]...

On p. 45
Eq. (D2) ... \xi (ERt^2/c)(f'_\alpha (x) + ... should be ... \xi^2 (ERt^2/c)(f'_\alpha (x) + ...

On p. 46
Eq. (D9) 3rd line from below:
... - c\omega'_\alpha (ct/2R)n^2 ... should be ... - c\omega'_\alpha (ct/2R)n^2 ...

On p. 46
Eq. (D9) 2nd line from below:
... + \tilde{A}_{26}^* 2n^2 t'_\gamma ... should be ... + \tilde{A}_{26}^* 2n t'_\gamma ...

On p. 47
Eq. (D12) ... + c\omega'_2 (ct/R)n^2 w'_o w'_2 = 0 ... should be ... + c\omega'_2 (ct/R)n^2 w'_o w'_2 = 0 ...

On p. 47
Eq. (D14) ... - (\tilde{A}_{66}^* + \tilde{A}_{12}^*) 4n^2 f'_\beta ... should be ... - (\tilde{A}_{66}^* + \tilde{A}_{12}^*) 4n^2 f'_\beta ...

On p. 48
Eq. (D14) ... + c\omega'_\beta ... should be ... + c\omega'_\beta ...

On p. 48
Eq. (D15) ... + 4\tilde{A}_{16}^* 8n^3 f'_\beta ... should be ... + \tilde{A}_{16}^* 8n^3 f'_\beta ...
On p. 50 Eq. (D29) \( \lambda = -N_0/(E\ell^2cR) \) should be \( \lambda = N_0/(E\ell^2/cR) \).

On p. 51 Eq. (D35) \( + W_x N_{xy} \) should be \( + W_{xy} N_{y} \).

On p. 53 Eq. (D47) \( - \bar{B}_{61}^* n_{11}' \) should be \( + \bar{B}_{61}^* n_{11}' \).

On p. 53 In the Postbuckling problem is missing equation for \( w_{\alpha} \) and \( f_{\alpha} \):

\[
\bar{B}_{11}^* w_{\alpha} + 2 R - \bar{B}_{21}^* f_{\alpha} = 0 \quad (D48)
\]

Reumber Eqs. (D48) \( \rightarrow \) (D49)

(D49) \( \rightarrow \) (D50)

(D50) \( \rightarrow \) (D51)

On p. 53 Eq. (D49) \( - \bar{B}_{61}^* 2n_{11}' \) should be \( + \bar{B}_{61}^* 2n_{11}' \).

...
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On p. 3  line 17 from above: ‘...can best the acquired.’ should be ‘...can best be acquired.’

On p. 7  Eq. (12): ‘...+ f_o^2(x)]’ should be ‘...+ f_o^2(x)]’

On p. 8  first line from above: ‘...constants.’ should be ‘...constant.’

On p. 9  first line of Eq. (22): ‘...+ A*_{11} f_1’ should be ‘...+ A*_{11} f_2’

On p. 10  first line of Eq. (27): ‘...+ c_2 f_1’ should be ‘+ c_2 f_1’

On p. 11  first line above Eq. (33): ‘...coefficients’ should be ‘...coefficients’

On p. 12  second line of Eq. (36): ‘...+ w_i^2 f_2)’ should be ‘...+ w_i^2 f_2’

On p. 18  first line of Eq. (70): ‘... = \frac{\partial \Delta}{\partial \Delta} \left( \Lambda = \Lambda_c \right)^{-1}’ should be ‘... = \frac{\partial \Delta}{\partial \Delta} \left( \Lambda = \Lambda_c \right)^{-1}’

On p. 20  second line from below: ‘...This...’ should be ‘...These...’

On p. 21  ‘Eq. (79)’ should be ‘Eq. (75)’

On p. 22  first line of Eq. (86): ‘... u^{(k-1)}’ should be ‘... u^{(k-1)}’

On p. 26  line 6 from below: ‘... (\overline{\tau}_x < 0) torsion...’ should be ‘... (\overline{\tau}_x < 0) torsion...’

On p. 38  line 10 from above: ‘N y = F_{xx}’ should be ‘N x = F_{xx}’

On p. 39  first line of Eq. (B5): ‘... \overline{\theta}_{12}’ should be ‘... \overline{\theta}_{22}’

On p. 67  last line of Table 3: ‘...D = E \frac{3}{4} c^2’ should be ‘...D = E \frac{3}{4} c^2’ and E = 5.83 x 10^6 psi, v = 0.363.

On p. 68  in the title of Table 4: ‘...coefficients.’ should be ‘...coefficients..’
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<td>(a)</td>
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<td>extensional stiffness matrix (see Eq. A9)</td>
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<td>nondimensional (A_{ij}^<em>) ((\bar{A}_{ij}^</em> = E \bar{A}_{ij}^*))</td>
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<td>(b)</td>
<td>second postbuckling coefficient (see Eq. 46)</td>
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<td>(B_{ij})</td>
<td>bending-stretching coupling matrix (see Eq. A10)</td>
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<td>(B_{ij}^*)</td>
<td>semi-inverted bending-stretching coupling matrix (see Eq. A19)</td>
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<td>(\bar{B}_{ij}^*)</td>
<td>nondimensional (B_{ij}^<em>) ((\bar{B}_{ij}^</em> = (2c/t)B_{ij}^*))</td>
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<td>(c)</td>
<td>= \sqrt{3(1-\nu^2)}</td>
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<td>(D_{ij})</td>
<td>flexural stiffness matrix (see Eq. A13)</td>
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<td>(D_{ij}^*)</td>
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<td>(E)</td>
<td>arbitrarily chosen reference Young's modulus</td>
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<td>prebuckling Airy stress function (see Eq. 12)</td>
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<td>(p(o), p(1), p(2))</td>
<td>zeroth order, first order, second order fields, respectively</td>
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<td>(h_k)</td>
<td>thickness of the (k)th layer</td>
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<td>(K_c)</td>
<td>slope of the variable load vs generalized displacement curve just before buckling</td>
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<td>(L_{A*})</td>
<td>linear operator defined by Eq. (4)</td>
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<td>(M_x, M_y, M_{xy})</td>
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<td>(n)</td>
<td>number of full waves in the circumferential direction (see Eqs. 19, 20, 31 and 32)</td>
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<td>(N_x, N_y, N_{xy})</td>
<td>stress resultants</td>
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<td>(P_a)</td>
<td>allowable buckling load</td>
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buckling load of the 'perfect' structure
axial load eccentricity measured from the midsurface of the
shell wall - positive inward
nondimensional load eccentricity ($\bar{q} = 4cRq/t^2$)
specially orthotropic laminar stiffness matrix (see Eq. A3)
generally orthotropic laminar stiffness matrix (see Eq. A5)
shell radius
shell wall-thickness
displacement components in the x and y directions, respectively
prebuckling radial displacement function (see Eq. 11)
buckling radial displacement functions (see Eq. 19)
prebuckling radial displacement functions (see Eq. 31)
radial displacement (positive inward)
zeroth order, first order, second order fields, respectively
initial radial imperfection ($\bar{W} = \bar{\bar{W}}$)
shape of the initial radial imperfection
axial Poisson's effect ($W_v = \bar{\bar{A}}_{12} \lambda/c$)
radial Poisson's effect ($W_p = \bar{\bar{A}}_{22} \bar{p}/c$)
circumferential Poisson's effect ($W_t = -\bar{\bar{A}}_{26} \bar{\tau}/c$)
axial and circumferential coordinates on the middle surface
of the shell, respectively
nondimensional coordinates ($\bar{x} = x/R$, $\bar{y} = y/R$)
unified vector variable (see Eqs. 73 and 77, respectively)
coordinate normal to the middle surface of the shell
(positive inward)
modified Batdorf parameter ($\bar{Z} = L^2/Rt$)

first imperfection form factor (see Eq. 53)
second imperfection form factor (see Eq. 54)
shearing strain
generalized displacement
membrane strain which corresponds to the applied variable
load
linear strain which corresponds to the variable applied
stress resultant (see Eq. 60)
normal strains
nonlinear strain (see Eq. 61)

generalized displacement (see Eq. 60)
ungcircumferential coordinate \( \theta = y/R \)
angle of the fundamental path \( \tilde{\theta} = \tan^{-1}(\epsilon_{MC}) \)
angle of the initial slope just after buckling \( \tilde{\theta}^* = \tan^{-1}(\epsilon_{MC}^*) \)
curvature components
nondimensional axial load parameter \( \lambda = (cR/Et^2)N_o \)
reliability based 'knockdown' factor
nondimensional variable load factor
nondimensional variable load factor evaluated at the bifurcation point
nondimensional variable load factor evaluated at the limit point
eigenvalue estimate (see Eq. 85)
arbitrarily chosen reference Poisson's ratio
perturbation parameter (see Eq. E11)
amplitude of the initial imperfection (see Eq. 58)
normalized variable load factor \( \rho_S = \Lambda_S/\Lambda_C \)
normal stresses
nondimensional torque parameter \( \tau = (cR/ Et^2)N_{xy} \)
- positive counter-clockwise
shearing stress
ANILISA - Computational Module for Koiter's Imperfection Sensitivity Theory

by

J. Arbocz and J.M.A.M. Hol

Delft University of Technology, The Netherlands

Abstract — The development of 'DISDECO', the Delft Interactive Shell Design Code is described. The purpose of this project is to make the accumulated theoretical, numerical and practical knowledge of the last 20 years readily accessible to users interested in the analysis of buckling sensitive structures. With this open ended, hierarchical, interactive computer code the user can access from his workstation successively programs of increasing complexity.

Included are modules that contain Koiter's imperfection sensitivity theory extended to anisotropic shell structures under combined loading. The nonlinear Donnell type anisotropic shell equations in terms of the radial displacement W and the Airy stress function F are used. The circumferential dependence is eliminated by Fourier decomposition. The resulting sets of ordinary differential equations are solved numerically via the 'Shooting Method'. Thus the specified boundary conditions can be enforced rigorously not only in the prebuckling but also in the buckling and the post-buckling problem. Initial results indicate that in order to obtain reliable results for anisotropic shells rigorous enforcing of the edge restraint and of the boundary conditions is in-deed a must.

1. INTRODUCTION

The central goal of the current shell research activities at the Faculty of Aerospace Engineering of the TU-Delft is the development of 'Improved Shell Design Criteria' for buckling sensitive structures, which incorporate all the theoretical knowledge accumulated in the last, say 30 years through intensive research in the Aerospace, the Nuclear and the Offshore fields and which make efficient use of the currently available interactive and (super) computing facilities.

All the current design manuals [1,2,3] adhere to the so-called 'Lower Bound Design Philosophy' that has already been in use 50 years ago. That is, they recommend the use of an empirical 'knockdown' factor $\gamma$, which is so chosen that when it is multiplied with the (classical) buckling load of the perfect shell a 'lower bound' to all available data is obtained (see Fig. 1).
The improvements in the currently recommended shell design procedures are primarily sought in a more selective approach at the choice of the 'knockdown' factor $\gamma$. Due to the pioneering efforts of Koiter [4], Budiansky and Hutchinson [5] and many other investigators initial geometric imperfections have been identified as the main cause for the wide experimental scatter observed in practice. Thus it is proposed that if a company takes the necessary care to produce all its shells according to well characterized standards, and if it can show via experiments that the boundary conditions are defined in such a way that during assembly no additional imperfections (especially at the shell edges) are introduced, then the use of an improved (higher) 'knockdown' factor $\gamma$ derived by the stochastic approach proposed by Elishakoff and Arboz [6,7] should be allowed. The proposed new 'Improved Shell Design Criteria' can be represented by the following formula

$$P_a \leq \frac{\lambda_a}{F.S.} P_c$$  \hspace{1cm} (1)

where $P_a$ = allowable buckling load, $P_c$ = buckling load of the 'perfect' structure calculated via one of the advanced shell codes like BOSOR-4 [8], $\lambda_a$ = reliability based improved (higher) 'knockdown' factor and F.S. = factor of safety.

The steps involved in the definition of the reliability based improved (higher) 'knockdown' factor $\lambda_a$ can be summarized as follows:

1. Compute the Fourier coefficients of the measured initial imperfection surveys of a relatively small sample (say 4) nominally identical shells produced by the same manufacturing process.

2. Calculate by ensemble averaging the mean vector and the variance-covariance matrix of the Fourier coefficients of the experimental sample.

3. Use the First Order Second Moment Analysis [7] to compute the reliability function $R(\lambda)$ of the buckling of shells with general random imperfections.

4. Determine the improved (higher) 'knockdown' factor $\lambda_a$ for the specified reliability from the plot of $R(\lambda)$ vs $\lambda$, where $\lambda$ is the normalized load parameter (see Fig. 2).
Notice that by using the First Order Second Moment Analysis to derive the reliability functions for shells produced by a certain manufacturing process one is combining the Lower Bound Design Philosophy with the notion of Goodness Classes. Thus shells made by a process which produces inherently a less damaging initial imperfection distribution will not be penalized because of the lower experimental results obtained with shells produced by another process, which generates a more damaging characteristic initial imperfection distribution.

2. DEVELOPMENT OF 'DISDECO'

The key to the success of any Stochastic Stability Analysis lies in the reliability and accuracy of the buckling load predictions made by the deterministic buckling analysis used. On the other hand the success of the deterministic buckling load analysis depends very heavily on the appropriate choice of the nonlinear model employed, which in turn requires considerable knowledge by the analyst as to the expected physical behaviour of imperfect shell structures. As has been pointed out by Arboz and Babcock [9] this knowledge can best the acquired by first using the series of imperfection sensitivity analysis of increasing complexity that have been published in the literature.

In order to make the search for the most appropriate nonlinear model feasible the development of 'DISDECO', the Delft Interactive Shell Design Code has been initiated. This 'open architecture' interactive code will combine shell computer programs of different degree of sophistication with the latest tools of information science such as Data Bases, Interactive Graphics and Expert Systems into an advanced hierarchical design and analysis system.

Analysing existing interactive systems for different types of applications has led to the identification of their major components. Based on the results of this study the conceptual design of the global architecture of DISDECO has been completed. The building blocks of the full scale system and their functional relations are displayed in Fig. 3.

The central part of the whole system is the 'Command and Control Processor'. Its function is to control and direct the activities of the system. It starts-up and winds-down the design system, processes user input
through execution of modules, creates a working environment and in general is the working partner of the user.

The link from the user to the command and control processor passes through the 'Man-Machine Interface'. Assuming that the user employs a terminal device or workstation which supports graphics, the man-machine interface controls the input stream from the user, analyses it, checks it for correct syntax, validates the commands and passes it in an interpretable format to the command and control processor. In a similar manner the output stream from the design and analysis system is converted to a meaningful output for the user.

To increase the flexibility of DISDECO the dependance on a specific data management system must be minimized. Therefore, a 'Generalized Database Interface' is needed to shield the global design and analysis system from the peculiarities of a commercially acquired RDBMS (Relational Data Base Management System).

The real work of DISDECO is done by a number of dedicated 'Analysis Modules'. Supplementary modules are required for pre- and postprocessing functions, remote batch processing of analysis tasks and general utility functions. An essential supplementary module is dedicated to the provision of on-line help information about system capabilities, usage details and at a later stage for literature retrieval.

Analysis modules can be of several types. Small analysis tasks can be done interactively under control of the design system, while large analysis tasks must be performed through batch processing either on the same computer or on a remote (super) computer. Medium size tasks can be done optionally interactively or through remote batch processing. This approach ensures optimum usage of available computing resources and frees the design system for other tasks. The layout of a generalized analysis module is shown in Fig. 4.

The development of the proposed design and analysis system DISDECO can only be concluded successfully through a step by step evolutionary effort. Thus initially a pilot system, containing the essential core features of the full scale system shown in Fig. 5, has been made operational [10]. It has been put together in accordance with a high-level of 'Development Standards'[11] so that it can serve as a starting base for the full-scale system. Test-site users have been asked to feed back their experience with the pilot system so as to incorporate their findings in the development of the full-scale system.
The final component of DISDECO is defined as an 'Expert System' (see Fig. 3). Its purpose is to make available the knowledge and experience of recognized experts. As such its function is parallel to that of the command and control processor, adding intelligence and reasoning capabilities to assist the user. Inclusion of a proven expert system has been opted for. However, acquisition and integration of an expert system requires major efforts for the generation of criteria and specifications. Collection and formalising of existing knowledge is another major issue requiring a dedicated effort. Thus, whereas the inclusion of an expert system is an essential step to improve the proposed system from a powerful toolbox to a full-fledged assistant, still its implementation will be delayed until the results and experiences with the pilot system have been evaluated.

3. DEVELOPMENT OF 'ANILISA'

When DISDECO is finished it will provide an easy access to most of the theoretical knowledge, that has been accumulated by the many scientists who have been active in the field of shell stability, via the advanced interactive and computational facilities offered by the modern high speed 32-bit workstations. Great care is being taken to present the results in a unified form so as to make it easy for the user to proceed step-by-step from the simpler approaches used by the early investigators to the more sophisticated analytical and numerical methods used presently.

This approach is illustrated by the extension of Koiter's b-factor method [4,12] to anisotropic shells loaded by axial compression, external pressure and torsion.

Anisotropic Shell Equations

Using the sign convention defined in Fig. 6 the Donnell type equations for perfect anisotropic shells\(^{[13]}\) can be written as

\[ L_{A*}(F) - L_{B*}(W) = - (1/R)W_{xx} - (1/2)L_{NL}(W,W) \]  
(2)

\[ L_{B*}(F) + L_{D*}(W) = (1/R)F_{xx} + L_{NL}(F,W) + p \]  
(3)
where the linear operators are

\[ L_{A^*} = A_{22}^* \cdot xxxx - 2A_{26}^* \cdot xxyy + (2A_{12}^* + A_{66}^*) \cdot xyy \]

\[-2A_{16}^* \cdot xyyy + A_{11}^* \cdot yyy \]  \hspace{1cm} (4)

\[ L_{B^*} = B_{21}^* \cdot xxxx + (2B_{26}^* - B_{61}^*) \cdot xxyy + (B_{11}^* + B_{22}^* - 2B_{66}^*) \cdot xyy \]

\[ + (2B_{16}^* - B_{62}^*) \cdot xyyy + B_{12}^* \cdot yyy \]  \hspace{1cm} (5)

\[ L_{D^*} = D_{11}^* \cdot xxxx + 4D_{16}^* \cdot xxyy + 2(D_{12}^* + 2D_{66}^*) \cdot xyy \]

\[ + 4D_{26}^* \cdot xyyy + D_{22}^* \cdot yyy \]  \hspace{1cm} (6)

and the nonlinear operator is

\[ L_{NL}(S,T) = S_{xxx} T_{yy} - 2S_{x} T_{xy} + S_{yy} T_{xx} \]  \hspace{1cm} (7)

Commas in the subscripts denote repeated partial differentiation with respect to the independent variables following the comma. The stiffness parameters \( A_{11}^*, B_{11}^*, D_{11}^*, A_{12}^*, \ldots \) etc. are defined in Appendix A. \( W \) is the component of displacement normal to the shell midsurface (here positive inward) and \( F \) is the Airy stress function.

Assuming that the eigenvalue problem for the buckling load \( \Lambda_c \) will yield a unique buckling mode \( W^{(1)} \) with the associated stress function \( F^{(1)} \), a solution, to be valid in the initial postbuckling regime, is sought in the form of the following asymptotic expansions

\[ \frac{\Lambda}{\Lambda_c} = 1 + a_\xi + b_\xi^2 + \ldots \]

\[ W = W^{(0)} + \xi W^{(1)} + \xi^2 W^{(2)} + \ldots \]  \hspace{1cm} (8)

\[ F = F^{(0)} + \xi F^{(1)} + \xi^2 F^{(2)} + \ldots \]
where $W^{(1)}$ will be normalized with respect to the shell thickness $t$ and $W^{(2)}$ is orthogonal to $W^{(1)}$ in some appropriate sense.

A formal substitution of this expansion into the nonlinear governing equations (2) and (3) generates a sequence of equations for the functions appearing in the expansions.

**Governing Equations of the $0^{th}$-order State (Prebuckling Problem)**

The set of governing equations for $W^{(0)}$ and $F^{(0)}$ are

\[
L_{A^*}(F^{(0)}) - L_{B^*}(W^{(0)}) = -(1/R)W^{(0)}_{xx} - (1/2)L_{NL}(W^{(0)}, W^{(0)})
\]

\[
L_{B^*}(F^{(0)}) + L_{D^*}(W^{(0)}) = (1/R)F^{(0)}_{xx} + L_{NL}(F^{(0)}, W^{(0)}) + p
\]

(9) \hspace{1cm} (10)

Since the external loading and the boundary conditions are axisymmetric, therefore the prebuckling solution will also be axisymmetric. Assuming

\[
W^{(0)} = t(W_{x} + W_{y} + W_{z}) + tw_{0}(x)
\]

\[
F^{(0)} = (Et^2/cR)[-(1/2) \lambda y^2 - (1/2)\tau_x^2 - \tau_{xy} + f_{O}(x)]
\]

(11) \hspace{1cm} (12)

then substitution into Eqs. (9) and (10) and regrouping yields

\[
\tilde{A}^{*}_{22}g^{iv}_{22} - (t/2R)\tilde{B}^{*}_{21}w^{iv}_{11} = -cw^m_{0}
\]

\[
2\tilde{B}^{*}_{21}f^{iv}_{21} + (t/R)\tilde{D}^{*}_{11}w^{iv}_{11} = (4cR/t)f^{m}_{O} - 4c\lambda w^m_{0}
\]

(13) \hspace{1cm} (14)

where $()' = R( )$, $x$. Equation (13) can be integrated twice yielding

\[
f^{m}_{O} = (t/2R)(\tilde{B}^{*}_{21}/\tilde{A}^{*}_{22})w^m_{11} - (c/\tilde{A}^{*}_{22})w_{11} + C_{1}\bar{x} + C_{2}
\]

(15)

where $\bar{x} = x/R$ and the constants of integration $C_{1}$ and $C_{2}$ are identically equal to zero because of the periodicity condition (see Appendix B for details). Eliminating $f^{m}_{O}$ between Eqs. (13) and (14) one obtains

\[
(\tilde{A}^{*}_{22}\tilde{D}^{*}_{11} + \tilde{B}^{*}_{21})w^{iv}_{11} + (4cR/t)(\tilde{A}^{*}_{22} \lambda - \tilde{B}^{*}_{21})w^m_{11} + (4c^2 R^2/t^2)w_{11} = 0
\]

(16)
A fourth order linear ordinary differential equation with constants coefficients which always admits an exponential solution. Closed form solutions for simply supported and for clamped boundary conditions have been published in the literature [14,15] (see also Appendix C). The quantities \( \bar{W}_v \), \( \bar{W}_p \) and \( \bar{W}_t \) are evaluated by enforcing the circumferential periodicity condition (see Appendix B for details).

**Governing Equations of the 1st-order State (Buckling Problem)**

The set of governing equations for \( \bar{W}^{(1)} \) and \( \bar{F}^{(1)} \) becomes

\[
L_A^*(F^{(1)}) - L_B^*(\bar{W}^{(1)}) = -(1/R)\bar{W}^{(1)}_{xx} - tw_{o,xx}\bar{W}^{(1)}_{yy}
\]

\[
L_B^*(F^{(1)}) + L_D^*(\bar{W}^{(1)}) = (1/R)F^{(1)}_{xx} + (ERt^2/c)f_{o,xx}\bar{W}^{(1)}_{yy} + tw_{o,xx}F^{(1)}_{yy}
\]

\[-(Et^2/cR)(\lambda\bar{W}^{(1)}_{xx} + \bar{p}\bar{W}^{(1)}_{yy} - 2\bar{t}\bar{W}^{(1)}_{xy})\]

These equations admit separable solutions of the form

\[
\bar{W}^{(1)} = t[w_1(x) \cos n\theta + w_2(x) \sin n\theta]
\]

\[
\bar{F}^{(1)} = (ERt^2/c) [f_1(x) \cos n\theta + f_2(x) \sin n\theta]
\]

where \( \theta = y/R \).

Substitution, regrouping and equating coefficients of like trigonometric terms results in the following system of 4 linear homogeneous ordinary differential equations with variable coefficients

\[
\bar{A}^*_{22} f^{iv}_{1} - (2\bar{A}^*_{12} + \bar{A}^*_{66})n^2f''_{1} + \bar{A}^*_{11}n^4f_{1} - 2\bar{A}^*_{26}nf'_{2} + 2\bar{A}^*_{16}n^3f'_{2}
\]

\[-(t/2R)[\bar{B}^*_{21}w^{iv}_{1} - (\bar{B}^*_{11} + \bar{B}^*_{22} - 2\bar{B}^*_{66})n^2w''_{1} + \bar{B}^*_{12}n^4w_{1} + (2\bar{B}^*_{26} - \bar{B}^*_{61})nw'_{2}]
\]

\[-(2\bar{B}^*_{16} - \bar{B}^*_{62})n^3w'_{2} + cw_{1} - (ct/R)n^2w''_{o1} = 0
\]
\[ \begin{align*}
\ddot{w}'_{22} & = - (2\dddot{w}_{12}' + \dddot{w}_{66}') n^2 f'' + \dddot{w}_{11}' + 2\dddot{w}'_{26} n \dddot{w}'_{1} - 2\dddot{w}'_{16} n \dddot{w}'_{1} \\
& - (t/2R)[\dddot{w}_{21}' + \dddot{w}_{11}'] - (2\dddot{w}_{12}' - 2\dddot{w}_{66}') n^2 f'' + \dddot{w}_{12}' n \dddot{w}'_{1} - (2\dddot{w}_{26}' - \dddot{w}_{61}') n \dddot{w}'_{1} \\
+ (2\dddot{w}_{10}' - \dddot{w}_{62}') n \dddot{w}'_{1}] \quad \text{(22)}
\end{align*} \]

\[ \begin{align*}
(2R/t)[\dddot{w}_{21}' + \dddot{w}_{11}] & - (2\dddot{w}'_{12} - 2\dddot{w}_{66}') n^2 f'' + \dddot{w}_{12}' n \dddot{w}'_{1} + (2\dddot{w}_{26}' - \dddot{w}_{61}') n \dddot{w}'_{1} \\
& - (2\dddot{w}_{16}' - \dddot{w}_{62}') n \dddot{w}'_{1} + 2(\dddot{w}_{12}' + \dddot{w}_{66}') n^2 \dddot{w}_{1} + \dddot{w}_{22}' n \dddot{w}_{1} + 4\dddot{w}_{16}' n \dddot{w}_{1} \\
& - 4\dddot{w}_{26}' n \dddot{w}_{1} - (4cR^2/t^2) f'' \\
+ (4cR/t)[\dddot{w}_{1} - \dddot{w}_{1} - 2n \dddot{w}_{2} + n^2 (f''_{1} + w''_{1} f_{1})] = 0 \\
\text{(23)}
\end{align*} \]

\[ \begin{align*}
(2R/t)[\dddot{w}_{21}' + \dddot{w}_{11}] & - (2\dddot{w}'_{12} - 2\dddot{w}_{66}') n^2 f'' + \dddot{w}_{12}' n \dddot{w}'_{1} - (2\dddot{w}_{26}' - \dddot{w}_{61}') n \dddot{w}'_{1} \\
+ (2\dddot{w}_{10}' - \dddot{w}_{62}') n \dddot{w}'_{1}] + \dddot{w}_{11}' - 2(\dddot{w}_{12}' + \dddot{w}_{66}') n^2 \dddot{w}_{1} + \dddot{w}_{22}' n \dddot{w}_{1} - 4\dddot{w}_{16}' n \dddot{w}_{1} \\
+ 4\dddot{w}_{26}' n \dddot{w}_{1} - (4cR^2/t^2) f'' \\
+ (4cR/t)[\dddot{w}_{2} - \dddot{w}_{2} + 2n \dddot{w}_{1} + n^2 (f''_{2} + w''_{2} f_{2})] = 0 \\
\text{(24)}
\end{align*} \]

Further, in order to be able to use the 'shooting method' of Ref. 16 to solve the governing equations of the 1st order state it is necessary, by considering Eqs. (21) and (23), to eliminate the \( w_{1}^{iv} \) term from Eq. (21) and the \( f_{1}^{iv} \) term from Eq. (23). Similarly, by considering Eqs. (22) and (24) one must eliminate the \( w_{2}^{iv} \) term from Eq. (22) and the \( f_{2}^{iv} \) term from Eq. (24). Finally, some regrouping makes it possible to write the resulting equations as

\[ f_{1}^{iv} = C_{17} f_{1}^{iv} - C_{18} f_{1} + C_{19} f_{1}^{iv} + C_{20} f_{1} + C_{21} f_{1}^{iv} + C_{22} f_{1} + C_{23} f_{1}^{iv} + C_{24} f_{1}^{iv} \\
+ C_{26} w_{1}^{w} f_{1} + C_{28} w_{1}^{w} f_{1} + C_{28} f_{1}^{w} + C_{30} w_{1}^{w} f_{1} - C_{31} w_{1}^{w} f_{1} - C_{28} w_{1}^{w} f_{1} \] (25)
$$f_{2}^{iv} = C_{17} f_{2}^{''} - C_{18} f_{2}^{'} - C_{19} f_{1}^{''} - C_{20} f_{1}^{'} + C_{21} w_{2}^{''} + C_{22} w_{2}^{'} - C_{23} w_{1}^{''} - C_{24} w_{1}^{'}$$

$$+ C_{26} w_{o}^{''} w_{2} + C_{28} \bar{w}_{2} - C_{29} f_{o}^{''} w_{2} - C_{30} \bar{w}_{2} - C_{31} w_{o}^{''} - C_{28} w_{o}^{''} f_{2}$$

(26)

$$w_{1}^{iv} = C_{1} f_{1}^{''} + C_{2} f_{1}^{'} - C_{3} f_{2}^{''} + C_{4} f_{2}^{'} + C_{5} w_{1}^{''} - C_{6} w_{1}^{'} - C_{7} w_{1}^{''} + C_{8} w_{1}^{'}$$

$$- C_{10} w_{o}^{''} w_{1} + C_{12} \bar{w}_{1} - C_{12} f_{o}^{''} w_{1} + C_{14} \bar{w}_{1} - C_{15} w_{o}^{''} - C_{12} w_{o}^{''} f_{1}$$

(27)

$$w_{2}^{iv} = C_{1} f_{2}^{''} + C_{2} f_{2}^{'} + C_{3} f_{1}^{''} - C_{4} f_{1}^{'} + C_{5} w_{2}^{''} - C_{6} w_{2}^{'} + C_{7} w_{2}^{''} - C_{8} w_{1}^{'}$$

$$- C_{10} w_{o}^{''} w_{2} + C_{12} \bar{w}_{2} - C_{12} f_{o}^{''} w_{2} - C_{14} \bar{w}_{1} - C_{15} w_{o}^{''} - C_{12} w_{o}^{''} f_{2}$$

(28)

The constants $C_1 - C_{31}$ are listed in Ref. 17; $f_{o}^{''}$ is given by Eq. (15).

This set of homogeneous differential equations with variable coefficients together with the appropriate boundary conditions listed in Appendix D form an eigenvalue problem which is solved numerically.

**Governing Equations of the 2nd-order State (Postbuckling Problem)**

The set of governing equations for $W^{(2)}$ and $F^{(2)}$ is

$$L_{A}^{*}(F^{(2)}) - L_{B}^{*}(W^{(2)}) = -(1/R)W_{xx}^{(2)} - tw_{o,xx}W_{yy}^{(2)}$$

$$+ (1/2)(t/R)^{2}n^{2}(w_{1}w_{1,xx}w_{1}^{'+} w_{1}^{',x}w_{1}^{'+} w_{2}^{'+} w_{1}^{'+} w_{2}^{'+} x^{'}w_{2}^{',x}$$

$$+ (w_{1}w_{1,xx}w_{1}^{'-} w_{1}^{',x}w_{2}^{'-} w_{2}^{',x} w_{1}^{'+} w_{2}^{',x}) \cos 2n\theta$$

$$+ (w_{1}w_{2}^{',xx}w_{2}^{',xx}w_{1}^{'+} w_{2}^{',xx} w_{1}^{'+} w_{2}^{',xx} \sin 2n\theta$$

(29)
\[ L^*_B(F^{(2)}) + L^*_D(\dot{w}^{(2)}) = (1/R)F^{(2)}_{xx} + \left(ERt^2/c\right)f^o_{x,xx} \dot{w}^{(2)}_{yy} + t \dot{w}^{(2)}_{x,xx} F^{(2)}_{yy} \]
\[- \left(ERt^2/cR\right)(AW^{(2)}_{xx} + pW^{(2)}_{yy} - 2\tau W^{(2)}_{xy}) \]
\[- \left(1/2\right)(ERt^3/cR)n^2 \left\{ (w^1_{f_1,xx} + 2w_1,xx + w_{1,xx} + w_{2,xx} + 2w_2,xx + 2w_{2,xx} + 2w_{1,xx} + 2w_{2,xx}) \right\} \cos 2\theta \]
\[ + [w^1_{f_1,xx} + w_{1,xx} - 2w_1,xx + w_{1,xx} - 2w_2,xx + w_{2,xx} + w_{2,xx}] \sin 2\theta \]
\[\text{These equations admit separable solutions of the form} \]
\[ \dot{w}^{(2)} = t[w^a(x) + w^b(x) \cos 2\theta + w^c(x) \sin 2\theta] \]
\[ F^{(2)} = \left(ERt^2/c\right)[f^a(x) + f^b(x) \cos 2\theta + f^c(x) \sin 2\theta] \]
Substituting, regrouping and equating coefficients of like trigonometric terms yields the following system of 6 linear inhomogeneous ordinary differential equations with variable coefficients
\[ \tilde{A}^* \tilde{f}^{iv}_{22} - \tilde{A}^* \tilde{f}^{iv}_{21} + c w^n = \left(\frac{ct}{2R}\right)n^2 (w_1, w_1, w_1, w_1, w_1, w_1, w_1, w_1, w_1) \]
\[ \tilde{B}^* \tilde{w}^{iv}_{21} - \frac{\tilde{B}^* \tilde{w}^{iv}_{21}}{12} + \frac{\tilde{A}^* \tilde{f}^{iv}_{16}}{16} + \frac{4\tilde{A}^* \tilde{f}^{iv}_{26}}{16} + 16\tilde{A}^* \tilde{f}^{iv}_{16} \]
\[ - \left(\frac{ct}{2R}\right)\left[\tilde{B}^* \tilde{w}^{iv}_{21} - \frac{\tilde{B}^* \tilde{w}^{iv}_{21}}{12} - \frac{2\tilde{B}^* \tilde{w}^{iv}_{26}}{12} + \frac{2\tilde{B}^* \tilde{w}^{iv}_{61}}{12} \right] n^2 \]
\[ = \left(\frac{ct}{2R}\right)n^2 (w_1, w_1, w_1, w_1, w_1, w_1, w_1, w_1, w_1) \]
\[ \tilde{f}^{1\nu}_{21} - (2\tilde{A}^{*}_{11} + \tilde{A}^{*}_{22})4n^2f^{\nu}_{\gamma} + \tilde{A}^{*}_{11} 16n^4f^{\nu}_{\gamma} + 4\tilde{A}^{*}_{26}nf^{\nu}_{\beta} = 16\tilde{A}^{*}_{11} 3n^2f^{\nu}_{\beta} \]

\[- (t/2R)[\tilde{B}^{*}_{11}\tilde{w}^{1\nu}_{\gamma} - (\tilde{B}^{*}_{11} + \tilde{B}^{*}_{22} - 2\tilde{B}^{*}_{66})4n^2w^{\nu}_{\gamma} + \tilde{B}^{*}_{12} 16n^4w^{\nu}_{\gamma} - (2\tilde{B}^{*}_{26} - \tilde{B}^{*}_{61})2nw^{\nu}_{\beta} + (2\tilde{B}^{*}_{16} - \tilde{B}^{*}_{62})8n^3w^{\nu}_{\beta}] + c\tilde{w}^{\nu}_{\gamma} = (ct/2R)n^2(w^{\nu}_{12}w^{\nu}_{21} - 2w^{\nu}_{12}w^{\nu}_{21}) \] (35)

\[ \tilde{B}^{*}_{21}\tilde{f}^{1\nu}_{\alpha} + (t/2R)\tilde{D}^{*}_{11}\tilde{w}^{1\nu}_{\alpha} - (2cR/t)f^{\nu}_{\alpha} + 2c\tilde{w}^{\nu}_{\alpha} \] (36)

\[ = - cn^2(w^{\nu}_{12}w^{\nu}_{12}f^{1\nu}_{\alpha} + w^{\nu}_{12}w^{\nu}_{12}f^{1\nu}_{\alpha} + w^{\nu}_{12}w^{\nu}_{12}f^{1\nu}_{\alpha} + w^{\nu}_{12}w^{\nu}_{12}f^{1\nu}_{\alpha}) \]

\[ \tilde{B}^{*}_{21}\tilde{f}^{1\nu}_{\beta} - (\tilde{B}^{*}_{11} + \tilde{B}^{*}_{22} - 2\tilde{B}^{*}_{66})4n^2f^{\nu}_{\beta} + \tilde{B}^{*}_{12} 16n^4f^{\nu}_{\beta} + (2\tilde{B}^{*}_{26} - \tilde{B}^{*}_{61})2nf^{\nu}_{\gamma} \]

\[- (2\tilde{B}^{*}_{16} - \tilde{B}^{*}_{62})8n^3f^{\nu}_{\gamma} + (t/2R)[\tilde{B}^{*}_{11}\tilde{w}^{1\nu}_{\beta} - 2(\tilde{B}^{*}_{12} + 2\tilde{D}^{*}_{66})4n^2w^{\nu}_{\beta} + \tilde{B}^{*}_{22} 16n^4w^{\nu}_{\beta} + 8\tilde{B}^{*}_{16}nw^{\nu}_{\gamma} - 32\tilde{B}^{*}_{26}n^3w^{\nu}_{\gamma}] - (2cR/t)f^{\nu}_{\beta} + 2c(\tilde{w}^{\nu}_{12} - 4n^2w^{\nu}_{\beta} - 4nw^{\nu}_{\gamma}) \]

\[ + 8cn^2(w^{\nu}_{12}f^{\nu}_{\beta} + f^{\nu}_{\beta}w^{\nu}_{\beta}) = - cn^2(w^{\nu}_{12}f^{\nu}_{\beta} - 2w^{\nu}_{12}f^{\nu}_{\beta} + w^{\nu}_{12}f^{\nu}_{\beta} + 2w^{\nu}_{12}f^{\nu}_{\beta} - w^{\nu}_{12}f^{\nu}_{\beta}) \] (37)

\[ \tilde{B}^{*}_{21}\tilde{f}^{1\nu}_{\gamma} - (\tilde{B}^{*}_{11} + \tilde{B}^{*}_{22} - 2\tilde{B}^{*}_{66})4n^2f^{\nu}_{\gamma} + \tilde{B}^{*}_{12} 16n^4f^{\nu}_{\gamma} - (2\tilde{B}^{*}_{26} - \tilde{B}^{*}_{61})2nf^{\nu}_{\beta} \]

\[ + (2\tilde{B}^{*}_{16} - \tilde{B}^{*}_{62})8n^3f^{\nu}_{\beta} + (t/2R)[\tilde{B}^{*}_{11}\tilde{w}^{1\nu}_{\gamma} - 2(\tilde{B}^{*}_{12} + 2\tilde{D}^{*}_{66})4n^2w^{\nu}_{\gamma} + \tilde{B}^{*}_{22} 16n^4w^{\nu}_{\gamma} - 8\tilde{B}^{*}_{16}nw^{\nu}_{\gamma} + 32\tilde{B}^{*}_{26}n^3w^{\nu}_{\gamma}] - (2cR/t)f^{\nu}_{\gamma} + 2c(\tilde{w}^{\nu}_{12} - 4n^2w^{\nu}_{\beta} + 4nw^{\nu}_{\gamma}) \]

\[ + 8cn^2(w^{\nu}_{12}f^{\nu}_{\gamma} + f^{\nu}_{\gamma}w^{\nu}_{\gamma}) = - cn^2(w^{\nu}_{12}f^{\nu}_{\gamma} - 2w^{\nu}_{12}f^{\nu}_{\gamma} + w^{\nu}_{12}f^{\nu}_{\gamma} + 2w^{\nu}_{12}f^{\nu}_{\gamma} - w^{\nu}_{12}f^{\nu}_{\gamma}) \] (38)

Equation (33) can be integrated twice to yield

\[ f^{\nu}_{\alpha} = (t/2R)(\tilde{B}^{*}_{21}/\tilde{A}^{*}_{22})w^{\nu}_{\alpha} - (c/\tilde{A}^{*}_{22})w_{\alpha} + (ct/4R)(n^2/\tilde{A}^{*}_{22})(w^{2}_{1}w^{2}_{2} + \tilde{c}_{3x} x + \tilde{c}_{4t}) \] (39)
where $\tilde{x} = x/R$ and the constants of integration $\tilde{C}_3$ and $\tilde{C}_4$ are identically equal to zero because of the periodicity condition (see Appendix B for details). Eliminating $f_\alpha$ between Eqs. (33) and (36) one obtains

$$w_{\alpha}^{iv} = (D_1 - D_2 \lambda)w_\alpha - D_3 w_\alpha + D_4 (w_1^2 + w_2^2) - D_5 (w_1 w_{11} + w_1 w_{11}^* + w_2 w_{22} + w_2 w_{22}^*) + D_8 (w_1 f_{11} + 2w_1 f_{11}^* + w_1 f_{11} + w_2 f_{22} + 2w_2 f_{22}^* + w_2 f_{22}^*)$$

(40)

Further, in order to be able to use the 'shooting method' of Ref. 16 to solve the governing equations of the 2nd order state it is necessary, by considering Eqs. (34) and (37), to eliminate the $w_{\beta}^{iv}$ term from Eq. (34) and the $f_{\beta}^{iv}$ term from Eq. (37). Similarly, by considering Eqs. (35) and (38) one must eliminate the $w_{\gamma}^{iv}$ term from Eq. (35) and the $f_{\gamma}^{iv}$ term from Eq. (38). Finally, some further regrouping makes it possible to write the resulting equations as

$$f_{\beta}^{iv} = D_9 f_{\beta} - (D_{10} + D_{17} w_\gamma) f_{\beta} + D_{11} f_{\gamma}^{iv} - D_{12} f_{\gamma}^{iv} - (D_{13} + D_{31} \lambda) w_\beta^{iv} - (D_{14} - D_{18} w_\gamma) w_\beta$$

- $D_{17} (f_{11}^{iv} - f_{01}^{iv}) w_{15}^{iv} + (D_{10} + D_{19} \tilde{\gamma}) w_{11}^{iv} + D_3 w_{11}^{iv} + w_{11}^{iv} - w_2 w_{22}^{iv} + w_{22}^{iv} + w_{22}^{iv}$

- $D_4 (w_1 f_{11}^{iv} - 2w_1 f_{11}^{iv} + w_1 f_{11}^{iv} + w_2 f_{22}^{iv} + 2w_2 f_{22}^{iv} - w_2 f_{22}^{iv})$  

(41)

$$f_{\gamma}^{iv} = D_9 f_{\gamma} - (D_{10} + D_{17} w_\gamma) f_{\gamma} - D_{11} f_{\beta}^{iv} + D_{12} f_{\beta}^{iv} - (D_{13} + D_{31} \lambda) w_\gamma^{iv}$$

- $(D_{14} - D_{18} w_\gamma) w_{15}^{iv} - D_{17} (f_{11}^{iv} - f_{01}^{iv}) w_{11}^{iv} + (D_{10} + D_{19} \tilde{\gamma}) w_{11}^{iv}$

+ $D_3 (w_1 w_{11}^{iv} + w_{22}^{iv} - 2w_1 w_{11}^{iv}) - D_2 (w_1 f_{11}^{iv} - 2w_1 f_{11}^{iv} + w_1 f_{11}^{iv} + w_2 f_{22}^{iv} - 2w_2 f_{22}^{iv} + w_2 f_{22}^{iv})$  

(42)

$$w_{\beta}^{iv} = - D_{20} f_{\beta}^{iv} - (D_{21} + D_{22} w_\gamma) f_{\beta} + (D_{23} + D_{24} \lambda) w_\beta^{iv} - (D_{24} + D_{17} w_\gamma) w_\beta$$

- $D_{22} (f_{11}^{iv} - f_{01}^{iv}) w_{25}^{iv} - D_{25} f_{11}^{iv} + D_{26} f_{11}^{iv} - D_{27} w_{11}^{iv} + (D_{28} + D_{29} \tilde{\gamma}) w_{11}^{iv}$

- $D_2 (w_1 w_{11}^{iv} - w_1 w_{11}^{iv} + w_2 w_{22}^{iv} + w_{22}^{iv}) - D_8 (w_1 f_{11}^{iv} - 2w_1 f_{11}^{iv} + w_1 f_{11}^{iv} - w_2 f_{22}^{iv} + 2w_2 f_{22}^{iv} - w_2 f_{22}^{iv})$  

(43)
\[ w_{IV} = -D_{20}f'' - (D_{21} + D_{22}w^o) f_Y + (D_{23} - D_{24})w'' + (D_{24} + D_{17}w^o)w_Y \]
\[ - D_{22}(f''^o - \tilde{p})w_Y + D_{26}f'' - D_{26}f'' + D_{27}w'' - (D_{28} + D_{29})w_Y \]
\[ - D_5(w_1^w + w_2^w - 2w_1^w - 2w_1^w) - D_8(w_1^w - 2w_1^w + w_2^w - w_2^w - w_1^w + w_2^w - w_1^w - w_2^w) \] (44)

The constants \( D_1 \) to \( D_{32} \) are listed in Ref. 17; \( f''^o \) is given by Eq. (15).

This set of inhomogeneous differential equations with variable coefficients together with the appropriate boundary conditions listed in Appendix D form a response problem which is solved numerically.

Postbuckling Coefficients and Imperfection Form Factors

For perfect shells one is interested in the variation of \( \lambda(\xi) \) with \( \xi \) in the vicinity of \( \lambda = \lambda_C \). Near the bifurcation point \( \lambda_C \) the asymptotic expansion given in Eq. (8) is valid. The postbuckling coefficients 'a' and 'b' are derived in Appendix E yielding

\[ a = - (3/2\lambda_C \tilde{\lambda}) P(1) \ast (W(1), W(1)) \] (45)

\[ b = - (1/\lambda_C \tilde{\lambda})[2P(1) \ast (W(1), W(2)) + P(2) \ast (W(1), W(1)) + a_\lambda C \tilde{\Pi}_1 + (1/2)(a_\lambda C)^2 \tilde{\Pi}_2] \] (46)

where

\[ \tilde{\lambda} = 2P(1) \ast (\ddot{W}_C(W(1)), \ddot{P}_C(W(1))) \] (47)

\[ \tilde{\Pi}_1 = P(1) \ast (\ddot{W}_C(W(1)), \ddot{P}_C(W(1))) + P(2) \ast (\ddot{W}_C(W(1))) + \ddot{P}_C(W(1), W(2)) \] (48)

\[ \tilde{\Pi}_2 = 2P(1) \ast (\dddot{W}_C(W(1)), \dddot{P}_C(W(1))) \] (49)

\[ (\ast)_C = \frac{\delta}{\partial \lambda} (\ast)_C \] (50)

The subscript \((\ast)_C\) denotes the fact that the prebuckling solution is evaluated at the bifurcation point. The short-hand notation used
\[ A^*(B,C) = \iint_s [A_{xx}B_{c}y + A_{yy}B_{c}x - A_{xy}(B_{c}y + B_{c}x)] dx dy \] (51)

has been first introduced by Hutchinson and Frauenthal[18].

For imperfect shells the variation of \( A(\xi, \tilde{\xi}) \) in the vicinity of the bifurcation point \( \lambda = \lambda_c \) is given by the following asymptotic expansion [19] (see also Fig. 7)

\[ (\lambda - \lambda_c)\xi = aA_c\xi^2 + bA_c\xi^3 + \ldots - aA_c\tilde{\xi} - \beta(\lambda - \lambda_c)\tilde{\xi} + O(\xi^5) \] (52)

The imperfection form factors 'a' and 'b' are derived in Appendix E yielding

\[ \alpha = (1/A_c)\left[ F_1^{(1)}(\tilde{W}_c, W_c) + F_1^{(1)}(\tilde{W}_c, W_c) \right] \] (53)

\[ \beta = (1/A_c)\left[ F_2^{(1)}(\tilde{W}_c, W_c) + F_3^{(1)}(\tilde{W}_c, W_c) + \eta_3 - aA_c[(1/2)\eta_4 + \eta_5] \right] \] (54)

where

\[ \eta_3 = \iint_s \left[ A_{11} \tilde{W}_{x,x} W_c + A_{12} \tilde{W}_{c,y} W_c + A_{16} (\tilde{W}_{c,x} W_c + \tilde{W}_{c,y}) \right] \tilde{W}_c dxdy \] (55)

\[ \eta_4 = 2F_1^{(1)}(\tilde{W}_c, W_c) + F_1^{(1)}(\tilde{W}_c, W_c) \] (56)

\[ \eta_5 = \iint_s \left[ A_{11} \tilde{W}_{x,x} W_c + A_{12} \tilde{W}_{c,y} W_c + A_{16} (\tilde{W}_{c,x} W_c + \tilde{W}_{c,y}) \right] \tilde{W}_c dxdy \] (57)

Since the initial imperfection is assumed to be
\[ \hat{\bar{W}} = \hat{\xi} \bar{W} \]  

(58)

therefore $\hat{\bar{W}}$ represents the shape of the initial imperfection. Notice that if the initial imperfection is assumed to be affine to the buckling mode then $\hat{\bar{W}} = \bar{W}^{(1)}$.

As can be seen from Fig. 7 the buckling load of the imperfect structure $\Lambda_S$ occurs at the 'limit point' of the prebuckling states. If the limit point is close enough to the bifurcation point then $\Lambda_S$, the maximum load that the structure can support prior to buckling, can also be evaluated from Eq. (52) by maximizing $\Lambda$ with respect to $\xi$. For the many practical applications where a unique buckling mode is associated with the lowest buckling load and the buckling and initial postbuckling behaviour are symmetric with respect to the buckling displacement, the first postbuckling coefficient 'a' is identically equal to zero. In this case using Eq. (52) to maximize $\Lambda$ with respect to $\xi$ leads to the Modified Koiter Formula\textsuperscript{[20]}

\[ (1-\rho_s)^{3/2} = (3/2) \sqrt{-3\alpha^2 b [1 - (\beta/\alpha)(1-\rho_s)]|\bar{\xi}|} \]  

(59)

where $\rho_s = \Lambda_S/\Lambda_C$ and $\bar{\xi}$ is the normalized amplitude of the initial imperfection. It should be emphasized that in all cases presented, $\bar{\xi}$ has been normalized with respect to the shell thickness $t$ and not some effective thickness of the stiffener-shell combination.

Generalized 'Load-Shortening' Relation

Information concerning the extent to which buckling can be expected to be gradual or sudden can be obtained from the postbuckling variation of the applied variable load $\Lambda$ with the generalized displacement $\Lambda$. Notice that $\Lambda^*\Lambda$ represents the decrease in potential energy of the applied variable loads. Thus

\[ \Lambda^*\Lambda = \int \int_{S} \bar{N}_{\alpha\beta} \epsilon_{\alpha\beta} \, dx \, dy \]  

(60)

where $\bar{N}_{\alpha\beta}$ is the variable applied stress resultant and $\epsilon_{\alpha\beta}$ is the corresponding linear strain. Notice that $\epsilon_{\alpha\beta}$ can be obtained from the nonlinear strain-displacement relations
\( E_{\alpha\beta} = c_{\alpha\beta} + \frac{1}{2} W_{\alpha} W_{\beta} \) \hspace{1cm} (61)

Using the perturbation expansions defined by Eq. (8) one gets upon substitution and regrouping

\[
\Lambda^{*\Lambda} = \int \int_s \tilde{N}_{\alpha\beta}[E^{(o)}_{\alpha\beta} - \frac{1}{2}(W_{\alpha} W_{\beta})]dxdy + \xi \{ \int \int_s \tilde{N}_{\alpha\beta}[E^{(1)}_{\alpha\beta} - \frac{1}{2}(W_{\alpha} W_{\beta}) + W_{\alpha} W_{\beta}]dxdy \} \hspace{1cm} (62)
\]

\[
+ \xi^2 \{ \int \int_s \tilde{N}_{\alpha\beta}[E^{(2)}_{\alpha\beta} - \frac{1}{2}(W_{\alpha} W_{\beta}) + W_{\alpha} W_{\beta}]dxdy \} + \ldots
\]

Since the applied external load and the prebuckling state is axisymmetric, therefore for an asymmetric buckling mode

\[
\int \int_s \tilde{N}_{\alpha\beta}[E^{(1)}_{\alpha\beta} - \frac{1}{2}(W_{\alpha} W_{\beta}) + W_{\alpha} W_{\beta}]dxdy = 0 \hspace{1cm} (63)
\]

Using Taylor series expansions at \( \Lambda = \Lambda_c \) for the prebuckling quantities in Eq. (62) and specializing the results to the cases where the first postbuckling coefficient 'a' is identically equal to zero one obtains after some regrouping

\[
\Lambda^{*\Lambda} = \Lambda^{*\Lambda}_c + (\Lambda - \Lambda_c) \{ \int \int_s \tilde{N}_{\alpha\beta}[E^{(o)}_{\alpha\beta} - \frac{1}{2}(W_{\alpha} W_{\beta}) + W_{\alpha} W_{\beta}]dxdy \}
\]

\[
+ (1/b\Lambda_c) \int \int_s \tilde{N}_{\alpha\beta}[E^{(2)}_{\alpha\beta} - \frac{1}{2}(W_{\alpha} W_{\beta}) + W_{\alpha} W_{\beta}] - \frac{1}{2}(W_{\alpha} W_{\beta}) dxdy \} + \ldots \hspace{1cm} (64)
\]

where

\[
\Lambda^{*\Lambda}_c = \int \int_s \tilde{N}_{\alpha\beta}[E^{(o)}_{\alpha\beta} - \frac{1}{2}(W_{\alpha} W_{\beta})]dxdy \hspace{1cm} (65)
\]

\[
E_{\alpha\beta} = E^{(o)}_{\alpha\beta} \left| \Lambda = \Lambda_c \right. \hspace{1cm} (66)
\]

\[
(\cdot)_c = \frac{\partial}{\partial \Lambda}(\cdot) \hspace{1cm} (67)
\]
Notice that if 'a' is identically equal to zero then from Eq. (8)

\[ \xi^2 = \frac{1}{b A_c} (\Lambda - \Lambda_c) \]  

(68)

Finally, the generalized displacement \( \Delta \) can be written as

\[
\Delta = \Delta_c + (p-1) \int \int (\bar{N}_{\alpha\beta}/\Lambda) [\hat{E}_{\alpha\beta} - (1/2)(W_{c,\alpha'c,\beta'} + W_{c,\alpha'c,\beta'})] \, dx \, dy
\]

\[ + \frac{1}{b} \int \int (\bar{N}_{\alpha\beta}/\Lambda) [\hat{E}^{(2)}_{\alpha\beta} - (1/2)(\bar{W}_{\alpha'c,\beta'} + \bar{W}_{c,\alpha'c,\beta'}) - (1/2)\bar{W}^{(1)}_{\alpha'c} \bar{W}^{(1)}_{\beta'}] \, dx \, dy \]  

(69)

where now \( p = \frac{\Lambda}{\Lambda_c} \), \( (\cdot)'' = \frac{\partial}{\partial \rho} \) and \( \Delta_c \) = generalized displacement just before buckling.

Computing the slope of the variable load vs generalized displacement curve just before buckling one obtains

\[
K_c = \left[ \frac{\partial \Delta}{\partial \Delta_c} \right]_{\Lambda = \Lambda_c} = \left[ \frac{\partial \Delta}{\partial \Delta_c} \right]_{\Lambda = \Lambda_c}^{-1}
\]

\[
= \left\{ \int \int (\bar{N}_{\alpha\beta}/\Lambda) [\hat{E}_{\alpha\beta} - (1/2)(W_{c,\alpha'c,\beta'} + W_{c,\alpha'c,\beta'})] \, dx \, dy \right\}^{-1}
\]

(70)

Conversely, the slope of the variable load vs generalized displacement curve just after buckling is given by

\[
K^* = \left[ \frac{\partial \Delta}{\partial \Delta} \right]_{\Lambda = \Lambda_c} = \left[ \frac{\partial \Delta}{\partial \Delta} \right]_{\Lambda = \Lambda_c}^{-1}
\]

\[
= \left\{ \int \int (\bar{N}_{\alpha\beta}/\Lambda) [\hat{E}_{\alpha\beta} - (1/2)(W_{c,\alpha'c,\beta'} + W_{c,\alpha'c,\beta'})] \, dx \, dy \right\}^{-1}
\]

\[ + \left\{ \int \int (\bar{N}_{\alpha\beta}/\Lambda) [\hat{E}^{(2)}_{\alpha\beta} - (1/2)(\bar{W}_{\alpha'c,\beta'} + \bar{W}_{c,\alpha'c,\beta'}) - (1/2)\bar{W}^{(1)}_{\alpha'c} \bar{W}^{(1)}_{\beta'}] \, dx \, dy \right\}^{-1}
\]

(71)

As can be seen from Fig. 8, the angle of the initial slope just after buckling \( \tilde{\theta}^* \) where
\( \tilde{\theta} = \tan^{-1}(\epsilon M C) \) \hspace{1cm} (72)

is indeed helpful in determining whether the buckling will be gradual or catastrophic. Notice that it is customary to normalize the generalized displacement \( \Delta \) by the appropriate membrane strain \( \epsilon M \) (the strain which corresponds to the applied variable load) so that for membrane prebuckling the angle of the fundamental path \( \tilde{\theta} = \tan^{-1}(\epsilon M C) = 45^\circ \).

4. NUMERICAL ANALYSIS

Introducing as a unified variable the 16-dimensional vector \( \tilde{y}^{(1)} \) defined as follows

\[
\begin{align*}
Y_1^{(1)} &= f_1, & Y_2^{(1)} &= f_2, & Y_3^{(1)} &= w_1, & Y_4^{(1)} &= w_2, & Y_5^{(1)} &= f_1', & \ldots, & Y_{16}^{(1)} &= w_2''
\end{align*}
\]  \hspace{1cm} (73)

then the system of equations (21)-(24) can be reduced to the following (nonlinear) eigenvalue problem

\[
\frac{d}{dx} y^{(1)} = \tilde{f}^{(1)}(x, y^{(0)}, \dot{y}^{(1)}; \lambda, \bar{p}, \bar{r}) \hspace{1cm} (74)
\]

\[
\tilde{B}_1^{(1)} y^{(1)}(x=0) + \tilde{B}_2^{(1)} y^{(1)}(x=L/R) = 0 \hspace{1cm} (75)
\]

where the components of the 8x16 matrices \( \tilde{B}_1^{(1)} \) and \( \tilde{B}_2^{(1)} \) depend on the boundary conditions at the shell edges. Notice that the 4-dimensional vector

\[
y^{(0)} = [w_o, w_o', w_o'', w_o''']^T \hspace{1cm} (76)
\]

contains the known solution of Eq. (16), the prebuckling problem.

Introducing further as another unified variable the 20-dimensional vector \( \tilde{y}^{(2)} \) defined as follows

\[
\begin{align*}
Y_1^{(2)} &= f_\beta, & Y_2^{(2)} &= f_\gamma, & Y_3^{(2)} &= w_\alpha, & Y_4^{(2)} &= w_\beta, & Y_5^{(2)} &= w_\gamma, & Y_6^{(2)} &= f_\beta', & \ldots, & Y_{20}^{(2)} &= w_\gamma''
\end{align*}
\]  \hspace{1cm} (77)
then the system of equations (40)-(44) can be reduced to the following inhomogeneous 2-point boundary value problem

\[
\frac{d}{dx} \tilde{y}(2) = \tilde{f}(2) (x, \tilde{y}(0), \tilde{y}(2); \lambda, \tilde{p}, \tilde{t}) + \tilde{f}(x, \tilde{y}(1))
\]

(78)

\[
\tilde{B}_1^{(2)} \tilde{y}(2)(x=0) + \tilde{B}_2^{(2)} \tilde{y}(2)(x=L/R) = 0
\]

(79)

where once again the components of the 10x20 boundary matrices \(\tilde{B}_1^{(2)}\) and \(\tilde{B}_2^{(2)}\) depend on the boundary conditions at the shell edges. Notice that here the 4-dimensional vector \(\tilde{y}^{(0)}\) contains the known solution of the prebuckling problem (Eq. 16) and that the 16-dimensional vector \(\tilde{y}^{(1)}\) is the eigenvector of the buckling problem (Eqs. 74-75).

Because of earlier successful experiences with the method \([21,22]\), it was decided to solve both the buckling problem (Eqs. 74-75) and the post-buckling problem (Eqs. 78-79) by the numerical technique known as 'parallel shooting over \(n\)-intervals' \([16]\).

**Solution of the Buckling Problem**

To solve the buckling problem a generalization of Stodola's method \([23]\) for the calculation of the asymmetric buckling loads and the corresponding buckling modes of circular cylindrical shells is used. This generalization was first published by Cohen \([24]\).

The applied loading consists of axial compression, internal or external pressure and clockwise or counter-clockwise torque. It is assumed to have a uniform spatial distribution and is divided into a fixed part and a variable part. The magnitude of the variable part is allowed to vary in proportion to a load parameter \(\Lambda\). This leads to an eigenvalue problem for the critical load \(\Lambda_c\). In Eq. (74) the user can select \(\Lambda_c\) to be the critical value of either the normalized axial load \(\tilde{\lambda}\), or the normalized external pressure \(\tilde{p}\) or the normalized torque \(\tilde{t}\).

Because of the nonlinear dependence of the prebuckling state on the variable load \(\lambda\), in general it is necessary to approach the critical eigenvalue (for a given circumferential wave number \(n\)) by the solution of a sequence of modified (linearized) eigenvalue problems. This equations are obtained by restricting the search for eigenvalues to a sufficiently small
neighbourhood of an estimate \( \Lambda = \Lambda_e \) so that in this neighbourhood the functions \( \bar{y}^{(o)} \) have a linear dependence on \( \Lambda \). Setting

\[
\Lambda = \Lambda_e + \mu
\]  

one has to first order in \( \mu \)

\[
\bar{y}^{(o)}(\Lambda) = \bar{y}^{(o)}(\Lambda_e) + \mu \bar{y}^{(o)}(\Lambda_e)
\]  

where

\[
\begin{align*}
\bar{y}^{(o)} &= \frac{d}{d\Lambda} \bar{y}^{(o)} \\
\end{align*}
\]

Substituting this expression into Eqs. (74)-(75) and using \( \lambda \) as the variable load yields the following modified (linearized) eigenvalue problem

\[
\frac{d}{dx} \bar{y}^{(1)} = \xi^{(1)}(\bar{x}, \bar{y}^{(o)}, \bar{y}^{(1)}; \Lambda_e, \bar{p}, \bar{r}) + \mu g^{(1)}(\bar{x}, \bar{y}^{(o)}, \bar{y}^{(1)})
\]

\[
\bar{B}_1^{(1)} \bar{y}^{(1)}(\bar{x}=0) + \bar{B}_2^{(1)} \bar{y}^{(1)}(\bar{x}=L/R) = 0
\]  

Notice that each of the 'effective load terms' is split into a part independent of \( \mu \) and a second part linear in \( \mu \). The iteration equations are obtained by setting \( \mu = 1 \) in the second parts of the 'effective load terms' and interpreting the buckling mode variables of these parts as being known inputs from the previous iteration. Thus the first parts of the 'effective load terms' become homogeneous terms and the second parts become inhomogeneous terms for the equivalent linearized problem of each iteration. Thus one must solve repeatedly

\[
\frac{d}{dx} \bar{y}^{(k)} = \xi^{(1)}(\bar{x}, \bar{y}^{(o)}, \bar{y}^{(k)}; \Lambda_e, \bar{p}, \bar{r}) + g^{(1)}(\bar{x}, \bar{y}^{(o)}, \bar{y}^{(k-1)})
\]

\[
\bar{B}_1^{(1)} \bar{y}^{(k)}(\bar{x}=0) + \bar{B}_2^{(1)} \bar{y}^{(k)}(\bar{x}=L/R) = 0
\]  

where
\( \gamma^{(k)} \) = buckling mode of the \( k \)th iteration
\( \gamma^{(k-1)} \) = buckling mode of the \((k-1)\)th iteration

After each iteration the corresponding eigenvalue estimate \( \mu^{(k)} \) is calculated by evaluating the following Rayleigh quotient

\[
\mu^{(k)} = (\sigma^{(k)}, u^{(k)}; \sigma^{(k-1)}, u^{(k-1)}) / (\sigma^{(k)}, u^{(k)}; \sigma^{(k)}, u^{(k)})
\]  \( (85) \)

where the inner products are defined as follows

\[
(\sigma^{(k)}, u^{(k)}; \sigma^{(k-1)}, u^{(k-1)}) = \int \int [N^{(k)}_x w^{(k)}_x w^{(k)}_x + N^{(k)}_y w^{(k)}_y w^{(k)}_y + N^{(k-1)}_y (w^{(k)}_x w^{(k)}_y + w^{(k)}_x w^{(k)}_y)] dx dy
\]
\[
+ \int \int [N^{(k)}_x w^{(k)}_x w^{(k)}_x + N^{(k)}_y w^{(k)}_y w^{(k)}_y + N^{(k-1)}_y (w^{(k)}_x w^{(k)}_y + w^{(k)}_x w^{(k)}_y)] dx dy
\]
\[
+ \int \int [N^{(k)}_x w^{(k)}_x w^{(k)}_x + N^{(k)}_y w^{(k)}_y w^{(k)}_y + N^{(k-1)}_y (w^{(k)}_x w^{(k)}_y + w^{(k)}_x w^{(k)}_y)] dx dy
\]

The iterations are continued until the sum \( \Lambda_e^{(k)} + \mu^{(k)} \) remains essentially constant at the value \( \Lambda_1 \). A suitable choice for the sequence \( \Lambda_e^{(k)} \) is \( \Lambda_e^{(1)} = 0 \) and \( \Lambda_e^{(k)} = \Lambda_e^{(k-1)} + (1/2)\mu^{(k-1)} \) for \( k > 1 \), where the 'relaxation factor' \( 1/2 \) is inserted in order to assure that at each stage \( \Lambda^{(k)} < \Lambda_1 \). Cohen [24] has shown that in order to insure that the eigenvalues \( \mu^{(k)} \) are real it is necessary that \( \Lambda_e^{(k)} < \Lambda_1 \). For further details of the solution procedure the interested reader should consult References 17,24.

Solution of the Postbuckling Problem

Once the prebuckling solution vector \( \gamma^{(0)} \) and the solution of the buckling problem, the eigenvalue \( \Lambda_c \) and the corresponding eigenvector \( \gamma^{(1)} \) are known, the postbuckling problem is governed by

\[
\frac{d}{dx} \gamma^{(2)} = \mathbf{f}^{(2)}(x, \gamma^{(0)}, \gamma^{(2)}; \Lambda_c, \mathbf{P}, \mathbf{c}) + \mathbf{f}(x, \gamma^{(1)})
\]
\[
(87)
\]

\[
\mathbf{B}_1^{(0)} \gamma^{(2)}(x=0) + \mathbf{B}_2^{(0)} \gamma^{(2)}(x=L/R) = 0
\]
\[
(88)
\]
Due to the often complicated functions of \( x \) represented by the known solution vectors \( Y^{(0)} \) and \( Y^{(1)} \), anything but a numerical solution of this linear, inhomogeneous 2-point boundary value problem is out of question. A detailed description of the method used is given in References 16 and 17. Parallel shooting over \( n \)-intervals is slower than a coarse standard finite difference or finite element scheme. However, if the length of the intervals of integration is chosen properly so that numerical instabilities are avoided, then this method gives more accurate results. Also since the step size is changed automatically so as to satisfy the chosen convergence criterion, a single run is sufficient to obtain a converged solution. Thus it is not necessary to repeat the solution with different step sizes to ascertain that a properly converged solution has been found, as is the recommended practice with the standard finite difference or finite element codes.

It is well known that for the linearized 2-point boundary value problem in principle Newton's method yields the correct initial value \( S^{(2)} \) directly without the need of iterations\[^{[16]}\]. In later work it was found that the numerical accuracy of the solution can be improved greatly by a few iterations, whereby the Jacobian is kept constant and only the right-hand side is varied\[^{[17]}\]. The solution of the associated initial value problems and the corresponding variational equations is done by the library subroutine DEQ from Caltech's Willis Booth Computer Center. DEQ uses the method of Runge-Kutta-Gill to compute starting values for an Adams-Moulton predictor-corrector scheme. As mentioned earlier, the program includes an option with variable interval size and it uses automatic truncation error control. For shells with \( L/R = 1.0 \) parallel shooting over 8 intervals is used. This actually involves the numerical integration of six 440 dimensional and two 220 dimensional vector equations. These high dimensions are due to the simultaneous integration of the variational equations and the corresponding initial value problem.

Finally, after the solution of the postbuckling problem has been obtained, one must evaluate the integrals involved in the definition of the postbuckling coefficients 'a' and 'b' (Eqs. 45-49), of the imperfection form factors 'a' and 'b' (Eqs. 53-57), of the angle of the fundamental path \( \tilde{\theta} \) and of the angle of the initial slope just after buckling \( \tilde{\theta}^* \) (Eqs. 70-71). It has been shown in Reference 25 that it is advantageous to evaluate the above integrals by solving initial value problems rather than using numerical
integration schemes. This same approach is used here. The interested reader should consult Reference 17 for further details.

5. NUMERICAL RESULTS

Thanks to the extensive NASA sponsored research programs carried out in the sixties and the early seventies it is known that the degree of imperfection sensitivity of thin-walled shell structures depends on the combination of shell geometry and the type of the applied loading. The use of Koiter's general elastic postbuckling theory has been widely explored and it was found that also boundary conditions [18] and nonlinear modal interactions [26] can have a profound effect on the imperfection sensitivity predictions. Furthermore it has been shown by Hutchinson and Frauenthal [18] for orthotropic cylinders and by Tennyson et al [27] for anisotropic shells that for reliable prediction of the postbuckling behaviour one must use a rigorous prebuckling analysis.

Thus, although the computational module ANILISA has the capability of using a membrane prebuckling analysis, in this paper only results of the rigorous prebuckling branch are included. To test the accuracy and reliability of ANILISA among others the following published results have been partially reproduced:

1. Hutchinson and Frauenthal's work on the postbuckling behaviour of stiffened cylindrical shells [18];
2. Tennyson et al's work on the buckling of imperfect anisotropic cylinders under combined loading [27].

As can be seen from Tables 1 and 2 the agreement is good, for the anisotropic shell even very good. The differences in the orthotropic shell results are due to the fact that the authors of Reference 18 treat nL/R as a continuous variable, whereas for ANILISA the length over radius is specified and n is treated as an integer during the search for the lowest eigenvalue.

To illustrate the capabilities of ANILISA one of the glass/epoxy (30°, 0°, -30°) composite cylindrical shells tested by Booton [14] is used. Its geometric and material data are given in Table 3. In the following the initial imperfection sensitivity computations are based on the assumption that the
shapes of the initial imperfections are affine to the corresponding buckling modes. Thus

$$\tilde{W} = \xi^t \tilde{W} = \xi^t (w_1 \cos n\theta + w_2 \sin n\theta)$$

(89)

Also in all cases only the imperfection sensitivity of the lowest buckling load is calculated.

**Axial compression**

The normalized buckling load $\lambda_c$ and the second imperfection sensitivity coefficient $\tilde{b} = a^2 b$ are plotted in Fig. 9 as a function of the modified Batdorf parameter $\tilde{Z} = L^2 / R t$ for simply supported $(N_x = -N_o, v = W = 0, M_x = 0)$ anisotropic shells. At the lower part of the figure the circumferential wave numbers $n$ at which the lowest buckling loads occur are indicated. Notice that sharp changes in the second imperfection sensitivity coefficient $\tilde{b} = a^2 b$ always occur at places where there is a change in the critical circumferential wave number $n$. In Fig. 10 the prebuckling, buckling and postbuckling mode shapes of a relatively short shell ($\tilde{Z} = 50$) and of a shell of moderate length ($\tilde{Z} = 400$) are displayed. The amplitudes of the buckling modes are normalized by the wall thickness $t$, and as has been proven by Booton [14] one of the components (here $w_1$) is symmetric and the other antisymmetric with respect to the mid-plane of the shell in the axial direction. Notice that for better illustration all 3 postbuckling modes are plotted normalized to one using as divisor their maximum amplitudes indicated in the figure.

To investigate the effect of different boundary conditions in Fig. 11 the normalized buckling load $\lambda_c$ and the second imperfection sensitivity coefficient $\tilde{b} = a^2 b$ are shown for fully clamped $(u = v = W = W_x = 0)$ anisotropic shells. At low $\tilde{Z}$ values the sharp changes in $\tilde{b} = a^2 b$ occur at either the locations where there is a jump in the critical circumferential wave number $n$, or as has been shown in Ref. 17 at $\tilde{Z}$ values where there are (nearly) simultaneous buckling modes. At the higher $\tilde{Z}$ values the fluctuation of $\tilde{b} = a^2 b$ can be attributed to the minimization of $\lambda_c$ with respect to discrete values of $n$. In Fig. 12 the prebuckling, buckling and postbuckling mode shapes are displayed for shells of $\tilde{Z} = 50$ and $\tilde{Z} = 400$, respectively.
Hydrostatic pressure

For simply supported anisotropic shells the normalized buckling pressure $\tilde{p}_c$ and the second imperfection sensitivity coefficient $\tilde{b}=a^2b$ are shown in Fig. 13, whereas in Fig. 14 the prebuckling, buckling and postbuckling mode shapes for shells of $\tilde{Z}=50$ and $\tilde{Z}=400$ are displayed. Notice that for $\tilde{Z}=50$ the $w_γ$-mode undergoes rapid changes close to the shell edge, a behaviour that would be missed completely by a coarse finite element discretization [28]. The decrease in $\tilde{b}=a^2b$ with increasing values of $\tilde{Z}$ is similar to earlier results obtained by Yamaki [29] for isotropic shells. Notice that discontinuities in $\tilde{b}=a^2b$ occur at places where there is a change in the critical circumferential wave number $n$. Also, a closer observation of Fig. 14 reveals that for the hydrostatic pressure loading both the buckling and the postbuckling modes exhibit very little skewedness.

Counter-clockwise vs. clockwise torsion

For the particular Booton shell chosen the only '16' and '26' terms in the constitutive equations listed in Table 3 that do not vanish are the $B_{26}^*$ and $B_{26}^*$ terms. These terms represent coupling between the bending and shear strain of the middle surface. Further, as can be seen from the constitutive equations, if the twisting of the cylinder is due to a positive (counter-clockwise) applied torque, then $γ_{xy}$ is also positive, which results in a negative bending moment $M_\gamma$. Considering the definition of $M_\gamma$ in Fig. 15 it is clear that a negative bending moment will produce an outward normal deflection at the mid-plane of the shell (at $\tilde{x} = 1/2(x/L)$), which is stabilizing [18]. Conversely, a negative torsional loading (clockwise torque) results in a positive bending moment $M_\gamma$, which in turn produces an inward normal deflection at $\tilde{x} = 1/2(x/L)$, which is destabilizing. Considering now the figures 16 and 17, which display the prebuckling, the buckling and the postbuckling modes for counter-clockwise ($\tilde{τ}_c>0$) and clockwise ($\tilde{τ}_c<0$) torsion, respectively, indeed the magnitudes (absolute values) of the critical normalized torsion parameter $\tilde{τ}_c$ are higher if the applied external torque is counter-clockwise. This phenomena was first described by Booton [14]. Notice also that for torsional loading the prebuckling deformations at the bifurcation point are about an order of magnitude smaller than for the other external loads considered.
From the plots of the second imperfection sensitivity coefficient $\tilde{b} = a^2 b$, shown in figures 18 and 19, it is clear that for shorter shells ($\tilde{Z} < 100$) if the applied torque is counter-clockwise one obtains a configuration which is more sensitive to initial imperfections affine to the buckling mode than if the applied external torque is clockwise. For increasing $\tilde{Z} = L^2 / R t$ values the difference in imperfection sensitivity is smaller and the curves become gradually nearly identical and thus independent of the direction of the applied torque. Notice also that discontinuities in $\tilde{b} = a^2 b$ occur only at places where there is a change in the critical circumferential wave number $n$.

Experimental evidence [30,31] indicates that the shapes of the dominant initial imperfections in real structures often do not coincide with the buckling modes of the perfect structure. With the introduction of the imperfection form factors $\alpha$ and $\beta$ the computational module ANILISA can also be used for an initial estimate of the imperfection sensitivity due to the occurrence of any single mode asymmetric imperfection. To illustrate this capability Tables 4 and 5 display data about the imperfection sensitivity of Booton's anisotropic shells under different external loads, whereby besides an initial imperfection affine to the respective buckling mode (see Eq. 89) also the following modal initial imperfection

$$\bar{W} = \tilde{\xi} \bar{W} = \tilde{\xi} t \sin \pi x / L \cos \theta \quad (90)$$

has been investigated. In all cases the circumferential wave number $n$ is chosen to be identical to the critical wave number (the circumferential wave number at which the lowest buckling load occurs). Notice also that the modal imperfection specified by Eq. (90) consists of a single half wave in the axial direction.

Considering now the tabulated results it appears that for axial compression, except for shorter shells, the modal imperfection specified by Eq. (90) does not affect the buckling load of the perfect anisotropic shells at all. On the other hand if the external load is hydrostatic pressure then both the single affine and the single modal imperfections, specified by Eqs. (89) and (90), produce about the same degree of imperfection sensitivity. Further also under counter-clockwise or clockwise torsional loading the single modal imperfection appears to cause very little imperfection sensitivity.
Finally, in the practice, the design engineers are not only interested in whether and how much the buckling load prediction of their proposed shell structure is sensitive to initial imperfections, they also want to obtain an estimate of the 'knockdown' factor $\gamma$ with which they must multiply the buckling load prediction of the perfect structure in order to arrive at the safe allowable load level. With the help of Eq. (59) such an estimate can be computed if besides the imperfection sensitivity coefficient $\delta = \alpha^2 b$ one also has an idea of the size of the amplitude $\xi$ of the single mode imperfection that is likely to occur. For such cases the predictions of Cohen's formula (Eq. 59) are conveniently summarized in Fig. 20.

6. CONCLUSIONS

It is by-now widely accepted that Koiter's General Theory of Elastic Stability [4] has greatly contributed to our understanding of the sometimes perplexing stability behaviour of thin-walled structures. However, due to its mathematical complexity it is not always easy for the practicing structural engineer to find the information he wants for the particular structure at hand from the many publications that are available. What he needs is a computational module that enables him to obtain the desired information readily.

It has been shown that within the context of Koiter's initial postbuckling theory the computational module ANILISA can be used successfully to investigate the imperfection sensitivity of the buckling loads of isotropic, orthotropic and of fully anisotropic cylindrical shells under combined axial compression, external or internal pressure and torsion taking into account the effect of different boundary conditions and of different initial imperfection shapes.

As a 'building block' of the hierarchical design and analysis system 'DISDECO' the computational module ANILISA makes the first step towards acquiring of detailed understanding of the expected instability behaviour of different cylindrical shell configurations feasible. This knowledge is a prerequisite for the development of discrete nonlinear computational models which can reliably predict the load carrying capability of the structure.

It must be stressed that the predictions of ANILISA provide only a first indication of the expected nonlinear behaviour and all its findings must be
evaluated within the context of the fundamental assumptions involved in the theory. Thus the fact that a single long wavelength modal imperfection does not appear to produce any significant imperfection sensitivity may be misleading, since especially under axial compression for the anisotropic shells considered there may exist several nearly simultaneous buckling modes, and it is known that in such situations nonlinear modal interaction may reduce the load carrying capability of the structure considerably. For these cases the user must switch to other more advanced computational modules available within 'DISDECO' which can handle the nonlinear interaction problem of multiple initial imperfection and response modes.

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REFERENCES


APPENDIX A

DERIVATION OF THE ANISOTROPIC CONSTITUTIVE EQUATIONS

The derivation of the stiffness coefficients for a layered anisotropic shell follows essentially the outline given by Booton[14], however, here the positive directions of $W$ and $\theta_k$ are defined differently. Using the sign convention shown in Fig. 6, with $W$ positive inward the numbering of the layers begins at the outer surface. Notice that the angle of rotation $\theta_k$ ($k = 1,2,\ldots,N$) of the individual layers is defined with respect to the x-axis of the shell. The shell reference surface coincides with the midsurface of the laminate. Thus if $t$ is the total thickness of the laminate defined by

$$t = \sum_{k=1}^{N} (h_k - h_{k-1})$$

then $h_0 = -t/2$ and $h_N = t/2$.

For thin shells ($R/t > 50$), if one neglects the fiber-resin geometry then each lamina may be considered a homogeneous orthotropic medium in a plane stress state. Thus the constitutive equations for the $k$th homogeneous orthotropic lamina are

$$\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix}_k =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & 0 & 2Q_{66}
\end{bmatrix}_k
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\frac{1}{\nu} \gamma_{12}
\end{bmatrix}_k$$

where

$$Q_{11} = E_{11}/(1 - \nu_{12} \nu_{21})$$

$$Q_{22} = E_{22}/(1 - \nu_{12} \nu_{21})$$

$$Q_{12} = \nu_{21} E_{11}/(1 - \nu_{12} \nu_{21}) = \nu_{12} E_{22}/(1 - \nu_{12} \nu_{21})$$

$$Q_{66} = G_{12}$$
Notice that there are 4 elastic constants $E_{11}$, $E_{22}$, $v_{12}$ and $G_{12}$. Since the stiffness matrix must be symmetrical therefore $v_{12}E_{22} = v_{21}E_{11}$. Thus the fifth elastic constant $v_{21}$ can be expressed in terms of the other constants.

Normally the lamina principal axes (1,2) do not coincide with the reference axes of the shell wall (x,y). Thus the constitutive equations for each individual lamina must be transformed to the shell wall reference axes in order to be able to determine the shell wall (or laminate) constitutive equations. This transformation yields

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}_k =
\begin{bmatrix}
\tilde{Q}_{11} & \tilde{Q}_{12} & \tilde{Q}_{16} \\
\tilde{Q}_{12} & \tilde{Q}_{22} & \tilde{Q}_{26} \\
\tilde{Q}_{16} & \tilde{Q}_{26} & \tilde{Q}_{66}
\end{bmatrix}_k
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\gamma_{xy}
\end{bmatrix}_k
\]  

(A4)

where

\[
\begin{align*}
\tilde{Q}_{11} &= Q_{11}C^4 + 2(Q_{12}+2Q_{66})C^2S^2 + Q_{22}S^4 \\
\tilde{Q}_{12} &= (Q_{11}+Q_{22}+4Q_{66})C^2S^2 + Q_{12}(C^4+S^4) \\
\tilde{Q}_{22} &= Q_{11}S^4 + 2(Q_{12}+2Q_{66})C^2S^2 + Q_{22}C^4 \\
\tilde{Q}_{66} &= (Q_{11}+Q_{22}+4Q_{66})C^2S^2 + Q_{66}(C^4+S^4) \\
\tilde{Q}_{16} &= (Q_{11}+Q_{12}+2Q_{66})C^3S + (Q_{12}+Q_{22}+2Q_{66})CS^3 \\
\tilde{Q}_{26} &= (Q_{11}+Q_{12}+2Q_{66})CS^3 + (Q_{12}+Q_{22}+2Q_{66})C^3S
\end{align*}

(A5)

and

$$C = \cos\theta_k, \quad S = \sin\theta_k$$

The $\tilde{Q}$ matrix is now fully populated and it appears that there are 6 elastic constants. However, $\tilde{Q}_{16}$ and $\tilde{Q}_{26}$ are merely linear combinations of the 4 basic elastic constants and are not independent\[32\].

The stress resultants (see Fig. 15) acting at the shell midsurface are given by
\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix}
= \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} \, dz
\]  \tag{A6}

where \( N \) is the number of laminae and the position of the \( k \)th lamina is defined by \( h_{k-1} < z < h_k \).

Recalling the Kirchhoff-Love hypothesis for a thin shell, the strain at any layer can be written in terms of the strain and curvature of the midsurface as

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
= \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} + z \begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]  \tag{A7}

Substituting these expressions into Eq. (A4) and introducing the resulting relations into Eq. (A6), followed by carrying out the indicated integrations yields the following constitutive equations between stress resultants and midsurface strains and curvatures

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} & A_{16} \\
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} + \begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{bmatrix} \begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]  \tag{A8}

where

\[
A_{ij} = \sum_{k=1}^{N} (\tilde{q}_{ij})_k (h_k - h_{k-1})
\]  \tag{A9}

for \( i,j = 1,2,6 \)

\[
B_{ij} = \frac{1}{2} \sum_{k=1}^{N} (\tilde{q}_{ij})_k (h_k^2 - h_{k-1}^2)
\]  \tag{A10}
Similarly the moment resultants (see Fig. 15) acting on the shell midsurface are defined as

\[
\begin{bmatrix}
    M_x \\
    M_y \\
    \frac{M_{xy} + M_{yx}}{2}
\end{bmatrix}
= \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \begin{bmatrix}
    \sigma_x \\
    \sigma_y \\
    \tau_{xy}
\end{bmatrix} zdz
\]  

(A11)

Substituting Eq. (A7) into Eq. (A4) and introducing the resulting expressions into Eq. (A11), and then carrying out the indicated integrations results in the following constitutive equations between moment resultants and midsurface strains and curvatures

\[
\begin{bmatrix}
    M_x \\
    M_y \\
    \frac{M_{xy} + M_{yx}}{2}
\end{bmatrix}
= \begin{bmatrix}
    B_{11} & B_{12} & B_{16} \\
    B_{12} & B_{22} & B_{26} \\
    B_{16} & B_{26} & B_{66}
\end{bmatrix}
\begin{bmatrix}
    \varepsilon_x \\
    \varepsilon_y \\
    \gamma_{xy}
\end{bmatrix}
+ \begin{bmatrix}
    D_{11} & D_{12} & D_{16} \\
    D_{12} & D_{22} & D_{26} \\
    D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix}
    \kappa_x \\
    \kappa_y \\
    \kappa_{xy}
\end{bmatrix}
\]  

(A12)

where

\[
D_{ij} = \frac{1}{3} \sum_{k=1}^{N} (\bar{Q}_{ij})_k (h^3_{k} - h^3_{k-1}) \quad \text{for } i, j = 1, 2, 6
\]  

(A13)

The constitutive equations (A8) and (A12) can be written in matrix form as

\[
[N] = [A] [\varepsilon] + [B] [\kappa]
\]  

(A14)

\[
[M] = [B] [\varepsilon] + [D] [\kappa]
\]  

(A15)

and after partial inversion as

\[
[\varepsilon] = [A^*] [N] + [B^*] [\kappa]
\]  

(A16)

\[
[M] = [C^*] [N] + [D^*] [\kappa]
\]  

(A17)
where

\[ [A^*] = [A]^{-1} \]  \hspace{1cm} (A18)

\[ [B^*] = - [A]^{-1}[B] \]  \hspace{1cm} (A19)

\[ [C^*] = - [B^*]^T \]  \hspace{1cm} (A20)

\[ [D^*] = [D] - [B] [A]^{-1}[B] \]  \hspace{1cm} (A21)

Notice that the 3x3 matrices A, B, D, A* and D* are symmetric. However, the 3x3 matrix B* is, in general, non-symmetric. Further, in the text, the following nondimensional stiffness parameters are employed

\[ \tilde{A}_{ij} = (1/Et)A_{ij} \quad ; \quad \tilde{B}_{ij} = (2c/Et^2)B_{ij} \quad ; \quad \tilde{D}_{ij} = (4c^2/Et^3)D_{ij} \]  \hspace{1cm} (A22)

and

\[ \tilde{A}_{ij}^* = Et \ A_{ij}^* \quad ; \quad \tilde{B}_{ij}^* = (2c/t)B_{ij}^* \quad ; \quad \tilde{D}_{ij}^* = (4c^2/Et^3)D_{ij}^* \]  \hspace{1cm} (A23)

where

\[ c^2 = 3(1-v^2) \]  \hspace{1cm} (A24)

and the quantities E and v are arbitrarily chosen reference values used for normalization purposes only.
APPENDIX B

PERIODICITY CONDITION

If the solution is to satisfy the periodicity requirement then, by definition,

\[
\int_0^{2\pi R} v_r y \, dy = 0 \quad (B1)
\]

must hold where

\[
v_r y = \varepsilon_y + (1/R)W - (1/2)W_y^2 \quad (B2)
\]

\[
\varepsilon_y = A_{12}^k N_x + A_{22}^k N_y + A_{26}^k N_{xy} + B_{21}^k \kappa_x + B_{22}^k \kappa_y + B_{26}^k \kappa_{xy} \quad (B3)
\]

further

\[
N_x = F_{,yy}, \quad N_y = F_{,xx}, \quad N_{xy} = -F_{,xy}
\]

and

\[
\kappa_x = -W_{,xx}, \quad \kappa_y = -W_{,yy}, \quad \kappa_{xy} = -2W_{,xy}
\]

Substituting for \( W \) and \( F \) the assumed perturbation expansion yields after regrouping and ordering by powers of \( \xi \)

\[
v_r y = (t/cR)\{(-A_{12}^\alpha+cw_{,\alpha})+(-p \bar{A}_{22}^\alpha+cw_t)+(\bar{A}_{26}^\alpha+cw_1^2+2w_{,2}^2)\}^{(t/2R)}B_{21}^\alpha w_{,\alpha} + \ldots
\]

\[
+ (t/cR)\xi[(\bar{A}_{22}^{\alpha-1}A_{12}^\alpha)^2 f_1 - \bar{A}_{26}^\alpha n_{f_1}^2]
\]

\[
- (t/2R)(\bar{B}_{21}^{\alpha-1}w_{,\alpha} + \bar{B}_{22}^{\alpha-1}w_{,\alpha} + 2\bar{B}_{26}^{\alpha-1}n_{w_{,\alpha}^2}) + \ldots
\]

\[
+ [\bar{A}_{22}^{\alpha-1}f_2 - \bar{A}_{22}^{\alpha-1}f_2]^{(t/2R)}B_{26}^\alpha w_{,\alpha} + \ldots
\]

\[
- (t/2R)(\bar{B}_{21}^{\alpha-1}w_{,\alpha} + \bar{B}_{22}^{\alpha-1}w_{,\alpha} + 2\bar{B}_{26}^{\alpha-1}n_{w_{,\alpha}^2}) + \ldots
\]

\[
+ (t/cR)\xi^2(\bar{A}_{22}^{\alpha-1}f_2 - (t/2R)\bar{B}_{21}^{\alpha-1}w_{,\alpha} + \ldots)
\]

\[
- (ct/4R)n^2(w_{,\alpha}^2 + w_{,\alpha}^2) + \ldots
\]
\[ \ldots + \left[ \tilde{A}^{*}_{22} \tilde{f}^{2} \tilde{A}^{*}_{12} \frac{4n^{2}f_{r}}{\beta_{26}} 2nf_{r}^{2} \right] \\
- \left( \frac{t}{2R(\tilde{B}^{*}_{21} \tilde{w}_{1}^{2} \tilde{B}^{*}_{21} \tilde{w}_{1}^{2} - 4\tilde{B}^{*}_{26} \tilde{w}_{1}^{2})} + c_{w} \right) + \left( \frac{ct}{4R} n^{2}(w_{1}^{2} - w_{2}^{2}) \cos 2n\theta \right) \\
+ \left[ \tilde{A}^{*}_{22} \tilde{f}^{2} \tilde{A}^{*}_{12} \frac{4n^{2}f_{r}}{\beta_{26}} 2nf_{r}^{2} \right] \\
- \left( \frac{t}{2R(\tilde{B}^{*}_{21} \tilde{w}_{1}^{2} \tilde{B}^{*}_{21} \tilde{w}_{1}^{2} - 4\tilde{B}^{*}_{26} \tilde{w}_{1}^{2})} + c_{w} \right) + \left( \frac{ct}{2R} n^{2}w_{1}^{2}w_{2} \right) \sin 2n\theta \right) \\
+ \left( \frac{t}{cR} \xi^{3} \right) \{- (ct/R)n^{2}\left[ (w_{1}w_{\beta} + w_{2}w_{r}) \cos 3n\theta - (w_{2}w_{\beta} - w_{1}w_{r}) \sin 3n\theta \right] \right) \\
+ \left( \frac{t}{cR} \xi^{4} \right) \{- (ct/R)n^{2}\left[ (w_{2}^{2} - w_{1}^{2}) \cos 4n\theta \right] \} \]

where \( \theta = y/R \). Substituting this expression into equation (B1) and carrying out the \( y \)-integration yields

\[
\{( - \lambda \tilde{A}^{*}_{12} + c_{w} ) + ( - \tilde{p} \tilde{A}^{*}_{12} + c_{w} ) + ( \tilde{r} \tilde{A}^{*}_{26} + c_{w} ) + \tilde{A}^{*}_{22} \tilde{f}^{2} \} \frac{(t/2R) \tilde{B}^{*}_{21} \tilde{w}_{1}^{2} + c_{w}}{2n^{2}w_{1}^{2}w_{2}} + c_{w} \right) \\
+ \xi^{2}\{( - \lambda \tilde{A}^{*}_{12} + c_{w} ) + ( - \tilde{p} \tilde{A}^{*}_{12} + c_{w} ) + ( \tilde{r} \tilde{A}^{*}_{26} + c_{w} ) \} \\
+ \xi^{4}\{- (ct/R)n^{2}(w_{1}^{2} + w_{2}^{2}) \} = 0
\]

Notice that the underlined terms vanish identically since they are equal to equations (15) and (34), respectively, with the constants \( \tilde{c}_{1} = \tilde{c}_{2} = 0 \) and \( \tilde{c}_{3} = \tilde{c}_{4} = 0 \). If one now lets

\[
W_{v} = \tilde{A}^{*}_{12} \lambda / c \\
W_{p} = \tilde{A}^{*}_{22} \tilde{p} / c \\
W_{t} = - \tilde{A}^{*}_{26} \tilde{t} / c
\]

then the periodicity condition (B1) is satisfied up to and including terms of the order \( \xi^{3} \).
APPENDIX C

SOLUTION OF THE AXISYMMETRIC PREBUCKLING STATE

The governing equation of the axisymmetric prebuckling state is given by Eq. (16) of the text

\[
(\bar{A}^{*} + \bar{B}^{*} + \bar{B}^{*} + \bar{B}^{*}) w^{iv} + (4cR/t)(\bar{A}^{*} \bar{B}^{*}) w^{ii} + (4c^2 R^2/t^2) w = 0
\]  

(C1)

This fourth order ordinary differential equation with constant coefficients always admits an exponential solution

\[
w_o = c_k e^{-\mu_k x}
\]  

(C2)

Substitution and regrouping yields the following characteristic equation

\[
\mu_k^4 + x_1 \mu_k^2 + x_2 = 0
\]  

(C3)

where

\[
x_1 = (4cR/t)(\bar{A}^{*} \bar{B}^{*}) / (\bar{A}^{*} \bar{B}^{*} + \bar{B}^{*} \bar{B}^{*})
\]  

(C4)

\[
x_2 = (4c^2 R^2/t^2) / (\bar{A}^{*} \bar{B}^{*} + \bar{B}^{*} \bar{B}^{*})
\]  

(C5)

Notice that from Eq. (C3)

\[
\mu_k^2 = (1/2)(-x_1 \pm \sqrt{x_1^2 - 4x_2})
\]  

(C6)

where the type of the roots depends on the sign of the discriminant

\[
x_1^2 - 4x_2 = (4cR/t)^2 [\bar{A}^{*} / (\bar{A}^{*} + \bar{B}^{*} \bar{B}^{*}) - (\bar{B}^{*} / \bar{A}^{*})] [\lambda^2 - 2\lambda (\bar{B}^{*} / \bar{A}^{*})]
\]  

(C7)

The critical value of the normalized axial load \( \lambda = \lambda^* \) can be obtained by setting the discriminant equal to zero. Then from

\[
\lambda^2 - 2\lambda (\bar{B}^{*} / \bar{A}^{*}) - (\bar{B}^{*} / \bar{A}^{*}) = 0
\]  

(C8)

one obtains
\[ \Lambda^* = \left( \tilde{B}_{21}^* / \tilde{A}_{22}^* \right) \pm \gamma \]  

(C9)

where

\[ \gamma^2 = \left( \tilde{A}_{22}^* \tilde{D}_{21}^* + \tilde{B}_{21}^* \tilde{B}_{21}^* \right) / \left( \tilde{A}_{22}^* \tilde{A}_{22}^* \right) \]  

(C10)

Notice that in Eq. (C9) only the positive sign is admissible since for compressive loading \( \Lambda^* > 0 \).

The general solution can be written as

\[ w_o = \sum_{k=4}^{4} C_k e^{\mu_k x} \]  

(C11)

where the \( 4 \) constants \( C_k \) are evaluated from the prescribed boundary conditions and where depending on the value of \( \Lambda \) the roots \( \mu_k \) have different forms.

Solution for \( \Lambda < \Lambda^* \)

In this case the discriminant \( \chi_1^2 - 4 \chi_2 < 0 \). Thus

\[ \sqrt{\chi_1^2 - 4 \chi_2} = i \sqrt{|\chi_1^2 - 4 \chi_2|} \]  

(C12)

Hence from Eqs. (C3) and (C6)

\[ \mu_k = \pm R(\alpha \pm i\beta) \quad (k = 1, 2, 3, 4) \]  

(C13)

where

\[ (\alpha R)^2 = (cR/t) \left[ \tilde{A}_{22}^* \gamma - (\tilde{A}_{22}^* \Lambda - \tilde{B}_{21}^*) \right] / \left( \tilde{A}_{22}^* \tilde{B}_{21}^* + \tilde{B}_{21}^* \tilde{B}_{21}^* \right) \]  

(C14)

\[ (\beta R)^2 = (cR/t) \left[ \tilde{A}_{22}^* \gamma + (\tilde{A}_{22}^* \Lambda - \tilde{B}_{21}^*) \right] / \left( \tilde{A}_{22}^* \tilde{B}_{21}^* + \tilde{B}_{21}^* \tilde{B}_{21}^* \right) \]  

(C15)

and the general solution (Eq. C11) can be written

\[ w_o = C_1 e^{aR \sin \beta R x} + C_2 e^{aR \cos \beta R x} + C_3 e^{-aR \sin \beta R x} + C_4 e^{-aR \sin \beta R x} \]  

(C16)
Solution for $\lambda > \lambda^*$

In this case the discriminant $x_1^2 - 4x_2 > 0$. Thus from Eq. (C6) both

\[ u_k^2 = (1/2)(-x_1 + \sqrt{x_1^2 - 4x_2}) = -(R_{p1}^*)^2 \]  \hspace{1cm} (C17)

and

\[ u_k^2 = (1/2)(-x_1 - \sqrt{x_1^2 - 4x_2}) = -(R_{p2}^*)^2 \]  \hspace{1cm} (C18)

are negative real numbers where

\[ (R_{p1}^*)^2 = (2cR/t)/(\tilde{A}_{22}^* \tilde{B}_{11}^* \tilde{B}_{21}^* \tilde{B}_{21}^*)[\tilde{A}_{22}^* \lambda - \tilde{B}_{21}^* \tilde{A}_{22}^* \sqrt{\lambda^2 - 2\lambda(\tilde{B}_{21}^*/\tilde{A}_{22}^*) - (\tilde{B}_{11}^*/\tilde{A}_{22}^*)}] \]  \hspace{1cm} (C19)

\[ (R_{p2}^*)^2 = (2cR/t)/(\tilde{A}_{22}^* \tilde{B}_{11}^* \tilde{B}_{21}^* \tilde{B}_{21}^*)[\tilde{A}_{22}^* \lambda - \tilde{B}_{21}^* \tilde{A}_{22}^* \sqrt{\lambda^2 - 2\lambda(\tilde{B}_{21}^*/\tilde{A}_{22}^*) - (\tilde{B}_{11}^*/\tilde{A}_{22}^*)}] \]  \hspace{1cm} (C20)

Finally, the general solution (Eq. C11) can now be written

\[ w_0 = C_1 \cos \rho_1 \bar{x} + C_2 \sin \rho_1 \bar{x} + C_3 \cos \rho_2 \bar{x} + C_4 \sin \rho_2 \bar{x} \]  \hspace{1cm} (C21)

The constants $C_1 + C_4$ are evaluated from the given boundary conditions whereby, depending on whether $\lambda < \lambda^*$ or $\lambda > \lambda^*$, respectively the solution given by Eq. (C16) or by Eq. (C21) is used.

Simply Supported Boundary Condition: $W = 0, M = -q N_x$ 

From Appendix D the corresponding reduced boundary conditions valid for the prebuckling solution at $\bar{x} = 0, L/R$ are

\[ w_0 = -(W_v + W_p + W_t) \]  \hspace{1cm} (C22)

\[ w_0'' = K_M \]  \hspace{1cm} (C23)

where

\[ K_M = -(2R/t)[(\tilde{A}_{12}^* \tilde{B}_{21}^* - \tilde{A}_{21}^* \tilde{B}_{21}^*) - q\tilde{A}_* - (\tilde{A}_{20}^* \tilde{B}_{21}^* - \tilde{A}_{21}^* \tilde{B}_{21}^*)]/(\tilde{A}_{22}^* \tilde{B}_{21}^* + \tilde{B}_{21}^* \tilde{B}_{21}^*) \]  \hspace{1cm} (C24)
and \( W_v, W_p \) and \( W_t \) are derived in Appendix B. Evaluating the constants yields for the case that \( \lambda < \lambda^* \)

\[
C_1 = \left( e^{-\alpha L/2\alpha R^2} \right) \left\{ K_M \cos \beta L + (W_v + W_p + W_t) \left[ (\alpha^2 - \beta^2)R^2 \cos \beta L - 2\alpha R^2 \sin \beta L \right] \right\} \tag{C25}
\]

\[
C_2 = \left( e^{-\alpha L/2\alpha R^2} \right) \left\{ K_M \sin \beta L + (W_v + W_p + W_t) \left[ (\alpha^2 - \beta^2)R^2 \sin \beta L + 2\alpha R^2 \cos \beta L \right] \right\} \tag{C26}
\]

\[
C_3 = \left( -1/2\alpha R^2 \right) \left\{ K_M + (W_v + W_p + W_t) (\alpha^2 - \beta^2)R^2 \right\} \tag{C27}
\]

\[
C_4 = -(W_v + W_p + W_t) \tag{C28}
\]

and for the case that \( \lambda > \lambda^* \)

\[
C_1 = \frac{\left[ (W_v + W_p + W_t) (\hat{\rho}_2 R)^2 - K_M \right]}{\left[ (\hat{\rho}_1 R)^2 - (\hat{\rho}_2 R)^2 \right]} \tag{C29}
\]

\[
C_2 = \frac{\left[ (W_v + W_p + W_t) (\hat{\rho}_2 R)^2 - K_M (1 - \cos \hat{\rho}_1 L) \right]}{\left[ (\hat{\rho}_1 R)^2 - (\hat{\rho}_2 R)^2 \right] \sin \hat{\rho}_1 L} \tag{C30}
\]

\[
C_3 = \frac{\left[ K_M - (W_v + W_p + W_t) (\hat{\rho}_1 R)^2 \right]}{\left[ (\hat{\rho}_1 R)^2 - (\hat{\rho}_2 R)^2 \right]} \tag{C31}
\]

\[
C_4 = \frac{\left[ K_M - (W_v + W_p + W_t) (\hat{\rho}_1 R)^2 \right]}{\left[ (\hat{\rho}_1 R)^2 - (\hat{\rho}_2 R)^2 \right] \sin \hat{\rho}_2 L} \tag{C32}
\]

It should be mentioned here that in the first case (when \( \lambda < \lambda^* \)) in order to arrive at simpler expressions use is made of the fact that for \( \alpha L \) large \( e^{-\alpha L} \) becomes vanishingly small as compared to one. Thus the solution is valid only for \( 0 < \bar{x} < (1/2)(L/R) \). However, with identical boundary conditions at the 2 shell edges the prebuckling solution is symmetrical with respect to \( \bar{x} = (1/2)(L/R) \).

**Clamped Boundary Condition: \( W = W_p = 0 \)**

From Appendix D the corresponding reduced boundary conditions valid for the prebuckling solution at \( \bar{x} = 0, L/R \) are

\[
w_o = -(W_v + W_p + W_t) \tag{C33}
\]

\[
w_o' = 0 \tag{C34}
\]
where $W_v$, $W_p$ and $W_t$ are derived in Appendix B. Evaluating the constants yields for the case that $\lambda < \lambda^*$

$$C_1 = e^{-\alpha L}(W_v + W_p + W_t) \left[ (\alpha/\beta) \cos \beta L - \sin \beta L \right]$$  \hspace{1cm} (C35)

$$C_2 = -e^{-\alpha L}(W_v + W_p + W_t) \left[ (\alpha/\beta) \sin \beta L + \cos \beta L \right]$$  \hspace{1cm} (C36)

$$C_3 = -(\alpha/\beta)(W_v + W_p + W_t)$$  \hspace{1cm} (C37)

$$C_4 = -(W_v + W_p + W_t)$$  \hspace{1cm} (C38)

and for the case that $\lambda > \lambda^*$

$$C_1 = -W_v \left[ (\hat{\rho}_1 R)(\hat{\rho}_2 R)(1 + \cos \hat{\rho}_1 L)(1 - \cos \hat{\rho}_2 L) - (\hat{\rho}_2 L)^2 \sin \hat{\rho}_1 L \sin \hat{\rho}_2 L \right] / \Lambda_c$$  \hspace{1cm} (C39)

$$C_2 = W_v \left[ (\hat{\rho}_2 L)^2 \sin \hat{\rho}_2 L (1 - \cos \hat{\rho}_1 L) - (\hat{\rho}_1 L)(\hat{\rho}_2 L) \sin \hat{\rho}_1 L (1 - \cos \hat{\rho}_2 L) \right] / \Lambda_c$$  \hspace{1cm} (C40)

$$C_3 = -W_v \left[ (\hat{\rho}_1 R)(\hat{\rho}_2 R)(1 - \cos \hat{\rho}_1 L)(1 + \cos \hat{\rho}_2 L) - (\hat{\rho}_1 L)^2 \sin \hat{\rho}_1 L \sin \hat{\rho}_2 L \right] / \Lambda_c$$  \hspace{1cm} (C41)

$$C_4 = W_v \left[ (\hat{\rho}_1 L)^2 \sin \hat{\rho}_1 L (1 - \cos \hat{\rho}_2 L) - (\hat{\rho}_1 L)(\hat{\rho}_2 L) \sin \hat{\rho}_2 L (1 - \cos \hat{\rho}_1 L) \right] / \Lambda_c$$  \hspace{1cm} (C42)

where

$$\Lambda_c = 2(\hat{\rho}_1 L)(\hat{\rho}_2 L)(1 - \cos \hat{\rho}_1 L \cos \hat{\rho}_2 L) - (\hat{\rho}_1^2 + \hat{\rho}_2^2)R^2 \sin \hat{\rho}_1 L \sin \hat{\rho}_2 L$$  \hspace{1cm} (C43)

Once again in the first case (when $\lambda < \lambda^*$) in order to arrive at simpler expressions the fact is used that for $\alpha L$ large $e^{-\alpha L}$ becomes vanishingly small as compared to one.
APPENDIX D

DERIVATION OF THE REDUCED BOUNDARY CONDITIONS

It is necessary to express the different boundary conditions in terms of the functions assumed for \( W(x,y) \) and \( F(x,y) \). Recalling that

\[
W(x,y) = t(W_v + W_p + W_t) + tw_o(x) + \xi t(w_1(x)\cos n\theta + w_2(x)\sin n\theta)
\]

\[
+ t_s^2\{w_\alpha(x) + w_\beta(x)\cos 2n\theta + w_\gamma(x)\sin 2n\theta\}
\]  

\[
F(x,y) = (Et^2/cR)[-(1/2)\lambda y^2-(1/2)\tau x^2-\tau xy + f_o(x)]
\]

\[
+ \xi (ERt^2/c)\{f_1(x)\cos n\theta + f_2(x)\sin n\theta\}
\]

\[
+ \xi (ERt^2/c)\{f_\alpha(x) + f_\beta(x)\cos 2n\theta + f_\gamma(x)\sin 2n\theta\}
\]

where \( \Theta = y/R \), the individual boundary conditions can be expressed as follows.

1. Boundary condition: \( u = 0 \)

Here one must express the condition \( u = 0 \) in terms of the variables \( W \) and \( F \). Eliminating \( v \) from the strain-displacement relations

\[
\varepsilon_y = v_{,y} - (1/R)W + (1/2)W_{,y}^2
\]

\[
\gamma_{xy} = u_{,x} + v_{,y} + W_{,x}W_{,y}
\]

results in the following equation

\[
u_{,yy} = \gamma_{xy,y} - \varepsilon_{y,x} - (1/R)W_{,x} - W_{,x}W_{,yy}
\]

Recalling the fact that if a function \( \phi(x,y) \) in an orthogonal reference frame \( x,y \) satisfies the condition

\[
\phi(x,y) = C \quad \text{at } x = x_o
\]

where both \( C \) and \( x_o \) are constants then
\[ \frac{\partial^2}{\partial r^2} \phi(x, y) = 0 \quad \text{at } x = x_0 \quad \text{(D6)} \]

for \( r = 1, 2, 3, \ldots \), one can simplify Eq. (D4) since \( u = 0 \) at \( x = 0, L \) implies that also \( u_{yy} = 0 \) at the shell edges. Thus Eq. (D3) becomes

\[ \gamma_{xy,y} - \epsilon_{y,x} - \frac{1}{(1/R)} W_{,x} - W_{,x} W_{,yy} = 0 \quad \text{(D7)} \]

Substituting for \( \epsilon_{y} \) and \( \gamma_{xy} \) from the semi-inverted form of the constitutive equation (A16) one gets

\[ \begin{align*}
& A_{16}^F \gamma_{yy} + A_{16}^F \gamma_{xy} - A_{26}^F \gamma_{xy} - B_{0y} W_{,x} - B_{0x} W_{,yy} - 2B_{26}^F W_{,xy} \\
& \quad - (A_{12}^F \gamma_{yy} + A_{22}^F \gamma_{xx} - A_{26}^F \gamma_{xy} - B_{12}^F \gamma_{xx} - B_{22}^F \gamma_{xy} - 2B_{26}^F \gamma_{xx}) \\
& \quad - (1/R) W_{,x} - W_{,x} W_{,yy} = 0
\end{align*} \quad \text{(D8)} \]

Introducing now for \( W \) and \( F \) the assumed perturbation expansions (Eqs. D1 and D2) and ordering by powers of \( \xi \) yields

\[ \begin{align*}
& - \tilde{A}_{22}^F \phi''_{0} + (t/2R) \tilde{B}_{21}^F \phi''_{0} - c w_{0} \\
& + \xi \left[ - \tilde{A}_{16}^F f^2_{-} + \tilde{A}_{16}^F n f_{-} + \tilde{A}_{16}^F n^2 f_{-} - (t/2R) \left( \tilde{B}_{0y} n w_{-} - \tilde{B}_{0x} n^2 w_{-} - 2\tilde{B}_{26}^F n^2 w_{-} \right) \\
& + \tilde{A}_{12}^F n^2 f_{-} - \tilde{A}_{22}^F n f_{-} + \tilde{A}_{26}^F n^2 f_{-} + (t/2R) \left( \tilde{B}_{12}^F n^2 w_{-} + 2\tilde{B}_{26}^F n^2 w_{-} \right) \\
& - c w_{1} + (ct/R) n^2 w_{1} \cos n \theta \right] \\
& + \xi^2 \left[ - \tilde{A}_{16}^F f^2_{-} + (t/2R) \tilde{B}_{21}^F \phi''_{0} - c w''_{0} \right] (ct/R) n^2 (w_{1} w_{-} + w_{2} w_{-}) \quad \text{(D9)}
\end{align*} \]
\[ \ldots - c w'_{\beta} + (ct/2R)n^2(8 w_{\alpha} w_0 + w_1 w_2 + w_{\gamma} w_{\gamma}) \cos 2\theta \]
\[ + [\bar{A}^*_{16} - 8 n^2 f_{12} + 4 n^2 f_{12} + (t/2R)(\bar{B}^*_{12} - 8 n^2 f_{12} + 8 n^2 f_{12})]
\[ + 4 n^2 f_{12} - 8 n^2 f_{12} + (t/2R)(\bar{B}^*_{12} - 8 n^2 f_{12} + 8 n^2 f_{12})]
\[ - c w'_{\gamma} + (ct/2R)n^2(8 w_{\alpha} w_0 + w_1 w_2 + w_{\gamma} w_{\gamma}) \sin 2\theta \}
\[ + O(\xi^3) \ldots = 0 \]

This expression must hold for all values of \( \theta \). The coefficients of the different powers of \( \xi \) yield the reduced boundary conditions for the Prebuckling problem:

\[ \bar{A}^*_{22} f''_{12} - (t/2R)\bar{B}^*_{21} w''_{12} + c w'_{\alpha} = 0 \]  
(D10)

Buckling problem:

\[ \bar{A}^*_{22} f''_{12} - (\bar{A}^*_{12} + \bar{A}^*_{16}) n^2 f_{12} - 8 n^2 f_{12} + \bar{A}^*_{16} n^2 f_{12} \]
(D11)
\[ - (t/2R)[\bar{B}^*_{21} w''_{12} + (2 \bar{B}^*_{12} - \bar{B}^*_{12}) n^2 w''_1 + (2 \bar{B}^*_{26} - \bar{B}^*_{26}) n^2 w''_2] \]
\[ + c w''_{12} - (ct/2R)n^2 w''_{12} = 0 \]

Postbuckling problem:

\[ \bar{A}^*_{22} f''_{12} - (t/2R)\bar{B}^*_{21} w''_{12} + c w'_{\alpha} - (ct/2R)n^2(w_1 w_1 + w_2 w_2) = 0 \]  
(D13)

\[ \bar{A}^*_{22} f''_{12} - (\bar{A}^*_{12} + \bar{A}^*_{16}) n^2 f_{12} - 8 n^2 f_{12} + \bar{A}^*_{16} n^2 f_{12} \]
(D14)
\[ - (t/2R)[\bar{B}^*_{21} w''_{12} + (2 \bar{B}^*_{12} - \bar{B}^*_{12}) n^2 w''_1 + (2 \bar{B}^*_{26} - \bar{B}^*_{26}) n^2 w''_2 + \bar{B}^{*\gamma} n^2 w''_2] + \ldots \]
\[ \ldots + c n_p \frac{(ct/2R)}{(ct/2R)}n^2 (8w_0 w_p + w_1 w_2 - w_1 w_2) = 0 \]

\[ \begin{aligned}
&= \bar{\bar{A}}^{*} f''_\gamma - \left( \bar{\bar{A}}^{*}_{12} + \bar{\bar{A}}^{*}_{12} \right) 4n^2 f'_\gamma + 4\bar{\bar{A}}^{*}_{16} F_p \\
&\quad - (t/2R) \left[ \bar{\bar{B}}^{*}_{21} w'' + (2\bar{\bar{B}}^{*}_{21} - 2\bar{\bar{B}}^{*}_{21}) 4n^2 w'_\gamma - (2\bar{\bar{B}}^{*}_{21} - 2\bar{\bar{B}}^{*}_{21}) 2nw'' + \bar{\bar{B}}^{*}_{21} 8n^3 w_p \right] \\
&\quad + c w'_\gamma - \left( \frac{(ct/2R)}{(ct/2R)} \right) n^2 (8w_0 w_p + w_1 w_2 + w_2 w_1) = 0
\end{aligned} \] (D15)

Notice that the equations (D10) and (D13) represent no new boundary conditions, since they reduce identically to zero if one differentiates equations (15) and (39) once and substitutes the resulting expressions for \( f''_0 \) and \( f''_\alpha \), respectively.

2. Boundary condition: \( v = 0 \)
To express the condition \( v = 0 \) in terms of the variables \( W \) and \( F \) one solves first for \( v_{,y} \) from the strain-displacement relation

\[ \varepsilon_y = v_{,y} - (1/R)W + (1/2)W_{,yy} \] (D3)

and uses the semi-inverted form of the constitutive equations (A16) to substitute for \( \varepsilon_y \) yielding the following equation

\[ \begin{aligned}
v_{,y} &= A^{*}_{12} F_{,yy} + A^{*}_{12} F_{,xx} - \bar{\bar{A}}^{*}_{21} W_{,xx} - \bar{\bar{B}}^{*}_{21} W_{,yy} - 2B^{*} W_{,xy} \\
&\quad + (1/R)W - (1/2)W_{,yy}^2
\end{aligned} \] (D16)

Next recalling the behaviour of a function \( \Phi(x,y) = C \) in an orthogonal reference frame (see Eqs. D5 and D6) one concludes that \( v = 0 \) at \( x = 0,L \) implies that also \( v_{,y} = 0 \) at the shell edges. Introducing further for \( W \) and \( F \) the assumed perturbation expansions (Eqs. D1 and D2), ordering by powers of \( \xi \) and then proceeding as in the previous case yields the following reduced boundary conditions for the

Prebuckling problem:

\[ \begin{aligned}
\bar{\bar{A}}^{*}_{22} f''_0 - (t/2R) \bar{\bar{B}}^{*}_{21} w''_0 + c W_0 + c(W_0 + W + W_0) - \bar{\bar{A}}^{*}_{12} \bar{\bar{A}}^{*}_{22} + \bar{\bar{A}}^{*}_{26} + \bar{\bar{A}}^{*}_{26} \tau = 0
\end{aligned} \] (D17)
Buckling problem:

\[ A^{*} f'' - A^{*} n f_{1} - A^{*} n f_{2} -(t/2R)(B^{*} w'' - B^{*} n w_{1} + 2B^{*} n w_{2}) + c w_{1} = 0 \]  (D18)

\[ A^{*} f'' - A^{*} n f_{2} + A^{*} n f_{1} -(t/2R)(B^{*} w'' - B^{*} n w_{2} - 2B^{*} n w_{1}) + c w_{2} = 0 \]  (D19)

Postbuckling problem:

\[ A^{*} f'' - (t/2R)B^{*} w'' + c w_{a} - (ct/4R)n^2(w_{1}^2 + w_{2}^2) = 0 \]  (D20)

\[ A^{*} f'' - A^{*} n^2 f - A^{*} 2n f_{1} - (t/2R)(B^{*} w'' - B^{*} 4n^2 w_{1} + B^{*} 4n w_{1} + B^{*} 4n w_{2}) + c w_{a} + (ct/4R)n^2(w_{1}^2 - w_{2}^2) = 0 \]  (D21)

\[ A^{*} f'' - A^{*} n^2 f + A^{*} 2n f_{1} - (t/2R)(B^{*} w'' - B^{*} 4n^2 w_{2} - B^{*} 4n w_{1} + B^{*} 4n w_{2}) + c w_{a} + (ct/2R)n^2 w_{1} w_{2} = 0 \]  (D22)

Notice that in this case equations (D17) and (D20) represent no new boundary conditions, since they reduce identically to zero if one substitutes for \( f'' \) and for \( W_{V}, W_{P} \) and \( W_{T} \) from Eqs. (15) and (B6) and for \( f'' \) from Eq. (39), respectively.

3. Boundary condition: \( W = 0 \)

By considering the assumed perturbation expansion (D1) for \( W \) then it follows by inspection that the reduced boundary conditions are for the

Prebuckling problem:

\[ w_{o} = - (W_{v} + W_{P} + W_{T}) \]  (D23)

Buckling problem:

\[ w_{1} = w_{2} = 0 \]  (D24)

Postbuckling problem

\[ w_{a} = w_{b} = w_{y} = 0 \]  (D25)

4. Boundary condition: \( W_{x} = 0 \)

Also in this case the reduced boundary conditions are obtained by inspection from the assumed perturbation expansion (D1) for \( W \) yielding for the
Prebuckling problem:
\[ w'_o = 0 \]  
(D26)

Buckling problem:
\[ w'_1 = w'_2 = 0 \]  
(D27)

Postbuckling problem:
\[ w'_\alpha = w'_\beta = w'_\gamma = 0 \]  
(D28)

5. Boundary condition: \( N_x = -N_o \)
In this case it follows by inspection from the assumed perturbation expansion (D2) for \( F \) that the reduced boundary conditions are for the
Prebuckling problem:
\[ \lambda = -N_o/(Et^2/cR) \]  
(D29)

Buckling problem:
\[ f_1 = f_2 = 0 \]  
(D30)

Postbuckling problem:
\[ f_\beta = f_\gamma = 0 \]  
(D31)

Notice that in this case equation (D29) is just the definition of the normalized axial load parameter. Thus it represents no new boundary condition.

6. Boundary condition: \( N_{xy} = T_o \)
Once again the reduced boundary conditions are obtained by inspection from the assumed perturbation expansion (D2) for \( F \) yielding for the
Prebuckling problem:
\[ \bar{\tau} = T_o/(Et^2/cR) \]  
(D32)

Buckling problem:
\[ f'_1 = f'_2 = 0 \]  
(D33)

Postbuckling problem:
\[ f'_\beta = f'_\gamma = 0 \]  
(D34)

Notice that in this case (D32) represents no new boundary condition since it is just the definition of the normalized torque \( \bar{\tau} \).
7. Boundary condition: \( H = 0 \)

By definition \([33]\), for a perfect shell

\[
H = M_{x,x} + (M_{xy} + M_{y,x})_{,y} + W_{,x}N_{x} + W_{,x}N_{xy} \tag{D35}
\]

Thus using the semi-inverted constitutive equations \((A17)\) this boundary condition can be written as

\[
B_{11}^{*}F_{,xy} + \left( B_{12}^{*} + B_{61}^{*} \right) F_{,xxx} - B_{21}^{*} F_{,xx} - B_{62}^{*} + D_{11}^{*} W_{,xxx} + D_{12}^{*} W_{,xyy} + 2D_{16}^{*} W_{,xyy} + 2D_{16}^{*} W_{,xyy} + 2D_{16}^{*} W_{,xyy} - W_{,x} F_{,yy} + W_{,y} F_{,xy} = 0
\tag{D36}
\]

Introducing now for \( W \) and \( F \) the assumed perturbation expansions \((D1)\) and \((D2)\), ordering by powers of \( \xi \) and then proceeding as in the first two cases yields the following reduced boundary conditions for the

Prebuckling problem:

\[
\tilde{B}_{11}^{*} W_{1}'' + (2R/t)\tilde{B}_{21}^{*} F_{1}'' + (4cR/t)\xi W_{0}' = 0 \tag{D37}
\]

Buckling problem:

\[
\tilde{B}_{11}^{*} W_{1}'' - (\tilde{B}_{12}^{*} - 4\tilde{B}_{66}^{*}) n_{1}^{2} w_{1}' + 4\tilde{B}_{16}^{*} n_{w}'' - 2\tilde{B}_{26}^{*} n_{w} w_{2}' - 2\tilde{B}_{36}^{*} n_{w} w_{2}' + (2R/t) \left[ (\tilde{B}_{21}^{*} - (\tilde{B}_{11}^{*} - 2\tilde{B}_{66}^{*}) n_{1} f_{1}'' + (2\tilde{B}_{26}^{*} - \tilde{B}_{61}^{*} n_{w} f_{1} - 2\tilde{B}_{16}^{*} n_{1} f_{2} \right] + (4cR/t) (\xi W_{1}'' + n_{w} f_{1}'' + n_{w} w_{2}') = 0 \tag{D38}
\]

\[
\tilde{B}_{11}^{*} W_{2}'' - (\tilde{B}_{12}^{*} + 4\tilde{B}_{66}^{*}) n_{1}^{2} w_{2}' - 4\tilde{B}_{16}^{*} n_{w}'' + 2\tilde{B}_{26}^{*} n_{w} w_{1}' + (2R/t) \left[ (\tilde{B}_{21}^{*} - (\tilde{B}_{11}^{*} - 2\tilde{B}_{66}^{*}) n_{1} f_{1}'' - (2\tilde{B}_{26}^{*} - \tilde{B}_{61}^{*} n_{w} f_{1} + 2\tilde{B}_{16}^{*} n_{1} f_{1} \right] + (4cR/t) (\xi W_{2}'' + n_{w} f_{2}'' + n_{w} w_{1}') = 0 \tag{D39}
\]

Postbuckling problem:

\[
\tilde{B}_{11}^{*} W_{1}'' + (2R/t)\tilde{B}_{21}^{*} F_{1}'' + (4cR/t)\xi W_{0}' + \ldots \tag{D40}
\]
\[ \ldots + (2cR/t) n^2(w'_1 f'_1 + w'_2 f'_2 + w'_3 f'_3) = 0 \]

\[ \bar{B}_{11} w''_\beta - (\bar{B}_{21} - 2\bar{B}_{66}) 4n^2 w'_\beta + 8\bar{B}_{16} n w'_\gamma - 16\bar{B}_{26} n^3 w'_\gamma \]  

\[ + (2R/t) [\bar{B}_{21} f''_\beta - (\bar{B}_{11} - 2\bar{B}_{66}) 4n^2 f''_\beta + (2\bar{B}_{26} - \bar{B}_{61}) 2n f''_\gamma - 16\bar{B}_{16} n^3 f''_\gamma] \]

\[ + (4cR/t)(\lambda w'_0 + 4n^2 w'_0 f'_0 - 2n w'_\beta) + (2cR/t) n^2 (w'_1 f'_1 - w'_1 f'_2 + w'_2 f'_2) = 0 \]

\[ \bar{B}_{11} w''_\gamma - (\bar{B}_{12} + 4\bar{B}_{66}) 4n^2 w'_\gamma - 8\bar{B}_{16} n w'_\beta + 16\bar{B}_{26} n^3 w'_\beta \]  

\[ + (2R/t) [\bar{B}_{21} f''_\gamma - (\bar{B}_{11} - 2\bar{B}_{66}) 4n^2 f''_\gamma - (2\bar{B}_{26} - \bar{B}_{61}) 2n f''_\beta + 16\bar{B}_{16} n^3 f''_\beta] \]

\[ + (4cR/t)(\lambda w'_1 + 4n^2 w'_1 f'_1 + 2n w'_\beta) + (2cR/t) n^2 (w'_1 f'_1 - w'_1 f'_2 + w'_2 f'_1 - w'_2 f'_2) = 0 \]

Notice that Eq. (D37) can be simplified further if one differentiates equation (15) and substitutes the resulting expression for \( f''_0 \). This yields upon some regrouping

\[ (\bar{A}_{22} \bar{B}_{11} + \bar{B}_{21} \bar{B}_{66}) w''_0 + (4cR/t)[\bar{A}_{22} \lambda - (1/2) \bar{B}_{21}] w'_0 = 0 \]  

(D43)

8. Boundary condition: \( M_x = -N_0 q \)

Here one must first express \( M_x \) in terms of the variables \( W \) and \( F \). Using the corresponding semi-inverted constitutive equation (A17) one gets upon substitution

\[ B_{11} F_{yy} + B_{21} F_{xx} - B_{26} F_{xy} + D_{11} W_{xx} + D_{12} W_{yy} + 2D_{16} W_{xy} - N_0 q = 0 \]  

(D44)

Introducing now for \( W \) and \( F \) the assumed perturbation expansions (D1) and (D2), ordering by powers of \( \xi \) and then proceeding as in the first two cases yields the following reduced boundary conditions for the

Pribuckling problem:

\[ \bar{B}_{11} w''_0 - [(2R/t)\bar{B}_{11} + q] \lambda + (2R/t)(\bar{B}_{21} f''_0 - \bar{B}_{21} f - \bar{B}_{61} n f'_0) = 0 \]  

(D45)

Buckling problem:

\[ \bar{B}_{12} w''_1 - \bar{B}_{12} n^2 w'_1 + 2\bar{B}_{66} n w'_2 + (2R/t)(\bar{B}_{21} f''_1 - \bar{B}_{21} n^2 f'_1 - \bar{B}_{61} n f'_2) = 0 \]  

(D46)
\[ \bar{D}_{11}^* w^{**} - \bar{D}_{12}^* n^2 w_2 - 2\bar{D}_{16}^* n w_1 + (2R/t)(\bar{B}_{21}^* f''_{11} - \bar{B}_{21}^* f''_{12} - \bar{B}_{61}^* f''_{11}) = 0 \quad (D47) \]

Postbuckling problem:

\[ \bar{D}_{11}^* w'' - \bar{D}_{12}^* 4n^2 w_\beta + 4\bar{D}_{16}^* n w'_\gamma + (2R/t)(\bar{B}_{21}^* f''_{11} - \bar{B}_{21}^* f''_{12} - \bar{B}_{61}^* 2f''_{11}) = 0 \quad (D48) \]

\[ \bar{D}_{11}^* w'' - \bar{D}_{12}^* 4n^2 w_\gamma - 4\bar{D}_{16}^* n w'_\beta + (2R/t)(\bar{B}_{21}^* f''_{11} - \bar{B}_{21}^* f''_{12} - \bar{B}_{61}^* 2f''_{11}) = 0 \quad (D49) \]

By substituting for \( f'' \) from Eq. (15) one can simplify equation (D45) yielding

\[ (\bar{A}_{22}^* \bar{B}_{11}^* + \bar{B}_{22}^* \bar{B}_{22}^* ) w'' - (2cR/t)\bar{B}_{21}^* w_0 \]

\[-(2R/t)\bar{A}_{22}^* (\bar{B}_{11}^* \lambda + \bar{B}_{21}^* \bar{B}_{21}^* - \bar{B}_{61}^* \bar{B}_{61}^* ) - \bar{q} \bar{A}_{22}^* \lambda = 0 \quad (D50) \]

where the nondimensional load eccentricity \( \bar{q} = (4cR/t^2)q \).
APPENDIX E

DERIVATION OF THE POSTBUCKLING COEFFICIENTS AND OF THE IMPERFECTION FORM FACTORS

Perfect Structure

The theory of initial postbuckling behaviour as developed by Koiter[4] is essentially a perturbation technique which relies on the principle of stationary potential energy. The alternate approach, proposed by Budiansky and Hutchinson[5] to arrive at equivalent results, writes the field equations directly in variational form with the aid of the principle of virtual work. Thus the equilibrium equations of the nonlinear Donnell type shell theory follow from the variational statement

\[ \int \int \left\{ N_x \delta \epsilon_x + N_y \delta \epsilon_y + N_{xy} \delta \gamma_{xy} + M_x \delta \kappa_x + M_y \delta \kappa_y + \frac{M_{xy} + M_{yx}}{2} \delta \kappa_{xy} \right\} dx dy \]

\[ \text{E1} \]

\[ \int N_x \bigg|_{x=L} dx \bigg|_{x=0} = 0 \]

where \( N_x \bigg|_{x=L} = - N_0 \) is the applied axial compression, \( p \) is the applied external pressure, \( N_{xy} \bigg|_{x=L} = T_0 \) is the applied counter-clockwise torque and \( q \) is the load eccentricity (positive inward) of the applied axial compression.

For \( W \) positive inward one has the following nonlinear strain-displacement relations

\[ \epsilon_x = u_x + \frac{1}{2} \omega_x^2 \]

\[ \kappa_x = - \omega_{xx} \]

\[ \epsilon_y = v_y - \frac{1}{2} R + \frac{1}{2} \omega_y^2 \]

\[ \kappa_y = - \omega_{yy} \]

\[ \gamma_{xy} = u_y + v_x + \omega_x \omega_y \]

\[ \kappa_{xy} = - 2 \omega_{xy} \]

(E2)

and for anisotropic shells the following constitutive equations
\[ [N] = [A][\varepsilon] + [B][\kappa] \]  
\[ [M] = [B][\varepsilon] + [D][\kappa] \]

(E3a) 
(E3b) 

Notice that the stiffness coefficients involved are defined in Appendix A.

For the sake of simplicity in the following the highly abbreviated notation introduced first by Budiansky\(^{34}\) and later adopted by many other authors\(^{19,35}\) will be used. Let \( u, \varepsilon, q \) and \( \sigma \) denote the generalized displacement, strain, load and stress variables. Then the nonlinear strain displacement relations Eq. (E2) and the constitutive equations (E3) can be written as

\[ \varepsilon = L_1(u) + (1/2)L_2(u) \]  
\[ \sigma = H(\varepsilon) \]

(E4) 
(E5) 

where \( L_1 \) and \( H \) are linear functionals and \( L_2 \) is a quadratic functional. The variational statement of equilibrium Eq. (E1) becomes

\[ \sigma \cdot \delta \varepsilon - q \cdot \delta u = 0 \]  

(E6) 

Notice that here \( \sigma \cdot \delta \varepsilon \) and \( q \cdot \delta u \) denote, respectively, the internal virtual work of the stress \( \sigma \) through the strain variation \( \delta \varepsilon \), and the external virtual work of the load \( q \) through the displacement variation \( \delta u \), both integrated over the entire shell middle surface. Further \( \delta \varepsilon \) is the first order strain variation produced by \( \delta u \). If one defines a bilinear functional \( L_{11} \) such that

\[ L_2(u+v) = L_2(u) + 2L_{11}(u,v) + L_2(v) \]

(E7) 

then it follows from (E4) that

\[ \delta \varepsilon = L_1(\delta u) + L_{11}(u,\delta u) \]

(E8) 

Next it is assumed that the variables \((u, \varepsilon, \sigma)\) of the postbuckling equilibrium state are expanded in the following perturbation series about the prebuckling
equilibrium state \((u_o, \varepsilon_o, \sigma_o)\) at the same value of the variable load parameter \(\lambda\)

\[ u = u_o + \xi u_1 + \xi^2 u_2 + \xi^3 u_3 + \ldots \]  \hspace{1cm} (E9a)

\[ \varepsilon = \varepsilon_o + \xi \varepsilon_1 + \xi^2 \varepsilon_2 + \xi^3 \varepsilon_3 + \ldots \]  \hspace{1cm} (E9b)

\[ \sigma = \sigma_o + \xi \sigma_1 + \xi^2 \sigma_2 + \xi^3 \sigma_3 + \ldots \]  \hspace{1cm} (E9c)

Notice that if one assumes that \(w = \xi^2 u_2 + \xi^3 u_3 + \ldots\) then the scalar parameter \(\xi\) can be defined as the 'measure of the participation' of \(u_1\) in the displacement which occurred after buckling, if one requires that \(u_1\) be orthogonal to \(w\) in some sense. Specifically it has been shown by Fitch\(^{[35]}\) that if \(Q_{11}\) is a bilinear operator satisfying the condition \(Q_{11}(u_1, u_1) \neq 0\), then the orthogonality condition

\[ Q_{11}(u_1, w) = 0 \]  \hspace{1cm} (E10)

implies

\[ \xi = Q_{11}(u-u_o, u_1)/Q_{11}(u_1, u_1) \]  \hspace{1cm} (E11)

Further, the variables \((u_o, \varepsilon_o, \sigma_o)\) and \((u, \varepsilon, \sigma)\) are, in general, nonlinear functions of \(\lambda = \lambda(\xi)\), whereas the expansion functions \((u_k, \varepsilon_k, \sigma_k)\) where \(k = 1, 2, \ldots\) are independent of \(\lambda\) and \(\xi\). The expansions by equations (E9) are assumed to be asymptotically valid in the neighbourhood of the critical point defined by \(\lambda = \lambda_c\) and \(\xi = 0\).

Substituting equations (E9) into equations (E4), (E5) and (E6) and subtracting from the resulting expressions equations (E4), (E5) and (E6) evaluated at the corresponding prebuckling state yield the following results

\[ \xi \{\varepsilon_1 - [L_1(u_1) + L_{11}(u_0, u_1)]\} + \xi^2 \{\varepsilon_2 - [L_1(u_2) + L_{11}(u_0, u_2) + (1/2)L_2(u_1)]\} \]

\[ + \xi^3 \{\varepsilon_3 - [L_1(u_3) + L_{11}(u_0, u_3) + L_{11}(u_1, u_2)]\} + \ldots = 0 \]  \hspace{1cm} (E12a)
\[
\xi\{\sigma_1-H(\epsilon_1)\} + \xi^2\{\sigma_2-H(\epsilon_2)\} + \xi^3\{\sigma_3-H(\epsilon_3)\} + \ldots = 0 \quad \text{(E12b)}
\]
\[
\xi\{\sigma_1 \cdot \delta\epsilon_1 + \sigma_0 \cdot L_{11}(u_1, \delta u)\} + \xi^2\{\sigma_2 \cdot \delta\epsilon_0 + \sigma_0 \cdot L_{11}(u_2, \delta u) + \sigma_1 \cdot L_{11}(u_1, \delta u)\}
\]
\[
+ \xi^3\{\sigma_3 \cdot \delta\epsilon_0 + \sigma_0 \cdot L_{11}(u_3, \delta u) + \sigma_1 \cdot L_{11}(u_2, \delta u) + \sigma_2 \cdot L_{11}(u_1, \delta u)\} + \ldots = 0 \quad \text{(E12c)}
\]

where
\[
\delta\epsilon_0 = L_{11}(\delta u) + L_{11}(u_0, \delta u) \quad \text{(E13)}
\]

As has been pointed out by Cohen\textsuperscript{[19]} the coefficients of the different powers of \(\xi\) cannot as yet be equated to zero, because the coefficients themselves are functions of \(\xi\). This is due to the presence of the prebuckling variables \((\sigma_0, u_0)\) which depend on \(\Lambda\).

Notice that dividing the above equations by \(\xi\) and then taking the limit as \(\xi \to 0\) one obtains the eigenvalue problem for the critical load \(\Lambda_c\) and the corresponding buckling mode \(u_1\)
\[
\epsilon_1 = L_1(u_1) + L_{11}(u_0, u_1) \quad \text{(E14a)}
\]
\[
\sigma_1 = H(\epsilon_1) \quad \text{(E14b)}
\]
\[
\sigma_1 \cdot \delta\epsilon_c + \sigma_0 \cdot L_{11}(u_1, \delta u) = 0 \quad \text{(E14c)}
\]

where the quantities with the subscript \((\cdot)_c\) are evaluated at \(\Lambda = \Lambda_c\) and
\[
\delta\epsilon_c = L_{11}(\delta u) + L_{11}(u_0, \delta u) \quad \text{(E15)}
\]

Next it is assumed that the prebuckling variables can be expanded in Taylor series
\[
u_0 = u_c + (\Lambda-\Lambda_c)^\ast u_c^\ast + (1/2)(\Lambda-\Lambda_c)^2 u_c^\ast + \ldots \quad \text{(E16)}
\]
\[
\sigma_0 = \sigma_c + (\Lambda-\Lambda_c)^\ast \sigma_c^\ast + (1/2)(\Lambda-\Lambda_c)^2 \sigma_c^\ast + \ldots
\]
where the dots represent differentiation with respect to \( \lambda \). In addition it will be assumed that \((\lambda - \lambda_c)\) admits the asymptotic expansion

\[
\lambda - \lambda_c = a\lambda_c \xi + b\lambda_c \xi^2 + \ldots
\]

(E17)

Substituting into equations (E12) and regrouping yields

\[
\xi\{\varepsilon_1 - [L_1(u_1) + L_{11}(u_c, u_1)]
\]

\[+ \xi^2\{\varepsilon_2 - [L_1(u_2) + L_{11}(u_c, u_2) + (1/2)L_2(u_1) + a\lambda_c L_{11}(u_c, u_1)]\}
\]

\[+ \xi^3\{\varepsilon_3 - [L_1(u_3) + L_{11}(u_c, u_3) + L_{11}(u_1, u_2) + b\lambda_c L_{11}(u_c, u_1) + a\lambda_c L_{11}(u_c, u_2) + (1/2)(\lambda - \lambda_c) L_{11}(u_c, u_1)]\} + \ldots = 0
\]

(E18a)

\[
\xi\{\sigma_1 - H(\varepsilon_1)\} + \xi^2\{\sigma_2 - H(\varepsilon_2)\} + \xi^3\{\sigma_3 - H(\varepsilon_3)\} + \ldots = 0
\]

(E18b)

\[
\xi\{\sigma_1 \cdot \delta \varepsilon_c + \sigma_c \cdot L_{11}(u_1, \delta u)\}
\]

\[+ \xi^2\{\sigma_2 \cdot \delta \varepsilon_c + \sigma_c \cdot L_{11}(u_2, \delta u) + \sigma_1 \cdot L_{11}(u_1, \delta u)\}
\]

\[+ a\lambda_c [\sigma_1 \cdot L_{11}(u_c, \delta u) + \sigma_c \cdot L_{11}(u_1, \delta u)]\}

\[+ \xi^3\{\sigma_3 \cdot \delta \varepsilon_c + \sigma_c \cdot L_{11}(u_3, \delta u) + \sigma_1 \cdot L_{11}(u_2, \delta u) + \sigma_2 \cdot L_{11}(u_1, \delta u)\}
\]

\[+ b\lambda_c [\sigma_1 \cdot L_{11}(u_c, \delta u) + \sigma_c \cdot L_{11}(u_1, \delta u)]
\]

\[+ a\lambda_c [\sigma_2 \cdot L_{11}(u_c, \delta u) + \sigma_c \cdot L_{11}(u_2, \delta u)]
\]

\[+ (1/2)(a\lambda_c)^2 [\sigma_1 \cdot L_{11}(u_c, \delta u) + \sigma_c \cdot L_{11}(u_1, \delta u)]\} + \ldots = 0
\]

(E18c)

Equating the coefficients of the various powers of \( \xi \) to zero yield the equations satisfied by the different terms in the expansions (E9). In equation (E18c) the coefficient of \( \xi \) yields the variational equation of the eigenvalue
problem. Notice that by letting $\delta u = u_n (n=1,2,\ldots)$ and using the expression $\delta c = L_1 (\delta u) + L_{11} (u_c, \delta u)$ equation (E14c) becomes

$$\sigma_1 \cdot [L_1 (u_n) + L_{11} (u_c, u_n)] + \sigma_c \cdot L_{11} (u_1, u_n) = 0$$  \hspace{1cm} (E19)

Using $\delta u = u_1$ in equation (E18c) one obtains

$$\xi^2 \{ \sigma_2 \cdot \epsilon_1 + \sigma_c \cdot L_{11} (u_2, u_1) + \sigma_1 \cdot L_{11} (u_1, u_1)$$

$$+ a_c \{ \sigma_1 \cdot L_{11} (\ddot{u}_c, u_1) + \ddot{c} \cdot L_{11} (u_1, u_1) \} \}

+ \xi^3 \{ \sigma_3 \cdot \epsilon_1 + \sigma_c \cdot L_{11} (u_3, u_1) + \sigma_1 \cdot L_{11} (u_2, u_1) + \sigma_2 \cdot L_{11} (u_1, u_1)$$

$$+ b_c \{ \sigma_1 \cdot L_{11} (\ddot{u}_c, u_1) + \ddot{c} \cdot L_{11} (u_1, u_1) \} \}

+ a_c \{ \sigma_2 \cdot L_{11} (\ddot{u}_c, u_1) + \ddot{c} \cdot L_{11} (u_2, u_1) \}

$$+ (1/2) (a_c)^2 \{ \sigma_1 \cdot L_{11} (\dddot{u}_c, u_1) + \dddot{c} \cdot L_{11} (u_1, u_1) \}) + \ldots = 0$$  \hspace{1cm} (E20)

Employing the reciprocity relation $\sigma_i \cdot \epsilon_j = \sigma_j \cdot \epsilon_i \ (i,j=1,2,\ldots)$ and Eq. (E19) simplifies the above equation further

$$\xi^2 \{ (1/2) \sigma_1 \cdot L_2 (u_1) + \sigma_1 \cdot L_{11} (u_1, u_1) + a_c \{ 2 \sigma_1 \cdot L_{11} (\ddot{u}_c, u_1) + \ddot{c} \cdot L_{11} (u_1, u_1) \} \}

+ \xi^3 \{ 2 \sigma_1 \cdot L_{11} (u_1, u_2) + \sigma_2 \cdot L_{11} (u_1, u_1) + b_c \{ 2 \sigma_1 \cdot L_{11} (\ddot{u}_c, u_1) + \ddot{c} \cdot L_{11} (u_1, u_1) \} \}

+ a_c \{ \sigma_1 \cdot L_{11} (\ddot{u}_c, u_2) + \sigma_2 \cdot L_{11} (\ddot{u}_c, u_1) + \ddot{c} \cdot L_{11} (u_1, u_2) \}

$$+ (1/2) (a_c)^2 \{ 2 \sigma_1 \cdot L_{11} (\dddot{u}_c, u_1) + \dddot{c} \cdot L_{11} (u_1, u_1) \}) + \ldots = 0$$  \hspace{1cm} (E22)

The formula for the first postbuckling coefficient 'a' is obtained by equating to zero the coefficient of the $\xi^2$ term yielding

$$a = - (3/2 \lambda_c) \sigma_1 \cdot L_{11} (u_1, u_1)$$  \hspace{1cm} (E23)

where
\[ \dot{A} = 2\sigma_1 \cdot L_{11}(u_c, u_1) + \sigma_c \cdot L_{11}(u_1, u_1) \]

Finally, equating to zero the coefficient of the \( \xi^3 \) term leads to the formula for the second postbuckling coefficient 'b', namely

\[
b = - (1/\Lambda_c) \left\{ 2\sigma_1 \cdot L_{11}(u_1, u_2) + \sigma_2 \cdot L_2(u_1) \right\} + a\Lambda_c \left[ \sigma_1 \cdot L_{11}(\ddot{u}_c, u_2) + \sigma_2 \cdot L_{11}(\ddot{u}_c, u_1) + \sigma_c \cdot L_{11}(u_1, u_2) \right] + (1/2)(a\Lambda_c)^2 \left[ 2\sigma_1 \cdot L_{11}(\ddot{u}_c, u_1) + \sigma_c \cdot L_2(u_1) \right]
\]

Comparing equations (45)-(51) with equations (E23) and (E24) the following equivalence in notations becomes evident

\[
A \ast (B, C) = a \cdot L_{11}(b, c)
\]

**Imperfect Structure**

If the initial imperfection is denoted by \( \xi \vec{u} \) where \( \xi \) is the imperfection amplitude, then the strain resulting from an additional displacement \( u \) is

\[
\epsilon = L_1(u) + (1/2)L_2(u) + \xi \ddot{L}_{11}(\ddot{u}, u)
\]

Since to the order to which the analysis is carried out terms such as \( \xi \ddot{u}_{11} \), etc. do not enter, therefore the expansion given by equation (E9a) is still valid. Thus

\[
u = u_0 + \xi u_1 + \xi^2 u_2 + \xi^3 u_3 + \ldots
\]

Substitution into equation (E26) yields

\[
\epsilon = L_1(u_0) + (1/2)L_2(u_0) + \xi \ddot{L}_{11}(\ddot{u}, u_0) + \xi[L_1(u_1) + L_{11}(u_0, u_1)]
\]

\[
+ \xi^2[L_1(u_2) + L_{11}(u_0, u_2) + (1/2)L_2(u_1)]
\]

\[
+ \xi^3[L_1(u_3) + L_{11}(u_0, u_3) + L_{11}(u_1, u_2)] + \theta(\xi \dot{\epsilon}) + \ldots
\]
Similarly the variation in strain can be written as

\[ \delta \epsilon = L_1(\delta u) + L_{11}(u_0, \delta u) + \xi L_{11}(\hat{u}, \delta u) \]

\[ + \xi^2 L_{11}(u_1, \delta u) + \xi^2 L_{11}(u_2, \delta u) + \xi^3 L_{11}(u_3, \delta u) + \ldots \]  

(E28)

Finally the constitutive equation (E5) becomes

\[ \sigma = H(\epsilon) = H(\epsilon_0) + \xi H[L_{11}(\hat{u}, u_0)] + \xi H[L_1(u_1) + L_{11}(u_0, u_1)] + \xi^2 H[L_1(u_2) + L_{11}(u_0, u_2) + (1/2)L_2(u_1)] + \xi^3 H[L_1(u_3) + L_{11}(u_0, u_3) + L_{11}(u_1, u_2)] + \theta(\xi \hat{\epsilon}) + \ldots \]  

(E29)

where \( \epsilon_0 = L_1(u_0) + (1/2)L_2(u_0) \).

Substituting these expressions into the variational statement of equilibrium (Eq. E6) one gets after some regrouping

\[ (\sigma \cdot \delta \epsilon_0 - q \cdot \delta u) + \xi \{ \sigma_0 \cdot L_{11}(\hat{u}, \delta u) + H[L_{11}(\hat{u}, u_0)] \cdot \delta \epsilon_0 \} \]

\[ + \xi \{ \sigma_0 \cdot L_{11}(u_1, \delta u) + H[L_1(u_1) + L_{11}(u_0, u_1)] \cdot \delta \epsilon_0 \} \]

\[ + \xi^2 \{ \sigma_0 \cdot L_{11}(u_2, \delta u) + H[L_1(u_2) + L_{11}(u_0, u_2) + (1/2)L_2(u_1)] \cdot \delta \epsilon_0 \}

\[ + H[L_1(u_1) + L_{11}(u_0, u_1)] \cdot L_{11}(u_1, \delta u) \}

\[ + \xi^3 \{ \sigma_0 \cdot L_{11}(u_3, \delta u) + H[L_1(u_3) + L_{11}(u_0, u_3) + L_{11}(u_1, u_2)] \cdot \delta \epsilon_0 \}

\[ + H[L_1(u_2) + L_{11}(u_0, u_2) + (1/2)L_2(u_1)] \cdot L_{11}(u_2, \delta u) \}

\[ + H[L_1(u_1) + L_{11}(u_0, u_1)] \cdot L_{11}(u_2, \delta u) \} + \theta(\xi \hat{\epsilon}) + \ldots = 0 \]

(E30)

where \( \delta \epsilon_0 = L_1(\delta u) + L_{11}(u_0, \delta u) \).

Notice that the first term vanishes identically since it constitutes the variational statement of equilibrium of the fundamental state. Expanding now the prebuckling variable \( u_0 \) in the usual Taylor series
\[ u_0 = u_c + (\Lambda - \Lambda_c) \ddot{u}_c + (1/2)(\Lambda - \Lambda_c) \dot{u}_c^2 + \ldots \]  

(E16)

and using the following modified asymptotic expansion [19]

\[ \xi(\Lambda - \Lambda_c) = a\Lambda_c \xi^2 + b\Lambda_c \xi^3 - a\Lambda_c \xi - \beta(\Lambda - \Lambda_c) \xi + \ldots \]  

(E31)

one can write the variational statement of equilibrium (Eq. E30) as

\[ \bar{\xi}\{\sigma_o \cdot L_{11}(u_1, \delta u) + H[L_{11}(\dot{u}_c, u_c)] \cdot \delta \epsilon_o\} + (\Lambda - \Lambda_c) \bar{\xi}\{H[L_{11}(\dot{u}_c, u_c)] \cdot \delta \epsilon_o\} \]

- \[a\Lambda_c \bar{\xi}\{H[L_{11}(\dot{u}_c, u_c)] \cdot \delta \epsilon_o\} - \beta(\Lambda - \Lambda_c) \bar{\xi}\{H[L_{11}(\dot{u}_c, u_c)] \cdot \delta \epsilon_o\} \]

- \[ (1/2)a\Lambda_c (\Lambda - \Lambda_c) \bar{\xi}\{H[L_{11}(\ddot{u}_c, u_c)] \cdot \delta \epsilon_o\} \]

+ \[\xi\{\sigma_1 \cdot \delta \epsilon_o + \sigma_o \cdot L_{11}(u_1, \delta u)\} \]

+ \[\xi^2\{\sigma_2 \cdot \delta \epsilon_o + \sigma_1 \cdot L_{11}(u_1, \delta u) + \sigma_o \cdot L_{11}(u_2, \delta u)\} \]

+ \[\xi^3\{\sigma_3 \cdot \delta \epsilon_o + \sigma_2 \cdot L_{11}(u_1, \delta u) + \sigma_1 \cdot L_{11}(u_2, \delta u) + \sigma_o \cdot L_{11}(u_3, \delta u)\} + \theta(\xi \xi^2) + \ldots = 0 \]

where

\[ \sigma_1 = H(\epsilon_1) = H[L_{11}(u_1) + L_{11}(u_c, u_2)] \]  

(E33a)

\[ \sigma_2 = H(\epsilon_2) = H[L_{11}(u_2) + L_{11}(u_c, u_2) + (1/2)L_{21}(u_1) + a\Lambda_c L_{11}(\dot{u}_c, u_1)] \]  

(E33b)

\[ \sigma_3 = H(\epsilon_3) = H[L_{11}(u_3) + L_{11}(u_c, u_3) + L_{11}(u_1, u_2)] \]

+ \[b\Lambda_c L_{11}(\dot{u}_c, u_1) + a\Lambda_c L_{11}(\dot{u}_c, u_2) + (1/2)(a\Lambda_c)^2 L_{11}(\ddot{u}_c, u_1)] \]

Next using the Taylor series expansion of the prebuckling variable \( \sigma_o \)

\[ \sigma_o = \sigma_c + (\Lambda - \Lambda_c) \sigma_c + (1/2)(\Lambda - \Lambda_c)^2 \sigma_c + \ldots \]  

(E16)

and the expanded form of the variation of strain

\[ \delta \epsilon_o = \delta \epsilon_c + (\Lambda - \Lambda_c)L_{11}(\dot{u}_c, \delta u) + (1/2)(\Lambda - \Lambda_c)^2 L_{11}(\ddot{u}_c, \delta u) + \ldots \]  

(E34)
where $\delta e_c = L_1(\delta u) + L_{11}(u_c, \delta u)$ one gets upon substitution into Eq. (E32) and using the asymptotic expansion Eq. (E31) whenever necessary

$$\tilde{\xi}[H[L_{11}(\hat{u}, u_c)] \cdot \delta e_c + \sigma_c \cdot L_{11}(\hat{u}, \delta u) - a \Lambda_c \cdot H[L_{11}(u_c, u_1)] \cdot \delta e_c + \sigma_1 \cdot L_{11}(u_c, \delta u)$$

$$+ \delta_c \cdot L_{11}(u_1, \delta u) \}$$

$$+ (\Lambda - \Lambda_c) \tilde{\xi} \{ H[L_{11}(\hat{u}, u_c)] \cdot \delta e_c + \delta_c \cdot L_{11}(\hat{u}, \delta u) + H[L_{11}(u_c, u_1)] \cdot \delta e_c + (1/2) \sigma_c \cdot L_{11}(u_1, \delta u)$$

$$- a \Lambda_c \cdot (1/2) \sigma_1 \cdot L_{11}(u_c, \delta u) + (1/2) H[L_{11}(u_c, u_1)] \cdot \delta e_c + (1/2) \sigma_c \cdot L_{11}(u_1, \delta u)$$

$$+ H[L_{11}(u_c, u_1)] \cdot L_{11}(u_c, \delta u)$$

$$- \beta \cdot H[L_{11}(u_c, u_1)] \cdot \delta e_c + \sigma_1 \cdot L_{11}(u_c, \delta u) + \delta_c \cdot L_{11}(u_1, \delta u) \}$$

$$+ \xi \{ \sigma_1 \cdot \delta e_c + \sigma_c \cdot L_{11}(u_1, \delta u) \} + \xi^2 \{ \sigma_2 \cdot \delta e_c + \sigma_c \cdot L_{11}(u_2, \delta u) + \sigma_1 \cdot L_{11}(u_1, \delta u)$$

$$+ a \Lambda_c \cdot [\sigma_1 \cdot L_{11}(u_c, \delta u) + \delta_c \cdot L_{11}(u_1, \delta u)]$$

$$+ \xi^3 \{ \sigma_3 \cdot \delta e_c + \sigma_c \cdot L_{11}(u_3, \delta u) + \sigma_1 \cdot L_{11}(u_2, \delta u) + \sigma_2 \cdot L_{11}(u_1, \delta u)$$

$$+ a \Lambda_c \cdot [\sigma_2 \cdot L_{11}(u_c, \delta u) + \delta_c \cdot L_{11}(u_2, \delta u)]$$

$$+ b \Lambda_c \cdot [\sigma_1 \cdot L_{11}(u_c, \delta u) + \delta_c \cdot L_{11}(u_1, \delta u)]$$

$$+ (1/2) (a \Lambda_c)^2 \{ \sigma_1 \cdot L_{11}(u_c, \delta u) + \delta_c \cdot L_{11}(u_1, \delta u) \} + \theta(\tilde{\xi}) + \ldots = 0$$

Notice that the coefficient of $\xi$ vanishes identically since it represents the variational equation of the eigenvalue problem. Next one sets $\delta u$ equal to $u_1$ and uses the generalized reciprocity relation $H[L_{11}(u_c, u_1)] \cdot e_1 = H(e_1) \cdot L_{11}(u_c, u_1)$ etc. whenever necessary to simplify the above equation to

$$\tilde{\xi} \{ \sigma_1 \cdot L_{11}(u_c, u_1) + \sigma_c \cdot L_{11}(u_1, u_1) - a \Lambda_c \cdot [2 \sigma_1 \cdot L_{11}(u_c, u_1) + \delta_c \cdot L_{11}(u_1, u_1)]$$

$$+ (\Lambda - \Lambda_c) \tilde{\xi} \{ \sigma_1 \cdot L_{11}(u_c, u_1) + \delta_c \cdot L_{11}(u_1, u_1) + H[L_{11}(u_c, u_1)] \cdot L_{11}(u_1, u_1) + \ldots$$
... - \alpha_c \sigma_c \cdot L_{11}(\dot{u}_c, u_1) + (1/2) \dot{\alpha}_c \cdot L_{11}(\dot{u}_c, u_1) + H[L_{11}(\dot{u}_c, u_1)] \cdot L_{11}(\dot{u}_c, u_1) \\
- \beta [2\sigma_1 \cdot L_{11}(\dot{u}_c, u_1) + \dot{\sigma}_c \cdot L_{11}(u_1, u_1)] \\
+ \xi^2 \{ \sigma_1 \cdot L_{11}(u_1, u_2) + \sigma_1 \cdot L_{11}(u_1, u_1) \} \\
+ \xi^3 \{ \sigma_1 \cdot L_{11}(u_1, u_3) + \sigma_1 \cdot L_{11}(u_1, u_2) \} \\
+ a \alpha_c \cdot L_{11}(\dot{u}_c, u_1) + \dot{\sigma}_c \cdot L_{11}(u_1, u_1) \\
+ \frac{1}{2} (a \alpha_c)^2 \cdot L_{11}(\dot{u}_c, u_1) + \sigma_c \cdot L_{11}(u_1, u_1) \} + \theta (\xi \dot{\xi}) + \ldots = 0 \\

But now

\sigma_2 \cdot \epsilon_1 + \sigma_c \cdot L_{11}(u_1, u_2) = \sigma_1 \cdot \epsilon_2 + \sigma_c \cdot L_{11}(u_1, u_2) \quad (E37) \\
= \sigma_1 \cdot [L_1(u_2) + L_{11}(u_c, u_2)] + (1/2) L_2(u_1) + a \alpha_c \cdot L_{11}(\dot{u}_c, u_1) + \sigma_c \cdot L_{11}(u_1, u_2) \\

\sigma_3 \cdot \epsilon_1 + \sigma_c \cdot L_{11}(u_1, u_3) = \sigma_1 \cdot \epsilon_3 + \sigma_c \cdot L_{11}(u_1, u_3) \quad (E38) \\
= \sigma_1 \cdot [L_1(u_3) + L_{11}(u_c, u_3) + L_{11}(u_1, u_2) + a \alpha_c \cdot L_{11}(u_c, u_2) \\
+ b \alpha_c \cdot L_{11}(\dot{u}_c, u_1) + (1/2) (a \alpha_c)^2 L_{11}(u_c, u_1) + \sigma_c \cdot L_{11}(u_1, u_3) \\

Notice that the underlined terms vanish identically because of equation (E19). 
Introducing Eqs. (E37) and (E38) into Eq. (36) yields 

\tilde{\xi} \{ \sigma_1 \cdot L_{11}(\dot{u}, u_c) + \sigma_c \cdot L_{11}(\dot{u}, u_1) - \alpha_c \cdot \{2 \sigma_1 \cdot L_{11}(\dot{u}, u_1) + \dot{\sigma}_c \cdot L_{11}(u_1, u_1) \} \\
+ (a - \alpha_c) \tilde{\xi} \{ \sigma_1 \cdot L_{11}(\dot{u}, u_c) + \sigma_c \cdot L_{11}(\dot{u}, u_1) + H[L_{11}(u_c, u_1)] \cdot L_{11}(\dot{u}, u_c) \\
- a \alpha_c \cdot \sigma_1 \cdot L_{11}(u_c, u_1) + (1/2) \sigma_c \cdot L_{11}(u_1, u_1) + H[L_{11}(u_c, u_1)] \cdot L_{11}(\dot{u}, u_1) \\
- \beta [2 \sigma_1 \cdot L_{11}(\dot{u}, u_1) + \dot{\sigma}_c \cdot L_{11}(u_1, u_1)] + \ldots \}

\ldots
\[ \cdots + \xi^2( (1/2) L_2(u_1) + \sigma_1 \cdot L_{11}(u_1,u_1) + a \Lambda_c [2 \sigma_1 \cdot L_{11}(u_c,u_1) + \dot{\sigma}_c \cdot L_{11}(u_1,u_1)] \]
\[ + \xi^3 (2 \sigma_1 \cdot L_{11}(u_1,u_2) + \sigma_2 \cdot L_{11}(u_1,u_1) + b \Lambda_c [2 \sigma_1 \cdot L_{11}(u_c,u_1) + \ddot{\sigma}_c \cdot L_{11}(u_1,u_1)] \]
\[ + a \Lambda_c [\sigma_1 \cdot L_{11}(u_c,u_2) + \sigma_2 \cdot L_{11}(u_c,u_1) + \dot{\sigma}_c \cdot L_{11}(u_1,u_2) \]
\[ + (1/2)(a \Lambda_c)^2 [2 \sigma_1 \cdot L_{11}(u_c,u_1) + \dot{\sigma}_c \cdot L_{11}(u_1,u_1)] \cdot \theta(\xi \bar{\xi}) + \cdots = 0 \]

The formula for the first imperfection form factor 'a' is obtained by equating to zero the coefficient of the \( \xi \) term yielding

\[ a = (1/\Lambda_c) \{ \sigma_1 \cdot L_{11}(u_c,u_1) + \sigma_c \cdot L_{11}(u,u_1) \} \]  \hspace{1cm} (E40)

where

\[ \hat{\Lambda} = 2 \sigma_1 \cdot L_{11}(u_c,u_1) + \dot{\sigma}_c \cdot L_{11}(u_1,u_1) \]

Similarly the formula for the second imperfection form factor 'b' is obtained by equating to zero the coefficient of the \( (\Lambda - \Lambda_c) \xi \) term yielding

\[ b = (1/\hat{\Lambda}) \{ \sigma_1 \cdot L_{11}(u_c,u_1) + \dot{\sigma}_c \cdot L_{11}(u,u_1) + H[L_{11}(u_c,u_1)] \cdot L_{11}(u_c,u_c) \} \]
\[ - a \Lambda_c [\sigma_1 \cdot L_{11}(u_c,u_1) + \sigma_c \cdot L_{11}(u_1,u_1) + H[L_{11}(u_c,u_1)] \cdot L_{11}(u_c,u_1)] \]

Finally, equating the coefficients of the \( \xi^2 \) and \( \xi^3 \) terms equal to zero yield the formulas for the first and the second postbuckling coefficients 'a' and 'b', which have already been obtained in equations (E23) and (E24), respectively.

Up to now no formal use of the orthogonality condition involving \( u_0, u_1, u_2 \) and \( u_3 \) has been made. Following the established practice whenever needed the following orthogonality condition is used

\[ \sigma_c \cdot L_{11}(u_i,u_1) = 0 \hspace{1cm} i = 0, 1, 2, \ldots \]  \hspace{1cm} (E42)
Table 1. Comparison of results using Hutchinson's orthotropic shell
\( A_s / d_s t = 1.0, EI_s / d_s D = 100, e_s / t = -6.0 \) (outside), \( GJ/d_s D = 0 \)

SS-3 Boundary condition \( (N_x = -N_0, v = W = 0, M_x = 0) \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( nL/R )</th>
<th>( \lambda_c )</th>
<th>( b )</th>
<th>( \bar{b} )</th>
<th>( \theta_c )</th>
<th>( \theta^*_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z=300</td>
<td>Ref. [18]</td>
<td>6.5</td>
<td>8.79</td>
<td>-0.012</td>
<td>-0.0042</td>
<td>36.0</td>
</tr>
<tr>
<td>ANILISA</td>
<td>8</td>
<td>6.24</td>
<td>8.813</td>
<td>-0.0119</td>
<td>-0.00379</td>
<td>28.1</td>
</tr>
<tr>
<td>Z=500</td>
<td>Ref. [18]</td>
<td>10.6</td>
<td>5.04</td>
<td>-0.029</td>
<td>-0.0130</td>
<td>43.0</td>
</tr>
<tr>
<td>ANILISA</td>
<td>11</td>
<td>11.08</td>
<td>5.046</td>
<td>-0.0270</td>
<td>-0.0113</td>
<td>37.5</td>
</tr>
<tr>
<td>Z=750</td>
<td>Ref. [18]</td>
<td>13.1</td>
<td>3.75</td>
<td>-0.034</td>
<td>-0.0210</td>
<td>44.0</td>
</tr>
<tr>
<td>ANILISA</td>
<td>11</td>
<td>13.57</td>
<td>3.752</td>
<td>-0.0333</td>
<td>-0.0204</td>
<td>40.1</td>
</tr>
<tr>
<td>Z=1000</td>
<td>Ref. [18]</td>
<td>14.3</td>
<td>3.25</td>
<td>-0.030</td>
<td>-0.0230</td>
<td>45.0</td>
</tr>
<tr>
<td>ANILISA</td>
<td>10</td>
<td>14.24</td>
<td>3.243</td>
<td>-0.0306</td>
<td>-0.0233</td>
<td>41.0</td>
</tr>
</tbody>
</table>

C-4 Boundary condition \( (u = v = W = 0, x = 0) \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( nL/R )</th>
<th>( \lambda_c )</th>
<th>( b )</th>
<th>( \bar{b} )</th>
<th>( \theta_c )</th>
<th>( \theta^*_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z=500</td>
<td>Ref. [18]</td>
<td>10.45</td>
<td>10.44</td>
<td>-0.056</td>
<td>-0.064</td>
<td>45.0</td>
</tr>
<tr>
<td>ANILISA</td>
<td>10</td>
<td>10.07</td>
<td>10.480</td>
<td>-0.0526</td>
<td>-0.0540</td>
<td>44.8</td>
</tr>
<tr>
<td>Z=1000</td>
<td>Ref. [18]</td>
<td>17.0</td>
<td>6.07</td>
<td>-0.046</td>
<td>-0.040</td>
<td>45.0</td>
</tr>
<tr>
<td>ANILISA</td>
<td>12</td>
<td>17.09</td>
<td>6.076</td>
<td>-0.0459</td>
<td>-0.0384</td>
<td>45.0</td>
</tr>
</tbody>
</table>

NOTE: \( \bar{b} = a^2 b \); \( \theta_c \) and \( \theta^*_c \) are given in degrees

Table 2. Comparison of results using Booton's anisotropic shell
\( (30^\circ, 0^\circ, -30^\circ) \), \( R/t = 100, Z = L^2 / Rt = 200, t = 0.0267 \)
C-4 Boundary condition \( (u = v = W = 0, x = 0) \)

<table>
<thead>
<tr>
<th></th>
<th>Booton [14]</th>
<th>ANILISA</th>
<th>UNITS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial pressure</td>
<td>6</td>
<td>-6785.</td>
<td>-6780.1</td>
</tr>
<tr>
<td>External pressure</td>
<td>9</td>
<td>27.34</td>
<td>27.348</td>
</tr>
<tr>
<td>Clockwise torsion</td>
<td>9</td>
<td>7523.</td>
<td>7522.3</td>
</tr>
<tr>
<td>Counter-clockwise torsion</td>
<td>9</td>
<td>8228.</td>
<td>8229.3</td>
</tr>
</tbody>
</table>
Table 3. Booton's Anisotropic Shell \((30^\circ, 0^\circ, -30^\circ)\), \(R/t=100\), \(Z=L^2/Rt=200\), \(t=0.0267\)

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = \frac{1}{Et} \begin{bmatrix}
1.3751 & -0.7582 & 0. \\
-0.7582 & 2.6292 & 0. \\
0. & 0. & 4.8885
\end{bmatrix} \begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} + \frac{t}{2c} \begin{bmatrix}
0. & 0. & 0.1785 \\
0. & 0. & -0.0096 \\
0.7430 & 0.1965 & 0.
\end{bmatrix} \begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy} + M_{yx}
\end{bmatrix} = \frac{t}{2c} \begin{bmatrix}
0. & 0. & -0.7430 \\
0. & 0. & -0.1965 \\
-0.1785 & 0.0096 & 0.
\end{bmatrix} \begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} + D \begin{bmatrix}
0.5634 & 0.2214 & 0. \\
0.2214 & 0.3898 & 0. \\
0. & 0. & 0.1856
\end{bmatrix} \begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]

where \(c = \sqrt{3(1-v^2)}\), \(D = Et^3/4c^2\)
<table>
<thead>
<tr>
<th>$\bar{Z}$</th>
<th>SIMPLY SUPPORTED</th>
<th>CLAMPED</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha$</td>
<td>$\bar{b} = a^2 b$</td>
</tr>
<tr>
<td></td>
<td>Affine</td>
<td>Modal</td>
</tr>
<tr>
<td>50</td>
<td>0.4317</td>
<td>0.2743</td>
</tr>
<tr>
<td>100</td>
<td>0.4244</td>
<td>0.1462</td>
</tr>
<tr>
<td>200</td>
<td>0.4952</td>
<td>0.1155</td>
</tr>
<tr>
<td>300</td>
<td>0.5087</td>
<td>-0.0097</td>
</tr>
<tr>
<td>400</td>
<td>0.4804</td>
<td>0.0047</td>
</tr>
<tr>
<td>500</td>
<td>0.5060</td>
<td>-0.0025</td>
</tr>
<tr>
<td>600</td>
<td>0.4883</td>
<td>-0.0015</td>
</tr>
<tr>
<td>700</td>
<td>0.5022</td>
<td>0.0056</td>
</tr>
<tr>
<td>800</td>
<td>0.4977</td>
<td>0.0052</td>
</tr>
<tr>
<td>900</td>
<td>0.5022</td>
<td>0.0144</td>
</tr>
<tr>
<td>( \hat{z} )</td>
<td>Affine</td>
<td>Modal</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>50</td>
<td>0.9701</td>
<td>0.9519</td>
</tr>
<tr>
<td>100</td>
<td>0.9914</td>
<td>0.9761</td>
</tr>
<tr>
<td>200</td>
<td>0.9975</td>
<td>0.9915</td>
</tr>
<tr>
<td>300</td>
<td>0.9982</td>
<td>0.9953</td>
</tr>
<tr>
<td>400</td>
<td>0.9989</td>
<td>0.9967</td>
</tr>
<tr>
<td>500</td>
<td>0.9993</td>
<td>0.9977</td>
</tr>
<tr>
<td>600</td>
<td>0.9995</td>
<td>0.9982</td>
</tr>
<tr>
<td>700</td>
<td>0.9997</td>
<td>0.9985</td>
</tr>
<tr>
<td>800</td>
<td>0.9996</td>
<td>0.9987</td>
</tr>
<tr>
<td>900</td>
<td>0.9997</td>
<td>0.9989</td>
</tr>
</tbody>
</table>
Fig. 1. Test data for isotropic cylinders under axial compression [1]

\[ \frac{P}{P_{cl}} = 1 - 0.902(1 - e^{-\frac{1}{1000}V/R}) \]

Fig. 2. Reliability function \( R(\lambda) \) for a given \( R/t \) ratio
Fig. 3. Global architecture of the interactive shell design and analysis system 'DISDECO'

Fig. 4. Generalized analysis module
Fig. 5. Schematic layout of the full scale system
Fig. 6. Notation and sign convention

Fig. 7. Equilibrium paths of perfect and imperfect systems
Fig. 8. Generalized 'load-shortening' curves

Fig. 9. Perfect shell buckling loads and imperfection sensitivity coefficients for simply supported anisotropic shells under axial compression
Fig. 10. Mode shapes of simply supported anisotropic shells under axial compression
Fig. 11. Perfect shell buckling loads and imperfection sensitivity coefficients for fully clamped anisotropic shells under axial compression.
Fig. 12. Mode shapes of fully clamped anisotropic shells under axial compression
Fig. 13. Perfect shell buckling loads and imperfection sensitivity coefficients for simply supported anisotropic shells under hydrostatic pressure.
Fig. 14. Mode shapes of simply supported anisotropic shells under hydrostatic pressure.
Fig. 15. Definition of stress- and moment resultants
Fig. 16. Mode shapes for simply supported anisotropic shells under counter-clockwise torsion
Fig. 17. Mode shapes for simply supported anisotropic shells under clockwise torsion
Fig. 18. Perfect shell buckling loads and imperfection sensitivity coefficients for simply supported anisotropic shells under counter-clockwise torsion
Fig. 19. Perfect shell buckling loads and imperfection sensitivity coefficients for simply supported anisotropic shells under clockwise torsion.
Fig. 20. Estimates of critical loads for imperfection sensitive structures [20]