NOTES ON THE UNSTEADY RECTILINEAR MOTION OF A PERFECT GAS

VI. On a transformation of Staniukovich and its generalization

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SUMMARY

A transformation, originally due to Staniukovich, which transforms a differential equation for $p(h,t)$ in a homentropic flow, into a differential equation for $p_1(h_1,t)$ in a flow of an LMS-gas is applied to the general theory of ref. [3]. It yields transformation rules for all the flow parameters.

Next the Staniukovich transformation is generalized resulting in a 3 parameter continuous transformation group. The infinitesimal operators of the group are obtained and the invariance of a number of differential equations under the transformation is established.

Applications to physical situations are not included in this report, but are expected to appear in due time.
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1. INTRODUCTION

In some earlier papers a so-called LMS-gas (LMS = Ludford, Martin, Staniukovich) has been considered. This is an ordinary ideal gas with constant specific heats. In contrast to the most common homentropic case, when every fluid element has the same entropy, every fluid element of the LMS-gas has a different entropy, which it will retain for all time provided it does not pass through a shock-wave. The choice of this entropy distribution is made in such a way that generalized Riemann-invariants exist.

Several distinct methods have been used to establish this special entropy distribution. Martin and Ludford [7] considered intermediate integrals for a second order nonlinear partial differential equation. Staniukovich [8] noticed by a change of variables, and for a distinct choice of entropy distribution, which characterizes the LMS-gas, that the problem reduces to an equation of Darboux type of the same form as applies in a homentropic flow. The present author [3,4] again used a different method to obtain the entropy distribution of the LMS-gas.

In this report we begin with the method of Staniukovich to obtain the LMS-gas. Apart from ref. [8] and an earlier paper [1] by the same author a very concise and clear presentation of this work has been given by Cabannes in ref. [2].

The method of Staniukovich concentrates on an equation for the pressure \( p \), as function of the Lagrangian variable \( h \) and the time \( t \). The equation also contains the arbitrary entropy distribution \( B(h) \). The variables \( p \) and \( h \) are now changed to new variables \( h_1 \) and \( p_1 \), related to \( h \) and \( p \) by

\[
\frac{h_1}{h} = \frac{1}{p}, \quad \frac{p_1}{h} = \frac{p}{h}.
\]

(1.1.)

It follows that the new equation for \( p_1(h_1, t) \) has the same form as the original equation for \( p(h, t) \) provided a new entropy distribution function \( B_1(h_1) \) is chosen such that

\[
B_1(h_1) = h_1^{3\gamma-1} B(h).
\]

(1.2.)

If the original entropy function \( B(h) \) is a constant \( B_0 \), so that we start from a homentropic flow, the new entropy function is

\[
B_1(h_1) = B_0 h_1^{3\gamma-1} = B_0 h_1^{-(3\gamma-1)}.
\]

(1.3.)

which represents the entropy distribution for the LMS-gas. The transformation composed of (1.1.) and (1.2.) will be called the Staniukovich transformation.

In the first part of this report the Staniukovich-transformation is applied to the general theory of unsteady rectilinear gas motion developed in ref. [3]. It is shown that the transformation applies equally well to the equation for \( \Phi(h, t) \) developed there. Since all the flow parameters can be written as derivatives of \( \Phi(h, t) \) it is then easily found how the different flow parameters transform if the Staniukovich-transformation is applied. In particular this gives the opportunity to derive the flow of an LMS-gas from the flow of a homentropic gas. Applications of the method have not been included in this report,
but are expected to appear in due time.

In the second part of the report, starting in section 6 an attempt is made to generalize the Staniukovich transformation, yielding the transformation (6.19.). It turns out that this generalization has the form of a 3 parameter continuous transformation group G3. In particular if the homentropic flow is an element of the group also the LMS-gas is an element of the group. Stated differently the homentropic flows and the flows of an LMS-gas have been embedded in one continuous transformation group G3. The theory of continuous transformation groups can now be brought to bear upon the problems. The infinitesimal operators of the group with several extended operators are constructed, together with the commutators. The transformation and the infinitesimal operators are applied to test the invariance of a number of differential equations describing the flows and developed mainly in ref. [3]. Again applications to concrete physical situations have not been included but are expected to appear in due time.

The report is divided in 16 sections. In section 2 the equation of Staniukovich and Cabannes for p(h,t) is derived and it is shown to be closely related to the equation for \( \Phi(h,t) \) developed in ref. [3]. In section 3 the Staniukovich-transformation is introduced and it is shown that a similar transformation applies to \( \Phi(h,t) \). In section 4 the flow parameters expressed as derivatives of \( \Phi \) are transformed together with the equations of motion and the equations for the characteristics. Section 5 is mainly spent on a discussion of the range of the different variables in the Staniukovich-transformation.

In section 6 the generalized Staniukovich-transformation is obtained and in section 7 some of its properties are discussed. This is followed in section 8 by the transformation of the different flow parameters and differential equations along the lines of section 4, but now by means of the generalized Staniukovich-transformation. In section 9 it is shown that the generalized Staniukovich-transformation forms a degeneration of the projective group on a line. In section 10 the infinitesimal operators of the group are constructed together with the commutators, and the first and second extensions of the infinitesimal operators. This is followed in section 11 by the construction of the transformation formulae for the third and fourth derivatives of \( \Phi \), together with the third and fourth extension of the infinitesimal operators. The sections 12-15 are used to study the invariance of the partial differential equations for p(h,t), for \( \Phi(h,t) \), for M(p,h) and t(L,u) under the generalized Staniukovich transformation. In section 16 finally some of the results and conclusions are summarized.
2. THE EQUATION OF STANIUKOVICH AND CABANNES

with an equation for the pressure p, expressed as function of the Lagrangian
mass coordinate h and the time t. This equation is obtained from the equations
of motion and has the form
\[ \frac{\partial^2 p}{\partial h^2} + b(h) \frac{\partial^2}{\partial t^2} \left( \frac{1}{\gamma} \right) = 0. \] (2.1.)

To construct this equation one may start from the equations of motion in the
Lagrangian variables
\[ \frac{\partial v}{\partial t} - \frac{\partial u}{\partial h} = 0, \] (2.2.)
\[ \frac{\partial u}{\partial t} + \frac{\partial p}{\partial h} = 0, \] (2.3.)
\[ p \gamma = \exp \left( \frac{S}{c_v} \right) = B(h) = b(h) \gamma. \] (2.4.)

Elimination of u from (2.2.) and (2.3.) yields
\[ \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 p}{\partial h^2} = 0, \] (2.5.)
and eliminating V from (2.5.) by means of (2.4.) the equation (2.1.) follows.

It is useful to present some alternative ways for obtaining (2.1.) which are
related to the general theory of [3].

Following [3] equations (2.2.) and (2.3.) are satisfied identically by the
functions E (= x) and K defined by
\[ dE = dx = V dh + u dt, \quad dK = -udh + p dt, \]
\[ V = x_h, \quad u = x_t = -K_h, \quad p = K_t, \] (2.6.)

with subscripts denoting partial derivatives.

The set of equations (2.2.) - (2.4.) can now be replaced by
\[ \frac{\partial x}{\partial t} + \frac{\partial K}{\partial h} = 0, \] (2.7.)
\[ \frac{\partial K}{\partial t} \left( \frac{\partial x}{\partial h} \right)^\gamma = B(h) = b(h) \gamma. \] (2.8.)

Elimination of x from these equations yields
\[ b(h) \frac{\partial}{\partial t} \left\{ \left( \frac{\partial K}{\partial t} \right)^2 - \frac{1}{h^2} \right\} + \frac{3}{2} \frac{\partial K}{\partial h} = 0. \]  

(2.9.)

Since \( p = K_t \) from (2.6.), differentiation of (2.9.) with respect to \( t \) and substitution of \( p \) results in (2.1.).

Returning once again to the general theory of [3] the function \( \Phi(h,t) \) may be introduced by writing

\[ d\Phi = x \, dh - K dt, \quad \Phi_h = x = E, \quad \Phi_t = -K, \]  

(2.10.)

By means of (2.10.) equation (2.7.) is satisfied identically and equation (2.8.) takes the form

\[ \Phi_{tt}(\Phi_{hh})^2 + B(h) = 0. \]  

(2.11.)

In [3] equation (2.11.) was considered as the fundamental equation of the problem. The parameters \( x \) and \( K \) are expressed as first derivatives of \( \Phi \) by (2.10.) and the other physical parameter \( p, V \) and \( u \) then appear as second derivatives of \( \Phi \), obtained from (2.6.) by

\[ V = \Phi_{hh}, \quad u = \Phi_{ht}, \quad p = -\Phi_{tt}. \]  

(2.12.)

To transform equation (2.11.) to the form (2.1.) one may rewrite (2.11.) in the form

\[ -\Phi_{hh} + b(h) \left( -\Phi_{tt} \right) \frac{1}{\gamma} = 0, \]  

(2.13.)

with \( b(h) \) taken from (2.4.).

Differentiating this equation twice with respect to \( t \) and substituting \( p \) from eq. (2.12.) again results in eq. (2.1.).
3. THE TRANSFORMATION OF STANIUKOVICH

Following the original authors equation (2.1.) is taken as point of departure. New variables \( h_1 \) and \( p_1 \) are defined by

\[
h_1 = \frac{1}{h}, \quad p_1 = \frac{p}{h},
\]

while the time \( t \) remains unchanged. Considering \( p_1 \) as function of \( h_1 \) and \( t \) one easily deduces

\[
\frac{\partial p_1}{\partial h} = p_1 - h_1 \frac{\partial p_1}{\partial h_1}, \quad \frac{\partial^2 p_1}{\partial h^2} = h_1^3 \frac{\partial^2 p_1}{\partial h_1^2}.
\]

Replacing \( p(h, t) \) in eq. (2.1.) by \( p_1(h_1, t) \) one finds by means of (3.2.)

\[
\frac{\partial^2 p_1}{\partial h_1^2} + b(h) h_1^{\gamma - 1} \frac{\partial}{\partial t} \left( \frac{1}{\gamma} p_1 \right) = 0.
\]

Extending the transformation (3.1.) to \( b(h) \) by introduction of the transformation rule

\[
b_1(h_1) = h_1^{\gamma - 1} b \left( \frac{1}{h_1} \right) = h^{\gamma - 1} b(h),
\]

the equation (3.3.) may be rewritten

\[
\frac{\partial^2 p_1}{\partial h_1^2} + b_1(h_1) \frac{\partial^2}{\partial t^2} \left( \frac{1}{\gamma} p_1 \right) = 0.
\]

This equation has the same form as eq. (2.1.) and it follows that eq. (2.1.) remains invariant under the transformation composed of (3.1.) and (3.4.).

Since equation (2.1.) was obtained from the equations of motion (2.2.) - (2.4.), one can imagine that eq. (3.5.) results from a similar set of equations of motion, obtained from (2.2.) - (2.4.) by adding subscripts "1" to the different parameters, and executing the required eliminations.

If a flow has been constructed in the \( h, t \) variables, satisfying (2.2.) - (2.4.) and hence also eq. (2.1.), application of the transformations (3.1.) and (3.4.) to this solution will generate a pressure distribution, and more generally, a flow, in the \( h_1, t \) variables for a gas with the entropy distribution \( b_1(h_1) \) of (3.4.), distinct from \( b(h) \), and satisfying the equation (3.5.) and in general also the equations of motion in the variables \( h_1 \) and \( t \).

In particular if the flow in the \( h, t \) variables is homentropic with \( b(h) = b_0 = \text{const} \), it follows from (3.4.) that the transformation generates a flow in the \( h_1, t \) variables with the entropy distribution.
\[ b_1(h_1) = b_0 h_1^{\frac{3\gamma - 1}{\gamma}} , \quad (3.6.) \]

or writing \( B_0 = b_0^\gamma \) and using the notation in (2.4.)

\[ B_1(h_1) = B_0 h_1^{-(3\gamma - 1)} . \quad (3.7.) \]

This is the entropy distribution for the LMS-gas. It follows that the transformation composed of (3.1.) and (3.4.) when applied to a homentropic flow will generate a flow of an LMS-gas.

To bring the transformation more in line with the general theory of [3] a slight modification will be introduced.

Therefore consider the equation (2.11.) and instead of (3.1.) and (3.4.) define the transformation by

\[ h_1 = \frac{1}{h} , \quad \phi_1 = \frac{1}{h} \phi , \quad B_1(h_1) = h^{3\gamma - 1} B(h) . \quad (3.8.) \]

Transformation of eq. (2.11.) to variables with subscript "1" as defined in (3.8.) then yields

\[ \frac{\partial^2 \phi_1}{\partial t^2} \left( \frac{\partial^2 \phi_1}{\partial h_1^2} \right)^\gamma + B_1(h_1) = 0 , \quad (3.9.) \]

and is the same as eq. (2.11.). It follows that the remarks made before with respect to \( p \) and eq. (2.1.) apply equally well to \( \phi \) and eq. (2.11.). In particular application of the transformation (3.8.) to \( \phi(h,t) \) valid for a homentropic flow with \( B(h) = B_0 = \text{const} \), will result in \( \phi_1(h_1,t) \) applicable to the flow of an LMS-gas.
4. FLOW PARAMETERS UNDER THE STANIUKOVICH–TRANSFORMATION

From section 2, or from [3] it is clear that all the parameters of the flow can be expressed as derivatives of \( \Phi \). One has

\[
\begin{align*}
x &= E = \Phi_h, \quad K = -\Phi_t, \\
V &= \Phi_{hh}, \quad u = \Phi_{ht}, \quad p = -\Phi_{tt},
\end{align*}
\]

(4.1.)

with subscripts denoting partial derivatives. The same expressions, where all the symbols (except \( t \)) have subscripts "1" will apply to the flow in the \( h_1, t \) variables. To obtain the relations between the parameters of the two flows one deduces from (3.8.)

\[
\begin{align*}
\frac{\partial \Phi_1}{\partial t} &= \frac{1}{h} \frac{\partial \Phi}{\partial t}, \quad \frac{\partial \Phi_1}{\partial h_1} = -h \frac{\partial \Phi}{\partial h} + \Phi, \\
\frac{\partial^2 \Phi_1}{\partial t^2} &= \frac{1}{h} \frac{\partial^2 \Phi}{\partial t^2}, \quad \frac{\partial^2 \Phi_1}{\partial h_1 \partial t} = -h \frac{\partial^2 \Phi}{\partial h \partial t} + \frac{\partial \Phi}{\partial t}, \\
\frac{\partial^2 \Phi_1}{\partial h_1^2} &= h^2 \frac{\partial^2 \Phi}{\partial h^2},
\end{align*}
\]

(4.2.)

and also, if desired

\[
\begin{align*}
\frac{d B_1}{d h_1} &= -h^{3Y} \left\{ h \frac{d B}{d h} + (3Y - 1) B_1 \right\}, \\
\frac{d^2 B_1}{d h_1^2} &= h^{3Y+1} \left\{ h^2 \frac{d^2 B}{d h^2} + 6Yh \frac{d B}{d h} + 3Y(3Y - 1) B_1 \right\}.
\end{align*}
\]

(4.3.)

Employing (4.1.) together with corresponding expressions in \( \Phi_1, h_1 \) etc. one obtains from (4.2.)

\[
\begin{align*}
K_1 &= \frac{1}{h} K, \quad x_1 = -hx + \Phi, \\
P_1 &= \frac{1}{h} p, \quad u_1 = -hu - K, \quad V_1 = h^3 V.
\end{align*}
\]

(4.4.)

It is clear from (4.4.) that the transformation (3.8.) includes the original transformation (3.1.).

From (3.8.) one deduces immediately

\[
\begin{align*}
h &= \frac{1}{h_1}, \quad \Phi = \frac{1}{h_1} \Phi_1, \quad B(h) = h_1^{3Y-1} B_1(h_1).
\end{align*}
\]

(4.5.)

which shows that parameters with and without subscripts can be interchanged. The same rule then applies to the derivatives and also to the relations in (4.4.). It can easily be verified for specific cases.
In [3] the different Legendre transformations of \( E (= x) \), \( K \) and \( \Phi \) are considered. Here we consider the Legendre transformations

\[
L(u,t) = hu + K(h,t), \quad \psi(x,t) = hx - \Phi(h,t),
\]

and corresponding expressions with subscripts "1".

From (4.4.) and (4.6.) one finds

\[
x_1 = -\psi(x,t), \quad u_1 = -L(u,t), \quad \psi_1(x_1,t) = -x, \quad L_1(u_1,t) = -u.
\]

(4.7.)

and after some simple manipulation also

(4.8.)

The relations (4.7.) and (4.8.) show that the transformation (3.8.) essentially interchanges \( x \) and \( \psi \), with a simultaneous change of sign. The same applies to \( u \) and \( L \).

Other flow parameters to be considered are the speed of sound \( a \) and the specific acoustic impedance \( \frac{a}{V} \). From the definition \( a^2 = \gamma pV \) one deduces

\[
a_1^2 = \gamma p_1 V_1 = h^2 \gamma pV = h^2 a^2,
\]

(4.9.)

and further from (4.9.) and (4.4.)

\[
\frac{a_1}{V_1} = \frac{1}{h^2} \frac{a}{V}.
\]

(4.10.)

If the flow in the \( h,t \) variables is homentropic the equations of the characteristics will admit the Riemann invariants

\[
r = u + \frac{2a}{\gamma - 1}, \quad s = u - \frac{2a}{\gamma - 1},
\]

(4.11.)

which are constant along the \( r \)-, respectively the \( s \)-characteristics. Application of the Stanikovich-transformation to a homentropic flow in \( h,t \) yields a flow of an LMS-gas in \( h_1,t \). From (4.4.), (4.8.) and (4.9.) one has

\[
u = -h_1 u_1 - K_1 = -L_1, \quad a = h_1 a_1,
\]

(4.12.)

and substitution in (4.11.) yields

\[
r = -L_1 + \frac{2}{\gamma - 1} h_1 a_1 = -K_1 - h_1 \left(u_1 - \frac{2a_1}{\gamma - 1}\right) = -s_1^*,
\]

(4.13.)
\[ s = -L_1 - \frac{2}{\gamma - 1} h_1 a_1 = -K_1 - h_1 \left( u_1 + \frac{2a_1}{\gamma - 1} \right) = -r_1^*. \] (4.13.)

The expressions (4.13.), apart from the -signs, have been found before as the generalized Riemann-invariants \( s^* \) and \( r^* \) for the flow of an EMS-gas (Cf [3,4]).

The relations between the flow parameters of two flows, connected by the Staniukovich-transformation have now been sufficiently clarified. It is not difficult to transform also the equations of motion (2.2.) - (2.4.) and the equations of the characteristics.

One finds:

\[ \frac{\partial v}{\partial t} + \frac{\partial u}{\partial h} = h_1^2 \left( \frac{\partial K_1}{\partial h_1} + u_1 \right) + h_1^3 \left( \frac{\partial v_1}{\partial t} - \frac{\partial u_1}{\partial h_1} \right) = 0 , \]

\[ \frac{\partial u}{\partial t} + \frac{\partial p}{\partial h} = \left( p_1 - \frac{\partial K_1}{\partial t} \right) - h_1 \left( \frac{\partial u_1}{\partial t} + \frac{\partial p_1}{\partial h_1} \right) = 0 . \]

(4.14.)

From the definition of \( K \) in (2.6.) it follows that the expressions in the first brackets in (4.14.) vanish. In a similar fashion one derives for the \( r \)-characteristics in the \( h, t \) variables

\[ dh - \frac{a}{V} dt = - \frac{1}{h_1} \left( dh_1 + \frac{a_1}{V_1} dt \right) = 0 , \]

\[ dp + \frac{a}{V} du = - \frac{p_1}{h_1} \left( dh_1 + \frac{a_1}{V_1} dt \right) + \frac{1}{h_1} \left( dp_1 - \frac{a_1}{V_1} du_1 \right) = 0 . \]

(4.15.)

The relation (4.15.) shows that the \( r \)-characteristics in \( h, t \) are transformed in \( s_1 \)-characteristics in \( h_1, t \). The same applies to the \( s \)- and \( r_1 \)-characteristics.
5. FURTHER DISCUSSION OF THE STANIUKOVICH-TRANSFORMATION

In this section some other properties of the Staniukovich-transformation will be discussed and some limitations are pointed out. Indications are given for the removal of some restrictions.

Consider the transformation in the form

\[ h_1 = \frac{1}{h}, \quad p_1 = \frac{p}{h}, \quad B_1(h_1) = h^{3Y-1} B(h), \]  \hspace{1cm} (5.1)

which generates a pressure distribution \( p_1 \), entropy distribution \( B_1 \) and a flow in the variables \( h_1, t \) when starting from a pressure distribution \( p \), entropy distribution \( B(h) \) and in general a flow expressed in the variables \( h, t \).

Suppose that the transformation is applied a second time transforming \( h_1, p_1(h_1, t) \) and \( B_1(h_1) \) into \( h_2, p_2(h_2, t) \) and \( B_2(h_2) \) by means of

\[ h_2 = \frac{1}{h_1}, \quad p_2 = \frac{p_1}{h_1}, \quad B_2(h_2) = B_1(h_1) \cdot h_1^{3Y-1}. \]  \hspace{1cm} (5.2)

Eliminating the parameters with subscript "1" from (5.1.) and (5.2.) one finds

\[ h_2 = h, \quad p_2 = p, \quad B_2(h_2) = B(h), \]  \hspace{1cm} (5.3)

indicating that the original situation in the variables \( h, t \) has returned. A transformation with this property is called involutory. It is of interest to modify the transformation in such a way that the involutory character is removed.

In geometry there occurs a special form of the projective transformation defined by

\[ x_1 = \frac{1}{x}, \quad y_1 = \frac{y}{x}, \]  \hspace{1cm} (5.4)

which is sometimes called the Lambert-involutio (Cf. p. 374 of [5]) after the German mathematician-scientist J.H. Lambert (1728-1777). Clearly the Lambert-involutio and the Staniukovich-transformation are closely related.

In the Staniukovich-transformation it has been assumed so far that both \( h \) and \( h_1 \) are positive, since otherwise negative pressures, specific volumes etc. would occur. The restriction \( 0 \leq h \leq +\infty, \quad 0 \leq h_1 \leq +\infty \), seems unnatural and we investigate whether this restriction can be removed.

The first expression in (5.1.) represents an equilateral hyperbola in the \( h_1 \)-plane, with branches in the first and third quadrant. So far only the branch of this curve in the first quadrant was used. To admit the branch in the third quadrant, with \( -\infty < h < 0, \quad -\infty < h_1 < 0 \), one needs to replace (5.1.), (and if desired also (3.8.)) by

\[ h_1 = \frac{1}{h}, \quad p_1 = -\frac{p}{h}, \quad B_1(h_1) = (-h)^{3Y-1} B(h). \]  \hspace{1cm} (5.5)
It may be checked that by this modification the invariance properties of
Section 3 are retained, together with the involutorial character of the trans-
formation, while pressures, specific volumes etc. remain positive.

The next question to be asked is whether the transformation has to be
restricted to a hyperbola with branches in the first and third quadrant. Consider
therefore the transformation

\[ h_1 = -\frac{1}{h}, \quad p_1 = \frac{p}{h}, \quad B_1(h_1) = B(h) h^{3\gamma - 1}, \tag{5.6} \]

valid for \( 0 < h < +\infty, -\infty < h_1 < 0 \), i.e. the fourth quadrant of the \( h h_1 \)-plane
and

\[ h_1 = -\frac{1}{h}, \quad p_1 = -\frac{p}{h}, \quad B_1(h_1) = B(h) (-h)^{3\gamma - 1}, \tag{5.7} \]

valid when \( -\infty < h < 0, 0 < h_1 < +\infty \), i.e. valid in the second quadrant of the
\( h h_1 \) plane.

It may be observed that for (5.1.) and (5.5.) we have \( \frac{dh_1}{dh} < 0 \), while for (5.6.)
and (5.7.) \( \frac{dh_1}{dh} > 0 \).

It is impossible now to apply the transformation (5.6.) twice in succession; a
positive \( h \) in (5.6.) yields a negative \( h_1 \) and the second transformation starting
with negative \( h \)-values should be of the type (5.7.) to yield positive pressures
etc. One may note however that application of (5.6.) and (5.7.) in succession
results in the original situation, or expressed differently, application of
(5.6.) and (5.7.) in succession yields the identity transformation.

Finally it may again be verified that the invariance properties of Section 3
are retained by both transformations (5.6.) and (5.7.).

The above discussion indicates that the values of \( h \) and \( h_1 \) in the Staniukovich-
transformation need not be restricted to positive values. Slight modifications,
especially a judicious introduction of some -signs, allows the extension of
the transformation to the entire \( h h_1 \)-plane, while retaining the invariance
properties of the equations (2.1.) and (2.11.) discussed in Section 3 and
yielding positive pressures etc.

Finally it may be noted that the Staniukovich-transformation owes its effective-
ness to the second relation in (3.2.). It shows that the second derivative of
\( p \) (or \( \Phi \)) with respect to \( h \) has to be multiplied with some powers of \( h \) to obtain
the second derivative in the new variables. In the next section a generalization
of the Staniukovich-transformation will be constructed, which retains this
feature and the invariance properties of Section 3.
6. A GENERALIZATION OF THE STANIUKOVICH-TRANSFORMATION

Starting from the last remark in the preceding section we attempt to generalize the Stanikovitch-transformation by writing

\[ h_1 = G(h), \quad (6.1.0) \]
\[ \phi_1(h_1, t) = F(h) \phi(h, t). \]

It follows that

\[ \frac{\partial \phi_1}{\partial t} = F(h) \frac{\partial \phi}{\partial t}, \]
\[ \frac{\partial \phi_1}{\partial h_1} = \left( F(h) \frac{\partial \phi}{\partial h} + F'(h) \phi \right) \frac{1}{G'(h)}, \quad (6.2.0) \]

and furthermore

\[ \frac{\partial^2 \phi_1}{\partial h_1^2} = \frac{F}{(G')^2} \frac{\partial^2 \phi}{\partial h^2} + \frac{2F'G' - G''F}{(G')^3} \frac{\partial \phi}{\partial h} + \frac{F'G' - G''F'}{(G')^3} \phi, \quad (6.3.0) \]

with dashes denoting derivatives with respect to \( h \). The terms with \( \phi \) and \( \frac{\partial \phi}{\partial h} \) in (6.3.) vanish if it is possible to select \( F \) and \( G \) in such a way that simultaneously

\[ F''G' - F'G'' = 0, \]
\[ 2F'G' - FG'' = 0. \quad (6.4.0) \]

Solutions of \( G' \) and \( G'' \), different from zero, will exist, if

\[ \begin{vmatrix} F'' & -F' \\ 2F' & -F \end{vmatrix} = -FF'' + 2(F')^2 = 0, \quad (6.5.0) \]

yielding

\[ F(h) = -\frac{1}{C_1 h + C_2}, \quad (6.6.0) \]

with \( C_1 \) and \( C_2 \) integration constants. Since \( C_1 \) and \( C_2 \) are arbitrary the -sign in (6.6.) is non-essential. Conditions of "physical reality", yielding always positive pressures etc. as discussed in Section 5 will usually determine the sign to be used. Once \( F(h) \) has been found \( G(h) \) can be determined from (6.4.). One finds
where $C_3$ and $C_4$ are again integration constants.

The formula (6.7.) appears in several branches of mathematics, for both real and complex values of the different parameters and is known alternately as the fractional linear transformation, the Moebius-transformation or the formula for the projective group on a line.

It is convenient to adjust the notation and to write the formula (6.7.) in the form:

$$h_1 = G(h) = \frac{a_{11}h + a_{12}}{a_{21}h + a_{22}}.$$  \hfill (6.8.)

Further we denote the coefficient matrix suggested by (6.8.) as

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$  \hfill (6.9.)

and the determinant of this matrix by $A$.

To show that (6.8.) is a generalization of the hyperbola in the first expression in (3.1.) and of the hyperbolae in (5.5.), (5.6.) and (5.7.) one may rewrite (6.8.) in the form

$$a_{21}h_1 - a_{11} = -\frac{A}{a_{21}h + a_{22}},$$  \hfill (6.10.)

and deduce also

$$\frac{dh_1}{dh} = \frac{A}{(a_{21}h + a_{22})^2}.$$  \hfill (6.11.)

Comparison of these expressions with the discussion in Section 5 shows that again we have orthogonal hyperbolae in the $h_1$-plane, but the origin and the asymptotes are determined by

$$a_{21}h_1 - a_{11} = 0, \quad a_{21}h + a_{22} = 0,$$  \hfill (6.12.)

instead of $h_1 = 0$, $h = 0$. Also for $A > 0$ the slope of the curves is positive, representing branches in the proper second and fourth quadrant, while $A < 0$ yields a negative slope corresponding to branches in the first and third quadrants.

Modifying (6.6.) in a similar fashion as (6.7.), we write for (6.6.)
\[ F(h) = \frac{\alpha}{a_{21}h + a_{22}} , \]  

(6.13.)

with \( \alpha \) representing a coefficient, selected in such a way that for all values of \( h \) we have \( F(h) > 0 \) and so positive pressures, specific volumes etc. This requires that \( \alpha \) changes sign for \( a_{21}h + a_{22} = 0 \). The second relation in (6.1.) then takes the form

\[ \Phi_1(h_1, t) = \frac{\alpha}{a_{21}h + a_{22}} \Phi(h, t) . \]  

(6.14.)

With the relations (6.8.) and (6.14.) the invariance of the eq. (2.11.) may now be considered. From the two expressions one derives, using also (6.10.)

\[ \frac{\partial^2\Phi}{\partial t^2} = \frac{a_{21}h + a_{22}}{\alpha} \frac{\partial^2\Phi_1}{\partial t^2} = \frac{-A}{\alpha(a_{21}h_1 - a_{11})} \frac{\partial^2\Phi_1}{\partial t^2} , \]

\[ \frac{\partial^2\Phi}{\partial h^2} = \frac{A}{\alpha(a_{21}h + a_{22})} \left( \frac{\partial^2\Phi_1}{\partial h_1^2} \right) = \frac{(a_{21}h_1 - a_{11})^3}{-\alpha A} \frac{\partial^2\Phi_1}{\partial h_1^2} . \]  

(6.15.)

The second derivatives of \( \Phi \) and \( \Phi_1 \) in (6.15.) represent \(-p\), \(-p_1\) respectively \( V \) and \( V_1 \) as comparison with (2.12.) shows. One checks that the condition \( F(h) > 0 \) associated with (6.13.) assures that \( p \) and \( V \) are positive, if \( p_1 \) and \( V_1 \) are positive and conversely.

Substitution of (6.15.) into (2.11.) yields

\[ \frac{\partial^2\Phi_1}{\partial t^2} \left( \frac{\partial^2\Phi_1}{\partial h_1^2} \right)^\gamma + \frac{\alpha^{\gamma+1}(a_{21}h + a_{22})^{3\gamma-1}}{A^{2\gamma}} B(h) = 0 , \]  

(6.16.)

and putting

\[ B_1(h_1) = \frac{\alpha^{\gamma+1}}{A^{2\gamma}} (a_{21}h + a_{22})^{3\gamma-1} - \frac{\alpha^{\gamma+1}(-A)^{\gamma-1}}{(a_{21}h_1 - a_{11})^{3\gamma-1}} B(h) , \]  

(6.17.)

the equation (6.16.) assumes the form

\[ \frac{\partial^2\Phi_1}{\partial t^2} \left( \frac{\partial^2\Phi_1}{\partial h_1^2} \right)^\gamma + B_1(h_1) = 0 , \]  

(6.18.)

identical with (3.9.) and also (2.11.).

We conclude then that the equation (2.11.) is invariant under the generalized Staniukovich-transformation defined by (6.8.), (6.14.) and (6.17.) or written together
\[
    h_1 = \frac{a_{11} h + a_{12}}{a_{21} h + a_{22}},
\]

\[
    \phi_1(h_1, t) = \frac{\alpha}{a_{21} h + a_{22}} \phi(h, t), \tag{6.19.}
\]

\[
    B_1(h_1) = \frac{\alpha^{\gamma+1}}{a_{21} h + a_{22}} (a_{21} h + a_{22})^{3\gamma-1} B(h).
\]

In the following sections of this report the discussion will be restricted to the transformation (6.19.) and some of its consequences. It may be noted however that the form of the equation (6.18.), (3.9.) and (2.11.) allows some further transformations which may be introduced if needed. They are associated with the properties that in (6.18.)

(i) \( t \) does not occur explicitly, neither does \( h \).

(ii) only second derivatives of \( \phi \) with respect to \( h \) and \( t \) occur.

It follows from (i) that changing \( t \) into \( a_{21} t + a_{32} \), results in an additional factor \( a_{31} \) in the transformation of \( B(h) \). Physically it involves a shift in the origin of \( t \), and measurement of \( t \) in different units. From (ii) it follows that an expression linear in \( t \) and \( h \) can be added to \( \phi \) without affecting the equation.
7. ELEMENTARY PROPERTIES OF THE GENERALIZED TRANSFORMATION

The formula (6.8.) or the first formula in (6.19.) is completely determined if a quadruplet \((a_{11}, a_{12}, a_{21}, a_{22})\) has been chosen. Since each \(a_{ij} \ (i,j = 1,2)\) can be chosen freely from the set of real numbers there are \(\omega^4\) of these quadruplets and formulae (6.8.). However the quadruplet \((a_{11}, a_{12}, a_{21}, a_{22})\) and \((\lambda a_{11}, \lambda a_{12}, \lambda a_{21}, \lambda a_{22})\) with \(\lambda \neq 0\) yields the same transformation. It follows that there exist \(\omega^3\) different transformations in the family represented by (6.8.). To obtain a unique set of \(a_{ij}\)'s for each transformation one additional condition between the 4 \(a_{ij}\)'s has to be imposed. One may reduce a non-zero \(a_{ij}\) to the value 1, or one may impose for example the condition

\[
A = a_{11}a_{22} - a_{12}a_{21} = \pm 1. \tag{7.1.}
\]

A condition similar to (7.1.) will be used later in this report. It should be noted that both values \(+1\) and \(-1\) are necessary and that the transformations separate in 2 families. For \(A = +1\) one has hyperbolae with positive slope, and branches in the 2nd and 4th quadrants, while for \(A = -1\) the slopes are negative and the branches in the appropriate 1st and 3rd quadrants (Cf. (6.10.) and (6.11.).)

The identity transformation

\[
h_1 = h_1, \tag{7.2.}
\]

is obtained from (6.8.) by putting \(a_{11} = a_{22}, a_{12} = a_{21} = 0\). With the condition (7.1.) it follows \(a_{11} = a_{22} = \pm 1\) and \(A = \pm 1\).

The original Staniukovich-transformation in (3.1.) appears if \(a_{11} = a_{22} = 0, a_{12} = a_{21}\), and with the condition (7.1.) this yields \(a_{12} = a_{21} = \pm 1\) and \(A = -1\). Also one may check that for the transformations in (5.6.) and (5.7.) one obtains with (7.1.) that \(A = +1\).

Solving \(h\) from (6.8.) one obtains

\[
h = \frac{a_{22}h_1 - a_{12}}{-a_{21}h_1 + a_{11}}, \tag{7.3.}
\]

and multiplying each coefficient with \(\frac{1}{A}\), we write

\[
[A^{-1}] = \frac{1}{A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \tag{7.4.}
\]

Multiplication of (6.9.) and (7.4.) then yields

\[
[A] [A^{-1}] = [A^{-1}] [A] = [I], \tag{7.5.}
\]

with
\[ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \] (7.6.)

For the determinant \( A^{-1} \) of the matrix (7.4.) one obtains
\[ A^{-1} = \frac{1}{A^2} A = \frac{1}{A}. \] (7.7.)

The above relations show that it is desirable to consider the coefficients of the inverse transformation to be given by (7.4.) that is \( \frac{1}{A} \times \) (the coefficients in (7.3.).)

Considering also the formulae of Section 6 in this way we write instead of (6.10.)
\[ -\frac{1}{A} a_{21}^{\prime} + \frac{1}{A} a_{11} = -\frac{a_{21} h_1 + a_{11}}{A} = \frac{1}{a_{21} h + a_{22}}. \] (7.8.)

In similar fashion, it is preferable, upon solving \( \Phi \) and \( B \) in (6.14.) and (6.17.)
to write
\[ \Phi = \frac{1}{\alpha} \frac{A}{-a_{21}^{\prime} + a_{11}} \Phi_1, \] (7.9.)
\[ B = \frac{A^{2y}}{\alpha^{y+1}} \left( \frac{-a_{21} h_1 + a_{11}}{A} \right)^{3y-1} B_1. \]

The relations (7.3.) and (7.9.) together represent the inverse transformation of (6.19.). To verify this properly imagine that (6.19.), which transforms \( (h, \Phi, B) \) into \( (h_1, \Phi_1, B_1) \) is followed by a transformation of the same class, which transfers \( (h_1, \Phi_1, B_1) \) into \( (h_2, \Phi_2, B_2) \). If this second transformation is
\[ h_2 = \frac{a_{22} h_1 - a_{12}}{-a_{21} h_1 + a_{11}}, \]
\[ \Phi_2 = \frac{1}{\alpha} \frac{A}{-a_{21}^{\prime} + a_{11}} \Phi_1, \] (7.10.)
\[ B_2 = \frac{A^{2y}}{\alpha^{y+1}} \left( \frac{-a_{21} h_1 + a_{11}}{A} \right)^{3y-1} B_1, \]

one may check upon eliminating \( h_1, \Phi_1 \) and \( B_1 \) from (6.19.) and (7.10.) that
\[ h_2 = h, \quad \Phi_2 = \Phi, \quad B_2 = B, \] (7.11.)

and so yields the identity transformation.

The parameter \( \alpha \), introduced in (6.13.) is necessary to make \( F(h) > 0 \). However, it also acts as a proportionality factor for \( \Phi \). Due to the form of eq. (2.11.), it appears again as a proportionality factor \( \alpha^{y+1} \) in the transformation (6.17.)
for B. In general this proportionality will not be considered and the values assigned to \( a \) will then simply be \( \pm 1 \) if possible.
8. The Behaviour of the Flow Parameters Under the General Staniukovich-Transformation

In section 4 the parameters of the problem, all expressed as derivatives of $\phi$ (and $B$) were studied, when subjected to the Staniukovich-transformation. In this section the process will be repeated for the transformation (6.19.). Things are slightly more complicated since the inverse transformations require a bit more care.

From (6.19.) one deduces, upon employing (6.11.), that

$$
\frac{\partial \phi}{\partial t} = \alpha \frac{\partial \phi}{a_{21} h + a_{22} \frac{\partial t}{\partial t}},
$$

$$
\frac{\partial \phi}{\partial h} = \frac{\alpha}{A} \left\{ (a_{21} h + a_{22}) \frac{\partial \phi}{\partial h} - a_{21} \phi \right\},
$$

$$
\frac{\partial^2 \phi}{\partial t^2} = \frac{\alpha}{a_{21} h + a_{22} \frac{\partial t}{\partial t}},
$$

$$
\frac{\partial^2 \phi}{\partial h \partial t} = \frac{\alpha}{A} \left\{ (a_{21} h + a_{22}) \frac{\partial^2 \phi}{\partial h \partial t} - a_{21} \frac{\partial \phi}{\partial t} \right\},
$$

$$
\frac{\partial^2 \phi}{\partial h^2} = \frac{\alpha}{A} \left\{ (a_{21} h + a_{22})^3 \frac{\partial^2 \phi}{\partial h^2} \right\}.
$$

Upon using the relations (4.1.) and the Legendre transformations (4.6.) the relations (8.1.) can be written

$$
K_1 = \frac{\alpha}{a_{21} h + a_{22}} K,
$$

$$
x_1 = \frac{a_{21} h + a_{22}}{a_{21} h + a_{22}} \frac{\partial \phi}{\partial h} = \frac{\alpha}{A} \left\{ (a_{21} h + a_{22}) x - a_{21} \phi \right\} = \frac{\alpha}{A} \left\{ a_{21} \psi + a_{22} x \right\},
$$

$$
P_1 = \frac{\alpha}{a_{21} h + a_{22}} P,
$$

$$
u_1 = \frac{a_{21} h + a_{22}}{a_{21} h + a_{22}} \frac{\partial \phi}{\partial h} = \frac{\alpha}{A} \left\{ (a_{21} h + a_{22}) u + a_{21} \psi \right\} = \frac{\alpha}{A} \left\{ a_{21} \psi + a_{22} u \right\},
$$

$$
V_1 = \frac{a_{21} h + a_{22}}{a_{21} h + a_{22}} \frac{\partial \phi}{\partial h} = \frac{\alpha}{A} \left\{ (a_{21} h + a_{22})^3 \frac{\partial \phi}{\partial h} \right\}.
$$

Clearly the formulae (8.2.) are the analogues of (4.4.) for the generalized Staniukovich-transformation. Upon employing (4.6.) one also deduces the transformation formulae for $\psi_1$ and $L_1$. One finds

$$
\psi_1 = \frac{\alpha}{A} \left\{ a_{11} \psi + a_{12} x \right\},
$$

$$
L_1 = \frac{\alpha}{A} \left\{ a_{11} L + a_{12} u \right\}.
$$

The transformations of $\psi_1$ and $x_1$ and those for $L_1$ and $u_1$ can now be taken to-
gather in the matrix form
\[
\begin{bmatrix}
\psi_1 \\
x_1
\end{bmatrix} = \frac{\alpha}{A} \left[ A \right] \begin{bmatrix}
\psi \\
x
\end{bmatrix},
\tag{8.4.}
\]
and
\[
\begin{bmatrix}
L_1 \\
u_1
\end{bmatrix} = \frac{\alpha}{A} \left[ A \right] \begin{bmatrix}
L \\
u
\end{bmatrix}.
\tag{8.5.}
\]
It is interesting to note that the generalized Stanikovich-transformation for the parameters \( h, \phi \) and \( B \) induces an ordinary linear transformation for the parameters \( \psi \) and \( x \), and also for \( L \) and \( u \).

The following step is to express the derivatives of \( \phi \) in parameters with subscripts "1". This may be done in a straightforward fashion, starting from (7.3.) and (7.9.) or by inverting the formulae (8.2.) upon using (7.8.). Omitting the details one finds
\[
K = \frac{1}{\alpha} \frac{A}{-a_{11}^{-1} + a_{11}^{-1}} K_1,
\]
\[
x = \frac{A}{\alpha} \left\{ \frac{-a_{21}^{-1} + a_{11}^{-1}}{A} x_1 + \frac{a_{21}^{-1}}{A} \phi_1 \right\} = \frac{1}{\alpha} \left( -a_{21}^{-1} \psi_1 + a_{11}^{-1} x_1 \right),
\]
\[
p = \frac{1}{\alpha} \frac{A}{-a_{21}^{-1} + a_{11}^{-1}} P_1,
\]
\[
u = \frac{A}{\alpha} \left\{ \frac{-a_{21}^{-1} + a_{11}^{-1}}{A} u_1 - \frac{a_{21}^{-1}}{A} K_1 \right\} = \frac{1}{\alpha} \left( -a_{21}^{-1} L_1 + a_{11}^{-1} u_1 \right),
\]
\[
v = \frac{A^2}{\alpha} \left( -a_{21}^{-1} + a_{11}^{-1} \right)^3 v_1.
\tag{8.6.}
\]
The relations for \( x \) and \( u \) in (8.6.) appear once again when (8.4.) and (8.5.) are inverted upon employing (7.4.) and (7.5.). This yields
\[
\begin{bmatrix}
\psi \\
x
\end{bmatrix} = \frac{\alpha}{A} \left[ A^{-1} \right] \begin{bmatrix}
\psi_1 \\
x_1
\end{bmatrix} = \frac{1}{\alpha} \begin{bmatrix}
a_{22} & -a_{12} \\
a_{21} & a_{11}
\end{bmatrix} \begin{bmatrix}
\psi_1 \\
x_1
\end{bmatrix},
\tag{8.7.}
\]
\[
\begin{bmatrix}
L \\
u
\end{bmatrix} = \frac{\alpha}{A} \left[ A^{-1} \right] \begin{bmatrix}
L_1 \\
u_1
\end{bmatrix} = \frac{1}{\alpha} \begin{bmatrix}
a_{22} & -a_{12} \\
a_{21} & a_{11}
\end{bmatrix} \begin{bmatrix}
L_1 \\
u_1
\end{bmatrix}.
\tag{8.8.}
\]
In addition to the derivatives of \( \phi \), the derivatives of \( B \) have to be considered. From (6.17.) or (6.19.) one finds
\[
\frac{dB_1}{dh_1} = \frac{a^{\gamma+1}}{A^{2\gamma+1}} (a_{21}h + a_{22})^{3\gamma} \left\{ \frac{(a_{21}h + a_{22})}{A} \frac{dB_1}{dh} + (3\gamma - 1) \frac{a_{21}B_1}{A} \right\},
\]
\[
\frac{d^2B_1}{dh_1^2} = \frac{a^{\gamma+1}}{A^{2(\gamma+1)}} (a_{21}h + a_{22})^{3\gamma+1} \left\{ \left( \frac{a_{21}h + a_{22}}{A} \right)^2 \frac{d^2B_1}{dh_1^2} + 
\right.
\]
\[+ 6\gamma a_{21} (a_{21}h + a_{22}) \frac{dB_1}{dh} + 3\gamma (3\gamma - 1) \frac{a_{21}B_1}{A} \right\},
\]

and the inverse relations

\[
\frac{dB}{dh} = \frac{A^{2\gamma+1}}{a^{\gamma+1}} \left( -\frac{a_{21}h + a_{11}}{A} \right)^{3\gamma} \left\{ \frac{-a_{21}h + a_{11}}{A} \frac{dB}{dh} - (3\gamma - 1) \frac{a_{21}}{A} B_1 \right\},
\]
\[
\frac{d^2B}{dh^2} = \frac{A^{2\gamma+2}}{a^{\gamma+1}} \left( -\frac{a_{21}h + a_{11}}{A} \right)^{3\gamma+1} \left\{ \left( \frac{-a_{21}h + a_{11}}{A} \right)^2 \frac{d^2B}{dh_1^2} - 
\right.
\]
\[- 6\gamma \frac{a_{21}}{A} \left( -\frac{a_{21}h + a_{11}}{A} \right) \frac{dB}{dh} + 3\gamma (3\gamma - 1) \left( \frac{a_{21}}{A} \right)^2 B_1 \right\}. \tag{8.10.}
\]

Clearly the formulae (8.10.) can be simplified, by dividing by some powers of \(A\), if desired. The form presented shows the close analogy with (8.9.) when the direct transformation (6.19.) is replaced by the inverse transformation (7.8.) etc.

Continuing the transformations we find for the speed of sound \(a\) and the specific acoustic impedance \(\frac{a}{V}\) that

\[
a_1 = -\frac{a}{A} (a_{21}h + a_{22}) a, \tag{8.11.}
\]

\[
\frac{a_1}{V_1} = -\frac{1}{(a_{21}h + a_{22})^2} \frac{a}{V}. \tag{8.12.}
\]

The condition on \(F\) in (6.13.) shows that \(a(a_{21}h + a_{22})\) is always positive. Since \(a\) and \(a_1\) together with \(a/V\) and \(a_1/V_1\), should always be positive it follows that for \(A > 0\) the +sign in (8.11.) is required and for \(A < 0\) the -sign. The inverse expressions are

\[
a = -\frac{A}{a} \left( -\frac{a_{21}h + a_{11}}{A} \right) a_1, \tag{8.12.}
\]

\[
\frac{a}{V} = -\frac{1}{A} \left( -\frac{a_{21}h + a_{11}}{a_{21}h_1 + a_{11}} \right)^2 \frac{a_1}{V_1}. \tag{8.12.}
\]

Next the transformations of the equations of motion and of the equations for the characteristics have to be considered. One finds from the relations (8.6.)

\[
\frac{\partial V}{\partial t} - \frac{3u}{3h} = A^2 \left( \frac{-a_{21}h + a_{11}}{A} \right)^3 \left( \frac{\partial V}{\partial t} - \frac{3u_1}{3h_1} \right), \tag{8.13.}
\]

\[
\frac{\partial V}{\partial t} - \frac{\partial u}{\partial h} = A^2 \left( \frac{-a_{21}h + a_{11}}{A} \right)^3 \left( \frac{\partial V_1}{\partial t} - \frac{\partial u_1}{\partial h_1} \right). \tag{8.14.}
\]
\[ \frac{\partial u}{\partial t} + \frac{\partial p}{\partial h} = \frac{a}{\alpha} \left( \frac{-a_{21} h_1 + a_{11}}{A} \right) \left( \frac{\partial u}{\partial t} + \frac{\partial p}{\partial h_1} \right). \] (8.13)

The equations for the \( r \)-characteristics are

\[ dh - \frac{a}{V} dt = 0, \quad dp + \frac{a}{V} du = 0, \] (8.14)

and for the \( s \)-characteristics

\[ dh + \frac{a}{V} dt = 0, \quad dp - \frac{a}{V} du = 0. \] (8.15)

Transformation of the equations (8.14.) for the \( r \)-characteristics yields

\[ dh - \frac{a}{V} dt = \frac{1}{A} \left( \frac{A}{-a_{21} h_1 + a_{11}} \right)^2 \left( dh_1 + \frac{a_1}{V_1} dt \right), \] (8.16)

with the \(-\) sign applicable for \( A > 0 \), and the \(+\) sign for \( A < 0 \). The second equation in (8.14.) yields

\[ dp + \frac{a}{V} du = \frac{1}{\alpha} \left( \frac{-a_{21} h_1 + a_{11}}{A} \right) \left( dp_1 + \frac{a_1}{V_1} du_1 \right) + \]

\[ + \frac{1}{\alpha} \frac{a_{21}}{A} \left( \frac{A}{-a_{21} h_1 + a_{11}} \right)^2 p_1 \left( dh_1 + \frac{a_1}{V_1} dt \right), \] (8.17)

where the same sign-rules apply.

Finally, assuming that the flow in \( h,t \) variables is homentropic, the transformation of the Riemann-invariants may be considered. We have then

\[ r = u + \frac{2a}{\gamma - 1}, \quad s = u - \frac{2a}{\gamma - 1}, \] (8.18)

and upon employing the relations (8.6.) and (8.12.) one easily deduces

\[ r = \frac{1}{\alpha} \left( -a_{21} L_1 + a_{11} u_1 \right) + \frac{A}{\alpha} \left( \frac{-a_{21} h_1 + a_{11}}{A} \right) \frac{2a_1}{\gamma - 1} = \]

\[ = \frac{-a_{21}}{\alpha} \left( L_1 + \frac{2a_1 h_1}{\gamma - 1} \right) + \frac{a_{11}}{\alpha} \left( u_1 + \frac{2a_1}{\gamma - 1} \right), \] (8.19)

and for \( s \)

\[ s = -\frac{a_{21}}{\alpha} \left( L_1 + \frac{2a_1 h_1}{\gamma - 1} \right) + \frac{a_{11}}{\alpha} \left( u_1 + \frac{2a_1}{\gamma - 1} \right). \] (8.20)

Clearly \( r \) and \( s \) in (8.19.) and (8.20.) are linear combinations of the classical invariants of homentropic flow defined by (8.18.) and the generalized Riemann-
invariants of the LMS-gas, which appeared in \(4.13\).

This completes the transformations. It may be verified that the relations in this section will reduce to the expressions in Section 4, if one takes \(a_{11} = a_{22} = 0, a_{12} = a_{21}, A < 0\) and \(\alpha = a_{21}\). Also the identity transformation may be verified.
9. THE GROUP OF TRANSFORMATIONS

In Section 8 we studied how the different flow parameters and differential equations are transformed once a particular transformation (6.19.) with specified values \( a_{ij} \ (i,j = 1,2) \) and \( \alpha \) has been chosen.

In this section we consider the entire class of transformations (6.19.) and in particular the relations between different members of the class. To begin with consider only the transformation of \( h \). Later also \( \Phi \) and \( B \) will be drawn in the discussion.

The transformation

\[
h_1 = \frac{a_{11} h + a_{12}}{a_{21} h + a_{22}},
\]

(9.1.)

an arbitrary member of the class of transformations, will be denoted symbolically by

\[
(h_1) = T_a(h).
\]

(9.2.)

Another transformation of the class

\[
h_2 = \frac{b_{11} h_1 + b_{12}}{b_{21} h_1 + b_{22}},
\]

(9.3.)

will be denoted similarly by

\[
(h_2) = T_b(h_1).
\]

(9.4.)

Application of the two transformations in succession, so that \( h \) is first transformed into \( h_1 \) and then into \( h_2 \), shows that \( h_2 \) can be obtained from \( h \) by one single transformation, which again has the form (9.1.) and (9.3.). If this single transformation is denoted by \( T_c \) we find

\[
(h_2) = T_b(h_1) = T_b(T_a(h)) = T_b T_a(h) = T_c(h),
\]

(9.5.)

or

\[
T_b T_a = T_c.
\]

(9.6.)

The property found here for the transformations of the form (9.1.) and (9.3.) is called the group-property.

Working out the details, that is substituting (9.1.) into (9.3.), so that \( h_1 \) is eliminated one finds for \( T_c \)
\[ h_2 = \frac{c_{11} h + c_{12}}{c_{21} h + c_{22}}, \]  
(9.7.)

with

\[
\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},
\]
(9.8.)

or with the notation introduced in (6.9.)

\[ [C] = [B] [A]. \]
(9.9.)

It may be appropriate at this stage to write down the definition of a group [6].
A class of transformations (or other things) \( T_a, T_b \) etc. are said to form a group under the following conditions

1° If \( T_a \) and \( T_b \) are elements of the class, then \( T_b T_a \) is uniquely defined and is an element of the class.

2° The associative law holds, thus

\[ T_{c(ba)} = (T_{cT_b}) \cdot T_a. \]

3° There is in the class an element \( I \) called the identity such that \( T_a I = I T_a = T_a \) for every element \( T_a \).

4° For each element \( T_a \) of the class there is an element \( T_a^{-1} \) such that

\[ T_a^{-1} T_a = T_a T_a^{-1} = I. \]

If only the conditions 1° and 2° of the above four are satisfied the transformations are said to form a semi-group.

It is now easily seen that the class of transformations (9.1.) forms a group. The condition 1° is satisfied by the discussion in connection with (9.1.) - (9.9.). Condition 2° may be verified directly or by appealing to the rules of matrix multiplication (9.8.) and (9.9.), which satisfy the same rule. The identity \( I \) is obtained for \( a_{11} = a_{22} \) and \( a_{12} = a_{21} = 0 \), and \( A > 0 \). Finally the discussion in Section 7 in connection with (7.3.) - (7.8.) verifies condition 4°.

Since the parameters \( a_{ij} \) (\( i,j = 1,2 \)) may vary over the real numbers and since the discussion in Section 7 has shown that essentially three degrees of freedom are involved the class of transformations is a continuous 3-parameter group \( G_3 \), called the projective group on a line.

Considering now also the functions \( \Phi \) and \( B \) and the transformations appearing in (6.19.) one may verify that (6.19.) followed by the transformation

\[ \Phi_2 = \frac{\beta}{B_{21} h_1 + B_{22}} \Phi_1, \quad B_2 = \frac{\beta^{\gamma+1} \Gamma}{B_{21} \Gamma(\beta_{21} h_1 + B_{22})^{3\gamma-1}} \Gamma_1, \]
(9.10.)
yields upon using (9.1.) and (9.8.)

\[ \phi_2 = \frac{\beta \alpha}{c_{21} h + c_{22}} \phi, \]

\[ B_2 = \frac{(\beta \alpha)^{Y+1}}{(\beta \alpha)^{2Y}} (c_{21} h + c_{22})^{3Y-1} B. \] (9.11.)

This represents again the group property, if we introduce the additional composition rules

\[ \beta \alpha = c, \] (9.12.)

(\( \gamma \) has been used as the isentropic exponent and is not available. Therefore \( c \) is chosen) and the product rule of determinants

\[ \beta A = C, \] (9.13.)

which follows from the matrix rule (9.9.) if \( B \) and \( C \) are the determinants of the matrices \([B]\) and \([C]\).

With the composition rules (9.12.) and (9.13.) it is clear that (9.11.) is again a single transformation of the form (6.19.) and (9.10.). The associative law \( 2^0 \) may be verified directly. The identity transformation found for \( h \) above turns (6.19.) into

\[ \phi_1 = \frac{\alpha}{a_{22}} \phi, \quad B_1 = \left( \frac{\alpha}{a_{22}} \right)^{Y+1} B. \] (9.14.)

and the additional requirement \( \alpha = a_{22} \) reduces (9.14.) also to the identity. The inversion of the transformation (6.19.) for \( \phi \) and \( B \) has been discussed already in connection with (7.9.) - (7.11.). The coefficient matrix for the inverse transformation of \( h \) is given by (7.4.), while in the inverse transformation for \( \phi \) and \( B \), we need \( A^{-1} \), instead of \( A \), in agreement with (7.7.) and also \( \alpha^{-1} \).

It is then clear that the entire transformation (6.19.) including \( \phi \) and \( B \) forms a transformation group \( G_3 \). Since the transformation of \( h \) forms the king pin of the transformation, while \( \phi \) and \( B \) essentially stay in line, one calls the group (6.19.) a "degeneration" (German: Ausartung) of the projective group on a line.

Once the relation of the transformation (6.19.) with the theory of continuous transformation groups has been established, this part of group theory can be brought to bear upon our problems. In the next sections we shall initiate these steps.
10. THE INFINITESIMAL OPERATORS OF THE GROUP

In this section the infinitesimal operators of the group will be written down. From the infinitesimal operators the commutators will be constructed together with the first and second extensions of the operators.

The infinitesimal operators appear as follows. Let

\( (h_1) = T_a(h) \), \hspace{1cm} (10.1.)

represent an arbitrary transformation of the group, where \( h \) may stand for the triplet \( (h, \phi, B) \). Giving small increments \( \delta a_{ij} \) to the parameters \( a_{ij} \) \( (i,j = 1,2) \) in \( T_a \) we obtain

\( (h_1 + \delta h_1) = T_{a+\delta a}(h) \), \hspace{1cm} (10.2.)

where \( \delta h_1 \) indicates the small change in \( h_1 \) caused by \( \delta a_{ij} \) and symbolically denoted by \( \delta a \).

The inverse transformation of (10.1.) may be written

\( (h) = T_a^{-1}(h_1) \). \hspace{1cm} (10.3.)

and substituting this in (10.2.) one obtains

\( (h_1 + \delta h_1) = T_{a+\delta a} T_a^{-1}(h_1) \). \hspace{1cm} (10.4.)

The operator

\[ T_{a+\delta a} T_a^{-1}, \] \hspace{1cm} (10.5.)

changes \( h_1 \) into a slightly changed value \( h_1 + \delta h_1 \) and is the symbolic form of the infinitesimal operator.

It is easily shown in this symbolic form that the infinitesimal operator is independent of the actual values \( a_{ij} \) selected as starting point. To demonstrate this consider that the \( T_a \) is followed by \( T_b \) to yield \( T_c \) as discussed in Section 9. Then we have

\( (h_2) = T_c(h) = T_b T_a(h) \). \hspace{1cm} (10.6.)

Suppose that the parameters \( b \) are changed to \( b + \delta b \). This also modifies the c's and we find

\( (h_2 + \delta h_2) = T_{c+\delta c}(h) = T_{b+\delta b} T_a(h) \). \hspace{1cm} (10.7.)
Inversion of (10.6.) yields

\[(h) = T^{-1}_c(h_2) = T^{-1}_a T^{-1}_b(h_2) , \tag{10.8.}\]

and substitution in (10.7.) then results in

\[(h_2 + \delta h_2) = T^{-1}_{c+6c} T^{-1}_c(h_2) = T^{-1}_{b+\delta b} T^{-1}_a T^{-1}_b(h_2) . \tag{10.9.}\]

Since \(T^{-1}_a\) results in the identity operation we then have

\[T^{-1}_{c+6c} T^{-1}_c = T^{-1}_{b+\delta b} T^{-1}_b , \tag{10.10.}\]

indicating that the infinitesimal operator for the arbitrary values \(b\) and \(c\) are the same.

To calculate the infinitesimal operators in detail, the expressions \(h_1, \delta_1\) and \(B_1\) are to be considered as functions of the coefficients \(a_{ij}\) \((i,j = 1,2)\). The group is a 3-parameter group and selecting \(\tau\) as one of these, it is convenient to consider increments in \(a_{ij}, h_1\) etc. to be due to increments in the parameter \(\tau\). Starting from

\[h_1 = \frac{a_{11}h + a_{12}}{a_{21}h + a_{22}} , \tag{10.11.}\]

one finds upon differentiation to \(\tau\)

\[\frac{\delta h_1}{\delta \tau} = \frac{(a_{11}h + a_{12})(\dot{a}_{11}h + \dot{a}_{12}) - (a_{11}h + a_{12})(\dot{a}_{21}h + \dot{a}_{22})}{(a_{21}h + a_{22})^2} , \tag{10.12.}\]

where \(\dot{a}_{ij} = \frac{\delta a_{ij}}{\delta \tau}\) and the symbol \(\delta\) is used instead of \(\partial\) to indicate differentiations within the group.

Substitution of the inverse transformation (7.3.) and using (7.8.) the \(h\) in the right-hand side of (10.12.) is replaced by \(h_1\). The result may be written

\[\frac{\delta h_1}{\delta \tau} = \lambda + \mu h_1 + \nu h_1^2 , \tag{10.13.}\]

with

\[\lambda = \frac{1}{A} (a_{11}\dot{a}_{12} - a_{12}\dot{a}_{11}) , \]

\[\mu = \frac{1}{A} (a_{21}\dot{a}_{22} - a_{22}\dot{a}_{21} + a_{12}\dot{a}_{21} - a_{11}\dot{a}_{22}) , \tag{10.14.}\]

\[\nu = \frac{1}{A} (a_{21}\dot{a}_{22} - a_{22}\dot{a}_{21}) .\]

It seems indicated now to consider \(\lambda, \mu\) and \(\nu\) as the three free parameters. In order to express the 4 \(\dot{a}_{ij}\)'s uniquely in the 3 parameters just chosen, the
relation $A = \text{const.}$ may be added. This relation was already suggested in the beginning of Section 7. Upon differentiation it yields

$$a_{22} \dot{A}_{11} - a_{21} \dot{A}_{12} - a_{12} \dot{A}_{21} + a_{11} \dot{A}_{22} = 0. \quad (10.15.)$$

From (10.14.) and (10.15.) one then obtains

$$a_{22} \dot{A}_{11} - a_{21} \dot{A}_{12} = \frac{A \mu}{2},$$

$$a_{12} \dot{A}_{11} - a_{11} \dot{A}_{12} = -A \lambda, \quad (10.16.)$$

$$a_{12} \dot{A}_{21} - a_{11} \dot{A}_{22} = \frac{A \mu}{2},$$

$$a_{22} \dot{A}_{21} - a_{21} \dot{A}_{22} = -A \nu,$$

and from (10.16.)

$$\dot{A}_{11} = \frac{1}{2} \mu a_{11} + \lambda a_{21},$$

$$\dot{A}_{12} = \lambda a_{22} + \frac{1}{2} \mu a_{12}, \quad (10.17.)$$

$$\dot{A}_{21} = -\frac{1}{2} \mu a_{21} - \nu a_{11},$$

$$\dot{A}_{22} = -\nu a_{12} - \frac{1}{2} \mu a_{22}.$$

For prescribed functions $\lambda(\tau)$, $\mu(\tau)$ and $\nu(\tau)$ one has to integrate this system to find $a_{ij}(\tau)$. However these calculations will not be needed. The result we do need is (10.13.) and the relations (10.16.).

We next consider the transformation of $\Phi$, i.e.

$$\Phi \frac{\alpha}{a_{21} h + a_{22}} = (10.18.)$$

Differentiation to $\tau$, followed by substitution of the inverse transformations of $\Phi$ and $h$ then yields

$$\frac{\delta \Phi}{\delta \tau} = -\alpha \frac{\dot{A}_{21} h + \dot{A}_{22}}{(a_{21} h + a_{22})^2} \Phi =$$

$$= -\frac{1}{A} \left( \dot{A}_{21} (a_{22} h_1 - a_{12}) + \dot{A}_{22} (-a_{21} h_1 + a_{11}) \right) \Phi =$$

$$= (\nu h_1 + \frac{1}{2} \mu) \Phi_1. \quad (10.19.)$$

In a similar fashion the transformation of $B$ yields

$$\frac{\delta B}{\delta \tau} = \frac{a^{\gamma+1}}{A^{\gamma}} (3\gamma - 1) (a_{21} h + a_{22})^{3\gamma-2} (\dot{A}_{21} h + \dot{A}_{22}) B =$$
\[
\begin{align*}
&= (3\gamma - 1) \frac{\dot{a}_{21}h + \dot{a}_{22}}{a_{21}h + a_{22}} B_1 = \\
&= -\mu \frac{3\gamma - 1}{2} B_1 - \nu \frac{3\gamma - 1}{2} h_1 B_1 .
\end{align*}
\]

(10.20.)

Since \( \lambda, \mu \) and \( \nu \) are three independent functions, we can consider the increments generated by one function at the time. Applying these increments to an arbitrary function \( F(h, \phi, B) \), with differential

\[
dF = \frac{\partial F}{\partial h} dh + \frac{\partial F}{\partial \phi} d\phi + \frac{\partial F}{\partial B} dB ,
\]

(10.21.)

we obtain the three infinitesimal operators of the group. One finds by considering \( \lambda(\tau) \), \( \mu(\tau) \) and \( \nu(\tau) \) in turn

\[
\frac{\delta F}{\delta \tau} = V_1 F = \frac{\partial F}{\partial h} ,
\]

\[
\frac{\delta F}{\delta \tau} = V_2 F = h \frac{\partial F}{\partial h} + \phi \frac{\partial F}{\partial \phi} - \frac{3\gamma - 1}{2} B \frac{\partial F}{\partial B} ,
\]

(10.22.)

\[
\frac{\delta F}{\delta \tau} = V_3 F = h^2 \frac{\partial F}{\partial h} + h\phi \frac{\partial F}{\partial \phi} - (3\gamma - 1) hB \frac{\partial F}{\partial B} .
\]

Once the infinitesimal operators are obtained we construct for future reference the commutators defined by

\[
(V_i V_j) F = V_i (V_j F) - V_j (V_i F) = -(V_j V_i) F ,
\]

(10.23.)

and find

\[
(V_i V_2) F = V_1 F ,
\]

\[
(V_2 V_3) F = V_3 F ,
\]

(10.24.)

\[
(V_1 V_3) F = 2V_2 F .
\]

The commutator table then takes the form

<table>
<thead>
<tr>
<th>( V_1 )</th>
<th>( V_2 )</th>
<th>( V_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_1 )</td>
<td>0</td>
<td>( 2V_2 )</td>
</tr>
<tr>
<td>( V_2 )</td>
<td>( -V_1 )</td>
<td>0</td>
</tr>
<tr>
<td>( V_3 )</td>
<td>( -2V_2 )</td>
<td>( -V_3 )</td>
</tr>
</tbody>
</table>
and employing Lie's second fundamental theorem in the form

\[(V_i V_j) = c^k_{ij} V_k,\]  \hspace{1cm} (10.25.)

we find for the structure constants \(c^k_{ij}\)

\[
\begin{align*}
  c^1_{12} &= 1, & c^2_{12} &= 0, & c^3_{12} &= 0, \\
  c^1_{23} &= 0, & c^2_{23} &= 0, & c^3_{23} &= 1, \\
  c^1_{31} &= 0, & c^2_{31} &= -2, & c^3_{32} &= 0.
\end{align*}
\hspace{1cm} (10.26.)

It is not sufficient to stop with the infinitesimal operators for \(h, \Phi\) and \(B\) since the physical parameters are expressed as derivatives of \(\Phi\). Also if the invariance of some differential equations has to be considered the same quantities, derivatives of \(\Phi\) etc. are needed. This requires the calculations of the different extended operators. In section 8 the transformations of the first and second derivatives of \(\Phi\) and of \(B\) were already considered. Application of the same differentiation procedures as applied in this section to \(h, \Phi\) and \(B\) then yields the infinitesimal increments of these first derivatives, and once these are obtained the first and second extension operators are easily written down.

Skipping over the (lengthy) details one obtains from (8.9.)

\[
\frac{\delta}{\delta \tau} (B') = -\mu \frac{3\gamma + 1}{2} B' - \nu \{(3\gamma + 1) hB' + (3\gamma - 1) B\},
\]

\[\frac{\delta}{\delta \tau} (B'') = -\mu \frac{3(\gamma + 1)}{2} B'' - \nu \{3(\gamma + 1) hB'' + 6\gamma B'\},
\]

and from (8.1.)

\[
\frac{\delta}{\delta \tau} (\Phi_c) = \mu \left(\frac{1}{2} \Phi_c\right) + \nu (h\Phi_c),
\]

\[
\frac{\delta}{\delta \tau} (\Phi_h) = -\mu \left(\frac{1}{2} \Phi_h\right) + \nu (\Phi - h\Phi_h),
\]

\hspace{1cm} (10.28.)

Together with the increments of the second derivatives

\[
\frac{\delta}{\delta \tau} (\Phi_{tt}) = \mu \left(\frac{1}{2} \Phi_{tt}\right) + \nu (h\Phi_{tt}),
\]

\[
\frac{\delta}{\delta \tau} (\Phi_{ht}) = -\mu \left(\frac{1}{2} \Phi_{ht}\right) + \nu (\Phi_t - h\Phi_{ht}),
\]

\[
\frac{\delta}{\delta \tau} (\Phi_{hh}) = -\mu \left(\frac{3}{2} \Phi_{hh}\right) + \nu (3h\Phi_{hh}).
\]

From (10.27.) and (10.28.) we find for the first extensions
\[ V^*_2 F = V_2 F + \frac{1}{2} \phi t \frac{\partial F}{\partial \phi_t} - \frac{1}{2} \phi h \frac{\partial F}{\partial \phi_h} - \frac{3}{2} \beta B' \frac{\partial F}{\partial B'}, \]

\[ V^*_3 F = V_3 F + h \phi t \frac{\partial F}{\partial \phi_t} + (\phi - h \phi_t) \frac{\partial F}{\partial \phi_h} - \{ (3\gamma + 1) h B' + (3\gamma - 1) B \} \frac{\partial F}{\partial B'}, \tag{10.30} \]

and from (10.27.) and (10.29.) for the second extensions

\[ V^{**}_2 F = V^*_2 F + \frac{1}{2} \phi tt \frac{\partial F}{\partial \phi_{tt}} - \frac{1}{2} \phi ht \frac{\partial F}{\partial \phi_{ht}} - \frac{3}{2} \phi hh \frac{\partial F}{\partial \phi_{hh}} - \frac{3\gamma + 3}{2} B'' \frac{\partial F}{\partial B''}, \tag{10.31} \]

\[ V^{**}_3 F = V^*_3 F + h \phi tt \frac{\partial F}{\partial \phi_{tt}} + (\phi_t - h \phi_{tt}) \frac{\partial F}{\partial \phi_{ht}} - 3h \phi hh \frac{\partial F}{\partial \phi_{hh}} - \{ (3\gamma + 1) h B'' + 6\gamma B' \} \frac{\partial F}{\partial B''}. \]

Since according to Section 2 or (4.1.) the derivatives of \( \phi \) represent the parameters \( x, K, p, u, V \) it is clear that the extended operators (10.30.) and (10.31.) may be written in several alternative forms. For example (10.30.) may be rewritten in the form

\[ V^*_2 F = V_2 F + \frac{1}{2} K \frac{\partial F}{\partial K} - \frac{1}{2} x \frac{\partial F}{\partial x} - \frac{3}{2} \beta B' \frac{\partial F}{\partial B'}, \]

\[ V^*_3 F = V_3 F + h K \frac{\partial F}{\partial K} + (\phi - hx) \frac{\partial F}{\partial x} - \{ (3\gamma + 1) h B' + (3\gamma - 1) B \} \frac{\partial F}{\partial B'}, \tag{10.32} \]

while for (10.31.) similar forms are easily written down.
12. THE INVARIANCE OF THE EQUATION OF STANIUKOVICH AND CABANNEs

The generalized Staniukovich-transformation constructed in Section 6 was obtained for the equation (2.11.) for the function $\Phi$. In this section we consider the equation (2.1.) under the generalized Staniukovich-transformation. The equation (2.1.) reads

$$
\frac{\partial^2 \xi}{\partial h^2} + b(h) \frac{\partial^2}{\partial t^2} (p \gamma) = 0.
$$

(2.1.)

From (7.9.) one obtains

$$
b(h) = \frac{A^2}{a} \left(\frac{-a_{21}h_1 + a_{11}}{A}\right)^3 \frac{1}{\gamma} b_1(h_1),
$$

(12.1.)

and from (8.6.)

$$
-\frac{1}{p \gamma} = \frac{1}{\gamma} \left(\frac{-a_{21}h_1 + a_{11}}{A} \right) b_1(h_1) \frac{1}{\gamma}.
$$

(12.2.)

It follows that

$$
b(h) \frac{\partial^2}{\partial t^2} (p \gamma) = \frac{A^2}{a} \left(\frac{-a_{21}h_1 + a_{11}}{A}\right)^3 b_1(h_1) \frac{\partial^2}{\partial t^2} (p \gamma).
$$

(12.3.)

From the third relation in (11.5.) we have

$$
\frac{\partial^2 p_1}{\partial h_1^2} = \frac{a}{A} \left(\frac{-a_{21}h_1 + a_{11}}{A}\right)^3 + \frac{3}{\partial^2 h^2},
$$

(12.4.)

and solving for $\frac{\partial^2 p}{\partial h^2}$ we obtain

$$
\frac{\partial^2 p}{\partial h^2} = \frac{a}{A} \left(\frac{-a_{21}h_1 + a_{11}}{A}\right)^3 \frac{\partial^2 p_1}{\partial h_1^2},
$$

(12.5.)

where we used (7.8.).

Substitution of (12.3.) and (12.5.) into (2.1.) then yields

$$
\frac{A^2}{a} \left(\frac{-a_{21}h_1 + a_{11}}{A}\right)^3 \left\{ \frac{\partial^2 p_1}{\partial h_1^2} + b_1(h_1) \frac{\partial^2}{\partial t^2} (p \gamma) \right\} = 0.
$$

(12.6.)

Since the factors outside the braces are in general different from zero, it follows that the expression in braces has to vanish and so we have again to satisfy (2.1.). Clearly the equation (2.1.) remains invariant under the generalized Staniukovich-transformation.

Since the finite transformations in a group can be considered as the repeated application of infinitesimal one's it follows that the infinitesimal transformation for the entire second term in (2.1.) is the same as the infinitesimal
transformation of $\frac{\partial^2 P}{\partial h^2}$. The latter appears as a term in the 2 operators (11.8.) and the verification of the invariance of eq. (2.1.) by means of the infinitesimal operators becomes trivial.

If one desires to employ the infinitesimal operators obtained so far one may expand (2.1.) in the form

$$\frac{\partial^2 P}{\partial h^2} - \frac{1}{Y} \frac{\partial P}{\partial t} = 0,$$

and apply the infinitesimal operators

$$V_2^{***} F = 2^y - 3^y - 1 B \frac{\partial F}{\partial B} + \frac{\partial t}{\partial t} + \frac{1}{2} \frac{\partial F}{\partial p} + \frac{1}{2} \frac{\partial F}{\partial p_t} + \frac{1}{2} \frac{\partial F}{\partial p_{tt}} - \frac{3}{2} \frac{\partial F}{\partial p_{ph}} - \frac{3}{2} \frac{\partial F}{\partial p_{hh}},$$

$$V_3^{***} F = -(3^y - 1) hB \frac{\partial F}{\partial B} + hF \frac{\partial F}{\partial p} + hF \frac{\partial F}{\partial p_t} + hF \frac{\partial F}{\partial p_{tt}} - 3hF \frac{\partial F}{\partial p_{ph}} - 3hF \frac{\partial F}{\partial p_{hh}},$$

which are obtained from (11.8.) upon omitting those terms that do not appear in (12.7.). It may be noticed from (12.8.) and (12.9.) that for these forms we have

$$V_3^{***} F = 2hV_2^{***} F.$$

Applying (12.8.) and (12.9.) to (12.7.) the form (12.7.) reappears and since the R.H.S. in (12.7.) is zero the same applies to the infinitesimal increment obtained from the infinitesimal operator.
13. THE INVARINCE OF THE EQUATION FOR $\Phi$

The generalized Staniukovich-transformation was obtained in Section 6 by insisting upon the invariance of the equation (2.11.) for $\Phi$. In this section we verify that for

$$F = \Phi_{tt} \Phi_{hh}^\gamma + B(h) = 0,$$  \hspace{1cm} (13.1.)

also the infinitesimal operators applied to $F$ yield zero as result. The operators required are

$$V_1 F = \frac{\partial F}{\partial h} = 0,$$

$$V_2^{**} F = -\frac{3\gamma - 1}{2} B \frac{\partial F}{\partial B} + \frac{1}{2} \Phi_{tt} \Phi_{tt} \Phi_{tt} - \frac{3}{2} \Phi_{tt} \Phi_{hh} \Phi_{hh}^\gamma = 0,$$  \hspace{1cm} (13.2.)

$$V_3^{**} F = -(3\gamma - 1) hB \frac{\partial F}{\partial B} + h\Phi_{tt} \Phi_{tt} \Phi_{tt} \Phi_{tt} - 3h\Phi_{tt} \Phi_{hh} \Phi_{hh}^\gamma = 0,$$

where similar to (12.8.) and (12.9.) only those terms in (10.31.) are retained, which appear in (13.1.). Also in (13.2.)

$$V_3^{**} F = 2hV_2^{**} F.$$  \hspace{1cm} (13.3.)

Application of (13.2.) to (13.1.) yields

$$\frac{\partial F}{\partial h} = 0, \quad \frac{\partial F}{\partial B} = 1, \quad \frac{\partial F}{\partial \Phi_{tt}} = \Phi_{hh}^\gamma, \quad \frac{\partial F}{\partial \Phi_{hh}^\gamma} = \gamma \Phi_{tt} \Phi_{hh}^{\gamma-1},$$  \hspace{1cm} (13.4.)

and so

$$V_2^{**} F = -\frac{3\gamma - 1}{2} \{B + \Phi_{tt} \Phi_{tt}^\gamma\} = -\frac{3\gamma - 1}{2} F.$$  \hspace{1cm} (13.5.)

Since from (13.1.) we have $F = 0$ it is clear from (13.5.) that $V_2^{**} F = 0$, and from (13.3.) and (13.1.) that $V_3^{**} F$ and $V_1 F = 0$.

This verifies by means of the infinitesimal operators of the group that equation (2.11.) remains invariant under the group.

In the following two sections we consider the behaviour of two other equations under the group. The first one is the equation for $M(p,h)$ considered originally by Martin and Ludford [7], the other equation is for the function $t(L,u)$, which allows the linear transformation (8.5.) to be applied.
14. THE EQUATION OF MARTIN-LUDFORD FOR M(p,h)

The equation considered by Martin and Ludford in ref. [7] has the form

\[
\frac{\partial^2 M}{\partial p^2} \frac{\partial^2 M}{\partial h^2} - \left( \frac{\partial^2 M}{\partial h \partial p} \right)^2 + \frac{v^2}{a^2} = 0 ,
\]

(14.1)

with \(M(p,h)\) representing a Legendre transformation of \(K(h,t)\) defined by

\[
M(h,p) = pt - K(h,t)
\]

(14.2)

and

\[
\frac{v^2}{a^2} = \frac{1}{B(h)Y} - \frac{\gamma + 1}{Y} .
\]

(14.3)

From the transformation (6.19.) and the transformations deduced in Section 8 it follows that the generalized Staniukovich-transformation yields for the parameters appearing here

\[
h_1 = \frac{a_{11}h + a_{12}}{a_{21}h + a_{22}}, \quad p_1 = \frac{a}{a_{21}h + a_{22}},
\]

(14.4)

\[
M_1 = \frac{\alpha}{a_{21}h + a_{22}} M, \quad B_1 = \frac{\alpha^{\gamma + 1}}{\alpha^{2\gamma}} (a_{21}h + a_{22})^{\gamma - 1} B .
\]

To investigate the application of (14.4.) to the equation (14.1.) we start from the equation in terms of \(M_1(p_1, h_1)\) and \(v_1^2/a_1^2\). In contrast to the transformations applied so far, both the independent variables \(h_1\) and \(p_1\) transform; up to now \(t\) was an independent variable not affected by the transformation.

It is straightforward then to obtain the relations

\[
\frac{\partial M_1}{\partial h_1} = \frac{\alpha}{A} \left\{ \left( a_{21}h + a_{22} \right) \frac{\partial M_1}{\partial h} + a_{21}p \frac{\partial M_1}{\partial p} - a_{21}M \right\} ,
\]

\[
\frac{\partial M_1}{\partial p_1} = \frac{\partial M}{\partial p} - \frac{\partial^2 M_1}{\partial p^2} = \frac{a_{21}h + a_{22}}{\alpha} \frac{\partial M}{\partial p} ,
\]

(14.5)

\[
\frac{\partial^2 M_1}{\partial h_1 \partial p_1} = \frac{a_{21}h + a_{22}}{A} \left\{ \left( a_{21}h + a_{22} \right) \frac{\partial^2 M_1}{\partial p \partial h} + a_{21}p \frac{\partial^2 M_1}{\partial p^2} \right\} ,
\]

\[
\frac{\partial^2 M}{\partial h_1^2} = \frac{\alpha}{A^2} \left( a_{21}h + a_{22} \right) \left\{ \left( a_{21}h + a_{22} \right) \frac{\partial^2 M}{\partial h^2} + \right.
\]

\[
+ 2a_{21}p \left( a_{21}h + a_{22} \right) \frac{\partial^2 M}{\partial p \partial h} + a_{21}^2 \frac{\partial^2 M}{\partial p^2} \} .
\]
It follows then easily that
\[
\frac{\partial^2 M_1}{\partial h_1^2} \frac{\partial^2 M_1}{\partial p_1^2} \left( \frac{\partial^2 M_1}{\partial p_1 \partial h_1} \right)^2 + \frac{1}{Y} \frac{1}{B_1} \frac{1}{P_1} \frac{1}{Y} = \frac{(a_{11} h + a_{22})^4}{A^2} \left\{ \frac{\partial^2 M}{\partial h^2} \frac{\partial^2 M}{\partial p^2} - \left( \frac{\partial^2 M}{\partial p \partial h} \right)^2 + \frac{1}{Y} \frac{1}{B} \frac{1}{p} \frac{1}{Y} \right\},
\]

(14.6.)

and so the original equation is reproduced. The invariance of (14.1.) under the transformation is thereby established.
15. THE EQUATION FOR $t(L,u)$ AND ITS BEHAVIOUR UNDER THE TRANSFORMATION

In ref. [3] it was found that the Legendre transformation $L(u,t)$ of $K(h,t)$ has to satisfy the partial differential equation

$$\frac{1}{B(u^t)} \left\{ L_{uu} - \frac{L^2}{L_t} \right\} + \frac{\gamma + 1}{L_t^\gamma} = 0,$$

(15.1.)

with subscripts denoting partial derivatives. In Appendix 1 of ref. [3] an equation for $u(L,t)$ was constructed which had been considered by Naylor [9]. Here we apply the same method to construct an equation for $t(L,u)$.

The Legendre transformation $L(u,t)$ appears first in (4.6.) and its differential is easily found to be

$$dL = hdu + pdt, \quad h = L_u, \quad p = L_t.$$

(15.2.)

Considering $t$ as function of $L$ and $u$, (15.2.) yields

$$dt = \frac{1}{p} dL - \frac{h}{p} du, \quad t_L = \frac{1}{p}, \quad t_u = -\frac{h}{p}.$$

(15.3.)

Comparison of (15.2.) and (15.3.) then yields

$$L_u = -\frac{t_u}{t_L} = h, \quad L_t = \frac{1}{t_L} = p.$$

(15.4.)

Next we write

$$dL_L = L_{tu} du + L_{tt} dt = -\frac{1}{2} dt_L = \frac{1}{t_L^2} (t_{LL} dL + t_{Lu} du) =$$

$$= \frac{t_{LL}}{t_L^2} dt + \frac{t_{uLL} + t_{tL} t_{Lu}}{t_L^2} du.$$

(15.5.)

In entirely the same way we then take

$$dL_u = L_{uu} du + L_{ut} dt = -d\left(\frac{t_u}{t_L}\right) = -\frac{t_{L} dt_u - t_u dt_L}{t_L^2} =$$

$$= \frac{(t_u t_{LL} - t_L t_{uL}) dL + (t_u t_{Lu} - t_L t_{tu}) du}{t_L^2} =$$

$$= \frac{1}{t_L^3} (t_u t_{LL} - t_L t_{uL}) dt + \frac{du}{t_L^3} (-t_L^2 t_{uu} + 2t_u t_L t_{uL} - t_u^2 t_{LL}).$$

(15.6.)

From (15.5.) and (15.6.) it follows
\[ L_{tt} = -\frac{t_{LL}}{3}, \quad L_{tu} = \frac{t_u t_{LL} - t_L t_{Lu}}{t_L^3}, \quad L_{uu} = -\frac{t_L^2 t_{uu} + 2t_u t_L t_{UL} - t_u^2 t_{LL}}{t_L^3}, \] (15.7)

and furthermore

\[ L_{tt} \cdot L_{uu} - L_{ut}^2 = \frac{1}{t_L} \left( t_{uu} t_{LL} - t_{uL}^2 \right). \] (15.8)

Employing also the expressions in (15.4.) it is then easily verified that (15.1.) transforms into

\[ B \left( -\frac{t_u}{t_L} \right)^Y \left( t_{uu} t_{LL} - t_{uL}^2 \right) + \gamma \frac{3Y - 1}{t_L} Y^Y = 0, \] (15.9)

which is the equation sought.

We now subject (15.9.) to the generalized Staniukovich-transformation (6.19.) and its further consequences obtained in Section 8. Since \( t \) is left unchanged during this transformation, it is in case of (15.9.) a matter of changing the independent variables and the function \( B \). The variables \( L \) and \( u \) transform according to (8.5.) and (8.8.). One easily finds for the first derivatives of \( t \)

\[ \frac{\partial t}{\partial L} = \frac{\alpha}{A} \left( a_{11} \frac{\partial t}{\partial L_1} + a_{21} \frac{\partial t}{\partial u_1} \right), \]
\[ \frac{\partial t}{\partial u} = \frac{\alpha}{A} \left( a_{12} \frac{\partial t}{\partial L_1} + a_{22} \frac{\partial t}{\partial u_1} \right), \] (15.10)

and for the second derivatives

\[ \frac{\partial^2 t}{\partial u^2} = \frac{\alpha^2}{A^2} \left( a_{12} \frac{\partial^2 t}{\partial L_1^2} + 2a_{12} a_{22} \frac{\partial^2 t}{\partial L_1 \partial u_1} + a_{22} \frac{\partial^2 t}{\partial u_1^2} \right), \]
\[ \frac{\partial^2 t}{\partial L^2} = \frac{\alpha^2}{A^2} \left( a_{21} \frac{\partial^2 t}{\partial L_1^2} + 2a_{21} a_{22} \frac{\partial^2 t}{\partial L_1 \partial u_1} + a_{22} \frac{\partial^2 t}{\partial u_1^2} \right), \] (15.11)
\[ \frac{\partial^2 t}{\partial L \partial u} = \frac{\alpha^2}{A^2} \left( a_{11} a_{12} \frac{\partial^2 t}{\partial L_1^2} + (a_{11} a_{22} + a_{12} a_{21}) \frac{\partial^2 t}{\partial L_1 \partial u_1} + a_{21} a_{22} \frac{\partial^2 t}{\partial u_1^2} \right). \]

From (15.11.) one then derives

\[ \frac{\partial^2 t}{\partial u^2} \frac{\partial^2 t}{\partial L^2} - \left( \frac{\partial^2 t}{\partial L \partial u} \right)^2 = \frac{\alpha^4}{A^2} \left( \frac{\partial^2 t}{\partial L_1^2} \frac{\partial^2 t}{\partial u_1^2} - \left( \frac{\partial^2 t}{\partial L_1 \partial u_1} \right)^2 \right). \] (15.12)
To transform B we use the inverse transformation of (6.19.), as represented in (7.9.). This yields
\[ B(h) = \frac{A^{2\gamma}}{a^{\gamma+1}} \left( \frac{-a_{21}h_1 + a_{11}}{A} \right)^{3\gamma-1} B_1(h_1), \]  
\[ (15.13.) \]
and upon using (15.4.)
\[ B \left( -\frac{t_u}{t_L} \right) = \frac{A^{2\gamma}}{a^{\gamma+1}} \left( \frac{a_{21}t_{u1} + a_{11}t_{L1}}{At_{L1}} \right)^{3\gamma-1} B_1 \left( -\frac{t_{u1}}{t_{L1}} \right), \]  
\[ (15.14.) \]
where we returned to the subscript notation for partial derivatives. Taking also \( L_t \) from (15.10.) it is then easily verified that
\[ B \left( -\frac{t_u}{t_L} \right) \frac{1}{t_L^{-\gamma}} = \frac{a^4}{A^2} \frac{1}{(t_{L1})^{-\gamma}} B_1 \left( -\frac{t_{u1}}{t_{L1}} \right)^{-\gamma}, \]  
\[ (15.15.) \]
Combining (15.12.) and (15.15.) by substituting into (15.9.) one verifies that (15.9.) transforms into the same equation multiplied with a factor \( \frac{a^4}{A^2} \).
It follows that (15.9.) also is invariant under the generalized Staniukovich transformation.
16. CONCLUSIONS

The Staniukovich-transformation is originally formulated to transform \( p(h,t) \) into \( p'_1(h'_1,t) \) where the differential equations for \( p \) and \( p'_1 \) have the same form. It is shown that a similar transformation applies to \( \Phi(h,t) \) which is the basis of the general theory of ref. [3]. Since the flow parameters are expressed as derivatives of \( \Phi \), transformation rules for all the flow parameters are obtained.

Next the Staniukovich-transformation is generalized and takes the form of a 3 parameter continuous transformation group. In particular a homentropic flow and the flow of IMS-gas can be embedded in this group.

The different infinitesimal operators of the group are constructed and the invariance of some differential equations under the transformation group are verified.
REFERENCES


