Fluid motions generated by the injection of an electric current

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A.J.M. Jansen

TU Delft
Faculty of Aerospace Engineering
Delft University of Technology
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ABSTRACT

In this thesis fluid motions induced by a Lorentz force due to the injection of an electric current and its associated magnetic field are studied.
For the analytical examination some simple configurations are considered: the semi-infinite point electrode configuration, also called the prototype model, and the semi-infinite disk electrode configuration.
The analysis shows that the behaviour of the fluid motion in the neighbourhood of an electrode of finite size is regular.
At large radial distance from the electrode and in the prototype model, the usual similarity method is not applicable for larger values of the hydrodynamic Reynolds number.
The derivation of the general inviscid and viscous solutions clarifies the inconsistencies appearing in the far field.
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LIST OF SYMBOLS

\(a\) \hspace{1cm} \text{radius of the disk electrode, see fig. 5.1.}

\(a_n\) \hspace{1cm} \text{coefficients used in analytical expressions of } g(c)

\(a_m\) \hspace{1cm} \text{constant used in (2.70)}

\(\text{Ai}(z)\) \hspace{1cm} \text{Airy function}

\(A_n(z)\) \hspace{1cm} \text{coefficients used in (D.8) - (D.14)}

\(h\) \hspace{1cm} \text{semidistance between the electrodes, see fig. 1.2.}

\(b_n\) \hspace{1cm} \text{coefficients used in analytical expressions of } f(c)

\(B\) \hspace{1cm} \text{magnetic field}

\(\text{Bi}(z)\) \hspace{1cm} \text{Airy function}

\(B_n(z)\) \hspace{1cm} \text{coefficients used in (D.8) - (D.14)}

\(c\) \hspace{1cm} = \cos(\theta)

\(\chi\) \hspace{1cm} \text{velocity of light}

\(c_0\) \hspace{1cm} = \cos(\theta_0)

\(d_n\) \hspace{1cm} \text{coefficients used in the analytical expression of } G_n(c), \text{ see (4.65) - (4.69)}

\(E\) \hspace{1cm} \text{electric field}

\(f(c)\) \hspace{1cm} \text{0-dependent part of the magnetic field component } B_\phi \text{ defined in (2.15)}

\(f_n(c)\) \hspace{1cm} \text{coefficients of series expansions of } f(c)

\(f_s\) \hspace{1cm} \text{function defined in (2.27)}

\(f_{se}\) \hspace{1cm} \text{function defined in (2.27)}

\(f_{sh}\) \hspace{1cm} \text{function defined in (2.27)}

\(F_L\) \hspace{1cm} \text{= } J \times B \text{ Lorentz force}

\(\mathcal{F}_{L,0}\) \hspace{1cm} \text{total Lorentz force defined in (5.29)}

\(\mathcal{F}_{L,a}\) \hspace{1cm} \text{total Lorentz force defined in (5.30)}

\(\mathcal{F}_{i,z}\) \hspace{1cm} \text{total inertia force defined in (5.135)}

\(\mathcal{F}_{L,z}\) \hspace{1cm} \text{total Lorentz force defined in (5.137)}
\( \vec{F}_{p,z} \) total pressure force defined in (5.136)

\( \vec{F}_{\eta,z} \) total viscous force defined in (5.138)

\( F_n \) polylogarithmic integrals defined in (4.94)

\( g(c) \) \( \theta \)-dependent part of the Stokes stream function defined in (2.14)

\( g_n(c) \) coefficients of series expansions of \( g(c) \)

\( g_s \) function defined in (2.28)

\( g_{se} \) function defined in (2.28)

\( g_{sh} \) function defined in (2.28)

\( G_\eta(c) \) function representing the effect of the Lorentz force defined in (2.51)

\( h \) function defined in (3.12)

\( \bar{a} \) unit vector, \( \bar{a} = x_1, x_2, r, \theta, \varphi, s, z, \eta, \zeta \)

\( I \) electric current defined in (5.22)

\( I_o \) total electric current supplied into the fluid through the electrode

\( I_{o,\text{max}} \) maximum value of \( I_o \), see table 4.1.

\( J \) electric current density

\( k \) interface introduced in fig. 1.1.

\( K_b \) effective magnetic Reynolds number defined in (2.16)

\( K_{b,\text{max}} \) maximum value of \( K_b \), see (3.30)

\( K_s \) parameter given in (2.30)

\( K_{se} \) parameter given in (2.30)

\( K_{sh} \) parameter given in (2.29)

\( K_{s,\text{max}} \) maximum value of \( K_s \), see table 4.1.

\( K_\eta \) inverse effective hydrodynamic Reynolds number defined in (2.17)

\( K_{\eta,\text{min}} \) minimum value of \( K_\eta \) defined in (4.90)

\( \bar{R}_{\eta,\text{min}}(m) \) successive approximations of \( K_{\eta,\text{min}} \) defined in (C.14), see appendix C
m  integer
M  Hartmann number
n  integer
N  interaction parameter
p(c)  pressure in the fluid
p_n(c)  coefficients of series expansions of p(c)
P_∞  uniform reference pressure at infinity, see (2.64)
P  integration constant in (2.45)
Pr  magnetic Prandtl number
p_n^a_b(x)  Jacobi polynomials, see (5.6) - (5.11)
Q  constant of integration, see (2.45)
r  radial distance from the origin, see fig. 2.1.
r  = r/a  dimensionless radial distance from the electrode, see (5.18)
r, θ, φ  spherical polar co-ordinate system
R  auxiliary function, see (5.7)
Re  hydrodynamic Reynolds number
Re,max  maximum value of Re, see table 4.1.
Rm  magnetic Reynolds number
R_s,bs  parameter given in (2.30)
s  = r\sqrt{1-c^2} = a \cosh(\eta) \sin(\zeta) , distance from the axis of symmetry
S  integration constant in (2.45)
T_{kin}  kinetic energy, see (3.66)
t  integration variable
T_n(c,c_o)  auxiliary function used in appendices A and B
U  electrical potential
U_∞  electrical potential at large radial distance, see (2.68)
u(c)  transformation function introduced in (4.77)
u_n  coefficients of series expansions of Airy functions, see (D.43) - (D.50)
\( v \)  velocity of the fluid

\( v(z) \) auxiliary function given in (D.2)

\( v_n \) coefficients of series expansions of Airy functions, see (D.43) - (D.50)

\( V_n \) auxiliary function, see (4.91) and appendix C

\( \tilde{V}_n(c) \) auxiliary function introduced in appendix C, see (C.11) - (C.13)

\( x = 1 - 2c^2 \), see (5.6) - (5.17)

\( x_1, x_2, \phi \) axisymmetric curvilinear co-ordinate system, see fig. 1.1.

\( X_{se} \) function defined in (2.27)

\( y = \overline{r}^2, \overline{r}^{-2} \), see (5.6) - (5.17)

\( z = rc = a \sinh(\eta) \cos(\zeta) \), distance along the axis of symmetry

\( z(c) \) auxiliary function, see (4.119) and appendix D

\( z, \xi, \varphi \) cylindrical polar co-ordinate system

\( Z(\overline{r}, c) \) function defined in (5.44)

\( a_n(c) \) coefficients of series expansion of \( Z(\overline{r}, c) \), see (5.47) - (5.51)

\( \tilde{a}_n(c) \) coefficients of series expansion of \( Z(\overline{r}, c) \), see (5.52) - (5.56)

\( a_{se} \) parameter given in (2.29)

\( \gamma \) free parameter introduced in (6.4)

\( \gamma_n \) free parameter introduced in chapter 5

\( \gamma^* \) special value of the free parameter \( \gamma \) defined in (6.29)

\( \gamma_{\text{min}} \) special value of the free parameter \( \gamma \) defined in (6.71)

\( \Gamma(z) \) gamma function

\( \delta(z) \) auxiliary function given in (D.6)

\( \varepsilon \) permittivity of the fluid

\( \zeta_n \) coefficients of series expansion of \( G_\eta(c) \), see (C.8) - (C.10)

\( \eta \) dynamic viscosity of the fluid

\( \eta, \xi, \varphi \) oblate spheroidal co-ordinate system

\( \theta \) meridional angle form the axis of symmetry, see fig. 2.1.
\( \theta_e \) edge of the viscous jet flow in the neighbourhood of the axis of symmetry, see fig. 4.5.

\( \theta_o \) apex angle of the right circular cone, see fig. 2.1.

\( \lambda = \frac{\gamma}{\gamma^*} \) ratio defined in (6.59)

\( \lambda = K_\eta^{-1} \) parameter defined in appendix D, see (D.3)

\( \tilde{\lambda} \) coefficient given in (5.87)

\( \lambda_{nu} \) parameter given in (2.30)

\( \lambda_{nu,\text{max}} \) maximum value of \( \lambda_{nu} \), see table 4.1.

\( \mu \) permeability of the fluid

\( \bar{\mu} \) coefficient given in (5.88)

\( \nu_n \) coefficients of series expansion, see (D.32) - (D.35)

\( \xi(c) \) auxiliary function, see (4.118) and appendix D

\( \xi_n \) coefficients used in series expansion of \( G_\eta(c)/(1-c^2)^2 \), see (D.68) - (D.69)

\( \Pi \) constant, see (D.19) - (D.21)

\( \rho \) density of the fluid

\( \rho_e \) space charge density

\( \rho_n \) coefficients of series expansion, see (D.36) - (D.39)

\( \sigma \) electrical conductivity of the fluid

\( \tau_n \) coefficients of series expansion of \( G_\eta(c) \), see (D.26) - (D.29)

\( \varphi \) azimuthal angle about the axis of symmetry, see fig. 2.1.

\( \Phi \) potential function, see (4.113)

\( \Phi(c) \) function defined in (6.14)

\( \psi \) Stokes stream function, see (2.11) - (2.14)

\( \tilde{\psi} \) dimensionless Stokes stream function, see (5.33) - (5.35)

\( \Psi \) mass flux of the fluid flow

\( \omega(c) \) vorticity

\( \omega(c) \) azimuthal component of the vorticity
\( \omega_n(c) \) : coefficients of series expansions of \( \omega(c) \)

\( \Omega \) : constant, see (D.19), (D.20), (D.23)

\( T \) : constant, see (D.19), (D.20), (D.22)
1. INTRODUCTION

Here we introduce the problem to be investigated, we present a general survey and brief discussion of the literature in particular that part which concerns the theoretical side of the problem and we give an outline of the thesis.

1.1. General introduction

In this thesis we consider fluid motions induced by the injection of an electric current. When an electric current is supplied through an interface into an electrically conducting fluid, the electric current distribution and the self-magnetic field result in a Lorentz force. The electromagnetic force will always generate a pressure distribution and in most cases also a fluid motion in the medium.

The conditions under which a fluid motion will be generated and especially the mathematical examination and calculation of analytical solutions of the flow field are the main features of the investigations to be presented here.

In contrast with problems usually considered in magnetohydrodynamics (MHD), in the present study the presence of an external magnetic field or an imposed pressure field or fluid motion are excluded. Thus the Lorentz force, which drives the fluid motion, is caused only by the electric current distribution in the fluid and its associated magnetic field.

The generation of fluid motion due to the injection of an electric current is observed in practical applications, e.g. the weld pool and plasma arc in the electric welding process, arc furnaces, ion propulsion systems for space flight, mercury arc rectifiers, plasma devices and electrochemistry.

The phenomenon is sometimes an attendant effect; however in other cases such as the stirring of liquid metals in arc furnaces and the electrical propulsion for space flight with ion rockets, the effect is applied intentionally.

In these applications the electric current usually enters the electrically conducting fluid at a more-or-less concentrated place, viz. an electrode or a plasma arc at a free surface or an electrode located in a rigid non-conducting wall. The electric current passes through the medium and leaves the fluid via a second electrode at some distance.

It appears that due to the injection of electric current the fluid motion in the neighbourhood of each electrode has in general a jet-flow structure. The generated fluid motion consists of an incoming flow directed along the boundary wall towards the tip of the electrode, which in the neighbourhood of the
electrode is accelerated and turned off away from the wall in an outwards
directed jet-flow along the axis of symmetry of the electrode.
In order to gain a better understanding of the mathematical and physical
background of this phenomenon occurring in practical applications, we restrict
ourselves here to a theoretical and especially analytical study for rather
simple configurations.
Theoretical considerations and practical experiments have shown that it is the
Lorentz force, due to the electric current distribution in the fluid and the
associated magnetic field, which causes the generation of fluid motion. Moreover
it turns out that this effect is essentially three-dimensional so that a two-
dimensional analogue does not exist. In a three-dimensional configuration the
fluid cannot remain at rest.
It has to be remarked that in general two effects may play an important role in
this problem. Primarily the effect of the generation of fluid motion by the
Lorentz force \( \mathbf{F}_L = \mathbf{j} \times \mathbf{B} \). Secondly the backwards effect of the disturbance of
the electric current distribution in the fluid by the induced fluid motion, i.e.
the electromagnetic induction. In general the effect of the Lorentz force is
dominant; however in some cases the effect of the electromagnetic induction may
overrule the primary effect nearly completely.
To clarify these statements we consider the three-dimensional axisymmetric
configuration with a curvilinear co-ordinate system \((x_1,x_2,\phi)\) as given in figure
1.1, which shows in detail the situation in the neighbourhood of an electrode.
In this configuration an electric current \( I_0 \) is supplied into the fluid by a
cylindrical electrode located at an interface \( k \). The electric current diverges
in the medium and flows to a second electrode at some distance.
In the case of overall symmetry as assumed here the electric current distribu-
tion flows in the \((x_1,x_2)\) plane so that the associated magnetic field is purely
azimuthal. The resulting Lorentz force \( \mathbf{F}_L = \mathbf{j} \times \mathbf{B} \), situated in the meridional
plane is perpendicular to the lines of constant electric current density.
The Lorentz force, which has the maximum value at the intersection of the elec-
trode and the interface \( k \), decreases at larger distance from the electrode and
at smaller meridional angle \( \theta \), approaching zero at the axis of symmetry.
In view of the direction and strength distribution of the Lorentz force, viz.
away from the electrode and in the direction of the axis of symmetry, we expect
a fluid motion generated as sketched at the left-hand side of figure 1.1 and as
mentioned already; i.e an outwards jet-flow along the axis of symmetry of the
electrode.
It should be noticed that the direction of the Lorentz force does not change when the electric current is not injected but extracted from the fluid. Then the direction of both the electric current density $\mathbf{J}$ and of the magnetic field $\mathbf{B}$ reverse, but not the direction of the Lorentz force $\mathbf{F}_L = \mathbf{J} \times \mathbf{B}$. Thus the effect also occurs with alternating current. This observation implies the conclusion that in a three-dimensional configuration the occurrence of an outwards jet-flow in principle may appear at every electrode.

The generated fluid motion in its turn induces an electric field $\mathbf{E} = \mathbf{v} \times \mathbf{B}$, which may alter the electric current distribution in the fluid. The left-hand side of the figure shows that the electromagnetic induction can only affect the electric current flow in the middle region of $\theta$ of the fluid domain; but not in the neighbourhood of the axis of symmetry and of the interface.

The behaviour of the fluid-flow and electromagnetic field quantities, in the model described here, is in general governed by two parameters, viz. the hydrodynamic and magnetic Reynolds numbers. The hydrodynamic Reynolds number $Re$ determines the viscous boundary layer at the interface and the viscous spreading of the jet-flow along the axis of symmetry. The magnetic Reynolds number $Rm$ is associated with the influence of the electromagnetic induction and with the appearance of electromagnetic boundary layers at
the interface and at the axis of symmetry.

Boundary layers of small thickness usually occur at larger magnitudes of respectively Re and Rm.

Also it turns out that the Reynolds numbers possess a typical local character throughout the flow field, viz. a relatively low Re and Rm behaviour in the neighbourhood of the electrode and a relatively high Re and Rm behaviour at larger distance from the electrode.

An analytical calculation of the general problem at arbitrary values of Re and Rm turns out to lead to insurmountable difficulties.

Therefore, as a first step to tackle this interesting problem, we consider in this thesis only the dominant effect of the generation of inviscid and viscous fluid motions induced by a Lorentz force caused by the injection of an electric current and its associated magnetic field.

In some cases only the weak perturbing effect of the fluid motion upon the electric current distribution and the mutual interaction, as it occurs at small Rm, is included in the calculations.

Hence we consider fluid motion at arbitrary value of Re and at small Rm. In terms of practical application this restriction applies to liquid metal situations. For example: the motion of molten metal in the weld pool in the electric welding process and the stirring of liquid metals in arc furnaces.

Experiments using liquid metals have confirmed the conclusion that the electromagnetic force is the primary cause of the fluid motion. These experiments have demonstrated also that contributions due to the thermal convection and to the influence of the arc plasma jet, as applied in electric welding, are small by comparison.

The Lorentz force induces in general a pressure distribution and a flow field in the fluid. Nevertheless it turns out that especially the rotationality of the Lorentz force is the essential source of the non-zero three-dimensional fluid motion. This statement is clarified by the following considerations.

The expressions of the Lorentz force and of its rotationality, expressed in circular-cylinder co-ordinates \( z, s, \phi \) yield

\[
\frac{J \times B}{- \frac{B^2}{\mu s}} \text{grad} (s) - \frac{1}{2\mu} \text{grad} \left( \frac{B^2}{\phi} \right),
\]

(1.1)

\[
\text{curl} \left( \frac{J \times B}{s} \frac{B}{\phi} \right) = \frac{2J \frac{B}{s}}{s} \frac{1}{\phi},
\]

(1.2)

where

\[
J_s = \frac{1}{s} \cdot \text{grad} (s).
\]

(1.3)
In these expressions $J$ is the electric current density, $B = B_{\varphi} = B_{\varphi}(\varphi)$ is the magnetic field, $\mu$ is the permeability, $s$ is the distance from the axis of symmetry, $z$ is the distance measured along the axis of symmetry, $\varphi$ is the azimuthal co-ordinate and $x_1, x_2$ are the meridional co-ordinates applied in figure 1.1.

It can easily be verified that in the case of a two-dimensional configuration the first term of the right-hand side of (1.1) vanishes, so that the right-hand side of (1.2) becomes identical to the zero vector in that case. This explanation clearly demonstrates the essentially three-dimensional character of the problem.

The above expressions also show that the normal component of the electric current distribution $J_s$ is responsible for the appearance of a fluid motion.

To illustrate this phenomenon in detail we consider the flow patterns of the electric current distribution in three different three-dimensional axisymmetric electrode-pair configurations at a mutual distance $2b$, see figure 1.2. They are: (a) two infinite electrode-plates, (b) two cylindrical electrodes of radius $a$, located in rigid insulating walls, (c) two curved electrodes being bodies of revolution. The electrodes carry a constant electric current $I_o$ through the medium.

\[\begin{array}{ccc}
\text{(a)} & \text{(b)} & \text{(c)} \\
\end{array}\]

Figure 1.2.
In the configuration 1.2.a, \( J_s \) is identical to zero throughout the entire fluid because the flat electrodes extend to infinity. In this three-dimensional model only a pressure distribution is generated but not a fluid motion.

In the configuration 1.2.b, \( J_s \) is non-zero due to the end-effect of the disk electrodes, so that in this model a fluid motion will be generated.

Also a fluid motion will be induced in the configuration 1.2.c, where \( J_s \) is unequal to zero due to the curvature of the surface of the electrodes.

In view of the direction of the Lorentz force, acting at each electrode as discussed before, we expect that in symmetric situations of figure 1.2 the main fluid-flow pattern will consist of a double circulation, as sketched in figure 1.2.b. Namely incoming flows from all sides along the solid walls and/or the surfaces of the electrodes, which in the neighbourhood of the electrodes are turned off and accelerated into outwards jet-flows, escaping into the regions of weaker Lorentz force and of weaker rotationality of the electromagnetic force.

Examination of the semi-infinite disk electrode configuration, see chapter 5, and of a configuration like figure 1.2.b with \( a = 0^+ \) indicates that the velocities in the flow field are respectively proportional to \( a^{-1} \) and \( b^{-1} \), where \( a \) is the radius of the cylindrical electrode and \( b \) is the semi-distance between the electrodes. Therefore in the case of electrodes of unequal dimensions or curvature, having an asymmetric electric current distribution, the fluid circulation caused by the smaller and/or more curved electrode will dominate or even suppress the circulation of the larger and/or less curved electrode.

As a consequence in semi-infinite single electrode configurations and in container configurations in most cases only a pure axisymmetric rotation of the fluid, being directed away from the smallest electrode, has been found in analytical and numerical calculations and in practical experiments. The calculated and observed fluid motions in container configurations also show that the dimension of the electrode is of greater influence than the curvature of the electrode surface.

As pointed out before, the main feature of the investigations to be presented here is the analytical calculation and examination of the generation of fluid motion due to the injection of an electric current. However, mathematically speaking, even symmetric models of configurations as shown in figure 1.2 are too complicated for that purpose.

Therefore semi-infinite configurations will be considered, because they are the only configurations which allow a rather simple mathematical treatment. In fact these models represent the situation in the neighbourhood of one electrode in detail, whereas the second electrode is thought to be situated at very large
distance.

From a mathematical point of view it appears that there is no essential fundamental difference when the interface is a rigid wall or a free surface. Since mathematically it can be treated more generally, only the case of an electrode located in a non-conducting solid wall will be studied here.

In this thesis two models of configuration will be considered:

The semi-infinite point electrode configuration
In this configuration the electric current is supplied into the fluid by a point electrode located at the apex of a non-conducting right circular cone of arbitrary vertical angle. This model is a generalization of the flat-wall point electrode model studied by several authors.

The semi-infinite disk electrode configuration
In this model the electric current is injected into the fluid via a cylindrical electrode of non-zero radius a located in an insulating flat wall.

In both configurations the fluid occupies the entire space outside the electrode-wall-boundary. Moreover in the special case of a flat wall the behaviour of the respective field quantities in both configurations coincides at large radial distance from the electrode.

Since it appears unnecessary to include viscosity in order to achieve a steady state, for simplicity and in order to render the problems more tractable mathematically, analytical solutions of steady inviscid and viscous fluid motions will be considered.

To that purpose in the point electrode configuration a similarity method will be applied, and in the disk electrode configuration series expansions in positive and negative powers of the radial distance r to describe respectively the near and far field of the fluid domain.

The point electrode configuration, which is the simplest model, leads to some virtual inconsistencies. By consideration of the more complicated disk electrode configuration these difficulties will be clarified and resolved for the greater part.

Much attention has been paid in the literature to the problem of fluid motions induced by an electric current and its associated magnetic field. In order to show the relation between our studies and the work done by other authors, in the next section a survey and a brief discussion of the literature concerning this interesting problem will be presented.
1.2. General review of the literature

In view of the theoretical treatment carried out in this thesis we start with a chronological survey and brief discussion of theoretical publications. In order to show the correlation between theory and practice the results and observations obtained from experiments will be discussed. Further some other applications, which are beyond the present examinations, will only be mentioned.

The theoretical study of fluid motions induced by the injection of an electric current and its associated magnetic field has attracted the attention of many authors.

Zhigulev (1960a) pointed out that the Lorentz force is the essential cause of the fluid motion. He introduced a similarity method which is applicable in the point electrode configuration. Unfortunately the paper does not contain further calculations of the inviscid and viscous problems. The author expressed the opinion that at large value of the magnetic Reynolds number $Rm$ the electric current flow would be wholly confined to the axis of symmetry. This assumption is not correct and is clarified here by considering the semi-infinite disk electrode configuration of $Rm \to \infty$.

Zhigulev (1960b) repeated the similarity method applicable in the point electrode model in relation with other local solutions applied in magneto-hydrodynamics.

Lundquist (1969) examined the slow viscous solution where at small value of the hydrodynamic Reynolds number the effect of the inertia force may be neglected in the Navier-Stokes equation. Analytical solutions of the solid flat wall problem and of the free surface problem are determined at small values of $Re$ and $Rm$. Consideration of the condition $Rm \ll 1$ leads to a maximum admissible electric current which exceeds the values obtained by other authors.

Shercliff (1970) was the first to explain in a clear and excellent way the physical and mathematical background of the problem of fluid motions induced by the injection of an electric current and its associated magnetic field. He calculated the analytical solution of the inviscid fluid motion in the semi-infinite point electrode configuration with a flat wall for $Rm \ll 1$. Further the weak perturbing effect of the fluid motion upon the electric current distribution at low value of the magnetic Reynolds number was studied. Shercliff also suggested the appearance of local behaviour of the Reynolds numbers which is confirmed in the disk electrode configuration to be considered in chapter 5 of this thesis.

Sozou (1971a) examined the viscous fluid motion in the semi-infinite point
electrode configuration with a flat wall at arbitrary value of the hydrodynamic Reynolds number and at low Rm. It turns out that in the general case where the effect of the inertia force is included in the Navier-Stokes equation the solution of the viscous fluid motion breaks down at a relatively small value of the viscous Reynolds number. His explanation of the behaviour of the flow field at and above the critical value of Re differs from our's given in chapter 4 of the thesis. Sozou's criticism of Shercliff's inviscid solution is without foundation. It cannot be expected that a strongly singular viscous solution in the limit Re → ∞ can be compared or even matched to an inviscid solution with a relatively weak singularity in the flow field at the axis of symmetry.

Sozou (1971b) studied the combined problem of viscous fluid motion induced by the injection of an electric current and by a point source of momentum. As function of the strength of the momentum source again at relatively low maximum value of the hydrodynamic Reynolds number breakdown of the fluid motion was found.

Narain & Uberoi (1971) investigated the viscous fluid motion, inside a semi-infinite conical fluid region with a solid boundary, induced by the injection of an electric current from a point electrode at the apex of the cone. The configuration is a simple model of the electric plasma arc as observed in arc welding and plasma devices. The authors derive the analytical solutions of the slow viscous fluid motion and of the pressure distribution at small value of the hydrodynamic Reynolds number and for arbitrary value of the apex-angle. Inclusion of the effect of the inertia force in the Navier-Stokes equation leads to the usual breakdown of the viscous fluid motion at relatively low critical value of Re.

Sozou (1972) considered the viscous fluid motion in a semi-infinite configuration due to the injection of an electric current supplied by a cylindrical electrode of finite non-zero radius located in a non-conducting flat wall. The first three terms of a series expansion of the Stokes stream function, being valid in the so-called far field of the fluid domain were calculated. Again the governing basic solution breaks down at relatively low value of Re.

Sozou & English (1972) studied the viscous fluid motion due to an electric current and to a point source of momentum in the semi-infinite point electrode configuration with a flat wall. Of particular interest is the examination of the effect of the electromagnetic induction at larger values of the magnetic Reynolds number Rm. The calculations indicate that the maximum critical value of the hydrodynamic Reynolds number increases at larger values of the magnetic Reynolds number.

Narain & Uberoi (1973) examined the inviscid flow in a conical fluid domain
induced by an electric current supplied through a point electrode located at the apex of the cone. Exact solutions of the inviscid fluid motion and of the pressure field are presented for $Rm \ll 1$. The authors reconsidered the solution of the viscous fluid motion which includes the effect of the inertia force and suggested that bounded solutions at large values of the hydrodynamic Reynolds number can be obtained only by a reformulation of the viscous problem. Sozou (1974) considered the semi-infinite point electrode configuration with a flat wall where the electric current is supplied in a conical region around the axis of symmetry. This configuration is a primitive model of the arc appearing in arc welding and plasma apparatus. The calculations show that the viscous fluid motion breaks down at a critical upper bound of the hydrodynamic Reynolds number. Sozou's opinion that the breakdown of the viscous fluid motion is caused by the inclusion of the nonlinear inertia terms in the Navier-Stokes equation is debatable.

Sozou & Pickering (1975) investigated the transient development of the viscous fluid motion to the steady state solution in the semi-infinite point electrode configuration with a flat wall. The calculations, carried out numerically at small value of the magnetic Reynolds number, show a closed circulation situated near the point electrode at small time which drifts away to larger radial distance at increasing time-values, developing a jet-flow along the axis of symmetry.

Sozou & Pickering (1976) studied the viscous fluid motion due to the injection of an electric current from a point electrode in a hemispherical bowl with a free surface. Application of the usual similarity method indicates that in this configuration the viscous fluid motion even breaks down at a lower value of the hydrodynamic Reynolds number than in the case of a configuration with a solid flat wall. At small Re the resulting fluid motion consists of a single circulation.

Butsenieks, Peterson, Sharamkin & Shcherbinin (1976) calculated numerically the viscous fluid motion generated in a cylindrical vessel due to the electric current flow between electrodes of different radii. Also the effect of the electromagnetic induction is included in the calculations. The results show a single toroidal circulation when the ratio of the radii of the electrodes is five or larger. It turns out that the maximum velocity at the axis of symmetry located near the smallest electrode moves a little away for larger values of the viscous Reynolds number. Although the inertia force was included in the Navier-Stokes equation no critical value of the hydrodynamic Reynolds number was mentioned in the paper.

Jansen (1977) studied the inviscid fluid motion, at arbitrary value of the
magnetic Reynolds number, in a semi-infinite point electrode configuration consisting of a non-conducting right circular cone of arbitrary apex-angle. The analytical examinations demonstrate that in the inviscid case no electric current inversion can occur at any value of $R_m$. It also turned out that the relatively weak singularity in the flow field at the axis of symmetry implies a singularity in the space charge density at that location. As a result, the normal component of the electric field cannot satisfy the boundary condition at the axis of symmetry.

Sozou & Pickering (1978) studied the slow viscous fluid motion in a hemispheroidal container with a free surface due to the injection of electric current supplied by a cylindrical electrode of finite size. The solutions of the flow field at low value of the hydrodynamic Reynolds number are calculated numerically. The results show that the intensity of the fluid motion is inversely proportional to the diameter of the disk electrode. When the radius of the electrode approaches the radius of the bath up to 80% and more a small eddy develops in the flow field at the rim of the electrode.

Moffatt (1978) reviewed some problems in the magnetohydrodynamics of liquid metals, including the weld pool problem. The author considered the total force imparted to a certain volume of fluid and concluded that the presence of the point electrode would imply local cavitation and intermittency of the electric current passing to the fluid. As an alternative to resolve the viscous problem he suggested to include the effect of the electromagnetic induction.

Andrews & Craine (1978) examined the slow viscous fluid motion in a hemisphere with a free surface induced by a distributed source of electric current. To approximate the electric current flow in the fluid several point electrode pair and point and ring sink electrode models were applied. These models only agree with practical situations when the surface of symmetry of the electric current pattern is located at the bottom or under the container (see A&C, p.287, case b), otherwise one obtains unrealistic fluid motions. Another disadvantage of these electrode configurations is the fact that only a small part of the total electric current passes through the fluid in the container.

Attney (1980) was the first one who carried out a numerical computation of the viscous fluid motion in a hemispherical container with a free surface at larger values of the hydrodynamic Reynolds number. The calculations, which include the nonlinear effect of the inertia force in the Navier-Stokes equation, did not show any breakdown of the flow field for electric currents up to 100 A, which corresponds to a value of 600 of the hydrodynamic Reynolds number.

Boyarevich (1981a) examined the viscous fluid motion in a semi-infinite point electrode configuration with a flat wall or a free surface. In order to obtain
bounded viscous solutions at values of the hydrodynamic Reynolds number which are larger than the usual relatively low critical values, some modifications of the configuration are considered. They are: isolation of the axis of symmetry by a small cone and supply of electric current to the point electrode via an isolated thin filament passing through the fluid. It was shown that for some variants of the model numerical solutions can be found at much higher values of Re.

Boyarevich (1981b) considered the viscous fluid motion generated in the semi-infinite point electrode configuration with a flat wall or a free surface. Matched asymptotic expansions have been applied to derive a formal solution in the form of a composite expansion of the viscous flow field at very large value of the hydrodynamic Reynolds number, viz. \( \text{Re} \to \infty \).

Jansen (1983) studied the inviscid fluid motion in the semi-infinite point electrode configuration with a solid non-conducting right circular cone of arbitrary apex-angle. Analytical and numerical calculations are carried out to determine the effect of the electromagnetic induction upon the original isotropic electric current distribution and upon the fluid motion together with their mutual interaction at small value of the magnetic Reynolds number. For that purpose asymptotic expansions including twelve terms of the regular perturbations at small \( \text{Rm} \) of all fieldquantities have been calculated. Also the forces exerted on a certain part of the fluid domain have been examined in detail.

Craine & Andrews (1984) calculated numerically the steady viscous fluid motion and the heat flow induced in the weld pool. The calculations indicate an increase of the depth-to-width ratio of the weld pool at larger and/or more concentrated supply of electric current, due to the effect of the fluid motion generated. At larger electric currents the intensity of the flow field increases resulting in a deeper weld pool than one might expect on grounds of the heat flux only.

Jansen (1984) presented a general review of analytical solutions of the inviscid and viscous fluid motions induced in the semi-infinite point electrode configuration with a solid non-conducting right circular cone of arbitrary apex-angle. The inviscid flow field is considered and the perturbation of the fluid motion and of the electric current density at small value of the magnetic Reynolds number \( \text{Rm} \). Examination of the slow viscous solution leads to regular perturbations of the fluid-flow and electromagnetic fieldquantities respectively at low and high values of the magnetic Prandtl number \( \text{Prm} \) and at low \( \text{Rm} \). Much attention is paid to the general viscous solution at arbitrary \( \text{Re} \) and low \( \text{Rm} \) and in particular to the behaviour of the flow field when the hydrodynamic Reynolds
number approaches the critical value. Then a semi-infinite line sink develops on the axis of symmetry which implies that the solution must be rejected as being physically unrealistic for \( Re > Re, \text{crit} \). A brief consideration of the disk electrode configuration enabled us to resolve some inconsistencies of the point electrode model and to suggest a reformulation of the general viscous problem, see chapter 5 and 6 of this dissertation.

Oreper & Szekely (1984) carried out a full numerical computation of the transient development of the fluid-flow and temperature fields in a more practical representation of the weld pool. In the calculations the effects of the electromagnetic-, buoyancy- and surface forces are included for a broadly distributed and for a sharply focussed heat flux and electric current distribution. Consideration of the single and combined effects of the respective forces and of the electric current and heat flow distributions demonstrates a great variety of possible phenomena that may occur in the weld pool at different time-scales.

Ajayi, Sozou & Pickering (1984) examined the steady nonlinear fluid motion in a hemispheroidal container with a free surface due to the injection of electric current supplied by a cylindrical electrode. The partly analytical, partly numerical calculations show that the viscous fluid motion breaks down when the hydrodynamic Reynolds number exceeds a certain critical value. In the case of an electrode of small radius and a nearly spherical container this critical value is relatively low. However for a shallow container and a large electrode the critical upper bound of \( Re \) turned out to be much higher. These observations are in excellent agreement with the near and far field calculations of the semi-infinite disk electrode configuration to be carried out in chapter 5.

The generation of fluid motion due to the Lorentz force caused by the injection of an electric current and its associated magnetic field has been investigated experimentally by several authors.

Woods & Milner (1971) have examined the applicability of the stirring of liquid metals by means of the fluid motion induced by an electric current and the self-magnetic field. These experiments are carried out in different cylindrical and hemispheroidal containers. Accurate measurements indicate that the magnitude of the velocity \( \nu \) at the axis of symmetry is respectively proportional to \( I_0^2 \) and \( a^{-1} \); where \( I_0 \) is the total electric current supplied into the fluid and \( a \) is the radius of the electrode to be considered. The second relation: \( \nu \propto a^{-1} \) demonstrates the fact that the main circulation in the bath is determined by the electrode with the smallest radius. When the electric current flow between the electrodes is symmetrical the fluid-flow pattern is a double circulation. How-
ever when the current path is asymmetric, the fluid-flow pattern corresponding to the smallest electrode dominates the fluid motion in the container; sometimes resulting in a single circulation. The experiments clearly showed that the Lorentz force is the primary cause of the fluid motion and that contributions due to the action of the plasma arc on the free surface and to the thermal convection are negligible.

Kublanov & Erokhin (1974) studied the stirring of molten metals in a hemispheroidal container under the action of the electromagnetic force and of the velocity head of an arc gas flow. The maximum supplied electric current into the fluid amounts to 700 A. The authors observed the usual toroidal circulation in the container with high velocities at the axis of symmetry away from the disk electrode and moreover a small eddy at the free surface near the container wall. At larger electric currents the centre of the main circulation moves sideways away from the electrode towards the surface of the container.

Further experiments concerning the electric welding process and the stirring of molten metals have been carried out by Apps & Milner (1963) and by Butsenieks, Kompan, Sharamkin, Shilova & Shcherbinin (1975).

The application of an external axial magnetic field turns out to be very effective to smooth down the vigorous fluid motion, induced by the injection of the electric current, resulting in a more stable and reproducible welding process. Theoretical calculations and practical experiments carried out by Craine & Weatherill (1980a+b), Willgoss (1981), Bojarevičs & Schcherbinin (1983) and Cook & Allen (1984) have confirmed that statement. The external axial magnetic field introduces an additional azimuthal velocity field resulting in a rotation of the fluid about the axis of symmetry.

Consideration by the author of the governing equations clearly shows that at every value of the hydrodynamic Reynolds number the usual breakdown of the solution of the viscous fluid motion can be avoided by an external axial magnetic field of the appropriate strength.

Stuhlinger (1964) and Au (1968) have investigated the applicability of the effect of the Lorentz force due to the injection of an electric current and the associated magnetic field for ion propulsion systems for space flight. Unfortunately the considerations are presented in terms of magnetic pressure without adequate recognition that it is the rotationality of the magnetic force that determines the fluid motion.

Hence in this field of application which becomes more important nowadays, much
work remains to be done. Especially the examination of the behaviour of the viscous fluid motion and the electric current distribution at larger values of the magnetic Reynolds number is of great interest.

1.3. Outline of the thesis

In this introduction the problem to be investigated is surveyed by explaining the relevancy and by referring to the theoretical and experimental papers dealing with this interesting and fundamental physical principle. For the examination of the generation of fluid motions induced by the injection of an electric current, in this thesis two models of configurations are considered. In chapters 2-4 and 6 the prototype model: the semi-infinite point electrode configuration with a solid non-conducting right circular cone of arbitrary apex-angle is applied, and in chapter 5 the more complicated semi-infinite disk electrode configuration consisting of a disk electrode of radius a located in a flat insulating wall is used.

In chapter 2 the semi-infinite point electrode problem is formulated with the limitations and assumptions.

By the introduction of a similarity method the governing partial differential equations of the curl of the Navier-Stokes equation and of the curl of Ohm's law are reduced to ordinary differential equations. The similar solutions also imply the appearance of two parameters $K_b$ and $K_\eta$ which govern the behaviour of the fluid-flow and electromagnetic field quantities. They can be recognized respectively as the effective magnetic Reynolds number and as the effective inverse hydrodynamic Reynolds number. Practical values of these parameters for different fluids and as function of the total electric current supplied into the fluid are presented in a figure.

Further the boundary conditions, and the integral equations for the functions $g(c)$ and $f(c)$, which represent respectively the meridional angle dependent part of the Stokes stream function $\psi$ and of the azimuthal magnetic field $B_\varphi$, together with the expressions of the other field quantities are derived in this chapter.

In chapter 3 the inviscid fluid motion in the semi-infinite point electrode configuration is examined. At first the behaviour of the inviscid solution at arbitrary value of the magnetic Reynolds number is considered. This leads to the conclusion that the inviscid solution can exist only in this type of configuration when the flow field contains a relatively weak singularity at the axis of symmetry. The
further analysis indicates that in the inviscid point electrode problem no inversion of electric current can occur for any value of the magnetic Reynolds number.

Analytical solutions of the inviscid fluid motion and of the electromagnetic field quantities are obtained at low value of the magnetic Reynolds number, together with the corresponding weak perturbing effect of the electromagnetic induction.

The chapter is concluded with a discussion about the behaviour of the field quantities when the magnetic Reynolds number tends to infinity.

Chapter 4 deals with the semi-infinite viscous point electrode problem.

Analytical solutions of the slow viscous fluid motion are calculated at small values of the magnetic and viscous Reynolds numbers and in addition the regular perturbations of the flow field at low and high values of the magnetic Prandtl number are presented.

The behaviour of the general viscous solution, which includes the nonlinear effect of the inertia force, at arbitrary values of the hydrodynamic Reynolds number is examined extensively; in particular the breakdown of the viscous fluid motion at a certain maximum value of Re as function of the apex-angle. It turns out that at and above the critical value of the hydrodynamic Reynolds number, viz. for \( Re > Re_{max} \), physically unrealistic phenomena enter the flow field from the axis of symmetry and the mass conservation equation is no longer satisfied.

Of particular interest is the abrupt breakdown of the edge of the viscous jet flow along the axis of symmetry when the hydrodynamic Reynolds number approaches the maximum critical value, suggesting that the respective forces cannot balance any longer.

In chapter 5 the semi-infinite disk electrode problem is considered with an configuration consisting of a cylindrical electrode of radius \( a \) located in a insulating flat wall.

It appears that in this more complicated configuration analytical calculation of the inviscid and viscous fluid motions is possible separately for the near and far field of the fluid domain in the form of series expansions of the Stokes stream function respectively in positive and negative powers of the radial distance \( r \).

The far field solutions approach the solutions obtained in the point electrode model in the case of a flat wall in the limit \( r \to \infty \) or \( a = 0^+ \).

In the near field fluid domain different series expansions of the Stokes stream function need to be derived for the inviscid and the viscous fluid motion. Analytical solutions for both fluids are calculated and show a behaviour which
differs strongly from that in the far field. The inviscid fluid motion in the near field does not contain any singularity in the flow field and moreover the viscous fluid motion exists for all values of the hydrodynamic Reynolds number. This chapter ends up with the consideration of the forces exerted on certain volumina of fluid and with the demonstration of the local behaviour of the hydrodynamic and magnetic Reynolds numbers throughout the flow field. By consideration of the disk electrode problem some inconsistencies arising in the point electrode model could be removed.

In chapter 6 the general solutions of the inviscid and viscous fluid motion in the point electrode configuration are considered.

In view of the order of the governing differential equation and the number of boundary conditions to be satisfied, it turns out that the general inviscid and viscous solutions should contain free parameters. This was already found in the near field of the disk electrode configuration.

The general solutions of the slow viscous, of the viscous and inviscid fluid motions, each with a free parameter $\gamma$, are calculated. It appears that non-zero values of the free parameter imply a relatively weak logarithmic singularity in the flow field at the axis of symmetry in the slow viscous and the viscous fluid motion. In addition the general inviscid solution also possesses a relatively weak singularity in the flow field at the surface of the cone, which is removed only when the free parameter satisfies Shercliff's condition.

From a physical point of view the appearance of the free parameter denotes a point source of momentum located at the origin or the exchange of momentum between different fluid regions. Therefore the general solutions represent in fact the generation of fluid motion due to the combined effect of the Lorentz force and of a point source of momentum.

Although the general solutions do not result in physically more realistic solutions of the inviscid and viscous fluid motion, they supply much more insight in the applicability of the similarity method for the point electrode model. Also they indicate the way both the inviscid and viscous problems need to be reformulated in order to obtain realistic flow solutions induced exclusively by the Lorentz force associated with the injection of an electric current.
2. FORMULATION OF THE POINT ELECTRODE PROBLEM

2.1. Introduction

In this chapter the point electrode problem will be formulated. We consider a point electrode located at the vertex of a non-conducting right circular cone of arbitrary apex angle and discuss different assumptions concerning the fluid and boundary conditions.

By introduction of a similarity method the basic M.H.D. vector equations for the velocity- and electromagnetic fields are reduced to integral equations for the functions $g$ and $f$, which represent respectively the $\theta$-dependent parts of the Stokes stream function and the azimuthal component of the self magnetic field. The similarity assumptions lead to the definition of two dimensionless parameters: $K_b$ and $K_\eta$, which govern the form of the equations. Their relations with well-known M.H.D. parameters are discussed and the actual behaviour, as function of the total electric current supplied into the fluid, are given for some electrically conducting liquids and gases.

The boundary conditions and physical assumptions are discussed and expressions for the other field quantities are presented.

2.2. The point electrode configuration

Consider a uniform, incompressible and electrically conducting fluid of constant density $\rho$ and electrical conductivity $\sigma$, occupying the entire space exterior to an electrically insulated right circular cone of arbitrary apex-angle $\theta_o$. In this semi-infinite axisymmetric configuration a spherical polar co-ordinate system $(r, \theta, \varphi)$ is chosen, with the origin at the vertex of the cone and the line $\theta = 0, \pi$ along the axis of the cone, see figure 2.1. Now $r$ is the radial distance from the vertex, $\theta$ is the angle between the radius vector and the positive $z$-axis, and $\varphi$ is the meridian angle about the axis of symmetry. It is convenient to use $c = \cos(\theta)$ instead of $\theta$ as independent variable. The positive $z$-axis is then described by $\theta = 0, c = 1$; the negative $z$-axis by $\theta = \pi, c = -1$; the surface of the cone by $\theta = \theta_o, c = c_o$ for $0 < \theta_o < \pi$, $-1 < c_o < 1$; and the fluid region by $r > 0, 0 < \theta < \theta_o, c_o < c < 1$, $0 < \varphi < 2\pi$. The semi-infinite flat wall configuration ($c_o = 0$), studied by several authors is a special case of this more general one.

In some cases it will be useful to apply cylindrical polar co-ordinates $(z, s, \varphi)$ with origin at the same place; where $z$ is the vertical distance with
respect to the $z = 0$ (i.e. $c = 0$) plane and $s$ is the distance from the axis of symmetry. The relations with spherical polar co-ordinates are $z = rc$, $s = r \sqrt{1 - c^2}$. The respective vector components are indicated by the subscripts $r, \theta, z, s, \varphi$.

![Diagram of a point electrode configuration](image)

Figure 2.1. The point electrode configuration.

A constant electric current of total strength $I_o$ is supplied through the axis of the cone at $\theta = \pi$ by a thin filament of negligible thickness. The electric current enters the fluid at the vertex of the cone, passes through the fluid, and leaves through a second electrode of spherical shape, centred at the origin, and located at large radial distance.

Assuming that an azimuthal component of the electric current density and its associated magnetic field are absent, and that the effect of the fluid motion upon the electric current density is omitted to begin with, the constant electric current $I_o$, supplied into the fluid, leads to an isotropic, purely radial current distribution $J_{\text{isot}}$ in the fluid, see figure 2.1. The electric current distribution and its associated magnetic field result in a Lorentz force
in the direction of $-\mathbf{e}_\theta$, see (2.63). For constant $r$ the Lorentz force has the largest value at the surface of the cone and is identical to zero at $c = 1$. For constant $\theta$, the Lorentz force decreases with increasing radial distance.

Since the rotationality of the Lorentz force is non-zero in a three-dimensional configuration, the magnetic force induces a fluid motion for every value of the electrical conductivity and viscosity. In most cases, examined in this thesis, the Lorentz force generates an incoming flow almost parallel to the cone surface. In the neighbourhood of the origin the inflow is rather abruptly turned off into an outwards jet flow along the axis of symmetry. As noted by Shercliff (1970), the fluid escapes into the region of weaker $\mathbf{J} \times \mathbf{B}$ - forces at large $z$.

The fluid motion induces an electric field $\mathbf{v} \times \mathbf{B}$ in the meridian plane, perpendicular to the streamlines of the flow. Especially in the case of large electrical conductivity the induced electric field will tend almost everywhere to reduce the original isotropic electric current distribution. Exceptions are small regions near the axis of symmetry and the surface of the cone, where the electric field is normal to the imposed radial current distribution. Since the total electric current is constant, this effect may involve shifting of electric current towards the axis of symmetry and to the surface of the cone, as shown in figure 2.1.

2.3. The basic equations

We consider steady fluid flow with overall symmetry about the axis of symmetry $(\partial/\partial \varphi = 0)$, leaving $r$ and $\theta$, respectively $c$, as the only independent variables. Since rotation of the fluid about the axis of symmetry and the azimuthal component of the current density are not excited or induced, they are assumed to be identically zero, viz. $\mathbf{v}_\varphi = 0$, $\mathbf{J}_\varphi = 0$. The magnetic field is now purely azimuthal: $\mathbf{B} = B_\varphi \mathbf{e}_\varphi$. Moreover the electric field is irrotational, see (2.5), yielding $E_\varphi = 0$.

The governing equations of steady viscous M.H.D. flow, after the usual approximations and elimination of $\mathbf{H}$ and $\mathbf{D}$ are

$$\text{div } \mathbf{v} = 0 \ , \quad (2.1)$$

$$\rho (\mathbf{v} \cdot \nabla) \mathbf{v} + \text{grad } p = \mathbf{J} \times \mathbf{B} + \eta \Delta \mathbf{v} \ , \quad (2.2)$$

$$\mathbf{J} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \ , \quad (2.3)$$
\[ \mu \mathbf{J} = \text{curl} \mathbf{B} , \]  
\[ \text{curl} \mathbf{E} = 0 , \]  
\[ \mathbf{E} = - \text{grad} U , \]  
\[ \rho_e = \varepsilon \text{ div} \mathbf{E} , \]

where \( \mathbf{v} \) is the velocity vector, \( \mathbf{J} \) the electric current density, \( \mathbf{B} \) the magnetic field, \( \mathbf{E} \) the electric field, \( p \) the pressure, \( U \) the electrical potential, \( \rho_e \) the space charge density, \( \rho \) the density of the fluid, \( \eta \) the dynamic viscosity, \( \sigma \) the electrical conductivity, \( \mu \) the permeability, and \( \varepsilon \) the permittivity. The physical parameters \( \sigma, \rho, \eta, \varepsilon, \mu \) are assumed to be uniform throughout the fluid. The fluid is supposed to be non-magnetic so that \( \mu \) takes its vacuum value.

In this semi-infinite configuration the total injected electric current \( I_0 \), the radial current density in the fluid and the associated magnetic field together generating the Lorentz forces, are considered as the origin of the fluid motion. The Navier-Stokes equation (2.2) indicates that the Lorentz force induces both a velocity- and a pressure distribution. Hence the other field quantities are not imposed in this problem. They result from the electric current density, the associated magnetic field, the fluid motion and the pressure distribution.

Upon elimination of \( p, \mathbf{E} \) and \( \mathbf{J} \), the equations, governing the generation of fluid motion and the effect of the electromagnetic induction upon the electric current distribution, become

\[ \text{div} \mathbf{v} = 0 , \]  
\[ \rho \mu \text{ curl} (\mathbf{v} \times \mathbf{v}) = \text{curl} (\text{curl} \mathbf{B} \times \mathbf{B}) + \]
\[ - \eta \mu \text{ curl} \text{ curl} \mathbf{v} , \]  
\[ \text{curl} \text{ curl} \mathbf{B} = \sigma \mu \text{ curl} (\mathbf{v} \times \mathbf{B}) , \]

where \( \omega \) is the vorticity, given by

\[ \omega = \text{curl} \mathbf{v} . \]
The solutions of $\mathbf{v}$ and $\mathbf{B}$ obtained from the above equations enable us to calculate the other field quantities. This will be discussed in section 2.8.

The mass conservation equation (2.1) can be satisfied identically by introducing the Stokes stream function $\psi$. In terms of spherical polar co-ordinates, the corresponding velocity components are

$$ v_r = -\frac{1}{r^2} \frac{\partial \psi}{\partial c}, $$

(2.11)

$$ v_\theta = -\frac{1}{r\sqrt{1-c^2}} \frac{\partial \psi}{\partial r}, $$

(2.12)

$$ v_\varphi = 0, $$

(2.13)

where $c = \cos(\theta)$.

The introduction of the Stokes stream function reduces the vector equations (2.8)-(2.10) to a system of two coupled partial differential equations for the functions $\psi$ and $B_\varphi$.

These partial differential equations cannot be solved analytically. However, solutions of $\psi$ and $B_\varphi$ can be obtained by introducing a similarity method.

2.4. Similar solutions

Shercliff (1970) and Sozou (1971a) pointed out that the semi-infinite point electrode problem does not contain any fundamental length - or velocity scale. The inviscid and viscous problem are characterized only by three, respectively four, physical parameters, namely $\rho \mu, \sigma \mu, \mu I_0$, and $\eta \mu$; where $\mu I_0$ originates from the boundary condition of the magnetic field at the surface of the cone. Nevertheless solutions of the velocity- and magnetic field can be obtained by introducing a similarity method, as formulated by Shercliff (1970) and Sozou (1971a). The introduction of the similarity method will lead to the definition of two characteristic dimensionless parameters, which also determine the form of the similar solutions. This statement is clarified by the following considerations.

The respective terms in the curl of Navier-Stokes equation (2.8) represent the inertia force, the Lorentz force, the viscous force, and in the curl of Ohm's law (2.9), the diffusion and convection of the magnetic field. The ratio of these quantities is measured by the following parameters: the hydrodynamic
Reynolds number Re, representing the ratio of the inertia force to the viscous force; the magnetic Reynolds number Rm measuring the ratio of magnetic convection to magnetic diffusion; the Hartmann number M, indicating the ratio of the magnetic body force to the viscous force; and finally the interaction parameter or magnetic force coefficient N, which is a measure for the ratio of the electromagnetic force to the inertia force.

In the curl of the Navier-Stokes equation always one parameter can be expressed in terms of the other two; so that with the curl of Ohm's law we obtained three independent dimensionless parameters. In order to obtain a minimum number of parameters in the problem, it is clear that one of the remaining M.H.D. parameters can be set equal to unity, without loss of generality. Which one will be set equal to unity is usually ascertained by the physics of the problem. It is then clear that the form of the solutions is finally determined by two dimensionless characteristic parameters. In general four sets of two parameters can be defined. Our choice is based upon the following considerations.

The analysis presented here will be especially concerned with applications to liquid metal, such as welding and arc stirring, using electric currents of several hundred amperes. In these applications the Lorentz force balances the inertia force over the greater part of the flow field. Only in the boundary layer at the surface of the cone and in the jet flow along the axis of symmetry, viscosity plays a significant role. We will consider both the inviscid- and viscous problem. In most liquid metal applications the effect of the electromagnetic induction upon the electric current distribution is negligible. Therefore we set here the interaction parameter N equal to unity.

The similar solutions of the Stokes stream function \( \phi \) and the azimuthal component of the magnetic field \( B_\varphi \) then take the form

\[
\phi = \frac{I_0}{2\pi} \frac{\mu}{\sqrt{\rho}} r g(c), \quad (2.14)
\]

\[
B_\varphi = \frac{\mu I_0}{2\pi} \frac{f(c)}{r \sqrt{1-c^2}}. \quad (2.15)
\]

The corresponding characteristic non-dimensional parameters, governing the form of the basic equations, are
\[ K_b = \frac{\sigma \mu I_0}{2\pi} \sqrt{\frac{\mu}{\rho}}, \quad (2.16) \]

\[ K_\eta = \frac{2}{I_0} \frac{\pi \eta}{\sqrt{\rho \mu}}, \quad (2.17) \]

Note that \( g \) and \( f \) are non-dimensional functions, in general depending on \( c, c_0, K_b \) and \( K_\eta \). The factor \( \frac{I_0}{2\pi \sqrt{\frac{\mu}{\rho}}} \), in (2.14), is effectively a diffusivity for the corresponding velocities, see (2.21) - (2.23), having the dimensions (length x velocity), like all diffusivities. The choice of the factor in the expression of \( B_\phi \), (2.15), yields a simple boundary condition for \( f \), independent of \( c_0 \), see section 2.6. It should be mentioned that the above form of the self-similar solutions has been suggested for the first time by Zhigulev (1960 a + b). A discussion of the characteristic parameters \( K_b \) and \( K_\eta \), their magnitudes in practical applications, and their relations with well-known M.H.D. parameters are presented in section 2.5.

Applying (2.14) and (2.15) to the equations (2.8) - (2.10), simplifies the equations to be solved. The azimuthal component of the curl of the Navier-Stokes equation now takes the form

\[ \frac{d^2 g^2}{dc^2} = \frac{4f}{1-c^2} \frac{df}{dc} + 2K_\eta \left\{ (1-c^2) \frac{d^4 g}{dc^4} - 4c \frac{d^3 g}{dc^3} \right\}, \quad (2.18) \]

where from left to right the respective terms originate from the inertia-, Lorentz-, and viscous forces.

The vorticity \( \omega \) is purely azimuthal, \( \omega = \omega \frac{d}{d\phi} \), with

\[ \omega = -\frac{I_0}{2\pi} \sqrt{\frac{\mu}{\rho}} \frac{\sqrt{1-c^2}}{r^2} \frac{d^2 g}{dc^2}. \quad (2.19) \]

The curl of Ohm's law, in azimuthal direction, is given by

\[ \frac{d^2 f}{dc^2} = K_b \left[ \frac{g}{1-c^2} \frac{df}{dc} + \left\{ \frac{2cg}{(1-c^2)^2} + \frac{2}{1-c^2} \frac{dg}{dc} \right\} f \right], \quad (2.20) \]
where the left-hand term represents the diffusion of the magnetic field and the right one the convection of the magnetic field due to electromagnetic induction. In this way the problem has been reduced to a system of two simultaneous nonlinear ordinary differential equations, in which the parameter $K_b$ is a measure of the effect of the induced fluid motion upon the electric current distribution, whereas the effect of viscosity is determined by the magnitude of $K_\eta$. Since $K_\eta$ and $K_b^{-1}$ multiply the highest order derivatives of the respective differential equations, we may expect the appearance of hydrodynamic boundary layers at small $K_\eta$ and of electromagnetic boundary layers in case of large $K_b$. In slightly different forms these differential equations have been derived by Shercliff (1970), Sozou (1971a) and Sozou & English (1972). The relations between the functions $f$ and $g$, and the parameters $K_b$ and $K_\eta$, as defined in (2.14) - (2.17), and those of the references are presented in section 2.5. From (2.4), (2.11) - (2.15), the components of the velocity and the electric current density become

$$v_r = -\frac{I_0}{2\pi} \sqrt{\frac{\mu}{\rho}} \frac{d g}{dr}$$  \hspace{1cm} (2.21)

$$v_\theta = -\frac{I_0}{2\pi} \sqrt{\frac{\mu}{\rho}} \frac{g}{r\sqrt{1-c^2}}$$  \hspace{1cm} (2.22)

$$v_\varphi = 0$$  \hspace{1cm} (2.23)

and

$$J_r = -\frac{I_0}{2\pi} \frac{1}{r^2} \frac{df}{dc}$$  \hspace{1cm} (2.24)

$$J_\theta = 0$$  \hspace{1cm} (2.25)

$$J_\varphi = 0$$  \hspace{1cm} (2.26)

It is clear from the preceding relations that the induced velocities and vorticity are proportional to the total electric current $I_0$, supplied into the fluid by the point electrode, when $g$ is of order unity. Moreover when both $f$ and $g$ are of order unity, the velocity turns out to be related to the Alfvén velocity $B_\varphi / \sqrt{\rho \mu}$. In addition, the introduction of the similar solutions implies a purely
radial electric current distribution; no cross currents can exist. As a consequence, the angular component of the electromagnetic induction term in Ohm's law (2.3) merely induces a $E_\theta$, but no $J_\theta$ in this prototype configuration. Hence the fluid motion generates a radial electric current density, which may change the isotropic current distribution for sufficiently large $K_b$, and an electric field involving the generation of a space charge density.

2.5. The parameters $K_b$ and $K_\eta$

In the previous section the introduction of the similar solutions involved the definition of two non-dimensional characteristic parameters,

\[ K_b = \frac{\sigma u I_0}{2\pi} \sqrt{\frac{\mu}{\rho}}, \]  

\[ K_\eta = \frac{2\eta}{I_0 \sqrt{\rho \mu}} \]  

(2.16)  

(2.17)

which appear in the basic equations (2.18), (2.20). It was noted by Shercliff (1970), that $K_b$ is a form of the magnetic Reynolds number $Rm$. At least at small $K_b$, the velocities are of order $\frac{I_0}{2\pi} \sqrt{\frac{\mu}{\rho}} \frac{l}{r}$, so that $Rm = \sigma u x \text{velocity} \times r$ becomes $\frac{\sigma u I_0}{2\pi} \sqrt{\frac{\mu}{\rho}} = K_b$. In the same way $K_\eta$ appears to be a form of the inverse hydrodynamic Reynolds number; $Re = \frac{\rho}{\eta} x \text{velocity} \times r$ becomes $\frac{I_0 \sqrt{\rho \mu}}{2\pi \eta} = \frac{1}{K_\eta}$.

Since there is no fundamental length- or velocity scale available in the point electrode problem, and since $K_b$ and $K_\eta^{-1}$ take over respectively $Rm$'s- and $Re$'s normal role, $K_b$ will be denoted as the effective magnetic Reynolds number and $K_\eta$ as the inverse effective hydrodynamic Reynolds number. The viscous- and magnetic Reynolds numbers, defined in this way, have the same value throughout the flow field. It should be noted that in chapter 5, where the disk electrode configuration is considered, we will find relatively high $Re$ and $Rm$ behaviour at large distance from the electrode and relatively low $Re$ and $Rm$ behaviour near the origin. Hence in that case we have so-called local Reynolds numbers.

The relations between $K_b$ and $K_\eta$ and other essential M.H.D. parameters is as follows. The Hartmann number $M$, which is the ratio of the magnetic force to the viscous force, satisfies $M = \frac{1}{K_\eta}$. The magnetic Prandtl number $Pm$, being a measure of the ratio of vorticity diffusion to magnetic diffusion, becomes
\[ Pm = \frac{q_{\text{m}} \eta}{\rho} = K_b \eta. \]

When the magnetic Prandtl number is small, the magnetic field diffuses much more rapidly than vorticity and magnetic boundary layers are much thicker than viscous ones. This justifies simplifications such as neglecting the viscosity in the magnetic boundary layer. In case of large \( Pm \), hydrodynamic boundary layers are much thicker than electromagnetic ones. An electromagnetic boundary layer is merely a variation on the skin-effect, so the electric current flows mainly in current sheets, shielding the magnetic field from penetrating the fluid. Note that the above relations between the parameters \( K_b \) and \( \eta \) and the essential M.H.D. parameters are determined by the choice to set the interaction parameter \( N \) equal to unity.

Values of \( K_b \) and \( \eta \) for different fluids, as function of \( I_o \), are given in figure 2.2.

![Figure 2.2. Values of \( K_b \) and \( \eta \) for different fluids, as function of \( I_o \).](image)

(1) mercury and molten steel; (2) liquid sodium; (3) an electrolytic solution; (4) an arc discharge; (5) dense hot plasma; (6) tenuous hot plasma.
The dash-dot lines in the figure show typical values of \( K_b \) and \( K_\eta \) for mercury or molten steel at \( I_0 = 500 \, \text{A} \); viz. \( K_b \approx 10^{-3} \) and \( K_\eta \approx 10^{-5} \). The dashed lines in figure 2.2 represent the upper- and lower bounds of \( K_b \) and \( K_\eta \) for liquid metals, see Hunt & Moreau (1976). Thus, in general \( K_b \) and \( K_\eta \) are very small compared to unity. Only in very high power devices involving liquid metals or in applications using very low electric currents, respectively \( K_b \) or \( K_\eta \) may exceed unity.

For liquid metal applications, using moderate electric currents \( (I_0 = 1 - 10^4 \, \text{A}) \), some conclusions can be drawn about both the electromagnetic and the flow field. Due to the low value of the effective magnetic Reynolds number, the effect of the induced fluid motion upon the electromagnetic field is negligible, and the diffusion of the magnetic field is dominant throughout the entire field. Hence expressions of \( J \) and \( B \) can be directly obtained from the purely electrostatic problem \( (K_b = 0) \) of this semi-infinite point electrode configuration.

The small magnitude of \( K_\eta \) restricts the effect of viscosity to small regions in the neighbourhood of the boundaries; viz. a viscous boundary layer on the surface of the cone and viscous spreading of the jet flow along the axis of symmetry. In the remainder of the flow field the Lorentz force balances the inertia force and the pressure distribution.

When the fluid is an ionized gas or a plasma, terms representing the Hall current and ion slip should be added. However it turns out that these terms can dominate only in the neighbourhood of the electrode and not at large distance. This fact becomes clear when the respective orders of the radial distance of the Hall- and ion slip current density is considered in relation to asymptotic expansions of the electric current distribution in the disk electrode configuration, see chapter 5. Moreover in these kinds of fluids the density \( \rho \) is not a constant. In general it varies throughout the entire flow field; so that the perfect gas law and the energy equation must be included to complete the basic equations. Nevertheless as a first step it will be useful to consider how the behaviour of the electromagnetic field and the fluid motion is effected by the parameters \( K_b \) and \( K_\eta \) in the for this problem somewhat inadequate incompressible model.

Figure 2.2 shows that for an ionized gas or a plasma the values of \( K_b \) and \( K_\eta \) are much larger than for liquid metals. Especially in case of high-temperature plasma the magnitudes may become very large compared to unity. In that case the low viscous Reynolds number indicates that the inertia force is negligible with respect to the viscous force. As a consequence no hydrodynamic boundary layers will occur at injection of moderate electric currents \( (I_0 = 1 - 10^4 \, \text{A}) \). Moreover a large value of the magnetic Reynolds number implies a strong effect of the
electromagnetic induction upon the electric current distribution, resulting in the appearance of electromagnetic boundary layers.

The magnitudes of $K_b$ and $K_\eta$ are completely determined by the value of $I_o$. The general tendency in the behaviour of $K_b$ and $K_\eta$ as function of $I_o$, see figure 2.2, indicates that at very low $I_o$ no boundary layers will occur ($K_b$, $K_\eta \ll 1$); whereas at sufficiently large values of $I_o$ both hydrodynamic- and electromagnetic boundary layers are always to develop ($K_b$, $K_\eta^{-1} \gg 1$). The appearance of boundary layers at moderate electric current ($I_o = 1 - 10^4$ A) is determined by the magnetic Prandtl number $Pm$, being independent of $I_o$.

$Pm$ is a measure of the ratio of the thickness of the respective boundary layers, see Shercliff (1965). In this way the magnitude of the magnetic Prandtl number determines for increasing $I_o$ which boundary layer thickness will be the larger. Typical values of $Pm$ are: $10^{-7} - 10^{-5}$ for liquid metals; $10^{-2} - 1$ for arc discharges and $10^3 - 10^5$ for high-temperature plasmas. Hence at increasing $I_o$, in liquid metals at first hydrodynamic boundary layers occur; in arc discharges both boundary layers will develop about together and in high-temperature plasmas the electromagnetic boundary layers appear first.

Several authors have studied the present problem. To compare the respective results, the relations between $f$, $g$, $K_b$, $K_\eta$ and dimensionless functions and parameters, defined by some authors, may be given, viz.

\begin{equation}
  f = 2\pi f_{sh} = 1 - f_s = 1 - f_{se} = -x_{se},
\end{equation}

\begin{equation}
  g = 2\pi g_{sh} = K_\eta g_s = K_\eta g_{se},
\end{equation}

\begin{equation}
  K_b = \frac{K_{sh}}{2\pi} = \frac{\alpha_{se}}{K_\eta},
\end{equation}

\begin{equation}
  K_\eta = \sqrt{\frac{2}{K_s}} = \sqrt{\frac{2}{K_{se}}} = \frac{1}{(1-c_o)\sqrt{\nu u}} = \frac{2\pi}{K_{s,bs}},
\end{equation}

where the subscript $sh$ refers to Shercliff (1970) for the flat wall problem $c_o = 0$; $s$ to Sozou (1971a) for $c_o = 0$; $se$ to Sozou & English (1972) for $c_o = 0$; $nu$ to Narain & Uberoi (1971, 1973) for arbitrary $c_o$; and $bs$ to Bojarevits & Shcherbinin (1983).
2.6. The boundary conditions

To satisfy the boundary conditions, the azimuthal component of the magnetic field has to be continuous across the surface of the cone and identical to zero at the axis of symmetry. An arbitrary finite electric current distribution, caused by injection of a total electric current $I_0$ into the fluid, implies

$$B_\varphi(r, c_0) = \frac{\mu I_0}{2\pi r \sqrt{1-c_0^2}}, \quad B_\varphi(r, 1) = 0. \quad (2.31)$$

From (2.15) and (2.31) we then obtain

$$f(c_0) = 1, \quad (2.32)$$

$$f(1) = 0, \text{ or } \lim_{c \to 1} \frac{f(c)}{\sqrt{1-c^2}} = 0. \quad (2.33)$$

Thus a finite electric current density throughout the field, involves $f(1) = 0(1-c)$, see (2.20), (2.24). Expressions (2.24) - (2.26) show that the boundary conditions of the current density are satisfied.

Excluding fluid sources or sinks, respectively blowing or suction through the cone wall, the boundary conditions of the flow field require that the transverse velocity is zero on the surface of the cone and on the axis of symmetry. It follows from (2.21) and (2.22) that

$$g(c_0) = 0, \quad (2.34)$$

$$g(1) = 0, \quad (2.35)$$

or formulating the latter condition more accurately, in the form

$$\lim_{c \to 1} \left[ \frac{cg}{\sqrt{1-c^2}} + \frac{dg}{dc} \right] = 0, \quad (2.36)$$

obtained from (2.21) and (2.22) to compose the transverse velocity. When the velocity along the axis remains finite, the last condition simplifies to
\[ g(1) = 0(1-c), \text{ and } \frac{dg}{dc} \bigg|_{c=1} = \text{constant.} \quad (2.37) \]

At the surface of the cone the viscous fluid has to satisfy the no-slip condition.

\[ \frac{dg}{dc} \bigg|_{c=c_0} = 0 \quad \text{for } K \eta \neq 0. \quad (2.38) \]

In order to verify the appropriateness of conditions (2.35) and (2.36), we consider a surface of revolution \( S^* \) about the axis of symmetry, generated by a meridian curve \( s^* \). From (2.11) - (2.13) it can be easily derived that in the case of \( v_\phi = 0 \), considered here, the relation between the velocity vector and the Stokes stream function satisfies

\[ \mathbf{v} = \frac{1}{r \sqrt{1-c^2}} \left\{ \text{grad} \ (\psi) \times \mathbf{i}_\phi \right\}. \quad (2.38a) \]

Upon using the above expression it is easy to show that the expression of the total mass flux of fluid \( \Psi \) through the surface \( S^* \) takes the form

\[ \Psi = \rho \iint_{S^*} \mathbf{v} \cdot \mathbf{n} \, dS^* = 2\pi \rho \int_{s^*} \text{grad} \ (\psi) \cdot \mathbf{t} \, ds^* = 2\pi \rho \int_{s^*} d\psi, \quad (2.38b) \]

or

\[ \Psi = 2\pi \rho \left\{ \psi (r_1, c_1) - \psi (r_2, c_2) \right\}, \quad (2.39) \]

where \( \psi (r_1, c_1) \) and \( \psi (r_2, c_2) \) denote the respective values of \( \psi \) at the endpoints of the curve \( s^* \). In (2.39) the mass flux is considered in the direction of the unit vector \( \mathbf{n} \) normal to the surface \( S^* \), satisfying \( \mathbf{n} = \mathbf{t} \times \mathbf{i}_\phi \); where \( \mathbf{t} \) is the unit vector tangential to the curve \( s^* \), directed from \( r_2, c_2 \) to the endpoint \( r_1, c_1 \). Note that (2.39) is also valid in case of a non-zero azimuthal velocity component.

The condition that neither the axis of symmetry or the flow field contains a fluid source or sink, nor that blowing or suction through the cone wall takes place is obviously that the mass flux through any closed surface \( S^* \) is identical to zero. If we now consider two different curves \( s^* \), respectively \( s^*_1 \), where the end-points join two distinct points on the axis of symmetry, and \( s^*_2 \) where one end-point is located on the axis and the other one on the cone-surface, it
follows from (2.14) and (2.34) that the surface of the cone and the axis of symmetry are represented by the streamline

\[ \psi = 0 \quad \text{at} \quad c = c_o, \quad 1. \quad \text{(2.40)} \]

Thus the function \( g(c) \) always has to obey both boundary conditions (2.35) and (2.36). If the conditions (2.34) - (2.36), (2.40) are not satisfied, mass is transferred from an external source, casu quo sink.

In my opinion an extra condition should be added in this very simple semi-infinite point electrode configuration in order to obtain stable and physically realistic solutions. In this prototype model the appearance of electric current inversion for some \( \Theta \), as found by Sozou & English (1972), will imply electrical short-circuiting. Therefore, especially in the point electrode problems, electric current inversion must be excluded for any \( \Theta \), so from (2.24) we require

\[ \frac{df}{dc} < 0. \quad \text{(2.41)} \]

As a consequence it follows from (2.32) and (2.33) that, for \( c_o < c < 1 \)

\[ 0 < f(c) < 1. \quad \text{(2.42)} \]

In chapter 5 where a configuration with an electrode of finite dimension is considered we are allowed to dispose of this severe condition.

The boundary conditions of the other field quantities are satisfied implicitly. Except one, usually neglected in the literature concerning this problem. This is the condition that the component of the electric field normal to the axis of symmetry needs to be identical to zero at that location, viz. see figure 2.1

\[ E_s = 0 \quad \text{at} \quad c = 1. \quad \text{(2.43)} \]

All field quantities contain singularities at \( r = 0 \). It should be noticed that these singularities are only caused by the simplicity of the model. A physically more realistic configuration is considered in chapter 5; there no singularities appear at the origin of the co-ordinate system.

2.7. Reduction to integral equations of \( g \) and \( f \)

In section 2.4 a sixth-order system of non-linear ordinary differential
equations (2.18) and (2.20) for the electromagnetic and velocity fields has been derived. The system can be considerably simplified by integration with respect to \( c \); resulting in integral expressions for the basic field quantities, which govern this problem.

Integration of Navier-Stokes equation (2.18), respectively two and three times in succession, some further integrations by parts, and using (2.32) yields the relations

\[
2g \frac{dg}{dc} = 2K_\eta \left\{ (1-c^2) \frac{d^2g}{dc^2} + 2g \right\} + (1+c) \int \frac{f^2(t)}{c_o (1+t)^2} dt + \\
+ (1-c) \int \frac{f^2(t)}{c_o (1-t)^2} dt - \frac{2(c-c_o)}{1-c_o^2} + 2Pc + Q ,
\]

(2.44)

and

\[
g^2 = 2K_\eta \left\{ (1-c^2) \frac{dg}{dc} + 2cg \right\} + \frac{(1+c)^2}{2} \int \frac{f^2(t)}{c_o (1+t)^2} dt + \\
- \frac{(1-c)^2}{2} \int \frac{f^2(t)}{c_o (1-t)^2} dt - \frac{(c-c_o)^2}{1-c_o^2} + Pc^2 + Qc + S ,
\]

(2.45)

where \( P, Q, S \) are constants of integration. For the special case \( f = 0 \) this integration was obtained by Sleizkin (1934).

The velocities are assumed to be finite throughout the flow field, except at \( r = 0 \) where a singularity is unavoidable; satisfying condition (2.37) on the axis of symmetry. Substitution of the boundary conditions (2.32) - (2.38) in (2.44) and (2.45) yields identical expressions of the constants \( P, Q, S \) for both the inviscid - and viscous fluid problem. Moreover note that the additional expression (2.49) is only valid for \( K_\eta \neq 0 \).

\[
P = \frac{1}{1-c_o^2} + \frac{2c_o}{(1-c_o)^2} \int \frac{f^2(t)}{c_o (1+t)^2} dt ,
\]

(2.46)
\[ Q = \frac{2c_o}{1-c^2_o} - \frac{2(1+c^2_o)}{(1-c^2_o)^2} \int \frac{f^2(t)}{c_o(1+t)^2} dt, \quad (2.47) \]

\[ S = \frac{c^2_o}{1-c^2_o} + \frac{2c_o}{(1-c^2_o)^2} \int \frac{f^2(t)}{c_o(1+t)^2} dt, \quad (2.48) \]

\[ K_\eta (1-c_o)^2 \left[ \frac{d^2 \theta}{dc^2} \right]_{c=c_o} = \int \frac{f^2(t)}{c_o(1+t)^2} dt \quad \text{for } K_\eta \neq 0. \quad (2.49) \]

Substitution of (2.46) - (2.48) in relation (2.45) leads to the equation of motion in the form of a Riccati differential equation (2.50), which in principle is valid for all \( K_\eta > 0 \), see Jansen (1977)

\[ g^2 = 2K_\eta \left\{ \left(1-c^2\right) \frac{dg}{dc} + 2cg \right\} - G_\eta(c), \quad (2.50) \]

where

\[ G_\eta(c) = \frac{2(c-c_o)(1-c_o)c}{(1-c^2_o)^2} \int \frac{f^2(t)}{c_o(1+t)^2} dt - \frac{(1+c^2)^2}{2} \int \frac{f^2(t)}{c_o(1+t)^2} dt + \]

\[ + \frac{(1-c^2)^2}{2} \int \frac{f^2(t)}{c_o(1-t)^2} dt, \quad (2.51) \]

and in addition,

\[ \left[ \frac{d^2 \theta}{dc^2} \right]_{c=c_o} = \frac{1}{K_\eta (1-c_o)^2} \int \frac{f^2(t)}{c_o(1+t)^2} dt \quad \text{for } K_\eta \neq 0. \quad (2.52) \]

In (2.50) the left-hand term originates from the inertia force; the term multiplied by \( K_\eta \), from the viscous force; and \( G_\eta(c) \) from the Lorentz force, which drives the fluid motion.

For non-zero values of \( K_\eta \), differential equation (2.50) can be integrated another time after dividing by \( (1-c^2)^2 \), resulting in the momentum equation in
integral form, see Jansen (1984). By substitution in (2.50), two integral
equations are obtained to calculate the vector components of the viscous
velocity field, viz.

\[
g(c) = \frac{(1-c^2)}{2K_\eta} \int_c^{\infty} \frac{g^2(t) + G_\eta(t)}{(1-t^2)^2} \, dt,
\]

for \( K_\eta > 0 \) (2.53)

\[
\frac{dg}{dc} = \frac{g^2(c) + G_\eta(c)}{2K_\eta(1-c^2)} - \frac{c}{K_\eta} \int_c^{\infty} \frac{g^2(t) + G_\eta(t)}{(1-t^2)^2} \, dt.
\]

(2.54)

These forms seem well suited to iteration.

By rewriting the expression of \( G_\eta(c) \) (2.51) into

\[
G_\eta(c) = \frac{2(1-c^2)}{(1-c_0)^2} c \int_c^{\infty} \frac{(t-c_0)(1-c_0 t)}{(1-t^2)^2} \, f^2(t) \, dt +
\]

\[
\frac{2(c-c_0)(1-c_0)}{(1-c_0)^2} \int_c^{\infty} \frac{f^2(t)}{(1+t)^2} \, dt,
\]

(2.55)

it is clear that \( G_\eta(c) \) is non-negative. Some further calculations yield

\[
G_\eta(c) > 0 \quad \text{on } c_0 < c < 1,
\]

(2.56)

\[
G_\eta(c_0) = 0(c-c_0), \quad G_\eta(1) = 0((1-c)^2).
\]

(2.57)

Now it follows from (2.37) and (2.57) that the integral equations (2.53), (2.54)
have a regular behaviour at \( c = 1 \).

Moreover, under the restriction of conditions (2.41), i.e. no appearance of
electric current inversion, it can be shown that the function \( G_\eta(c) \) possesses a
maximum at \( c = c_m \) and an inflection point at \( c = c_b \), for \( c_0 < c_m < c_b < 1 \).
In addition that \( G_\eta(c) \) is concave on \( c_0 < c < c_b \) and convex on \( c_b < c < 1 \).
Since \( G_\eta < f^2 \) at the maximum \( c = c_m \), the function \( G_\eta(c) \) is usually rather small
compared to unity.

At this stage already some important conclusions can be drawn from the results
obtained.
When the fluid is inviscid ($K_\eta = 0$), the Riccati differential equation (2.50) degenerates to

$$g^2(c) = - g_\eta(c) \quad \text{for} \ K_\eta = 0,$$  \hspace{1cm} (2.58)

where the right-hand side is given by (2.51). From (2.56) it is evident now, that in the inviscid fluid no real stream function $\psi$ and $g(c)$ can be found under the assumption of condition (2.37). This fact implies that an inviscid fluid motion with finite velocities throughout the field for $r > 0$ does not exist in this model. In chapter 3 a solution of the inviscid fluid motion will be derived with a relatively weak singularity on the axis of symmetry.

As a consequence, equations (2.50) - (2.52) are only valid for a viscous medium $K_\eta > 0$.

From (2.53) - (2.57) it follows that $g(c) > 0$ on $\eta_o < c < 1$ for $K_\eta > 0$. This typical behaviour of $g(c)$ indicates that in the viscous flow $\theta_\eta < 0$ on $\eta_o < c < 1$, $\nu_r < 0$ at $c = \eta_o$ and $\nu_r > 0$ at $c = 1$, see (2.21) - (2.23). Hence, real and finite solutions of $g$, which satisfy (2.18), (2.20), (2.50) - (2.54) for distinct values of $K_b$ and $K_\eta$ will yield flow patterns that differ only little. They consist of an incoming flow almost parallel to the cone wall, being accelerated in the neighbourhood of the point electrode by the increasingly rotational $\mathbf{j} \times \mathbf{B}$ force field. As a consequence the flow is rather abruptly turned off into an outwards jet flow along the axis of symmetry. As noted by Shercliff (1970) the fluid escapes into the region of weaker $\mathbf{j} \times \mathbf{B}$ forces. Since $g(c)$ does not contain any zero on $\eta_o < c < 1$ an outwards fluid discharge at another angle than $\theta = 0$ cannot exist. Solutions of the viscous flow in the semi-infinite point electrode configuration are presented in chapter 4, and different forms appear in chapter 6.

Expressions of $f$ and $\frac{df}{dc}$, which determine the angular behaviour of the radial current distribution $J_r$ and the magnetic field $B_\varphi$, see (2.15) and (2.24), are obtained from Ohm's law (2.20). The equation has been integrated twice; substitution of the boundary conditions (2.32) - (2.35) and some manipulations yield, see Jansen (1977, 1983)

$$f = \frac{1-c}{1-c_o} + \frac{K_b(1-c)}{1-c_o} \int \frac{g}{c_o} \left[ \frac{2(1-c_o)t}{1-t^2} + \left( t-c_o \right) \frac{df}{dt} \right] dt + \left( t-c_o \right) \frac{df}{dt} \bigg|_0^t$$
- \frac{K_b(c-c_o)}{1-c_o} \int_c^1 \frac{g}{1+t} \left\{ \frac{2f}{1-t^2} - \frac{df}{dt} \right\} dt,

(2.59)

and

\frac{df}{dc} = - \frac{1}{1-c_o} + \frac{2K_b g f}{1-c^2} - \frac{K_b c}{c_o} \int_c^{1-c_o} \frac{2(1-c_o) f}{1-t^2} \left( \frac{t-c_o}{1-t^2} \right) dt + \frac{df}{dt},

(2.60)

When the effect of the fluid motion upon the electromagnetic field is negligible, as it is usually in liquid metal applications, the expressions for \( J_r \) and \( B_\phi \) become

\[ J_r = \frac{I_o}{2\pi r^2(1-c_o)}, \]

(2.61)

for \( K_b \ll 1 \)

\[ B_\phi = \frac{\mu I_o}{2\pi(1-c_o)} \sqrt{\frac{1-c}{1+c}}. \]

(2.62)

As expected \( J_r \) is independent of \( \theta \), whereas the magnetic field varies with \( \tan \left( \frac{\theta}{2} \right) \). The Lorentz force, which drives the fluid motion, is in general in the direction of \(- \frac{\mathbf{J}}{\mu} \), viz. \( F_L = J \times B = F_L, \theta = \frac{\pi}{2} \). In case of low magnetic Reynolds number the angular component of the Lorentz force becomes

\[ F_{L, \theta} = - \frac{\mu I_o^2}{4\pi^2 r^3(1-c_o)^2} \sqrt{\frac{1-c}{1+c}} \quad \text{for} \quad K_b \ll 1, \]

(2.63)

indicating that the Lorentz force has the largest value on the surface of the cone, slowly decreases for smaller \( \theta \), and is identical to zero on the axis of symmetry, see figure 2.1. Note that the rotationality of the Lorentz force possesses the same \( c \) and \( c_o \) behaviour, viz. \( \text{curl} (J \times B) = \frac{2}{r} (J \times B) \times \frac{\mathbf{1}}{r} \).

In general, solutions of \( g, \frac{dg}{dc}, f, \frac{df}{dc} \) for arbitrary values of \( K_b \) and \( K_\eta \) \((K_\eta > 0)\) are obtained by simultaneous numerical calculation of four integral equations, given by (2.53), (2.54), (2.59), (2.60). However, the attention in
This thesis is focussed more upon calculation of analytical solutions and on derivation of generally valid properties of the fluid- and electromagnetic fieldquantities for special values of the parameters \( K_b \) and \( K_\eta \). In addition, we consider the difficulties arising in the viscous flow at small \( K_\eta \), as found by Sozou (1971a) et al.

### 2.8. Expressions of the other fieldquantities

Since the constant electric current \( I_0 \), supplied by the point electrode into the fluid, is considered here as the origin of the induced fluid motion, of the electric current distribution and of the magnetic field, the other fieldquantities are determined by the velocity- and electromagnetic fields.

The pressure generated both by the fluid motion and the magnetic field is obtained from the Navier-Stokes equation (2.2). Integration of the respective vector components of the equation with respect to \( r \) and \( c \) leads to an expression of the pressure distribution for \( K_\eta > 0 \), see Jansen (1984)

\[
p = p_\infty - \frac{\mu I_0^2}{16 \pi^2 r^2} \left[ \frac{d^2 p}{dc^2} + \frac{2a^2}{1-c^2} - 2K_\eta \left[ (1-c^2) \frac{d^3 g}{dc^3} - 2c \frac{d^2 g}{dc^2} \right] \right] \text{ for } K_\eta > 0 \tag{2.64}
\]

where \( p_\infty \) is a uniform reference pressure at infinity. Note that in the above expression the well-known term of magnetic origin has been eliminated on using (2.18), resulting in an expression valid for all \( K_\eta > 0 \).

The vector components of the electric field are obtained from Ohm's law (2.3). They are

\[
E_r = \frac{I_0}{2 \pi \sigma} \left\{ - \frac{df}{dc} + K_b \frac{g}{1-c^2} \right\}, \tag{2.65}
\]

\[
E_\theta = -\frac{I_0}{2 \pi \sigma} \frac{1}{r^2} K_b \frac{f}{\sqrt{1-c^2}} \frac{dg}{dc}. \tag{2.66}
\]

The azimuthal component \( E_\phi \) is identical to zero because the electric field is irrotational, see (2.5). The boundary condition of the electric field, (2.43) requires that the normal component \( E_n \) vanishes at the axis of symmetry. This involves an extra condition for the electromagnetic and fluid field, viz.
\[ E_s = \frac{I_o}{2\pi \sigma} \frac{1}{r^2} \left\{ -\sqrt{1-c^2} \frac{df}{dc} + \frac{K_b f}{\sqrt{1-c^2}} (g - c \frac{dg}{dc}) \right\} = 0 \quad \text{at } c = 1 \] (2.67)

The electric potential \( U \), associated with the electric field by the equation

\[ \vec{E} = -\text{grad} \ U , \] (2.6)

satisfies

\[ U = U_\infty + \frac{I_o}{2\pi \sigma} \frac{1}{r} \left\{ -\frac{df}{dc} + K_b \frac{gf}{1-c^2} \right\} , \] (2.68)

where \( U_\infty \) is the potential at large radial distance.

The space charge density, derived from (2.3) and (2.7) becomes, see Jansen (1983)

\[ \rho_e = \frac{\varepsilon I_o}{2\pi \sigma} \frac{1}{r^3} K_b \left\{ f \frac{d^2 g}{dc^2} + f \frac{dg}{dc} \right\} . \] (2.69)

Note that the space charge distribution is generated by the electromagnetic induction \( \gamma \times \mathbf{B} \). In magnetohydrodynamics its influence is usually very small and the charge distribution in itself is of no interest. However in inviscid and some viscous electrically-driven flows the effect of the space charge density might be important; especially in the neighbourhood of the axis of symmetry.

2.9. Some limiting cases when no fluid motion is generated

It was mentioned in the introduction that for the three-dimensional axisymmetric configuration the Lorentz forces are the reason why the fluid cannot remain at rest. Nevertheless there exist three limiting cases when no fluid motion is generated. They appear for \( K_\eta = \infty \), \( K_b = \infty \).

When \( K_\eta \) is infinite the medium is a solid so that as a matter of course no fluid motion can occur. The electromagnetic fieldquantities then satisfy (2.61) and (2.62).
Secondly when the fluid is a perfect conductor \( (\sigma = \infty) \), \( K_b \) is infinite. This implies that the right-hand side of Ohm's law (2.20) needs to be identical to zero. Integrating with respect to \( c \), results in

\[
fg^2 = a_\infty (1-c^2) \quad \text{for} \quad K_b = \infty, \tag{2.70}
\]

where \( a_\infty \) is a constant of integration. The boundary conditions (2.32) - (2.36), (2.38) require that \( a_\infty = 0 \), and as a consequence \( g = 0 \). Consideration of the respective equations of motion for the inviscid and viscous flows, see chapters 2, 3, 4, 6, indicates that the only remaining and also physically realistic solutions of \( f \) and \( g \) are

\[
g = 0 \quad \text{on} \quad c_0 < c < 1, \tag{2.71}
\]

\[
f = 0 \quad \text{on} \quad c_0 < c < 1, \quad \text{for} \quad K_b = \infty
\]

\[
f(c_0) = 1
\]

Now the total electric current \( I_0 \) supplied by the point electrode is forced to flow in a surface current layer on the surface of the cone towards the second electrode at large radial distance. The current sheet prevents the magnetic field to enter the fluid, so that no fluid motion can be generated. It should be noticed that this solution is at variance with Zhigulev's suggestion (1960a), that the current would be confined wholly to the axis of symmetry at high \( K_b \).

When both \( K_\eta \) and \( K_b \) tend to infinity, for \( K_\eta = \infty \) again no fluid motion can exist because the medium behaves like a solid. However in contrast to the former cases the result depends upon the sequence of the respective limits whether the electromagnetic field is shielded from the fluid or frozen in.
3. THE INVISCID POINT ELECTRODE PROBLEM

3.1. Introductory remarks

In the previous chapter it has been shown that solutions of the inviscid flow with finite velocities throughout the field for $r > 0$ cannot be obtained. Here solutions of the inviscid flow will be derived by permitting a relatively weak singularity in the flow field on the axis of symmetry. A general analysis of the electromagnetic- and flow fields at arbitrary values of the effective magnetic Reynolds number $K_b$ provides some interesting features. Asymptotic expansions at low magnetic Reynolds number are carried out, resulting in analytical expressions of the basic solutions of $f$ and $g$. A discussion of the problems, arising in the calculation of the electromagnetic- and flow fields at larger values of the magnetic Reynolds number, concludes this chapter.

The present investigation corresponds with practical liquid metal applications, where $K_\eta$ and $K_b$ are usually small compared to unity. Omitting the viscosity at first is a common simplification to render the mathematical problem more tractable.

3.2. The momentum equation of the inviscid flow in integral form

When the fluid is a liquid metal, the effective viscous Reynolds number is usually very large compared to unity ($K_\eta \ll 1$), see figure 2.2. The effect of the viscous force is then restricted to small regions, viz. the viscous boundary layer on the surface of the cone and the fluid jet along the axis of symmetry. It is then a common simplification to neglect the effect of viscosity at first to render the problem more tractable mathematically.

Shercliff (1970) pointed out that it is not necessary to include viscosity to achieve a steady state. In this context the author mentioned the subduing effect on the flow as a result of the continually shortening of the vorticity lines, being azimuthal circles, by the in-flow.

In section 2.7 it has been proved, see (2.58), that solutions of the inviscid flow ($K_\eta = 0$) with finite velocities throughout the field for $r > 0$ do not exist. Of course the no-slip condition was omitted in that consideration.

In the present examination we assume that the tangential velocity along the surface of the cone is bounded. Substitution of the boundary conditions (2.34) and (2.35) in expression (2.45) and of (2.34) in (2.44) then yield expressions of the constants of integration $P$, $Q$, $S$, valid for $K_\eta = 0$, which differ from
(2.46) - (2.48) for \( K_\eta > 0 \) and read

\[
p = \frac{1}{1-c^2} - \frac{2}{(1-c_0)^2} \int \frac{f^2(t)}{c_0 (1+t)^2} \, dt,
\]

for \( K_\eta = 0 \)

\[
Q = - \frac{2c_0}{1-c^2} + \frac{4c_0}{(1-c_0)^2} \int \frac{f^2(t)}{c_0 (1+t)^2} \, dt,
\]

\[
S = \frac{c_0^2}{1-c^2} - \frac{2c_0^2}{(1-c_0)^2} \int \frac{f^2(t)}{c_0 (1+t)^2} \, dt.
\]

Substitution of (3.1) - (3.3) in (2.44), (2.45) indicates that \( g^2(1) = 0 \), but that \( \left. \frac{dg}{dc} \right|_{c=1} \) is a constant. Hence the assumption of finite velocities on the axis of symmetry (2.37) cannot be satisfied and from the above considerations a real inviscid fluid motion is obtained when we introduce, see Jansen (1977)

\[
g(c) = \sqrt{1-c^2} \, h(1+o(1)) \quad \text{for } c \uparrow 1, \ K_\eta = 0,
\]

where \( h \) will be a positive constant with respect to \( c \).

The proposed behaviour of \( g(c) \) when \( c \) approaches unity, as given by (3.4), is a limiting case of what is physically admissible. The more accurate condition (2.36) for the non-existence of fluid sources or sinks on the axis of symmetry is satisfied if \( g(1) = O((1-c)^\alpha) \) for \( \alpha > \frac{1}{2} \).

The solution (3.4) obtained from a mathematical argument is in full agreement with the solution of the flat wall configuration \( c_0 = 0 \), suggested by Shercliff (1970) from a more physical point of view. Shercliff noted that if the flow upstream (near the surface of the cone) approaches an irrotational flow, then the flow downstream (along the axis of symmetry) is necessarily singular on the axis of symmetry.

Although Sozou (1971a, 1972, 1974) and Sozou & English (1972) considered this solution of the inviscid flow to be physically unrealistic, we must accept the inviscid flow with a relatively weak singularity on the axis of symmetry as the only one that is available within the framework of the similarity method. This is in particular the case since no physical principle is violated. Moreover analytical investigation of the inviscid flow at arbitrary values of the effective magnetic Reynolds number \( K_b \) shows that a real solution of the inviscid
flow always exists and that physically non-realistic phenomena, such as electric current inversion do not occur, see Jansen (1977). In section 3.3 we will present a brief review of that examination.

Introduction of (3.4) induces a strong jet flow along the axis of symmetry, with a relatively weak singularity at \( c = 1 \), viz.

\[
\frac{d g}{d c} = - \frac{c h}{\sqrt{1-c^2}} \to -\infty , \tag{3.5}
\]

\[
v_r = \frac{I_o}{2\pi} \sqrt{\frac{\mu}{\rho}} \frac{c h}{r\sqrt{1-c^2}} \to +\infty , \tag{3.6}
\]

\[
v_\theta = -\frac{I_o}{2\pi} \sqrt{\frac{\mu}{\rho}} \frac{h}{r} , \quad \text{for } c \to 1 \tag{3.7}
\]

\[
v_s = 0(1-c^2) \to 0 , \tag{3.8}
\]

\[
v_z = \frac{I_o}{2\pi} \sqrt{\frac{\mu}{\rho}} \frac{h}{r\sqrt{1-c^2}} \to +\infty . \tag{3.9}
\]

In spite of the singularity, the boundary conditions (2.35), (2.36), (2.40) are still satisfied at \( c = 1 \), so that the inviscid solution does not involve fluid sources or sinks on the axis of symmetry.

The mass flow \( \Psi \) through a small circle of radius \( s = r\sqrt{1-c^2} \) with its centre on and located in a plane perpendicular to the axis of symmetry becomes

\[
\Psi = \rho \int \int v \cdot n \, dS = I_o \sqrt{\rho \mu} \frac{h}{r} \sqrt{1-c^2} \quad \text{for } s = 0 , \tag{3.10}
\]

indicating that the mass flow remains finite and approaches zero with the radius of the circle.

By substituting (3.1) - (3.3) in expressions (2.44) - (2.45) for \( K_\eta = 0 \) and including (3.4) in the calculation, the momentum equation of the inviscid flow in integral form takes the form, see Jansen (1977)
\[ g^2 = \frac{(1+c)^2}{2} \int_0^c \frac{f^2(t)}{(1+t)^2} \, dt - \frac{(1-c)^2}{2} \int_0^c \frac{f^2(t)}{(1-t)^2} \, dt + \]

\[ \frac{2(c-c_o)^2 h^2}{1-c_o^2} , \quad \text{for } K_\eta = 0 \] (3.11)

with the constant \( h \), as defined in (3.4), denoting the expression

\[ h^2 = \frac{1+c}{1-c_o} \int_0^c \frac{f^2(t)}{(1+t)^2} \, dt , \] (3.12)

where \( h \) is in fact a parameter, being only dependent on \( c_o \) and \( K_b \), see (2.16), (2.59), (2.60).

The derivative of \( g^2 \) is given by

\[ \frac{dg^2}{dc} = (1+c) \int_0^c \frac{f^2(t)}{(1+t)^2} \, dt + (1-c) \int_0^c \frac{f^2(t)}{(1-t)^2} \, dt + \]

\[ - \frac{4(c-c_o)^2 h^2}{1-c_o^2} , \quad \text{for } K_\eta = 0 \] (3.13)

The above expressions show that the sign of \( g \) and so the direction of the inviscid flow is indeterminate. Since the direction of the Lorentz force \( \mathbf{J} \times \mathbf{B} \), see (2.63), is purely meridional and positive in the direction of \( -\mathbf{z} \), and by consideration of the flow fields, found in experiments by Woods & Milner (1971), Kulaev & Erkhojin (1974), Bojarevišs & Shcherbinin (1983), et al., an outward jet flow along the axis of symmetry, directed away from the electrode has been chosen; i.e. \( v_z \) and \( g \) positive, see (3.4) - (3.9). Since the Lorentz force creates vorticity, Shercliff (1970) concluded that flow from the region of zero vorticity (along the surface of the cone) is the natural one to choose. Also Jansen (1977) showed that the chosen direction of the inviscid flow agrees with the direction of the expected viscous flow, see (2.53) and (2.56).
This choice of sign of $g$ also implies that $h$ and $\frac{dg}{dc} \Big|_{c=c_0}$ are positive. So we propose

$$g = + \sqrt{g^2},$$

$$\frac{dg}{dc} = \frac{1}{2g} \frac{dg^2}{dc},$$

and at the surface of the cone, $c = c_0$

$$\frac{dg}{dc} \Big|_{c=c_0} = + \sqrt{\frac{1-h^2}{1-c_0^2}}.$$

(3.14)

(3.15)

(3.16)

From (2.21), (2.22), (3.6), (3.7), (3.16), it is obvious that the constant $h$ couples the velocities along the surface of the cone and along the axis of symmetry.

Simultaneous numerical computation of the four integral equations (2.59), (2.60), (3.11) - (3.13) and using (3.4), (3.5), (3.14) - (3.16) will result in solutions of the electromagnetic field and of the inviscid flow field, viz. $g$, $\frac{dg}{dc}$, $f$, $\frac{df}{dc}$, for arbitrary values of the effective magnetic Reynolds number $K_b$.

3.3. General analysis of the electromagnetic and fluid field quantities at arbitrary values of the magnetic Reynolds number

The fluid motion, generated by Lorentz forces, may alter the originally isotropic electric current distribution due to electromagnetic induction, i.e. the $v \times B$ term in Ohm's law (2.3), see figure 2.1. In case of low magnetic Reynolds number $K_b$, which is usual in liquid metal applications, the effect of the electromagnetic induction upon the electric current distribution is small. For the inviscid point electrode problem this effect has been studied by Shercliff (1970) for the flat wall configuration $c_0 = 0$ and by Jansen (1983) for arbitrary value of $c_0$, $-1 < c_0 < 1$. In report LR-256, see Jansen (1983), it has been shown that within the validity range of an asymptotic expansion at low magnetic Reynolds number no electric current inversion occurs, viz. condition (2.41) is always satisfied. The typical low $K_b$ behaviour of the electromagnetic
and fluid field quantities at arbitrary $c_\circ$ will be discussed in section 3.4. Upon using the properties of the fluid and electromagnetic fields at low magnetic Reynolds number, especially the condition $\frac{df}{dc} < 0$ for $c_\circ < c < 1$, i.e. (2.41), a qualitative analysis has been carried out by Jansen (1977) for larger and arbitrary values of $K_b$. The investigations showed that for all finite values of $K_b$ a real inviscid fluid motion exists, with a flow pattern almost similar to the one obtained at low $K_b$, see section 3.4. Moreover it has been proved that inversion of electric current cannot occur for any $K_b$. For a detailed analysis the reader is referred to report LR-228, see Jansen (1977). Here a brief review of the main properties of the functions $f$ and $g$ derived for all $K_b$ and $\eta = 0$ will be presented.

By application of (2.41) and the resulting condition (2.42) upon integral expressions of $g^2$ and its derivatives, see e.g. (3.11) - (3.13) and by calculation of the values of the functions at the boundaries $c = c_\circ$ and $c = 1$, it follows that $g^2$ is non-negative and behaves as shown in figure 3.1.a.

![Figure 3.1. a + b. Qualitative pictures of the behaviour of the functions $g$, $g^2$, $f$ as function of $c$.](image-url)
This conclusion is based upon the following considerations. From (2.18), (2.41), (2.42) it is clear that $\frac{d^3g}{dc^3} < 0$ on $c_o < c < 1$. Hence $\frac{d^2g}{dc^2}$ is a monotonically decreasing function, and $\frac{dg}{dc}$ is concave. Since $\frac{d^2g}{dc^2} |_{c=c_o} > 0$ and $\frac{d^2g}{dc^2} |_{c=1} < 0$, the function $\frac{d^2g}{dc^2}$ has only one zero on $c_o < c < 1$, say at $c = c_b$. The concave shape of $\frac{dg}{dc}$ on the interval $c_o < c < 1$ and the fact that $\frac{d^2g}{dc^2} |_{c=c_o} > 0$, $\frac{d^2g}{dc^2} |_{c=c_m} = 0$ and $\frac{d^2g}{dc^2} |_{c=1} < 0$ indicate that $\frac{dg}{dc}$ has only one zero on $c_o < c < 1$, say at $c = c_m$. This implies that $\frac{dg}{dc} > 0$ on $c_o < c < c_m$ and $\frac{dg}{dc} < 0$ on $c_m < c < 1$, whereas $\frac{dg}{dc} = 0$ at $c = c_o$ and $c = c_m$. From these considerations and the boundary conditions $g^2(c_o) = g^2(1) = 0$, it follows that $g^2$ is non-negative on $c_o < c < 1$ and that $g^2$ has precisely one maximum at $c = c_m$ and one inflexion point at $c = c_b$, where $c_o < c_b < c_m < 1$. As a consequence $g^2$ is convex on $c_o < c < c_b$ and concave on $c_b < c < 1$, as shown in figure 3.1.a.

Some further calculations, including series expansions of the functions $f$ and $g$ in powers of $c-c_o$ and $(1-c)^{\frac{1}{2}}$, respectively valid in the neighbourhood of $c = c_o$ and $c = 1$, yield the properties

$$g - (c-c_o) \frac{dg}{dc} > 0,$$  \hspace{1cm} (3.17)

$$cg + (1-c^2) \frac{dg}{dc} > 0,$$  \hspace{1cm} (3.18)

$$\frac{d^2g}{dc^2} < 0,$$  \hspace{1cm} (3.19)

for $c_o < c < 1$, and

$$0 < h^2 < \frac{1}{2},$$  \hspace{1cm} (3.20)

$$\frac{d^2f}{dc^2} |_{c=c_o} > 0, \quad \frac{d^2f}{dc^2} |_{c=1} < 0,$$  \hspace{1cm} (3.21)

$$\frac{d^3f}{dc^3} |_{c=c_o} < 0, \quad \frac{d^3f}{dc^3} |_{c=1} < 0.$$  \hspace{1cm} (3.22)
The above results and (3.14), (3.16) imply that the function \( g \) is non-negative on \( c_o < c < 1 \), with a maximum at \( c = c_m > 0 \), like \( g^2 \). In addition that \( g(c) \) is concave on \( c_o < c < 1 \), as depicted in figure 3.1.a.

Examination of the maximum value of \( g^2 \) at \( c = c_m \) indicates that \( g^2 \) is always smaller than a half, so that with (3.14)

\[
0 < g^2(c) < g(c) < \frac{1}{2} \sqrt{2} \quad \text{for } c_o < c < 1.
\]  

(3.23)

In section 2.6 it was found that the stream function \( \psi \) needs to be identical to zero on the surface of the cone and the axis of symmetry. Also the expressions of the partial derivatives of \( \psi \) with respect to \( s = r \cdot \sqrt{1-c^2} \) and \( z = rz \) satisfy for \( K_\eta > 0 \)

\[
\frac{\partial \psi}{\partial s} = \frac{I_0}{2\pi} \sqrt{\frac{\mu}{\rho}} \cdot \sqrt{1-c^2} \left\{ g - c \frac{dg}{dc} \right\},
\]  

(3.24)

\[
\frac{\partial \psi}{\partial z} = \frac{I_0}{2\pi} \sqrt{\frac{\mu}{\rho}} \left\{ cg + (1-c^2) \frac{dg}{dc} \right\}.
\]  

(3.25)

Hence from (3.18) and the above considerations about the behaviour of \( g \), it is clear now that in the inviscid flow \( \frac{\partial \psi}{\partial s} > 0 \) for \( c_m < c < 1 \) and \( \frac{\partial \psi}{\partial z} > 0 \) for \( c_o < c < 1 \). In addition upon using (3.4), we find that in the neighbourhood of the axis of symmetry the derivatives approach

\[
\frac{\partial \psi}{\partial s} = \frac{I_0}{2\pi} \sqrt{\frac{\mu}{\rho}} \cdot h,
\]  

(3.26)

for \( c \uparrow 1 \), \( K_\eta = 0 \)

\[
\frac{\partial \psi}{\partial z} = O((1-c)^{3/2}) \rightarrow 0,
\]  

(3.27)

indicating that in the inviscid case the incoming flow, being almost parallel to the surface of the cone, converges into a parallel outward jet flow along the axis of symmetry for all values of \( K_b \).

By consideration of Ohm's law (2.20) and inequality (3.18), it can easily be proved that the function \( f \) is non-oscillatory on \((c_o,1)\), see Ince (1956). This
means that \( f \) has no more than one zero on \((c_o, 1)\). However, if we suppose that \( f \) has a zero on \((c_o, 1)\), say at \( c = c_t \), the boundary condition \( f(1) = 0 \) requires that \( f \) possesses at least one negative minimum on \((c_t, 1)\), which is excluded by (2.20) and (3.18). Therefore we must conclude that \( f(c) \) is positive on \([c_o, 1)\).

From (3.21) it follows that \( f \) is convex in the neighbourhood of \( c = c_o \) and concave near \( c = 1 \). Shercliff (1970) and Jansen (1983) found that in case of small \( K_b \) the function \( f \) possesses one inflexion point, viz. \( \frac{d^2 f}{dc^2} = 0 \) in \( c = c_a \), separating the convex and concave parts of the function. On assuming that \( f \) could show more than one inflexion point when \( K_b \) exceeds a certain value, there must be a special value of \( K_b = K_b^* \) where \( \frac{df}{dc} < 0 \), \( \frac{d^2 f}{dc^2} = \frac{d^3 f}{dc^3} = \frac{d^4 f}{dc^4} = 0 \), \( \frac{d^5 f}{dc^5} < 0 \) at \( c = c_a \); see figure 8 of report LR-228, Jansen (1977). Extensive calculation showed that this assumption was not allowed, so that the function \( f \) always contains just one inflexion point at \( c = c_a \), with \( \frac{d^2 f}{dc^2} > 0 \) on \( c_o < c < c_a \), \( \frac{d^2 f}{dc^2} = 0 \) at \( c = c_a \), and \( \frac{d^2 f}{dc^2} < 0 \) on \( c_a < c < 1 \) for all \( K_b \).

In a similar way it has been proved that the equations and derived relations do not allow that \( \frac{df}{dc} = 0 \) on \([c_o, 1)\) for a certain value of \( K_b \). Hence the derivative of \( f \) remains negative for all \( K_b \),

\[
\frac{df}{dc} < 0 \quad \text{on} \quad c_o < c < 1 \quad \text{for} \quad K_b^* = 0 ,
\]  

(3.28)

indicating that \( f \) is a monotonically decreasing function, as depicted in figure 3.1.b, satisfying (2.42). Note that the dashed line in that figure represents the behaviour of \( f \) when \( K_b \) approaches zero, see (2.59). In addition some further calculations yield

\[
\frac{d^3 f}{dc^3} < 0 \quad \text{on} \quad c_o < c < 1 ,
\]  

(3.29)

indicating that \( \frac{df}{dc} \) is concave on the interval.

Note that the properties of \( f \) and \( g \), as derived in this section, have been obtained under the assumption that \( \frac{df}{dc} < 0 \) on the interval. Since this leads to (3.28), viz. \( \frac{df}{dc} < 0 \), as it is at low \( K_b \), we must conclude that these properties are valid for all \( K_b \). Note that \( \frac{df}{dc} = 0 \) is only possible for \( K_b = \infty \) on \( c_o < c < 1 \), see (2.71). Hence for all finite values of the magnetic Reynolds number \( K_b \) a real solution of the inviscid fluid motion exists, with a relatively weak singularity on the axis of symmetry. Moreover in case of an inviscid fluid
motion no electric current inversion, see (2.24) and (3.28), can occur for any \( K_b \) and conditions (2.41) and (2.42) are always satisfied. In this context it should be noticed that the expression of \( \frac{\partial \psi}{\partial z} \), see (3.25), and the term multiplying \( f \) in Ohm's law (2.20) indicate that inversion of electric current only may occur when \( \frac{\partial \psi}{\partial z} < 0 \). Therefore electric current inversion may exist at larger values of \( K_b \) in the diverging viscous jet flow along the axis of symmetry, see Sozou & English (1972).

At large \( K_b \) the electromagnetic- and flow fields show typical boundary layer behaviour. Investigation of development of boundary layers has been carried out by setting \( K_b^{-1} = 0 \) in the equations of \( f \) and \( g \), and by examination whether the reduced equations are able to satisfy the respective boundary conditions.

Consideration of the differential equation of the function \( g \), see (3.43) of LR-228, Jansen (1977), indicates that at increasing value of the magnetic Reynolds number the incoming flow is almost parallel with and concentrated near the surface of the cone, whereas the strength of the jet flow along the axis of symmetry slowly decreases. From a mathematical point of view it is clear that only at the surface of the cone a boundary layer develops. Note that this inviscid boundary layer strongly differs from the well-known viscous velocity boundary layer. It is more similar to the so-called skin layer of electric current, namely the maximum value coincides with the interface.

Consideration of Ohm's law (2.20), the solution of the reduced equation (2.70) for \( K_b^{-1} = 0 \) and (3.4) show that the magnetic field will contain boundary layers at the surface of the cone and at the axis of symmetry. As a consequence at increasing \( K_b \) the electric current is shifted to the axis and the cone-surface; whereas in the middle region of \( \theta \) the electric current flow is strongly reduced by the effect of the electromagnetic induction. This effect was first noticed by Shercliff (1970). As discussed in section 2.9 the electric current will be entirely concentrated in a surface current-sheet on the surface of the cone, when the magnetic Reynolds number approaches infinity. Thus the behaviour as described above appears at large, but intermediate values of \( K_b \). The problems arising in the calculation of the electromagnetic- and flow fields at larger values of \( K_b \) will discussed in section 3.5.

Investigation of the behaviour of the other fieldquantities showed that the transverse component of the electric field does not satisfy the boundary condition (2.67) at the axis of symmetry, namely \( E_{x} \neq 0 \) at \( c = 1 \), and that the space charge density contains a weak singularity at the axis of symmetry. It is evident that these inconsistencies arise from the relatively weak singularity in the flow field at the axis of symmetry, which must be allowed there in order to obtain a real solution of the inviscid fluid motion.
Though the behaviour of $\rho_e$ and $E_s$ does not involve the appearance of local electric sources or sinks, it would imply that in the inviscid fluid model the effect of the Coulomb force $\rho_e E$ in Euler's equation (2.2) and of the convection current density $\rho_e v$ in Ohm's law (2.3) cannot be neglected in a region near the axis of symmetry. Moreover the usual M.H.D. approximation: $|v| \ll c_L$, where $c_L$ is the velocity of light, is not satisfied at that place. However, on using (3.20), it turns out that in liquid metals, e.g. molten steel at $I_0 = 10,000$ A these effects only become dominant at a distance normal to the axis of symmetry less than $10^{-9}$ m. Therefore they can be neglected in a theoretical treatment based upon the continuum hypothesis, see Batchelor (1967). One would expect that adding viscosity to the problem may resolve the mathematical difficulties and physical inconsistencies.

3.4. Asymptotic expansions at small magnetic Reynolds number

In practical applications using liquid metals at non-excessively large electric currents the magnetic Reynolds number $K_b$ will be usually small compared to unity, see figure 2.2. In that case the electric current and its associated magnetic field are only slightly disturbed by the induced fluid motion. The electric current then flows almost isotropically outwards from the point electrode to the second electrode at large radial distance. In order to calculate the generated fluid motion and the effect of the weak mutual interaction between the flow- and electromagnetic fields, it is useful to carry out straightforward expansions of the functions, appearing in the governing equations, at small values of $K_b$. In extensive form this has been done by Jansen (1983) for all field quantities appearing in (2.1) - (2.7). Here we will confine ourselves to the main features of the fluid- and electromagnetic field-quantities. For more details and the behaviour of the other field quantities the reader is referred to report LR-256, see Jansen (1983).

In general the asymptotic expansions are of the form

$$t(c) = \sum_{n=0}^{\infty} K_b^n t_n(c) \text{ for } K_b \ll K_{b,\text{max}},$$

(3.30)

where $t$ represents the function to be expanded, e.g. $f, \frac{df}{dc}, g, \frac{dg}{dc}, g^2, \frac{dg^2}{dc}$, etcetera.
The magnitude of $K_{b, \text{max}}$ depends on the function to be expanded and on the value of $c_0$. $K_{b, \text{max}}$ is more-or-less determined by the condition $|t_{n+1}| < |t_n|$ for $c_0 < c < 1$ and $n = 0, 1, 2, \ldots, N$; where $N$ is the number at which the series is truncated. It has been found by numerical calculation that $K_{b,\text{max}} \sim \frac{6}{1-c_0^2 |c_0|}$ for $30^\circ < \theta_0 < 150^\circ$ and $N = 12$. To be on the safe side a slightly smaller value of $K_{b,\text{max}}$, viz. $K_{b,\text{max}}$ is chosen, namely: $K_{b,\text{max}} = 10$ for $c_0 = \frac{1}{2} \sqrt{2} (\theta_0 = 45^\circ)$, $K_{b,\text{max}} = 5$ for $c_0 = 0 (\theta_0 = 90^\circ)$ and $K_{b,\text{max}} = 2$ for $c_0 = -\frac{1}{2} \sqrt{2} (\theta_0 = 135^\circ)$. Shercliff (1970), when considering the flat wall problem ($c_0 = 0$), showed that an analytical calculation of the zero-order regular perturbation solutions of $f$ and $g$ is possible. Appying (3.30) to the integral expressions (2.59), (2.60), (3.11) - (3.13) and some elementary calculations yield the expressions of the zero-order perturbation solutions for arbitrary value of the vertical angle of the right circular cone, viz. $-1 < c_0 < 1$, see Jansen (1983).

\[ f_0 = \frac{1-c}{1-c_0}, \quad (3.31) \]

\[ \frac{df_0}{dc} = -\frac{1}{1-c_0}, \quad (3.32) \]

\[ g_0 = a_0 + a_1 c + a_2 c^2 + a_3 (1+c)^2 \ln (1+c), \quad (3.33) \]

\[ \frac{dg_0^2}{dc} = (a_1 + a_3) + (2a_2 + a_3) c + 2a_3 (1+c) \ln (1+c), \quad (3.34) \]

\[ h_0^2 = \frac{1}{2} (a_0 - a_2 - 2a_3), \quad (3.35) \]

for $c_0 < c < 1$. The coefficients $a_0 - a_3$, being only dependent on $c_0$, are given by the expressions

\[ a_0 = -\frac{2c_0 (1+c_0)}{(1-c_0)^3} + \frac{2(1+c_0)(1-3c_0)}{(1-c_0)^4} \ln (1+c_0) + \frac{8c_0^2}{(1-c_0)^4} \ln (2), \quad (3.36) \]
\[ a_1 = \frac{2(1+c_o)^2}{(1-c_o)^3} + \frac{4(1+c_o)^2}{(1-c_o)^4} \ln \left(1+c_o\right) - \frac{16c_o}{(1-c_o)^4} \ln \left(2\right), \]  
(3.37)

\[ a_2 = -\frac{2(1+c_o)}{(1-c_o)^3} - \frac{2(1+c_o)(3-c_o)}{(1-c_o)^4} \ln \left(1+c_o\right) + \frac{8}{(1-c_o)^4} \ln \left(2\right), \]  
(3.38)

\[ a_3 = -\frac{2}{(1-c_o)^2}. \]  
(3.39)

It turns out that expressions (3.33) and (3.35) are also bounded and single-valued for \( c_o = 1 \). Note that the coefficients \( a_0 - a_3 \) are generally used in analytical expressions of \( g \) or \( g^2 \). Therefore they are not related to those used for the viscous problem, see (4.10) - (4.15). It may also be mentioned that the general solution of \( g_o^2(c) \) (3.33), being valid for \( -1 < c_o < 1 \), was found by the author independently from Narian & Uberoi (1973), which has been referred to by Andrews & Craine (1978).

Expressions of \( g_o^2(c) \) for some special values of \( c_o \) take the form

\[ g_o^2(c) = 0 \quad \text{for} \quad c_o = 1, \]  
(3.40)

\[ g_o^2(c) = \frac{(c-c_o)^2(1-c)}{3(1-c_o)^2} \left(1 + o(1)\right) \quad \text{for} \quad c_o = 1, \]  
(3.41)

\[ g_o^2(c) = 2c(1-c) + 8c^2 \ln \left(2\right) - 2(1+c)^2 \ln \left(1+c\right) \quad \text{for} \quad c_o = 0, \]  
(3.42)

\[ g_o^2(c) = -\frac{1}{2} \left(1+c\right)^2 \ln \left(\frac{1+c}{2}\right) \quad \text{for} \quad c_o = -1. \]  
(3.43)

For \( c_o = 1 \) \( (\theta_o = 0) \) the general solution (3.33) satisfies the condition that no fluid motion can exist when no fluid region is available. In case of the flat wall configuration, i.e. \( c_o = 0 \), the expression of \( g_o^2(c) \) (3.42) becomes identical to the solution found by Shercliff (1970) on using the relations (2.27) - (2.29) with the subscript sh. Numerical calculation for different
values of \( c_0 \) indicates that \( g_0^2(c) \) acquires the largest magnitude at \( c_0 = -1 \) \((\theta_0 = 180^\circ)\). In addition analytical calculation shows that \( g_0^2(c) \) is bounded for \( c_0 = -1 \). In this respect the above inviscid solution differs from the analytical solution of the slow viscous flow, see section 4.2.

From (3.31) - (3.39) solutions of \( g_0, \frac{dg_0}{dc}, h_0 \) and as a result of \( v_{r,o}, v_{\theta,o}, J_{r,o}, B_{\varphi,o}, \varphi_o \) can be obtained on using (2.14), (2.15), (2.21) - (2.24), (3.4) - (3.9), (3.14) - (3.16).

Investigation of the basic solutions shows that the functions display the properties mentioned in the previous section and as derived in Jansen (1977). It can be proved, see appendix A, that the basic solutions \( g_0^2(c) \) and \( h_0^2 \) are bounded by

\[
0 < g_0^2(c) < \frac{1}{2} \quad \text{for } c_0 < c < 1, \quad -1 < c_0 < 1, \quad (3.44)
\]

\[
\frac{1}{6} < h_0^2 < \frac{1}{2} \quad \text{for } -1 < c_0 < 1. \quad (3.45)
\]

These results are in agreement with the general conditions (3.20) and (3.23). Hence we conclude that at small \( K_0 \) a real inviscid fluid motion always exists.

Values of \( h_0^2 \) for some special values of \( c_0 \) are

\[
h_0^2 = \frac{1}{6} \quad \text{for } c_0 = 1, \quad (3.46)
\]

\[
h_0^2 = \frac{(5-c_0)}{24} (1 + o(1)) \quad \text{for } c_0 \to 1, \quad (3.47)
\]

\[
h_0^2 = 3 - 4 \ln(2) = 0.2274 \quad \text{for } c_0 = 0, \quad (3.48)
\]

\[
h_0^2 = \frac{1}{2} \quad \text{for } c_0 = -1. \quad (3.49)
\]

Since \( \frac{dh_0^2}{dc} < 0 \), see theorem 2 of appendix A, it follows from (3.9) that the strength of the inviscid jet flow at the axis of symmetry slowly decreases for smaller values of \( \theta_0 \). From (3.4) it is clear that the results \( g_0^2(c) = 0 \) and \( h_0^2 \neq 0 \) for \( c_0 = 1 \) do not clash.

On the surface of the cone the velocity of the incoming flow is determined by,
see (2.21) and (3.16)

\[
\frac{dg_o}{dc} \bigg|_{c=c_o} = \sqrt{\frac{1-2h^2_o}{1-c^2_o}}.
\]

(3.50)

For some special values of \(c_o\), \(\frac{dg_o}{dc} \bigg|_{c=c_o}\) satisfies

\[
\frac{dg_o}{dc} \bigg|_{c=c_o} = 0 \quad \text{for } c_o = 1,
\]

(3.51)

\[
\frac{dg_o}{dc} \bigg|_{c=c_o} = \frac{1}{\sqrt{3(1-c_o)}} (1+o(1)) + + \infty \quad \text{for } c_o \to 1,
\]

(3.52)

\[
\frac{dg_o}{dc} \bigg|_{c=c_o} = -5 + 8 \ln(2) = 0.7384 \quad \text{for } c_o = 0,
\]

(3.53)

\[
\frac{dg_o}{dc} \bigg|_{c=c_o} = -\frac{1}{2} \ln (1+c_o) (1+o(1)) + + \infty \quad \text{for } c_o \to -1,
\]

(3.54)

where (3.51) follows from (3.40). These expressions indicate that the inwards directed radial velocity on the surface of the cone tends to infinity when \(c_o\) approaches \(\pm 1\). Note that the singularity at \(c_o = -1\) is weaker than the one for \(c_o = 1\). In contrast with \(g_o^2(c)\) and \(h_o^2\), the behaviour of \(\frac{dg_o}{dc} \bigg|_{c=c_o}\) for respectively \(c_o = 1\) and \(c_o = -1\) is discontinuous. From (3.45) and (3.50) it is clear that \(\frac{dg_o}{dc} \bigg|_{c=c_o}\) is positive for \(-1 < c_o < 1\). Numerical calculation of the function for different values of \(c_o\) shows that the behaviour of \(\frac{dg_o}{dc} \bigg|_{c=c_o}\) is smooth for intermediate values of \(c_o\); with a minimum value of 0.7250 at \(c_o = -0.2209 (\theta_o \sim 103^\circ)\), see Jansen (1983).

Calculation of the velocity normal to the surface of the cone in the neighbourhood of the boundary indicates that the transverse velocity component of the incoming flow (being identical to zero on the cone-surface) is directed away
from the cone for \(0 < \theta_o < 130.5^\circ\) and towards the cone for larger \(\theta_o\) up to \(180^\circ\), see report LR-256 (eq. (3.22), figs 2 & 3), Jansen (1983). Figure 3.2. presenting flow patterns for respectively \(\theta_o = 45^\circ, 90^\circ, 135^\circ\), shows that the effect is rather small. The effect may become significant only in figures showing in detail streamlines of constant low-value \(\bar{\psi}_o\) near the surface of the cone.

In order to examine the effect of the electromagnetic induction upon the electric current distribution and the weak mutual interaction between the flow- and electromagnetic fields, the higher-order perturbation solutions need to be determined. However it turns out that analytical calculation of the higher-order solutions leads to insurmountable complexities. Therefore numerical computations have been carried out on expressions, obtained by application of the series expansion in powers of \(K_b\), see (3.30), upon the functions appearing in the integral expressions of \(f, \frac{df}{dc}, g^2\) and \(\frac{dg^2}{dc}\), see (2.59), (2.60), (3.11) - (3.13).

For figures presenting the behaviour of the regular perturbation solutions of \(f, \frac{df}{dc}, g^2, \frac{dg^2}{dc}\) for \(c_o = 0, \pm \frac{1}{2} \sqrt{2} (\theta_o = 45^\circ, 90^\circ, 135^\circ)\) and \(n = 0 - 3\) we refer to Jansen (1983).

To present here figures of the flow field and the electric current distribution we introduce

\[
\phi = \frac{I_o}{2\pi} \sqrt{\frac{\mu}{\rho}} \{\bar{\psi}_o + \Delta \bar{\psi}\},
\]  

(3.55)

\[
J_r = \frac{I_o}{2\pi r^2} \{\bar{J}_{r,0} + \Delta \bar{J}_r\},
\]  

(3.56)

where respectively the terms with suffix zero represent the basic solutions and the \(\Delta\) terms sums of an in general infinite, but here limited number of higher-order perturbation solutions, viz. see (2.14), (2.24), (3.30), (3.32)

\[
\bar{\psi}_o = r g_o(c),
\]  

(3.57)

\[
\Delta \psi = r \sum_{n=1}^{N} K^n_b g_n(c),
\]  

(3.58)

\[
\bar{J}_{r,0} = -\frac{df}{dc} = \frac{1}{1-c_o},
\]  

(3.59)
\[ \Delta \vec{J}_r = - \sum_{n=1}^{N} k_b^n \frac{df}{dc} \ Quad (3.60) \]

As noticed before up to \( k_b = k_{b,\text{max}} \), where \( k_{b,\text{max}} \) is respectively 10, 5, 2 for \( c_0 = \frac{1}{2} \sqrt{2}, 0, -\frac{1}{2} \sqrt{2} \), it has been found that a number of twelve terms (\( N = 12 \)) amply suffices for all field quantities to be considered.

In figure 3.2, streamlines of constant modified stream functions \( \Phi_o \) and \( \Delta \Phi \) at \( k_b = 1 \) are depicted for \( c_0 = 0, \pm \frac{1}{2} \sqrt{2} \). The graphs of constant \( \Phi_o \) represent the flow field as generated by the isotropic current distribution and its associated magnetic field. The figures of constant \( \Delta \Phi \) streamlines show the effect of the mutual interaction. The summation of \( \Phi_o \) and \( \Delta \Phi \) equals the total flow field at \( k_b = 1 \).

The graphs of \( \Delta \Phi \) show that the perturbation part of the flow field is divided into two parts, separated by a streamline \( \Delta \Phi = 0 \) at an angle \( \theta \approx 0.8 - 0.9 \theta_o \). Moreover, they indicate that the velocities of the base flow are reduced in almost the entire flow field; except near the surface of the cone, where the tangential velocity directed towards the point electrode increases. The effect increases for larger \( k_b \), with the region of increasing velocity concentrated closer to the cone wall. At \( k_b = k_{b,\text{max}} \) the influence of \( \Delta \Phi \) upon the base flow \( \Phi_o \) is at most about 10 per cent.

In figure 3.3 the modified isotropic current distribution \( \vec{J}_{r,\phi} \) and the total perturbation part \( \Delta \vec{J}_r \) at \( k_b = 1 \) are sketched for \( c_0 = 0, \pm \frac{1}{2} \sqrt{2} \). As given in \( (3.56) \), summation of both terms yields the total electric current distribution at \( k_b = 1 \). Similar to the perturbed part of the flow field, \( \Delta \vec{J}_r \) turns out to be almost proportional to \( k_b \) for \( k_b < 1 \), and less than proportional for larger values up to \( k_{b,\text{max}} \). Hence calculation of the first two terms of the asymptotic expansions of \( f \) and \( g \) will suffice in order to determine the flow field and the electromagnetic field for \( k_b < 1 \) with a reasonable accuracy.

Graphs of \( \Delta \vec{J}_r \) show the effect of the fluid motion upon the isotropic current distribution \( \vec{J}_{r,\phi} \).

---

Figure 3.2. Streamlines of constant \( \Phi_o \) and \( \Delta \Phi \) at \( k_b = 1 \) and \( k_\eta = 0 \) for \( \theta_0 = 45^\circ, 90^\circ, 135^\circ \). The numbers by the curves are the values of \( \Phi_o \) and \( \Delta \Phi \), see \( (3.55) \), measured in the same length unit as \( r \) (see page 58).
Figure 3.3. The modified electric current distributions $\bar{J}_{r,0}$ and $\Delta \bar{J}_{r}$ at $K_{b} = 1$ and $K = 0$, as defined by (3.56), for $\theta_{0} = 45^\circ, 90^\circ, 135^\circ$. 
As perceived by Shercliff (1970), the electromagnetic induction tends to shift the total current flow towards the axis of symmetry and to the surface of the cone. It has been found that the shifting of electric current becomes more pronounced for increasing values of the effective magnetic Reynolds number $K_b$, see Jansen (1977, 1983).

In order to examine the shifting of the electric current flow towards the axis of symmetry and to the surface of the cone for different values of $K_b$ and $\theta_o$, special angles $\theta_1$ and $\theta_2$ ($\theta_1 < \theta_2$) are introduced at which the perturbation current changes from being outwards to being inwards; ibidem $\Delta J_r = 0$, see figure 3.4.

![Figure 3.4. Perturbation currents at low $K_b$.](image)

Upon writing $\theta_1 = \lambda_1 \theta_o$ and $\theta_2 = \lambda_2 \theta_o$, it has been found, see Jansen (1983), that the ratios $\frac{\theta_1}{\theta_o}$ and $\frac{\theta_2}{\theta_o}$ are next to constant for small values of $K_b$, viz. $\lambda_1 = 0.29 - 0.30$ and $\lambda_2 = 0.81 - 0.82$ at $K_b < 1$. For larger and increasing $K_b$, up to $K_{b,\text{max}}$, the angles $\theta_1$ and $\theta_o - \theta_2$ become smaller, by which $\theta_1$ decreases faster than $\theta_o - \theta_2$. In addition we define $I_a$ and $I_w$ as being the total outward directed electric currents respectively in the neighbourhood of the axis of symmetry for $0 < \theta < \theta_1$ and near the surface of the cone for $\theta_2 < \theta < \theta_o$. It is evident now that both $I_a$ and $I_w$ can be separated into two terms, viz. $I_a = I_{a,o} + \Delta I_a$ and $I_w = I_{w,o} + \Delta I_w$; where $I_{a,o}$ and $\Delta I_a$ are respectively caused by
the isotropic - and total perturbation part of the current distribution between 
\( \theta = 0 \) and \( \theta = \theta_1 \). In the same way \( I_{w,0} \) and \( \Delta I_w \) are the respective electric 
currents at \( \theta < \theta < \theta_0 \).

It turns out that for increasing \( K_b \) the ratio \( \frac{I_{w,0}}{I_{a,0}} \) increases, whereas \( \frac{\Delta I_w}{\Delta I_a} \) decreases. Nevertheless the fact that \( \frac{I_w}{I_a} \) increases does indicate that near the 
surface of the cone a greater part of the outward electric current is concen-
trated than near the axis of symmetry. Note that this behaviour is not affected 
by the fact that the ratio \( \frac{I_w}{I_a} \) decreases for larger \( \theta_0 \) but constant \( K_b \), because 
in that case the effect of the shifting of electric current towards the surface 
of the cone becomes more pronounced, see figure 3.3. In other words at constant 
\( K_b \), \( \frac{I_w}{I_a} \) decreases faster than \( \frac{\Delta I_w}{\Delta I_a} \) for larger values of \( \theta_0 \). The behaviour is 
clearly indicated by the ratio \( \frac{I_w}{I_a} / \frac{I_{w,0}}{I_{a,0}} \), as given in table 3.1, for different 
values of \( \theta_0 \) and \( K_b \).

<table>
<thead>
<tr>
<th>( \theta_0 )</th>
<th>( K_b = 10^{-2} )</th>
<th>( K_b = 1 )</th>
<th>( K_b = K_{b,\max} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>45°</td>
<td>1.00</td>
<td>1.01</td>
<td>1.06</td>
</tr>
<tr>
<td>90°</td>
<td>1.00</td>
<td>1.05</td>
<td>1.11</td>
</tr>
<tr>
<td>135°</td>
<td>1.00</td>
<td>1.15</td>
<td>1.24</td>
</tr>
</tbody>
</table>

Table 3.1. Values of \( \frac{I_w}{I_a} / \frac{I_{w,0}}{I_{a,0}} \) for different values of \( K_b \) and \( \theta_0 \).

The values of \( K_{b,\max} \) are respectively 10, 5, 2 for \( \theta_0 = 45°, 90°, 135° \).

The behaviour of the perturbation currents at low \( K_b \), up to \( K_{b,\max} \), is in 
agreement with the electric current distribution to be expected when \( K_b \) tends to 
infinity, viz. approaching a current sheet upon the surface of the cone. However 
it is not yet quite clear whether the behaviour at small \( K_b \) continues for inter-
mediate values of $K_b$. Note that due to the stronger shifting of electric currents at increasing values of $K_b$, up to $K_b_{\text{max}}$, less vorticity is generated by the Lorentz force in the greater part of the flow field, but more in the neighbourhood of the cone wall. This concurs with the behaviour of the flow field as discussed before.

Since the Lorentz force generates both a fluid motion and a pressure field in the inviscid medium, it will be useful to consider also the pressure distribution. From (2.64), (3.30) and (3.33), or (4.3), (4.4) of report LR-256, see Jansen (1983), it follows that the angular behaviour of the basic solution of the pressure distribution yields

$$p_0(c) = -\frac{1}{1-c^2} \left\{ (2a_0 + 2a_2 + 3a_3) + 2a_1 c + 3a_3 c^2 + 4a_3 (1+c) \ln (1+c) \right\} ,$$

(3.61)

where $a_0 - a_3$ are given by (3.36) - (3.39). It can easily be proved that $p_0(c)$ is a monotonically increasing function of $c$ with $p_0(c_0) < 0$ and $p_0(1) > 0$. Hence the pressure field is divided into two parts separated at an angle $\theta_3 \approx 0.55 - 0.82 \theta_0$ where the pressure equals the reference pressure at large radial distance $p_\infty$; i.e. $p_0(c_3) = 0$. For $0 < \theta < \theta_3$ the induced pressure distribution is larger, whereas for $\theta_3 < \theta < \theta_0$ it is smaller than $p_\infty$. Extreme but in general finite values appear on the surface of the cone and on the axis of symmetry, with the latter one being the larger of the two for $c_0 = \frac{1}{2} \sqrt{2}$, 0 but the smaller for $c_0 = -\frac{1}{2} \sqrt{2}$.

For some special values of $c_0$, the base solution $p_0(c)$ takes the form

$$p_0(c) = + \infty, \text{ viz. } \lim_{c_0 \uparrow 1} \lim_{c \uparrow 1} p_0(c) = + \infty \quad \text{for } c_0 = 1 ,$$

(3.62)

$$p_0(c) = -\frac{2(1-3c+2c_0)}{3(1-c_0)^2} (1 + o(1)) \quad \text{for } c_0 \to 1 ,$$

(3.63)

$$p_0(c) = \frac{2(5+3c)}{1+c} - \frac{8}{1-c^2} \left\{ 2 \ln (2) - (1+c) \ln (1+c) \right\} \quad \text{for } c_0 = 0 ,$$

(3.64)

$$p_0(c) = \frac{3}{2} + \frac{2}{1-c} \ln \left( \frac{1+c}{2} \right) \quad \text{for } c_0 = -1 .$$

(3.65)
Note that the singular behaviour of \( p_o(c) \) for \( c_o = 1 \) and at \( c = -1 \) for \( c_o = -1 \) are caused by the singularities of the Lorentz force in these points, see (2.63).

For a further discussion including graphs we refer to report LR-256, see Jansen (1983), where the basic- and perturbation solutions of all fluid- and electromagnetic field quantities appearing in (2.1) - (2.7) have been calculated and sketched for \( K_b = 10^{-2} \), 1, \( K_{b,\text{max}} \) and \( c_o = \frac{1}{2} \sqrt{2} \), 0 , \( -\frac{1}{2} \sqrt{2} \). As denoted before the corresponding values of \( K_{b,\text{max}} \) are respectively 10, 5, 2 at \( c_o = \frac{1}{2} \sqrt{2} \), 0 , \( -\frac{1}{2} \sqrt{2} \). Moreover in that report the respective forces exerted on the fluid and the surface of the cone have been examined extensively. In particular the singularities arising in expressions of the forces and in most cases also of the field quantities at \( r = 0 \) and \( r \to \infty \), as mentioned and overestimated by Moffatt (1978), were examined. The latter subject will be discussed in detail in chapter 5 of this thesis.

We will finish this section by considering the kinetic energy induced in the inviscid fluid by the rotationality of the Lorentz force. This is of particular interest in arc furnace applications where the kinetic energy is a measure of the extent of the stirring of liquid metals. The kinetic energy generated in a volume \( V \) consisting of a sphere-sector, centred on the origin, with radius \( r \) and angles variables \( 0 < \theta < \theta_o \) and \( 0 < \varphi < 2\pi \) satisfies, see (2.21), (2.22)

\[
T_{\text{kin}} = \frac{1}{2} \iint_V \rho (y, y) \, dv = \frac{\mu I^2}{4\pi} \int_0^{\theta_o} \left[ \frac{g_o^2}{c_o} \right] \, dc + \left[ \frac{\left( \frac{dg_o}{dc} \right)^2}{c_o} \right] \, dc . \tag{3.66}
\]

From (3.5) it is clear that the second term in the integrand becomes singular of order \((1-c)^{-1}\) at \( c = 1 \); whence the kinetic energy induced by the inviscid fluid motion is logarithmically singular for all \( c_o \) satisfying \(-1 < c_o < 1\). Nevertheless by excluding that singularity we are able to determine a finite part of the kinetic energy generated by the base inviscid fluid motion for \( K_b \ll 1 \), namely

\[
T_{\text{kin},0} = \tilde{T}_{\text{kin},0} + \Delta T_{\text{kin},0} , \tag{3.67}
\]

where

\[
\tilde{T}_{\text{kin},0} = \frac{\mu I^2}{4\pi} \int_0^{\theta_o} \left[ \frac{g_o^2}{c_o} \right] \, dc + \int_{\Delta} \left( \frac{dg_o}{dc} \right)^2 \, dc +
\]

\[
\Delta T_{\text{kin},0} = \frac{\mu I^2}{4\pi} \int_0^{\theta_o} \left[ \frac{g_o^2}{c_o} \right] \, dc + \int_{\Delta} \left( \frac{dg_o}{dc} \right)^2 \, dc .
\]
\[
\Delta T_{\text{kin},o} = \frac{\mu I_o^2}{4\pi} r \frac{h_o^2}{2} \lim_{\epsilon \to 0} \ln \left( \frac{\Delta}{1-\epsilon} \right).
\]

\[+ \frac{h_o^2}{2} \ln \left( \frac{2}{2-\Delta} \right) \]

\[\frac{\tilde{T}_{\text{kin},o}}{[\tilde{T}_{\text{kin},o}]_{c_0=0}} \quad (a)\]

\[\frac{\Delta \tilde{T}_{\text{kin},o}}{[\Delta \tilde{T}_{\text{kin},o}]_{c_0=0}} \quad (b)\]

Figure 3.5. The behaviour of the ratios \(\tilde{T}_{\text{kin},o}/[\tilde{T}_{\text{kin},o}]_{c_0=0}\) and \(\Delta T_{\text{kin},o}/[\Delta T_{\text{kin},o}]_{c_0=0}\) as function of \(c_0\). \(\tilde{T}_{\text{kin},o}\) is the finite part of the kinetic energy generated by the base inviscid fluid motion and \(\Delta T_{\text{kin},o}\) is the singular part of the kinetic energy due to the weak singularity in the velocity field at the axis of symmetry.
Here $\Delta$ is the equidistant stepsize of the numerical integration procedure defined by $\Delta = \frac{1-c_0}{n}$, where $n$ is the number of integration steps. Numerical computation of $T_{\text{kin},0}^c$ for $n = 500, 1000, 2000$ shows that the results strongly depend on the chosen stepsize. Nevertheless it turns out that the ratio $T_{\text{kin},0}^c / \{T_{\text{kin},0}^c\}_{c_0=0}$ is next to independent of $n$. For equal stepsize $\Delta$ and by taking the limit $c \uparrow 1$ in an identical way for all values of $c_0$, we are able also to consider the ratio $\Delta T_{\text{kin},0}^c / \{\Delta T_{\text{kin},0}^c\}_{c_0=0} = \frac{h^2(c_0)}{h^2(0)}$. Although the kinetic energy itself is singular, both ratios give some indication of the behaviour of the kinetic energy as function of $c_0$. In figure 3.5 the variation of both ratios as function of $c_0$ is presented.

From (3.40) and (3.51) it is clear that $T_{\text{kin},0}^c = 0$ for $c_0 = 1$. In addition in view of the increasing magnitude of $g_0(c)$ for larger $\theta_0$ as noted before, the ratios of the finite and singular parts of the kinetic energy become larger at increasing values of $\theta_0$. At $c_0 = -1 (\theta_0 = 180^\circ)$ the ratio of the singular parts of the kinetic energy is bounded, whereas the ratio of the finite parts becomes unbounded. Note from (3.66) that the kinetic energy is proportional to $I_0^2$ and $r$, i.e. the radius of the sphere-sector to be considered.

3.5. A note on investigations at large values of $K_b$

A useful extension of the present work on fluid motions generated by an electromagnetic force will be the investigation of the behaviour of the electromagnetic- and flow fields at larger values of the effective magnetic Reynolds number $K_b$. In that case the backward effect of the electromagnetic induction ($\mathbf{v} \times \mathbf{B}$ term in Ohm's law) alters the originally isotropic electric current distribution and as a consequence also the generated flow pattern, see sections 2.9 and 3.3. Despite the relatively weak singularity at the axis of symmetry, the behaviour of the inviscid flow is simple. This is the case mainly since in the inviscid fluid no electric current inversion can occur as mentioned in section 3.3., in contrast with viscous flows.

In practice the effect of the electromagnetic induction upon the electric current distribution and the fluid motion becomes dominant only in liquid metal devices using very high electric currents and in plasma applications, e.g. the arc in the TIG welding process and ion rockets, see Au (1968). This can also be seen from figure 2.2. Especially in plasma applications the theory is frequently described incompletely in terms of magnetic pressure only. As noted by Shercliff (1970) without adequately recognizing that the fluid pressure also acts and that
therefore it is the rotationality of the magnetic force that determines the fluid motion.

It turns out that an analytical approach for large values of \( K_b \) involves a great number of mathematical difficulties. Therefore we have to confine here the discussion to the results and conclusions obtained so far.

At large \( K_b \) the electromagnetic field is the dominating field quantity. Even when \( K_b \) tends to infinity, it is the only one that remains, see section 2.9. Hence Ohm's law, see (2.20), is the governing equation, which determines the fluid motion by Euler's equation, see (2.18) with \( K_\eta = 0 \).

Despite several attempts, it appears to be impossible to derive a full expression for the differential equation of \( f(c) \) from Ohm's law and Euler's equation by eliminating \( g(c) \) and its derivatives from the equations. It failed because of its size. Nevertheless an integro-differential equation of \( f(c) \) and in addition a differential equation of \( g(c) \) can rather easily be derived, see Jansen (1977, p. 37, 59). From the differential equations of \( f \) and \( g \) and from Ohm's law, reduced equations have been obtained by setting \( K_b^{-1} = 0 \). The so-called limiting solutions of the respective reduced equations show that the outer solutions of \( f \) and \( g \) satisfy

\[
f^{(o)}(c) = \sqrt{2} h, \tag{3.70}
\]

\[
g^{(o)}(c) = h \sqrt{1-c^2}, \tag{3.71}
\]

where \( h^2 \) is given by (3.12).

These outer solutions satisfy the combined limiting solution derived from Ohm's law, see (2.70), so that \( a_\infty = \sqrt{2} h^3 \). Also it is of interest to notice that the outer solutions do satisfy identically the form of Euler's equation as presented by (2.18) for \( K_\eta = 0 \). However they do not satisfy its integrated form, see Jansen (1977, p. 24). This discrepancy is caused by the fact that for the derivation of the latter equation the boundary conditions at \( c = c_0 \) or \( c = 1 \) have been used, see (3.4), (3.5), (3.16). This fact points to the appearance of a boundary layer in the electromagnetic field at the axis of symmetry for large values of \( K_b \), as stated hereafter, see O'Malley (1974).

Some further examination of the equations and the boundary conditions of \( f \) and \( g \), see (2.32) - (2.36), indicates that at large values of \( K_b \) the electromagnetic field quantities possess boundary layers on the cone wall and at the axis of symmetry, while the fluid motion contains a boundary layer on the surface of the cone only. It should be pointed out that the behaviour of this inviscid boundary
layer differs from the well-known viscous boundary layers. It is merely a local concentration of the flow that still can be generated in the fluid when $K_b$ becomes large or tends to infinity.

It follows from (3.11), (3.12), (3.70), (3.71) that $f(c_o) = 1$, $f(c) = 0$ on $c_o < c < 1$, $g(c) = 0$ on $c_o < c < 1$ and $h = 0$ when $K_b$ equals infinity, as stated by (2.71) in section 2.9. Thus when the fluid is perfectly conducting, the electric current $I_o$ supplied by the point electrode into the fluid is wholly concentrated in a current sheet on the surface of the cone. The current sheet prevents the magnetic field to enter the fluid, so that no fluid motion can be generated in this limit case. This conclusion is different from Zhigulev's opinion. He suggests that the electric current would be wholly confined to the axis of symmetry at large $K_b$, see Zhigulev (1960a) and Shercliff (1970).

The usual singular perturbation techniques cannot be applied to the integro-differential equation of $f(c)$ or to the differential equation of $g(c)$, because the precise behaviour of $h$ at larger values of $K_b$ is unknown. Formal inner expansions of $f(c)$ and $g(c)$ at $c = c_o$ can only be determined from Ohm's law and Euler's equation. However, they result in equations for the inner solutions which are analytically unsolvable. Moreover, inner expansions of the combined equations at $c = 1$ cannot be determined uniquely, again because of the unknown behaviour of $h$ as function of $K_b$.

A purely numerical calculation of $f(c)$ and $g(c)$ at larger values of $K_b$ has been carried out using the general integral equations and accessory relations, see (2.59), (2.60), (3.11) - (3.16). In this case the analytical expressions of $f$ and $g$ at small $K_b$, see (3.31) - (3.39), are used as start values. By iteration and step-wise increase of the value of $K_b$ it has been endeavoured to calculate the behaviour of $f(c)$ and $g(c)$ and their first derivatives at larger values of $K_b$. However, the applied numerical program showed a rather poor convergence and moreover it became unstable at $K_b > 10$. An improvement of the numerical program belongs certainly to the possibilities. Here a combined analytical and numerical approach may resolve the mathematical difficulties. This may especially be the case since recently from Ohm's law a composite asymptotic expansion of $f(c)$ as function of $c$, $g(c)$ and $K_b$ has been derived, which is valid on the interval $c_o < c < 1$.

The aim of this thesis is mainly the investigation and calculation of the prime effect: the generation of fluid motion due to a Lorentz force caused by an injected electric current and its associated magnetic field. Therefore, further investigations and calculations of the inviscid flow problem at larger values of $K_b$ are considered to be beyond the scope of the thesis. Nevertheless, the effect of the electromagnetic induction upon the electric current distribution and the
fluid motion is an interesting feature, which will be a subject of further examination.
4. THE VISCOUS POINT ELECTRODE PROBLEM

4.1. Introductory remarks

In this chapter we investigate the viscous flow with finite velocities throughout the field at large and at arbitrary values of the inverse effective hydrodynamic Reynolds number \( K_\eta \). The effective magnetic Reynolds number \( K_b \) is assumed to be small.

At large \( K_\eta \) and small \( K_b \) the so-called slow viscous solution can be calculated analytically by introduction of a series expansion in negative powers of \( K_\eta \). Also application of a formal regular perturbation at small \( K_b \) and large \( K_\eta \) leads to an analytical solution which represents the weak perturbation of the electromagnetic field quantities due to the electromagnetic induction; i.e. the fluid motion.

For arbitrary values of \( K_\eta \) the calculations need to be carried out by numerical computation. At small values of \( K_\eta \) we obtain a critical value of \( K_\eta; K_\eta, \text{min} \) in the same way as found first by Sozou (1971a). Due to our definitions it is a lower bound of the parameter (i.e. a maximum value of \( \text{Re} \)). It will be shown that for \( K_\eta < K_\eta, \text{min} \) the viscous flow field, generated by the Lorentz force, becomes essentially singular in a conical region around the axis of symmetry.

The cause of the singular behaviour of the viscous flow is discussed and examined. To that purpose a composite asymptotic expansion at small \( K_\eta \) has been derived. Also a series expansion method has been developed in order to calculate the critical value \( K_\eta, \text{min} \) as function of \( c_o \). It turns out that this method leads to results in an easier way than the usual numerical iteration of the integral equation, derived from the governing differential equation.

We finish this chapter with a discussion of the possibilities in which way the breakdown of the viscous fluid motion at relatively low values of the viscous Reynolds number may be resolved.

4.2. The slow viscous solution for \( K_\eta > 1 \)

In this section we consider the viscous fluid motion, induced by the Lorentz force, at large values of \( K_\eta \). We start with the introduction of a general series expansion in negative powers of \( K_\eta \) for the function \( g(c) \) representing the angle dependent part of the Stokes stream function, see (4.1) - (4.3).

In the case of small \( K_b \) an analytical expression of the governing solution \( g_o(c) \) at arbitrary and some special values of \( c_o \) can be found, see (4.10) -
(4.20).

Also the weak perturbing effect of the electromagnetic induction upon the fluid motion and the electromagnetic field quantities and their mutual interaction have been investigated. Since series expansions in negative powers of $K_\eta$ and positive powers of $K_b$ become too complicated, we will present only regular perturbations of the functions $g(c)$ and $f(c)$ valid at large value of $K_\eta$ and small value of $K_b$, see (4.30), (4.31).

It is clear that these formal regular perturbations represent the leading terms of the more general but complicated series expansions.

The asymptotic expansions of $g(c)$ and $f(c)$ imply two different perturbations of the flow field and lead to an analytical expression of the perturbation of the electromagnetic field quantities due to the electromagnetic induction, i.e. the $(\mathbf{v} \times \mathbf{B})$-term in Ohm's law (2.3). The respective results are presented in figures 4.2 and 4.3.

In case of small viscous Reynolds number flow, i.e. $K_\eta > 1$, which may occur in liquid metals or in hot plasmas under certain conditions, see figure 2.2, it is useful to carry out an asymptotic expansion in negative powers of $K_\eta$ to solve the Navier-Stokes equation (2.53) by itself. It turns out that the asymptotic expansion takes the form

$$g(c) = \sum_{n=0}^{\infty} \frac{1}{K_\eta^{2n+1}} g_n(c) \quad \text{for } K_\eta > 1,$$

(4.1)

where

$$g_0(c) = \frac{(1-c^2)}{2} \int_c^{c_0} \frac{G_\eta(t)}{t^2} \, dt,$$

(4.2)

$$g_n(c) = \frac{(1-c^2)}{2} \sum_{m=0}^{n-1} \int_c^{c_0} \frac{g_m(t) g_{n-m-1}(t)}{(1-t^2)^2} \, dt,$$

(4.3)

for $n = 1, 2, 3, ...$
Upon substitution of the expression of $G_\eta(c)$, see (2.51), the expression of $g_0(c)$ can be rewritten in the form

$$g_0(c) = \frac{(1-c^2)}{2} \int \frac{f^2(t)}{c_0(1-t^2)^2} \, dt - \frac{(1+c)}{4} \int \frac{f^2(t)}{c_0(1+t)^2} \, dt +$$

$$- \frac{(1-c)}{4} \int \frac{f^2(t)}{c_0(1-t)^2} \, dt + \frac{(c-c_0)^2}{2(1-c_0)^2} \int \frac{f^2(t)}{c_0(1+t)^2} \, dt. \quad (4.4)$$

At arbitrary values of $K_b$ and $K_\eta > 1$, the expressions (4.3), (4.4), (2.59), (2.60) lead to a system of three coupled integral equations for each term of the asymptotic expansion. In this way the behaviour of $f$ and $g$ and therefore of the electromagnetic- and flow fields can generally be determined at large $K_\eta$ and arbitrary $K_b$.

Since $G_\eta(c)$ is non-negative on $c_0 < c < 1$, see (2.55) - (2.57), so is $g_0(c)$. Successive application of the results obtained in (4.3) proves that all functions $g_n(c)$ are positive on the open interval and identical to zero at the boundaries $c = c_0, 1$.

Upon assuming that $f(c)$ is bounded on $c_0 < c < 1$, say $|f(c)| < 1$, see (2.32) and (2.33), it can easily be found that all $g_n(c)$ are bounded, viz.

$$0 < g_n(c) < g_{n,\text{max}} \quad \text{for} \quad n = 0, 1, 2, \ldots, \quad c_0 < c < 1, \quad -1 < c_0 < 1, \quad (4.5)$$

where on the assumption that $|f(c)| < 1$ for $n = 0$ the upper bound is of the form

$$g_{0,\text{max}} = -\frac{(3+c_0)}{4(1-c_0)^2} \left[ 2(1-c_0) + (3+c_0) \ln \left( \frac{1+c_0}{2} \right) \right]. \quad (4.6)$$

One may note from (4.1) - (4.3) and (4.6) that $g_n(c) \equiv 0$ for $n = 0, 1, 2, \ldots$ in the case of $c_0 = 1$, and that $g_{0,\text{max}}$ becomes logarithmically singular for $c_0 = -1$. In general we cannot exclude electric current inversion here, see Sozou & English (1972). The above assumed upper and lower bounds of $f(c)$ exclude only excessive inversion of electric current.

Note that $g_0(c_0) = 0((c-c_0)^2)$ and that $g_n(1) = 0(1-c)$ for $n = 0, 1, 2, \ldots$, as the boundary conditions (2.34), (2.35), (2.38) require. The fact that
\[ g_n(c_o) = 0((c-c_o)^{3n+2}) \] for \( n = 0,1,2, \ldots \) indicates that the behaviour of \( g(c) \) near \( c = c_o \) is almost completely determined by \( g_o(c) \), while for smaller \( K_\eta \) (\( K_\eta > 1 \)) the behaviour of \( g(c) \) on \( c_o < c < 1 \) is determined by more terms of the asymptotic expansion. Also it can be shown that the function \( g_o(c) \) always possesses a double zero at \( c = c_o \), an inflexion point at \( c = c_b \), a maximum value at \( c = c_m > 0 \) and a single zero at \( c = 1 \), where \( c_o < c_b < c_m < 1 \).

The behaviour of \( g_o(c) \) is convex on \( c_o < c < c_b \) and concave on \( c_b < c < 1 \). The effect of higher order terms of the expansion does not alter the global behaviour essentially. Therefore \( g(c) \) shows a similar behaviour as \( g_o(c) \); only for different \( K_\eta \) the magnitude of \( g(c) \) changes and other values of \( c_b \) and \( c_m \) will be found. It should be noticed that the form of \( g_o(c) \) and \( g(c) \) at large \( K_\eta \) is similar to that of \( g^2(c) \) at \( K_\eta = 0 \), see figure 3.1.a.

In section 2.7 it has been found that the viscous fluid motion consists of an incoming flow along the surface of the cone, which is rather abruptly turned off into an outwards jet flow along the axis of symmetry. Due to viscous spreading the jet flow diverges. The edge of the jet can be defined as the place where the streamlines are at their minimum distance from the axis, see Batchelor (1967), and it is readily seen from (3.25) that this edge occurs at \( \theta = \theta_e (c = c_e) \) where \( \frac{\partial \psi}{\partial z} = 0 \). By consideration of the viscous fluid motion represented by \( g_o(c) \), it can be proved that \( \frac{\partial \psi}{\partial z} > 0 \) for \( c_o < c < c_e \) and \( \frac{\partial \psi}{\partial z} < 0 \) for \( c_e < c < 1 \).

It was already mentioned in section 3.3, that as a consequence we may expect the appearance of electric current inversion in the region where the viscous jet spreads out \( (c_e < c < 1) \), when \( K_b \) exceeds a certain large value. This topic has been investigated by Sozou & English (1972) for the flat wall configuration \( c_o = 0 \). The relations between their functions and parameters and the ones defined here are given by (2.27) - (2.30), where the subscript \( e \) refers to quantities used by the above-mentioned authors. However from the results of Sozou & English it is not clear whether electric current inversion may occur under the restrictions of the asymptotic expansion (4.1), i.e. \( K_\eta < M_a \) where \( M_a = O(1) - O(10) \), see (4.30), (4.31).

The Lorentz force generates both a viscous fluid motion and a pressure distribution in the medium. Substitution of (4.1) into (2.64) yields the corresponding asymptotic expansion of the pressure field

\[ p = p_\infty + \frac{\mu I^2}{8\pi^2 r^2} \sum_{n=0}^{\infty} \frac{1}{K_\eta^{2n}} p_n(c), \]  

(4.7)
where

\[ p_n(c) = (1-c^2) \frac{d^3 g_n}{dc^3} - 2c \frac{d^2 g_n}{dc^2} + \]

\[ - \sum_{m=0}^{n-1} \left[ g_m \frac{d^2 g_{n-m-1}}{dc^2} + \frac{dg_m}{dc} \frac{dg_{n-m-1}}{dc} + \frac{g_m g_{n-m-1}}{1-c^2} \right], \tag{4.8} \]

for \( K_\eta > 1 \), and \( p_\infty \) represents a reference pressure at large radial distance. Note that the series in (4.8) only appears for \( n > 1 \). By substituting (4.4) the expression of \( p_0(c) \) takes the form

\[ p_0(c) = - \frac{f^2}{1-c^2} + 2c \int \frac{f^2(t)}{c_0 (1-t^2)} \frac{dt}{(1-c_0)^2} - \frac{2c}{1} \int \frac{f^2(t)}{c_0 (1+t)^2} \frac{dt}{(1-c_0)^2}. \tag{4.9} \]

Some further manipulation shows that \( p_0(c) \), being negative on the surface of the cone, increases monotonically to a positive value on the axis of symmetry \( \theta = 0 \).

When the effective magnetic Reynolds number \( K_b \) is small compared to unity, viz. \( K_b \ll 1 \), we are able to calculate the slow-viscous flow solution \( g_0(c) \) analytically. From (2.59) and (4.4) we obtain, see Jansen (1984)

\[ g_0(c) = a_0 + a_1 c + a_2 c^2 + a_3 (1+c) \ln (1+c), \tag{4.10} \]

\[ \frac{dg_0}{dc} = (a_1 + a_3) + 2a_2 c + a_3 \ln (1+c), \tag{4.11} \]

for \( c_o < c < 1, -1 < c_o < 1 \). The coefficients \( a_0 - a_3 \) being only dependent on \( c_o \), are given by

\[ a_0 = \frac{c_o}{(1-c_o)^3} - \frac{(1-2c_o-c_o^2)}{(1-c_o)^4} \ln (1+c_o) - \frac{2c_o^2}{(1-c_o)^4} \ln (2), \tag{4.12} \]
\[ a_1 = - \frac{(1+c_o)}{(1-c_o)^3} - \frac{(1+c_o)^2}{(1-c_o)^4} \ln(1+c_o) + \frac{4c_o}{(1-c_o)^4} \ln(2), \]  
\[ (4.13) \]

\[ a_2 = \frac{1}{(1-c_o)^3} + \frac{2}{(1-c_o)^4} \ln(1+c_o) - \frac{2}{(1-c_o)^4} \ln(2), \]  
\[ (4.14) \]

\[ a_3 = \frac{1}{(1-c_o)^2}, \]  
\[ (4.15) \]

for \( K_\eta > 1 \), \( K_\beta < 1 \).

For all values of \( c_o \), \(-1 < c_o < 1\), \( g_o(c) \) shows the behaviour mentioned. It may be remarked that the above general solution for arbitrary values of \( c_o \) was found by the author independently from Narain & Uberoi (1971). It should be mentioned that the above expressions of \( a_0 - a_3 \) are not related to the same ones used for the analytical inviscid solution in the preceding chapter, see (3.33) - (3.39).

Expressions of \( g_o(c) \) for some special values of \( c_o \) take the form

\[ g_o(c) = 0 \quad \text{for} \quad c_o = 1, \]  
\[ (4.16) \]

\[ g_o(c) = \frac{(c-c_o)^2(1-c)}{24(1-c_o)^2} (1 + o(1)) \quad \text{for} \quad c_o \rightarrow 1, \]  
\[ (4.17) \]

\[ g_o(c) = -c(1-c) - 2 \ln(2) c^2 + (1+c) \ln(1+c) \quad \text{for} \quad c_o = 0, \]  
\[ (4.18) \]

\[ g_o(c) = -\frac{1}{8} \left[ -c^2 + (1+c)^2 \ln(2) + (1-c^2) \ln(1+c) + \right. \]
\[ - 2(1+c) \ln(1+c) \left] (1 + o(1)) \right. \quad \text{for} \quad c_o \rightarrow -1, \]  
\[ (4.19) \]

\[ g_o(c) = +\infty \quad \text{for} \quad c_o = -1. \]  
\[ (4.20) \]
For \( c_0 = 1 \) the solution of \( g_o(c) \) is identical to zero, as expected. In case of the flat wall configuration \( c_0 = 0 \) the expression of \( g_o(c) \), (4.18), is identical to the solution found by Lundquist (1969).

To consider the analogy with the analytical solution for the inviscid problem \( K_\eta = 0 \), see (3.33) - (3.43); for \( K_\eta \gg 1 \) the function \( g_o(c) \) achieves the largest magnitude for \( c_0 = -1 \). However in contrast with the inviscid solution, the slow-viscous solution becomes unbounded for \( c_0 = -1 \).

It can easily be proved, see appendix B, that the basic solution \( g_o(c) \), as presented by (4.10) - (4.15), is bounded by

\[
0 < g_o(c) < \overline{g}_{o,\text{max}} \quad \text{for } c_0 < c < 1, \quad -1 < c_0 < 1, \quad (4.21)
\]

where

\[
\overline{g}_{o,\text{max}} = -\frac{1}{4(1-c_0)^2} \left\{ 2(1-c_0) + (3+c_0) \ln \left( \frac{1+c_0}{2} \right) \right\} < \frac{1}{4(1+c_0)}, \quad (4.22)
\]

for \( K_b \ll 1 \).

The expressions (4.19) and (4.22) clearly indicate that \( g_o(c) \) becomes logarithmically singular when \( c_0 \) approaches \(-1\). Hence from (4.1) - (4.3) it follows that \( g_n(c) \) are singular at \( c_0 = -1 \) and the same then applies to the complete solution \( g(c) \). Also note the difference between the above upper bound of \( g_o(c) \) and the more general one given by (4.6), which is valid also for larger values of \( K_b \) with the restriction that \( f(c) \) is bounded, viz. \( |f(c)| < 1 \). It has to be remarked that although the function \( g_o(c) \) is logarithmically unbounded on \(-1 < c < 1 \) for \( c_0 = -1 \), the boundary conditions \( g_o(c) = 0 \) at \( c = c_0, 1 \) and \( \{ \frac{dg_o}{dc} \}_{c=c_0} = 0 \) are still satisfied.

The pressure distribution induced by the basic flow- and electromagnetic fields now satisfies, see (4.7) - (4.10)

\[
p_o(c) = -a_3 - 4a_2c, \quad (4.23)
\]

where \( a_2 \) and \( a_3 \) are given by (4.14), (4.15). It can easily be shown that \( p_o(c) \) is a monotonically increasing function of \( c \), with \( p_o(c_0) < 0 \) and \( p_o(1) > 0 \).
Hence the pressure field is divided into two parts separated at an angle \( \theta_3 \approx 0.50 - 0.58 \theta_0 \) for \( 0 < \theta_0 < 180^\circ \) where \( p_0(c_3) = 0 \), viz. the pressure equals the reference pressure at large radial distance \( p_\infty \). For \( 0 < \theta < \theta_3 \) the pressure is larger and for \( \theta_3 < \theta < \theta_0 \) it is smaller than \( p_\infty \). Extreme but in general finite values are situated on the surface of the cone and on the axis of symmetry, whereby \( |p_0(c_o)| > |p_0(1)| \).

Note that the pressure distribution for large \( K_\eta \) is almost similar to that for \( K_\eta = 0 \).

For some special values of \( c_o \) the expression of \( p_0(c) \) yields

\[
p_0(c) = +\infty, \text{ viz. } \lim_{c_0 \to 1} \lim_{c \to 1} p_0(c) = +\infty \quad \text{for } c_0 = 1, \quad (4.24)
\]

\[
p_0(c) = -\frac{(2+c_0-3c)}{3(1-c_0)^2}(1 + o(1)) \quad \text{for } c_0 = 1, \quad (4.25)
\]

\[
p_0(c) = -1 - 4 \{1 - 2 \ln(2)\} c \quad \text{for } c_0 = 0, \quad (4.26)
\]

\[
p_0(c) = -\frac{1}{2} \ln \left(\frac{1+c_0}{2}\right) c (1 + o(1)) \quad \text{for } c_0 = -1, \quad (4.27)
\]

\[
p_0(c) = \pm \infty \quad \text{for } c_0 = -1. \quad (4.28)
\]

The expression for \( c_0 = 0 \), see (4.26), is identical to the one found first by Lundquist (1969). Like the velocity field, see (4.20), the pressure distribution becomes logarithmically singular in the entire field for \( c_0 = -1 \). The singular behaviour of both field quantities over the entire field is caused by the Lorentz force. Nevertheless the electromagnetic force and its rotationality are unbounded only at \( c = -1 \) for \( c_0 = -1 \), see (2.63). So we observe that a boundary singularity in the Lorentz force results in unbounded solutions of \( g_o(c) \) and \( p_0(c) \) for all \( c, c_0 < c < 1 \).

The kinetic energy generated by the base solution of the slow viscous fluid
motion in a sphere-sector, centred on the origin with radius \( r \) and angles \( 0 < \theta < \theta_0 \), \( 0 < \varphi < 2\pi \), satisfies, see (3.66), (4.1)

\[
T_{\text{kin},0} = \frac{\mu I_o^2}{4\pi K_\eta^2} r \int \frac{1}{c_o} \left[ \frac{g_o^2}{1-c_o^2} + \left( \frac{dg_o}{dc} \right)^2 \right] dc.
\]

The behaviour of \( T_{\text{kin},0} \) as function of \( c_o \) is presented in figure 4.1.

![Graph showing the behaviour of kinetic energy](image)

**Figure 4.1.** The behaviour of the kinetic energy generated by the base solution of the slow-viscous flow as function of \( c_o \).

The figure shows an exponential variation of the magnitude of \( T_{\text{kin},0} \) as function of \( c_o \); which is in contrast with the behaviour of the kinetic energy in the
inviscid flow, see figure 3.5. Also here the kinetic energy increases with increasing $\Theta_o$. Note that $T_{\text{kin},o}=0$ for $c_o=1$ and unbounded for $c_o=-1$. From (2.17) and (4.29), it follows that the kinetic energy of the low hydrodynamic Reynolds number flow is proportional to $T_o^4$ and $r$, i.e. the radius of the sphere-sector to be considered.

A useful extension of the present investigation of the slow viscous flow at large value of the inverse effective hydrodynamic Reynolds number $K_\eta$ will be the examination of the additional effect of the electromagnetic induction (i.e. the influence of the $(\nu \times B)$-term in Ohm's law (2.3)) upon the fluid motion, the electromagnetic fieldquantities and their mutual interaction at small values of the effective magnetic Reynolds number $K_b$. To that purpose also the function $f(c)$, representing the angle-dependent part of the magnetic field, see (2.15), needs to be expanded in positive powers of $K_b$. However in view of the non-linear character of the integral forms of respectively the curl of Navier-Stokes equation and the curl of Ohm's law, see (2.53) - (2.55), (2.59), (2.60), it is clear that general series expansions of $g(c)$ and $f(c)$ valid for large value of $K_\eta$ and small value of $K_b$ will become complicated. In addition to the usual terms in negative powers of $K_\eta$ and positive powers of $K_b$ in the respective series expansions of $g(c)$ and $f(c)$, there appears an infinite number of terms due to the mutual correlation of the respective terms of both series expansions. Therefore we present here only regular perturbations of both functions, which represent the leading terms of the respective full series expansions.

In the case of large $K_\eta$ and small $K_b$ the asymptotic expansions of the fluid-flow and electromagnetic fieldquantities take the form, see Jansen (1984)

$$f = f_o + \frac{K_b}{K_\eta^4} f_1 + \cdots ,$$

$$g = \frac{1}{K_\eta^4} g_o + \frac{1}{K_\eta^3} g_1 + \frac{K_b}{K_\eta^2} g_2 + \cdots ,$$

for $K_\eta > 1$ and $K_b < 1$, where respectively $f_o$, $\frac{df_o}{dc}$, $g_o$, $\frac{dg_o}{dc}$ are given by (3.31), (3.32), (4.10), (4.11), and
\[
f_1 = \frac{(1-c)}{(1-c_o)^2} \int_{c_o}^{c} \left( \frac{2+c_o+t}{(1+t)^2} \right) g_o(t) \, dt - \frac{(c-c_o)}{(1-c_o)^2} \int_{c}^{(3+t)} g_o(t) \, dt , \quad (4.32)
\]

\[
\frac{df_1}{dc} = \frac{2 g_o(c)}{(1-c_o)(1+c)} - \frac{1}{(1-c_o)^2} \int_{c_o}^{c} \frac{(2+c_o+t)}{(1+t)^2} g_o(t) \, dt + \frac{1}{c} \int_{c_o}^{(3+t)} g_o(t) \, dt , \quad (4.33)
\]

\[
g_1 = \frac{(1-c^2)}{2} \int_{c_0}^{c} \frac{g^2_o(t)}{(1-t^2)^2} \, dt , \quad (4.34)
\]

\[
\frac{dg_1}{dc} = \frac{g^2_o(c)}{2(1-c^2)} - c \int_{c_0}^{c} \frac{g^2_o(t)}{(1-t^2)^2} \, dt , \quad (4.35)
\]

\[
g_2 = -\frac{1}{(1-c_o)} \int_{c_o}^{c} \frac{(c-t)^2 f_1(t)}{(1+t)^2 (1-t)} \, dt + \frac{(c-c_o)}{(1-c_o)^3} \int_{c_o}^{(1-t)} f_1(t) \, dt , \quad (4.36)
\]

\[
\frac{dg_2}{dc} = -\frac{2}{(1-c_o)} \int_{c_o}^{c} \frac{(c-t) f_1(t)}{(1+t)^2(1-t)} \, dt + \frac{2(c-c_o)}{(1-c_o)^3} \int_{c_o}^{(1-t)} f_1(t) \, dt . \quad (4.37)
\]

It is easy to verify that the perturbation solutions satisfy the boundary conditions (2.32) - (2.38), viz. \( f_1(c) = g_1(c) = g_2(c) = 0 \) for \( c = c_o \), and \( \frac{dg_1}{dc} \bigg|_{c=c_o} = \frac{dg_2}{dc} \bigg|_{c=c_o} = 0 \). The inhomogeneous boundary condition of \( f(c) \) at
c = c_0 is satisfied by the basic solution f_0(c). Also note the difference between the complete asymptotic expansion (4.1) - (4.3), valid for $K_\eta > 1$ and the above formal expansion for $K_\eta > 1$ and $K_b < 1$; in particular note that $g_0$ and $g_1$ of both expansions are identical for $K_b < 1$. From the expression of $g_1$, (4.34), it is clear that $g_1 > 0$ on $c_0 < c < 1$. Numerical computation for different values of $c_0$ shows that $f_1$ is an alternating function with always a single zero on $c_0 < c < 1$, say at $c = c_2$; so that $f_1(c) < 0$ on $c_0 < c < c_2$ and $f_1(c) > 0$ on $c_2 < c < 1$. Also it turns out that $g_2(c) < 0$ on $c_0 < c < 1$.

The isotropic current distribution and the associated magnetic field expressed by $f_0$ and $\frac{df_0}{dc}$ generates the basic solution of the fluid motion: $g_0$, $\frac{dg_0}{dc}$. Due to the electromagnetic induction this fluid motion disturbs the isotropic electric current distribution represented by $f_1$ and $\frac{df_1}{dc}$. In its turn the perturbation of the electromagnetic field alters the induced fluid motion, where $g_1$, $\frac{dg_1}{dc}$ are generated when the viscous force balances the interia force and $g_2$, $\frac{dg_2}{dc}$ when the viscous force balances the Lorentz force.

Which perturbation of the basic fluid motion dominates, depends on the magnitude of the magnetic Prandtl number, satisfying $Pm = K_b K_\eta$. In case of liquid metal the magnetic Prandtl number is usually very small compared to unity: $Pm \approx 10^{-6}$, so that $g_1$ is dominant, see (4.31). In case of large $Pm$, as it is in a high-temperature plasma with $Pm \approx 10^4$, $g_2$ dominates, whereas for a low-temperature plasma or an arc discharge where $Pm$ is about of order unity both terms disturb the generated basic fluid motion in the same measure.

It turns out that here in contrast with the inviscid problem the perturbation of the electromagnetic field $f_1(c)$ and $\frac{df_1}{dc}$, due to the weak effect of the electromagnetic induction, can be calculated analytically. The expressions, obtained by substituting (4.10) - (4.15) into (4.32) and (4.33), are

$$f_1(c) = b_0 + b_1 c + b_2 c^2 + b_3 \ln (1+c) + b_4 (1+c) \ln (1+c) + b_5 (1+c) \ln^2 (1+c),$$

(4.38)

$$\frac{df_1}{dc} = (b_1 + b_4) + 2b_2 c + \frac{b_3}{1+c} + (b_4 + 2b_5) \ln (1+c) + b_5 \ln^2 (1+c),$$

(4.39)
for $c_0 < c < 1$, $-1 < c_0 < 1$. The coefficients $b_0 - b_5$, being only dependent on $c_0$, are given by

$$b_0 = \frac{3c_0}{2(1-c_0)^4} + \frac{c_0(5+2c_0)}{(1-c_0)^5} \ln (1+c_0) - \frac{c_0(5+2c_0)}{(1-c_0)^5} \ln (2) +$$

$$- \frac{2(1+6c_0+c_0^2)}{(1-c_0)^6} \ln (1+c_0) \ln (2) + \frac{(1+c_0)(5+2c_0+c_0^2)}{2(1-c_0)^6} \ln^2 (1+c_0) +$$

$$\frac{c_0(7+2c_0-c_0^2)}{(1-c_0)^6} \ln^2 (2), \quad (4.40)$$

$$b_1 = - \frac{3(1+c_0)}{2(1-c_0)^4} - \frac{(1+c_0)(3+2c_0)}{(1-c_0)^5} \ln (1+c_0) + \frac{5(1+c_0)}{(1-c_0)^5} \ln (2) +$$

$$+ \frac{2(5+2c_0+c_0^2)}{(1-c_0)^6} \ln (1+c_0) \ln (2) - \frac{(1+c_0)(5+2c_0+c_0^2)}{2(1-c_0)^6} \ln^2 (1+c_0) +$$

$$- \frac{(7+2c_0-c_0^2)}{(1-c_0)^6} \ln^2 (2), \quad (4.41)$$

$$b_2 = \frac{3}{2(1-c_0)^4} + \frac{3}{(1-c_0)^5} \ln (1+c_0) - \frac{3}{(1-c_0)^5} \ln (2), \quad (4.42)$$
\[ b_3 = \frac{2(1+c_o)}{(1-c_o)^4} + \frac{2(1+c_o)^2}{(1-c_o)^5} \ln (1+c_o) - \frac{2(1+c_o)^2}{(1-c_o)^5} \ln (2), \quad (4.43) \]

\[ b_4 = -\frac{2(1+c_o)}{(1-c_o)^4} + \frac{(5+2c_o+c_o^2)}{(1-c_o)^5} \ln (1+c_o) + \frac{4(1+c_o)}{(1-c_o)^5} \ln (2), \quad (4.44) \]

\[ b_5 = \frac{1}{2(1-c_o)^3}. \quad (4.45) \]

For some special values of \( c_o \) the expression of \( f_1(c) \) reduces to

\[ f_1(c) = 0 \quad \text{for } c_o = 1, \quad (4.46) \]

\[ f_1(c) = \frac{(c-c_o)(1-c)}{288(1-c_o)^3} \left[ 1 - 3c_o - c_o^2 + 5c_o c + c - 3c^2 \right] (1 + o(1)) \]

\[ \quad \text{for } c_o \neq 1, \quad (4.47) \]

\[ f_1(c) = -\frac{1}{2} \left[ 3 - 10 \ln (2) + 14 \ln^2(2) \right] c + \frac{3}{2} \left[ 1 - 2 \ln (2) \right] c^2 + \]

\[ + 2\left[ 1 - \ln (2) \right] \ln (1+c) - 2\left[ 1 - 2 \ln (2) \right] (1+c) \ln (1+c) + \]

\[ + \frac{1}{2} (1+c) \ln^2(1+c) \quad \text{for } c_o = 0, \quad (4.48) \]

\[ f_1(c) = -\frac{1}{32} \left[ 3(1-c^2) + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]

\[ \left. + 3(1-c^2) \ln \left( \frac{1+c_o}{2} \right) + 4(1+c) \ln \left( \frac{1+c}{2} \right) \ln \left( \frac{1+c}{2} \right) + \right. \]
\[-2(1+c) \ln^2\left(\frac{1+c}{2}\right) (1 + o(1)) \quad \text{for } c_o = -1, \quad (4.49)\]

\[f_1(c) = \pm \infty \quad \text{for } c_o = -1. \quad (4.50)\]

Since the general analytical expressions of \(g_1(c)\) and \(g_2(c)\) contain dilogarithm functions, which in general cannot be expressed in elementary transcendental functions, they are omitted, see Abramowitz & Stegun (1972, p. 1004). We shall present only the behaviour of \(g_1(c)\) and \(g_2(c)\) for \(c_o = 1\), obtained by substitution of (4.17) and (4.47) into respectively (4.34) and (4.36) and yielding

\[g_1(c) = 0 \quad \text{for } c_o = 1, \quad (4.51)\]

\[g_1(c) = \frac{(c-c_o)^5(1-c)}{11520(1-c_o)^4} (1+o(1)) \quad \text{for } c_o + 1, \quad (4.52)\]

together with

\[g_2(c) = 0 \quad \text{for } c_o = 1, \quad (4.53)\]

\[g_2(c) = -\frac{(c-c_o)^2(1-c)}{69120(1-c_o)^4} \left\{4 - 16c_o + 19c_o^2 - 4c_o^3 + 4c - 6c_o c - 7c_o^2 c + \right.\]

\[+ 10c_o^2 c^2 - c^2 - 3c^3\} (1+o(1)) \quad \text{for } c_o + 1. \quad (4.54)\]

To obtain values or figures of the perturbed flow- and electromagnetic field-quantities the integral- or the analytical expressions need to be calculated numerically. In view of the complexity of the analytical expressions of \(f_1\), \(g_1\) and \(g_2\) a straightforward numerical computation of the integral expressions (4.32) - (4.37) is preferable.

In order to present figures of the generated and perturbed fluid motion and current distribution we introduce some modified forms of these quantities, defined by...
\[
\psi = \frac{I_0}{2\pi} \sqrt{\frac{\mu}{\rho}} \bar{\psi},
\]  
(4.55)

with

\[
\bar{\psi} = \frac{1}{K_\eta} \bar{\psi}_o + \frac{1}{K_\eta^3} \Delta \bar{\psi}_1 + \frac{K_b}{K_\eta^2} \Delta \bar{\psi}_2 + \cdots,
\]  
(4.56)

where

\[
\bar{\psi}_o = r g_0(c),
\]  
(4.57)

\[
\Delta \bar{\psi}_i = r g_i(c) \quad \text{for } i = 1, 2.
\]  
(4.58)

and

\[
J_r = \frac{I_0}{2\pi r^2} \bar{J}_r,
\]  
(4.59)

with

\[
\bar{J}_r = \bar{J}_{r,o} + \frac{K_b}{K_\eta} \Delta \bar{J}_{r,1} + \cdots,
\]  
(4.60)

where

\[
\bar{J}_{r,o} = - \frac{df_o}{dc} = \frac{1}{1-c_o},
\]  
(4.61)

\[
\Delta \bar{J}_{r,1} = - \frac{df_1}{dc}.
\]  
(4.62)

Figure 4.2. Streamlines of constant \( \bar{\psi}_o \) and the perturbation of the electric current distribution \( \Delta \bar{J}_{r,1} \), see (4.55) - (4.62), at \( K_\eta > 1 \) and \( K_b < 1 \) for \( \theta_o = 45^\circ, 90^\circ, 135^\circ \) (see page 85).
Figure 4.3. Streamlines of constant $\Delta \tilde{\psi}_1$ and $\Delta \tilde{\psi}_2$ for $\theta_0 = 45^\circ$, $90^\circ$, $135^\circ$, being the perturbations of the basic fluid motion $\tilde{\psi}_0$, see (4.55) - (4.58). $\Delta \tilde{\psi}_1$ is dominant for $Pm \ll 1$ and $\Delta \tilde{\psi}_2$ for $Pm \gg 1$. 
Note the difference between above modified functions and those given in the inviscid case by (3.55) - (3.60); viz. the latter ones include a finite number of perturbation terms multiplied by the respective power of the parameter \( K_b \).

Graphs of \( \Psi_0, \Delta \Psi_1, \Delta \Psi_2 \) and \( \Delta \bar{J}_{r,1} \), for \( c_o = 0, \pm 1/2 \sqrt{2} (\theta_o = 45^\circ, 90^\circ, 135^\circ) \) are sketched in figures 4.2 and 4.3. For the behaviour of \( \bar{J}_{r,0} \), at different \( c_o \), being identical of course at \( K_b < 1 \), we refer to figure 3.3.

The graphs of the basic fluid motion \( \Psi_0 \) clearly show the jet-like structure of the flow field and the viscous spreading of the diverging jet along the axis of symmetry. The dashed lines indicate the edge of the jet defined by \( \frac{\partial \Psi_0}{\partial z} = 0 \) at \( \theta = \theta_e \) (The edge of the jet is defined as the place where the streamlines are at their minimum distance from the axis).

It has been found that \( \theta_e \approx 0.44 - 0.47 \theta_o \) for \( 30^\circ < \theta_o < 150^\circ \); within the definition area of \( c_o, \theta_e < 90^\circ \).

The mass flow \( \Psi \) through a small circle of radius \( s = r \sqrt{1-c^2} \) with the centre on and perpendicular to the axis of symmetry, of the viscous jet represented by \( g_o(c) \) yields

\[
\Psi = \frac{I_o \sqrt{\rho \mu \eta}}{K \eta} r (1-c) \quad \text{for } s = 0 ,
\]  

(4.63)

where from (4.10) - (4.15) it can be derived that

\[
h_\eta = -\{a_1 + 2a_2 + a_3 + a_3 \ln (2)\} > 0 .
\]  

(4.64)

Comparison of the mass flow caused by the inviscid- and viscous jet at \( s = 0 \) shows that \( \Psi = O(s) \) for \( \eta = 0 \), see (3.10), and \( \Psi = O\left(\frac{s^2}{z}\right) \) for \( \eta > 1 \); where \( s \) is the distance to the axis and \( z \) the vertical distance to the point electrode measured along the axis of symmetry. Also for \( c_o = 0, K_b < 1 \) and at constant \( I_o, \rho, \mu, s \) the mass flow of the slow viscous jet is a factor \( \frac{s}{12K_\eta z} \) smaller than the one obtained for the inviscid jet. It is clear that this is due to the spreading of the viscous jet.

The graphs of \( \Delta \bar{J}_{r,1} \) show the effect of the electromagnetic induction, i.e. the \( \gamma \times B \)-term in Ohm's law see (2.3), upon the isotropic electric current distribution \( \bar{J}_{r,0} \). The generated viscous fluid motion tends to shift the electric current flow towards the cone wall and the axis of symmetry. The measure of shifting of electric current at low \( \frac{K_b}{K_\eta} \) is defined by the angles where \( \Delta \bar{J}_{r,1} \)
equals zero, i.e. \( \frac{df_1}{dc} = 0 \) at \( c = c_1, c_2 \), with \( \theta_1 < \theta_2 \), see figure 3.4. By introducing \( \theta_1 = \lambda_1 \theta_0 \) and \( \theta_2 = \lambda_2 \theta_0 \), we respectively obtained \( \lambda_1 \approx 0.30 - 0.32 \) and \( \lambda_2 \approx 0.77 - 0.80 \), being very close to the values found in the inviscid problem, see section 3.4. In analogy with the inviscid problem, it turns out that at small \( K_b \) the perturbation of the isotropic current distribution is larger at the surface of the cone than at the axis of symmetry; especially at larger values of \( \theta_0 \). It should be noticed that in the viscous flow \( \frac{d^2f}{dc^2} = 0 \) at \( c = c_0 \), which may enlarge this effect.

The perturbations \( \Delta \Xi^1 \) and \( \Delta \Xi^2 \) of the base fluid motion \( \Xi \) are presented in figure 4.3. The perturbation \( \Delta \Xi^1 \), which dominates compared to \( \Delta \Xi^2 \) in case of liquid metals with \( \mathrm{Pm} = K_b K_\eta \ll 1 \), increases the velocities in the entire field and will reduce the edge of the jet, whereas the typical high-temperature plasma (\( \mathrm{Pm} \gg 1 \)) perturbation term \( \Delta \Xi^2 \) decreases the velocities in the entire flow field, increasing the edge of the jet flow. When the magnetic Prandtl number is about of order unity as it may be in a low-temperature plasma or an arc discharge, the effects of both perturbed fluid motions \( \Delta \Xi^1 \) and \( \Delta \Xi^2 \) will more or less cancel each other so that then the basic fluid motion will be only slightly disturbed. Consideration of the respective magnitudes of \( g_1(c) \) and \( g_2(c) \) indicates that a complete cancellation over the entire flow field is not possible. Namely, when the total effect of \( \Delta \Xi^1 \) and \( \Delta \Xi^2 \) upon \( \Xi \) is negligible in the middle region of \( \theta \), \( \Delta \Xi^1 \) predominates over \( \Delta \Xi^2 \) near the axis of symmetry, whereas \( \Delta \Xi^2 \) prevails in the neighbourhood of the surface of the cone.

Finally note the differences in the perturbations of the basic inviscid- and slow viscous fluid motions, see figures 3.2 and 4.3.

It is clear that in the regular perturbations, presented in this section, the value of \( K_b \) may exceed the value of \( K_\eta \) up to a certain upper bound. However when \( K_b \) becomes much larger than \( K_\eta \) the asymptotic expansions will diverge as usual. This conjecture is clarified by consideration of the regular perturbation (4.30) and (4.31). Therefore in that case the integrated form of the curl of the Navier-Stokes equation (2.53) has to be applied, together with (2.51), (2.59), (2.60), instead of the asymptotic expansions (4.1) - (4.3), (4.30), (4.31).

4.3. The viscous flow solution at arbitrary value of \( K_\eta \) and small \( K_b \); i.e. \( K_\eta > K_\eta, \text{min} \)

In this section we consider the viscous fluid motion at arbitrary value of \( K_\eta \).
and at small value of $K_b$. In order to calculate solutions of the viscous flow field, integral expressions are derived, from which solutions are obtained for different values of $K_\eta$ and $c_o$ by numerical computation. When $K_\eta$ becomes equal or smaller than a certain critical value: $K_\eta, \text{min}$, which depends on $c_o$, singularities and as a consequence physically unrealistic phenomena enter the flow field from the axis of symmetry. The relation between the critical value $K_\eta, \text{min}$ and those found by other authors will be discussed. Values of $K_\eta, \text{min}$ as function of $c_o$ are calculated by numerical iteration and by a straightforward series expansion method. It turns out that the latter method leads to more accurate results in an easy way.

To investigate the viscous flow at arbitrary value of $K_\eta$ we return to the Riccati differential equation (2.50) and the integral equation of $g$ (2.53), viz.

$$g^2 = 2K_\eta \left[ (1-c^2) \frac{dg}{dc} + 2cg \right] - G_\eta(c), \quad (2.50)$$

$$g = \frac{(1-c^2)}{2K_\eta} \int_{c_o}^{c} \frac{g^2(t)}{(1-t^2)^2} \, dt + \frac{(1-c^2)}{2K_\eta} \int_{c_o}^{c} \frac{G_\eta(t)}{(1-t^2)^2} \, dt, \quad (2.53)$$

where $G_\eta(c)$ is given by (2.51). These equations have been obtained by integrating the curl of the Navier-Stokes equation (2.18) respectively three and four times with respect to $c$; and by substituting the boundary conditions (2.32) - (2.38) together with the assumption that the velocity along the axis of symmetry is finite at $c = 1$, see section 2.7 and chapter 6. It is clear that the applied similarity solution method (2.14), (2.15) couples the viscous-, inertia- and Lorentz forces for all $c$, $c_o < c < 1$, $-1 < c_o < 1$.

From the behaviour of $G_\eta(c)$, see (2.56), (2.57) as derived from (2.55), and the integral equation of $g$ (2.53) in section 2.7 we were able to predict the general behaviour of the viscous flow field generated. It was found to consist of an incoming flow along the surface of the cone which is turned off into an outwards diverging jet flow along the axis of symmetry. At small $K_b$, to be considered here, we are able to specify that behaviour more precisely. For $K_b \ll 1$ the weak perturbing effect of the fluid motion upon the electromagnetic field is negligible, so that an analytical expression of $G_\eta(c)$ can be derived. By substitution of $f = f_o$, where $f_o$ is given by (3.31), in the integral expression (2.51), $G_\eta(c)$ takes the form
\[ G_\eta(c) = d_o + d_1 c + d_2 c^2 + d_3 (1+c)^2 \ln (1+c), \quad (4.65) \]

where
\[ d_o = -\frac{4c_o}{(1-c_o)^3} - \frac{2(1+c_o)^2}{(1-c_o)^4} \ln (1+c_o) + \frac{8c_o}{(1-c_o)^4} \ln (2), \quad (4.66) \]

\[ d_1 = \frac{4(1+c_o)}{(1-c_o)^3} + \frac{4(1+c_o)^2}{(1-c_o)^4} \ln (1+c_o) - \frac{8(1+c_o^2)}{(1-c_o)^4} \ln (2), \quad (4.67) \]

\[ d_2 = -\frac{4}{(1-c_o)^3} - \frac{2(1+c_o)^2}{(1-c_o)^4} \ln (1+c_o) + \frac{8c_o}{(1-c_o)^4} \ln (2), \quad (4.68) \]

\[ d_3 = \frac{2}{(1-c_o)^2}, \quad (4.69) \]

for \( c_o < c < 1, -1 < c_o < 1 \).

The function \( G_\eta(c) \) satisfies the properties stated by (2.56), (2.57) viz.,
\( G_\eta(c) > 0 \) on \( c_o < c < 1 \), \( G_\eta(c_o) = 0(c-c_o) \), \( G_\eta(1) = 0((1-c)^2) \).
Also it can be proved that \( G_\eta(c) \) is bounded by
\[ 0 < a_1 \frac{(c-c_o)(1-c)^2}{(1-c_o)^5} < G_\eta(c) < a_2 \frac{(c-c_o)(1-c)^2}{(1-c_o)^5} < \frac{8}{27}, \quad (4.70) \]

for \( c_o < c < 1, -1 < c_o < 1 \), where

\[ a_1 = -\{ (1+3c_o)(1-c_o) + 2(1+c_o^2) \ln \left(\frac{1+c_o}{2}\right) \} > 0 , \quad (4.71) \]

\[ a_2 = 2 \{ (3+c_o)(1-c_o) + 4(1+c_o) \ln \left(\frac{1+c_o}{2}\right) \} > 0 , \quad (4.72) \]
on applying the same techniques as used in appendices A and B. The upper- and lower bounds indicate that $G_\eta(c)$ is bounded on $c_0 < c < 1$ for $-1 < c_0 < 1$ and that $G_\eta(c)$ reaches the largest magnitude for $c_0 = -1$.

For some special values of $c_0$ the general expression of $G_\eta(c)$ reduces to

$$G_\eta(c) = 0 \quad \text{for } c_0 = 1,$$  \hspace{1cm} (4.73)

$$G_\eta(c) = \frac{(c-c_0)(1-c)^2}{3(1-c_0)^2} \quad \text{for } c_0 = 1,$$  \hspace{1cm} (4.74)

$$G_\eta(c) = 4c(1-c) - 8 \ln(2) c + 2(1+c)^2 \ln(1+c) \quad \text{for } c_0 = 0,$$  \hspace{1cm} (4.75)

$$G_\eta(c) = \frac{1}{2} (1-c^2) + \frac{1}{2} (1+c)^2 \ln\left(\frac{1+c}{2}\right) \quad \text{for } c_0 = -1,$$  \hspace{1cm} (4.76)

indicating that $G_\eta(c)$ is bounded for all $c$ and $c_0$.

Although the integral equation of $g$, (2.53), seems suitable for iteration, it is more convenient to introduce the usual transformation

$$g = -\frac{2K_\eta (1-c^2)}{u} \frac{du}{dc},$$  \hspace{1cm} (4.77)

which transforms the non-linear first order Riccati differential equation (2.50) into a linear homogeneous second order differential equation of $u$, see Sozou (1971a),

$$\frac{d^2u}{dc^2} + \frac{G_\eta(c)}{4K_\eta^2 (1-c^2)^2} u = 0.$$  \hspace{1cm} (4.78)

From (2.56) and (2.57) it is evident that the coefficient multiplying $u$ in the differential equation does not contain any singularity at $c = 1$; the coefficient is positive on $c_0 < c < 1$ and it possesses a simple zero at $c = c_0$, which is a so-called turning point.

The boundary conditions of $g(c)$, see (2.34) - (2.38), require that $u(c_0)$ is a constant and that the derivative of $u$ is identical to zero at $c = c_0$. Since $u$ may be multiplied by an arbitrary constant, different from zero, which follows from the transformation, we choose
\[ u(c_o) = 1, \quad (4.79) \]
\[
\left( \frac{du}{dc} \right)_{c=c_o} = 0. \quad (4.80)
\]

The boundary condition of \( g \) at \( c = 1 \) is satisfied by the transformation. It is clear that (4.77) transforms the boundary value problem (2.50) into the initial value problem (4.78) - (4.80), which facilitates the numerical computation. From (4.77) and (4.78) it can be found that the derivative of \( g \) yields

\[
\frac{dg}{dc} = \frac{2K_\eta(1-c^2)}{u^2} \left( \frac{du}{dc} \right)^2 + \frac{4K_\eta c}{u} \frac{du}{dc} + \frac{C(c)}{2K_\eta(1-c^2)}. \quad (4.81)
\]

From (2.57), (4.79), (4.80) it is observed that the typical viscous boundary condition on the surface of the cone is satisfied; i.e. \( \left( \frac{dg}{dc} \right)_{c=c_o} = 0 \). Therefore the behaviour of the viscous boundary layer on the surface of the cone is determined by the second derivative of \( g \) at \( c = c_o \), see (2.52), which for \( K_b \ll 1 \) is of the form

\[
\left( \frac{d^2g}{dc^2} \right)_{c=c_o} = \frac{1}{K_\eta(1+c_o)(1-c_o)^4} \left\{ (3+c_o)(1-c_o) + 4(1+c_o) \ln \left( \frac{1+c_o}{2} \right) \right\} > 0. \quad (4.82)
\]

Upon substitution of (4.65) - (4.69) in (2.53) and (2.54) and some algebra, for \( K_b \ll 1 \) the integral equations of \( g \) and \( \frac{dg}{dc} \) take the form

\[
g(c) = \frac{(1-c^2)}{2K_\eta} \int_{c_o}^{c} \frac{g^2(t)}{(1-t^2)^2} \, dt - \frac{(c-c_o)(1-c)}{K_\eta(1-c_o)^3} + \frac{(1+2c-2c_o-c_o^2)(1-c)}{K_\eta(1-c_o)^4} \ln \left( \frac{1+c_o}{2} \right) + \frac{(1+c)}{K_\eta(1-c_o)^2} \ln \left( \frac{1+c}{2} \right), \quad (4.83)
\]

and

\[
\frac{dg}{dc} = -\frac{c}{K_\eta} \int_{c_o}^{c} \frac{g^2(t)}{(1-t^2)^2} \, dt + \frac{g^2(c)}{2K_\eta(1-c^2)} + \frac{2(c-c_o)}{K_\eta(1-c_o)^3} +
\]

\[
\frac{g^2(c)}{2K_\eta(1-c^2)} + \frac{2(c-c_o)}{K_\eta(1-c_o)^3}.
\]
\[
- \frac{(1-4c+2c_0+c_0^2)}{K_\eta (1-c_0)^4} \ln \left( \frac{1+c_0}{2} \right) + \frac{1}{K_\eta (1-c_0)^2} \ln \left( \frac{1+c}{2} \right)
\]

for \( c_0 < c < 1 \), \(-1 < c_0 < 1 \).

At this stage we may draw already some conclusions concerning the behaviour of \( g(c) \) in the limit cases \( c_0 = \pm 1 \), \((\theta_o = 0; 180^\circ)\). From (4.73), (4.77) - (4.80) it is obvious that \( g(c) \) is identical to zero for \( c_0 = 1 \), as expected. In addition in the limit \( \theta_o + 180^\circ \), (4.83) reduces to

\[
g(c) = \frac{(1-c^2)}{2K_\eta} \int_{c_0}^c \frac{g^2(t)}{(1-t^2)^2} \, dt - \frac{(1-c^2)}{8K_\eta} + \frac{(1-c^2)}{8K_\eta} \ln \left( \frac{1+c}{2} \right) + \frac{(1+c)}{4K_\eta} \ln \left( \frac{1+c}{2} \right) + \frac{(1+c)}{4K_\eta} \ln \left( \frac{1+c}{2} \right) + \infty
\]

for \( c_0 < -1 \), \( c_0 < c < 1 \).

The third term of the right-hand side is positive for \(-1 < c_0 < 1 \) and is unbounded for \( c_0 = -1 \), \( c \neq \pm 1 \). Since the other terms and the positive integral of \( g^2 \) cannot cancel this singularity, we conclude that \( g(c) \) becomes positive singular on \( c_0 < c < 1 \) for \( c_0 = -1 \).

This statement is in agreement with the behaviour of \( \frac{d^2g}{dc^2} \) at \( c_0 = -1 \), see (4.82). Hence we obtain

\[
g(c) = 0 \quad \text{for} \quad c_0 = 1 , \quad (4.86)
\]

\[
g(c) = +\infty \quad \text{on} \quad c_0 < c < 1 \quad \text{for} \quad c_0 = -1 , \quad (4.87)
\]

where \( g(c) \) still satisfies the viscous boundary conditions at \( c_0 = -1 \).

Due to the complexity of the coefficient multiplying \( u \) in (4.78), solutions of \( u \) and \( \frac{du}{dc} \) can be obtained only by numerical computation. For that purpose the differential equation of \( u \) has been replaced by integral equations of \( u \) and \( \frac{du}{dc} \). Integration with respect to \( c \) and by parts and substitution of the boundary conditions of \( u \) and \( \frac{du}{dc} \) lead to

\[
u = 1 - \frac{1}{4K_\eta^2} \int_{c_0}^c \frac{(c-t) \mathcal{G}(t) u(t)}{(1-t^2)^2} \, dt , \quad (4.88)
\]
\[
\frac{du}{dc} = -\frac{1}{4K^2_\eta} \int_0^{c_c} \frac{G_\eta(t) u(t)}{(1-t^2)^2} \, dt,
\]

(4.89)

where at small \(K_b\), \(G_\eta(c)\) is given by (4.65) - (4.69).

To avoid difficulties in the numerical calculation at \(c = 1\), we used there the value of \(G_\eta(c)\) obtained from a series expansion in powers of \(1-c\).

For given values of \(K_\eta\) and \(c_o\), the numerical calculation has been carried out by an iteration process. Starting with \(u(c) = 1\) for \(c < c < 1\), the calculation was terminated when a relative error of less than \(10^{-3}\) at each integrated point on \(c_o < c < 1\) had been reached. The convergence of the process turned out to be rapid. The chosen numbers of equidistant integration steps are \(n = 1000\) for \(c_o > 0\) and \(n = 2000\) for \(c_o < 0\). With the accurate solution of \(u\), the solution of \(\frac{du}{dc}\) is determined by a straightforward numerical integration. The flow field components \(g\) and \(\frac{dg}{dc}\) are then calculated from (4.77), (4.81). It should be noted that the above procedure to calculate numerically the viscous flow at arbitrary \(K_\eta\) was formulated by Sozou (1971a). By repeated differentiation of \(\frac{dg}{dc}\) and \(G_\eta(c)\), see (4.65) - (4.69), (4.81) and substitution of (4.77), (4.78), higher order derivatives of \(g(c)\) can be determined and from these with (2.64) the pressure field, generated by the fluid motion and the electromagnetic field at specific values of \(K_\eta\) and \(c_o\).

The generated fluid motion and from (2.59), (2.60) also the shifting of electric current flow has been calculated numerically at \(K_\eta = 1\) and small \(K_b\) for \(c_o = 0, \pm \frac{1}{2}, \pm \frac{1}{2} \sqrt{2}, \pm \frac{1}{2} \sqrt{3}\) (\(30^\circ < \theta_o < 150^\circ\)). Since the fluid motion and the shifting of electric current show only small relative differences with the values obtained earlier from the asymptotic expansions at large \(K_\eta\) (4.1) - (4.3) and (4.30) - (4.33), there is no need to depict the \(K_\eta = 1\) behaviour in separate figures. For \(c_o = 0\) the maximum relative differences amount to \(10^{-3}\) for respectively \(g_o\), \(g\) and \(10^{-2}\) for \(\frac{df_1}{dc}\), increasing a little for larger values of \(\theta_o\).

In the same way the edge of the viscous jet flow does not change much. At \(K_\eta = 1\) we found that \(\theta_o \approx 0.44 - 0.46 \theta_o\) for \(30^\circ < \theta_o < 150^\circ\).

In addition on applying only the first two terms of the asymptotic expansion (4.1) - (4.3), viz. \(g_o\) and \(g_1\) and by using the relations between our quantities and those defined by Sozou, see (2.27) - (2.30), for \(K_\eta = \sqrt{2}\), values of \(g_s\) have been found identical to those given by Sozou (1971a, p. 29) in table 1 for \(K_s = 1\), which were obtained also by numerical computation.

The above observations justify the conclusion that at small \(K_b\) and \(K_\eta > 1\) it makes little difference whether the flow field is calculated numerically from (4.77), (4.81), (4.88), (4.89) or analytically from the asymptotic expansions.
(4.1) - (4.3) or (4.30) - (4.37), truncated after a sufficient number of terms. In particular this is the case since both asymptotic expansions originally have been derived for large values of $K_\eta$; however it turns out that they are applicable at least as far as $K_\eta = 1$.

Examination of the behaviour of $u$ and $\frac{du}{dc}$ as obtained from (4.88) and (4.89) for $K_\eta > 1$ shows that $u(c) > 0$ and $\frac{du}{dc} < 0$ on $c_o < c < 1$ with $u(c_o) = 1$ and $\frac{du}{dc}_{c=c_o} = 0$. At increasing values of $c$ both functions monotonically decrease, reaching their lowest values at $c = 1$, where $u(1) > 0$ and $\frac{du}{dc}_{c=1} < 0$. At smaller $K_\eta > 1$ the decay of the functions is stronger. The behaviour of $u$ and $\frac{du}{dc}$ shown here agrees with $g(c) > 0$ on $c_o < c < 1$, see (4.77), as already derived in section 2.7.

When we take $K_\eta$ smaller than unity we find a certain value of $K_\eta$ where $u(1)$ becomes identical to zero and consequently, as appears from (4.77) and (4.81), the solution of the viscous flow field becomes critical. This particular value of $K_\eta$ is defined as

$$K_\eta = K_{\eta, \text{min}},$$

when

$$u(c) > 0 \quad \text{for } c_o < c < 1,$$

$$u(1) = 0.$$  \hspace{1cm} (4.90)

The appearance of a critical lower bound of $K_\eta$ in the viscous flow field, to be considered here, will be discussed in some detail. In the following part of this section we will present a figure and a table of values of $K_{\eta, \text{min}}$ as function of $c_o$. Also we shall consider the relations between the critical value $K_{\eta, \text{min}}$ found here and those found by other authors.

Usually the value of $K_{\eta, \text{min}}$ is calculated by numerical iteration. It turns out that for a converging iteration lower and upper values, being very close to $K_{\eta, \text{min}}$, need to be used as start values for the numerical computation. The procedure to obtain these start values appears to be very laborious. Therefore a straightforward series expansion method has been developed which results in very accurate results in an easier way due to the very rapid convergence of the series expansion.

In the next section 4.4 we consider the phenomena and their physical significance, that occur in the viscous flow field when $K_\eta < K_{\eta, \text{min}}$. 
It appears from (4.90) and the covering text that the critical value of $K_{\eta}$ i.e. $K_{\eta, \text{min}}$ turns out to be a lower bound. Here small values of $K_b$ are considered; in general the value of $K_{\eta, \text{min}}$ for a fixed value of $c_o$, is also associated with a certain fixed value of $K_b$. Therefore the value of $K_{\eta, \text{min}}$ as function of $c_o$ at small value of $K_b$, as presented hereafter, is in fact an asymptotic value for $K_b \rightarrow 0^+$. In view of (2.30) the minimum value of $K_\eta$ is analogous to the maximum value of $K_s$ as found by Sozou (1971a) for the flat wall configuration $c_o = 0$, namely $K_\eta = \frac{2}{K_s}$. It appears from figure 4.4 that the value of $K_{\eta, \text{min}}$ is strongly dependent on the vertical angle $\theta_o$ of the cone wall configuration.

![Graph](image_url)

**Figure 4.4.** Values of $K_{\eta, \text{min}}$ as function of $c_o$ for small $K_b$.

Numerical values of $K_{\eta, \text{min}}$ for a number of values of $c_o$ are given in table 4.1. For comparison the values of $K_s$ and $\lambda_{nu}$ as defined by Sozou (1971a) and Narain & Uberoi (1973), where the subscripts refer to the respective authors, see (2.30), are added and since $K_\eta$ is in fact the inverse effective viscous Reynolds number we give the corresponding maximum value: $Re, \text{max}$. In addition it has been found
that $K_\eta = 6.28 \times 10^{-2}/I_o$ for molten steel and $K_\eta = 7.5 \times 10^{-2}/I_o$ for mercury, representing more or less average values of $K_\eta$ as function of $I_o$ for liquid metals, see figure 2.2. Therefore a mean maximum value of the total electric current supplied by the point electrode into the viscous fluid, obtained from $I_{o,max} = 7 \times 10^{-2}/K_{\eta,min}$, is included in the table.

As mentioned by Andrews & Craine (1978) the maximum admissible electric currents are very low compared with the normal welding- or stirring currents of several hundred amperes. Only at very small values of $\theta$, $I_{o,max}$ and $J_{r,max}$ possess larger upper bounds.

<table>
<thead>
<tr>
<th>$c_o$</th>
<th>$\theta_o$</th>
<th>$K_{\eta,min}$</th>
<th>$K_{s,max}$</th>
<th>$\lambda_{nu,max}$</th>
<th>$Re,max$</th>
<th>$I_o,max$</th>
</tr>
</thead>
<tbody>
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<td>0.99</td>
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<td>5.17x10^{-3}</td>
<td>7.47x10^{4}</td>
<td>3.74x10^{8}</td>
<td>1.93x10^{2}</td>
<td>1.35x10^{1}</td>
</tr>
<tr>
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<td>1.17x10^{-2}</td>
<td>1.45x10^{4}</td>
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<tr>
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<tr>
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<td>1.50x10^{2}</td>
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Table 4.1. Values of $K_{\eta,min}$, $K_{s,max}$, $\lambda_{nu,max}$, $Re,max$ and $I_{o,max}$ at typical values of $c_o$ for $K_\eta \ll 1$.

The values of $K_{\eta,min}$, presented in figure 4.4 and table 4.1 have been calculated by numerical iteration. Starting with chosen values respectively a little larger and a little smaller than the expected value of $K_{\eta,min}$, the iteration process has been continued until $|u(1)| < 10^{-8}$. In view of the large range of the value
of $K_{\eta, \min}$ as function of $c_o$, see figure 4.4, this shoot- and iteration method is rather laborious. In this way we found that at $c_o = 0$ in fact $K_{\eta, \min} = 8.163417 \times 10^{-2}$ and so from (2.30) the corresponding value of $K_{s, \max} = 3.001138 \times 10^2$, being equivalent to the value found by Sozou (1971a). The upper bound of $\lambda_{nu} : \lambda_{nu, \max}$, see (2.30), at $c_o = \frac{1}{2} \sqrt{3} (\theta_o = 30^\circ)$ as given by Narain & Uberoi (1973) turns out to be a bit smaller, viz. $1.41739 \times 10^5$. Moreover their approximate formula for the critical value of $\lambda_{nu}$ appears to be moderately accurate for $-\frac{1}{4} < c_o < \frac{1}{2} \sqrt{3}$; but very inaccurate for other values of $c_o$.

A very accurate but much more complicated approximation of $K_{\eta, \min}$, $\tilde{K}_{\eta, \min}$ for $-1 < c_o < 1$, which has been derived from a straightforward calculation of a series expansion of $K_{\eta, \min}$, see appendix C, will be presented hereafter, see (4.97) - (4.99).

For a finite hemisphere with a free surface Sozou & Pickering (1976) found at small $K_b$ even a smaller maximum value, viz. $K_{s, \max} = 94.1$; corresponding with $K_{\eta, \min} = 0.1458$.

As mentioned before in general the minimum value of $K_{\eta}$ is associated with a certain value of $K_b$. For the semi-infinite flat wall configuration Sozou & English (1972) obtained a critical value $K_{se} = 813$ when $a = 1$. Expressed in our quantities this amount to $K_{\eta, \min} = 0.0496$ when $K_b = 20.16$. Note that in practical liquid metal applications, as welding and stirring using rather large electric currents (500 A), the actual values of the parameters are about: $K_{\eta} = 10^{-4}$ and $K_b = 10^{-3}$.

Purely numerical calculations which also include the effect of the inertia force in Navier-Stokes equation have been carried out by Atthey (1980) for a finite hemispherical container with a free surface and by Craine & Andrews (1984) for an actual weld pool model. They found no instabilities or singularities in the flow field for electric currents up to 100 A, corresponding with $K_{\eta} = 7 \times 10^{-4}$.

As denoted by Atthey (1980), the numerical results suggest that the flows are still laminar for these values of the viscous Reynolds number ($Re = K_{\eta}^{-1}$). However in view of the results that will be presented in chapters 5 and 6 it is suggested that the effect of numerical diffusion plays an important role in the calculations, see Atthey (1980), Roache (1972), and also that it strongly depends on the chosen size of the mesh; especially at the axis of symmetry.

An alternative straightforward method to calculate $K_{\eta, \min}$ as function of $c_o$ is presented in appendix C. This method, derived from (4.88) and (4.90), results in an expression of $K_{\eta, \min}$ as an infinite series expansion,
\[ K_{\eta,\text{min}}^2 = \frac{1}{4} \sum_{n=0}^{\infty} \frac{V_n}{2^{2n} K_{\eta,\text{min}}^{2n}} , \]  

(4.91)

where analytical expressions of the first two terms of the series yield

\[ V_0 = \frac{2}{(1-c_0^2)^2} - \frac{2(3+c_0)}{(1-c_0^3)3} \ln \left( \frac{1+c_0}{2} \right) - \frac{2(1+c_0)^2}{(1-c_0^4)^4} \ln^2 \left( \frac{1+c_0}{2} \right) + \frac{2}{(1-c_0^2)^2} F_0 , \]  

(4.92)

\[ V_1 = \frac{2(11+c_0)}{(1-c_0^5)^5} + \frac{12(7+4c_0+c_0^2)}{(1-c_0^6)^6} \ln \left( \frac{1+c_0}{2} \right) + \frac{8(2+c_0)^2(3+c_0)}{(1-c_0^7)^7} \ln^2 \left( \frac{1+c_0}{2} \right) + \]  

\[ + \frac{16(2+c_0)^2(1+c_0)^2}{(1-c_0^8)^8} \ln^3 \left( \frac{1+c_0}{2} \right) + \frac{4}{(1-c_0^4)^4} F_1 - \frac{1}{2} V_0^2 , \]  

(4.93)

where \( F_n \), \( n = 0,1 \) are polylogarithmic integrals

\[ F_n = \int_{c_0}^{1} \frac{1}{1-t} \ln^{n+1} \left( \frac{1+t}{2} \right) \, dt . \]  

(4.94)

It turns out that the decay in magnitude of the coefficients of the series is tremendous with increasing order, see tables C.1 - C.4 of appendix C. Therefore only a few terms of the series expansion are required to achieve very accurate values of \( K_{\eta,\text{min}} \). The derivation, convergence of the coefficients and extended tables of values of different approximations of \( K_{\eta,\text{min}} \) for a large number of values of \( c_0 (-0.99 < c_0 < 0.99) \) are presented in appendix C.

In the calculation of \( K_{\eta,\text{min}} \) this method deserves preference over the time-consuming shoot- and iteration method used so far, see Sozou (1971a), Narain & Uberoi (1971,1973), Sozou (1974), Jansen (1984) and especially Sozou & English (1972). Likewise this straightforward method will be very useful in the calculation of the lower bound of \( K_{\eta} \) in case of a free parameter, see chapter 6, and at arbitrary value of \( K_{b} \), see Sozou & English (1972).

Truncation of the infinite series expansion of \( K_{\eta,\text{min}}^2 \) (4.91) after \( m \) terms leads to the order \( m \) approximation of \( K_{\eta,\text{min}} \) denoted by \( \tilde{K}_{\eta,\text{min}}^{(m)} \). The order one and two approximations yield
\[ \tilde{K}_{\eta, \text{min}}^{(1)} = \frac{1}{2} \sqrt{\nu_o} \]  
(4.95)

\[ \tilde{K}_{\eta, \text{min}}^{(2)} = \frac{1}{8} \left( \frac{\nu_o}{\sqrt{\nu^2 + 4\nu_1}} \right) \]  
(4.96)

From these approximations an accurate analytical approximation formula of \( K_{\eta, \text{min}} \) can be obtained, viz.

\[ K_{\eta, \text{min}} = \tilde{K}_{\eta, \text{min}} \]  
(4.97)

where

\[ \tilde{K}_{\eta, \text{min}} = \alpha \tilde{K}_{\eta, \text{min}}^{(1)} + (1-\alpha) \tilde{K}_{\eta, \text{min}}^{(2)} \]  
(4.98)

with

\[ \alpha = 0.125 \left( 1 + c_o \right)^{0.4} \]  
(4.99)

It turns out that the relative error of \( \tilde{K}_{\eta, \text{min}} \) is less than \( 10^{-3} \) for all values of \( c_o, -0.99 < c_o < 0.99 \).

The series expansion of \( K_{\eta, \text{min}} \) enables us to draw some conclusions about the behaviour of \( K_{\eta, \text{min}} \) in the limit cases \( c_o \to \pm 1 \). From (4.91) - (4.96), it can be found

\[ K_{\eta, \text{min}} = \delta \sqrt{1-c_o} \left( 1+o(1) \right) \to 0 \quad \text{for} \ c_o \to 1 \]  
(4.100)

\[ K_{\eta, \text{min}} = \sqrt{-\frac{1}{8 \ln \left( \frac{1+c_o}{2} \right)} \left( 1+o(1) \right) \to +\infty} \quad \text{for} \ c_o \to -1 \]  
(4.101)

where an accurate value of \( \delta = 14\pi / (\sqrt{3} (49\pi^2 + 10)) = 5.144 \times 10^{-2} \) is obtained from (D.72) of appendix D. The behaviour of \( K_{\eta, \text{min}} \) indicates that in the limit cases \( c_o = \pm 1 \) no real viscous fluid motions exist. In the trivial case \( c_o = 1 \) no fluid region is available, while at \( c_o = -1 \) the maximum admissible value of the viscous Reynolds number is identical to zero. Note that this conclusion can also be drawn from (4.86), (4.87).
It may be mentioned that any relation between $K_{\eta, \min}$ and the velocity in the neighbourhood of the surface of the cone or near the axis of symmetry, represented respectively by $\frac{d^2 g}{dc^2}$, see (4.82), or $\frac{du}{dc_{c=1}}$, see (4.88), (4.89), could not be found. The origin of the appearance of a lower bound of $K_\eta$ in the mathematics of the viscous problem will be discussed in next section 4.4.

To examine the phenomena that occur in the fluid near the axis of symmetry when $K_\eta$ approaches $K_{\eta, \min}$, we calculated the flow field and in particular the edge of the viscous jet, defined by $\frac{\partial \psi}{\partial z} = 0$ for $\theta = \theta_e$, see (3.25), at distinct values of $K_\eta$, related to $K_{\eta, \min}$. Namely at $K_\eta = \tilde{\alpha} K_{\eta, \min}$, where respectively $\tilde{\alpha} = 6, 3, 2, 1.6, 1.35, 1.15, 1.08, 1.03, 1.005$ for different values of $c_0$, viz. $c_0 = 0, \pm \frac{1}{2}, \pm \frac{1}{2} \sqrt{2}, \pm \frac{1}{2} \sqrt{3}$. Of particular interest is the behaviour of the ratio of $\frac{\theta_e}{\theta_e, \max}$, where $\theta_e, \max$ is the value at large $K_\eta$, as function of $\tilde{\alpha} = K_\eta/K_{\eta, \min}$, see figure 4.5.

![Figure 4.5](image)

Figure 4.5. The behaviour of $\frac{\theta_e}{\theta_e, \max}$ as function of $\frac{K_\eta}{K_{\eta, \min}}$ for $30^\circ < \theta_e < 150^\circ$. 
Since the relative deviation of $\frac{\theta_e}{\theta_{e,\text{max}}}$ turns out to be less than $2.5 \times 10^{-2}$ for $-\frac{1}{2} \sqrt{3} < c_o < \frac{1}{2} \sqrt{3}$, the figure shows only a single line. It is worth noticing that $\theta_e = \theta_{e,\text{max}}$ for $K_\eta > 4 K_{\eta,\text{min}}$, $\theta_e = 0.8 \theta_{e,\text{max}}$ at $K_\eta = 1.4 K_{\eta,\text{min}}$, and in addition that $\theta_e = 0.5 \theta_{e,\text{max}}$ at $K_\eta = 1.1 K_{\eta,\text{min}}$. Hence we observe a very abrupt breakdown of the spreading of the viscous jet flow along the axis of symmetry, being identical over the range $30^\circ < \theta_o < 150^\circ$, when $K_\eta$ approaches $K_{\eta,\text{min}}$. Note that for $K_\eta > K_{\eta,\text{min}}$ the general behaviour of $\frac{\partial \psi}{\partial z}$ is as follows: $\frac{\partial \psi}{\partial z} > 0$ for $\theta_e < \theta < \theta_o$, $\frac{\partial \psi}{\partial z} < 0$ for $0 < \theta < \theta_e$ with $\frac{\partial \psi}{\partial z} = 0$ for $\theta = 0, \theta_e, \theta_o$. To examine this subject in more detail we consider a value of $K_\eta$ slightly larger than $K_{\eta,\text{min}}$, viz. $K_\eta = K_{\eta,\text{min}} + \delta$, where $\delta > 0$, $\delta \to 0^+$. In that case $u(1)$ is still positive, hence we suppose

$$u(c) = a(1+\varepsilon-c)(1+o(1)) \quad \text{for } c = 1, K_\eta \sim K_{\eta,\text{min}},$$  \hfill (4.102)

where

$$a = -\frac{d u}{d c} \bigg|_{c=1} > 0,$$  \hfill (4.103)

and $\varepsilon > 0$, $\varepsilon \to 0^+$. 

Upon substituting (4.102) and (4.103) into (4.77), (4.81) and using (2.57), (3.25), we find that the edge of the viscous jet, defined as usually by $\frac{\partial \psi}{\partial z} = 0$, is located at

$$\frac{\partial \psi}{\partial z} = 0 \quad \text{at } c = c_e = 1 - \varepsilon \quad \text{for } c = 1, K_\eta \sim K_{\eta,\text{min}},$$  \hfill (4.104)

for $\varepsilon = \varepsilon(\delta) > 0$.

In section 4.4 it will be shown that in the viscous flow field $\frac{\partial \psi}{\partial z} = 0$ at $c = c_e = 1$ never can occur. There it will become evident that $\frac{\partial \psi}{\partial z} > 0$ on $c_o < c < 1$ for $K_\eta = K_{\eta,\text{min}}$, and on $c_o < c < c_s$ for $K_\eta < K_{\eta,\text{min}}$, see (4.121).

This behaviour contrasts with the inviscid problem, where $\frac{\partial \psi}{\partial z} > 0$ on $c_o < c < 1$ with $\frac{\partial \psi}{\partial z} = 0$ at $c = 1$ for all finite values of $K_b$, indicating a convergence of the inviscid flow into a parallel outwards jet flow along the axis of symmetry. It has to be remarked that the behaviour of the inviscid jet flow, see chapter 3, (besides the relatively weak singularity in the velocity field at the axis of
symmetry) agrees better with observations from experiments, carried out by Woods & Milner (1971) and Kublanov & Erokhin (1974), than the viscous jet flow solution found in this chapter.

A further examination of the behaviour of the viscous flow at larger values of the effective magnetic Reynolds number \( K_b \) for \( \eta > \eta_{\text{min}} \) and at different values of \( c_0 \) is considered as beyond the scope of the present investigations; especially in view of the additional viscous solutions that will be derived in chapter 6. These calculations and examinations have been carried out by Sozou & English (1972) for the flat wall configuration \( c_0 = 0 \). It should be noticed that the authors observed inversion of the electric current flow in some regions of \( \Theta \) at larger values of \( K_b \), before singularities enter the viscous flow field. The physically unrealistic phenomena that appear in the viscous flow field in the case of \( \eta < \eta_{\text{min}} \) will be discussed in the next section 4.4.

### 4.4. Examination of the phenomena that occur in the viscous flow field for \( \eta < \eta_{\text{min}} \)

In this section we consider in detail the phenomena that occur in the viscous fluid motion when \( \eta \) becomes equal or smaller than the critical value \( \eta_{\text{min}} \).

It will be shown that for \( \eta < \eta_{\text{min}} \) the phenomena are physically unrealistic, which leads to the conclusion that the solution of the viscous flow field must be rejected for these particular values of \( \eta \).

When \( \eta = \eta_{\text{min}} \), as defined in (4.90), \( u(1) \) becomes identical to zero at \( c = 1 \). From (4.70), (4.89), (4.90) it follows that it will be a single zero; therefore we introduce

\[
u(c) = \alpha(1-c)(1+o(1)) \quad \text{for } c = 1, \quad \eta = \eta_{\text{min}}
\]

where

\[
\alpha = \left. \frac{du}{dc} \right|_{c=1} > 0 .
\]

Substitution in (4.77) and (4.81) and some calculations yield

\[
g(1) = 4 \, K_{\eta},
\]
\[ \left[ \frac{dG}{dc} \right]_{c=1} = 2 K_\eta, \quad \text{for } K_\eta = K_{\eta,\text{min}} \]  

(4.108)

\[ \left[ \frac{d^2G}{dc^2} \right]_{c=1} = -\frac{c_2}{6K_\eta} < 0, \]  

(4.109)

where \( G_\eta(c) \equiv c_2 (1-c)^2 (1+o(1)) \) for \( c = 1 \) as it follows from the series expansion of \( G_\eta(c) \) in powers of \( 1-c \), see (C.8) - (C.10) of appendix C.

The above expressions indicate that at \( K_\eta = K_{\eta,\text{min}} \) the no fluid source or sink condition (2.35), (2.36) on the axis of symmetry is not satisfied any longer. From (2.14), (2.35), (2.40), (4.107) we find at \( c = 1 \)

\[ \psi(r,1) = 0 \quad \text{for } K_\eta > K_{\eta,\text{min}}, \]  

(4.110)

\[ \psi(r,1) \neq 0 \quad \text{for } K_\eta = K_{\eta,\text{min}}. \]  

(4.111)

The Stokes stream function \( \psi \) is defined in order to satisfy the mass conservation equation identically, see (2.1), (2.11) - (2.13). In mathematical terms the Stokes stream function needs to be twice continuously differentiable. It is obvious that this condition is not satisfied at \( c = 1 \) for \( K_\eta = K_{\eta,\text{min}} \); indirectly it follows from (2.39), (2.40), (4.110), (4.111) and the covering text, but directly from the discontinuous behaviour of \( \frac{\partial \psi}{\partial z} \) at \( c = 1 \) when \( K_\eta \) approaches \( K_{\eta,\text{min}} \). Namely \( \left[ \frac{\partial \psi}{\partial z} \right]_{c=1} = 0 \) for \( K_\eta > K_{\eta,\text{min}} \) and \( \left[ \frac{\partial \psi}{\partial z} \right]_{c=1} = \frac{4\eta}{\rho} \) for \( K_\eta = K_{\eta,\text{min}} \), see section 4.3 and (4.112). Hence on that occasion the mass conservation equation (2.1) is not satisfied on the axis of symmetry, viz. \( \text{div} \, \mathbf{v} \neq 0 \) at \( c = 1, K_\eta = K_{\eta,\text{min}} \).

Actually the axis of symmetry now contains a semi-infinite line sink for \( z > 0 \), with a Stokes stream function

\[ \psi = \frac{4\eta}{\rho} z \quad \text{at } c = 1, z > 0, K_\eta = K_{\eta,\text{min}}, \]  

(4.112)

where \( z \) is the distance from the point electrode along the axis of symmetry, viz. \( z = rc \).

Since at \( c = 1 \) the curl \( \mathbf{v} = 0 \), see (2.19), (4.109), we locally introduce a potential function \( \Phi \), defined by \( \mathbf{v} = \text{grad} \, \Phi \), which satisfies, see Jansen (1984)
\[ \Phi = -\frac{4\eta}{\rho} \ln(s) \]  
\[ \text{at } c = 1, z > 0, k_\eta = k_{\eta,\text{min}}, \]  
(4.113)

being the potential function of a line sink of strength \( 4I_0 \sqrt{\frac{\mu}{\rho}} k_\eta = \frac{8\pi\eta}{\rho} \), see Batchelor (1967, p.91). Note that in the above expression \( s \) is the distance from the axis of symmetry, \( s = r \sqrt{1 - c^2} \).

Moreover from (2.21), (2.22), (4.107), (4.108) we find that \( v_r(r,1) \) is negative but finite and that \( v_\theta(r,1) \) tends to minus infinity, whereas respectively for the slow viscous flow, the viscous flow for \( k_\eta > k_{\eta,\text{min}} \) and the inviscid flow \( (k_\eta = 0) \) we always obtained \( v_r(r,1) > 0 \) and \( v_\theta(r,1) < 0 \). Although the direction of \( v_r(r,1) \) is reversed at \( k_\eta = k_{\eta,\text{min}} \), nevertheless due to the singularity in \( v_\theta \) at \( c = 1 \) we retain an outwards "jet flow" along the axis of symmetry directed from the point electrode, and with the velocity components

\[ v_s = -\frac{4\eta}{\rho} \frac{1}{s}, \]
\[ \text{at } c = 1, k_\eta = k_{\eta,\text{min}} \]  
(4.114)

\[ v_z = \frac{2\eta}{\rho} \frac{1}{r}, \]
\[ \text{for } k_\eta = k_{\eta,\text{min}} \]  
(4.115)

The expressions given in (4.107) - (4.109) can also be obtained by series expansion in powers of \( 1 - c \) from the Riccati differential equation (2.50) as well as from (4.77), (4.78), (4.81), upon using the expression of \( \Gamma_\eta(c) \) at small \( k_b \) as represented in (4.65) - (4.69).

When we consider the outwards mass flow \( \Psi \) in positive \( i_r \) direction through a sphere-sector of radius \( r \) centred in the origin and bounded by the surface of the cone, viz. \( r = r_1 = r_2, c_1 = c_0 \) and \( c_2 = 1 \) in (2.39), we find from (2.40) that the net mass flow through the sphere-sector is identical to zero for \( k_\eta > k_{\eta,\text{min}} \), as to be expected. However when \( k_\eta = k_{\eta,\text{min}} \) we obtain from (4.112)

\[ \Psi = -8\pi\eta z \]
\[ \text{for } k_\eta = k_{\eta,\text{min}} \]  
(4.116)

being caused by the line sink located on the positive part of the axis of symmetry; where \( z = rc \). Since fluid sources and sinks were excluded in our problem, the behaviour at \( c = 1 \) implies that the solution at \( k_\eta = k_{\eta,\text{min}} \) needs to be rejected as being physically unrealistic.

The behaviour of \( u(c) \) at small \( k_\eta \) including \( k_\eta < k_{\eta,\text{min}} \) has been derived by means of a singular perturbation technique applied to the differential equation of \( u(c) \), see (4.78); which is presented in appendix D. At small \( k_\eta \) the asymptotic solution of \( u(c) \) takes the form, see (D.4), (D.51), (D.52)
\[ u(c) = \sqrt{\pi} z - \frac{1}{4} \left( \frac{dz}{dc} \right)^2 - \frac{1}{2} \left( \frac{dz}{dc} \right)^2 \frac{3}{2} K_\eta \left[ -2 \left( \frac{dz}{dc} \right)^2 c = c_0 \right] \sin^2 \left( \xi + \frac{\pi}{12} \right) + \]
\[ + \frac{1}{12} \frac{dz}{dc} \left. \frac{dz}{dc} \right|_{c = c_0} A_1(0) \frac{2}{3} + \]
\[ \frac{5}{48 z^2} + B_0(c) \right] A_1(0) \sin \left( \xi - \frac{\pi}{12} \right) K_\eta + \]
\[ \frac{5}{48 z^2} + B_0(c) \right] A_1(0) \cos \left( \xi + \frac{\pi}{12} \right) K_\eta + \]
\[ + O \left( \frac{z^2}{K_\eta} \right) \]

for \( K_\eta \ll 1 , c \neq c_0 \), \hspace{1cm} (4.117)

where

\[ \xi(c) = \frac{1}{2K_\eta} \int_{c_0}^{c} \frac{\sqrt{G_\eta(t)}}{1-t^2} \, dt, \hspace{1cm} (4.118) \]
and \( z(c) \) is defined by

\[ z(c) = \left[ \frac{3}{4} \int_{c_0}^{c} \frac{\sqrt{G_\eta(t)}}{1-t^2} \, dt \right]^2 \frac{2}{3}. \hspace{1cm} (4.119) \]

For expressions of other terms in (4.117) we refer to appendix D. The asymptotic expansion clearly shows that \( K_\eta = K_{\eta,\text{min}} \) with \( u(1) = 0 \) is just a special case. When \( K_\eta \) decreases to smaller values than \( K_{\eta,\text{min}} \) the function \( u(c) \) will possess an increasing number of zeros on the interval \( c_0 < c < 1 \). This fact is also indicated by the respective approximations of \( K_{\eta,\text{min}} : K^{(i)}_{\eta,\text{min}} \) see (D.53) - (D.60) of appendix D, being derived from the asymptotic expansion of \( u(c) \), viz.

\[ K_\eta = K_{\eta,\text{min}} = \tilde{K}^{(i)}_{\eta,\text{min}} \text{ for } i = I, II, III, IV, \]

with \hspace{1cm} (4.120)
\[
\kappa_{\eta,\text{min}}^{(1)} = \frac{6}{(7+12k)\pi + 12\alpha^{(1)}} \int_{c_0}^{1} \frac{\sqrt{g(t)}}{1-t^2} \, dt \quad \text{for } k = 0,
\]

where \( i \) is the number of terms of (4.117) taken into account to satisfy \( u(1) = 0 \) and the respective expressions of \( \alpha^{(1)} \) are given by (D.55) - (D.60).

For non-zero values of \( k \), respectively \( k = 1, 2, 3, \ldots \) distinct approximate values of \( K_\eta \) are found where an additional zero of \( u(c) \) enters the interval \( c_0 < c < 1 \) via \( c = 1 \). The first zero of \( u(c) \): \( c = c_0 \) is defined as follows

\[
u(c_0) = 0 \quad \text{for } K_\eta < K_\eta,\text{min} \quad (4.121)
\]

\[
u(c) > 0 \quad \text{for } c_0 < c < c_0
\]

Integration of the transformation (4.77) produces an integral relation between \( u \) and \( g \),

\[
u(c) = \nu(c_0) \exp \{- \frac{1}{2K_\eta} \int_{c_0}^{c} \frac{g(t)}{1-t^2} \, dt \} , \quad (4.122)
\]

where \( \nu(c_0) = 1 \) as the boundary condition (4.79) requires.

In section 2.7 we were able already to predict from the integral equation of \( g(c) \) (2.53), (2.55) that \( g(c) > 0 \) on \( c_0 < c < 1 \) for \( K_\eta > 0 \), which agrees with \( u(c) > 0 \) in the above integral relation. However when \( u = 0 \) in \( c = c_0 \neq 1 \) for \( K_\eta < K_\eta,\text{min} \), the above identity can be satisfied only when either \( \nu(c_0) = 0 \) or \( \int_{c_0}^{c} \frac{g(t)}{1-t^2} \, dt = +\infty \). The first condition \( \nu(c_0) = 0 \) is not possible since it involves that \( \frac{d^n\nu}{dc^n} = 0 \) for all \( n = 0, 1, 2, \ldots \), implying that \( \nu(c) \equiv 0 \) on \( c_0 < c < 1 \), which contradicts (4.117). Hence the only remaining possibility is that \( g(c) \) becomes positive and unbounded in \( c = c_0 \). In addition from the integral equation of \( g(c) \) (2.53) it then follows that \( g(c) \) remains positive unbounded on the remainder of the interval, viz. \( c_0 < c < 1 \).

Mathematically speaking it means that the transformation (4.77), given the relation between \( u \) and \( g \), decouples as soon as \( u(c) \) becomes identical to zero. The preceding considerations clearly show this feature; namely \( g(c) \) is positive singular on \( c_0 < c < 1 \), whereas the solution of \( u(c) \) is finite and behaves oscillatory, see Ince (1956, p.224), which appears from (4.117), (4.118) and the calculations presented in appendix D. Note that \( K_\eta = K_\eta,\text{min} \) with \( c_0 = 1 \) is a limit case where (4.122) is still satisfied identically; viz. \( g(1) \) is non-zero
but finite, see (4.107), whereas the integral of \( g(c)/1-c^2 \) becomes singular at \( c = c_s = 1 \).

It has to be remarked that the above interpretation is at variance with the explanation given in most references concerning the viscous problem. In following Sozou's explanation (1971a), they assumed that at decreasing \( K_\eta < K_\eta,\text{min} \), an increasing number of distinct singularities enters the flow field from the axis of symmetry.

From the preceding considerations we must conclude that such a description is very debatable, since it has been derived from the function \( u \), which is not related to \( g(c) \) in that part of the interval.

A similar discrepancy has been observed in a paper of Dong-Jian Wang (1983) considering an analogous problem of jets associated with radio sources of galaxies and quasars.

Also it should be remarked that Sozou's (mis)interpretation tempted him to the conclusion that the solution of the inviscid problem, as presented by Shercliff (1970), is based on an amalgamation of singularities. From the above considerations it is evident that this statement cannot be maintained. Shercliff's inviscid solution, containing a relatively weak singularity in the flow field at the axis of symmetry, belongs to a class of analytical solutions of which Shercliff's one is the most physically realistic solution that can be found, see Jansen (1977) and chapters 3 and 6 of this thesis.

The behaviour of the ratio \( \theta_s(c_s = \cos(\theta_s)) \), see (4.121), to \( \theta_0 \) has been calculated at distinct values of \( K_\eta < K_\eta,\text{min} \). Namely at \( K_\eta = K_\eta,\text{min} \), where respectively \( \alpha = 0.99, 0.98, 0.96, 0.94, 0.92, 0.9, 0.8, 0.6, 0.4, 0.2, 0.1, 0.08, 0.06, 0.04, 0.02, 0.01 \) and different values of \( c_0 \), viz. \( c_0 = 0, \pm \frac{1}{4}, \pm \frac{1}{2}, \pm \frac{1}{2}\sqrt{2}, \pm \frac{1}{2}\sqrt{3} \). The behaviour of \( \theta_s/\theta_0 \) as function of \( K_\eta/K_\eta,\text{min} \) is shown in figure 4.6.

Since the relative deviation of \( \theta_s/\theta_0 \) is less than \( 2.0 \times 10^{-2} \) for \(-\frac{1}{2}\sqrt{3} < c_0 < \frac{1}{2}\sqrt{3}\), the figure only shows a single line. The figure shows that for \( K_\eta = K_\eta,\text{min} \times 10^{-2} \) the solution of \( g(c) \) is singular in nearly the entire flow field; \( c_0 < c < 1 \).

For \( K_\eta = K_\eta,\text{min} \) as well as \( K_\eta < K_\eta,\text{min} \) the net mass flow \( \dot{M} \) through a sphere-sector, as considered before, see (4.116), is not identical to zero. The value is negative, indicating the appearance of a line- or concical sink situated respectively on and around the axis of symmetry.

Therefore it follows that the solutions for \( K_\eta < K_\eta,\text{min} \) must be rejected as being physically unrealistic. The viscous flow solutions as presented in section 4.3 only exist for \( K_\eta > K_\eta,\text{min} \).
4.5. Discussion of chapter 4

In this chapter we considered the viscous flow solutions with finite velocity at the axis of symmetry in the point electrode configuration. At large value of the inverse effective hydrodynamic Reynolds number $K_\eta$ and at small value of the effective magnetic Reynolds number $K_b$ analytical solutions of the basic viscous flow field and of the regular perturbations of the fluid-flow and electromagnetic field quantities have been derived. In the calculation for arbitrary value of $K_\eta$ and at low $K_b$ we observed a breakdown of the viscous flow when $K_\eta$ becomes equal or smaller than a certain critical value $K_{\eta,\text{min}}$. It turned out that this critical value strongly depends
on the apex-angle of the right circular cone.

Examination of the phenomena that occur in the viscous flow field for $K_\eta < K_{\eta,\text{min}}$ leads to the conclusion that the viscous solutions with finite velocity at the axis of symmetry only exist for $K_\eta > K_{\eta,\text{min}}$. This fact implies that the viscous solutions obtained exist only for relatively low values of the viscous Reynolds number $Re$ and moreover that the maximum admissible electric current, that can be injected into the fluid, is very low compared to the large electric currents applied in practical applications using liquid metals.

Batchelor (1967, p. 346) noted that steady laminar three-dimensional jets always become unstable when the hydrodynamic Reynolds number exceeds a certain critical value. However in view of the very low values as given in table 4.1, the question remains whether it is the cause of the breakdown of the laminar viscous flow. The fact that the inviscid flow solution exists for all values of the effective magnetic Reynolds number $K_b$ and also that the edge of the viscous jet flow decreases very quickly when $K_\eta$ approaches $K_{\eta,\text{min}}$, see figure 4.5, suggest that the viscous force cannot balance the inertia- and Lorentz forces any longer in the way as prescribed by the similarity method introduced by Zhigulev (1960a + b).

Attempts to find other similar solutions or to resolve the viscous problem by adding a non-zero azimuthal velocity $v_\varphi$ or current density $J_\varphi$ also failed. It can be shown that a temporary induced azimuthal electric current distribution $J_\varphi$ cannot be maintained by its associated magnetic field and the viscous fluid motion. Also it turns out that a possible additional azimuthal electric current distribution is unable to resolve the viscous problem.

In addition an azimuthal velocity distribution $v_\varphi$, e.g. caused by a weak asymmetry in the geometry, leads to a semi-infinite vortex line situated on the positive part of the axis of symmetry; i.e. for $z > 0$. This would imply that the problem is in fact dominated by a swirling vortex and that the meridional fluid motion is generated by the Lorentz force and by the secondary flows induced by the vortex. Since the aim of this thesis is primarily to investigate purely meridional fluid motions, generated by an electric current, injected into the fluid, and its self-magnetic field, as observed in practical applications, this subject is beyond the scope of the present investigations.

Bojarevišić & Shcherbinin (1983) and Jansen (1984) noted that the appearance of a non-zero and finite azimuthal velocity distribution throughout the flow field is possible only when it is caused or initiated by an external source. The application of an external magnetic field in order to stabilize the arc in fusion welding and the observed vigorous motion in the weld pool has been studied by Craine & Weatherill (1980a + b), Bojarevišić & Shcherbinin (1983) and

It has been verified by the author that an external axial magnetic field of sufficiently large magnitude and correct distribution induces an additional rotating fluid motion \( \nabla \phi \) which is capable to stabilize the viscous fluid motion in the meridional plane. Also it has been found that an externally induced azimuthal flow is always accompanied by a relatively weak azimuthal electric current distribution which hardly affects the other fieldquantities. However in view of the limitations of our investigations, as mentioned before, this subject is beyond the scope of the calculations to be presented in this thesis.

It should be noticed that in the case of a converging current flow, i.e. \( J_s < 0 \) in stead of a diverging one, finite solutions of the flow field for all values of \( K_\eta \) would be obtained here. However, as noted by Shercliff (1970), the geometry prescribes a diverging current flow from the point electrode to the spherical shaped electrode at large radial distance.

In view of the behaviour of \( \theta_e \) and \( \theta_s \), see figures 4.5 and 4.6, and the singularities arising at the axis of symmetry, the author suggested that the viscous problem needs to be resolved by derivation of an inner expansion valid in the neighbourhood of the axis of symmetry, which must be matched to the outer solution representing the solution obtained by the similarity method, see Jansen (1984).

However recently new viscous flow solutions have been found by the author that satisfy the Navier-Stokes equation (2.2), (2.8), (2.18) on applying the similar solution method (2.14), (2.15). These solutions which admit all values of \( K_\eta \), including the ones appearing in practical liquid metal applications with \( K_\eta \sim 10^{-4} \), will be presented in chapter 6.

Some authors suggested that the singularities appearing in the inviscid- and viscous point electrode configurations arise from the point electrode and/or the singularity in the fieldquantities at the vertex of the cone.

Therefore in chapter 5 we will consider first a more realistic semi-infinite three-dimensional axisymmetric electrode configuration, consisting of a cylindrical electrode of finite and non-zero radius \( a \), situated in a non-conducting flat wall. This examination will present some interesting features.
5. THE DISK ELECTRODE PROBLEM

5.1. Introduction

Some authors: Sozou (1974), Sozou & Pickering (1976, 1978), Moffatt (1978), Andrews & Craine (1978), Craine & Weatherill (1980 a + b) suggest that the singularities appearing in the inviscid- and viscous point electrode problems arise from the point electrode to be considered and/or from the singularities in the fieldquantities at the vertex of the cone.

Therefore in this chapter we will consider a semi-infinite three-dimensional axisymmetric configuration consisting of a cylindrical electrode of radius $a$, situated in a non-conducting flat wall.

It will be shown that the far field solution of the flow field of this configuration approaches the solution of the flat wall point electrode configuration in case of $r \gg a$. This fact was already found by Sozou (1972) from a simple straightforward two-term expansion at large radial distance. However the conclusions that can be drawn from the complete calculations of the far- and near fields to be presented here turn out to be at variance with those of the above-mentioned authors.

In this chapter we present complete series expansions of both the far ($\frac{r}{a} > 1$) - and near ($\frac{r}{a} < 1$) fields of the electromagnetic-, fluid-, and vorticity field-quantities. Especially we calculate analytically the inviscid- and viscous near field solutions of the flow field for $\frac{r}{a} = 0$ at small value of the magnetic Reynolds number; i.e. when neglecting the effect of the fluid motion upon the electric current distribution.

We observe an essentially different behaviour of the fieldquantities and parameters between the two regions, that resolves some inconsistencies of the point electrode problem. Nevertheless the breakdown of the viscous fluid motion at relatively low value of the viscous Reynolds number as found in section 4.3 for the point electrode model or in the far field of the present disk electrode configuration cannot be resolved here. For that we refer to chapter 6.

The calculations presented show that the point electrode model is a very useful first step to describe analytically the far field behaviour in a semi-infinite configuration with an electrode of finite dimensions. In particular because it renders the problem more tractable mathematically.

The investigations clearly show that from the point electrode model no conclusion can be drawn about the behaviour of the flow field in the neighbourhood of the electrode. This is likewise caused by the fact that the applicability of the similarity method, see (2.14, (2.15), is restricted to the farthest field
Since the derivations are very laborious and complicated we present only the main features of the calculations and results. Moreover we restrict ourselves to the calculation of the flow field. Detailed derivations, calculations and examinations of the behaviour of all field quantities will be presented in forthcoming reports and publications.

5.2. Formulation of the disk electrode problem

5.2.1. The disk electrode configuration

We consider a semi-infinite, three-dimensional, axisymmetric configuration of a disk electrode of radius \( a \), located in an electrically insulating flat wall, see figure 5.1.

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**Figure 5.1. The disk electrode configuration.**
The electrode supplies a constant electric current $I_0$ into the fluid, occupying the exterior of the configuration, which leaves the fluid through a spherically shaped electrode at infinity.

In this model we use an oblate spheroidal co-ordinate system $(\eta, \zeta, \varphi)$ to calculate the electromagnetic field quantities and respectively spherical polar co-ordinates $(r, \theta, \varphi)$ and cylindrical polar co-ordinates $(z, s, \varphi)$ to calculate analytical solutions of the generated fluid motion. The origins of the three co-ordinate systems are located at the centre of the disk electrode surface. The axis of symmetry for $z > 0$ is given by $\zeta = \theta = s = 0$; the insulating flat wall by $\zeta = \frac{\pi}{2}$, $\theta = \frac{\pi}{2}$ for $r > a$, $z = 0$ for $s > a$, and the surface of the disk electrode by $\eta = 0$, $\theta = \frac{\pi}{2}$ for $0 < r < a$, $z = 0$ for $0 < s < a$. The respective values of the azimuthal co-ordinate $\varphi$ of the three co-ordinate systems coincide. Now the fluid region is defined by $0 < \eta < \infty$, $0 < \zeta < \frac{\pi}{2}$; $0 < r < \infty$, $0 < \theta < \frac{\pi}{2}$; $0 < z < \infty$, $0 < s < \infty$ with $0 < \varphi < 2\pi$.

The relations between the respective co-ordinates yield

$$z = rc = a \sinh(\eta) \cos(\zeta), \quad (5.1)$$

$$s = r \sqrt{1 - c^2} = a \cosh(\eta) \sin(\zeta), \quad (5.2)$$

where $c = \cos(\theta)$.

In analogy with the preceding calculations for the point electrode configuration we assume an overall symmetry about the axis of symmetry, i.e. $\frac{\partial}{\partial \varphi} = 0$; in addition the azimuthal components of the velocity field and of the electric current density are identical to zero, viz. $v_\varphi = 0$, $J_\varphi = 0$.

5.2.2. The behaviour of the electromagnetic field quantities and the force distributions

In the case of small value of the magnetic Reynolds number $Rm$ to be considered here, the effect of the fluid motion upon the electric current distribution, i.e. the electromagnetic induction term $\nabla \times \mathbf{B}$ in Ohm's law (2.3), is negligible. From (2.3) - (2.5), leading to curl curl $\mathbf{B} = 0$ for $Rm \ll 1$, it can be found that the expressions of the electric current distribution and of the magnetic field expressed in oblate spheroidal co-ordinates then take the form
\[ J_\eta = \frac{I_0}{2\pi a^2} \frac{1}{\sqrt{\cosh^2(\eta) - \sin^2(\zeta) \cosh(\eta)}} \]  

(5.3)

\[ J_\zeta = 0 , \]  

(5.4)

\[ B_\varphi = \frac{\mu I_0}{2\pi a} \frac{(1 - \cos(\zeta))}{\cosh(\eta) \sin(\zeta)} . \]  

(5.5)

It is clear that the system of oblate spheroidal co-ordinates is the best suited co-ordinate system to calculate the electromagnetic field quantities in this particular configuration, in the same way as the spherical polar co-ordinate system is best fitted to the point electrode model. In both cases the cross currents are identical to zero, see (2.25), (5.4).

In the case of \( Rm << 1 \), the electric current distribution is isotropic in the semi-infinite point electrode configuration, whereas in the semi-infinite disk electrode model the electric current distribution is anisotropic, especially in the neighbourhood of the disk electrode. At larger distances the anisotropic weakens, approaching an isotropic current distribution when \( \eta \) tends to infinity.

Application of the oblate spheroidal co-ordinates to the expression of the curl of the Navier-Stokes equation (2.8) and substitution of the expressions of the electromagnetic field quantities (5.3) - (5.5) evidently show that analytical calculation of the induced fluid motion is not possible. Nevertheless analytical solutions of the generated flow field can be obtained by expressing the curl of the Navier-Stokes equation and the expressions of the electromagnetic field quantities into spherical polar co-ordinates together with applying the following generating function relation, see Abramowitz & Stegun (1972, p. 783)

\[ \sum_{n=0}^{\infty} \frac{p_n^{(\alpha,\beta)}(x)}{n!} y^n = \frac{2}{R} \frac{\alpha+\beta}{(1 - y + R)^{\alpha} (1 + y + R)^{-\beta}} \text{ for } |y| < 1 , \]  

(5.6)

with

\[ R = \sqrt{1 - 2xy + y^2} , \]  

(5.7)

where \( p_n^{(\alpha,\beta)}(x) \) are Jacobi polynomials given by
\[ p_n^{(\alpha, \beta)}(x) = 2^{-n} \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \binom{n+\beta}{m} (-1)^m (1-x)^m (1+x)^{n-m} \quad (5.8) \]

with

\[ p_0^{(\alpha, \beta)}(x) = 1 \quad (5.9) \]

\[ p_n^{(\alpha, \beta)}(-1) = (-1)^n \binom{n+\beta}{n} \quad (5.10) \]

\[ p_n^{(\alpha, \beta)}(1) = \binom{n+\alpha}{n} \quad (5.11) \]

see Magnus, Oberhettinger & Soni (1966).

The restricted convergence radius of the generating function concept, see (5.6), involves two different series expansions of the rewritten expressions for the electromagnetic field quantities. By setting \( x = 1 - 2c^2 \) and \( y = \frac{r}{a} - 2 \) and some elementary analysis one obtains expressions of the electromagnetic field quantities in the form of numerical series being valid in the so-called far field \( \frac{r}{a} > 1 \), viz. \( a < r < \infty \)

\[ J_r = \frac{I_o}{2\pi a^2} \sum_{n=1}^{\infty} \frac{p_n^{(-\frac{1}{2}, 0)}(x)}{\frac{r}{2n}} \quad (5.12) \]

\[ J_\theta = -\frac{I_o}{2\pi a^2} c \sqrt{1-c^2} \sum_{n=2}^{\infty} \frac{p_n^{(\frac{1}{2}, 1)}(x)}{\frac{r}{2n}} , \quad \text{for} \frac{r}{a} > 1 \quad (5.13) \]

\[ B_\varphi = \frac{\mu I_o}{2\pi a} \frac{1}{\sqrt{y} \sqrt{1-c^2}} \left\{ (1-c) - \frac{c(1-c^2)}{2} \sum_{n=1}^{\infty} \frac{p_n^{(\frac{1}{2}, 1)}(x)}{\frac{r}{2n}} \right\} . \quad (5.14) \]

for \( \frac{r}{a} > 1 \) with \( x = 1 - 2c^2 \).

In the same way one finds for \( x = 1 - 2c^2 \) and \( y = \frac{r}{a} - 2 \) numerical series being valid in the near field \( \frac{r}{a} < 1 \), viz. \( 0 < r < a \)}}
\[ J_r = \frac{I_0}{2\pi a^2} c \sum_{n=0}^{\infty} \frac{p_n(\frac{1}{2}, 0)(x)}{r} \sim 2n, \quad (5.15) \]

\[ J_\theta = -\frac{I_0}{2\pi a^2} \sqrt{1 - c^2} \sum_{n=0}^{\infty} \frac{p_n(\frac{1}{2}, 1)(x)}{r} \sim 2n, \quad \text{for } r < 1 \quad (5.16) \]

\[ B_\varphi = \frac{\mu I_0}{4\pi a} \sqrt{1 - c^2} \sum_{n=0}^{\infty} \frac{p_n(\frac{1}{2}, 1)(x)}{r} \sim 2n+1, \quad (5.17) \]

for \( r < 1 \) and \( x = 1 - 2c^2 \).

In the above expressions \( \bar{r} \) is the dimensionless radial distance from the origin defined by

\[ \bar{r} = \frac{r}{a}. \quad (5.18) \]

The above series expansions (5.12) - (5.17) indicate that the electric current density is proportional to \( a^{-2} \) and that the magnetic field varies with \( a^{-1} \), according to (5.3) - (5.5).

The conversion of the expressions of the electromagnetic field quantities into series expansions enables us to derive from the curl of the Navier-Stokes equation analytical solutions of the flow field in the form of asymptotic expansions, which are distinct for the far field \( \bar{r} > 1 \) and for the near field \( \bar{r} < 1 \). These calculations will be carried out in sections 5.3 - 5.5.

It is worth remarking that many other sets of series expansions can be derived which divide the fluid domain in a different way into two regions. If analytical solutions of these flow fields can be found, they will provide very useful overlap domains for the present series expansion solutions.

It will be of particular interest to investigate the behaviour of the electromagnetic field quantities and the resulting Lorentz force in some special regions of the fluid domain, e.g. at \( c = 1 \), at \( c = 0 \) for \( \bar{r} < 1 \) and \( \bar{r} > 1 \), at \( \bar{r} = 0 \), \( \bar{r} = 1 \), \( \bar{r} \to \infty \). However, a detailed examination at all those regions is beyond the scope of the present investigations. Therefore, we confine ourselves here to the behaviour at the electrode and mention only some further features without derivations.
On the surface of the disk electrode the electromagnetic field quantities satisfy

\[ J_r = J_s = 0, \]  \hspace{1cm} (5.19)

\[ J_\theta = - J_z = - \frac{I_0}{2\pi a^2 \sqrt{1 - \frac{r^2}{a^2}}}, \quad \text{for } c = 0, \quad \bar{r} < 1 \]  \hspace{1cm} (5.20)

\[ B_\varphi = \frac{\mu_0 I_0}{2\pi a} \left[ 1 - \sqrt{1 - \frac{r^2}{a^2}} \right], \]  \hspace{1cm} (5.21)

which can be found from (5.1) - (5.5) and some elementary analysis.

The distribution of electric current injected into the fluid due to the above current density at the surface of the electrode yields

\[ I = \iint_S J \cdot n \, dS = I_0 \left[ 1 - \sqrt{1 - \frac{r^2}{a^2}} \right] \quad \text{for } c = 0, \quad 0 < \bar{r} < 1 \]  \hspace{1cm} (5.22)

The behaviour of \( J_z \), \( I \) and \( \frac{dI}{dr} \) on the electrode surface is sketched in figure 5.2.

It has to be remarked that this electric current distribution differs from the Gaussian current distribution as applied by Atthey (1980) and Craine & Andrews (1984) and moreover with the source-sink distribution model used by Andrews & Craine (1978) and Craine & Weatherill (1980a).

The vector components of the Lorentz force \( F_L = J \times B \) on the surface of the electrode take the form

\[ F_{L,s} = - \frac{\mu_0 I_0^2}{4\pi^2 a^3} \left[ 1 - \sqrt{1 - \frac{r^2}{a^2}} \right] \frac{1}{\bar{r} \sqrt{1 - \frac{r^2}{a^2}}} \]  \hspace{1cm} (5.23)

for \( c = 0 \), \( \bar{r} < 1 \)

\[ F_{L,z} = 0, \]  \hspace{1cm} (5.24)
Figure 5.2. The behaviour of $J_z$, $I$, $\frac{dI}{dr}$ on the surface of the electrode.

and on the surrounding insulating flat wall

$$F_{L,s} = 0,$$  \hspace{1cm} (5.25)

for $c = 0$, $\bar{r} > 1$
\[ F_{L,z} = \frac{\mu I_o^2}{4 \pi^2 a^3} \frac{1}{\bar{r} \sqrt{\bar{r}^2 - 1}}. \]  

(5.26)

Hence we observe that the resulting Lorentz force on the surface of the electrically non-conducting flat wall is in purely axial direction, whereas the resulting Lorentz force on the electrode surface is directed towards the axis of symmetry, see figure 5.3. The latter phenomenon is general, since the surface of any electrode of arbitrary shape is of equal electrical potential. Also it should be noted that the tangential Lorentz force on the surface of an electrode will strongly affect the formation and detachment of droplets of molten metal from the electrode, as occurring in the electric welding process, see Amson (1965).

![Figure 5.3. The behaviour and directions of the resulting Lorentz force respectively on the surface of the electrode (\( \bar{r} < 1 \)) and on the insulating flat wall (\( \bar{r} > 1 \)), see (5.22) - (5.26).](image-url)
A detailed examination indicates that the electric current density and the Lorentz force components contain a relatively weak but integrable singularity of order $c^{-1}$ at the rim of the electrode for $\overline{r} = 1$, whereas $B_\varphi$ and the injected electric current, see (5.22), are bounded there.

Some authors, Sozou & Pickering (1978) and Andrews (1982), suggest that this weak singularity in the electric current density and in the Lorentz force and as a result also in the right-hand side of the curl of the Navier-Stokes equation, see (5.41) - (5.46), will invoke singularities in the generated flow field-quantities at that location.

However any reader who has difficulties with the weak singular behaviour at the rim of the electrode that appear or may appear in this mathematical model, may resolve that problem by replacing the surface of the disk electrode, here defined by $\eta = 0$, by $\eta = \varepsilon$ where $\varepsilon \to 0^+$. In that case the surface of the electrode becomes a very flat axisymmetric semi-oblate spheroid with semi-axes $s = a\sqrt{1+\varepsilon^2}$ and $z = a\varepsilon$. Despite the fact that the latter electrode configuration better approaches situations as they appear in practical applications, in my opinion it does not contribute to an essentially better understanding of the phenomena that occur in this problem. Moreover it has to be noted that the series expansion solutions of the flow field, to be derived in section 5.3 - 5.5, exclude the hemisphere $\overline{r} = 1$.

A very interesting feature is the behaviour and especially the direction of the Lorentz force at different radial distances from the origin as function of $\Theta$. A detailed examination shows that the resulting Lorentz force in the neighbourhood of the origin $\overline{r} = 0$ is directed towards the axis of symmetry $c = 1$; on the hemisphere $\overline{r} = 1$, towards the intersection $\overline{r} = 1, c = 1$ and that at large radial distance $\overline{r} \to \infty$ the Lorentz force is in purely meridional direction, viz. directed from the surface of the flat wall at $\Theta = \frac{\pi}{2}$ and towards the axis of symmetry for smaller values of $\Theta$, see figure 5.4. At constant $\overline{r}$, the Lorentz force has the largest magnitude at $c = 0$ and tends to zero at the axis of symmetry $c = 1$.

From (5.1) - (5.5) it can be found that at low magnetic Reynolds number $R_m$ the electromagnetic field quantities and the vector components of the Lorentz force yield in the entire fluid domain $s > 0$, $z > 0$, $0 < \varphi < 2\pi$

$$B_\varphi > 0, \ J_r > 0, \ J_\theta < 0, \ J_s > 0, \ J_z > 0,$$  \hspace{1cm} (5.27)

for $I_0 > 0$ and $R_m \ll 1$, resulting in
Figure 5.4. The behaviour of the resultant vector components of the Lorentz force on hemispheres of the respective radii $\bar{r} = 0$, $\bar{r} = 1$, $\bar{r} \to \infty$.

\[ F_{L,r} < 0, \ F_{L,\theta} < 0, \ F_{L,s} < 0, \ F_{L,z} > 0, \quad (5.28) \]

for $I_o < 0$ or $I_o > 0$ and $Rm \ll 1$.

The directions of the respective Lorentz force components clearly corroborate Shercliff's statement that due to the Lorentz force the generated fluid motion will consist of a converging incoming flow along the flat wall, being directed towards the electrode and the axis of symmetry, which is accelerated and rather abruptly turned off in the neighbourhood of the electrode into an outwards jet flow along the axis of symmetry, see Shercliff (1970).

Of particular interest is the consideration of the total force exerted by the Lorentz forces upon a body of fluid and in particular the comparison of the different results obtained in the point electrode configuration and in the disk electrode model.

The total Lorentz force $\mathbf{F}_{L,0}$ imparted to a volume of fluid situated between two sphere sectors: $r_1 < r < r_2$, $0 < \theta < \theta_o$, $0 < \varphi < 2\pi$, in the point electrode model yields for arbitrary electric current distribution, viz. for all $K_b > 0$, see Jansen (1983, 1984)
\[ \vec{F}_{L,0} = \iiint_{V_0} (J \times B) \, dz \, dv = \frac{\mu \eta^2}{4\pi} \ln \left( \frac{r_2}{r_1} \right) \quad \text{for } a = 0. \]  

(5.29)

Shercliff (1970) and Moffatt (1978) noted that the total Lorentz force diverges logarithmically as \( r_1 \to 0 \) or \( r_2 \to \infty \). This fact implies an unbounded flux of momentum in the fluid when \( r_2 \) approaches infinity. Also it may imply local cavitation and as a result intermittency of the electric current injected into the fluid when \( r_1 \) tends to zero, as remarked by Moffatt (1978). It will become clear that the latter difficulty is principally due to the application of a rather simple model for the configuration.

By calculation of the total Lorentz force \( \vec{F}_{L,a} \) exerted on a volume of fluid situated between two oblate hemispheroids: \( \eta_1 < \eta < \eta_2 \), \( 0 < \zeta < \frac{\pi}{2} \), \( 0 < \varphi < 2\pi \), in the disk electrode configuration, see figure 5.1, we find with (5.3) - (5.5), see Jansen (1983, 1984)

\[ \vec{F}_{L,a} = \iiint_{V_a} (J \times B) \, \frac{dz}{a} \, dv = \frac{\mu \eta^2}{4\pi} \ln \left( \frac{\cosh (\eta_2)}{\cosh (\eta_1)} \right) \quad \text{for } a \neq 0. \]  

(5.30)

When \( \eta_1 \) tends to zero, the total force \( \vec{F}_{L,a} \) now approaches a finite upper bound; whereas in the case of \( \eta_2 \to \infty \) a logarithmic singularity is obtained identical to the one found with (5.29). The corresponding asymptotic expansions, which can be derived from (5.1) and (5.2), clarify these observations, viz.

\[ \cosh(\eta) = 1 + \frac{r^2}{2a^2} \cos^2(\theta) (1 + o(1)) \quad \text{for } \frac{r}{a} \to 0, \]  

(5.31)

\[ \cosh(\eta) = \frac{r}{a} + \frac{a}{2r} \cos^2(\theta) (1 + o(1)) \quad \text{for } \frac{r}{a} \to \infty. \]  

(5.32)

Hence by replacing the point electrode by an electrode of finite size, which better approaches situations occurring in practice, or by considering the point electrode configuration as the limit case \( a \to 0^+ \) of the present configuration, the difficulties appearing at the point electrode are removed. Therefore it is unnecessary to consider phenomena that might be caused by the presence of a hypothetical point electrode in the mathematical model, see Jansen (1983). Nevertheless the relatively weak logarithmic singularities in the expressions of
the total Lorentz forces, when respectively \( r_2 \) or \( \eta_2 \) tend to infinity, cannot be removed. However this singular behaviour does not offend any physical principle since an infinite volume of fluid of finite density \( \rho \) needs an infinite total Lorentz force for its acceleration, implying an infinite flux of momentum. It is also clear that this difficulty vanishes when a configuration of finite dimensions is considered.

5.2.3. The fluid flow equations

In this section we define a dimensionless Stokes stream function \( \tilde{\Psi} \) and derive the partial differential equations of the curl of the Navier–Stokes equation and of the curl of Ohm's law.

In contrast with the point electrode configuration, the disk electrode configuration possesses a characteristic length scale, viz. the radius \( a \) of the disk electrode. This fact enabled us to introduce a dimensionless radial distance from the origin: \( \bar{r} = \frac{r}{a} \), see (5.18), and to define a dimensionless Stokes stream function \( \tilde{\Psi} \), which identically satisfies the mass conservation equation (2.1) by

\[
\nu_\rho = - \frac{I_0}{2\pi a} \sqrt{\frac{\mu}{\rho}} \frac{1}{r_2} \frac{\partial \tilde{\Psi}}{\partial c},
\]

(5.33)

\[
\nu_\theta = - \frac{I_0}{2\pi a} \sqrt{\frac{\mu}{\rho}} \frac{1}{r\sqrt{1-c^2}} \frac{\partial \tilde{\Psi}}{\partial \bar{r}}.
\]

(5.34)

Comparison with (2.11), (2.12) shows that the above dimensionless Stokes stream function \( \tilde{\Psi} \) and the one defined in the point electrode model are related by

\[
\phi = \frac{sI_0}{2\pi} \sqrt{\frac{\mu}{\rho}} \tilde{\Psi}.
\]

(5.35)

The expressions (5.33), (5.34) show that the velocity components are proportional to \( a^{-1} \). This behaviour agrees more-or-less with results obtained by Woods & Milner (1971), but not completely since the authors considered the flow phenomena in a container of finite dimensions with a free surface.

The boundary and initial conditions for the fluid flow require that the normal components of the velocity are zero at the axis of symmetry, on the surface of
the electrode and on the insulating flat wall. In addition the viscous flow has to satisfy the no-slip condition on the electrode- and wall surfaces, viz.

\[ v_s (\vec{r}, 1) = 0 \quad \text{for} \quad K_\eta > 0 , \quad (5.36) \]
\[ v_z (\vec{r}, 0) = 0 \quad \text{for} \quad K_\eta > 0 , \quad (5.37) \]
\[ v_s (\vec{r}, 0) = 0 \quad \text{for} \quad K_\eta > 0 . \quad (5.38) \]

In analogy with the point electrode configuration, see (2.40), we will find here that \( \tilde{\phi} = 0 \) at \( c = 0,1 \).

The corner \( \vec{r} = 0 \) is a stagnation point, which has to satisfy the condition that the vorticity is single valued there, see Atthey (1980), Craine & Andrews (1984) and (5.57), (5.58), being equivalent to

\[ \omega = 0 \quad \text{at} \quad \vec{r} = 0 , \quad \text{for} \quad 0 < c < 1 \quad \text{and} \quad K_\eta > 0 \quad (5.39) \]
\[ \frac{\partial \omega}{\partial \vec{r}} = 0 \quad (5.40) \]

Although in this configuration a characteristic length scale is available, it will appear that in consequence of the absence of a significant characteristic velocity, the basic equations are governed by the same characteristic dimensionless parameters \( K_b \) and \( K_\eta \) as defined in chapter 2, viz.

\[ K_b = \frac{\sigma \mu I}{2\pi \rho} \sqrt{\frac{\mu}{\rho}} , \quad (2.16) \]
\[ K_\eta = \frac{2\pi \eta}{I_0 \sqrt{\rho \mu}} . \quad (2.17) \]

In the point electrode configuration these parameters represent respectively the effective magnetic- and the effective inverse hydrodynamic Reynolds number. However in the present configuration the Reynolds numbers show a typical local behaviour which will become apparent in the further discussions in this chapter, see section 5.6. It should be noticed that the possibility of a local behaviour is already mentioned by Shercliff (1970).

The induced fluid motion generated by the Lorentz force is determined by the curl of the Navier-Stokes equation. Substitution of (5.33), (5.34) into (2.8),
(2.10) result in a fourth order non-linear partial differential equation of the Stokes stream function \( \widetilde{\psi} \) that governs the generation of fluid motion, viz.

\[
K_\eta \overline{r} (1-c^2) \left[ -\frac{4}{\overline{r}^4} \frac{\partial^4 \widetilde{\psi}}{\partial \overline{r}^4} + 2 \frac{\overline{r}^2}{(1-c^2)} \frac{\partial^3 \widetilde{\psi}}{\partial \overline{r}^2 \partial c^2} - 4 \frac{\overline{r}}{(1-c^2)} \frac{\partial^3 \widetilde{\psi}}{\partial \overline{r} \partial c^4} + 4(1-c^2) \frac{\partial^2 \widetilde{\psi}}{\partial c^2} - 4c (1-c^2) \frac{\partial \widetilde{\psi}}{\partial c} + (1-c^2)^2 \frac{\partial^4 \widetilde{\psi}}{\partial c^4} \right] + \\
+ \frac{\overline{r}^3}{3} (1-c^2) \frac{\partial \widetilde{\psi}}{\partial c} \frac{\partial^3 \widetilde{\psi}}{\partial \overline{r}^3} - \frac{\overline{r}^3}{3} (1-c^2) \frac{\partial \widetilde{\psi}}{\partial c} \frac{\partial^2 \widetilde{\psi}}{\partial \overline{r}^2 \partial c} - 2 \frac{\overline{r}^3}{c} \frac{\partial \widetilde{\psi}}{\partial c} \frac{\partial^2 \widetilde{\psi}}{\partial \overline{r} \partial c^2} + \\
- 2 \frac{\overline{r}^2}{(1-c^2)} \frac{\partial \widetilde{\psi}}{\partial c} \frac{\partial^2 \widetilde{\psi}}{\partial \overline{r}^2} - 4(1-c^2)^2 \frac{\partial \widetilde{\psi}}{\partial c} \frac{\partial^2 \widetilde{\psi}}{\partial \overline{r} \partial c^2} + \frac{\overline{r}}{(1-c^2)^2} \frac{\partial \widetilde{\psi}}{\partial c} \frac{\partial^3 \widetilde{\psi}}{\partial \overline{r} \partial c^3} = Z(\overline{r}, c) ,
\]

or in more compact form

\[
K_\eta D^4 \widetilde{\psi} + \frac{\partial}{\partial \overline{r}} \left[ \frac{1}{\overline{r}^2} \frac{\partial \widetilde{\psi}}{\partial c} D^2 \widetilde{\psi} \right] - \frac{(1-c^2)}{\overline{r}^2} \frac{\partial}{\partial c} \left[ \frac{1}{1-c^2} \frac{\partial \widetilde{\psi}}{\partial \overline{r}} D^2 \widetilde{\psi} \right] = \frac{Z(\overline{r}, c)}{\overline{r}^5 (1-c^2)} ,
\]

where \( D^2 \) is the operator

\[
D^2 = \frac{\partial^2}{\partial \overline{r}^2} + \frac{(1-c^2)}{\overline{r}^2} \frac{\partial^2}{\partial c^2}.
\]

In these expressions the linear terms of \( \widetilde{\psi} \) multiplied by \( K_\eta \) represent the viscous force and the non-linear terms of \( \widetilde{\psi} \) the inertia force; in fact their rotationality multiplied by some factor, see (5.44).

The right-hand side of the above equations, representing the effect of the Lorentz force, satisfies
\[ Z(\bar{r}, c) = \frac{4\pi^2 a^4}{\mu I_o^2} \bar{r}^6 (1 - c^2)^{3/2} \{ \text{curl} (\mathbf{j} \times \mathbf{B}) \} \cdot \frac{1}{\bar{r}^2} \phi , \quad (5.44) \]

where

\[ \{ \text{curl} (\mathbf{j} \times \mathbf{B}) \} \cdot \frac{1}{\bar{r}^2} = \frac{2J_s B_s \phi}{s} = \frac{2}{a r^2 \sqrt{1 - c^2}} \left\{ \sqrt{1 - c^2} J_r B_r + c J_\theta B_\theta \right\} \cdot \quad (5.45) \]

By rewriting the expressions of the electromagnetic field quantities (5.3) - (5.5) into spherical polar co-ordinates on application of (5.1), (5.2) and by substituting into (5.44), (5.45), the general form of the expression of \( Z(\bar{r}, c) \), valid in the entire fluid domain, takes the complicated form

\[ Z(\bar{r}, c) = \frac{2\bar{r}^3}{\sqrt{(\bar{r}^2 - 1)^2 + 4\bar{r}^2 c^2}} \left[ 1 - \sqrt{\frac{1}{2} (\bar{r}^2 - 1) + \frac{1}{2} \sqrt{(\bar{r}^2 - 1)^2 + 4\bar{r}^2 c^2}} + \frac{c}{\sqrt{\frac{1}{2} \{ \bar{r}^2 (\bar{r}^2 - 1) + (3\bar{r}^2 + 1)c^2 \} + \frac{1}{2} (\bar{r}^2 + c^2) - \sqrt{(\bar{r}^2 - 1)^2 + 4\bar{r}^2 c^2}} \right] . \quad (5.46) \]

It is clear that this expression is impracticable for an analytical calculation of the generated flow field. Nevertheless it turned out that upon application of the series expansion solutions of the electromagnetic field quantities (5.12) - (5.17), the generating function concept (5.6) - (5.11) and using the basic relations of the Jacobi polynomials, see Magnus et al. (1966) to (5.44), (5.45) and some elementary analysis, full series expansions of the expression of \( Z(\bar{r}, c) \) can be derived respectively for \( \bar{r} > 1 \) and \( \bar{r} < 1 \).

The series expansion of \( Z(\bar{r}, c) \) for the far field \( \bar{r} > 1 \) becomes

\[ Z(\bar{r}, c) = 2\bar{r}^2 (1 + c)^2 (1 - c)^2 \sum_{n=0}^{\infty} \frac{\alpha_n(c)}{\bar{r}^{2n}} \quad \text{for} \quad \bar{r} > 1 , \quad (5.47) \]

where \( \alpha_0(c) = 1 \) and in general

\[ \alpha_n(c) = \frac{p_{n+1}(-1, 1)(x) - c p_{n+1}(0, 1)(x)}{1 - c} \quad \text{for} \quad n = 0, 1, 2, \ldots \quad (5.48) \]
with $x = 1 - 2c^2$.

Expressions of $\alpha_0(c) - \alpha_2(c)$ are evaluated with the help of the series expression of the Jacobi polynomials, see (5.8)

$$\alpha_0(c) = 1,$$  \hspace{1cm} (5.49)

$$\alpha_1(c) = \frac{1}{2} (1 - c - 6c^2),$$  \hspace{1cm} (5.50)

$$\alpha_2(c) = \frac{1}{8} (3 - 5c - 47c^2 + 17c^3 + 80c^4),$$  \hspace{1cm} (5.51)

for $\bar{r} > 1$.

In the same way we obtained for the near field $\bar{r} < 1$

$$Z(r,c) = \bar{r}^8 c (1-c^2) \sum_{n=0}^{\infty} \bar{\alpha}_n(c) \bar{r}^{2n}$$  \hspace{1cm} (5.52)

for $\bar{r} < 1$

where $\bar{\alpha}_0(c) = 1$ and for arbitrary values of $n$

$$\bar{\alpha}_n(c) = \frac{2}{1-c^2} \left\{ p_{\frac{1}{2},1}^{(1)}(x) - r_{\frac{1}{2},1}^{(0,1)}(x) \right\}$$  \hspace{1cm} (5.53)

for $n = 0,1,2,...$

with $x = 1 - 2c^2$.

On using (5.8), expressions of $\bar{\alpha}_0(c) - \bar{\alpha}_2(c)$ become of the form

$$\bar{\alpha}_0(c) = 1,$$  \hspace{1cm} (5.54)

$$\bar{\alpha}_1(c) = \frac{1}{4} (7 - 19c^2),$$  \hspace{1cm} (5.55)

$$\bar{\alpha}_2(c) = \frac{1}{8} (19 - 126c^2 + 155c^4),$$  \hspace{1cm} (5.56)

for $\bar{r} < 1$.

This procedure enables us to derive analytical solutions of the flow field in the form of series expansions, separately for the far field $\bar{r} > 1$ and for the near field $\bar{r} < 1$.

It may be remarked that (5.47) and (5.48) in first instance have been found by the author by function recognition of an eleven terms expansion of (5.46), which also involved the derivation of (5.52) and (5.53). Later on it appeared to be possible to derive these expressions in a straightforward fashion from (5.6) - (5.17) by some laborious elementary calculations. The latter procedure is
preferable in view of the calculation of the pressure field. The expression of the vorticity generated by the Lorentz force in the fluid takes the form, see (2.10), (5.33), (5.34)

$$\omega = \text{curl } \mathbf{v} = \omega \mathbf{I}_{\varphi}, \quad (5.57)$$

where

$$\omega = -\frac{1}{2\pi a^2} \sqrt{\frac{\mu}{\rho}} \frac{1}{r \sqrt{1-c^2}} \left[ \frac{\partial^2 \psi}{\partial r^2} + \frac{(1-c^2)}{r^2} \frac{\partial^2 \psi}{\partial c^2} \right]. \quad (5.58)$$

The partial differential equation of the curl of Ohm's law yields in general form, see (2.9), (2.16), (5.33), (5.34)

$$\frac{\partial}{\partial r} \left[ r \sqrt{1-c^2} B_{\varphi} \right] + \frac{(1-c^2)}{r^2} \frac{\partial}{\partial c} \left[ r \sqrt{1-c^2} B_{\varphi} \right] =$$

$$= K_b (1-c^2) \left[ \frac{\partial}{\partial r} \frac{\partial}{\partial c} \left( \frac{B}{r \sqrt{1-c^2}} \right) - \frac{\partial}{\partial c} \frac{\partial}{\partial r} \left( \frac{B}{r \sqrt{1-c^2}} \right) \right], \quad (5.59)$$

where the left-hand side represents the diffusion of the magnetic field and the right-hand side the convection of the magnetic field due to the electromagnetic induction. In the case of small $K_b$ the magnetic field $B_{\varphi}$ satisfies (5.5), (5.14), (5.17).

In the further examinations in this chapter we confine ourselves to the analytical calculation of the series expansion solutions of the fluid motion and vorticity and in general only to those solutions that are actually generated by an appropriate term of the respective series expansions of the curl of the Lorentz force.

It is clear that from these solutions, the expressions of the other field quantities in the form of series expansions can be found.

5.3. The fluid motion in the far field $r > 1$

5.3.1. Introductory remarks
In this section we present the series expansions of respectively the Stokes stream function \( \tilde{\psi} \), the velocity components, and the vorticity and moreover the expressions of the differential equations that govern the series solutions of the flow field. These expansions and expressions are only valid in the fluid domain \( \tilde{r} > 1 \), here designated as the far field.

It turns out that the derivation can be executed simultaneously for the inviscid- and viscous fluid. This fact is in contrast with the near field \( \tilde{r} < 1 \), where the inviscid- and viscous flow must be considered separately, see sections 5.4 and 5.5.

In the last subsection 5.3.3 we only briefly consider the behaviour of the flow field at large radial distance, viz. for \( \tilde{r} \to \infty \). It will be shown that in that case all series expansions reduce to single expressions, being identical to the ones found in the point electrode configuration for the case of a flat wall \( c_o = 0 \). Note that the limit \( \tilde{r} \to \infty \) is equivalent with \( r \to \infty \) or \( a \to 0^+ \), see (5.18).

5.3.2. Derivation of the series expansions and governing differential equations for \( \tilde{r} > 1 \) and \( \kappa_\eta > 0 \)

The restriction that only series solutions of the flow field are considered which are actually induced by an appropriate term of the series expansion representing the curl of the Lorentz force, see (5.41), (5.47), leads to the following series expansion of the Stokes stream function

\[
\tilde{\psi} (\tilde{r}, c) = \sum_{n=0}^{\infty} \frac{g_n(c)}{\tilde{r}^{2n-1}} \quad \text{for } \tilde{r} > 1 \,, \, \kappa_\eta > 0 \quad (5.60)
\]

It can be easily verified that this series expansion of the Stokes stream function provides at larger values of \( \kappa_b \) an extra contribution in the equations (5.41) - (5.47) due to the electromagnetic induction term \( \mathbf{v} \times \mathbf{B} \) in Ohm's law. This contribution appears in the form of a series expansion containing terms of the same order of \( \tilde{r} \) as obtained already in the series expansion of the magnetic field at low \( \kappa_b \), see (5.14), (5.59).

It should be noticed that the above general series expansion of the dimensionless Stokes stream function in odd powers of \( \tilde{r} \) differs from the simple two-term expansion used by Sozou (1972).

Substitution of (5.60) into (5.33), (5.34), (5.58) presents the respective
series expressions of the velocity components

\[ \nu_r = \frac{I_0}{2\pi a} \sqrt{\frac{\mu}{\rho}} \sum_{n=0}^{\infty} \frac{1}{r^{2n+1}} \frac{dg_n}{dc}, \]  

(5.61)

for \( r > 1, K_\eta > 0 \)

\[ \nu_\theta = \frac{I_0}{2\pi a} \sqrt{\frac{\mu}{\rho}} \sum_{n=0}^{\infty} \frac{(2n-1) g_n(c)}{r^{2n+1} \sqrt{1-c^2}}, \]  

(5.62)

and of the vorticity distribution

\[ \omega = -\frac{I_0}{2\pi a} \sqrt{\frac{\mu}{\rho}} \sum_{n=0}^{\infty} \frac{\omega_n}{r^{2n+2}}, \]  

(5.63)

where

\[ \omega_n = \frac{\sqrt{1-c^2}}{dc^2} \frac{d^2 g_n}{dc^2} + \frac{2n(2n-1)}{\sqrt{1-c^2}} g_n(c). \]  

(5.64)

From (5.18), (5.35) it is clear that in the limit \( a \to 0^+ \) the series expressions (5.60) - (5.64) degenerate respectively to the similar solution (2.14) and to the expressions (2.19), (2.21), (2.22), where the basic solution \( g_o(c) \) of (5.60) equals the function \( g(c) \), as used in the point electrode configuration.

Upon substitution of (5.47) and (5.60) into (5.41), the partial differential equation representing the curl of the Navier-Stokes equation is converted into an infinite series of non-linear ordinary differential equations of the functions \( g_n(c) \)

\[ K_\eta \left\{ (1-c^2)^2 \frac{d^4 g_n}{dc^4} - 4c(1-c^2) \frac{d^3 g_n}{dc^3} + 4n(2n+1)(1-c^2) \frac{d^2 g_n}{dc^2} + \right. \]

\[ + 4n(n+1)(4n^2-1) g_n(c) \} \right. \]

\[ \left. + \sum_{m=0}^{n} \left\{ (2m-1)(1-c^2) g_m(c) \frac{d^3 g_{n-m}}{dc^3} + \right. \]

\[ \left. - (2n-2m+3)(1-c^2) \frac{d g_m}{dc} \frac{d^2 g_{n-m}}{dc^2} + 2(2m-1)(2n^2-4nm-n-2m) g_m(c) \frac{d g_{n-m}}{dc} + \right. \]
\[ + \frac{2n(2n-1)(2n-2m-1)c}{1-c^2} g_m(c) g_{n-m}(c) = 2(1-c) \alpha_n(c), \] (5.65)

for \( n = 0,1,2, \ldots, \tilde{r} > 1 \) and \( K_\eta > 0 \).

The expressions of \( \alpha_n(c) \) are given by (5.48) - (5.51).

The above expression clearly shows the direct coupling between the viscous-, inertia- and Lorentz forces for every value of \( n; n = 0,1,2, \ldots \). In the same way an infinite set of ordinary differential equations can be derived from the partial differential equation of the curl of the Ohm's law (5.59), which also shows a direct coupling between the terms representing the diffusion of the magnetic field and the electromagnetic induction for every value of \( n \) at larger values of the parameter \( K_b \).

The corresponding boundary conditions of the functions \( g_n(c) \), as they follow from (5.36) - (5.38), (5.61), (5.62) yield

\[ g_n(0) = 0 \quad \text{for} \quad K_\eta > 0, \] (5.66)

\[ \left( \frac{dg_n}{dc} \right)_{c=0} = 0 \quad \text{for} \quad K_\eta > 0, \] (5.67)

\[ g_n(1) = 0 \quad \text{for} \quad K_\eta > 0, \] (5.68)

in such a way that

\[ \lim_{c \to 1} \left( \sqrt{1-c^2} \frac{dg_n}{dc} - \frac{(2n-1)c}{\sqrt{1-c^2}} g_n(c) \right) = 0 \quad \text{for} \quad K_\eta > 0, \] (5.69)

for \( n = 0,1,2, \ldots \) and \( \tilde{r} > 1 \).

The behaviour of the basic solution \( g_0(c) \) and its consequences is discussed in next section 5.3.3.

5.3.3. A short note about the behaviour of the basic flow solution in the far field \( \tilde{r} > 1 \)

The differential equation of the curl of the Navier-Stokes equation, that governs the behaviour of the basic flow field solution \( g_0(c) \) in the far field \( \tilde{r} > 1 \), yields, see (5.49), (5.65)
\[ K_\eta \left\{ (1-c^2)^2 \frac{d^4 g_o}{dc^4} - 4c(1-c^2) \frac{d^3 g_o}{dc^3} \right\} - (1-c^2) g_o(c) \frac{d^3 g_o}{dc^3} + \]

\[ - 3(1-c^2) \frac{d g_o}{dc} \frac{d^2 g_o}{dc^2} = 2(1-c) . \]  

(5.70)

This equation is evidently identical to the one found in the point electrode configuration in the case of a flat wall \( c_o = 0 \) for \( K_b \ll 1 \), see (2.18), (3.31), (3.32). In the same way identical expressions can be derived for the curl of Ohm's law and for the pressure distribution, see (2.20), (2.64). Hence we conclude that the behaviour of the field quantities in the point electrode model for \( c_o = 0 \) is identical to the farthest field behaviour in the semi-infinite disk electrode configuration; i.e. at large radial distance \( \bar{r} \) from the origin.

Since the limit \( a \to 0 \) in the far field expansions for \( \bar{r} > 1 \) excludes the point \( r = 0 \), the singularities appearing in all field quantities in the point electrode configuration at \( r = 0 \) are excluded when the point electrode configuration is considered as the limiting case of the disk electrode configuration by taking the radius of the disk electrode a identical to zero, see Jansen (1984). This approach also disposes of the severe condition that the appearance of electric current inversion must be excluded in order to prevent electrical short-circuiting, as stated in section 2.6, see (2.41).

The present calculations clearly indicate that the breakdown of the viscous flow at relatively low value of the hydrodynamic Reynolds number, as found in chapter 4, is not caused by the application of the prototype model as suggested by some authors. The viscous fluid motion in the point electrode configuration and the phenomenon of the breakdown of the viscous flow at low Re will be reconsidered and resolved in chapter 6.

In principle other solutions of the series expansions for values of \( n > 0 \) of the inviscid-, viscous- and slow viscous fluid motions can be determined by analytical calculation or by numerical computation. However a further calculation will not be carried out here. It is of more interest to investigate the near field and to examine the essential differences in the behaviour of the flow field quantities and parameters for \( \bar{r} > 1 \) and \( \bar{r} < 1 \).

Finally note the fundamental lack of one boundary or initial condition in the calculation of both the inviscid and viscous flow fields, caused by the infinite extent of the fluid domain, see (2.18), (2.34) - (2.38), (5.65) - (5.69), while
the solutions obtained are fully defined, see chapters 3 and 4. That inconsistency also will be resolved in chapter 6.

5.4. The inviscid fluid motion in the near field $r < 1$

5.4.1. Introductory remarks

In contrast with the far field $r > 1$, in the near field fluid domain $r < 1$ the calculation of the series expansions of the flow field quantities and the reduction of the governing partial differential equation of the curl of the Navier-Stokes equation into an infinite series of ordinary differential equations need to be carried out separately for the inviscid and viscous fluid. This is due to the fact that both fluids turn out to have different series expansions of the dimensionless Stokes stream function $\tilde{\psi}$.

The examinations further demonstrate that the differential equations obtained for the inviscid flow and in section 5.5 for the viscous fluid motion are different from the ones derived for the far field $r > 1$ and thus also from those found in the point electrode configuration.

Some basic solutions of the generated inviscid fluid motion will be presented. It turns out that the solutions contain free parameters, as may be expected from a mathematical point of view. The physical relevancy of these free parameters will be discussed in section 5.7.

5.4.2. Derivation of the series expansions and governing differential equations for $r < 1$ and $K_\eta = 0$

Consideration of the curl of Euler's equation (5.41 with $K_\eta = 0$) and of the series expansion of the right-hand side of electromagnetical origin in the case of $r < 1$, see (5.52) - (5.56), which governs the generation of the inviscid fluid motion, suggests a series expansion of the dimensionless Stokes stream function $\tilde{\psi}$ in positive powers of $r$ in the near field $r < 1$. Rigorous application of the statement that every term of the series expansion of the Stokes stream function should be generated by an appropriate term of the series expansion representing the curl of $(J \times B)$-term, leads to a series expansion of the form: $\tilde{\psi}(r, c) = r^4 g_0(c) + r^6 g_1(c) + \ldots$ for $r < 1$ and $K_\eta = 0$. Examination of the accessory basic solution $g_0(c)$ results in a purely imaginary so-called singular solution, see e.g. Ince (1956, p. 355). Some further analysis of a more general form of
the solution of \( g_0(c) \) leads to the conclusion that this solution is not applicable, and as a consequence that the assumed series expansion of the Stokes stream function must be rejected.

A correct full series expansion of the Stokes stream function \( \tilde{\psi} \), representing the inviscid fluid motion generated in the near field fluid domain \( \tilde{r} < 1 \), turns out to be of the form

\[
\tilde{\psi}(\tilde{r},c) = \sum_{n=0}^{\infty} g_n(c) \tilde{r}^{2n+3} \quad \text{for} \quad \tilde{r} < 1, \ K_\eta = 0 \quad (5.71)
\]

In this case the base solution \( g_0(c) \) is not induced by an appropriate term of the series expansion of the curl of the Lorentz force. Since the Lorentz force generates vorticity into the medium, the solution \( g_0(c) \) represents an additional irrotational flow, as we will discuss, see (5.94).

It should be noticed that the present series expansion of the Stokes stream function (5.71) does not involve the appearance of extra terms of different order of \( \tilde{r} \) in the series expansions of the electromagnetic- and fluid field-quantities at larger values of \( K_b \), see (5.17), (5.59), (5.71). Also note that combination of (5.60) and (5.71) results in a continuous series expansion of the Stokes stream function in negative and positive odd powers of \( \tilde{r} \) in the case of an inviscid fluid.

Substitution of (5.71) in (5.33), (5.34), (5.58) results in the respective series expansions of the velocity components

\[
v_r = -\frac{I_0}{2\pi a} \sqrt{\frac{\mu}{\rho}} \sum_{n=0}^{\infty} \frac{d g_n}{dc} \tilde{r}^{2n+1}, \quad (5.72)
\]

for \( \tilde{r} < 1, \ K_\eta = 0 \)

\[
v_\theta = -\frac{I_0}{2\pi a} \sqrt{\frac{\mu}{\rho}} \sum_{n=0}^{\infty} \frac{(2n+3) g_n(c)}{\sqrt{1-c^2}} \tilde{r}^{2n+1}, \quad (5.73)
\]

and of the vorticity

\[
\omega = -\frac{I_0}{2\pi a^2} \sqrt{\frac{\mu}{\rho}} \sum_{n=0}^{\infty} \omega_n \tilde{r}^{2n}, \quad (5.74)
\]

where

\[
\omega_n = \sqrt{1-c^2} \frac{d^2 g_n}{dc^2} + \frac{2(n+1)(2n+3)}{\sqrt{1-c^2}} g_n(c). \quad (5.75)
\]

for \( \tilde{r} < 1, \ K_\eta = 0 \)
It is clear that the convergence radius $\bar{r} < 1$ of the above series expansions (5.71) - (5.75) does not admit to take the limit $a \to 0$.

Comparison with (5.61) and (5.62) shows the difference in the behaviour of the inviscid velocity components respectively in the neighbourhood of the origin and at large radial distance, viz. $\nu = O(\bar{r})$ for $\bar{r} = 0$ and $\nu = O(\bar{r}^{-1})$ for $\bar{r} \to \infty$ in the case of $K_\eta = 0$.

At the origin, which is a stagnation point, the velocity components become identical to zero, as required. In addition, since the basic solution $g_0(c)$ represents an additional potential flow, we will find later on that $\omega = 0$, see (5.94), so that also the conditions (5.39), (5.40) will be satisfied at that location.

Upon substitution of (5.52) and (5.71) into (5.41), the partial differential equation representing the curl of Euler's equation reduces to an infinite set of ordinary differential equations of the functions $g_n(c)$

$$
\sum_{m=0}^{n} \left\{ (2m+3) g_n(c) \frac{d^3 g_{n-m}}{dc^3} - (2n-2m-1) \frac{d g_m}{dc} \frac{d^2 g_{n-m}}{dc^2} + 
\right.
$$

$$
\left. + \frac{2(2m+3)(2n^2-4nm+5n-6m+4)}{1-c^2} g_m(c) \frac{dg_{n-m}}{dc} + 
\right\} = -c \bar{\alpha}_{n-1}(c), \quad (5.76)
$$

for $n = 0, 1, 2, \ldots$, $\bar{r} < 1$ and $K_\eta = 0$, with $\bar{\alpha}_{-1}(c) \equiv 0$; whereby the expressions of $\bar{\alpha}_n(c)$ are given by (5.53) - (5.56).

Since $\bar{\alpha}_{-1}(c) = 0$, which also follows from (5.9), (5.53), the above expressions of the differential equations of $g_n(c)$ clearly show that all solutions $g_n(c)$ are excited by the Lorentz force, except the basic solution $g_0(c)$. Moreover, note that the base solution $g_0(c)$ has to satisfy a non-linear homogeneous differential equation; whereas the higher order solutions $g_n(c)$ for $n = 1, 2, 3, \ldots$ are determined by linear inhomogeneous differential equations. Some basic solutions of the differential equations that govern the inviscid fluid motion in the neighbourhood of the origin will be presented in next section 5.4.3.

The boundary conditions of the functions $g_n(c)$, obtained from (5.36), (5.37), (5.72), (5.73), yield
\[ g_n(0) = 0, \quad \text{(5.77)} \]
\[ g_n(1) = 0, \quad \text{(5.78)} \]
for \( r < 1, \quad K_\eta = 0 \)

or
\[ \lim_{c \to 1} \left\{ \sqrt{1-c^2} \frac{dg_n}{dc} + \frac{(2n+3)c}{\sqrt{1-c^2}} g_n(c) \right\} = 0, \quad \text{(5.79)} \]
for \( n = 0, 1, 2, \ldots, \quad r < 1 \) and \( K_\eta = 0 \).

Together with the behaviour observed in the far field \( r > 1 \), the present near field expressions (5.71), (5.77), (5.78) show that the entire axis of symmetry \( c = 1 \), the origin \( r = 0 \), the surface of the disk electrode and the insulating flat wall coincide with the streamline \( \tilde{\psi} = 0 \), which is to be expected from (2.39), (2.40). It should be noted that in consequence of (2.39) also the rim of the electrode \( r = 1, \quad c = 0 \) is included, see Soozou & Pickering (1978).

The differential equations (5.76) derived here for \( r < 1 \) and \( K_\eta = 0 \) are valid only in the case of \( a \neq 0 \) and are different from the ones found in the far field \( r > 1 \) of the disk electrode configuration and in the point electrode model, see (2.18), (3.31), (3.32), (5.65) for \( K_\eta = 0 \) and \( c_o = 0 \).

Hence we expect to find a different behaviour of the inviscid fluid motion in the near field \( r < 1 \).

In the next section 5.4.3 we will derive this behaviour in the neighbourhood of the origin for \( r = 0 \).

5.4.3. The inviscid fluid motion in the neighbourhood of the origin \( \tilde{r} = 0 \)

In this section we derive from the differential equations and boundary conditions some basic solutions \( g_n(c) \) of the series expansion of the Stokes stream function \( \tilde{\psi} \), applied in the near field \( r < 1 \) and for the inviscid fluid, viz. \( K_\eta = 0 \), see (5.71). From these solutions, which appear to contain free parameters, the corresponding series solutions of the vorticity will be calculated.

For some special values of the free parameter \( \gamma_o \) the flow patterns generated by the base solutions \( g_o(c) \) and \( g_1(c) \) will be shown in figures, representing the inviscid fluid motion in the neighbourhood of the origin.

From the infinite set of ordinary differential equations, that replaces the partial differential equation of the curl of Euler's equation, see (5.76), it follows that the basic solution \( g_o(c) \) satisfies the non-linear homogeneous
differential equation

\[ 3g_o(c) \frac{d^3g_o}{dc^3} + \frac{dg_o}{dc} \frac{d^2g_o}{dc^2} + \frac{24}{1-c^2} g_o(c) \frac{dg_o}{dc} + \frac{36c}{(1-c^2)^2} g_o^2(c) = 0, \quad (5.80) \]

which can be rewritten into

\[ \frac{1}{3}(c) \left[ \frac{d^2g_o}{dc^2} + \frac{6}{1-c^2} g_o(c) \right] = A_o, \quad (5.81) \]

where \( A_o \) is an arbitrary constant of integration.

It was pointed out in section 5.4.2 that the base solution \( g_o(c) \) represents an additional potential flow which is not generated by an appropriate term of the series expansion of the Lorentz force. But it needs to be added in order to obtain a realistic solution of the inviscid fluid motion in the near field fluid domain. The independence of the effect of the Lorentz force is also apparent from (5.80); since the right-hand side is identical to zero.

Since the Lorentz force generates vorticity into the fluid, the expression of the corresponding basic vorticity, see (5.74), (5.75), (5.94) needs to be identical to zero, implying that the constant \( A_o \) in (5.81) is equal to zero. Also note that the boundary conditions of the inviscid flow and the conditions imposed upon the vorticity in the neighbourhood of the origin, which is a stagnation point, do not permit a non-zero basic vorticity distribution, see (5.36) - (5.40).

Calculation of the complementary function of the remaining homogeneous second order differential equation of (5.81) and substitution of the boundary conditions (5.77) - (5.79) results in a basic solution of the form

\[ g_o(c) = \gamma_o c(1-c^2) \quad \text{for } \bar{r} < 1, \ K_\eta = 0, \quad (5.82) \]

where \( \gamma_o \) is a non-zero free parameter of arbitrary constant value. The appearance of this free parameter is caused by the fact that we have a third order differential equation which has to satisfy only two boundary conditions.

The differential equation of the solution \( g_1(c) \) is obtained from (5.54) and (5.76). Substitution of (5.82) and some elementary algebra yield
\[
\frac{d^3g_1}{dc^3} - \frac{(1-3c^2)}{3c(1-c^2)} \frac{d^2g_1}{dc^2} + \frac{20}{1-c^2} \frac{dg_1}{dc} - \frac{20(1-9c^2)}{3c(1-c^2)^2} g_1(c) = -\frac{1}{3\gamma_0(1-c^2)}. \tag{5.83}
\]

The fact that the right-hand side of above differential equation is unequal to zero, indicates here that the solution \(g_1(c)\) is induced by the Lorentz force. This linear inhomogeneous differential equation can be transformed into a second order differential equation, see Kamke (1943, p. 512, 3.25), of the form

\[
\frac{d^2g_1}{dc^2} + \frac{20}{1-c^2} g_1(c) = -\frac{c}{2\gamma_0} + B_o \frac{1}{c(1-c^2)} \frac{1}{3}, \tag{5.84}
\]

where \(B_o\) is an arbitrary constant.

Calculation of the homogeneous solution, application of the method of variation of parameters in order to determine the particular solution and substitution of the boundary conditions (5.77) - (5.79) leads to the following solution of \(g_1(c)\)

\[
g_1(c) = -\frac{c(1-c^2)}{28\gamma_0} + \gamma_1 c(1-c^2)(3-7c^2) \quad \text{for } \tilde{r} < 1, \quad \gamma_1 = 0, \tag{5.85}
\]

where \(\gamma_1\) is again an additional free parameter.

The differential equation that governs the behaviour of the solution \(g_2(c)\) is obtained from (5.55), (5.76), (5.82), (5.85), viz.

\[
\frac{d^3g_2}{dc^3} - \frac{(1-3c^2)}{c(1-c^2)} \frac{d^2g_2}{dc^2} + \frac{42}{1-c^2} \frac{dg_2}{dc} - \frac{42(1-5c^2)}{c(1-c^2)^2} g_2(c) = \frac{\lambda + \mu c^2}{1-c^2}, \tag{5.86}
\]

where

\[
\bar{\lambda} = -\frac{1}{84\gamma_0^3} (2 + 49\gamma_0^2 - 168\gamma_0\gamma_1), \tag{5.87}
\]

\[
\bar{\mu} = \frac{1}{84\gamma_0^3} (1 + 133\gamma_0^2 - 280\gamma_0\gamma_1). \tag{5.88}
\]

In this case the non-zero right-hand side represents the combined effect of the Lorentz force and of the base solutions \(g_0(c)\) and \(g_1(c)\).
Again application of the method given by Kamke (1943, p. 512, 3.25) reduces the order of the differential equation by one,

\[
\frac{d^2 g_2}{dc^2} + \frac{42}{1-c^2} g_2(c) = \frac{(\lambda + \mu)}{2} c + \lambda c(1-c^2)\ln(c) - \frac{\lambda}{2} c(1-c^2)\ln(1-c^2) + C_0 c(1-c^2),
\]

where \(C_0\) is a constant of integration.

Calculation of the complementary function and of the particular integral, leads after substitution of the boundary conditions (5.77) - (5.79) to the eigenvalue \(\tilde{\lambda} = 0\); a situation which may appear in boundary value problems as considered here. The solution of \(g_2(c)\) then becomes

\[
g_2(c) = \frac{\tilde{\mu}}{72} c(1-c^2) + \gamma_2 c(1-c^2)(7-9c^2) + \gamma_3 c(1-c^2)(5-30c^2+33c^4)
\]

for \(\tilde{r} < 1\), \(K_\eta = 0\), (5.90)

with in consequence of \(\tilde{\lambda} = 0\),

\[
\tilde{\mu} = -\frac{1}{36\gamma_0^3} + \frac{11}{18\gamma_0},
\]

\[
\gamma_1 = \frac{1}{84\gamma_0} + \frac{7\gamma_0}{24},
\]

(5.91) (5.92)

where \(\gamma_2\) and \(\gamma_3\) are free parameters.

Despite the fact that the boundary conditions require \(\tilde{\lambda}\) identical to zero, so that one free parameter, i.e. \(\gamma_1\) vanishes, we still have three solutions: \(g_0(c)\), \(g_1(c)\), \(g_2(c)\) with three free parameters \(\gamma_0\), \(\gamma_2\), \(\gamma_3\). From a mathematical point of view it is the lack of one boundary condition that implies the appearance of the free parameters, a situation not unusual in problems concerning series expansions solutions with a limited convergence radius or in case of semi-infinite configurations.

In accordance with (5.92), the expression of \(g_1(c)\) now becomes of the form

\[
g_1(c) = -\frac{c^2(1-c^2)}{12\gamma_0} + \frac{7}{24} \gamma_0 c(1-c^2)(3-7c^2)
\]

for \(\tilde{r} < 1\), \(K_\eta = 0\). (5.93)

In view of the behaviour and the direction of the Lorentz force in the near field, especially on the surface of the disk electrode, see (5.23) (5.24) and figures 5.3, 5.4 and with regard to the generated viscous flow, see (5.113), we
choose $\gamma_0 > 0$. Thus we assume an incoming flow along the surface of the disk electrode being turned off in the neighbourhood of the origin into an outwards flow along the axis of symmetry, escaping in the direction of zero rotationality of the Lorentz force.

The corresponding contributions of the vorticity generated by the Lorentz force into the fluid yield, see (5.74), (5.75), (5.82), (5.90), (5.93)

$$\omega_0 = 0, \quad (5.94)$$

$$\omega_1 = -\frac{c\sqrt{1-c^2}}{2\gamma_0}, \quad \text{for } \bar{r} < 1, \ K_{\eta} = 0 \quad (5.95)$$

$$\omega_2 = -\frac{(1-22\gamma_0^2)}{72\gamma_0^3} c \sqrt{1-c^2} + 198\gamma_2 c(1-c^2)^{3/2} . \quad (5.96)$$

These expressions indicate that $g_0(c)$ represents an irrotational flow as stated before in order to satisfy the conditions imposed to the vorticity at the stagnation point $\bar{r} = 0$, see (5.39), (5.40), (5.74). Also the terms respectively multiplied by $\gamma_1$ in (5.85) and by $\gamma_3$ in (5.90) represent additional potential flows; in contrast with the term multiplying $\gamma_2$ in (5.90) which is an additional rotational flow. From (5.42), (5.43), (5.57), (5.58) it is clear that the curl of the inertia force does not exclude additional flows with non-zero vorticity.

In order to be able to present figures of the base fluid motions, represented by the solutions $g_0(c)$ and $g_1(c)$, we generally introduce, see (5.71), (5.102)

$$\tilde{\psi}(\bar{r},c) = \sum_{n=0}^{\infty} \tilde{\psi}_n(\bar{r},c), \quad (5.97)$$

with

$$\tilde{\psi}_n(\bar{r},c) = \bar{r}^{2n+e} g_n(c), \quad (5.98)$$

for $n = 0,1,2,...$ where $e = 3$ for $K_{\eta} = 0$ and $e = 7$ for $K_{\eta} > 0$.

Flow patterns of streamlines of constant $\tilde{\psi}_0$ and $\tilde{\psi}_1$ for $\gamma_0 = 0.1, 1, 10$ are presented in figure 5.5.

The respective flow patterns, corresponding to the basic solutions $g_0(c)$ and $g_1(c)$, see (5.82), (5.93), for different values of the free parameter $\gamma_0$, 
Figure 5.5. Streamlines of constant $\tilde{\Psi}_0$ and $\tilde{\Psi}_1$, see (5.82), (5.93), (5.97), (5.98), representing the induced inviscid fluid motion in the neighbourhood of the origin $\bar{r} = 0$ for $\gamma_o = 0.1, 1, 10$. 
turn out to be quite different. The graphs of \( \tilde{\Phi}_o \), being always dominant at \( \tilde{r} = 0 \), show a typical corner flow in the neighbourhood of a stagnation point; whereas the flow pattern of \( \tilde{\Phi}_1 \) is divided into two parts separated by a streamline \( \tilde{\phi}_1 = 0 \) at an angle \( \theta_1 \), satisfying

\[
\tilde{\phi}_1 = 0 \quad \text{at} \quad \theta_1 = \arccos \left\{ \frac{\sqrt{21 \gamma_o^2}}{2 + 49 \gamma_o^2} \right\} \quad \text{for} \quad \tilde{r} < 1, \quad K_\eta = 0, \quad (5.99)
\]

which follows from (5.93).

When \( \gamma_o \) tends to zero, the angle \( \theta_1 \) approaches 90°, resulting in a flow pattern of \( \tilde{\phi}_1 \) which is similar but of opposite direction to that of \( \tilde{\phi}_o \). The minimum value of \( \theta_1 = 49.10^\circ \), is reached when \( \gamma_o \) tends to infinity. The graphs of \( \tilde{\phi}_1 \) show that already for \( \gamma_o = 1.10 \) the corresponding values of \( \theta_1 \) are very near its minimum value. It is apparent from (5.99), that the described behaviour of \( \theta_1 \) as function of \( \gamma_o \) is independent of the sign of \( \gamma_o \).

Comparison of the respective expressions of \( g_1(c) \), see (5.85), (5.93), indicates that the part of the flow pattern of \( \tilde{\phi}_1 \) near the surface of the disk electrode, viz. between \( \theta_1 < \theta < 90^\circ \) is evidently caused by the additional potential flow, i.e. the term multiplied by \( \gamma_1 \) in the expression (5.85). This term in its turn is again related to the basic potential flow, which follows from (5.92), due to the eigenvalue \( \lambda = 0 \) of solution \( g_2(c) \).

The relative essentially non-additional contributions in the expressions of \( g_0(c) \), \( g_1(c) \), \( g_2(c) \), generated by the Lorentz force, are obtained by setting \( \gamma_1 = \gamma_2 = \gamma_3 = 0 \), where for simplicity the relation (5.92) is left out of consideration. The behaviour of \( \tilde{\phi}_o \) and \( \tilde{\phi}_1 \) at different values of \( \gamma_o \) clearly indicate that we may expect the appearance of one or more eddies in the inviscid fluid motion in the neighbourhood of the origin. This behaviour is not unusual in corner flows and implies that the axis of symmetry, the surface of the disk electrode or even the insulating flat wall may contain extra stagnation points.

In this regard it is worth observing the different behaviour of both series solutions of the inviscid fluid motion at different values of \( \gamma_o \). Namely the figures show that at larger values of \( \gamma_o \) the basic flow \( \tilde{\phi}_o \) dominates over \( \tilde{\phi}_1 \), also at larger radial distance, whereas for small values, e.g. \( \gamma_o = 0.1 \) the flow pattern of \( \tilde{\phi}_1 \) overrules that of \( \tilde{\phi}_o \) already at small radial distance from the origin.

In an analogous way an expression of the pressure field in the near field \( \tilde{r} < 1 \) for \( K_\eta = 0 \) can be derived in the form of a series expansion in positive powers.
of $\overline{r}^2$. It turns out that the governing pressure distribution, being dominant in the neighbourhood of the origin is caused by the basic flow solution $g_0(c)$ and an appropriate term of the series expansion of the Lorentz force. This behaviour is different from that in the far field for $\overline{r} \to \infty$ or in the point electrode model where the pressure is determined by the generated fluid motion only, see (2.64).

The series expansion of the magnetic field, valid in the near field $\overline{r} < 1$, see (5.17), can be rewritten in a more general form

$$B = \frac{\mu I_0}{2\pi a} \sum_{n=0}^{\infty} \frac{f_n(c)}{\sqrt{1-c^2}} \frac{2n+1}{r} \quad \text{for } \overline{r} < 1,$$

(5.100)

where for $K_b \ll 1$

$$f_n(c) = \frac{(1-c^2)^{\frac{n}{2}}(1,1)(x)}{2(n+1)},$$

(5.101)

for $n = 0, 1, 2, \ldots$, with $x = 1-2c^2$.

By substitution of (5.71) and (5.100) into (5.59), the partial differential equation, which represents the curl of Ohm's law, is transformed into an infinite series of ordinary non-linear differential equations of the functions $f_n(c)$.

Some further calculations indicate that in the case of inviscid flow $K_\eta = 0$, the base solutions $f_0(c)$ and $f_1(c)$ of (5.100), (5.101) can never be affected by the electromagnetic induction due to the induced fluid motion at any bounded value of $K_b$. This behaviour implies in fact a relatively low magnetic Reynolds number behaviour in the neighbourhood of the origin which will be discussed in detail in section 5.6.

The onset of deviation from the original electric current distribution at $K_b \ll 1$ due to the effect of the inviscid fluid motion upon higher order terms of (5.100) shows that at larger values of $K_b$ less electric current is supplied into the fluid in axial direction, i.e. the electric current distribution tends to shift towards the insulating flat wall at larger radial distance.

The most interesting feature of the present calculations of the inviscid flow field in the neighbourhood of the origin and thus in the middle of the disk electrode surface is the fact that the solutions of the inviscid fluid motion do not contain any singularity in the near field fluid domain. This is in contrast with the far field solutions, see chapters 3 and 6. In addition any weak singular behaviour that might occur at the rim of the electrode, caused by the
present mathematical model, can easily be resolved in the way mentioned in section 5.2.2. in the text between figures 5.3 and 5.4.

5.5. The viscous fluid motion in the near field $\bar{r} < 1$

5.5.1. Introductory remarks

In this section we shall derive a series expansion of the dimensionless Stokes stream function $\tilde{\psi}$ for the case of viscous fluid. The partial differential equation of the curl of the Navier-Stokes equation will be replaced by an infinite set of ordinary differential equations and we will calculate some basic solutions of the viscous fluid motion in the near field fluid domain $\bar{r} < 1$. Similar to the inviscid fluid motion discussed in section 5.4., it turns out that the near field behaviour of the viscous flow strongly differs from that in the far field $\bar{r} > 1$.
The solutions of the viscous flow field, to be presented, also contain free parameters, which will be discussed in section 5.7. It turns out that these solutions do not contain any singularity in the fluid domain and moreover that their existence is not limited by a maximum value of the hydrodynamic Reynolds number, as found in the far field of the disk electrode configuration for $\bar{r} \to \infty$ or in the point electrode model, see chapter 4 and 6.
The viscous fluid motion in the neighbourhood of the origin shows a typical slow viscous behaviour to be clarified in section 5.6.

5.5.2. Derivation of the series expansions and governing differential equations for $\bar{r} < 1$ and $K_\eta > 0$

A careful examination of the curl of the Navier-Stokes equation (5.41) and of the series expansion of the right-hand side (5.52), representing the curl of the $(\mathcal{E} \times \mathcal{B})$-force, indicates that the correct series expansion of the dimensionless Stokes stream function $\tilde{\psi}$ in the case of viscous fluid must be of the form

$$\tilde{\psi}(\bar{r},c) = \sum_{n=0}^{\infty} g_n(c) \bar{r}^{2n+7} \quad \text{for } \bar{r} < 1, \ K_\eta > 0 \ , \quad (5.102)$$

especially in view of the condition that the viscous force should dominate the
inertia force in the neighbourhood of the origin and on the surface of the disk electrode.

This series expansion also satisfies the condition that every term of the expansion is actually generated by an appropriate term of the series expansion of the Lorentz force, as stated before. In the present viscous case also the basic solution \( g_0(c) \) satisfies that condition.

In contrast with the near field inviscid fluid problem there is no necessity here to invoke an additional potential flow as the leading term of the series expansion of the Stokes stream function. Although it is possible in principle, it is beyond the scope of the present calculations.

Another advantage of the above series expansions is the fact that it does not imply the appearance of additional terms with different powers of \( \tilde{r} \) in the near field series expansions of the fluid-flow and electromagnetic field quantities at higher values of the parameter \( K_\eta \).

It follows from (5.33), (5.34), (5.58), (5.102) that the respective series expansions of the velocity components yield

\[
v_r = -\frac{I_0}{2\pi a} \sqrt{\frac{\mu}{\rho}} \sum_{n=0}^{\infty} \frac{dg_n}{dc} \frac{1}{\tilde{r}^{2n+5}}, \tag{5.103}
\]

for \( \tilde{r} < 1, K_\eta > 0 \)

\[
v_\theta = -\frac{I_0}{2\pi a} \sqrt{\frac{\mu}{\rho}} \sum_{n=0}^{\infty} \frac{g_n(c)}{\sqrt{1-c^2}} \frac{1}{\tilde{r}^{2n+5}}, \tag{5.104}
\]

and of the vorticity

\[
\omega = -\frac{I_0}{2\pi a^2} \sqrt{\frac{\mu}{\rho}} \sum_{n=0}^{\infty} \omega_n \frac{1}{\tilde{r}^{2n+4}}, \tag{5.105}
\]

for \( \tilde{r} < 1, K_\eta > 0 \)

\[
\omega_n = \sqrt{1-c^2} \frac{d^2 g_n}{dc^2} + \frac{2(n+3)(2n+7)}{\sqrt{1-c^2}} g_n(c). \tag{5.106}
\]

Comparison with (5.61) and (5.62) demonstrates the difference in the behaviour of the viscous velocity components at large radial distance and in the neighbourhood of the origin; namely \( v = 0(\tilde{r}^{-1}) \) for \( \tilde{r} \to \infty \) and \( v = 0(\tilde{r}^5) \) for \( \tilde{r} = 0 \) in case of \( K_\eta > 0 \). Also note the difference with the inviscid fluid for
\( \bar{r} < 1 \) and \( K_\eta = 0 \), see (5.72) and (5.73).
At the origin, which is a stagnation point, the viscous velocity components become identical to zero, as the boundary conditions require, while the vorticity satisfies there the conditions (5.39) and (5.40).
The infinite set of ordinary differential equations is obtained by substitution of the series expansion (5.102) into the partial differential equation (5.41) for the curl of the Navier-Stokes equation, while using also (5.52) leads to

\[
K_\eta \left\{ (1-c^2)^2 \frac{d^4 g_n}{dc^4} - 4c(1-c^2) \frac{d^3 g_n}{dc^3} + 4(n+3)(2n+5)(1-c^2) \frac{d^2 g_n}{dc^2} + 4(n+2)(n+3)(2n+5)(2n+7) g_n(c) \right\} + \sum_{m=0}^{n-3} \frac{d^3 g_{n-m-3}}{dc^3} + \frac{(2n-2m-3)(1-c^2)}{1-c^2} \frac{dg_m}{dc} \frac{d^2 g_{n-m-3}}{dc^2} - 2(2m+7)(2n^2-4nm+n-10m-9) \frac{dg_m}{dc} \frac{d^2 g_{n-m-3}}{dc^2} + \frac{2(n+3)(2m+7)(2n-2m+1)c}{1-c^2} g_m(c) g_{n-m-3}(c) = c(1-c^2) \bar{\alpha}_n(c),
\]

(5.107)

for \( n = 0,1,2,\ldots \), \( \bar{r} < 1 \) and \( K_\eta > 0 \); where the expressions of \( \bar{\alpha}_n(c) \) are given by (5.53) - (5.56).
The above expression of the differential equations of the functions \( g_n(c) \) of the series expansion of the Stokes stream function indicates that all solutions \( g_n(c) \) are induced by a corresponding term of the series expansion of the curl of the Lorentz force.
For the first three governing solutions \( g_0(c) - g_2(c) \) the Lorentz force balances merely the viscous force. In higher order solutions; i.e. at larger distance from the origin, the effect of the inertia force plays a role. Hence in the neighbourhood of the origin the viscous forces dominate the inertia forces which can be expected in view of the local behaviour of the viscous Reynolds number, discussed in section 5.6.
This behaviour and the form of the differential equations for \( \bar{r} < 1 \) and \( K_\eta > 0 \) is entirely different from that in the far field for \( \bar{r} > 1 \) and \( K_\eta > 0 \), see (5.65). The most interesting feature is the fact that the above differential equations are linear for every solution \( g_n(c) \), whereas the ones that govern the behaviour of the viscous flow in the far field \( \bar{r} > 1 \), see (5.65), are non-linear when the effect of the inertia forces is included.
Some solutions governing the viscous fluid motion in the neighbourhood of the origin will be calculated in section 5.5.3. The boundary conditions of the viscous fluid motion in the near field fluid domain \( \bar{r} < 1 \), derived from (5.36) - (5.38), (5.103), (5.104), are

\[
g_n(0) = 0 \quad , \quad (5.108)
\]

\[
\left( \frac{dg_n}{dc} \right)_{c=0} = 0 \quad , \quad (5.109)
\]

for \( \bar{r} < 1 \), \( K_\eta > 0 \)

\[
g_n(1) = 0 \quad , \quad (5.110)
\]

or

\[
\lim_{c \to 1} \left\{ \sqrt{1-c^2} \frac{dg_n}{dc} + \frac{(2n+7)c}{\sqrt{1-c^2}} g_n(c) \right\} = 0 \quad , \quad (5.111)
\]

for \( n = 0,1,2, \ldots \), \( \bar{r} < 1 \) and \( K_\eta > 0 \).

In the same way as found in the inviscid problem we are now able to conclude that the entire axis of symmetry \( c = 1 \), the origin \( \bar{r} = 0 \), the surface of the disk electrode, the rim of the electrode and the insulating flat wall \( c = 0 \), coincide with the streamline \( \tilde{\psi} = 0 \).

In the next section 5.5.3, we calculate the solutions \( g_o(c) - g_3(c) \), where only in the case of \( g_3(c) \) the effect of the inertia force enters the equations.

5.5.3. The viscous fluid motion in the neighbourhood of the origin \( \bar{r} = 0 \)

In this section we derive from the respective governing differential equations and the boundary conditions some basic solutions \( g_n(c) \) of the series expansion of the Stokes stream function \( \tilde{\psi} \), as applied in the near field \( \bar{r} < 1 \) in the case of viscous fluid, viz. \( K_\eta > 0 \), see (5.102). In order to examine the effect of the inertia force, the calculation will be executed upto and including \( g_3(c) \).

From these solutions, containing free parameters as in the inviscid near field case, some series solutions of the vorticity will be also calculated.

For special values of the free parameters, the flow patterns of the base
solutions \( g_0(c) \) and \( g_1(c) \) will be shown in figures, representing the viscous fluid motion in the neighbourhood of the origin.

From the infinite set of ordinary differential equations that replaces the partial differential equation of the curl of the Navier-Stokes equation, see (5.54), (5.107), it follows that the basic solution \( g_0(c) \) satisfies the linear inhomogeneous ordinary differential equation

\[
(1-c^2) \frac{d^4 g_0}{dc^4} - 4c \frac{d^3 g_0}{dc^3} + 60 \frac{d^2 g_0}{dc^2} + \frac{840}{1-c^2} g_0(c) = \frac{c}{K_\eta}.
\] (5.112)

Calculation of the complementary function and of the particular solution and substitution of the boundary conditions (5.108) - (5.111) result in a basic solution of the form

\[
g_0(c) = \frac{e^5(1-c^2)}{120 K_ \eta} + \gamma_0 c^3 (1-c^2)(5-9c^2) \quad \text{for} \quad r < 1, K_\eta > 0,
\] (5.113)

where \( \gamma_0 \) is a free parameter, in the same way as it has been found in the inviscid near field case, see section 5.4.3. It is evident that there is no reason for the values of the free parameters to be identical in the case of inviscid and viscous fluid motion; especially in view of the different series expansions of the Stokes stream function, see (5.71), (5.102). Only for simplicity we use the same symbol, see (5.82).

In view of the behaviour of the Lorentz force on the surface of the disk electrode and near the origin, see figures 5.3 and 5.4, we may expect that the governing solution of the viscous fluid motion in the neighbourhood of the origin consists of an incoming flow about parallel to the surface of the electrode being turned off at the origin into an axial jet flow along the axis of symmetry. In that case the solution \( g_0(c) \) will be positive on the open interval and as a consequence the value of the free parameter \( \gamma_0 \) will be bounded by the following inequality:

\[
g_0(c) > 0 \quad \text{on} \quad 0 < c < 1,
\]

if

\[
0 < \gamma_0 < \frac{1}{480 K_\eta}.
\] (5.114)
The differential equation of the solution $g_1(c)$, obtained from (5.55) and (5.107), yields

\[
(1-c^2) \frac{d^4 g_1}{dc^4} - 4c \frac{d^3 g_1}{dc^3} + 112 \frac{d^2 g_1}{dc^2} + \frac{3024}{1-c^2} g_1(c) = \frac{c}{4K\eta} (7-19c^2). \quad (5.115)
\]

Application of the boundary conditions of $g_1(c)$, (5.108) - (5.111), to the homogeneous and particular solutions, leads to the following expression of the solution $g_1(c)$

\[
g_1(c) = \frac{c^5(1-c^2)}{10080K\eta} (147-239c^2) + \gamma_1 c^3(1-c^2)(35-154c^2+143c^4)
\]

for $\bar{r} < 1$, $K\eta > 0$, \quad (5.116)

where $\gamma_1$ is the free parameter belonging to this solution due to the absence of an additional boundary condition.

For $n = 2$ the differential equation of the next solution $g_2(c)$, see (5.56) and (5.107), becomes after dividing by the factor $K\eta(1-c^2)$

\[
(1-c^2) \frac{d^4 g_2}{dc^4} - 4c \frac{d^3 g_2}{dc^3} + 180 \frac{d^2 g_2}{dc^2} + \frac{7920}{1-c^2} g_2(c) = \frac{c}{8K\eta} (19-126c^2+155c^4). \quad (5.117)
\]

By calculating the complementary function and the particular integral and upon substituting the boundary conditions (5.108) - (5.111), the expression of the solution $g_2(c)$ takes the form

\[
g_2(c) = \frac{c^5(1-c^2)}{60480K\eta} (1197-4554c^2+3881c^4) + \gamma_2 c^3(1-c^2)(105-819c^2+1755c^4-1105c^6)
\]

for $\bar{r} < 1$, $K\eta > 0$. \quad (5.118)

where $\gamma_2$ is the usual accessory free parameter.

The differential equation of the solution $g_3(c)$ strongly differs from the previous ones of $g_0(c) - g_2(c)$, because now for the first time the effect of the inertia force enters the equation. From (5.8), (5.53), (5.107) and substi-
tution of (5.113) it follows that the solution \( g_3(c) \) is determined by the differential equation

\[
(1-c^2) \frac{d^4 g_3}{dc^4} - 4c \frac{d^3 g_3}{dc^3} + 264 \frac{d^2 g_3}{dc^2} + \frac{17160}{1-c^2} g_3(c) = \\
= \frac{c}{64K} (187-2201c^2+6241c^4-4867c^6) + \frac{60c^3}{K_D} \left\{ \frac{c^4}{7200K^2} + \frac{\gamma_0 c^2}{30K} \right\} (4-7c^2) + \\
- \gamma_0^2 (5+14c^2-35c^4) ,
\]

where the first term at the right-hand side represents the Lorentz force and the second one with the braces the effect of the inertia force, originating from the basic flow solution \( g_0(c) \).

Calculation of the four homogeneous solutions of the homogeneous differential equation and of the respective particular solutions and substitution of the boundary conditions (5.108) - (5.111) result in the expression of the viscous solution \( g_3(c) \), which includes now the effect of all forces in the curl of the Navier-Stokes equation, of the form

\[
g_3(c) = \frac{c^5(1-c^2)}{8870400K} (215985-1412235c^2+2713755c^4-1588321c^6) + \\
+ \frac{c^7(1-c^2)}{1386K} \left\{ \frac{7c^4}{4800K^2} + \frac{\gamma_0 c^2}{12K} (44-63c^2) - 3\gamma_0^2 (165-220c^2-21c^4) \right\} + \\
+ \gamma_3 c^3(1-c^2)(231-2772c^2+10098c^4-14212c^6+6783c^8) \quad \text{for} \quad \bar{r} < 1 , \quad K_D > 0 ,
\]

where \( \gamma_3 \) is the additional free parameter.

In this expression the first term of the right-hand side represents the effect of the Lorentz force, the second term containing the free parameter \( \gamma_0 \) also, shows the influence of the inertia force due to the governing solution \( g_0(c) \).
and the third term multiplied by the free parameter $\gamma_3$ satisfies the homogeneous differential equation of $g_3(c)$.

Despite the usually non-linear behaviour of the inertia force leading usually to a singular behaviour of the far field solutions at the axis of symmetry, both for viscous and inviscid cases, see chapters 3, 4, 6, we observe in the near field $\bar{r} < 1$ that the inviscid and viscous velocity components remain bounded in the entire fluid domain $\bar{r} < 1$, including the axis of symmetry. Moreover the near field viscous fluid motion turns out to exist for all bounded values of the parameter $K_\eta$, which is related to the hydrodynamic Reynolds number, see section 5.6. Especially the latter fact is in contrast with the behaviour of the viscous fluid motion in the far field of the disk electrode configuration for $\bar{r} \to \infty$ or in the point electrode model, see chapters 4 and 6.

Nevertheless in considering boundary value problems we always may expect, in higher order viscous solutions and in the near field, that eigenvalues do occur, as found in the derivation of the function $g_2(c)$ for $\bar{r} < 1$ and $K_\eta = 0$, see section 5.4.3. Therefore we always have to take into account that such phenomena can imply the appearance of a relatively weak singularity in the inviscid and/or viscous flow field at the axis of symmetry in the higher order solutions, i.e. at larger radial distance ($\bar{r} < 1$) from the origin. In this connection it has to be remarked that the complementary functions of the general inviscid and viscous near field solutions in principle should show a relatively weak singular logarithmic behaviour of the velocity fields at the axis of symmetry $c = 1$ for $\bar{r} < 1$. However until now that singular behaviour is removed from the ultimate solutions by the boundary conditions $g_n(0) = g_n(1) = 0$.

It is clear that the fact that the influence of the inertia force enters the series expansion solutions of the Stokes stream function, see (5.102), (5.120), only at the fourth term in the viscous case, is due to the local relatively low Re behaviour in the neighbourhood of the origin, see section 5.6.

Finally it should be noted that we cannot set any viscous solution $g_n(c)$, obtained until now, identically to zero for some special value of the corresponding free parameter $\gamma_n$ on the entire interval $0 < c < 1$. In considering the behaviour of the complementary functions, being sums of associated Legendre polynomials, and of the respective particular solutions we do not expect that it will be possible for any viscous solution $g_n(c)$.

The vorticity contributions generated by the Lorentz force in the viscous fluid in the neighbourhood of the origin satisfy, see (5.105), (5.106), (5.113), (5.116),
\[ \omega_0 = \frac{c_0^3}{1 - c^2} - \frac{c_0^2}{6K_\eta} + 10\eta c^2 \sqrt{1 - c^2} (3 - 7c^2) , \] (5.121)
for \( \bar{r} < 1 , K_\eta > 0 \)

\[ \omega_1 = \frac{c_0^3}{120K_\eta} (35 - 67c^2) + 42\eta c^2 \sqrt{1 - c^2} (5 - 30c^2 + 33c^4) . \] (5.122)

In the same way as we will find for the viscous velocity field at \( \bar{r} \sim 0 \), the vorticity is proportional to \( I^2 \) when we ignore the possible influence of the additional free parameters \( \gamma_0 \) and \( \gamma_1 \), see (2.17), (5.103) - (5.106), (5.113), (5.116). However it turns out that the viscous velocity field is proportional to \( a^{-1} \) and the corresponding vorticity to \( a^{-2} \), where \( a \) is the radius of the disk electrode. In addition it can easily be verified that the vorticity satisfies the extra boundary conditions (5.39), (5.40), as required at the stagnation point \( \bar{r} = 0 \). The basic vorticity contribution \( \omega_0 \) is likewise \( g_0(c) \) positive on the open interval \( 0 < c < 1 \) when the free parameter \( \gamma_0 \) satisfies the condition (5.114).

In order to present figures of the basic viscous fluid motions, determined by the solutions \( g_0(c) \) and \( g_1(c) \), see (5.113), (5.116), in the special case \( \gamma_0 = \gamma_1 = 0 \), we apply (5.97), (5.98), (5.102) to obtain modified forms of the respective dimensionless Stokes stream functions \( \tilde{\psi}_0 \) and \( \tilde{\psi}_1 \).

Flow patterns of streamlines of constant \( K_\eta \tilde{\psi}_0 \) and \( K_\eta \tilde{\psi}_1 \) are presented in figure 5.6.

The flow pattern of \( \tilde{\psi}_0 \), which is always dominant in the neighbourhood of the origin for \( \bar{r} = 0 \), shows a typical viscous corner flow near a stagnation point. It should be noted that the direction of this basic viscous fluid motion, composed of an incoming flow about parallel to the surface of the disk electrode which is turned off into an outward flow along the axis of symmetry, being directed away from the electrode, is in agreement with numerical results obtained by Sozou & Pickering (1978), Atthey (1980), Craine & Andrews (1984), Ajayi, Sozou & Pickering (1984).

The flow pattern of \( \tilde{\psi}_1 \) shows a fluid motion which is divided into two parts separated by a streamline \( \tilde{\psi}_1 = 0 \) at an angle \( \theta_1 = 38.3^\circ \) in the case \( \gamma_1 = 0 \).

Note from figures 5.5 and 5.6 the similar behaviour of the first two base solutions of the inviscid and viscous fluid motions in the neighbourhood of the origin for \( \bar{r} = 0 \).
Figure 5.6. Streamlines of constant $K_\eta \tilde{\Phi}_0$ and $K_\eta \tilde{\Phi}_1$, see (5.97), (5.98), (5.113), (5.116), representing the induced viscous fluid motion in the neighbourhood of the origin for $\bar{r} = 0$ in the special case $\gamma_0 = \gamma_1 = 0$.

It is worth remarking that the velocity components of the viscous fluid motion in the neighbourhood of the origin is proportional to $K_\eta^{-1}$ and especially to $I_o^2$, which follows from (2.17), (5.103), (5.104), (5.113), (5.116), when not taking account of the possible effect of the free parameters $\gamma_0$ and $\gamma_1$. This conclusion based upon a theoretical calculation is in full agreement with results obtained from experiments carried out by Woods & Milner (1971).

In an analogous way an expression of the pressure distribution in the near field $\bar{r} < 1$ for $K_\eta > 0$ can be derived in the form of a series expansion in positive powers of $\bar{r}^2$. It turns out that the basic contribution of the pressure field, apart from the constant stagnation pressure at the origin, is caused by the Lorentz force only. This behaviour differs from the far field for $\bar{r} \to \infty$ where the pressure field is determined by the inviscid or viscous fluid motion and also from the inviscid near field case for $\bar{r} = 0$ where the pressure distribution balances the generated fluid motion and the Lorentz force.

Substitution of (5.100) and (5.102) into the partial differential equation of
the curl of the Ohm's law (5.59) results in an infinite set of ordinary non-linear differential equations of the functions \( f_n(c) \) and shows that in the case of viscous fluid motion \( K_\eta > 0 \) the base solutions \( f_0(c) - f_3(c) \) can never be affected by the electromagnetic induction for any bounded value of the parameter \( K_b \). In fact this behaviour indicates a relatively low magnetic Reynolds number behaviour in the neighbourhood of the origin, see section 5.6.

Investigation of the primary effect of the fluid motion upon the electric current distribution at larger values of \( K_b \) shows a less pronounced shifting of electric current towards the insulating flat wall at larger radial distance from the electrode, than in the inviscid near field case.

The most important conclusion to be drawn from the present calculations is that the viscous fluid motion is bounded in the neighbourhood of the origin and moreover that it exists for every bounded value of the parameter \( K_\eta \); a conclusion also valid when the effect of the inertia force is invoked.

The viscous solutions contain each a free parameter due to the absence of an additional boundary condition. Their physical relevance will be discussed in section 5.7.

A very interesting feature of the near field inviscid and viscous flows is the appearance of the local behaviour of viscous and magnetic Reynolds numbers. This phenomenon, which already has been suggested by Shercliff (1970), will be discussed in the next section 5.6.

5.6. The local behaviour of the hydrodynamic and magnetic Reynolds numbers

In this section we demonstrate that the viscous Reynolds number \( Re \) and the magnetic Reynolds number \( Rm \) possess a local behaviour in the fluid domain and especially in the neighbourhood of the origin for \( \bar{r} = 0 \) and at large radial distance for \( \bar{r} \to \infty \).

In the point electrode configuration and in the disk electrode configuration we found that no significant characteristic velocity scale is available. Only in the disk electrode problem, considered in this chapter, the radius \( a \) of the disk electrode may be regarded as a characteristic length scale. Since the configurations do not contain a characteristic velocity scale, the hydrodynamic and magnetic Reynolds numbers show a typical local behaviour throughout the flow field.

In section 2.5 it has been denoted that the magnetic Reynolds number \( Rm \) and the viscous Reynolds number \( Re \) are defined as
\[ Rm = \sigma \mu x \text{velocity} \times r, \quad (5.123) \]
\[ Re = \frac{p}{\eta} \times \text{velocity} \times r. \quad (5.124) \]

At least at low value of the so-called effective magnetic Reynolds number \( K_b \), see (2.16), the angle-dependent part \( g(c) \) of the Stokes stream function \( \psi \) in the case of the point electrode problem, see (2.14), and the corresponding function \( g_0(c) \) of the leading term of the respective series expansions of the dimensionless Stokes stream function \( \tilde{\psi} \) in the disk electrode configuration, see (5.60), (5.71), (5.102), can be regarded as to be of order unity in order to demonstrate the local behaviour of the magnetic and viscous Reynolds numbers. As already found in section 2.5, for the point electrode model and also from (5.61), (5.62), (5.123), (5.124), for the far field of the inviscid and viscous fluid domain in the disk electrode configuration, the magnetic and hydrodynamic Reynolds number satisfy at large radial distance and in the case of a point electrode \( a = 0^+ \)

\[ Rm = K_b \quad \text{for } \tilde{r} \to \infty, \ K_\eta > 0 \quad (5.125) \]
\[ Re = \frac{1}{K_\eta} \quad \text{for } \tilde{r} \to \infty, \ K_\eta > 0 \quad (5.126) \]

where the so-called effective magnetic Reynolds number \( K_b \) and the inverse hydrodynamic Reynolds number \( K_\eta \) yield

\[ K_b = \frac{q\mu I_o}{2\pi} \sqrt{\frac{\mu}{\rho}}, \quad (2.16) \]
\[ K_\eta = \frac{2\pi\eta}{I_o \sqrt{\rho \mu}}. \quad (2.17) \]

In the inviscid and viscous near field fluid domain of the semi-infinite disk electrode configuration the magnetic Reynolds number takes the forms

\[ Rm = K_b \tilde{r}^2 \quad \text{for } \tilde{r} \to 0, \ K_\eta = 0 \quad (5.127) \]
\[ Rm = K_b \tilde{r}^6 \quad \text{for } \tilde{r} \to 0, \ K_\eta > 0 \quad (5.128) \]
which is easily derived from (5.72), (5.73), (5.103), (5.104), (5.123).

From the definition of Re, see (5.124) and (5.103), (5.104), it follows that the local behaviour of the hydrodynamic Reynolds number in the neighbourhood of the centre of the disk electrode is of the form

\[
Re = \frac{\bar{r}^6}{K\eta} \quad \text{for } \bar{r} \to 0, \ K\eta > 0
\]  

(5.129)

In liquid metal applications, using very large electric currents of several hundred amperes, the values of \(K_b\) and \(K\eta\) are usual very small compared to unity. For example at \(I_o = 500\ A\) their magnitudes are: \(K_b \sim 10^{-3}\) and \(K\eta \sim 10^{-4}\), see figure 2.2.

Therefore we conclude from (5.125) - (5.129) that the local behaviour of the magnetic and hydrodynamic Reynolds numbers clearly shows a relatively high \(Rm\) and \(Re\) behaviour at large radial distance from the electrode and a relatively low \(Rm\) and \(Re\) behaviour near the origin and the electrode.

The possibility of this local behaviour of the characteristic parameters has already been suggested by Shercliff (1970).

Note that in the above consideration we did not use the fact that the angle-dependent part of the Stokes stream function \(g(c)\) or \(g_o(c)\) is of order \(K\eta^{-1}\) for the slow viscous flow \((K\eta \gg 1)\) at large radial distance, see (4.1), (4.10) - (4.15), and in the viscous flow in the neighbourhood of the origin in the disk electrode configuration, see (5.113), when the effect of the additional free parameter \(\gamma_o\) is omitted. When we include this behaviour, the dimensionless parameters become of the form

\[
Rm = \frac{K_b}{K\eta} \quad \text{for } \bar{r} \to \infty, \ K\eta \gg 1
\]  

(5.130)

\[
Re = \frac{1}{K\eta^2} \quad \text{for } \bar{r} \to \infty, \ K\eta \gg 1
\]  

(5.131)

and

\[
Rm = \frac{K_b}{K\eta} \bar{r}^6 \quad \text{for } \bar{r} \to 0, \ K\eta > 0
\]  

(5.132)

\[
Re = \frac{\bar{r}^6}{K\eta^2} \quad \text{for } \bar{r} \to 0, \ K\eta > 0
\]  

(5.133)
It is clear that these additional expressions do not violate the conclusion about the local behaviour of $\text{Re}$ and $\text{Rm}$, as stated before.

5.7. The physical significance of the free parameters

The appearance of the free parameters in the series solutions of the inviscid and viscous near field fluid motions is from a mathematical point of view due to the absence of one boundary condition in order to make the solutions unique. The respective third and fourth order governing differential equations have to satisfy only two respectively three boundary conditions. The absence of one boundary condition is caused primarily by the mathematical treatment to make an analytical calculation possible; i.e. consideration of near field and far field series expansion solutions, separated by the hemisphere $\tilde{r} = 1$. A second reason is that we consider a semi-infinite configuration. Another series expansion at small distance $s = r/\sqrt{1-c^2}$ from the axis will be helpful to connect the present near and far field solutions in the neighbourhood of the axis of symmetry, but in principle this will not result in a well-posed problem. For that we ought to consider a configuration with finite dimensions or an extra condition has to be added in the semi-infinite configuration. For example the assumption that the flux of momentum in axial direction is constant, as applied in the jet flow problem caused by the efflux from an orifice, see Schlichting (1979, p. 230). Unfortunately in the present problem such an assumption cannot be gained because of the overall effect of the Lorentz force generating fluid motions and pressure distributions in the entire fluid domain.

In order to investigate the physical meaning of the free parameters, appearing in the inviscid and viscous near field flow solutions, we consider the momentum equation in integral form of the respective terms of the Navier-Stokes equation. From (2.2), (2.4), by application of Gauss divergence theorem and with some elementary calculations, it can be derived that in a volume $V$ of fluid around the origin, symmetrical about the axis of symmetry, only the axial components of the respective forces, appearing in the Navier-Stokes equation are unequal to zero. They satisfy

$$\tilde{F}_{i,z} + \tilde{F}_{p,z} = \tilde{F}_{L,z} + \tilde{F}_{\eta,z},$$

(5.134)

with

$$\tilde{F}_{i,z} = \rho \iiint_V \nabla \cdot \nabla z \, dV = \rho \oint_S (\mathbf{v} \cdot \mathbf{n}) \, v_z \, dS,$$

(5.135)
\[
\tilde{F}_{p,z} = \iiint_V \mathbf{t} \cdot \mathbf{dV} = \oint_S \mathbf{t} \cdot (\mathbf{n} \times \mathbf{z}) \, dS,
\]
(5.136)

\[
\tilde{F}_{L,z} = \iiint_V (J \times \mathbf{B}) \cdot \mathbf{z} \, dV = -\frac{1}{2\mu} \oint_S \mathbf{B} \cdot (\mathbf{n} \times \mathbf{z}) \, dS,
\]
(5.137)

\[
\tilde{F}_{\eta,z} = \eta \iiint_V \Delta \mathbf{v} \cdot \mathbf{z} \, dV = \eta \oint_S \mathbf{n} \cdot \mathbf{v} \, dS,
\]
(5.138)

where \( S \) is the closed boundary surface of the volume \( V \).

Consideration of a closed surface \( S \) consisting of a hemisphere of small and constant radius \( \tilde{r} \ll 1 \) with the centre in the origin and accompanied by the appropriate part of the surface of the disk electrode, provides explicit expressions in the form of series expansions of the respective forces acting in the hemispherical fluid domain around the origin.

In the case of inviscid flow, where of course the viscous force \( \tilde{F}_{\eta,z} \) is identical to zero, the effect of the basic solutions \( g_0(c) \) and \( g_1(c) \) is taken into consideration. Substitution of (5.8), (5.17), (5.72), (5.73), (5.82), (5.93) and of the series solutions of the pressure field for \( \tilde{r} < 1 \), \( K_\eta = 0 \) into (5.135) – (5.137) and some elementary calculations result in the expressions

\[
\tilde{F}_{L,z} = \frac{\mu I^2}{4\pi} \left\{ \frac{\gamma_0^2}{36} \tilde{r}^6 + O(\tilde{r}^8) \right\} \quad \text{for } K_\eta = 0,
\]
(5.139)

\[
\tilde{F}_{p,z} = \frac{\mu I^2}{4\pi} \left\{ -\frac{\gamma_0^2}{72} \tilde{r}^4 + \frac{(11 + 70\gamma_0^2)}{72} \tilde{r}^6 + O(\tilde{r}^8) \right\} \quad \text{for } K_\eta = 0,
\]
(5.140)

with

\[
\tilde{F}_{L,z} = \frac{\mu I^2}{4\pi} \left( \frac{\tilde{r}^6}{24} + O(\tilde{r}^8) \right) \quad \text{for } K_\eta > 0,
\]
(5.141)

for constant \( \tilde{r} \sim 0 \).

In the case of viscous fluid motion, for simplicity only the effect of the basic solution \( g_0(c) \) is considered. Substitution of (5.103), (5.104), (5.113) and of the series expansion solution of the pressure distribution for \( \tilde{r} < 1 \), \( K_\eta > 0 \) into (5.135) – (5.138) yields after some algebra
\[ F_{i, z} = \frac{\mu I^2}{4\pi} \left( \left( \frac{1}{43200K^2} - \frac{13\gamma}{720K\eta} + 5\gamma^2 \right) \tilde{r}^{12} + O(\tilde{r}^{14}) \right), \]  
\[ \tilde{F}_{p, z} = \frac{\mu I^2}{4\pi} \left( \left( \frac{5}{72} + \frac{10K\gamma}{3\eta} \right) \tilde{r}^6 + O(\tilde{r}^8) \right), \]  
\[ \tilde{F}_{\eta, z} = \frac{\mu I^2}{4\pi} \left( \left( \frac{1}{36} + \frac{10K\gamma}{3\eta} \right) \tilde{r}^6 + O(\tilde{r}^8) \right), \]

for \( K\eta > 0 \) and constant \( \tilde{r} \sim 0 \), where \( \tilde{F}_{L, z} \) is given by (5.141).

It is clear that the magnitude of the free parameter \( \gamma_0 \), as used in the inviscid flow, see (5.139), (5.140), in general differs from the one applied in the viscous case, see (5.142) – (5.144). Only for simplicity we have used the same symbol.

The above expressions evidently show that the respective inviscid and viscous flow- and pressure fields, being generated by the Lorentz force, are also affected by the behaviour of the surrounding fluid motion and pressure distribution. Namely \( \tilde{F}_{i, z} \) represents the outwards axial directed convective flux of momentum across the surface of the hemisphere; \( \tilde{F}_{p, z} \) is the resultant thrust on the fluid contained within the closed surface \( S \); \( \tilde{F}_{L, z} \) is the total Lorentz force exerted on the fluid inside volume \( V \) and on the surface of the electrode and the expression of \( \tilde{F}_{\eta, z} \) states the resultant force at the boundary arising from the shear stress. It should be noticed that in the above consideration also the effect of the pressure- and Lorentz forces upon the surface of the electrode is included. The other forces do not contribute at that place.

Therefore the magnitudes of the respective free parameters are in general a measure of the respective quantities \( \tilde{F}_{i, z} \), \( \tilde{F}_{p, z} \), \( \tilde{F}_{\eta, z} \) and may be regarded as the rate of exchange of momentum between the fluid inside a certain volume and the surrounding fluid outside that volume.

The inviscid expressions (5.139) – (5.141) clearly show that in the neighbourhood of the origin the inertia force balances the pressure force caused by the additional potential flow that must be allowed in that case in order to obtain a realistic inviscid solution in the near field \( \tilde{r} < 1 \). At larger radial distance the Lorentz force balances the inertia- and pressure forces.

In the viscous case, see (5.141) – (5.144), the expressions clearly demonstrate that in the neighbourhood of the origin the Lorentz force balances the generated
pressure and viscous forces and that the effect of the inertia force only plays a role at larger radial distance from the origin.

It is easy to verify that in both cases the series expansion solutions of the respective forces satisfy the momentum equation in integral form (5.134) identically in the respective powers of \( \tilde{r} \), as required.

The differences in the present expression of the Lorentz force (5.141) imparted to a certain volume of fluid around the origin and \( \tilde{F}_{L,o} \), \( \tilde{F}_{L,a} \), see (5.29)–(5.32), are caused by the difference in the local behaviour of the Lorentz force and by the fact that different volumina of fluid have been considered.

5.8. Discussion of chapter 5

In this chapter we examined in detail the behaviour of the fluid-flow and the electromagnetic field quantities in a semi-infinite configuration consisting of a disk electrode of radius \( a \) located in an insulating flat wall.

We have been able to show that the differential equations which govern the behaviour of the fluid motion at large radial distance \( \tilde{r} \) from the electrode become identical to the one found in the point electrode configuration for \( c_o = 0 \) when \( \tilde{r} \to \infty \).

It turned out that the inviscid- and viscous fluid motions in the neighbourhood of the electrode are determined by differential equations which have a form different from those for the far field \( \tilde{r} > 1 \). Moreover with one exception, the differential equations of the inviscid and viscous solutions in the near field, derived from Euler's and Navier-Stokes equation, appear to be linear, in contrast with the ones in the far field.

In addition in view of the order of the respective differential equations and the number of boundary conditions to be satisfied, the inviscid and viscous near field solutions turned out to contain free parameters. However the present calculations gained the conclusions that the solutions of the inviscid and viscous fluid motions in the neighbourhood of the origin in the near field \( \tilde{r} \leq 1 \) are bounded and moreover that they exist for every value of the parameter \( K \eta \). This shows that the near field behaviour differs essentially from the far field behaviour.

It has to be remarked that these results are in excellent agreement with numerical calculations of steady viscous fluid motions induced in a hemispheroidal container, carried out by Ajayi, Sozou & Pickering (1984). It turns out that in the case of an electrode of small radius and a nearly spherical container the fluid motion breaks down at a relatively small value of the hydrodynamic
Reynolds number, as usual. However, for a relatively shallow container and a large electrode numerical solutions of the induced fluid motion have been obtained at much higher values of Re. It is clear that this behaviour is caused by the fact that in the latter case the largest part of the fluid domain is situated in the so-called near field \( \tilde{r} < 1 \).

We also examined the behaviour and the direction of the Lorentz force, which generates the fluid motion and the pressure distribution in the medium and found that the hydrodynamic and magnetic Reynolds numbers have a typical local behaviour throughout the fluid domain.

Examination of the momentum equation in integral form showed that the behaviour of the generated fluid motion besides the Lorentz force is also determined by the rate of exchange of momentum between the flows in different regions of the fluid domain; i.e., by the form of the configuration.

The present calculations of the near and far field solutions of the inviscid and viscous fluid motions show an interesting feature. In chapters 3 and 4 we calculated the inviscid and viscous far field solutions and obtained unique expressions. In chapter 5 we calculated the inviscid and viscous near field solutions and found solutions which are not unique due to the appearance of a free parameter in each series expansion solution. This was caused by the absence of an additional boundary condition.

However when we consider the order of the governing differential equations in the far field, see (5.65), and the respective number of boundary conditions to be satisfied for \( K_\eta = 0 \) and \( K_\eta > 0 \), see (5.66) - (5.69), we observe also the shortage of one boundary condition.

Hence we have to conclude that in general the series expansion for the inviscid and viscous far field solutions also should contain each a free parameter, including the respective governing solutions for \( \tilde{r} \to \infty \). This discrepancy will be investigated in the next chapter 6.

Finally it should be noticed that the present configuration, or even a more general axisymmetric form, enables us to prove that the fluid-flow and electromagnetic field quantities become identical to zero in the fluid when the parameter \( K_b \) approaches infinity; i.e., when the fluid becomes perfectly electrically conducting.
6. THE GENERAL SOLUTIONS OF THE VISCOUS AND INVISCID FLUID MOTIONS IN THE POINT ELECTRODE CONFIGURATION

6.1. Introduction

In the previous chapter we have considered the near and far field behaviour of the inviscid and viscous flow fields generated by the injection of an electric current in a disk electrode configuration. We found that the solutions of the inviscid and viscous flow fields in the near field contain free parameters. This could be expected in view of the order of the governing differential equations, derived from the curl of the Navier-Stokes and Euler's equations, and of the respective number of boundary conditions to be satisfied.

Consideration of the order of the differential equations and the number of boundary conditions that govern the fluid-flow behaviour in the far field shows that in general the far field solutions should also contain free parameters. However the inviscid and viscous solutions obtained so far in the literature and in chapters 3 and 4 do not contain a free parameter.

To resolve this inconsistency we return to the semi-infinite point electrode model with a right circular cone of arbitrary apex-angle $\theta_0$. The behaviour of this model of configuration is identical to the farthest field behaviour of the disk electrode configuration in the special case of a flat wall ($\theta_0 = 90^\circ$, $c_0 = 0$).

The rederivation, presented in this chapter, from the governing curl of Navier-Stokes equation yields extended differential and integral equations of the functions $g(c)$ and $u(c)$ with additional terms multiplied by a free parameter $\gamma$. This procedure leads to general solutions of the inviscid and viscous fluid motions which contain the free parameter $\gamma$ to be expected on purely mathematical grounds.

In the case of the viscous fluid three additional solutions will be found which each contain a relatively weak singularity in the flow field at the axis of symmetry, resulting in an unbounded velocity along the axis of symmetry. These extra solutions yield inwards and outwards jet flows at different angles of $\theta$, including the axis of symmetry and the surface of the cone. The viscous solution with a bounded axial velocity at the axis of symmetry, as discussed in chapter 4, represents the special case $\gamma = 0$.

The general inviscid solution contains a weak singularity in the flow field at the surface of the cone in addition to the usual relatively weak singularity at the axis of symmetry. Only for a certain special value of the free parameter (i.e. when Shercliff's condition is satisfied) the extra singularity vanishes,
resulting in the inviscid solution as discussed in chapter 3.

The essential difference in the derivation of the governing equations carried out in chapter 2 and in this chapter is in the axial velocity along the axis of symmetry which will be allowed to become infinite here, while in chapter 2 it was implicitly assumed to remain bounded.

Consideration of the differential equations of the general inviscid and viscous solutions shows that the free parameter $\gamma$ represents the appearance of an additional point source of momentum located at the origin in the semi-infinite point electrode configuration or in terms of a configuration with an electrode of finite non-zero dimensions the exchange of momentum between fluid motions in different parts of the fluid domain.

Therefore in the general case the fluid motions are in fact generated by the Lorentz force due to the injection of an electric current and by the effect of exchange or of a source of momentum.

The reconsideration of the point electrode problem leads to general solutions of the inviscid and viscous fluid motions with additional weak singularities in the flow field.

In the viscous case even a solution is obtained which exists for all values of the hydrodynamic Reynolds number. Although the mass flow and the kinetic energy are bounded throughout the flow field, due to the appearance of the weak singularity in the flow field at the axis of symmetry, the viscous problem is not resolved yet.

Nevertheless the present examination of the inviscid and viscous problems yields a better insight in the mathematical behaviour and background, in the applicability of the similarity method at larger values of the hydrodynamic Reynolds number and especially in the way the respective problems may be reformulated or resolved.

### 6.2. Rederivation of the governing general equations

This section is concerned with the derivation of the differential and integral equations of the general solutions of the viscous and inviscid flow fields in the point electrode configuration.

In chapter 5 it appeared that in view of the order of the respective differential equations and the corresponding number of boundary conditions to be satisfied, the inviscid and viscous solutions should contain each a free parameter; the same applied to the solutions of the inviscid and viscous fluid motions in the neighbourhood of the disk electrode for $\bar{r} = \frac{r}{a} < 1$.
It will appear that the inviscid and viscous solutions of the point electrode model, being identical to the farthest field solutions of the disk electrode configuration for the special case \( c_o = 0 \), as shown in chapters 3 and 4, are special cases of the more general solutions to be presented here. Namely the inviscid solution of chapter 3 satisfies the so-called Shercliff condition, i.e. the velocity along the surface of the cone is bounded, which results in a special value of the free parameter \( \gamma \) and in the case of the viscous solution given in chapter 4 the corresponding free parameter is identical to zero.

For the present derivation of the general solutions the same governing differential equations are used as before; viz. the curl of the Navier-Stokes or Euler's equation (2.18) which determines the fluid motion generated by the injection of an electric current and the curl of Ohm's law (2.20) which represents the effect of the fluid motion upon the electric current distribution; in addition the respective integrated forms of the momentum equation (2.44), (2.45) and the corresponding boundary conditions (2.34) - (2.38) are used.

In general the derivations and investigations will be carried out for arbitrary values of \( K_b \), whereas analytical expressions and numerical calculations are calculated at small \( K_b (K_b \ll 1) \).

The only difference with the previous calculations is the fact that as far as the boundary conditions \( g(c_o) = g(1) = 0 \) permit, we do not assume here bounded velocities respectively along the axis of symmetry for \( K_\eta > 0 \) and along the surface of the cone for \( K_\eta = 0 \).

Since in the following section 6.3 the slow viscous flow will be examined first, it is necessary to investigate here the general behaviour of the viscous solution at \( \gamma \neq 0 \) rather extensively.

Integration of the curl of Navier-Stokes equation (2.18) respectively two and three times in succession, some further integrations by parts, and using (2.32) result in the expressions

\[
2g \frac{dg}{dc} = 2K_\eta \left\{(1-c^2) \frac{d^2g}{dc^2} + 2g\right\} + (1+c) \int \frac{f^2(t)}{c_o (1+t)^2} dt +
\]

\[
+ (1-c) \int \frac{f^2(t)}{c_o (1-t)^2} dt - \frac{2(c-c_o)}{1-c^2} + 2P_c + Q ,
\tag{2.44}
\]

and

\[
g^2 = 2K_\eta \left\{(1-c^2) \frac{dg}{dc} + 2cg\right\} + \frac{(1+c)^2}{2} \int \frac{f^2(t)}{c_o (1+t)^2} dt +
\]
\[
- \frac{(1-c)^2}{2} \int_{c_0}^{c} \frac{f^2(t)}{(1-t)^2} \, dt - \frac{(c-c_0)^2}{1-c_0^2} + P c^2 + Q c + S, \tag{2.45}
\]

where \( P, Q, S \) are constants of integration.

Substitution of the boundary conditions \( g(c_0) = 0 \) for \( K_\eta > 0 \) and \( \frac{dg}{dc}\bigg|_{c=c_0} = 0 \) for \( K_\eta > 0 \), see (2.34), (2.38), into (2.44) and (2.45) yields

\[
P c_0^2 + Q c_0 + S = 0, \tag{6.1}
\]

\[
2K_\eta (1-c_0^2) \left\{ \frac{d^2 g}{dc^2} \right\}_{c=c_0} = -2P c_0 - Q \quad \text{for } K_\eta > 0. \tag{6.2}
\]

Substitution of the boundary condition \( g(1) = 0 \) for \( K_\eta > 0 \), see (2.35), into (2.45) results in

\[
P + Q + S = \frac{1-c_0}{1+c_0} - 2 \int_{c_0}^{1} \frac{f^2(t)}{(1+t)^2} \, dt, \tag{6.3}
\]

where we assume that \( \lim_{c \to 1} (1-c^2) \frac{dg}{dc} = 0 \) as the boundary condition \( g(1) = 0 \) requires.

In contrast with the derivation in section 2.7, here no condition at \( c = 1 \) is set to equation (2.44), since \( \lim_{c \to 1} \left\{ g \frac{dg}{dc} \right\} = 0(1) \) in case of \( K_\eta = 0 \) and as we will find later on \( \lim_{c \to 1} \left\{ (1-c^2) \frac{d^2 g}{dc^2} \right\} = 0(1) \) for \( K_\eta > 0 \) and \( \gamma \neq 0 \). It has to be remarked that in section 2.7 the same equations (6.1) - (6.3) have been obtained. However the application of the assumption that the viscous velocity along the axis of symmetry is bounded provided an additional equation which made the expressions of \( P, Q, S \) unique, see (2.46) - (2.48). In the present derivation that assumption is not made, resulting in the appearance of a free parameter.

In order to obtain the known expression of \( G_\eta(c) \), see (2.51), which represents the effect of the Lorentz force, in the differential equation we may introduce without loss of generality

\[
P = \frac{1}{1-c_0^2} + \frac{2c_0}{(1-c_0)^2} \int_{c_0}^{1} \frac{f^2(t)}{1+t^2} \, dt - \gamma, \tag{6.4}
\]
where $\gamma$ is an arbitrary constant.

This procedure leads to the following expressions of $Q$ and $S$

\[
Q = -\frac{2c_o}{1-c_o^2} - \frac{2(1+c_o^2)}{(1-c_o^2)^2} \int_{c_o}^{t} \frac{f^2(t)}{c_o (1+t)^2} \, dt + \gamma(1+c_o), \tag{6.5}
\]

\[
S = \frac{c^2}{1-c^2} + \frac{2c}{(1-c^2)} \int_{c}^{t} \frac{f^2(t)}{c_o (1+t)^2} \, dt - \gamma c_o. \tag{6.6}
\]

Comparison of the above expressions of $P, Q, S$ with respectively the ones derived in sections 2.7 and 3.2, see (2.46) - (2.48), (3.1) - (3.3), indicates that the viscous solution found in chapter 4 agrees with the particular case $\gamma = 0$ and that in the case of the inviscid solution of chapter 3, the free parameter $\gamma$ has a special value.

Substitution of (6.4) - (6.6) into (2.45) and (6.2) results in the general Riccati differential equation of the function $g(c)$

\[
g^2 = 2K_\eta \left( (1-c^2) \frac{dg}{dc} + 2cg \right) - G_\eta(c) + \gamma(c-c_0)(1-c) \quad \text{for } K_\eta > 0, \tag{6.7}
\]

with

\[
\frac{d^2g}{dc^2} |_{c=c_0} = \frac{1}{2K_\eta (1-c_0^2)} \left[ \frac{dG_\eta}{dc} |_{c=c_0} - \gamma(1-c_0) \right] \quad \text{for } K_\eta > 0, \tag{6.8}
\]

where

\[
G_\eta(c) = \frac{2(c-c_0)(1-c_0c)}{(1-c_0^2)^2} \int_{c_0}^{c} \frac{f^2(t)}{c_o (1+t)^2} \, dt - \frac{(1+c)^2}{2} \int_{c_0}^{c} \frac{f^2(t)}{c_o (1+t)^2} \, dt + \frac{(1-c)^2}{2} \int_{c_0}^{c} \frac{f^2(t)}{c_o (1-t)^2} \, dt, \tag{2.51}
\]

\[
\frac{dG_\eta}{dc} |_{c=c_0} = \frac{2(1+c)}{(1-c_0^2)} \int_{c_0}^{c} \frac{f^2(t)}{c_o (1+t)^2} \, dt. \tag{6.9}
\]

In the above expressions the constant $\gamma$ represents the free parameter. Since in the present derivation no limit condition has been imposed to the velocity along
the axis of symmetry at \( c = 1 \), it is evident that the general Riccati differential equation (6.7) is valid for both viscous and inviscid fluids, i.e. \( K_\eta > 0 \), and that differentiating three times with respect to \( c \) yields again the expression of the curl of the Navier-Stokes equation (2.18). It should be noticed that the above differential equation (6.7) differs from the ones derived by Sozou (1971b) and Sozou & English (1972).

In the same way as done in section 5.7, it can be shown that the free parameter \( \gamma \) can be interpreted physically as a measure of the rate of exchange of momentum between the fluid at large radial distance and the fluid situated at other parts of the flow field, including a possible source of momentum at the origin. Therefore the present problem is similar to the problem of a steady jet generated by a point source of momentum, investigated by Landau (1944), Yatseyev (1950), Squire (1951) and reviewed by Whitham (1963) and Batchelor (1967).

Integration of equation (6.7) with respect to \( c \) leads to the momentum equation in integral form and after some further manipulation to the following integral expressions of the vector components of the viscous velocity field

\[
g(c) = \frac{(1-c^2)}{2K_\eta} c \int_{c_0}^c \frac{g^2(t) + G_\eta(t)}{(1-t^2)^2} \, dt + \]

\[+ \frac{\gamma}{8K_\eta} \left[ 2(c-c_0)(1-c) - (1-c_0)(1-c^2) \ln \left( \frac{(1+c)(1-c_0)}{(1-c)(1+c_0)} \right) \right], \tag{6.10}
\]

\[
\frac{dg}{dc} = -\frac{c}{K_\eta} \int_{c_0}^c \frac{g^2(t) + G_\eta(t)}{(1-t^2)^2} \, dt + \frac{g^2(c) + G_\eta(c)}{2K_\eta(1-c^2)} + \]

\[-\frac{\gamma}{4K_\eta} \left[ 2(c-c_0) - c(1-c_0) \ln \left( \frac{(1+c)(1-c_0)}{(1-c)(1+c_0)} \right) \right], \tag{6.11}
\]

for \( K_\eta > 0 \), where \( G_\eta(c) \) is given by (2.51).

Solutions of the fluid-flow and electromagnetic field quantities \( g(c) \) and \( f(c) \) at arbitrary but fixed values of \( K_\eta \), \( K_\eta \) and \( \gamma \) can be obtained now from (2.51), (2.59), (2.60), (6.7), (6.10), (6.11) by numerical computation. Since in the present considerations the free parameter \( \gamma \) enters the equations, the solutions of \( g(c) \) and \( f(c) \) are in general not unique, as found also in the near field of the disk electrode problem, see section 5.5.3.
Although the integral equation of $g(c)$ for $K_\eta > 0$ (6.10) seems to be suitable for numerical iteration, the examination of the behaviour of the function $g(c)$ and of the appearance of limit values of the parameters $K_\eta$, $K_\eta', \gamma$ makes it preferable to apply the usual transformation

$$g(c) = -\frac{2K_\eta(1-c^2)}{u}\frac{du}{dc}, \quad (4.77)$$

with the condition

$$u(c) > 0 \quad \text{on} \quad c_0 < c < 1, \quad (6.12)$$

in order to prevent a strong singular behaviour in the flow field as observed in section 4.3 for $K_\eta < K_\eta, \text{min}$.

This relation transforms the nonlinear differential equation of Riccati (6.7) into a linear second order differential equation for the function $u(c)$

$$\frac{d^2u}{dc^2} + \frac{\Phi(c)}{4K_\eta^2(1-c^2)^2} u(c) = 0, \quad (6.13)$$

with

$$\Phi(c) = G_\eta(c) - \gamma(c-c_0)(1-c). \quad (6.14)$$

In view of (2.34), (2.35), (2.38), (4.77) the function $u(c)$ has to satisfy the boundary conditions (4.79), (4.80) and from (2.57), (6.13), (6.14)

$$u(c_0) = 1, \quad \left. \frac{du}{dc} \right|_{c=c_0} = 0, \quad \left. \frac{d^2u}{dc^2} \right|_{c=c_0} = 0. \quad (6.15)$$

Hence the slope of $u(c)$ in the neighbourhood of $c = c_0$ is determined by the third order derivative yielding

$$\left. \frac{d^3u}{dc^3} \right|_{c=c_0} = -\frac{1}{2K_\eta(1-c_0^2)} \left. \frac{d^2g}{dc^2} \right|_{c=c_0}. \quad (6.16)$$

The expression of the derivative of $g(c)$, which follows from (4.77), (6.13), (6.14), now takes the form

$$\frac{dg}{dc} = \frac{2K_\eta(1-c^2)}{u^2} \left( \frac{du}{dc} \right)^2 + \frac{4K_\eta c}{u} \frac{du}{dc} + \frac{G_\eta(c)}{2K_\eta(1-c^2)} - \frac{\gamma(c-c_0)}{2K_\eta(1+c)}. \quad (6.17)$$
For arbitrary value of $K_b$, the function $G_\eta(c)$ is positive on $c_o < c < 1$, with a single zero at $c = c_o$ and a double zero at $c = 1$, see (2.55) - (2.57). Therefore the term multiplying $u(c)$ in the differential equation (6.13) is identical to zero at $c = c_o$ and positive and bounded on $c_o < c < 1$ when $\gamma = 0$, as discussed in chapter 4.

However in the case of arbitrary non-zero value of the free parameter $\gamma$, the term multiplying $u(c)$ can be positive and/or negative on the open interval, is identical to zero at $c = c_o$ and is singular for $c = 1$. This different behaviour leads to additional viscous flow solutions with different limit-values of the parameters $K_b$, $K_\eta$ and $\gamma$, which will be examined in detail in section 6.4.

A detailed investigation of the behaviour at $c = 1$ shows that the differential equation (6.13), (6.14) for $\gamma \neq 0$ reduces to the Coulomb wave equation which has a regular singularity at $c = 1$, see Abramowitz & Stegun (1972,p.538). As a consequence the general solutions of $u(c)$ approach the Coulomb wave functions when $c$ tends to unity. From the behaviour of these functions, a further extensive calculation has been carried out with double series expansions in powers of $1-c$ and $\ln(1-c)$, applied to the differential equation of $u(c)$ (6.13), (6.14), to the transformation (4.77), to the curl of Ohm's law (2.20), to the expression of $G_\eta(c)$ (2.51) and to the expression of the pressure distribution (2.64). The results of this examination show that the flow fieldquantities contain a relatively weak logarithmically singular behaviour at $c = 1$ of the form

$$u(c) = u(1) + \frac{\gamma(1-c_o)u(1)}{16\kappa^2} (1-c) \ln(1-c)(1+o(1)) ,$$  

(6.18)

$$\frac{du}{dc} = -\frac{\gamma(1-c_o)u(1)}{16\kappa^2} \ln(1-c)(1+o(1)) ,$$  

(6.19)

$$g(c) = \frac{\gamma(1-c_o)}{4K_\eta} (1-c) \ln(1-c)(1+o(1)) ,$$  

(6.20)

$$\frac{dg}{dc} = -\frac{\gamma(1-c_o)}{4K_\eta} \ln(1-c)(1+o(1)) ,$$  

(6.21)
\[
\nu_r = \frac{\mu I_o^2}{16\pi^2 \eta} \frac{\gamma(1-c_0)}{r} \ln(1-c)(1+o(1)), \quad (6.22)
\]

\[
\nu_\theta = -\frac{\mu I_o^2}{16\pi^2 \eta} \frac{\gamma(1-c_0)}{r} \sqrt{\frac{1-c}{1+c}} \ln(1-c)(1+o(1)), \quad (6.23)
\]

\[
\nu_s = \frac{\mu I_o^2}{16\pi^2 \eta} \frac{\gamma(1-c_0)}{r} \sqrt{\frac{1-c}{1+c}} \ln(1-c)(1+o(1)), \quad (6.24)
\]

\[
\nu_z = \frac{\mu I_o^2}{16\pi^2 \eta} \frac{\gamma(1-c_0)}{r} \ln(1-c)(1+o(1)), \quad (6.25)
\]

\[
p(r,c) = p_\infty + \frac{\mu I_o^2}{16\pi^2 r^2} \gamma(1-c_0) \ln(1-c)(1+o(1)), \quad (6.26)
\]

\[
f(c) = -(1-c) \left[ \frac{df}{dc} \right]_{c=1} - \frac{K_b \gamma(1-c_0)}{8K_\eta} \left[ \frac{df}{dc} \right]_{c=1} (1-c)^2 \ln(1-c)(1+o(1)), \quad (6.27)
\]

for \( c = 1 \), \( \gamma \neq 0 \) and \( K_\eta > 0 \).

The above expressions clearly indicate that the particular behaviour of the flow field quantities at \( c = 1 \) does not essentially affect the electromagnetic field quantities. Namely the magnetic field and the electric current distribution remain bounded throughout the field; only the second order derivative of \( f(c) \) now becomes logarithmically singular at \( c = 1 \).

The present weak singular behaviour does not violate the boundary conditions of the fluid-flow and electromagnetic field quantities, including \( E_s = 0 \) at \( c = 1 \), see (2.67) and it does not invoke inconsistencies like fluid sources or sinks.

The mass flow \( \Psi \) through a small circle of radius \( s = r \sqrt{1-c^2} \) around the axis of symmetry yields

\[
\Psi(r,c) = \frac{\rho \mu I_o^2}{8\pi \eta} r \gamma(1-c_0)(1-c) \ln(1-c)(1+o(1)), \quad (6.28)
\]
for \( c = 1 \), \( \gamma \neq 0 \) and \( K_\eta > 0 \), indicating that the mass flow remains finite and approaches zero when the radius of the circle tends to zero.

In addition from (3.66) it can be found that the kinetic energy of the generated viscous fluid motion is bounded throughout the flow field; despite the fact that the velocity along the axis of symmetry is singular at \( c = 1 \).

Since the viscous boundary conditions require: \( g(c_\circ) = 0 \) and \( \frac{dg}{dc} |_{c=c_\circ} = 0 \), the behaviour and the direction of the fluid motion near the surface of the cone are determined by the second order derivative \( \frac{d^2g}{dc^2} |_{c=c_\circ} \), see (6.8), (6.9). It is clear that for arbitrary value of \( \gamma \) the value of \( \frac{d^2g}{dc^2} \) may be positive or negative. Hence in the general case considered here, the viscous flow along the surface of the cone can be directed towards or away from the point electrode.

A special value of \( \gamma \), marked by \( \gamma^* \), is obtained when the second order derivative of \( g(c) \) becomes identical to zero at the cone surface, viz.

\[
\frac{d^2g}{dc^2} |_{c=c_\circ} = 0 \quad \text{for} \quad \gamma = \gamma^*
\]

where

\[
\gamma^* = \frac{1}{1-c_\circ} \frac{dg}{dc} |_{c=c_\circ} = \frac{2(1+c_\circ)}{(1-c_\circ)^2} \int \frac{f^2(t)}{c_\circ} \frac{1}{(1+t)^2} \, dt,
\]

for arbitrary value of \( K_b \). It is clear that \( \gamma^* \) is always positive. In case of small \( K_b \), the expression of \( \gamma^* \) takes the form

\[
\gamma^* = \frac{2(3+c_\circ)}{(1-c_\circ)^3} + \frac{8(1+c_\circ)}{(1-c_\circ)^4} \ln\left(\frac{1+c_\circ}{2}\right) \quad \text{for} \quad K_b \ll 1.
\]

The behaviour of \( \gamma^* \) as function of \( c_\circ \) for \( K_b \ll 1 \) is shown in figure 6.1.

Some special values at low \( K_b \) are: \( \gamma^* = 0.5 \) for \( c_\circ = -1 \), \( \gamma^* = 0.4548 \) for \( c_\circ = 0 \); whereas \( \gamma^* \) tends to infinity in the way \( \gamma^* = (3(1-c_\circ))^{-1}(1+\circ(1)) \) when \( c_\circ \) approaches unity. Moreover it turns out that \( \gamma^* = 0.5 \) for \( c_\circ = -1 \) is valid for all values of \( K_b \).

From (6.8), (6.9), (6.29) it follows now that in the case \( \gamma < \gamma^* \) the fluid motion along the surface of the cone is always directed towards the point electrode, whereas for \( \gamma > \gamma^* \) the flow is in opposite direction.

In the special case \( \gamma = \gamma^* \), the direction of the flow near the cone surface is determined by the third order derivative of \( g(c) \), viz.
Figure 6.1. The behaviour of $\gamma^*$ as function of $c_0$ in case of small value of $K_b$

\[
\frac{d^3 \gamma}{dc^3} \bigg|_{c=c_0} = \frac{1}{K_\eta(1-c_0^2)^2} \left( -1 + \gamma^*(1-c_0) \right) \quad \text{for } \gamma = \gamma^*, \tag{6.31}
\]

where $\gamma^*$ is given by (6.29), (6.30).

It can be shown that the third order derivative at $c = c_0$ and $\gamma = \gamma^*$ is negative when one of the following conditions is satisfied:

(i) $f(c)$ is a monotonically decreasing and non-negative function on $c_0 < c < 1$,
(ii) when $|f(c)| < 1$ on $c_0 < c < 1$ in case of low $K_b$,
(iii) in general when $\int_{c_0}^{1} \frac{f^2(t)}{(1+t)^2} \, dt < \frac{1-c_0}{2(1+c_0)}$. 
Hence especially in the case of $K_b \ll 1$, to be considered in section 6.4, we may conclude that also in case of $\gamma = \gamma^*$ the viscous fluid motion along the surface of the cone is directed from the point electrode, as for $\gamma > \gamma^*$. In view of the completely or partially oscillatory or exponential behaviour of the solutions of the differential equation of $u(c)$, see (6.13), (6.14), depending upon the sign of $\Phi(c)$ on the interval $c_0 < c < 1$ and related to the appearance of lower bounds of $K_\eta$ and $\gamma$, we distinguish four different cases: $\gamma < 0$, $\gamma = 0$, $0 < \gamma < \gamma^*$, $\gamma > \gamma^*$, which will be examined in detail in section 6.4.

We conclude this section with a discussion of the two limit cases of the configuration: $c_0 = \pm 1$.

Since in (6.18) - (6.27) respectively $\gamma = 0$ or $c_0 = 1$ leads to the same result, we conclude from the calculation of the viscous solution at $\gamma = 0$, carried out in section 4.3, see (4.86), and from (4.73), (6.7) that

$$g(c) = 0 \quad \text{for} \quad c_0 = 1, K_\eta > 0,$$

(6.32)

and arbitrary value of $\gamma$.

The investigation of the behaviour of the viscous flow solution for $c_0 = -1$ and for arbitrary value of $K_b$ shows some interesting features.

Division of the Riccati differential equation (6.7) by $1 - c^2$ and using the boundary condition of the magnetic field on the surface of the cone $f(-1) = 1$, clearly demonstrate that the viscous boundary conditions $g(-1) = 0$ and $\left. \frac{dg}{dc} \right|_{c=-1} = 0$ can be satisfied simultaneously only if $\gamma = \gamma^* = \frac{1}{2}$. As mentioned before $\gamma^* = \frac{1}{2}$ at $c_0 = -1$ for all values of $K_b$; $0 < K_b < \infty$, as appears also from (6.29), (6.35).

Some further examination of the Riccati differential equation and Ohm's law (2.20) yields the behaviour of the functions $g(c)$, $u(c)$, $f(c)$ in the neighbourhood of the boundary, viz.

$$g(c) = -\frac{1}{4K_\eta} \left. \frac{df}{dc} \right|_{c=-1} (1+c)^2 \ln(1+c)(1+o(1)),$$

(6.33)

$$u(c) = 1 + \frac{1}{32K_b^2} \left. \frac{df}{dc} \right|_{c=-1} (1+c)^2 \ln(1+c)(1+o(1)),$$

(6.34)

$$f(c) = 1 + (1+c) \left. \frac{df}{dc} \right|_{c=-1} - \frac{3K_b}{16K_\eta} \left. \frac{df}{dc} \right|_{c=-1} (1+c)^2 \ln(1+c)(1+o(1)),$$

(6.35)
for $c = -1$ at $c_o = -1$, $\gamma = \gamma^* = \frac{1}{2}$, $0 < K_b < \infty$.

This distinct behaviour at $c = -1$, $c_o = -1$ shows that not all conditions, derived before, can be valid in this particular case. The main differences are: the second order derivatives of $g(c)$ and $u(c)$ at $c_o = -1$, $\gamma = \gamma^* = \frac{1}{2}$, have a weakly logarithmic singularity at $c = -1$, instead of being identical to zero as in the case $-1 < c_o < 1$, see (6.15), (6.29).

From the behaviour of $f(c)$ at $c = \pm 1$ for arbitrary value of $K_b$, see (6.27), (6.35), we conclude that the functions $f(c)$ and $\frac{df}{dc}$ and as a consequence the magnetic field and the electric current distribution are bounded in the entire fluid domain, including the axis of symmetry $c = 1$. Nevertheless as expected, the magnetic field $B_\phi$ contains a relatively weak singularity at $c = -1$, see (2.15), (2.32), due to the infinitely thin filament which carries a total electric current $I_o$ towards the point electrode at the apex of the cone of zero vertex-angle ($\theta_o = \pi$).

Integration of the Riccati differential equation (6.7) with the general expression of $G_\eta(c)$ (2.51) at $c_o = -1$ results in integral equations of $g(c)$ and $\frac{dg}{dc}$ for $c_o = -1$, $\gamma = \gamma^* = \frac{1}{2}$, $0 < K_b < \infty$. Since these expressions do not contain any term which become singular at the endpoints $c = \pm 1$ (besides the usual one of $\frac{dg}{dc}$ at $c = 1$ for $\gamma \neq 0$), we conclude that in this special case a bounded solution of $g(c)$ exists, viz.

$$|g(c)| < \infty \quad \text{on} \quad -1 < c < 1, \quad (6.36)$$

for $c_o = -1$, $\gamma = \gamma^* = \frac{1}{2}$, and in principle for all values of $K_b$, $0 < K_b < \infty$, as far as the eigenvalues of this boundary value problem permit.

Also it turns out that in case of low $K_b \ll 1$, the function $g(c)$ exists for all values of $K_\eta$ and satisfies

$$-\infty < g(c) < 0 \quad \text{on} \quad -1 < c < 1, \quad 0 < K_\eta < \infty, \quad (6.37)$$

for $c_o = -1$, $\gamma = \gamma^* = \frac{1}{2}$, $K_b \ll 1$.

In this particular case the generated viscous fluid motion consists of a flow along the entire axis of symmetry in $\frac{1}{2}$ directions; the case $\gamma > \gamma^*$ is similar and will be discussed in section 6.4.

With the general equations, expressions and relations, derived in this section, in the sections 6.3 - 6.5 we will investigate successively the general slow viscous solution at $K_\eta \gg 1$, the general viscous solution at arbitrary value of $K_\eta$ for the four different cases: $\gamma < 0$, $\gamma = 0$, $0 < \gamma < \gamma^*$, $\gamma > \gamma^*$, and the
general inviscid solution. In general the investigations of the general beha-
viour will be taken at arbitrary values of \( K_b \), whereas the actual analytical
and numerical calculations will be executed at low value of \( K_b \).

6.3. The general slow viscous solution

In this section we briefly consider the general solution of the viscous fluid
motion at arbitrary value of the free parameter \( \gamma \) and at large value of \( K_\eta \);
i.e. at low value of the hydrodynamic Reynolds number \( Re \). The special case
\( \gamma = 0 \) has been discussed in section 4.2.

Substitution of the series expansion in negative powers of \( K_\eta \) (4.1) into the
integral equation (6.10) yields the slow viscous flow solution at arbitrary
values of \( K_b \) and \( \gamma \). The expression of the governing solution \( g_0(c) \) of the
series expansion consists of the summation of the right-hand side of (4.4) and
of the term multiplied by \( \gamma \) of (6.10). The higher order solutions for \( n =
1, 2, 3, \ldots \) satisfy (4.3)

In the case of low \( K_b \), \( K_b < 1 \), the basic solution \( g_0(c) \) and the derivative
become

\[
g_0(c) = a_0 + a_1 c + a_2 c^2 + a_3 (1+c) \ln(1+c) + a_4 (1+c)^2 \ln(1+c) +
\
+ a_4 (1-c^2) \ln(1-c),
\]

(6.38)

\[
\frac{dg_0}{dc} = (a_1 + a_3) + 2 a_2 c + a_3 \ln(1+c) + 2 a_4(1+c) \ln(1+c) - 2 a_4 c \ln(1-c),
\]

(6.39)

where

\[
a_0 = \frac{c_0}{(1-c_0)^3} - \frac{(1-2c_0 - c_0^2)}{(1-c_0)^4} \ln(1+c_0) - \frac{2c_0^2}{(1-c_0)^4} \ln(2) +
\
- \frac{\gamma c_0}{4} + \frac{\gamma(1-c_0)}{8} \ln \left( \frac{1+c_0}{1-c_0} \right),
\]

(6.40)

\[
a_1 = \frac{(1+c_0)}{(1-c_0)^3} - \frac{(1+c_0)^2}{(1-c_0)^4} \ln(1+c_0) + \frac{4c_0}{(1-c_0)^4} \ln(2) + \frac{\gamma(1+c_0)}{4},
\]

(6.41)
\[ a_2 = \frac{1}{(1-c_o)^3} + \frac{2}{(1-c_o)^4} \ln(1+c_o) - \frac{2}{(1-c_o)^4} \ln(2) + \]
\[ - \frac{\gamma}{4} - \frac{\gamma(1-c_o)}{8} \ln \left( \frac{1+c}{1-c_o} \right), \] (6.42)

\[ a_3 = \frac{1}{(1-c_o)^2} - \frac{\gamma(1-c_o)}{4}, \] (6.43)

\[ a_4 = \frac{\gamma(1-c_o)}{8}, \] (6.44)

for \( K_\eta > 1 \), \( K_b < 1 \) and arbitrary value of \( \gamma \).

It can be verified easily that the above expressions satisfy the boundary conditions
\[ \frac{dg_o}{dc_o} = 0, \left. \frac{dg_o}{dc} \right|_{c = c_o} = 0, \quad g_o(1) = 0 \] for \(-1 < c_o < 1\. In the same way as the general solution at arbitrary value of \( K_\eta \), the present slow viscous solution for \( K_\eta > 1 \) possesses at \( c = 1 \) a relatively weak, logarithmic singularity for \( \gamma \neq 0 \), see (6.20), (6.21).

For some special values of \( c_o \) the expression of \( g_o(c) \) takes the form

\[ g_o(c) = 0 \quad \text{for} \quad c_o = 1, \quad -\infty < \gamma < \infty, \] (6.45)

\[ g_o(c) = \left[ \frac{(c-c_o)^2(1-c)}{24(1-c_o)^2} + \frac{\gamma(1-c)}{4} \left\{ (c-c_o) + (1-c_o) \ln \left( \frac{1+c}{1-c_o} \right) \right\} \right] (1+o(1)) \]

for \( c_o = 1 \), (6.46)

\[ g_o(c) = -c(1-c) - 2 \ln(2)c^2 + (1+c) \ln(1+c) + \]
\[ + \frac{\gamma}{8} \left\{ 2c(1-c) - (1-c^2) \ln \left( \frac{1+c}{1-c} \right) \right\} \quad \text{for} \quad c_o = 0, \] (6.47)

\[ g_o(c) = -\frac{1}{8} \left[ 1-c^2 + (1-c^2) \ln(1+c_o) + (1+c)^2 \ln(2) + \right. \]
\[ - 2(1+c) \ln(1+c) - 2\gamma \left\{ 1-c^2 + (1-c^2) \ln\left( \frac{1+c}{2} \right) + \right. \]

\[ + \frac{\gamma}{8} \left\{ 2c(1-c) - (1-c^2) \ln \left( \frac{1+c}{1-c} \right) \right\} \right. \quad \text{for} \quad c_o = 0, \] (6.47)
\[-(1-c^2) \ln\left(\frac{1+c}{1-c}\right)\right] (1+o(1)) \quad \text{for } c_o = -1, \quad (6.48)\]

\[g_o(c) = +\infty \quad \text{on } -1 < c < 1 \quad \text{for } c_o = -1, \gamma < \frac{1}{2}, \quad (6.49)\]

\[g_o(c) = -\infty \quad \text{on } -1 < c < 1 \quad \text{for } c_o = -1, \gamma > \frac{1}{2}, \quad (6.50)\]

\[g_o(c) = \frac{(1+c)}{8} \left[(1+c) \ln\left(\frac{1+c}{2}\right) + (1-c) \ln\left(\frac{1-c}{2}\right)\right] \quad \text{for } c_o = -1, \gamma = \frac{1}{2}. \quad (6.51)\]

Likewise in the general case at arbitrary value of $K_\eta$, as considered in the previous section, it appears that a bounded solution of the slow viscous fluid motion at $c_o = -1$ exists only if $\gamma = \gamma^* = \frac{1}{2}$. This solution (6.51) is negative on the interval, viz. $-\infty < g_o(c) < 0$ on $-1 < c < 1$ for $c_o = -1, \gamma = \gamma^* = \frac{1}{2}$, $K_b < 1$, see (6.36). In addition it should be noticed that the expression (6.45) is only valid if $\gamma = 0((1-c_o)^{\alpha})$ with $\alpha > -2$ for $c_o = 1$.

The basic solution of the series expansion of the pressure distribution, see (4.7), is at arbitrary value of $\gamma$ of the form

\[p_o(c) = -(a_3 - 6a_4) - 4a_2c - 4a_4c \ln\left(\frac{1+c}{1-c}\right), \quad (6.52)\]

for $K_\eta > 1$, $K_b < 1$ and arbitrary value of $\gamma$, where $a_2 - a_4$ are given by (6.42) - (6.44).

It can easily be verified that the governing slow viscous solutions $g_o(c)$ and $p_o(c)$ possess the identical weakly logarithmically singular behaviour at the axis of symmetry $c = 1$ for $-1 < c_o < 1$, as the general solutions at arbitrary value of $K_\eta$, see (6.20) - (6.26).

In the special case $c_o = -1$ and $\gamma = \gamma^* = \frac{1}{2}$ the above expression of the base solution of the pressure distribution reduces to

\[p_o(c) = \frac{3}{4} - \frac{c}{2} \ln\left(\frac{1+c}{1-c}\right) \quad \text{for } c_o = -1, \gamma = \frac{1}{2}, \quad (6.53)\]

indicating that $p_o(c)$ is weakly singular at both endpoints $c = \pm 1$ for $c_o = -1$.

The slow viscous solutions with as little singular behaviour as possible are of course the solutions with $\gamma = 0$ for $-1 < c_o < 1$, as considered in section 4.2,
and with $\gamma = \frac{1}{2}$ for $c_o = -1$. At small value of $K_b$, the $\gamma = 0$ solution consists of an incoming flow along the surface of the right circular cone, which is turned off in the neighbourhood of the electrode into an outwards jet flow along the axis of symmetry; whereas the special solution for $c_o = -1$ with $\gamma = \frac{1}{2}$ represents a flow in opposite direction with a weakly singular behaviour at the axis of symmetry, respectively of the velocity field for $z > 0$ and of the pressure distribution for $z > 0$ and $z < 0$. The transition between these two different flows is formed by the solutions with $0 < \gamma < \gamma^*$ for $-1 < c_o < 1$. The latter solutions, with a weak singularity at $c = 1$, consist of incoming flows along the axis of symmetry and the surface of the cone wall and of an outwards jet flow at a certain angle $\theta_2$, on $0 < \theta_2 < \theta_0$, see figure 6.4.

It has to be remarked that in the special case $c_o = -1$ the behaviour of the fluid motion in the point electrode model is no longer identical to the far field behaviour of a disk electrode configuration where the side of the cylindrical electrode of non-zero radius $a$ is electrically insulated. Therefore we may conclude that the observed difference between the slow viscous flows for $-1 < c_o < 1$ and $c_o = -1$ is caused by the weak singularity in the magnetic field and the Lorentz force at $c = -1$, see (2.15), (2.63), which occurs in the point electrode configuration but not in the disk electrode model, as described above.

It is noteworthy that in the inviscid case no inversion of the direction of the fluid motion occurs for $c_o = -1$, see (3.14), (3.43).

### 6.4. The general viscous solutions

In this section we consider the general solutions of the viscous fluid motion at arbitrary values of $K_\eta$ and the free parameter $\gamma$ and in general at low value of $K_b$. Besides the solution for $\gamma = 0$, as found in section 4.3, here three additional viscous solutions for special values of $\gamma$ will be derived. These extra solutions provide different flow patterns.

The behaviour of the viscous solutions is determined by the Riccati differential equation (6.7), where $G_\eta(c)$ is given by (2.51) and by (4.65) - (4.69) in case of low $K_b$.

Although the integral equations of $g(c)$ and $\frac{dg}{dc}$, see (6.10), (6.11), seem suitable for iteration, it is preferable in most cases to apply the usual integral equations of the functions $u(c)$ and $\frac{du}{dc}$ for the numerical computation. These expressions, derived by respectively one and two times integration with respect to $c$ of the differential equation of $u(c)$, see (6.13), (6.14), yield
\[ u(c) = 1 - \frac{1}{4K^2} \int_{c_0}^{c} \frac{(c-t) G_\eta(t) u(t)}{(1-t^2)^2} \, dt + \frac{\gamma}{4K^2} \int_{c_0}^{c} \frac{(c-t)(t-c_0) u(t)}{(1+t)^2(1-t)} \, dt, \quad (6.54) \]

\[ \frac{du}{dc} = -\frac{1}{4K^2} \int_{c_0}^{c} \frac{G_\eta(t) u(t)}{(1-t^2)^2} \, dt + \frac{\gamma}{4K^2} \int_{c_0}^{c} \frac{(t-c_0) u(t)}{(1+t)^2(1-t)} \, dt, \quad (6.55) \]

where \( u(c_0) = 1 \), see (6.15). It is clear that the integrands of the above integral equations easily can be expressed in the function \( \Phi(c) \), as defined by (6.14).

The corresponding behaviour of the functions \( g(c) \) and \( \frac{dg}{dc} \), representing the angular behaviour of the velocity field, is obtained now from the transformation (4.77) and the expression (6.17).

In the present boundary value problem the existence of the viscous solutions at arbitrary values of \( K_b, K_\eta \), and \( \gamma \) is restricted in general to certain eigenvalues of the parameters. The appearance of these limit values of the parameters \( K_b, K_\eta \) and \( \gamma \) is demonstrated clearly by the differential equation of \( u(c) \), see (6.13), (6.14), in contrast with the Riccati differential equation of \( g(c) \) (6.7) or the integral equation of \( g(c) \) (6.10). This statement is clarified by the following considerations.

In particular it is the sign or the change of sign on the interval \( c_0 < c < 1 \) of the function \( \Phi(c) \), appearing in the second order differential equation of \( u(c) \), which determines whether the behaviour of the solution of \( u(c) \) is respectively completely or partially, oscillatory and/or exponential (nonoscillatory) on the interval. In this context it is important to remember that \( G_\eta(c) > 0 \) on \( c_0 < c < 1 \), with \( G_\eta(c_0) = 0(c-c_0) \), and \( G_\eta(1) = 0((1-c)^2) \), so that the sign of \( \Phi(c) \) strongly depends on the sign and the value of the free parameter \( \gamma \).

In addition the behaviour of the solution of \( u(c) \) provides the behaviour and the existence of the solution of \( g(c) \), as it follows from the transformation (4.77). In this regard it is clear that the solution of \( g(c) \) can exist only if the condition \( u(c) > 0 \) on \( c_0 < c < 1 \) is satisfied, see (6.12). If \( u(c) = 0 \) on \( c_0 < c < 1 \), a strong singular behaviour, including the occurrence of fluid sources or sinks, enters the viscous flow field, as found in section 4.4 for \( \gamma = 0 \). Therefore the limit values and bounds of the parameters are obtained when \( u(1) = 0 \) with \( u(c) > 0 \) on \( c_0 < c < 1 \) or in general when \( u(c_1) = 0 \) for \( c_0 < c_1 < 1 \) with \( u(c) > 0 \) for \( c \neq c_1 \).

On the surface of the cone the boundary conditions and some further relations
require: $g(c_0) = 0$, \( \frac{dg}{dc}_{c=c_0} = 0 \), $u(c_0) = 1$, \( \frac{du}{dc}_{c=c_0} = 0 \), \( \frac{d^2u}{dc^2}_{c=c_0} = 0 \),

for $-1 < c_0 < 1$, see (2.34), (2.38), (6.15), so that the behaviour of $g(c)$ and $u(c)$ in that neighbourhood is determined by \( \frac{d^2g}{dc^2}_{c=c_0} \) and \( \frac{d^3u}{dc^3}_{c=c_0} \). From (6.16) and (6.29) it is clear that these functions change sign for $\gamma < \gamma^*$ and $\gamma > \gamma^*$ and that they become identical to zero for $\gamma = \gamma^*$; introducing a special positive value of the free parameter: $\gamma^*$, see (6.29), (6.30).

In view of the above considerations for arbitrary value of $\gamma$, we have to distinguish between the following four different cases: $\gamma < 0$, $\gamma = 0$, $0 < \gamma < \gamma^*$, $\gamma > \gamma^*$. Only for completeness we also include here the $\gamma = 0$ case, being already examined extensively in section 4.3.

6.4.1. Case 1: $\gamma < 0$, oscillatory solution

Since $G_\eta'(c) > 0$ on $c_0 < c < 1$, with $G_\eta(c_0) = 0(c-c_0)$ and $G_\eta(1) = 0((1-c)^2)$, see (2.56), (2.57), it follows from (6.14) that in case of $\gamma < 0$

$$\Phi(c) = G_\eta'(c) - \gamma(c-c_0)(1-c) > 0 \quad \text{on } c_0 < c < 1,$$

(6.56)

with $\Phi(c_0) = \Phi(1) = 0$ for arbitrary value of $K_B$.

This fact implies that the solution of $u(c)$ is oscillatory; in about the same way as found in the case $\gamma = 0$ considered in section 4.3. Some further analysis of (6.54) - (6.56) shows that $\frac{du}{dc} < 0$ on $c_0 < c < 1$ implying that $u(c)$ is a monotonically decreasing function on the interval. When the condition (6.12) to prevent strong singular behaviour in the flow field is satisfied, $u(c)$ decreases from unity at $c = c_0$ to a certain positive value at $c = 1$, viz. $0 < u(1) < 1$; whereby $\frac{du}{dc}_{c=1} = -\infty$ as it follows from (6.19).

From the above consideration it is clear that the parameters $K_\eta$ and $\gamma$ (at fixed e.g. low value of $K_B$) reach their limit values when $u(1)$ becomes identical to zero, i.e.

$$K_\eta = K_{\eta, \text{min}}^{(i)} \quad \text{at fixed } \gamma < 0$$

or

$$K_\eta = K_{\eta, \text{min}}^{(i)} \quad \text{for } i = 1, 2$$
\[ \gamma = \gamma_{\text{min}} \quad \text{at fixed } K_\eta > 0 \]

when

\[ u(c) > 0 \quad \text{on } c_0 < c < 1 \]

\[ u(1) = 0 \]

where \( i \) denotes the number of the case \((i = 1 \text{ for } \gamma < 0 \text{ and } i = 2 \text{ for } \gamma = 0)\).

For respectively fixed values of \( K_\eta \) or \( \gamma \) the oscillatory solution of \( u(c) \) satisfies the condition \( u(c) > 0 \) on \( c_0 < c < 1 \), see (6.12), and as a consequence the solutions of \( g(c) \) and \( \frac{dg}{dc} \) exist if

\[ K_{\eta,\text{min}}^{(1)} < K_\eta < \infty \quad \text{at fixed value of } \gamma < 0 \]

or

\[ \gamma_{\text{min}}^{(1)} < \gamma < 0 \quad \text{at fixed value of } K_\eta > 0 \] (6.58)

In view of the calculations carried out in section 4.3, see figure 4.4, it is clear that the respective values of \( K_{\eta,\text{min}}^{(1)} \), \( \gamma \) and of \( K_{\eta,\text{min}}^{(1)} \) strongly depend on the value of \( c_0 \). Moreover since a negative value of \( \gamma \) of larger magnitude makes the solution of \( u(c) \) to oscillate more rapidly, it is obvious from (6.13), (6.14), (6.56) that \( K_{\eta,\text{min}}^{(1)} > K_{\eta,\text{min}}^{(2)} \), where \( K_{\eta,\text{min}}^{(2)} \) is identical to the lower bound \( K_{\eta,\text{min}} \) as found in section 4.3 for the case \( \gamma = 0 \).

Solutions of \( u(c) \) that satisfy (6.58) yield \( \frac{du}{dc} < 0 \) and \( u(c) > 0 \) on \( c_0 < c < 1 \) and in consequence \( g(c) > 0 \) on \( c_0 < c < 1 \) as it follows from the transformation (4.77). Therefore in this case the viscous fluid motion shows the usual behaviour of an incoming flow along the surface of the cone and an outwards jet flow along the axis of symmetry. The only difference with the viscous solution for \( \gamma = 0 \), considered in section 4.3, is the fact that the present solution breaks down at a larger lower bound of \( K_\eta \) and moreover that the viscous flow field contains here a relatively weak, logarithmic singularity at the axis of symmetry. The expressions (6.25), (6.28) clearly show that \( v_z(r,1) = +\infty \), whereas the resulting mass flow through a small circle perpendicular to and around the axis of symmetry is bounded and approaches zero when the radius of the circle tends to zero; even the kinetic energy of the generated viscous fluid motion is bounded.

In figure 6.2 the behaviour of the functions \( u(c) \) and \( g(c) \) on \( c_0 < c < 1 \) and the corresponding generated viscous fluid motion are presented for \( K_\eta = 10^{-1} \).
Figure 6.2. The behaviour of the functions $u(c)$ and $g(c)$ on $c_o < c < 1$ and the generated viscous fluid motion for $K_\eta = 10^{-1}$, $K_b \ll 1$, $c_o = 0$, $\lambda = \gamma/\gamma^* = -0.2$. 
\[ K_b \ll 1, c_o = 0, \lambda = -0.2, \] where for simplicity the value of \( \gamma \) is related to the value of \( \gamma^* \), see (6.29), (6.30), by the ratio

\[ \lambda = \frac{\gamma}{\gamma^*}. \] (6.59)

In section 4.3 it has been observed that the edge of the jet flow along the axis of symmetry tends to zero when \( K_\eta \) approaches the lower bound \( K_{\eta, min} \), see figure 4.5. In this regard it will be useful to compare the respective edges for \( K_\eta = 10^{-1}, c_o = 0 \) and respectively \( \gamma < 0 (\lambda = -0.2) \) and \( \gamma = 0 (\lambda = 0) \), see figures 6.2 and 6.3.

Numerical calculation of the minimum value of \( \lambda \) at \( K_\eta = 10^{-1}, c_o = 0, K_b \ll 1 \) yields \( \lambda_{min}^{(1)} = -0.231670 \) which corresponds to \( \gamma_{min}^{(1)} = -0.105369 \), since \( \gamma^* = 0.454823 \) at \( c_o = 0 \) and \( K_b \ll 1 \).

It should be noticed that also in this case the respective minimum values of \( \gamma \) (or \( \lambda \)) and \( K_\eta \) easily can be calculated by an analogous procedure, as described in appendix C.

6.4.2. Case 2: \( \gamma = 0 \), oscillatory solution

This special case already has been examined extensively in sections 4.3 and 4.4. In contrast with the other cases the velocity along the axis of symmetry is now bounded there. Only for completeness and comparison the figure 6.3 has been added here to show the behaviour of the functions \( u(c) \) and \( g(c) \) and the corresponding viscous fluid motion for \( K_\eta = 10^{-1}, K_b \ll 1, c_o = 0, \lambda = \gamma = 0 \).

6.4.3. Case 3: \( 0 < \gamma < \gamma^* \), partially oscillatory and exponential solution

In this case the free parameter \( \gamma \) satisfies \( \gamma > 0 \) and \( \gamma < \gamma^* \) where \( \gamma^* = \frac{1}{1-c_o} \cdot \frac{dG}{dc} \bigg|_{c = c_o}, \) see (6.29). Hence with \( G_\eta(c) > 0 \) on \( c_o < c < 1 \) and \( G_\eta(c_o) = 0, G(1) = 0(1-c)^2 \), see (2.56), (2.57), it can be shown easily that \( \Phi(c) > 0 \) at \( c = c_o \) and \( \Phi(c) < 0 \) at \( c = 1 \) for arbitrary value of \( K_b \).

In view of the behaviour of \( G_\eta(c) \) as described in section 2.7 below (2.57) at low \( K_b \) or in the absence of electric current inversion, viz. \( \frac{df}{dc} < 0 \) on \( c_o < c < 1 \), see (2.41) and in addition from the behaviour of the function multiplying \( \gamma \) in (6.14) it follows that the function \( \Phi(c) \) has only one zero on the open interval, say at \( c = c_1 \), viz.
Figure 6.3. The behaviour of the functions $u(c)$ and $g(c)$ on $c_0 < c < 1$ and the generated viscous flow field for $K_\eta = 10^{-1}$, $K_b \ll 1$, $c_o = 0$, $\lambda = \gamma = 0$. 
\( \Phi(c) = G_\eta(c) - \gamma(c-c_0)(1-c) \),

with

\[
\begin{align*}
\Phi(c) &> 0 & \text{on } c_0 < c < c_1, \\
\Phi(c) &< 0 & \text{on } c_1 < c < 1, \\
\Phi(c) &= 0 & \text{at } c = c_0, c_1, 1,
\end{align*}
\]

for \( 0 < \gamma < \gamma^* \) and low value of \( K_b \) or \( \frac{df}{dc} < 0 \) on \( c_0 < c < 1 \).

It has to be remarked that \( \Phi(c) \) may have more than one zero on \( c_0 < c < 1 \) at arbitrary value of \( K_b \); especially when electric current inversion occurs.

On the assumption that the solution of \( g(c) \) exists, i.e. that the condition \( u(c) > 0 \) on \( c_0 < c < 1 \) is satisfied, see (6.12), it follows from the differential equation of \( u(c) \) that \( \frac{d^2u}{dc^2} = 0 \) with \( \frac{du}{dc} < 0 \) on \( c_0 < c < c_1 \) and \( \frac{d^2u}{dc^2} > 0 \) on \( c_1 < c < 1 \). Therefore the behaviour of the solution of \( u(c) \) is oscillatory and concave on the interval \( c_0 < c < c_1 \) and nonoscillatory and convex on \( c_1 < c < 1 \).

From respectively (6.55) and (6.60), the present behaviour of \( u(c) \) on the interval and \( \frac{du}{dc} \bigg|_{c=1} = +\infty \) for \( u(1) > 0 \), see (6.19), it is clear now that at a certain point \( c_2 > c_1 \), \( \frac{du}{dc} \bigg|_{c=c_2} = 0 \) with \( \frac{du}{dc} < 0 \) on \( c_0 < c < c_2 \) and \( \frac{du}{dc} > 0 \) on \( c_2 < c < 1 \), where \( c_0 < c_1 < c_2 < 1 \). Since \( \frac{d^2u}{dc^2} > 0 \) on \( c_1 < c < 1 \), it is obvious that \( u(c) \) can have only one minimum on the interval, i.e. at \( c = c_2 \).

In this case the function \( g(c) \) is no longer positive on the entire open interval, viz. \( g(c) > 0 \) on \( c_0 < c < c_2 \) and \( g(c) < 0 \) on \( c_2 < c < 1 \) with \( g(c) = 0 \) at \( c = c_0, c_2, 1 \), as the transformation (4.77) indicates.

The above consideration implies a flow pattern different from the one usually found; namely incoming flows along the surface of the cone and along the axis of symmetry, which both are turned off in the neighbourhood of the electrode into an outwards jet flow at some angle \( \theta = \theta_2(c = c_2) \), \( 0 < \theta_2 < \theta_0 \), see figure 6.4.

As usual for non-zero values of the parameter \( \gamma \), the velocity of the incoming flow along the axis of symmetry is weakly singular at that place: \( v_z(r,1) = -\infty \), see (6.25).
In figure 6.4 the behaviour of the functions $u(c)$ and $g(c)$ on $c_0 < c < 1$ and the corresponding viscous fluid motion are shown for $K_\eta = 10^{-2}$, $K_b \ll 1$, $c_0 = 0$, $\lambda = 0.75$ (i.e. $\gamma = 0.75 \gamma^*$, see (6.59)). The figure of $u(c)$ clearly shows in which way the present viscous solution might break down, i.e. the condition (6.12), which represents the existence of the solution of $g(c)$, is violated. For decreasing values of $K_\eta$ or $\gamma$ the oscillatory character of the solution of $u(c)$ increases on $c_0 < c < c_1$, until the minimum value of $u(c)$ at $c = c_2$ becomes identical to zero. Some further analysis including repeated differentiation of the differential equation of $u(c)$, see (6.13), indicates that in this particular case the function $u(c)$ and all its derivatives are identical to zero at $c = c_2$.

Therefore in this case the limit values of lower bounds of the parameters $K_\eta$ and $\gamma$ are defined by

$$K_\eta = K_\eta^{(3)} \quad \eta, \min \quad \text{at fixed value of } \gamma$$

or

$$\gamma = \gamma^{(3)} \quad \eta, \min \quad \text{at fixed value of } K_\eta > 0$$

when

$$u(c) > 0 \quad \text{on } c_0 < c < c_2$$

$$\left\{ \frac{d^n u}{dc^n} \right\}_{c = c_2} = 0 \quad \text{for } n = 0, 1, 2, \ldots$$

for $0 < \gamma < \gamma^*$ and low value of $K_b$.

It has to be remarked that in this limit case the Taylor's series expansion does not exist at $c = c_2$ and as a consequence the behaviour of $u(c)$ on $c_0 < c < c_2$ is not related at all to that on $c_2 < c < 1$; the two parts are completely decoupled now.

Therefore in this case the solutions of $g(c)$ and $\frac{dg}{dc}$ exist if

$$K_\eta^{(3)} \quad \eta, \min < K_\eta < \infty \quad \text{at fixed value of } \gamma$$
Figure 6.4. The behaviour of the functions $u(c)$ and $g(c)$ on $c_o < c < l$ and the resulting viscous fluid motion for $K_\eta = 10^{-2}$, $K_b \ll 1$, $c_o = 0$, $\lambda = \gamma/\gamma* = 0.75$. 
or

\[ \gamma^{(3)}_{\min} < \gamma < \gamma^* \quad \text{at fixed value of } K_{\eta} \]

for \( 0 < \gamma < \gamma^* \).

Since the positive value of \( \gamma \) smoothes down the partially oscillatory behaviour of the solution of \( u(c) \), it is clear that \( K_{\eta,\min}^{(3)} < K_{\eta,\min}^{(2)} \) where \( K_{\eta,\min}^{(2)} = K_{\eta,\min} = 0.081634 \) for \( c_0 = 0 \) and \( K_{b} << 1 \).

When \( K_{\eta} \) or \( \gamma \) satisfies the condition (6.62) a bounded solution of \( g(c) \) exists in the entire fluid domain. However when one of the parameters becomes identical to the respective lower bound suddenly in a discontinuous transition \( g(c) \) becomes essentially singular in the greater part of the fluid domain near the axis of symmetry: \( 0 < \theta < \theta_2 \), which follows from the transformation (4.77) and the integral equations of \( g(c) \) and \( \frac{dg}{dc} \), see (6.10), (6.11), resulting in a breakdown of the present solution of the viscous fluid motion.

This particular discontinuous breakdown behaviour of the viscous fluid motion strongly differs from the continuous transition to singular behaviour in the flow field as observed in the cases 1 and 2. In the latter cases the strong singular behaviour in the flow field starts at the axis of symmetry only, when one of the parameters becomes equal to the critical lower bound, enlarging continuously to a greater part of the fluid domain out of the axis of symmetry for decreasing value of the parameter concerned, see figure 4.6.

It is of interest to notice that in cases 1 and 2 the difficulties arise in the outwards jet flows at the axis of symmetry \( c = 1 \), whereas in the present case 3 it appears in the outwards jet flow at the angle \( \theta = \theta_2 \). Hence we have to conclude that the breakdown of the viscous fluid motion is related in one way or another to some critical behaviour of the respective outwards jet flows.

The calculation of the minimum value of \( \lambda \) has been carried out by numerical iteration. It turns out to be: \( \lambda_{\min}^{(3)} = 0.702550 \) for \( K_{\eta} = 10^{-2} \), \( c_0 = 0 \), \( K_{b} << 1 \), corresponding to \( \gamma_{\min}^{(3)} = 0.319536 \), see (6.30), (6.59). Since \( u(c) \) is very flat in the neighbourhood of \( c = c_2 \), as it follows from the definition (6.61), the numerical calculation appeared to be rather troublesome; especially since the value of \( c_2 \) changes very rapidly from 0.5279 to 0.8022 when \( \lambda \) decreases from 0.705 to the minimum value 0.702550.

In principle the calculation of the respective lower bounds of \( K_{\eta} \) and \( \gamma \) might be executed by a modified and extended procedure analogous to the one described in appendix C. Nevertheless at the moment it is not clear whether this approach will result in a better converging method.
6.4.4. Case 4: $\gamma > \gamma^*$, nonoscillatory solution

At low value of $K_b$ or in the absence of electric current inversion, i.e. $\frac{df}{dc} < 0$ on $c_o < c < 1$, see (2.41), it can be derived from (2.51), (6.29) that in the case of $\gamma > \gamma^*$ the function $\Phi(c)$ multiplying $u(c)$ in the differential equation of $u(c)$, see (6.13), (6.14), satisfies

$$\Phi(c) = G_\eta(c) - \gamma(c-c_o)(1-c) < 0 \quad \text{on } c_o < c < 1$$

(6.63)

with $\Phi(c_o) = \Phi(1) = 0$.

This fact implies that the solution of $u(c)$ is nonoscillatory on the entire interval and in addition that $u(c) > 1$, $\frac{du}{dc} > 0$, $\frac{d^2u}{dc^2} > 0$ on $c_o < c < 1$, as it follows from (6.13), (6.14), (6.54), (6.55), where the values at $c = c_o$ are given by (6.15). Thus $u(c)$ increases monotonically from $u(c_o) = 1$ to a much larger positive value at $c = 1$.

In this case the function $u(c)$ is unable to become identical to zero at any point of the interval, implying that the condition (6.12), which states the existence of the solution of $g(c)$ is satisfied for all values of $K_\eta$, yielding

$$K_{\eta, \text{min}} = 0$$

(6.64)

for $\gamma > \gamma^*$ and $\Phi(c) < 0$ on $c_o < c < 1$.

Hence the solutions of $g(c)$ and $\frac{dg}{dc}$ exist for all values of $K_\eta$, viz.

$$0 < K_\eta < \infty \quad \text{at } \gamma > \gamma^*$$

(6.65)

In view of the behaviour of $u(c)$, i.e. $u(c) > 1$, $\frac{du}{dc} > 0$, $\frac{d^2u}{dc^2} > 0$ on $c_o < c < 1$, the transformation (4.77) leads to the conclusion that $g(c) < 0$ on $c_o < c < 1$.

Therefore in this particular case the induced viscous fluid motion consists of an incoming flow along the axis of symmetry, which is turned off in the neighbourhood of the electrode into an outwards flow along the surface of the right circular cone, see figure 6.5. As usual for $\gamma \neq 0$ the velocity field contains a weak logarithmic singularity at the axis of symmetry $c = 1$, viz. $v_z(r, 1) = -\infty$, see (6.25).

In figure 6.5 the behaviour of the functions $u(c)$ and $g(c)$ on $c_o < c < 1$ and the corresponding viscous fluid motion are presented for $K_\eta = 2.5 \times 10^{-3}$, $K_b \ll 1$, $c_o = 0$, $\lambda = \gamma/\gamma^* = 1.1$.

Comparison of the respective graphs of the functions $u(c)$ and $g(c)$ in figures 6.4 and 6.5 clearly demonstrates that the maximum value of $u(c)$ at $c = 1$ in-
Figure 6.5. The behaviour of the functions $u(c)$ and $g(c)$ on $c_o < c < 1$ and the corresponding viscous flow pattern for $K_\eta = 2.5 \times 10^{-3}$, $K_0 \ll 1$, $c_o = 0$, $\lambda = \gamma/\gamma* = 1.1$.
creases exponentially for decreasing values of \( K_\eta \); whereas the magnitudes of the corresponding functions \( g(c) \) remain small and of equal order. The behaviour of \( u(c) \) and \( \frac{du}{dc} \) leads to overflow problems in the numerical calculation of the integral equations of \( u(c) \) and \( \frac{du}{dc} \), see (6.54), (6.55). Therefore it is preferable to apply the integral equations of \( g(c) \) and \( \frac{dg}{dc} \), see (6.10), (6.11), for the numerical computation at smaller values of \( K_\eta \); where the present solution of \( g(c) \) at \( K_\eta = 2.5 \times 10^{-3} \) can be used as starting value of the iteration.

In this section finally a solution of the viscous fluid motion has been found valid for all values of the hydrodynamic Reynolds number \( (0 < \text{Re} < \infty, 0 < K_\eta < \infty) \). Unfortunately the flow field contains a relatively weak singularity on the axis of symmetry, although the kinetic energy and the mass flow remain bounded there. Moreover the direction of the generated flow is opposite to what should be expected in view of the direction of the driving Lorentz force, see (2.63).

6.4.5. Evaluation of the results

The preceding examination of the general solutions of the viscous fluid motion of cases 1 - 4 shows that any outward directed jet flow not situated at a solid boundary becomes strongly singular, implying physically unrealistic phenomena like fluid sources or sinks, when the hydrodynamic Reynolds number exceeds a certain critical value. It has to be noticed that this particular behaviour is not unusual for laminar free jet flows, see Batchelor (1967, p.346) and Birkhoff & Zarantonello (1957, p.274).

It can easily be verified that the electric field satisfies the boundary condition \( E_S = 0 \) at \( c = 1 \), see (2.67), (6.20), (6.21), (6.27) for all values of \( \gamma \), as required. Nevertheless the space charge density appears to contain a relatively weak logarithmic singularity at the axis of symmetry, see (2.69), (6.20), (6.27), viz.

\[
\rho_e = -\frac{\varepsilon_0}{8\pi\sigma} \frac{K_b \gamma(1-c_0)}{r^3} \frac{dE}{dc} \bigg|_{c=1} \ln(1-c)(1+o(1)),
\]

(6.66)

for \( c = 1, \gamma \neq 0 \).
The relatively weak singular behaviour of \( \nu_e \) and \( \rho_e \) at \( c = 1 \), see (6.25), (6.66), suggests that in further examinations also the respective effects of the Coulomb force \( \rho_e E \) in the Navier-Stokes equation (2.2) and of the convection current density \( \rho_e \dot{\nu} \) in Ohm's law (2.3) need to be taken into consideration. On the other hand the weak singular behaviour of the space charge density can also be explained simply as to be caused by an analogous behaviour of the flow field. For a further discussion we refer to section 6.6.

It is of interest to recall that the present viscous flow solutions of cases 1,3,4 are actually generated by both the Lorentz force and by the exchange of momentum between the fluid-flow in the far field and the fluid motion in other regions of the fluid domain. The measure of exchange of momentum is represented by the free parameter \( \gamma \). In the prototype point electrode model \( \gamma \) may even be regarded as related to the strength of a point source of momentum located at the origin.

Only Sozou's solution, see Sozou (1971a), of the viscous fluid motion (case 2 for \( \gamma = 0 \)) represents a flow field induced exclusively by the rotationality of the Lorentz force. Unfortunately, as observed in section 4.3, see e.g. table 4.1, this viscous solution breaks down at relatively low value of the hydrodynamic Reynolds number.

The incorporation of non-zero values of the momentum parameter \( \gamma \) in the present general solutions of the viscous fluid motion always implies a weak logarithmically singular behaviour in the flow field at the axis of symmetry \( c = 1 \). A detailed examination of the general solutions of inviscid and viscous fluid motions induced by a point source of momentum clarifies this behaviour.

In spite of the relatively weak singular behaviour in the viscous flow field at \( c = 1 \), the mass flow and the kinetic energy remain bounded throughout the flow field.

From the preceding examination and calculations we have to conclude that the similarity method is not applicable for the calculation of the viscous fluid motion at larger values of the hydrodynamic Reynolds number. For a further discussion we refer to section 6.6.

The general solutions of the inviscid fluid motion will be considered in the next section.

6.5. The general inviscid solution

In this section the general solution of the inviscid fluid motion caused by the injection of an electric current is considered. The examination shows that the
solutions in general contain relatively weak singularities in the flow field at the axis of symmetry and at the surface of the right circular cone. It turns out that the inviscid solutions as formulated by Shercliff (1970) and Narain & Uberoi (1973) are special cases where for a special value of the free parameter $\gamma$ the weak singularity at $c = c_0$ is eliminated.

In the inviscid fluid ($K_\eta = 0$), the expression of the function $g(c)$ yields, see (6.7)

$$g^2(c) = -G_\eta(c) + \gamma(c-c_0)(1-c) \quad \text{for } K_\eta = 0,$$  \hspace{1cm} (6.67)

where $G_\eta(c)$ which represents the effect of the Lorentz force is given by (2.51). Since $G_\eta(c) > 0$ on $c_0 < c < 1$, with $G_\eta(c_0) = 0(c-c_0)$ and $G_\eta(1) = 0((1-c)^2)$, see (2.56), (2.57), it is evident that real solutions of $g(c)$ exist only for positive values of the free parameter $\gamma$, viz. $\gamma_{\min} < \gamma < \infty$, with $\gamma_{\min} > 0$.

The behaviour of the flow field near the axis of symmetry is determined only by the value of the free parameter, viz.

$$g(c) = \pm \sqrt{\gamma(1-c_0)} \sqrt{1-c} \ (1+o(1)) \quad \text{for } c = 1, \gamma_{\min} < \gamma < \infty$$ \hspace{1cm} (6.68)

The behaviour of the flow field in the neighbourhood of the surface of the cone admits two possibilities.

The general one with a relatively weak singularity

$$g(c) = \pm \sqrt{\gamma(1-c_0) - \left(\frac{dG_\eta}{dc}\right)_{c=c_0} \sqrt{c-c_0} \ (1+o(1))} \quad \text{for } c = c_0, \gamma_{\min} < \gamma < \infty,$$ \hspace{1cm} (6.69)

and a special one with a regular behaviour of the flow field at the cone,

$$g(c) = \pm \sqrt{\gamma - \frac{1}{2} \left(\frac{d^2G_\eta}{dc^2}\right)_{c=c_0} (c-c_0)(1+o(1))} \quad \text{for } c = c_0, \gamma = \gamma_{\min} \quad (6.70)$$

by satisfying what we will call the condition of Shercliff: $\gamma = \gamma_{\min}$, where see (6.9)

$$\gamma_{\min} = \frac{1}{1-c_0} \left(\frac{dG_\eta}{dc}\right)_{c=c_0} \frac{2(1+c_0)}{(1-c_0)^2} \frac{1}{c_0} \int \frac{f^2(t)}{(1+t)^2} \ dt,$$ \hspace{1cm} (6.71)
at arbitrary value of $K_b$.

It can easily be verified that the square root in (6.70) is real for $\gamma = \gamma_{\text{min}}$ and $|f(c)| < 1$ on $c_o < c < 1$ with $f(c_o) = 1$.

At small value of $K_b$ the expression of $\gamma_{\text{min}}$ takes the form

$$\gamma_{\text{min}} = \frac{2(3+c_o)}{(1-c_o)^3} + \frac{8(1+c_o)}{(1-c_o)^4} \ln \left( \frac{1+c_o}{2} \right) \quad \text{for } K_b \ll 1. \quad (6.72)$$

In section 3.2, see (3.4), (3.10), it has been shown from the expression of the mass flow that the relatively weak singularity in the flow field at the axis of symmetry does not imply physically unrealistic phenomena like fluid sources or sinks.

For the general inviscid solution for $\gamma_{\text{min}} < \gamma < \infty$, see (6.69), this fact has to be verified also at the surface of the cone. The mass flow $\Psi$ through a small sphere-sector of radius $r$ with the centre at the origin and situated upon and near the cone surface between $c$ and $c_o$ becomes

$$\Psi = I_o \sqrt{\rho \mu} \alpha r \sqrt{c-c_o} (1+o(1)) \quad \text{for } c = c_o, \gamma_{\text{min}} < \gamma < \infty \quad (6.73)$$

where

$$\alpha = \pm \sqrt{\gamma(1-c_o) - \left( \frac{dG}{dc} \right)_{c=c_o}}, \quad (6.74)$$

indicating that the mass flow remains bounded there and tends to zero when the control surface approaches zero; i.e. $c \to c_o$. Hence we observe that both weak singularities in the general inviscid flow field do not involve inconsistencies like fluid sources or sinks.

It has to be remarked that the expressions of the respective special values $\gamma_{\text{min}}$ for $K_\eta = 0$ and $\gamma^*$ for $K_\eta > 0$ are identical for all values of $K_b$, see (6.29), (6.30), (6.71), (6.72).

In view of the following facts:

(i) the identity $\gamma_{\text{min}} = \gamma^*$,

(ii) the solutions of the general inviscid fluid motion exist only for $\gamma_{\text{min}} < \gamma < \infty$,

(iii) the direction of the viscous fluid motion for $\gamma > \gamma^*$, see figure 6.5 and section 6.4.4,
we have to conclude that for the general inviscid solutions both directions of the fluid motion are admissible, viz.

\[ g(c) = g_1(c), g_2(c), \]

where

\[ g_1(c) = + \sqrt{g^2(c)}, \]
\[ g_2(c) = - \sqrt{g^2(c)}. \]

for \( K_\eta = 0 \), \( \gamma_{\min} < \gamma < \infty \) \hspace{1cm} (6.75)

This statement differs from (3.14) and Shercliff's interpretation. Thus in the general inviscid case the jet flow along the axis of symmetry is directed from or towards the point electrode, as sketched roughly in figure 6.6.

Figure 6.6. The possible flow directions of the general inviscid fluid motion.
Since $\gamma$ represents the momentum transfer between fluid motions in different fluid regions, which also may be affected by the configuration, or the effect of a point source of momentum at the apex of the cone, the general inviscid solutions for $\gamma_{\text{min}} < \gamma < =$ are always composed of the combined effect of the Lorentz force and of the exchange or supply of momentum. For $\gamma = 0$ the inviscid solution does not exist.

In the special case $\gamma = \gamma_{\text{min}}$, i.e. when Shercliff's condition is satisfied, the magnitude of $\gamma$ is related to the Lorentz force in a fixed prescribed way.

The general inviscid solution at arbitrary value of $K_b$ is of the form, see (2.51), (6.67)

$$
\begin{align*}
g^2(c) &= \frac{(1+c)^2}{2} \int_0^c \frac{f^2(t)}{c_0 (1+t)^2} \, dt - \frac{(1-c)^2}{2} \int_0^c \frac{f^2(t)}{c_0 (1-t)^2} \, dt + \\
&- \frac{2(c-c_o)(1-c) c_0}{(1-c)^2} \int_0^c \frac{f^2(t)}{c_0 (1+t)^2} \, dt + \gamma(c-c_o)(1-c),
\end{align*}
$$

(6.76)

for $\gamma_{\text{min}} < \gamma < =$, where $\gamma_{\text{min}}$ is given by (6.71).

At low value of $K_b$ the analytical expression of the general inviscid solution becomes, see (4.65) - (4.69), (6.67),

$$
g^2(c) = a_0 + a_1 c + a_2 c^2 + a_3 (1+c)^2 \ln(1+c),
$$

(6.77)

where

$$
a_0 = \frac{4c_o}{(1-c_o)^3} + \frac{2(1+c_o)^2}{(1-c_o)^4} \ln(1+c_o) - \frac{8c_o}{(1-c_o)^4} \ln(2) - \gamma c_o,
$$

(6.78)

$$
a_1 = -\frac{4(1+c_o)}{(1-c_o)^3} - \frac{4(1+c_o)^2}{(1-c_o)^4} \ln(1+c_o) + \frac{8(1+c_o)^2}{(1-c_o)^4} \ln(2) + \gamma(1+c_o),
$$

(6.79)

$$
a_2 = \frac{4}{(1-c_o)^3} + \frac{2(1+c_o)^2}{(1-c_o)^4} \ln(1+c_o) - \frac{8c_o}{(1-c_o)^4} \ln(2) - \gamma,
$$

(6.80)

$$
a_3 = -\frac{2}{(1-c_o)^2},
$$

(6.81)
for \( K_b \ll 1 \), \( \gamma_{\text{min}} \ll \gamma \ll \infty \), where \( \gamma_{\text{min}} \) at \( K_b \ll 1 \) is given by (6.72).

When Shercliff's condition is satisfied, i.e. \( \gamma = \gamma_{\text{min}} \), the general expression (6.76) reduces to (3.11), (3.12), and the special one at small \( K_b \) (6.77) - (6.81) to (3.33), (3.36) - (3.39), whereby the relation between \( \gamma_{\text{min}} \) and \( h \) yields

\[
\gamma_{\text{min}} = \frac{2h^2}{1-c_0}.
\]

(6.82)

A remaining disadvantage of the present general solutions of the inviscid fluid motion is the fact that the component of the electric field normal to the axis of symmetry is not identical to zero at that location, viz. \( E_\theta(r,l) \neq 0 \), whereas the boundary condition (2.67) requires that \( E_\theta \) is identical to zero there. Moreover for \( \gamma > \gamma_{\text{min}} \), \( E_\theta \) is singular at the surface of the cone.

As a coherent result the space charge distribution contains a relatively weak singularity at \( c = 1 \), see Jansen (1977, p.62), and for the general solution at \( \gamma > \gamma_{\text{min}} \) also at the surface of the cone. Here the same arguments about these phenomena can be used as given in section 6.4.5 for the general viscous solutions at \( \gamma \neq 0 \). Namely they can be considered as caused by the singularities in the flow field.

An attempt to establish a relation between the inviscid fluid-flow field-quantities in the near and far fields \((\bar{r} < 1, \bar{r} > 1)\), by application of the usual MHD variant of Bernoulli's theorem failed. It appeared that the theorem is only valid along the streamline \( \psi = 0 \) at the axis of symmetry \( c = 1 \) and not in the entire fluid domain. The failure is mainly caused by the necessity to apply curvilinear co-ordinate systems in the configurations considered.

Finally it should be noticed that due to the singularities in the flow field also in the general case the kinetic energy, induced in the inviscid flow field by the Lorentz force and the momentum parameter \( \gamma \), is not bounded, see (3.66), (6.68) - (6.70); whereas the mass flow is finite throughout the fluid domain and tends to zero at the particular places.

The examination of the general solutions of the inviscid fluid motions shows that the inviscid solution derived by Shercliff (1970) is the one with the least singularities in the flow field. Moreover this particular solution approaches practical situations best and therefore it is within the limitation of inviscid fluid motions the most physically realistic one.

However in view of the singularities appearing in the other field quantities at \( c = c_0, 1 \), especially in the space charge density, a further examination of solutions for the inviscid fluid motion will be desirable.

Nevertheless the investigations of the general inviscid and viscous solutions,
as carried out so far and presented in sections 6.4 and 6.5, contributes essentially to new insight in the mathematical background of the respective solutions obtained.

With the calculation and examination of the general solution of the inviscid fluid motion in the prototype model, our investigations about the applicability of the similarity method in the semi-infinite point electrode configuration and about the analogous behaviour in the far field of the fluid domain of the disk electrode configuration have come to an end.

The results obtained and the conclusions will be discussed in the following final section.

6.6. Discussion and concluding remarks

In this thesis inviscid and viscous fluid motions induced by the injection of an electric current in some simple models of configurations have been investigated extensively.

The configurations considered are: the semi-infinite point electrode configuration, also called the prototype model, and the semi-infinite disk electrode configuration, for which the mathematical treatment is more complex.

The simplification of the configurations is imposed in order to make an analytical treatment of the problem possible, which is the main feature of the present investigations. The configurations do reflect however actual situations as they occur in practical applications.

The conditions imposed upon the Lorentz force, due to the electric current distribution in the fluid and its associated magnetic field, and upon the configuration to be able to generate a fluid motion have been discussed in detail in the introduction given in chapter 1.

The semi-infinite point electrode problem has been formulated in chapter 2. The introduction of the similarity method leads to the reduction of the governing partial differential equations of the curl of the Navier-Stokes equation and of the curl of the Ohm's law to ordinary differential equations.

Since the semi-infinite point electrode problem does not contain any fundamental length- or velocity scale, the characteristic dimensionless parameters $\eta$ and $K_0$ are introduced. They represent respectively the inverse effective hydrodynamic Reynolds number and the effective magnetic Reynolds number.

In the derivation and conversion of the governing equations it is assumed
implicitly that the velocity along the axis of symmetry is bounded. In this way we followed the literature concerning the analytical calculation in the prototype model. It turns out then to be impossible to calculate the inviscid solution from this derivation in a straightforward way. Also in the viscous point electrode problem it implies an extra condition, resulting in a unique solution.

Further the boundary conditions and the expressions of the other fieldquantities appearing in the vector equations are determined in this chapter.

The inviscid fluid motion in the semi-infinite point electrode configuration has been considered in chapter 3. It turns out that a real solution of the inviscid fluid motion can be obtained only, when a relatively weak singularity in the flow field at the axis of symmetry is admitted. Analytical solutions of the inviscid fluid motion and numerical solutions of the weak perturbations of the electric current distribution and of the fluid motion, due to the effect of the electromagnetic induction, are calculated at small value of the magnetic Reynolds number. Consideration of the behaviour of the inviscid flow and the electromagnetic fieldquantities at arbitrary value of the magnetic Reynolds number demonstrates, that this inviscid flow solution exists for all values of the magnetic Reynolds number and moreover that inversion of the electric current distribution in the fluid cannot occur.

The relatively weak singularity in the flow field at the axis of symmetry gives rise to an unbounded velocity along the axis of symmetry. Due to this singular behaviour the normal component of the electric field is not identically zero at the axis of symmetry, whereas the boundary condition requires that it equals zero there. Also it appears that the space charge density contains a relatively weak singularity at \( c = 1 \).

The latter fact suggests that in the governing equations the effect of the Coulomb force and of the convection current density must be included. However it turns out that the similarity method can no longer be applied in that case. It may be remarked that the singularity in the space charge density and the impossibility of the normal component of the electric field to satisfy the boundary condition at the axis of symmetry are due to the weak singular behaviour of the inviscid fluid motion there. However in my opinion a reformulation of the inviscid point electrode problem is preferable in order to investigate whether inviscid solutions exist without or with a weaker singularity in the flow field and for which especially the boundary condition of the electric field at the axis of symmetry can be satisfied.
The semi-infinite viscous point electrode problem has been studied in chapter 4. The analytical solutions of the slow viscous flow at small value of the hydrodynamic Reynolds number, i.e. at large $K_\eta$, are determined. Also the weak perturbations of the fluid motion at small and large values of the magnetic Prandtl number and of the electric current distribution at low value of the magnetic Reynolds number, due to the electromagnetic induction and the effect of the inertia force, are presented.

In contrast with the inviscid flow solution, it appears that the basic slow viscous solution is unbounded on $c_0 < c < 1$ for $c_0 = -1$. On the other hand the weak perturbation of the electromagnetic field quantities, due to the electromagnetic induction, could be calculated analytically here.

The calculation of the viscous fluid motion at arbitrary value of $K_\eta$, which includes the nonlinear effect of the inertia force, shows that the solution breaks down at relatively low values of the hydrodynamic Reynolds number $Re = K_\eta^{-1}$. The critical values of $K_\eta$ as function of the apex-angle of the right circular cone are presented in a figure and a table.

The behaviour of the viscous solution when $K_\eta$ approaches and decreases below the critical value $K_{\eta, \text{min}}$ has been examined extensively. It turns out that at $K_\eta = K_{\eta, \text{min}}$ a semi-infinite line sink appears in the flow field at the axis of symmetry, which unfolds to a conical sink region for smaller values of $K_\eta$.

Of particular interest is the observed and very abrupt reduction of the spreading of the viscous jet flow along the axis of symmetry when $K_\eta$ tends to $K_{\eta, \text{min}}$. This behaviour indicates that the viscous force cannot balance the inertia and Lorentz forces any longer in that case.

The peculiar phenomena entering the flow field and the fact that the mass conservation equation is not satisfied lead to the conclusion that the viscous solution for $K_\eta < K_{\eta, \text{min}}$ or $Re > Re_{\text{max}}$ must be rejected as being physically unrealistic.

In order to examine and resolve some inconsistencies occurring in the inviscid and viscous point electrode problems, a configuration with an electrode of finite non-zero dimensions should be considered. Therefore in chapter 5 the fluid motion induced in the semi-infinite disk electrode configuration, consisting of a cylindrical electrode of radius a located in an insulating flat wall has been studied.

It turns out that for an analytical examination of the induced fluid motion the fluid domain needs to be separated into two regions: the near field $\tilde{r} < 1$ and the far field $\tilde{r} > 1$. Then the Stokes stream function can be expressed in different series expansions of respectively positive and negative powers of
\( \bar{r} = \frac{r}{a} \), where each term is multiplied with a function dependent on the variable \( c \) only.

Consideration of the basic series expansion solution in the far field (\( \bar{r} > 1 \)) shows that the behaviour of the fluid motion in the farthest field (\( \bar{r} \to \infty \) or \( a \to 0^+ \)) of the semi-infinite disk electrode configuration is identical to the flow in the semi-infinite point electrode configuration for the case of a flat wall.

The investigation of the behaviour of the fluid motion in the near field (\( \bar{r} < 1 \)) shows some interesting features. It turns out that a solution of the inviscid fluid motion exists, with finite velocities throughout the flow field, when the basic solution of the series expansion of the Stokes stream function is not generated by the Lorentz force but only by the exchange of momentum between fluid motions in different regions of the near field fluid domain.

In addition the solutions of the viscous fluid motion in the neighbourhood of the electrode appear to exist for all values of the hydrodynamic Reynolds number; this also holds true when the nonlinear inertia forces are included.

The analytical series expansion solutions of the inviscid and viscous fluid motion obtained in the near field all possess a free parameter as expected in view of the order of the respective differential equations and of the number of boundary conditions to be satisfied.

The physical significance of these free parameters can be clarified as the measure of the exchange of momentum between fluid motions in different regions of the fluid domain or when the prototype model is considered as the presence of a point source of momentum located at the origin. An alternative explanation is that the free parameters determine the ratios of the kinetic energy and the pressure distribution generated by the Lorentz force in the fluid at different locations.

From a mathematical point of view the appearance of the free parameters is due to the lack of one boundary condition in the consideration of semi-infinite configurations. In configurations of finite extent the solutions of the fluid motion are unique.

The examination of the hydrodynamic and magnetic Reynolds numbers in the near and far field clearly demonstrates a local behaviour of these parameters throughout the flow field. Namely a relatively low \( \text{Re} \) and \( \text{Rm} \) behaviour in the neighbourhood of the electrode and a relatively high \( \text{Re} \) and \( \text{Rm} \) behaviour at large radial distance from the electrode.

The main conclusions to be drawn from the investigation of the semi-infinite disk electrode configuration are the following ones.

The model cannot resolve some problems arising in the point electrode configu-
ration, such as the relatively weak singularity in the inviscid flow field at the axis of symmetry and the breakdown of the viscous fluid motion at relatively low value of the hydrodynamic Reynolds number.

The expressions of all field quantities derived for the point electrode model do not hold at the point electrode; as a consequence the apparent singularities at \( r = 0 \) do not exist.

The total Lorentz force exerted upon a certain volume of fluid does not possess a singular behaviour at the point electrode in fact. It follows that there is no reason to expect the occurrence of phenomena like local cavitation in the fluid and intermittency of the electric current passing to the fluid at that location, as suggested in the literature.

The calculations confirm Shercliff's hypothesis about the local behaviour of the Reynolds numbers.

The above conclusions enable us to state that the presence of the point electrode cannot be considered in any way as the cause of the problems arising in the semi-infinite point electrode configuration, as suggested by many authors.

The main feature of the investigation of the disk electrode model is the observation of the quite different behaviour of the inviscid and viscous fluid motions in the neighbourhood of the electrode and at large radial distance from the electrode.

The appearance of the free parameters in the near field solutions of the semi-infinite disk electrode configuration leads to a reconsideration of the semi-infinite point electrode configuration carried out in chapter 6.

The derivation of the governing equations and the substitution of the boundary conditions without any implicit assumption about the behaviour of the flow field result in general solutions of the inviscid and viscous fluid motion with an additional free parameter \( \gamma \), which represents the measure of exchange or the strength of a point source of momentum.

Arbitrary non-zero values of the free parameter imply a relatively weak logarithmic singularity in the viscous flow field at the axis of symmetry, resulting in an infinite velocity along the axis of symmetry. The general inviscid flow solution turns out to contain a relatively weak singularity at the surface of the cone, in addition to the one already present at the axis of symmetry. These weak singularities in the respective flow fields do not imply fluid sources or sinks; the respective mass flows are bounded and approach to zero at the places required.
The analytical solution of the general slow viscous solution has been calculated. In this general treatment also for $c_0 = -1$ a bounded solution of the flow field on $c_0 < c < 1$ could be found for the special value $\gamma = \frac{1}{2}$.

The examination of the general viscous solution, which includes also the nonlinear effect of the inertia force, yields four types of fluid motions with different flow patterns, i.e. for $\gamma < 0$, $\gamma = 0$, $0 < \gamma < \gamma^*$, $\gamma > \gamma^*$. Here $\gamma^*$ is a special value of the free parameter $\gamma$ at which the fluid motion along the surface of the cone changes its direction.

The general viscous solutions for $\gamma < 0$ and $\gamma = 0$ show similarity with the flow patterns found in chapter 4 and break down at relatively small values of the hydrodynamic Reynolds number.

For $0 < \gamma < \gamma^*$ the general viscous solution yields a flow pattern that consists of incoming flows along the axis of symmetry and the surface of the right circular cone and of an outwards jet flow at a certain angle of $\theta$, depending on $\gamma$. This solution breaks down at larger values of the hydrodynamic Reynolds number than for $\gamma < 0$.

The flow pattern of the general viscous solution for $\gamma > \gamma^*$ consists of an incoming flow along the axis of symmetry and an outwards directed flow along the surface of the cone. The most important feature of this particular solution is the fact that it exists for all values of the hydrodynamic Reynolds number.

These four types of viscous flow solutions clearly show the dominant effect of the strength of the point source of momentum at larger positive values of the parameter $\gamma$, which overrules the effect of the Lorentz force in the entire flow field when $\gamma$ becomes identical to or exceeds $\gamma^*$. Therefore the general viscous solution for $\gamma \neq 0$ is in fact generated by the Lorentz force and by the effect of the exchange or of a source of momentum.

The present study evidently demonstrates that the breakdown of the viscous fluid motions occurs in the flow field at the outwards jet flows, which are not situated at the surface of the cone, when the hydrodynamic Reynolds number exceeds a certain critical value.

Therefore it is clear that the Lorentz force itself is the essential cause of the breakdown of the viscous fluid motion and not the viscous force or the inertia force or the presence of the point electrode, as suggested by some authors.

Despite the fact that the kinetic energy is bounded and that the mass flow is finite throughout the flow field and approaches to zero at the axis of symmetry, we have to conclude that solutions of the viscous fluid motion with a relatively weak logarithmic singularity in the flow field at the axis of symmetry are not physically realistic and therefore must be rejected.
Hence only the viscous solution for $\gamma = 0$, as examined in chapter 4, remains. Nevertheless it has to be remarked that the viscous solution obtained for $\gamma > \gamma^*$, which exists for all values of the hydrodynamic Reynolds number, always contains a much weaker singularity in the flow field at the axis of symmetry, than the singularities which enter the flow field for $\gamma < \gamma^*$ when the hydrodynamic Reynolds number exceeds a certain critical value.

Perhaps the present general viscous solutions for $\gamma \neq 0$ may be interpreted as purely mathematical intermediate solutions, indicating the transition from laminar to turbulent flow which always occurs in jet flows at some distance from the origin above a certain critical value of the hydrodynamic Reynolds number.

Finally it should be noted that due to the local singular behaviour of the viscous fluid motion for $\gamma \neq 0$, the space charge density shows a relatively weak singularity at the axis of symmetry. In contrast with the general inviscid solution the boundary condition of the electric field at $c = 1$ can be satisfied here. The singular behaviour of the space charge density at $c = 1$ leads to the discrepancy that extra terms should be taken into account in the governing equations. However this is not possible within the framework of the similarity method applied.

The consideration of the general inviscid fluid motion shows that real solutions of the flow field can exist only for values of the free parameter $\gamma$ that satisfy $0 < \gamma_{\text{min}} < \gamma < \infty$.

The special case $\gamma = \gamma_{\text{min}}$; i.e. when the condition of Shercliff is satisfied, is studied in chapter 3. It is worth remarking that in this special case the strength of the point source of momentum is strongly related to the apex-angle of the right circular cone and in particular to the magnitude and the distribution of the magnetic field in the fluid.

For $\gamma > \gamma_{\text{min}}$ the inviscid fluid motion also possesses a relatively weak singularity in the flow field at the surface of the cone.

Due to the singular behaviour in the inviscid flow field for $\gamma > \gamma_{\text{min}}$, the space charge density is singular at $c = c_0$ and $c = 1$, the electric field cannot satisfy the boundary condition at the axis of symmetry and it is singular at the surface of the cone. With respect to these phenomena and their resolvability the same arguments can be used as before in the discussion of the special inviscid fluid motion examined in chapter 3.

Since a real solution of the inviscid fluid motion does not exist for $\gamma = 0$, the solutions obtained are in fact generated by the combined effect of the Lorentz force and of the exchange or of a point source of momentum, in the same way as found for the inviscid solution in the near field fluid domain of the
semi-infinite disk electrode configuration. Note that solutions of the viscous fluid motion do exist for \( \gamma = 0 \); however only for a very limited range of values of the hydrodynamic Reynolds number.

It is clear that Shercliff's inviscid solution for \( \gamma = \gamma_{\text{min}} \) with the least singularities in the flow field is the best inviscid solution available within the framework of the similarity method applied. Nevertheless since in general \( \gamma_{\text{min}} = \gamma^* \) and the effect of the point source of momentum dominates the Lorentz force throughout the entire flow field, viz. \( C_{\eta}(c) < \gamma_{\text{min}}(c-c_0)(1-c) \), certainly both directions of the inviscid fluid motion are admissible.

The introduction of the similarity method, as proposed by Zhigulev (1960a), in the semi-infinite point electrode configuration and of the series expansions in the semi-infinite disk electrode configuration yields the conclusions that physically realistic solutions of the inviscid and viscous fluid motion exist in the neighbourhood of the electrode. However at large radial distance from the electrode and in the prototype model, the only physically realistic solutions that are available from the analytical examination and calculation are:

(i) an inviscid solution with a relatively weak singularity in the flow field at the axis of symmetry,
(ii) a slow viscous solution which does not exist for \( c_0 = -1 \),
(iii) a viscous solution which breaks down at relatively small value of the hydrodynamic Reynolds number.

The investigation of the general viscous solution leads to a viscous solution which exists for all values of the hydrodynamic Reynolds number. Unfortunately this solution contains a very weak singularity in the flow field at the axis of symmetry, which causes the velocity along the axis of symmetry to be very weakly singular at that place. Due to this singular behaviour this particular solution cannot be regarded as being physically realistic. This observation demonstrates a great discrepancy between the purely analytical calculations and the numerical computations carried out by Butsenieks, Peterson, Sharankin & Scherbinin (1976), Arthey (1980), Craine & Andrews (1984) and Ajayi, Sozou & Pickering (1984).

The difference in the configurations considered for the numerical computation is that the authors mentioned above calculated the viscous fluid motion, induced by the injection of an electric current, in finite container configurations with a free surface. In this connection it is worth remarking that from a mathematical point of view there is no essential difference in the behaviour of the solutions
when the configuration contains a solid insulating wall or a free surface, for so far it concerns the farthest field solution. In addition the free parameters $\gamma_n$, appearing in $n$ terms of an extended series expansion solution of the Stokes stream function in the far field $r > 1$, will be able to satisfy the boundary conditions at the surface of the container at $n$ places.

The examination carried out by Ajayi, Sozou & Pickering (1984) is executed partly analytically and partly numerically. The calculations show that in the case of an electrode of small radius and a large spherical container the viscous fluid motion breaks down at small values of the hydrodynamic Reynolds number. This behaviour contrasts strongly with that for a shallow container with a large electrode where the breakdown occurs also but only at much larger critical values of the hydrodynamic Reynolds number.

The difference in the behaviour in the two configurations is clarified by the fact that in the latter case the greater part of the fluid is situated in the so-called near field fluid domain, as defined in our consideration of the semi-infinite disk electrode configuration, which results in bounded viscous solutions for all values of the hydrodynamic Reynolds number. Hence we note here a good agreement between the respective calculations.

The conclusions drawn from this partly analytical and partly numerical calculation and our purely analytical examination differ strongly from the results obtained by the purely numerical computations, carried out by the other authors mentioned before. In these numerical calculations no breakdown of the viscous fluid motion was observed or stated at any value of the hydrodynamic Reynolds number.

The cause of the discrepancy between the analytical and numerical calculations can be clarified by the following statements.

Some further analytical calculations in the far field $r > 1$ show that also the higher order solutions $g_n(c)$ seem to possess a similar weak logarithmic singularity in the flow field at the axis of symmetry like the basic governing solution $g_0(c)$. This singularity can be observed only in a numerical computation when a very dense mesh is applied in the neighbourhood of the axis. The number of mesh points used in the numerical calculations considered is too small to be able to recognize the appearance of a logarithmic singularity at the axis of symmetry. Especially since the magnitude of the logarithmic function $\ln(s)$, being the weakest singularity, increases very slowly when $s$ tends to zero.

In the numerical computations the value of the Stokes stream function in the flow field is determined. Our analytical calculations indicate that in the neighbourhood of the axis of symmetry the behaviour of the flow field quantities
is as follows: $\psi = 0(s^2 \ln(s)), \frac{\partial \psi}{\partial s} = 0(s \ln(s)), v_z = \frac{1}{s} \frac{\partial \psi}{\partial s} = 0(\ln(s)),$ where $s$ is the distance from the axis of symmetry.

In some calculations $\delta \theta$ and $\delta \phi$ are used as the respective distance between the mesh points, so that the value of $\delta s$ to the axis is minimal in the neighbourhood of the electrode, increasing to larger values at larger axial distance from the electrode.

In addition, the application of a Gaussian electric current distribution to represent the injection of electric current into the fluid, as done by some authors, implies in fact the introduction of a very stable near field behaviour over the larger part of the fluid domain.

Due to the distance between the respective mesh points and the fact that $\psi = \frac{\partial \psi}{\partial s} = 0$ at $s = 0$ and $\frac{\partial^2 \psi}{\partial s^2} = 0(\ln(s))$, it is clear that a logarithmic behaviour in the flow field at the axis of symmetry, if present, cannot be observed in these numerical calculations.

The above considerations clearly show that the conclusion, whether the viscous fluid motion contains a weak singularity in the flow field at the axis of symmetry at larger distance from the electrode, as found from the work presented in this thesis and corroborated by the calculation carried out by Ajayi, Sozou & Pickering (1984), or that the velocity along the axis of symmetry remains finite in the viscous fluid motion, cannot be drawn from the numerical computations, carried out so far.

This statement leads to the conclusion that a further analytical and numerical calculation and examination of the problem is needed in order to obtain a better understanding of what actually happens in the viscous fluid motion on the axis of symmetry at larger distance from the electrode.

Some attempts to resolve or to reformulate the viscous problem have been made already.

The incorporation of the effect of the electromagnetic induction, especially in the neighbourhood of the axis of symmetry, as suggested by Moffatt (1978), turns out to fail also in the case of the general solutions.

The introduction of a non-zero azimuthal velocity component $v_\phi$ and/or current density $J_\phi$ cannot resolve the problem. In the case of $v_\phi \neq 0$, an additional singularity enters the flow field at the axis of symmetry which implies that the fluid motion is actually generated by the Lorentz force and by a swirling vortex.

Attempts to derive another similarity method in the prototype model did not
succeed. Then the effect of the inertia force always dominates the viscous and Lorentz forces in the curl of the Navier-Stokes equation for $r \to \infty$. The solution of the resultant differential equation cannot satisfy the boundary conditions or it involves a singular behaviour in the flow field at a boundary or at large radial distance.

It has to be remarked that analytical solutions of the viscous fluid motion with finite velocities throughout the entire flow field can be found when an external axial magnetic field is applied. These solutions even do exist for all values of the hydrodynamic Reynolds number at an appropriate value of the strength of the external magnetic field. In this case the meridional fluid motion is accompanied by an azimuthal rotational flow about the axis of symmetry. However the consideration of the application of an external magnetic field is beyond the scope of the present investigation.

Although the derivation of the general solutions, carried out in chapter 6, does not result in solutions which are physically more realistic, one does gain a much better understanding of the mathematical behaviour of the solutions and of the background of this interesting problem. In particular it establishes the limited applicability of the similarity method applied to describe the inviscid and viscous fluid motion in the prototype model and at large radial distance from the electrode in the disk electrode configuration.

Therefore in order to obtain better analytical solutions, if they exist, a reformulation of the whole problem is needed; for example in the way as suggested by Jansen (1984).

Finally it should be noted that even if the laminar problem is fully understood, in practical situations the laminar jet flow will become unstable at some distance from the electrode when the hydrodynamic Reynolds number exceeds some critical value, resulting in transition to a turbulent state.
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APPENDIX A

Theorems concerning the bounds of the basic solutions \( g^2_o(c) \) and \( h^2_o \) of the inviscid point electrode problem \( K_\eta = 0 \) for \( K_\theta \ll 1 \)

In section 3.4 we state that the basic solutions \( g^2_o(c) \) and \( h^2_o \) are bounded by

\[
0 < g^2_o(c) < \frac{1}{2} \quad \text{for} \quad c_o < c < 1, \quad -1 < c_o < 1, \tag{3.44}
\]

\[
\frac{1}{6} < h^2_o < \frac{1}{2} \quad \text{for} \quad -1 < c_o < 1. \tag{3.45}
\]

In order to prove these statements from (3.33) a series expansion of \( g^2_o(c) \) in powers of \( 1-c \) is derived

\[
g^2_o(c) = \sum_{n=1}^{\infty} \beta_n (1-c)^n, \tag{A.1}
\]

for \( c_o < c < 1, \quad -1 < c_o < 1, \quad c \neq -1 \).

The coefficients yield

\[
\beta_1 = a_o - a_2 - 2a_3, \tag{A.2}
\]

\[
\beta_2 = a_2 + \frac{a_3}{2} \left\{ 3 + 2 \ln(2) \right\}, \tag{A.3}
\]

\[
\beta_n = -\frac{8a_3}{2^n n(n-1)(n-2)} \quad \text{for} \quad n = 3, 4, 5, \ldots, \tag{A.4}
\]

where \( a_o - a_3 \) are given by (3.36) - (3.39).

Substitution of the expressions of \( a_o - a_3 \) and expansion of the coefficients \( \beta_1 \) and \( \beta_2 \) in powers of \( 1-c_o \) lead to

\[
\beta_1 = 2 \sum_{n=0}^{\infty} \frac{(1-c_o)^n}{2^n (n+2)(n+3)} > 0 \quad \text{for} \quad -1 < c_o < 1, \tag{A.5}
\]

\[
\beta_2 = -2 \sum_{n=0}^{\infty} \frac{(1-c_o)^{n-1}}{2^n (n+1)(n+3)} < 0 \quad \text{for} \quad -1 < c_o < 1, \tag{A.6}
\]
\[
\beta_n = \frac{16}{2^n n(n-1)(n-2)(1-c_o)^2} > 0 \quad \text{for } -1 < c_o < 1, \quad n = 3, 4, 5, \ldots \tag{A.7}
\]

Combining (A.1) and (A.5) - (A.7) results in

\[
g^2_o(c) = \frac{4(1-c)}{(1-c_o)^2} \sum_{n=1}^{\infty} \frac{T_n(c, c_o)}{2^n n(n+1)(n+2)} \tag{A.8}
\]

where

\[
T_n(c, c_o) = n(1-c_o)^n(1-c)(1-c_o)^n + (1-c)^{n+1}, \tag{A.9}
\]

for \( c_o < c < 1, -1 < c_o < 1 \).

It should be noticed that although the series expansions (A.1), (A.5), (A.6) and the coefficients (A.7) exclude some distinct values of \( c \) and \( c_o \); nevertheless the final result (A.8), (A.9) appears to be valid for all \( c \) and \( c_o \), satisfying \( c_o < c < 1, -1 < c_o < 1 \).

**Theorem 1:** The basic inviscid solution \( g^2_o(c) \), as given by (3.33), (3.36) - (3.39), satisfies \( 0 < g^2_o(c) < \frac{1}{2} \) for \( c_o < c < 1, -1 < c_o < 1 \).

**Proof:** The expression of \( T_n(c, c_o) \) can be rewritten as

\[
T_n(c, c_o) = (1-c_o)^n(n+1)\left[1 - \frac{c-c_o}{1-c_o} \right] - 1 + \frac{(n+1)(c-c_o)}{1-c_o}. \tag{A.10}
\]

Since \( 0 < \frac{c-c_o}{1-c_o} < 1 \) for \( c_o < c < 1 \) and \( n+1 > 2 \), we are able to apply Bernoulli's inequality, viz.

\[
[1 - \frac{c-c_o}{1-c_o}]^{n+1} > 1 - \frac{(n+1)(c-c_o)}{1-c_o} \quad \text{for } n > 1. \tag{A.11}
\]

Hence it follows that \( T_n(c, c_o) > 0 \) for \( n > 1 \) and that \( g^2_o(c) \) is non-negative for \( c_o < c < 1, -1 < c_o < 1 \). Note that \( T_n(c, c_o) = 0 \) for \( c = c_o \) and \( g^2_o(c) = 0 \) for \( c = c_o \).

To prove the upper bound of \( g^2_o(c) \), we use the inequality
\[(1-c)^{n+1} < (1-c)(1-c_0)^n, \quad (A.12)\]

for \(c_0 < c < 1, -1 < c_0 < 1, n > 1\), which can be proved by finite induction. Applying (A.12) to (A.8), (A.9) gives

\[g_o^2(c) < \frac{4(c-c_0)(1-c)}{(1-c_0)^2} \sum_{n=1}^{\infty} \frac{(1-c_0)^n}{2^n(n+1)(n+2)}. \quad (A.13)\]

Some elementary calculations show that

\[0 < \frac{4(c-c_0)(1-c)}{(1-c_0)^2} < 1, \quad (A.14)\]

for \(c_0 < c < 1, -1 < c_0 < 1\). Since \(\frac{1-c_0}{2} < 1\) for \(-1 < c_0 < 1\), it is evident that the series expansion reaches the largest value at \(c_0 = -1\), so that

\[\sum_{n=1}^{\infty} \frac{(1-c_0)^n}{2^n(n+1)(n+2)} < \frac{1}{2}, \quad (A.15)\]

on using relation 0.2431 of Gradshteyn & Ryzhik (1980). Hence from (A.13) - (A.15) it is clear now that \(g_o^2(c)\) is bounded by

\[0 < g_o^2(c) < \frac{1}{2} \quad \text{for } c_0 < c < 1, -1 < c_0 < 1. \quad (A.16)\]

Note that the actual upper bound of \(g_o^2(c)\), obtained by numerical computation for different values of \(c_0\), appears to be 0.3679 at \(c_0 = -1\) and \(c = c_m = 0.2131\). Analytical calculation of the series expansion (A.8), (A.9) and of the analytical expression of \(g_o^2(c)\), see (3.33), (3.36) - (3.39), shows that \(g_o^2(c) = 0\) for \(c_0 = 1\), see (3.40).

**Theorem 2:** The function \(h_o^2\), as given by (3.35) - (3.39), satisfies \(\frac{1}{6} < h_o^2 < \frac{1}{2}\) for \(-1 < c_0 < 1\).
Proof: On using (3.35), (A.2), (A.5), $h_o^2$ can be rewritten as a series expansion

$$h_o^2 = \sum_{n=0}^{\infty} \frac{(1-c_o)^n}{2^n(n+2)(n+3)}$$

for $-1 < c_o < 1$. \hspace{1cm} (A.17)

Since $\frac{dh_o^2}{dc_o} < 0$ for $-1 < c_o < 1$, it is clear that the minimum value of $h_o^2$ coincides with $c_o = 1$. Then only the $n = 0$ term of the series remains, so that the lower bound of $h_o^2$ equals $\frac{1}{6}$. Since $\frac{1-c_o}{2} < 1$ for $-1 < c_o < 1$, the upper bound of the series expansion becomes, see (A.15)

$$h_o^2 < \sum_{n=0}^{\infty} \frac{1}{2^n(n+2)(n+3)} = \frac{1}{2}.$$ \hspace{1cm} (A.18)

Hence we obtain

$$\frac{1}{6} < h_o^2 < \frac{1}{2}$$

for $-1 < c_o < 1$, \hspace{1cm} (A.19)

which proves the theorem.
APPENDIX B

Proofs of the upper- and lower bounds of the base slow-viscous solution $g_o(c)$ for $K_\eta > 1, K_b << 1$

In section 4.2 we state that the basic solution $g_o(c)$, as given by (4.10) - (4.15) is bounded by

$$0 < g_o(c) < g_{o,\text{max}}$$

for $c_o < c < 1, -1 < c_o < 1$, (4.21)

where

$$g_{o,\text{max}} = -\frac{1}{4(1-c_o)^2} \left[ 2(1-c_o) + (3+c_o) \ln \left( \frac{1+c_o}{2} \right) \right] < \frac{1}{4(1+c_o)}. \quad (4.22)$$

To prove the lower- and upper bounds of $g_o(c)$ from (4.10) - (4.15) a composite series expansion in powers of 1-c and 1-c_o is derived, in the same way as done in appendix A. Here it takes the form

$$g_o(c) = \frac{1-c}{4(1-c_o)^2} \sum_{n=0}^{\infty} \frac{T_n(c,c_o)}{2^n(n+2)(n+3)}, \quad (B.1)$$

where

$$T_n(c,c_o) = (n+1)(1-c_o)^{n+2} - (n+2)(1-c)(1-c_o)^{n+1} + (1-c)^{n+2}, \quad (B.2)$$

for $c_o < c < 1, -1 < c_o < 1$.

Theorem 3: The basic slow-viscous solution $g_o(c)$, as given by (4.10) - (4.15), satisfies $0 < g_o(c) < g_{o,\text{max}}$ for $c_o < c < 1, -1 < c_o < 1$, where

$$g_{o,\text{max}} = -\frac{1}{4(1-c_o)^2} \left[ 2(1-c_o) + (3+c_o) \ln \left( \frac{1+c_o}{2} \right) \right] < \frac{1}{4(1+c_o)}. \quad (4.22)$$

Proof: The expression $T_n(c,c_o)$ can be rewritten as

$$T_n(c,c_o) = (1-c_o)^{n+2} \left[ (1- \frac{c-c_o}{1-c_o})^{n+2} - 1 + \frac{(n+2)(c-c_o)}{1-c_o} \right]. \quad (B.3)$$
Since \( 0 < \frac{c - c_o}{1 - c_o} < 1 \) for \( c < c_o < 1 \) and \( n+2 > 2 \), application of Bernoulli's inequality results in

\[
\left[ 1 - \frac{c - c_o}{1 - c_o} \right]^{n+2} > 1 - \frac{(n+2)(c - c_o)}{1 - c_o} \quad \text{for } n > 0 ,
\]

which proves that \( T_n(c, c_o) > 0 \) for \( n > 0 \) and as a consequence that \( g_o(c) \) is non-negative on the open interval \((c_o, 1)\). Note that \( T_n(c, c_o) = 0 \) for \( c = c_o \); and from (B.1) that \( g_o(c) = 0 \) for \( c = c_o, 1 \) as the boundary conditions require.

To prove the upper bound we apply the inequality

\[
(1-c)^{n+2} < (1-c)(1-c_o)^{n+1} ,
\]

for \( c_o < c < 1, -1 < c_o < 1, n > 0 \).

Substitution of (B.5) in (B.2) leads with (B.1) to

\[
g_o(c) < \frac{(c-c_o)(1-c)}{4(1-c_o)^2} \sum_{n=0}^{\infty} \frac{(n+1)(1-c_o)^{n+1}}{2^n(n+2)(n+3)} ,
\]

for \( c_o < c < 1, -1 < c_o < 1 \).

It is easy to verify that

\[
0 < \frac{(c-c_o)(1-c)}{4(1-c_o)^2} < \frac{1}{16} \quad \text{for } c_o < c < 1, -1 < c_o < 1 .
\]

Since \( 1 - c_o < 1 \), it is clear that the series expansion reaches the largest magnitude at \( c_o = -1 \). However due to the \( (n+1) \)-term in the numerator the result is not bounded, but the expression becomes logarithmically singular.

Therefore the series expansion is reduced to transcendental functions.

From (B.7) and some elementary calculations we obtain for \( c_o < c < 1, -1 < c_o < 1 \)

\[
0 < g_o(c) < g_o,_{\text{max}} , \quad (4.21)
\]

where

\[
g_o,_{\text{max}} = -\frac{1}{4(1-c_o)^2} \left[ 2(1-c_o) + (3+c_o) \ln \left( \frac{1+c_o}{2} \right) \right] . \quad (4.22)
\]

On using an upper bound for the logarithmic term in the expression of \( g_o,_{\text{max}} \), viz. (4.1.34) of Abramowitz & Stegun (1972), we find
\[ g_{o, \text{max}} < \frac{1}{4(1+c_o)}, \] (4.22)

for \(-1 < c_o < 1\), which proves the theorem.
APPENDIX C

Derivation of a series expansion and approximations of $K_{\eta, \min}$

The definition of $K_{\eta, \min}$, see (4.90), involves $u(c) > 0$ for $c_0 < c < 1$ and $u(1) = 0$. Hence from (4.88) it follows that the expression of $K_{\eta, \min}$ yields

$$K_{\eta, \min}^2 = \frac{1}{4} \int_{c_0}^{1} \frac{G_\eta(t)}{(1-t)(1+t)^2} u(t) \, dt,$$

where at $K = K_{\eta, \min}$, $u(c)$ is given by

$$u(c) = 1 - \frac{1}{4K_{\eta, \min}^2} \int_{c_0}^{1} \frac{G_\eta(t)}{(1-t)^2} \frac{G_\eta(t)}{(1-t^2)^2} u(t) \, dt.$$

Repeated substitution of (C.2) into (C.1) leads to a series expansion of $K_{\eta, \min}$ of the form

$$K_{\eta, \min}^2 = \frac{1}{4} \sum_{n=0}^{\infty} \frac{V_n}{2^n K_{\eta, \min}^{2n}},$$

where

$$V_0 = \int_{c_0}^{1} \frac{G_\eta(t)}{(1-t)^2} \, dt,$$

$$V_1 = -\int_{c_0}^{1} \frac{G_\eta(t)}{(1-t)^2} \int_{c_0}^{t} \frac{G_\eta(p)}{(1-p)^2} \, dp \, dt,$$

$$V_2 = \int_{c_0}^{1} \frac{G_\eta(t)}{(1-t^2)^2} \int_{c_0}^{t} \frac{G_\eta(p)}{(1-p^2)^2} \int_{c_0}^{p} \frac{G_\eta(s)}{(1-s^2)^2} \, ds \, dp \, dt,$$

or in general at arbitrary $n = 0, 1, 2, ...$

$$V_n = (-1)^n \int_{c_0}^{1} \frac{G_\eta(t_1)}{(1-t_1^2)^2} \int_{c_0}^{t_1} \frac{G_\eta(t_2)}{(1-t_2^2)^2} \int_{c_0}^{t_1-t_2} \frac{G_\eta(t_3)}{(1-t_3^2)} \, dt_3.$$
\[
\begin{align*}
&\frac{t_3}{c_o} \int (t_3 - t_4) \frac{G_{\eta}(t_4)}{(1-t_4^2)^2} \cdots \int (t_n - t_{n+1}) \frac{G_{\eta}(t_{n+1})}{(1-t_{n+1}^2)^2} \times \\
&x \ dt_{n+1} dt_n \cdots \ dt_2 dt_1 ,
\end{align*}
\]
(C.7)

where at small \(K_b\) as to be considered here \(G_{\eta}(c)\) is given by (4.65) - (4.69).

Note that at \(c = 1\) the integrands do not supply any contribution to the integrals as it follows from the series expansion of \(G_{\eta}(c)\) in powers of \(1-c\), viz.

\[
G_{\eta}(c) = \sum_{n=2}^{\infty} \zeta_n (1-c)^n \quad \text{for} \quad c_o < c < 1 , \quad -1 < c_o < 1 ,
\]
(C.8)

where

\[
\zeta_2 = - \frac{(1+3c_o)}{(1-c_o)^3} - \frac{2(1+c_o)^2}{(1-c_o)^4} \ln \left( \frac{1+c_o}{2} \right) > 0 ,
\]
(C.9)

\[
\zeta_n = - \frac{16}{2^n (n-1)(n-2)(1-c_o)^2} \quad \text{for} \quad n = 3, 4, 5, \ldots
\]
(C.10)

From (4.70) it is clear that \((-1)^n V_n > 0\) for \(-1 < c_o < 1\), i.e. the series expansion of \(K_{\eta, \min}^n\) as given by (C.3) is alternating. Moreover it can be proved that \(V_n = 0((1-c_o)^{n+1})\) for \(c_o \rightarrow 1\).

The general expression of \(V_n\) is a multidimensional integral of order \(n+1\). In order to reduce the number of numerical integrations to calculate \(V_n\) it will be useful to introduce the function \(\tilde{V}_n(c)\), satisfying

\[
\tilde{V}_n(c) = \int_{c_o}^c (c-t) \frac{G_{\eta}(t)}{(1-t^2)^2} \tilde{V}_{n-1}(t) \ dt , \quad \text{for} \quad n = 0, 1, 2, \ldots
\]
(C.11)

with

\[
\tilde{V}_{-1}(c) = 1 ,
\]
(C.12)

and where \(\tilde{V}_n(c)\) is related to \(V_n\) by

\[
V_n = (-1)^n \tilde{V}_n(1) .
\]
(C.13)

In this way the number of numerical integrations in order to calculate \(V_n\) is strongly reduced; namely from \((n+1)(n+2)/2\) to \(n+1\). The numerical calculations to calculate \(V_n\) for \(n = 0 - 7\) have been executed in respectively 1000, 2000 and
4000 equidistant steps on \([c_0,1]\) for \(0.2 < c_0 < 1\), \(-0.8 < c_0 < 0.2\) and \(-1 < c_0 < -0.8\). In tables C.1 - C.4 values of \(V_n\) are given as function of \(c_0\).

The tables clearly show the overall strong decay in magnitude of higher order terms with respect to \(V_o\). Due to the strong decay of higher order terms underflow problems (i.e. values smaller than \(10^{-72}\)) arose in the numerical program, especially at \(c = c_o\) for \(c_o + 1\). By temporary multiplication of \(\tilde{V}_n(c)\) by a large number (10\(^50\) or 10\(^60\)) these problems have been resolved.

By truncation of the infinite series expansion of \(K_{\eta, \min}\), see (C.3), after a certain number of terms, different orders of approximations of \(K_{\eta, \min}\): \(\hat{K}_m\) are defined in the following way

\[
(\hat{K}_m(\eta, \min)) = \frac{1}{4} \sum_{n=0}^{m-1} \frac{V_n}{2^n (\hat{K}_n(\eta, \min))^{2n}} \quad \text{for} \quad m = 1, 2, 3, \ldots \tag{C.14}
\]

Values of the respective approximations of \(K_{\eta, \min}\), \(m = 1 - 8\), as function of \(c_0\) for \(-0.99 < c_0 < 0.99\) are presented in tables C.5 - C.8. The results show a tremendous convergence of higher order approximations. For all values of \(c_0\), \(-0.99 < c_0 < 0.99\), \(\hat{K}_8(\eta, \min)\) turns out to be accurate to eight decimal places.

In view of the excellent results obtained by this straightforward calculation of \(K_{\eta, \min}\), attempts have been made to calculate analytical expressions of the coefficients \(V_n\). It could be done for the first two terms. Substitution of (4.65) - (4.69) into (C.4) and (C.5) and some algebra results in the following analytical expressions of \(V_o\) and \(V_1\), viz.

\[
V_o = -\frac{2}{(1-c_0)^2} - \frac{2(3 + c_0)}{(1-c_0)^3} \ln \left(\frac{1+c_0}{2} \right) +
\]

\[
- \frac{2(1+c_0)^2}{(1-c_0)^4} \ln^2 \left(\frac{1+c_0}{2} \right) + \frac{2}{(1-c_0)^2} F_o , \tag{C.15}
\]

\[
V_1 = \frac{2(1 + c_0)}{(1-c_0)^5} + \frac{12(7 + 4c_0 + c_0^2)}{(1-c_0)^6} \ln \left(\frac{1+c_0}{2} \right) +
\]

\[
+ \frac{8(2+c_0)(3+c_0)}{(1-c_0)^7} \ln^2 \left(\frac{1+c_0}{2} \right) + \frac{16(2+c_0)(1+c_0)^2}{(1-c_0)^8} \ln^3 \left(\frac{1+c_0}{2} \right) +
\]

\[
+ \frac{4}{(1-c_0)^4} F_1 - \frac{1}{2} V_0^2 , \tag{C.16}
\]
where \( F_n \) are polylogarithmic integrals of the form

\[
F_n = \int_0^1 \frac{1}{1-t} \ln^{n+1} \left( \frac{1+t}{2} \right) \, dt, \quad \text{for } n = 0, 1, 2, \ldots \tag{C.17}
\]

Properties and values of the polylogarithmic integrals are given by Abramowitz & Stegun (1972, p.1004) and van Wijngaarden (1954).

From the series expansion of the analytical expressions of \( V_0 \) and \( V_1 \), in powers of \( 1-c_0 \) and from composite series expansions of \( C_n(c)/(1-c^2)^2 \) and \( \tilde{V}_n(c) \) for \( n = -1, 0, 1, 2, \ldots \) in powers of \( 1-c \) and \( 1-c_0 \), it can be proved by finite induction that \( V_n = O((1-c_0)^{n+1}) \) for \( c_0 + 1 \), as mentioned before.

From the analytical expressions of \( V_0 \) and \( V_1 \), the first- and second order approximations of \( K_{\eta, \min} \) can be determined. They satisfy, see (C.14)

\[
\tilde{K}_{\eta, \min}^{(1)} = \frac{\sqrt{V_0}}{2}, \tag{C.18}
\]

\[
\tilde{K}_{\eta, \min}^{(2)} = \sqrt{\frac{V_0}{8} + \frac{1}{8} \sqrt{\frac{V_0}{8} + 4V_1}}, \tag{C.19}
\]

where the correct root of the quadratic equation of \( \tilde{K}_{\eta, \min}^{(2)} \) is derived from the expression of \( \tilde{K}_{\eta, \min}^{(1)} \). On the interval \(-0.99 < c_0 < 0.99\) the relative errors of respectively \( \tilde{K}_{\eta, \min}^{(1)} \) and \( \tilde{K}_{\eta, \min}^{(2)} \) turn out to be less than \( 1.5 \times 10^{-1} \) and \(-2.8 \times 10^{-2}\), whereby the magnitude of the relative errors becomes maximal at \( c_0 \sim 1 \) and minimal at \( c_0 \sim -1 \).

Higher order approximations of \( K_{\eta, \min} \) can be calculated numerically more easily with the help of successive approximations. On using Newton's method, see Scheid (1968, p.310), and \( \tilde{K}_{\eta, \min}^{(m-1)} \) as startvalue, the results converge very quickly to the proper value of \( \tilde{K}_{\eta, \min}^{(m)} \).
<table>
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<th>$c_o$</th>
<th>$V_o$</th>
<th>$V_1$</th>
</tr>
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<tbody>
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<td>$-3.90839194 \times 10^{-9}$</td>
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<td>$7.18974734 \times 10^{-4}$</td>
<td>$-1.02964333 \times 10^{-7}$</td>
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<td>$-4.40232628 \times 10^{-7}$</td>
</tr>
<tr>
<td>$\frac{1}{2}\sqrt{3}$</td>
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<td>$-8.27402751 \times 10^{-7}$</td>
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<td>0.8</td>
<td>$3.20536825 \times 10^{-3}$</td>
<td>$-2.02010828 \times 10^{-6}$</td>
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<td>$\frac{1}{2}\sqrt{2}$</td>
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<td>$-4.94759649 \times 10^{-6}$</td>
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Table C.8.
APPENDIX D

The behaviour of the function $u(c)$ at small $K_\eta$

In order to determine an asymptotic solution of the function $u(c)$ at small values of $K_\eta$ a singular perturbation technique is applied to the differential equation of $u(c)$, (4.78)

$$\frac{d^2u}{dc^2} + \frac{G_\eta(c)}{4K_\eta^2(1-c^2)^2} u = 0 .$$  \hspace{1cm} (D.1)

The behaviour of $G_\eta(c)$, see (2.56), (2.57), (4.70) - (4.72), and the series expansion of $G_\eta(c)$, see (C.8) - (C.10) of appendix C, indicate that the coefficient multiplying $u$ is positive and bounded on $c_0 < c < 1$ with a single zero at $c = c_0$. Hence the boundary $c = c_0$ is a so-called turning point.

Application of the usual WKBJ approximation to determine asymptotic solutions at small $K_\eta$ fails in the neighbourhood of the turning point, with the consequence that the complementary function of $u(c)$ cannot satisfy the boundary conditions, see (4.79), (4.80). Nevertheless uniformly valid expansions for all $c \in [c_0, 1]$ can be obtained by expressing the asymptotic expansions in terms of non-elementary functions which have the same qualitative features as the solutions of the differential equation of $u(c)$.

Application of the so-called Langer transformation, see Nayfeh (1973,p.339), isolates the turning point character of the differential equation.

Therefore we introduce the new independent variable $z = z(c)$, the new dependent variable $v(z)$, related to $u(c)$ by

$$v(z) = \left(\frac{dz}{dc}\right)^2 u(c) ,$$  \hspace{1cm} (D.2)

and for convenience the new parameter

$$\lambda = \frac{1}{K_\eta} ,$$  \hspace{1cm} (D.3)

which is assumed to be large.

Following Nayfeh (1973,p.340) the new independent variable equals

$$z(c) = \left\{\frac{3}{4} \int_{c_0}^c \sqrt{\frac{G_\eta(t)}{1-t^2}} dt \right\}^\frac{2}{3} > 0 .$$  \hspace{1cm} (D.4)
From (4.70) - (4.72) it is clear that \( z = 0 \) at \( c = c_0 \) and that \( z(c) \) is positive and bounded on \( c_0 < c < 1 \).

The differential equation of \( v(z) \) now takes the form

\[
\frac{d^2 v}{dz^2} + \lambda^2 z v = \delta(z) v, \tag{D.5}
\]

where

\[
\delta(z) = \frac{1}{4} \left( \frac{dz}{dc} \right)^{-4} \left[ 2 \frac{dz}{dc} \frac{d^2 z}{dc^2} - 3 \left( \frac{d^2 z}{dc^2} \right)^2 \right]. \tag{D.6}
\]

On applying (D.4), the series expansion of \( G_\eta(c) \) in powers of \( 1-c \), see (C.8) - (C.10) of appendix C, and the series expansion of \( z(c) \) in powers of \( c-c_0 \), see (D.36) - (D.39), it will become evident that \( \delta(z) \) is bounded on \( c_0 < c < 1 \).

In order to derive asymptotic solutions at large \( \lambda \), i.e. small \( K_\eta \), we introduce the general solution of (D.5) of the form

\[
v(z) = C v_1(z) + D v_2(z), \tag{D.7}
\]

with solutions of the form as given by Olver (1974,p.408)

\[
v_1(z) = A_i(-\lambda^{2/3} z) \sum_{n=0}^\infty \frac{A_n(z)}{\lambda^{2n}} + A_i'(-\lambda^{2/3} z) \sum_{n=0}^\infty \frac{B_n(z)}{\lambda^{4n/3}}, \tag{D.8}
\]

and

\[
v_2(z) = B_i(-\lambda^{2/3} z) \sum_{n=0}^\infty \frac{A_n(z)}{\lambda^{2n}} + B_i'(-\lambda^{2/3} z) \sum_{n=0}^\infty \frac{B_n(z)}{\lambda^{4n/3}}, \tag{D.9}
\]

where \( C \) and \( D \) are constants to be determined by the boundary conditions of \( u(c) \) and \( A_i \) and \( B_i \) are Airy functions, see Olver (1974,p.392) and Abramowitz & Stegun (1972,p.446). The coefficients \( A_n(z) \) and \( B_n(z) \) are determined by \( \delta(z) \), see (D.6).

Note that the prime denotes differentiation to the argument, i.e. \( A_i'(x) = \frac{d}{dx} (A_i(x)) \).

Differentiating of respectively (D.8) or (D.9), using the differential equation of the Airy functions \( A_i \) and \( B_i \): \( A_i''(-\lambda^{2/3} z) = -\lambda^{2/3} z A_i(-\lambda^{2/3} z) \), substituting in the differential equation of \( v(z) \), see (D.5) and comparing like powers of \( \lambda \) result in coupled differential equations of the coefficients \( A_n(z) \) and \( B_n(z) \)

\[
\frac{d^2 A_n}{dz^2} - \delta(z) A_n(z) + 2z \frac{dB_n}{dz} + B_n(z) = 0, \tag{D.10}
\]
and
\[
\frac{d^2 B_{n-1}}{dz^2} - \delta(z) B_{n-1}(z) - 2 \frac{dA_n}{dz} = 0 ,
\]  
(D.11)

for all \( z \) and \( n = 0,1,2, \ldots \).

It should be noticed that in the above equations and in further expressions given in this appendix coefficients with negative suffix are interpreted as being identical to zero.

Rewriting into the independent variable \( c \), using (D.6) and integration of these two equations produce expressions of the coefficients \( A_n(c) \) and \( B_n(c) \) in the form of integral equations,

\[
A_0(c) = 1 ,
\]  
(D.12)

\[
A_{n+1}(c) = A_{n+1}(c_0) + \frac{c}{c_0} \int c_0 \left[ \frac{dz}{dt} \right] - \frac{1}{2} \frac{d^2}{dt^2} \left[ \frac{dz}{dt} \right] - \frac{1}{2} B_n(t) \] \( dt \),  
(D.13)

\[
B_n(c) = - \frac{1}{2\sqrt{z(c)}} \int c_0 \frac{1}{\sqrt{z(t)}} \left[ \frac{dz}{dt} \right] - \frac{1}{2} \frac{d^2}{dt^2} \left[ \frac{dz}{dt} \right] - \frac{1}{2} A_n(t) \] \( dt \),  
(D.14)

for \( n = 0,1,2, \ldots \).

Successive application of (D.12) - (D.14) in a sequence \( A_0, B_0, A_1, B_1, \) etcetera determines the coefficients. As pointed out by Olver (1974,p.410) we may set \( A_0(c) \) equal to unity without loss of generality; in the same way we choose \( A_n(c_0) = 1 \).

From (D.2), (D.7) and some elementary calculations it is clear that the boundary conditions of \( u(c) \), see (4.79), (4.80) are satisfied when \( C \) and \( D \) obey the relations

\[
C v_1(0) + D v_2(0) = \left( \frac{dz}{dc} \right)^{1/2} \] \( c=c_0 \),  
(D.15)

\[
C \left( \frac{dv_1}{dz} \right)_{z=0} + D \left( \frac{dv_2}{dz} \right)_{z=0} = \frac{1}{2} \left( \frac{dz}{dc} \right)^{3/2} \frac{d^2z}{dc^2} \] \( c=c_0 \),  
(D.16)

leading to the following expressions of the constants \( C \) and \( D \).
\[
C = \frac{\left[ \frac{dz}{dc} \right]^{1/2} \frac{dv_2}{dz}}{\left[ \frac{dz}{dc} \right]^{1/2} v_2(0) - \frac{1}{2} \frac{dz}{dc} \frac{d^2 v_2}{dc^2} \frac{v_2(0)}{v_2(0)}}, \quad (D.17)
\]

\[
D = \frac{\left[ \frac{dz}{dc} \right]^{1/2} \frac{dv_1}{dz}}{\left[ \frac{dz}{dc} \right]^{1/2} v_1(0) - \frac{1}{2} \frac{dz}{dc} \frac{d^2 v_1}{dc^2} \frac{v_1(0)}{v_1(0)}}. \quad (D.18)
\]

By use of (D.8), (D.9), the differential equation and the Wronskian of the Airy functions and some algebra we obtain

\[
C = \frac{\pi (Bi(0) \Pi + Bi'(0) T)}{2 \frac{dz}{dc}^{1/2} \Omega}, \quad (D.19)
\]

\[
D = -\frac{\pi (Ai(0) \Pi + Ai'(0) T)}{2 \frac{dz}{dc}^{1/2} \Omega}, \quad (D.20)
\]

where

\[
\Pi = \lambda^{-2/3} \sum_{n=0}^{\infty} \frac{1}{\lambda^{2n}} \left[ \frac{d^2 z}{dc^2} A_n(c_o) - 2 \frac{dz}{dc} \frac{dA}{dc} \right], \quad (D.21)
\]

\[
T = \sum_{n=0}^{\infty} \frac{1}{\lambda^{2n}} \left[ 2 \frac{dz}{dc} \frac{d^2 z}{dc^2} A_n(c_o) + \frac{dB}{dc} \frac{B_{n-1}(c_o)}{dc} - 2 \frac{dz}{dc} \frac{dB_{n-1}}{dc} \right], \quad (D.22)
\]

\[
\Omega = \sum_{n=0}^{\infty} \frac{1}{\lambda^{2n}} \left[ \frac{dz}{dc} \Sigma_{m=0}^{n} A_m(c_o) A_{n-m}(c_o) - \Sigma_{m=0}^{n-1} \frac{dA}{dc} \right] + \frac{dB_{n-1}}{dc} \right] c=c_o, \quad (D.23)
\]

with, see Abramowitz & Stegun (1972, p.446)
\[ A_1(0) = \frac{B_1(0)}{\sqrt{3}} = \frac{1}{3^{2/3} \Gamma(\frac{2}{3})} \]  
\[ (D.24) \]

\[ A_1'(0) = -\frac{B_1'(0)}{\sqrt{3}} = -\frac{1}{3^{1/3} \Gamma(\frac{1}{3})} \]  
\[ (D.25) \]

and \( A_n(c_o) = 1 \), see \((D.12), (D.13)\).

We restate that coefficients with negative suffix should be interpreted as being identical to zero.

Unfortunately, the expressions of \( \frac{dz}{dc} \bigg|_{c=c_o} \) and \( \frac{d^2z}{dc^2} \bigg|_{c=c_o} \), appearing in the preceding expressions, have to be calculated in a rather cumbersome and laborious way. Starting from the analytical expression of \( G_\eta(c) \), see \((4.65) - (4.69)\), a series expansion of \( G_\eta(c) \) in powers of \( c-c_o \) has been derived, which takes the form

\[ G_\eta(c) = \sum_{n=1}^{\infty} \tau_n (c-c_o)^n \quad \text{for} \quad c_o < c < 1+2c_o, \quad -1 < c_o < 1 \]  
\[ (D.26) \]

where

\[ \tau_1 = \frac{2(3+c_o)}{(1-c_o)^2} + \frac{8(1+c_o)}{(1-c_o)^3} \ln \left( \frac{1+c_o}{2} \right) \]  
\[ (D.27) \]

\[ \tau_2 = -\frac{(1+3c_o)}{(1-c_o)^3} - \frac{8c_o}{(1-c_o)^4} \ln \left( \frac{1+c_o}{2} \right) \]  
\[ (D.28) \]

\[ \tau_n = -\frac{4(-1)^n}{n(n-1)(n-2)(1-c_o)^2(1+c_o)^n} \quad \text{for} \quad n = 3, 4, 5, \ldots \]  
\[ (D.29) \]

Moreover, expansion of the coefficients \( \tau_1 \) and \( \tau_2 \) in powers of \( 1-c_o \) yields

\[ \tau_1 = 2 \sum_{n=0}^{\infty} \frac{(1-c_o)^n}{2^n(n+2)(n+3)} > 0 \quad \text{for} \quad -1 < c_o < 1, \]  
\[ (D.30) \]

\[ \tau_2 = -\frac{1}{2} \sum_{n=-1}^{\infty} \frac{(n+5)(1-c_o)^n}{2^n(n+3)(n+4)} < 0 \quad \text{for} \quad -1 < c_o < 1, \]  
\[ (D.31) \]
which determine the signs of the coefficients.

Above series expression enables us to evaluate a series expansion of the function \( \sqrt{\frac{G_n(c)}{1-c^2}} \), appearing in the integral expression of \( z(c) \), see (D.4), in powers of \( c-c_o \), satisfying

\[
\sqrt{\frac{G_n(c)}{1-c^2}} = \sum_{n=0}^{\infty} v_n (c-c_o)^{n+\frac{1}{2}},
\]

(D.32)

where \( v_n \) is related to \( \tau_n \) by

\[
\tau_n = (1-c_o^2)^n \sum_{m=0}^{n-1} v_m v_{n-m-1} - 4c_o (1-c_o^2) \sum_{m=0}^{n-2} v_m v_{n-m-2} +
\]

\[- 2(1-3c_o^2) \sum_{m=0}^{n-3} v_m v_{n-m-3} + 4c_o \sum_{m=0}^{n-4} v_m v_{n-m-4} + \sum_{m=0}^{n-5} v_m v_{n-m-5},
\]

for \( n = 1, 2, 3, \ldots \)

(D.33)

and series with a negative upper bound, i.e. smaller than the lower bound \( m=0 \), are interpreted as zero.

The first two terms are found to be

\[
v_0 = \frac{\sqrt{\tau_1}}{1-c_o^2} > 0 \quad \text{for} \quad -1 < c_o < 1 ,
\]

(D.34)

\[
v_1 = \frac{4c_o \tau_1 + (1-c_o^2) \tau_2}{2(1-c_o^2)^3 v_o} < 0 \quad \text{for} \quad -1 < c_o < 1 .
\]

(D.35)

Substitution of this series expansion into the expression of \( z(c) \) and integration with respect to \( c \) lead to a series expansion of \( z(c) \) in powers of \( c-c_o \) of the form

\[
z(c) = \sum_{n=1}^{\infty} \rho_n (c-c_o)^n,
\]

(D.36)

where \( \rho_n \) is related to \( v_n \) by
\[ \sum_{m=0}^{n} \rho_{m} \prod_{k=0}^{n-m} \rho_{k+1} \rho_{m-k+1} = \frac{9}{4} \sum_{m=0}^{n} \frac{v_m v_{n-m}}{(2m+3)(2n-2m+3)} \quad \text{for } n = 0, 1, 2, \ldots \] (D.37)

The first two coefficients are found to be

\[ \rho_1 = \frac{v_1}{2} = \frac{\tau_1^{1/3}}{2^{2/3} (1-c_o^2)^{2/3}} > 0 \quad \text{for } -1 < c_o < 1, \] (D.38)

\[ \rho_2 = \frac{v_1}{5} \frac{v_0}{2} - 1/3 = \frac{4c_o \tau_1 + (1-c_o^2)\tau_2}{5.2^{2/3} (1-c_o^2)^{5/3} \tau_1^{2/3}} < 0 \quad \text{for } -1 < c_o < 1. \] (D.39)

In view of the finite convergence radius of the series expansion (D.26) from which the series expansion of \( z(c) \) has been derived, there is no reason to expect that the convergence radius of the latter expansion will be much larger. Nevertheless it will be sufficient to determine the derivatives of \( z(c) \) at \( c = c_o \), as they appear in expressions (D.15) - (D.23).

Differentiation of the series expansion of \( z(c) \), see (D.36), finally results in expressions of \( \frac{dz}{dc} \big|_{c=c_o} \) and \( \frac{d^2z}{dc^2} \big|_{c=c_o} \), viz.

\[ \frac{dz}{dc} \big|_{c=c_o} = \rho_1 = \frac{\tau_1^{1/3}}{2^{2/3} (1-c_o^2)^{2/3}} > 0, \] (D.40)

\[ \frac{d^2z}{dc^2} \big|_{c=c_o} = 2\rho_2 = \frac{2^{1/3} [4c_o \tau_1 + (1-c_o^2)\tau_2]}{5(1-c_o^2)^{5/3} \tau_1^{2/3}} < 0, \] (D.41)

for \( -1 < c_o < 1 \), where \( \tau_1 \) and \( \tau_2 \) are given by (D.27) and (D.28).

In the same laborious way higher order derivatives of \( z(c) \) at \( c = c_o \) can be determined, generally satisfying

\[ \frac{d^n z}{dz^n} \big|_{c=c_o} = n! \rho_n \quad \text{for } n = 1, 2, 3, \ldots \] (D.42)

Note that only \( z(c_o) = 0 \), whereas the higher order derivatives of \( z(c) \) are bounded and in principle non-zero at \( c = c_o \) for \( -1 < c_o < 1 \).

The complete asymptotic expansion of \( u(c) \), being uniformly valid on the entire interval \( c_o \leq c < 1 \), is now obtained by combining the results given in (D.2) -
In order to determine the leading terms of the asymptotic expansion of $u(c)$ at small $K_n$, we make use of the following asymptotic expansions of the Airy functions and their first derivatives valid for large negative value of the argument, see Olver (1974, p. 392-393)

\[
\text{Ai}(-\lambda^{2/3}z) = \frac{\lambda^{-1/6} - 1/4}{\sqrt{\pi}} \left[ \cos \left( \xi - \frac{\pi}{4} \right) \sum_{n=0}^{\infty} \frac{(-1)^n u_{2n}}{\xi^{2n}} + \right.

+ \sin \left( \xi - \frac{\pi}{4} \right) \sum_{n=0}^{\infty} \frac{(-1)^n u_{2n+1}}{\xi^{2n+1}} \bigg] \right),
\]

(D.43)

\[
\text{Ai}'(-\lambda^{2/3}z) = \frac{\lambda^{1/6} z^{-1/4}}{\sqrt{\pi}} \left[ \sin \left( \xi - \frac{\pi}{4} \right) \sum_{n=0}^{\infty} \frac{(-1)^n v_{2n}}{\xi^{2n}} + \right.

- \cos \left( \xi - \frac{\pi}{4} \right) \sum_{n=0}^{\infty} \frac{(-1)^n v_{2n+1}}{\xi^{2n+1}} \bigg] \right),
\]

(D.44)

\[
\text{Bi}(-\lambda^{2/3}z) = \frac{\lambda^{-1/6} - 1/4}{\sqrt{\pi}} \left[ -\sin \left( \xi - \frac{\pi}{4} \right) \sum_{n=0}^{\infty} \frac{(-1)^n u_{2n}}{\xi^{2n}} + \right.

+ \cos \left( \xi - \frac{\pi}{4} \right) \sum_{n=0}^{\infty} \frac{(-1)^n u_{2n+1}}{\xi^{2n+1}}, \bigg] \right),
\]

(D.45)

\[
\text{Bi}'(-\lambda^{2/3}z) = \frac{\lambda^{1/6} z^{-1/4}}{\sqrt{\pi}} \left[ \cos \left( \xi - \frac{\pi}{4} \right) \sum_{n=0}^{\infty} \frac{(-1)^n v_{2n}}{\xi^{2n}} + \right.

+ \sin \left( \xi - \frac{\pi}{4} \right) \sum_{n=0}^{\infty} \frac{(-1)^n v_{2n+1}}{\xi^{2n+1}} \bigg] \right),
\]

(D.46)

where
\[ \xi = \frac{2}{3} \lambda z^{3/2} = \frac{\lambda}{2} \int_{c_0}^{c} \frac{\sqrt{G_\eta(t)}}{1-t^2} \, dt, \]  
(D.47)

\[ u_o = v_o = 1, \]  
(D.48)

and for \( n = 1, 2, 3, \ldots \)

\[ u_n = \frac{\Gamma(3n + \frac{1}{2})}{(54)^n n!} \frac{\Gamma(n + \frac{1}{2})}{(216)^n n!}, \]  
(D.49)

\[ v_n = -\frac{(6n+1)}{(6n-1)} u_n, \]  
(D.50)

for \( \frac{3\lambda}{4} \int_{c_0}^{c} \frac{\sqrt{G_\eta(t)}}{1-t^2} \, dt \gg 1. \)

In view of the above validity condition it is clear that the asymptotic expansions of the Airy functions exclude the turning point \( c = c_0 \).

Application of the asymptotic expansions of the Airy functions to the expressions of \( v_1(z) \) and \( v_2(z) \), see (D.8), (D.9) and employing (D.2), (D.3), (D.7), (D.19) - (D.25) effect an asymptotic expansion solution of \( u(c) \) at small \( K_\eta \), being valid on \( c_0 < c < 1 \).

It can be verified that the first four terms of this expansion are of the form

\[ u(c) = \sqrt{\pi} z^{-1/4} \left( \frac{dz}{dc} \right)^{1/2} \left( \frac{dz}{dc} \right)^{-3/2} K^{1/6} \eta \left[ -2 \left( \frac{dz}{dc} \right)^2 c = c_0 \right. \]  

\[ \left. + \left( \frac{d^2z}{dc^2} \right) c = c_0 \right. \]  

\[ \left. \right. \left. A_1(0) \sin \left( \xi + \frac{\pi}{12} \right) K^{2/3} + \right. \]

\[ -2 \sqrt{z} \left( \frac{dz}{dc} \right)^2 c = c_0 \left[ \frac{5}{48z^2} \right. + B_{o(c)} \]  

\[ \left. \right. \left. \right. \left. A_1(0) \sin \left( \xi - \frac{\pi}{12} \right) K_\eta + \right. \]

\[ -\sqrt{z} \left( \frac{d^2z}{dc^2} \right) c = c_0 \left[ \frac{5}{48z^2} \right. + B_{o(c)} \]  

\[ \left. \right. \left. \right. \left. A_1(0) \cos \left( \xi + \frac{\pi}{12} \right) K^{5/3} + \right. \]
\[ + O(\kappa_\eta^2) \]
for \( \kappa_\eta \ll 1, c \neq c_0 \), \hspace{1cm} (D.51)

where
\[ \xi = \frac{1}{2K_\eta} \int_{c_0}^{c} \frac{\sqrt{G(t)}}{1-t^2} \, dt, \hspace{1cm} (D.52) \]

and for expressions of the other terms in (D.51) we refer to the preceding text of this appendix.

It has to be repeated that due to the asymptotic expansions of the Airy functions, the above asymptotic solution of \( u(c) \) is not uniformly valid; namely \( c = c_0 \) needs to be excluded. An asymptotic solution of \( u(c) \) valid at \( c = c_0 \) can be derived from series expressions of the Airy functions valid for small values of the argument, see e.g. Abramowitz & Stegun (1972, p.446).

From the four leading terms of the asymptotic expansion of \( u(c) \) and the definition of \( K_{\eta,\min} \), with \( u(1) = 0 \) and \( u(c) > 0 \) on \( c_0 < c < 1 \) see (4.90), we are able to derive several approximations of \( K_{\eta,\min} \): \( \tilde{K}_{\eta,\min}^{(i)} \) in concerning one to four terms \( (i = I, II, III, IV) \) of the asymptotic expansion (D.51). Note that these approximations of \( K_{\eta,\min} \), being only valid at small \( \kappa_\eta \) as the asymptotic expansion of \( u(c) \) prescribes, differ significantly from the general approximations of \( K_\eta \) as given in appendix C which are in principle valid for all values of \( K_{\eta,\min} \).

They take in general the forms
\[ K_{\eta,\min} = \tilde{K}_{\eta,\min}^{(i)} \quad \text{for } i = I, II, III, IV, \hspace{1cm} (D.53) \]

with
\[ \tilde{K}_{\eta,\min}^{(i)} = \frac{6}{(7+12k)\pi + 12a^{(i)}} \int_{c_0}^{c} \frac{\sqrt{G(t)}}{1-t^2} \, dt, \hspace{1cm} (D.54) \]

for \( k=0, K_{\eta,\min} \ll 1 \),

where
\[ a^{(I)} = 0, \hspace{1cm} (D.55) \]
\[ a^{(II)} = - \arctan \left\{ \sqrt{3} \beta_1 (\tilde{K}_{\eta,\min}^{(II)})^{2/3} \over 1 - \beta_1 (\tilde{K}_{\eta,\min}^{(II)})^{2/3} \right\}, \hspace{1cm} (D.56) \]
\[ \alpha(\text{III}) = - \arctan \left[ \frac{\sqrt{3} \beta_1(\tilde{K}_{\eta,\min}^{(\text{III})})^{2/3} - \beta_2(\tilde{K}_{\eta,\min}^{(\text{III})})}{1 - \beta_1(\tilde{K}_{\eta,\min}^{(\text{III})})^{2/3}} \right] \]  

\[ \alpha(\text{IV}) = - \arctan \left[ \frac{\sqrt{3} \beta_1(\tilde{K}_{\eta,\min}^{(\text{IV})})^{2/3} - \beta_2(\tilde{K}_{\eta,\min}^{(\text{IV})}) + \beta_1 \beta_2(\tilde{K}_{\eta,\min}^{(\text{IV})})^{5/3}}{1 - \beta_1(\tilde{K}_{\eta,\min}^{(\text{IV})})^{2/3} + \sqrt{3} \beta_1 \beta_2(\tilde{K}_{\eta,\min}^{(\text{IV})})^{5/3}} \right] \]  

with
\[ \beta_1 = \left. \frac{d^2z}{dc^2} \right|_{c=c_0} \frac{A_i(0)}{A_i'(0)} > 0 \]  

\[ \beta_2 = \sqrt{z(1)} \left\{ \frac{5}{48z^2(1)} + B_0(1) \right\} \]  

An advantage of the expression of \( \tilde{K}_{\eta,\min}^{(1)} \) in the form as represented by (D.54) is the fact that it presents an approximated value of \( K_{\eta,\min} \) in case of \( k = 0 \). But additionally it produces for non-zero values of \( k = 1, 2, 3, \ldots \) distinct approximated values of \( K_{\eta} \), when an additional zero of \( u(c) \) enters the interval \( c_0 < c < 1 \) via \( c = 1 \). In other words the expression clearly shows the oscillatory behaviour of \( u(c) \) at sufficiently small values of \( K_{\eta} \).

From figure 4.4 and table 4.1, see section 4.3, it is evident that \( K_{\eta,\min} \) only becomes really small when \( c_0 \) approaches unity. Application of (4.74) to (D.54), calculation and expansion in powers of \( 1-c_0 \) of the terms appearing in (D.55) - (D.60) and much elementary algebra enable us to derive approximate analytical expressions of (D.54) - (D.60) for the limit case \( c_0 \to 1 \) of the form

\[ K_{\eta,\min} = \tilde{K}_{\eta,\min}^{(i)} \]  

for \( i = I, \ II, \ III, \ IV \),

\[ \tilde{K}_{\eta,\min}^{(i)} = \frac{4 \sqrt{1-c_0}}{(7\pi + 2\alpha^{(1)}(1)) \sqrt{3} (1+c_0)} (1 + o(1)) \]  

where
\[ \alpha^{(1)} = 0 \]
\[ a^{(II)} = -\sqrt{3} \beta_1 \left( \frac{\eta}{\eta_{\text{min}}} \right)^{2/3} (1 + o(1)), \]  
(D.64)

\[ a^{(III)} = a^{(IV)} = \frac{5(1+c)}{84\pi} (1 + o(1)), \]  
(D.65)

with

\[ \beta_1 = \frac{9\sqrt{\pi}}{40 \Gamma(\frac{5}{6})} (1-c_o^2)^{2/3} (1 + o(1)), \]  
(D.66)

\[ \beta_2 = \frac{5\sqrt{3}}{24\sqrt{1-c_o}} (1 + o(1)), \]  
(D.67)

for \( c_o + 1 \).

Note that for the calculation of (D.65) we made use of (D.67), (D.72).

An asymptotic expression which describes accurately the behaviour of \( K_{\eta, \text{min}} \) when \( c_o \) tends to unity has been obtained in the following way. From (4.65) - (4.69) a series expansion of \( G_{\eta}(c)/(1-c^2)^2 \) in powers of \( 1-c \) has been derived, satisfying

\[ \frac{G_{\eta}(c)}{(1-c^2)^2} = \sum_{n=0}^{\infty} \xi_n (1-c)^n \quad \text{for} \quad c_0 < c < 1, -1 < c_o < 1, \]  
(D.68)

where

\[ \xi_n = -\frac{(n+1)}{2(n+2)(1-c_o)^3} \left[ (n+1) + (n+3) c_o \right] - \frac{(n+1)(1+c_o)}{2(1-c_o)^4} \ln \left( \frac{1+c_o}{2} \right), \]  
(D.69)

for \( n = 0, 1, 2, \ldots \), from which a series expansion of \( \sqrt{G_{\eta}(c)}/1-c^2 \) in powers of \( 1-c \) has been deduced.

Substitution into the integral appearing in (D.54), evaluation of a certain number of terms and recognizing of the general form of the terms of this series expansion establish an asymptotic expression of \( K_{\eta, \text{min}} \), being valid when \( c_o \) approaches unity, viz.

\[ K_{\eta, \text{min}} = \delta_1 \sqrt{1-c_o} (1 + o(1)) \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad c_o + 1, \]  
(D.70)

where \( \delta_1 \) and \( \delta_2 \), respectively applying (D.63) and (D.65), take the forms
\[ \delta_1 = \frac{2}{7\pi \sqrt{3}}, \quad (D.71) \]

\[ \delta_2 = \frac{14\pi}{(49\pi^2 + 10)\sqrt{3}}. \quad (D.72) \]

It turns out that the value of \( \delta_2 \) agrees the best with values obtained by extrapolation of numerical results for \( c_0 \sim 1 \).

It should be noticed that without numerical integration and especially formula manipulation higher order terms of this singular perturbation expansion at arbitrary value of \( c_0 \) cannot be obtained. The main problems arise in the calculation of the coefficients \( A_n(c) \) and \( B_n(c) \), see (D.8), (D.9), (D.13), (D.14).