Korn Inequalities for a Reinforced Plate

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Abstract Asymptotically optimal Korn inequalities are derived for a composite material that consists of two families of stiff rods surrounded by a homogeneous soft material. The composite plate is fixed through the protruding stiff rods only. The asymptotic behaviour is shown to be crucially different for families of connected rods and for those where the rods are isolated.

Keywords Korn inequality · Homogenisation · Reinforced plate

Mathematics Subject Classification (2000) 74K30 · 35Q72 · 74K10

1 Introduction

1.1 Definition of a Reinforced Plate

We consider a composite material that consists of a perpendicular mesh of two families of stiff fibres surrounded by a matrix of homogeneous softer material. These stiff fibres pro-

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Fig. 1 A sketch of the composite material containing two families of fibers in a matrix of another material. The matrix of which the band is the lateral surface is supposed to consist of softer material

Fig. 2 Isolated rods and welded rods

Fig. 3 A periodicity cell with disjoint rods

Fig. 4 A periodicity cell with welded rods

trude from the soft elastic material and are fixed at their ends. See Fig. 1. We will compare the asymptotic behaviour for two different cases, namely, 1) the case that the two families of fibers are only connected through the surrounding softer material, and 2) the fibers themselves are connected whenever they cross. This last situation can be considered as the two families of rods being welded. See Fig. 2. Away from the sides the material is periodic in both horizontal directions. For a typical periodicity cell see Figs. 3 and 4.

On first sight both situations might seem quite similar. A closer observation however will show that the asymptotic behaviour in these two situations will differ crucially. The most striking difference is that the limit problem in the first case is no longer elliptic.

The precise setting is as follows.

- The parameter $h \ll 1$ will denote the thickness of the plate. Moreover h will also be the distance of two consecutive rods and we also fix the diameter of the rods as $\mathcal{O}(h)$.
- The open cross-sections of the unscaled rods we name $\pi_1, \pi_2 \subset \mathbb{R}^2$ with $\overline{\pi_1} \subset (-\frac{1}{2}, \frac{1}{2}) \times [0, \frac{1}{2})$, and $\overline{\pi_2} \subset (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, 0]$. We assume that the two sets of rods will not intersect but might have a common face (a ‘welded’ connection) at level $z = 0$.

We define for each $j \in \mathbb{Z}$ the infinite rods

$$\Pi_h^{1,j} := \{(y_1, y_2, z) \in \mathbb{R}^3; h^{-1}(y_2 - j, z) \in \pi_1\},$$

$$\Pi_h^{2,j} := \{(y_1, y_2, z) \in \mathbb{R}^3; h^{-1}(y_1 - j, z) \in \pi_2\}.$$

In principle these rods could also ‘intersect’; the result will be identical except for the constants. It would slightly complicate the model. We do not consider rods that touch in a set with Hausdorff dimension less than 2.

- For the entire plate we take the cylinder

$$\Omega_h := \omega \times \left(-\frac{1}{2}h, \frac{1}{2}h\right), \quad (1.1)$$

where $\omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain. We will also need the h -neighborhood of ω :

$$\omega(h) := \{y \in \mathbb{R}^2; \text{dist}(y, \omega) < h\}.$$

- The rods are supposed to be fixed outside of $\omega(h) \times (-\frac{1}{2}h, \frac{1}{2}h)$, that is, we consider

$$\Pi^{i,j}(h) := \Pi_h^{i,j} \cap \left(\omega(h) \times \left(-\frac{1}{2}h, \frac{1}{2}h\right)\right).$$

For the end of the rods we write

$$\Gamma^{i,j}(h) := \Pi_h^{i,j} \cap \left(\partial\omega(h) \times \left(-\frac{1}{2}h, \frac{1}{2}h\right)\right).$$

So the volume that each family of rods fills up is

$$\Pi^i(h) := \bigcup_{j \in \mathbb{Z}} \Pi^{i,j}(h) \quad \text{with } i = 1, 2.$$

We will consider the junction to be fixed at the set

$$\Gamma^i(h) := \bigcup_{j \in \mathbb{Z}} \Gamma^{i,j}(h) \quad \text{with } i = 1, 2.$$

The number of rods that each family $\Pi^i(h)$ contains is of order h^{-1} .

- The matrix of the soft composite is but the set

$$\Theta(h) = \Omega_h \setminus \left(\overline{\Pi^1(h)} \cup \overline{\Pi^2(h)}\right).$$

The full junction that we study is now described by

$$\Xi(h) = \Theta(h) \cup \Pi^1(h) \cup \Pi^2(h). \quad (1.2)$$

1.2 The Elastic Energy Functional

We use a matrix notation in defining the relations of the linear elasticity theory (see e.g., [17, 21]). The displacement vector is interpreted as the column $u = (u_1, u_2, u_3)^T$, where $\top$ denotes the transposition. Let $D(\nabla_x)$ be the following (6×3) -matrix of first-order differential operators:

$$D(\nabla_x)^T = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & \frac{1}{\sqrt{2}}\partial_3 & \frac{1}{\sqrt{2}}\partial_2 \\ 0 & \partial_2 & 0 & \frac{1}{\sqrt{2}}\partial_3 & 0 & \frac{1}{\sqrt{2}}\partial_1 \\ 0 & 0 & \partial_3 & \frac{1}{\sqrt{2}}\partial_2 & \frac{1}{\sqrt{2}}\partial_1 & 0 \end{pmatrix},$$

where $\nabla_x = (\partial_1, \partial_2, \partial_3)^T$ and $\partial_j = \partial/\partial x_j$. Then, using the definition of the cartesian components of the strain tensor

$$\varepsilon_{mn}(u) = \frac{1}{2} \left(\frac{\partial u_n}{\partial x_m} + \frac{\partial u_m}{\partial x_n} \right), \quad m, n \in \{1, 2, 3\},$$

we obtain the formula

$$\varepsilon(u) = D(\nabla_x)u$$

for the strain column

$$\varepsilon(u) = (\varepsilon_{11}(u), \varepsilon_{22}(u), \varepsilon_{33}(u), \sqrt{2}\varepsilon_{23}(u), \sqrt{2}\varepsilon_{13}(u), \sqrt{2}\varepsilon_{12}(u))^T.$$

The factors $\sqrt{2}$ are introduced in order to have equal norms for the tensor and the columns representing them.

For any bounded domain $\Lambda \in \mathbb{R}^3$, we consider the functional

$$\mathcal{E}(u; \Lambda) = \sum_{m,n=1}^3 \int_{\Lambda} |\varepsilon_{mn}(u)|^2 dx = \|\varepsilon(u); L_2(\Lambda)\|^2 \quad (1.3)$$

as a replacement of the elastic energy functional. For a standard elastic material the elastic energy functional contains, next to strain terms, terms depending on the stress. With the linear relation between stress and strain in the present situation such an energy functional is equivalent to (1.3).

The column $\varepsilon(u)$ and the functional $\mathcal{E}(u; \Lambda)$ vanish on the linear space $\mathbf{R}$ of rigid motions

$$\mathbf{R} = \{u : \mathbb{R}^3 \rightarrow \mathbb{R}^3 ; u(x) = d(x)b, b \in \mathbb{R}^6\},$$

where $d(x)$ is the (6×3) -matrix

$$d(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{\sqrt{2}}x_3 & -\frac{1}{\sqrt{2}}x_2 \\ 0 & 1 & 0 & -\frac{1}{\sqrt{2}}x_3 & 0 & \frac{1}{\sqrt{2}}x_1 \\ 0 & 0 & 1 & \frac{1}{\sqrt{2}}x_2 & -\frac{1}{\sqrt{2}}x_1 & 0 \end{pmatrix}. \quad (1.4)$$

In this paper we consider the functional

$$\mathcal{E}_\gamma(u; \Xi(h)) = \sum_{i=1,2} \mathcal{E}(u; \Pi^i(h)) + \gamma \mathcal{E}(u; \Theta(h)), \quad (1.5)$$

where $\gamma > 0$ is a parameter characterizing the relative rigidity of the matrix $\Theta(h)$ compared with the fibers $\Pi^i(h)$. The case $\gamma \ll 1$ corresponds to soft fibers while $\gamma \gg 1$ represents hard ones. We obtain weighted Korn inequalities for the junction $\Xi(h)$ with weight factors that depend on the geometrical parameter h and the physical parameter γ . The constants themselves in these inequalities will be independent of h and γ .

1.3 Anisotropic and Weighted Korn Inequalities

Known approaches to prove the Korn inequality

$$\|\nabla_x u; L_2(\Omega)\|^2 \leq k(\Omega)(\mathcal{E}(u; \Omega) + \|\nabla_x u; L_2(B)\|^2), \quad (1.6)$$

where B is a subdomain of Ω (cf. [6, 7, 15, 16, 26, 27] and others), cannot directly describe the dependence of the Korn constant $k(\Omega)$ on the specific geometry of the domain Ω . For domains Ω and B , which are star-shaped with respect to all points in a ball $B = B_r$ of radius $r > 0$, it was discovered in [13] (see also [14]) that

$$k(\Omega) \leq C (r^{-1} \text{diam } \Omega)^3, \quad (1.7)$$

where C is an absolute constant. However, for a thin domain this result is not optimal, since $r = O(h)$ and $\text{diam } \Omega_h = O(1)$ while $k(\Omega_h)$, B_r is known to be $O(h^{-2})$, not $O(h^{-3})$ as would follow from (1.7) and (1.6).

To the authors' best knowledge, it was in [28] where the concept of an anisotropic Korn inequality was introduced in order to justify the Kirchhoff's plate theory, namely, the Sobolev norm of u in $H^1(\Omega)$ has to be supplied with weight factors that differ for transversal and longitudinal displacements, and their derivatives in those directions (see (2.3) with $\rho_h = 1$). Then, in [18] (see also [21, Chap. 3], [23, Sect. 2]) an additional weight of type (2.2) was put into the Sobolev norm. This weight uses the distance of a point x to the clamped part of the plate. Such a weight factor becomes a crucial tool in the derivation of an asymptotically sharp (with respect to the small parameter) form of Korn's inequality for junctions of thin plates (see [11]). Concerning junctions of massive bodies and thin rods we refer to [2–5, 8–10, 12, 19, 24] and the review [23]. We emphasize that the final form of an asymptotically sharp weighted anisotropic Korn inequality is always very sensitive to the global geometry of a junction including the type of boundary conditions.

In the present paper we provide a weighted anisotropic norm $|u|_{\Xi(h)}$ for u in $H^1(\Xi(h))$ such that the constant c in the inequality

$$|u|_{\Xi(h)}^2 \leq c\mathcal{E}(u; \Xi(h))$$

is independent of both the parameters $h \in (0, 1)$ and $\gamma \in (0, +\infty)$. To do so, we efficiently employ several techniques such as weighted anisotropic Korn inequalities for individual rods and the plate, a decomposition of the displacement field into the sum of two components, i.e. a rigid motion $u^0 \in \mathbf{R}$ and an "orthogonal" component $u^\perp$, processed by different tools, the relation (1.6) for tiny parts (the rod and plate periodicity cells), and then coupling the individual Korn inequalities and passing them from one composite fragment to another. Those techniques, developed and used in [12, 18, 22, 24] and others, are described and explained in review [23]. The typical scheme in the paper to improve step by step the quality

of Korn's inequality, i.e., to enlarge the weights on certain components, becomes

1.4 Structure of the Paper

In Sect. 4 we formulate anisotropic weighted Korn inequalities for isolated rods and for a plate, which are known to be asymptotically sharp (see [18, 28] and [23]). In Sect. 3 we formulate the final result for the elastic junction and explain how it differs with the Korn inequality for a junction where the family of rods are welded together (see the periodicity cells in Figs. 3 and 4). We finish the section by sketching some generalizations of our approach.

The proofs are presented in Sects. 5 and 6. We consistently process all the different cases of the relations between the stiffnesses of the rods and the plate which will realize the steps in (1.8). Each of these relations is supplied with an example of a particular displacement field that will demonstrate that the weights we have derived are indeed optimal (see Sects. 5.2, 5.3, 5.4 and 6.2).

2 The Korn Inequalities for Isolated Components

2.1 The Homogeneous Plate

We start by considering the vector field u defined on the set Ω_h and supplied with the condition

$$u(x) = 0, \quad x \in \Upsilon_h, \quad (2.1)$$

where

$$\Upsilon_h = \{(y, z) = (y_1, y_2, z) : y \in \partial\omega, z \in (-h/2, h/2)\}$$

is the lateral surface of Ω_h . The condition (2.1) means that the lateral surface Υ_h of the plate Ω_h is fixed. Korn inequalities for fixed plates were proved by B.A. Shokhet [28]. In [18] (see also [21, Chap. 3], [23, Sect. 2]) these inequalities are improved by introducing the weight factor

$$\rho_h(y) = \min\{1, h + \text{dist}(y, \partial\omega)\} \leq 1. \quad (2.2)$$

This weight factor is of order h in the vicinity of the fixed lateral side Υ_h of the plate Ω_h and of order $1 = h^0$ further away from Υ_h (see Fig. 5).

In order to formulate the resulting inequality we introduce the weighted norm $|\cdot|_{\Omega_h|_0}$ in the space $H^1(\Omega_h)^3$ (cf. [18], [21, Chap. 3], [23, Sect. 2]):

Fig. 5 A graph for the weight factor $y \mapsto \rho_h(y)$ as a function of $\text{dist}(y, \partial\omega)$

$$|u; \Omega_h|_0 := \left(\int_{\Omega_h} \left\{ \sum_{j=1}^2 \left[\left| \frac{\partial u_j}{\partial y_1} \right|^2 + \left| \frac{\partial u_j}{\partial y_2} \right|^2 + \frac{h^2}{\rho_h^2} \left(\left| \frac{\partial u_j}{\partial z} \right|^2 + \left| \frac{\partial u_3}{\partial y_j} \right|^2 \right) + \frac{1}{\rho_h^2} |u_j|^2 \right] + \left| \frac{\partial u_3}{\partial z} \right|^2 + \frac{h^2}{\rho_h^4} |u_3|^2 \right\} dy dz \right)^{1/2}. \quad (2.3)$$

Theorem 2.1 (See [5, 18, 21, 23]) *For $u \in H_0^1(\Omega_h)^3$ the following inequality holds*

$$|u; \Omega_h|_0^2 \leq C \mathcal{E}(u; \Omega_h), \quad (2.4)$$

where C is a constant independent of u and $h \in (0, h_0]$.

It is known (see, e.g., [18, 21, 23]) that the distribution of weights in the Sobolev norm (2.3) is optimal, namely it is not possible to enlarge the order in h and ρ_h in any of the weights. Let us comment on this distribution. In the vicinity of the fixed lateral side of the plate, weights on derivatives become of order $1 = h^0$ and on displacements of order h^{-2} . Inside the plate, where $\rho_h(y) = O(1)$, the longitudinal displacements u_1, u_2 and their derivatives in longitudinal directions keep weights of order 1, but the transversal displacements u_3 and the derivatives $\partial u_3 / \partial y_1, \partial u_3 / \partial y_2$ get weights of order h^2 . The latter means that, with the same order of the energy functional the transversal deformation of the plate can be much larger than its longitudinal deformation. One knows this fact from the everyday practice: it is much easier to bend a thin body than to stretch it.

2.2 Rods

Let us now consider the vector field u defined on the rod $\Pi^{i,j}(h)$ and supplied with the condition

$$u(x) = 0, \quad x \in \Gamma(h), \quad (2.5)$$

i.e. all ends of the rods $\Pi^{i,j}(h) \subset \Pi^i(h)$ are fixed. For this separated fixed rod $\Pi^{i,j}(h)$, we introduce the weighted Sobolev norm (cf. [18, 20] and others)

$$|u; \Pi^{i,j}(h)|_i := \left(\int_{\Pi^{i,j}(h)} \left\{ \left| \frac{\partial u_1}{\partial y_1} \right|^2 + \left| \frac{\partial u_2}{\partial y_2} \right|^2 + \left| \frac{\partial u_3}{\partial z} \right|^2 + \frac{h^2}{\rho_h^2} \left(\left| \frac{\partial u_2}{\partial y_1} \right|^2 + \left| \frac{\partial u_3}{\partial y_1} \right|^2 + \left| \frac{\partial u_1}{\partial y_2} \right|^2 + \left| \frac{\partial u_3}{\partial y_2} \right|^2 + \left| \frac{\partial u_1}{\partial z} \right|^2 + \left| \frac{\partial u_2}{\partial z} \right|^2 \right) + \frac{1}{\rho_h^2} |u_i|^2 + \frac{h^2}{\rho_h^4} (|u_{3-i}|^2 + |u_3|^2) \right\} dy dz \right)^{1/2}. \quad (2.6)$$

Theorem 2.2 (See [20, 21, 23]) For $u \in H_0^1(\Pi^{i,j}(h))^3$ there holds the inequality

$$|u; \Pi^{i,j}(h)|_i^2 \leq C\mathcal{E}(u; \Pi^{i,j}(h)), \quad (2.7)$$

where C is a constant independent of u and $h \in (0, h_0]$.

The distribution of weight that is known to be optimal, is a bit different than in the norm (2.3) for the isolated plate. Again, due to the behavior of ρ_h the factors for the displacements and their derivatives take respectively the orders h^{-2} and h^0 in the vicinity of the clamped end of a rod. In the middle part of a rod the two transversal displacements get the weight $O(h^2)$ (in analogy with u_3 in the plate) but the longitudinal displacements the weight $O(1)$ (in analogy with u_1 and u_2 in the plate). All derivatives have the same weight $O(h^2)$, except for $\partial u_p / \partial x_p$ which are directly present in the elastic energy functional (1.3).

Remark 2.1 Rods with the cross-section $\omega(h)$ and with the length independent of the small parameter h are considered in [5, 20, 21, 23]. In general, the lengths that appear in $\Pi^{i,j}(h)$ depend on h . However, (2.7) can be derived from the above-mentioned results since we may extend the rod $\Pi^{i,j}(h)$ to the left and to the right up to a fixed length and combine the inequalities for those extended rods in order to obtain the necessary weight.

2.3 A Plate with Welded Rods

We consider a construction of a plate with welded rods (see Fig. 3) in the case $\gamma = h$. The Korn inequalities for welded rods in this construction is obtained in [1]:

$$|u; \Pi^i(h)|_{\times,i}^2 \leq C\mathcal{E}_h(u; \Omega_h), \quad (2.8)$$

where

$$\begin{aligned} |u; \Pi^i(h)|_{\times,i} := & \left(\int_{\Pi^i(h)} \left\{ \sum_{j=1}^2 \left[\left| \frac{\partial u_j}{\partial y_1} \right|^2 + \left| \frac{\partial u_j}{\partial y_2} \right|^2 + \frac{h^2}{\rho_h^2} \left(\left| \frac{\partial u_j}{\partial z} \right|^2 + \left| \frac{\partial u_3}{\partial y_j} \right|^2 \right) + \frac{1}{\rho_h^2} |u_j|^2 \right] \right. \right. \\ & \left. \left. + \left| \frac{\partial u_3}{\partial z} \right|^2 + \frac{h^2}{\rho_h^4} |u_3|^2 \right\} dy dz \right)^{1/2} \end{aligned} \quad (2.9)$$

and C is a constant independent of u and $h \in (0, h_0]$. The distribution of the weights on the system of welded rods is just the same as in the isolated cylindrical plate (see Sect. 2.1).

For the matrix that we derive in Sect. 5, the following inequality holds:

$$|u; \Theta(h)|_{\times,0}^2 \leq C\mathcal{E}_h(u; \Omega_h), \quad (2.10)$$

where

$$\begin{aligned} |u; \Theta(h)|_{\times,0} := & \left(\int_{\Theta(h)} \left\{ \sum_{j=1}^2 \left[h \left| \frac{\partial u_j}{\partial y_1} \right|^2 + h \left| \frac{\partial u_j}{\partial y_2} \right|^2 + \min \left\{ h, \frac{h^2}{\rho_h^2} \right\} \left(\left| \frac{\partial u_j}{\partial z} \right|^2 + \left| \frac{\partial u_3}{\partial y_j} \right|^2 \right) \right. \right. \right. \\ & \left. \left. + \min \left\{ \frac{1}{h}, \frac{1}{\rho_h^2} \right\} |u_j|^2 \right] + h \left| \frac{\partial u_3}{\partial z} \right|^2 + \min \left\{ \frac{1}{h}, \frac{h^2}{\rho_h^4} \right\} |u_3|^2 \right\} dy dz \right)^{1/2}. \end{aligned} \quad (2.11)$$

3 The Main Results

3.1 The Statement

In order to formulate the Korn inequality for the matrix $\Theta(h)$ we introduce the norm

$$\begin{aligned} |u; \Theta(h)|_{\gamma,0} &:= \left(\int_{\Theta(h)} \left\{ \beta_1(h, \gamma, y) (|u_1|^2 + |u_2|^2) + h^2 \beta_2(h, \gamma, y) |u_3|^2 \right. \right. \\ &\quad + \beta_1(h, \gamma, y) h^2 \left(\left| \frac{\partial u_1}{\partial z} \right|^2 + \left| \frac{\partial u_2}{\partial z} \right|^2 + \left| \frac{\partial u_3}{\partial y_1} \right|^2 + \left| \frac{\partial u_3}{\partial y_2} \right|^2 \right) \\ &\quad \left. \left. + \min\{1, \gamma\} \left(\left| \frac{\partial u_1}{\partial y_1} \right|^2 + \left| \frac{\partial u_2}{\partial y_2} \right|^2 + \left| \frac{\partial u_3}{\partial z} \right|^2 + \left| \frac{\partial u_1}{\partial y_2} \right|^2 + \left| \frac{\partial u_2}{\partial y_1} \right|^2 \right) \right\} dy dz \right)^{1/2}, \end{aligned} \quad (3.1)$$

where

$$\beta_1(h, \gamma, y) = \min\{\rho_h(y)^{-2}, \gamma h^{-2}\}, \quad \beta_2(h, \gamma, y) = \min\{\rho_h(y)^{-4}, \gamma h^{-4}\}. \quad (3.2)$$

In Sect. 5 we prove the next result:

Theorem 3.1 *For $u \in H_0^1(\Xi(h))^3$ the following inequality holds*

$$|u; \Theta(h)|_{\gamma,0}^2 \leq c \mathcal{E}_\gamma(u; \Xi(h)), \quad (3.3)$$

where c is a constant independent of u and $h \in (0, h_0]$.

Let us consider the fibers $\Pi^1(h)$. We set the norm

$$\begin{aligned} |u; \Pi^i(h)|_{i,\gamma} &= \left(\sum_{j: \Pi^i, j(h) \in \Pi^i(h)} \int_{\Pi^i, j(h)} \left\{ \left| \frac{\partial u_1}{\partial y_1} \right|^2 + \left| \frac{\partial u_2}{\partial y_2} \right|^2 + \left| \frac{\partial u_3}{\partial z} \right|^2 \right. \right. \\ &\quad + \left(\frac{h^2}{\rho_h^2} + \min\{1, \gamma\} \right) \left(\left| \frac{\partial u_1}{\partial y_2} \right|^2 + \left| \frac{\partial u_2}{\partial y_1} \right|^2 \right) \\ &\quad + \frac{h^2}{\rho_h^2} \left(\left| \frac{\partial u_3}{\partial y_1} \right|^2 + \left| \frac{\partial u_3}{\partial y_2} \right|^2 + \left| \frac{\partial u_1}{\partial z} \right|^2 + \left| \frac{\partial u_2}{\partial z} \right|^2 \right) \\ &\quad \left. \left. + \frac{h^2}{\rho_h^4} |u_3|^2 + \frac{1}{\rho_h^2} |u_i|^2 + \left(\frac{h^2}{\rho_h^4} + \beta_1 \right) |u_{3-i}|^2 \right\} dy dz \right)^{1/2} \end{aligned} \quad (3.4)$$

and prove in the Sect. 6 the statement:

Theorem 3.2 *For $u \in H_0^1(\Xi(h))^3$ the following inequality holds*

$$|u; \Pi^i(h)|_{i,\gamma} \leq c \mathcal{E}_\gamma(u; \Xi(h)), \quad (3.5)$$

where c is a constant independent of u and $h \in (0, h_0]$.

3.2 Discussion

Korn's inequality is the only tool to justify the asymptotic formulae for elastic fields on junctions and for functionals calculated on these fields such as energy, mean rigid motion, eigenfrequencies and so on. At the same time, being asymptotically exact in weights and anisotropy, Korn's inequality helps to predict the asymptotic ansatz for the displacements, strains and stresses in the junction.

Let us analyze the difference in the asymptotic ansatz for the composite plates with the periodicity cells drawn in Figs. 3 and 4, which results from the Korn inequalities (2.8), (3.3), (3.5) above, and the results of [1].

We consider the most interesting case, namely $\gamma = ch$, when the bending rigidity of the homogeneous plate and an individual rod are comparable.

Let us collect the weights in the Korn inequalities for reinforced plates in Table 1. For simplification we set $\rho_h = 1$, i.e. we consider the value of the coefficients of the weighted norm at a distance from the boundary Γ_h^i .

At first sight, the difference in weight is minor, namely only on three positions we observe a reduced weight factor, that is, for $\partial_2 u_1$, $\partial_1 u_2$ and u_{3-i} in the rods families, $i = 1, 2$. However, this small change effects *crucially* the whole system of differential equations in the models for the reinforced plates with welded and isolated periodic families of fibers. As shown in [1], [23, Sect. 2] the homogenization procedure of the plate with welded reinforcing lead to the Dirichlet problem for an elliptic Agmon-Douglis-Nirenberg system with the ADN-indices as in Table 2.

Remark 3.1 Writing a system of partial differential equations $Lu = f$ as $\sum_k L_{jk} u_k = f_j$, the ADN-indices denote the order of the differential operator in the component L_{kj} . Whenever the system is properly elliptic these indices will determine the coefficients s_k and t_j such that $L : \prod_k W^{m+s_k} \rightarrow \prod_j W^{m-t_j}$ is Fredholm. A consistent table is necessary for ellipticity.

The limiting homogenized system for the isolated reinforcing keeps the ADN-structure in Table 2 but loses the ellipticity! In forthcoming papers (cf. [25]) we shall confirm this

Table 1 Weight factors for $i = 1, 2$ and $j = 1, 2, 3$

	welded rods		disjoint rods		
	rods	plate	rod 1	rod 2	plate
$\frac{\partial u_j}{\partial x_j}$	1	h	1	1	h
$\frac{\partial u_1}{\partial x_2}, \frac{\partial u_2}{\partial x_1}$	1	h	h	h	h
$\frac{\partial u_i}{\partial x_3}, \frac{\partial u_3}{\partial x_i}$	h^2	h^2	h^2	h^2	h^2
u_1	1	1	h	1	1
u_2	1	1	1	h	1
u_3	h^2	h^2	h^2	h^2	h^2

Table 2 The Agmon-Douglis-Nirenberg indices

s_k	t_j		
	1	1	2
1	2	2	3
1	2	2	3
2	3	3	4

announcement and also show ways how this situation can be improved using a different physical configuration.

To detect the difference in the norms (2.9) and (2.11) we would like to give a heuristical explanation. To do so, we kill the weight factor ρ_h in all norms leaving only factors h^q . Then we take the particular field

$$\begin{aligned} u_1(x) &= y_2, & u_2(x) &= y_1, & u_3(x) &= 0 \\ \varepsilon_{12}(u) &= 1 & \text{and other strain vanish} \end{aligned}$$

This field corresponds to shear deformation in longitudinal directions. Then we have

	energy	weighted norm on rods
welded rods	$O(h)$	$O(h)$
isolated rods	$O(h)$	$O(h^2)$

This means that the shear deformation of the composite plate in Fig. 1 can be very large, e.g., of order $h^{-1/2}$ with the energy of order 1. In mathematical terms this means that the homogenized system cannot be elliptic and in mechanical terms that the reinforcing with isolated rods does not work in the case of shear longitudinal loading. Any construction engineer can predict the latter and usually the reinforcing elements ought to be jointed somehow. Since such jointing needs one extra step in the construction process, it is often avoided.

3.3 Possible Generalizations

The obtained results remain valid for more general assumptions on the geometry of the construction $\Xi(h)$. For instance, we fixed the same unit spacing both for the reinforcement of the fibers $\Pi^1(h)$ as well as for the fibers $\Pi^2(h)$. We can however extend the results to the construction with different spacing q_1 and q_2 of the reinforcement. To derive Korn inequalities we introduce the coordinates $(\tilde{x}_1, \tilde{x}_2) = (x_1/a_1, x_2/a_2)$ and substitute (u_1, u_2) by $(u_1, \frac{a_2}{a_1}u_2)$ in order to keep the form for the replacement $\mathcal{E}(u; \Xi(h))$ of the elastic energy functional. Indeed, by means of this substitution we keep the form of the term $|\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}|$.

All results also hold for reinforced plates with more general geometrical structures, first in the case when π_i are Lipschitz domain and, secondly, for a plate with two reclining families of stiff fibres and for a plate with more than two families of fibres (see Fig. 6).

One may also extend the results of the present paper to local periodic thin plates and fibres with more general geometry, such as for slightly curved rods and rods with periodically varying cross-section.

Fig. 6 Other periodicity cells

4 Korn Inequalities for a Relatively Rigid Plate

4.1 Auxiliary Statements

To begin with, we recall the result of [13] for Korn inequalities on star-shaped domains.

Lemma 4.1 (See [13]) *Let Λ be a domain, which is star-shaped with respect to every point in the ball $\mathbb{B}_R = \{x \in \mathbb{R}^3; |x| < R\} \subset \Lambda$. Then for $u \in H^1(\Lambda)^3$ the following inequalities hold*

$$\|\nabla_x u; L_2(\Lambda)\|^2 \leq C_1 S^3 R^{-3} (\mathcal{E}(u; \Lambda) + \|\nabla_x u; L_2(\mathbb{B}_R)\|^2), \quad (4.1)$$

$$S^{-2} \|u; L_2(\Lambda)\|^2 \leq C_2 S R^{-1} (\|\nabla_x u; L_2(\Lambda)\|^2 + R^{-2} \|u; L_2(\mathbb{B}_R)\|^2), \quad (4.2)$$

where $2S$ is the diameter of Λ and C_1, C_2 are constants independent of S, R , and u .

We will introduce the following norm for a domain Λ depending on the parameter $S > 0$:

$$\|u; \Lambda\|_S^2 := \|\nabla_x u; L_2(\Lambda)\|^2 + S^{-2} \|u; L_2(\Lambda)\|^2. \quad (4.3)$$

Combining (4.1) with (4.2), we then come to the estimate

$$\|u; \Lambda\|_S^2 \leq C S^4 R^{-4} (\mathcal{E}(u; \Lambda) + R^{-2} \|u; L_2(\mathbb{B}_R)\|^2). \quad (4.4)$$

We derive inequalities of Korn type for the junction $\Xi(h)$ using systematically the resulting estimate (4.4).

Our goal in this section is an inequality for the junction Ξ which corresponds to the estimate (2.4) for the plate Ω_h . Korn's inequality (2.4) holds for a vector field supplied with the boundary condition (2.1) on the lateral surface of the plate. However, we consider a vector field defined on the junction Ξ and supplied with the condition (2.5) on the surface $\Gamma(h)$. In order to obtain a Korn type inequality for the junction Ξ , we will fix in a more precise way the properties of the domain ω in the next definition.

Definition 4.1 We call $\omega \subset \mathbb{R}^2$ a *strong Lipschitz domain* if there exists $M \in \mathbb{R}$, such that we can cover the boundary $\partial\omega$ by finitely many open sets $\{\mathcal{U}_i\}_{i=1}^n$ with

$$\omega \cap \mathcal{U}_i = \{t_2^i > \varphi_i(t_1^i)\} \cap \mathcal{U}_i,$$

where (t_1^i, t_2^i) is a local cartesian coordinate system and where φ_i satisfies the Lipschitz condition

$$|\varphi_i(s) - \varphi_i(t)| \leq M|s - t|. \quad (4.5)$$

We start the derivation of a Korn type inequality for the junction $\Xi(h)$ with one auxiliary geometrical statement. Let $\mathcal{G}_h^{i,j}, \mathcal{G}^i(h)$ denote the projections of $\Pi_h^{i,j}, \Pi^i(h)$ on the y -plane and let $B(P; r)$ be the circle with radius r and center at the point P in that y -plane. Let us introduce the subdomains

$$\mathcal{H}_h^{i,j}(A, B) = \left\{ y \in \mathcal{G}_h^{i,j}; y_i^A < y_i < y_i^B \right\},$$

where $A = (y_1^A, y_2^A), B = (y_1^B, y_2^B)$. Each domain $\mathcal{H}_h^{i,j}(A, B)$ consists of at most finitely many connected components, say $\mathcal{H}_h^{i,j,k}(A, B)$.

Lemma 4.2 *There exist $h_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0)$ and $P_0 \in \omega$, $Ch < \text{dist}(P_0, \partial\omega) < 2Ch$ we can find numbers i, j, k and points $A_+ \in \omega$, $A_- \notin \omega$, such that $\text{dist}(A_+, \partial\omega) < ch$ and $\text{dist}(A_-, \partial\omega) < ch$ and such that for sufficiently small $\delta > 0$ we have*

$$H_h^{i,j,k}(A_+, A_+ + \delta h) \subset B(P_0, Ch), \quad H_h^{i,j,k}(A_-, A_- - \delta h) \cap \omega = \emptyset. \quad (4.6)$$

Proof If $P_0 \in \mathcal{G}_1 \cup \mathcal{G}_2$ the assertion becomes evident. We consider the case where $P_0 \notin \mathcal{G}_1 \cup \mathcal{G}_2$. Let $Q_0 = (t_1^0, t_2^0) \in \partial\omega$ be the point such that

$$|Q_0 P_0| = \inf_{Q \in \partial\omega} |Q P_0|,$$

i.e. $|Q_0 P_0|$ is a distance from P_0 to $\partial\omega$. Let (t_1, t_2) be a coordinate system in the vicinity of P_0 where the curve $\partial\omega$ is defined by the equality $t_2 = \varphi(t_1)$ and φ satisfies condition (4.5).

There exist a point $Q_m = (y_1^m, y_2^m) \in \mathcal{G}_h^{m,p} \cap \partial\omega$ in the open set $\mathcal{U} \in \{\mathcal{U}_i\}_{i=1}^n$ such that the estimate $|t_1^0 - t_1^m| < \sqrt{2}h$ holds, where (t_1^0, t_2^0) and (t_1^m, t_2^m) are the local coordinates of the points Q_0 and Q_m . According to the Lipschitz condition in (4.5) with constant M , we can find a point $Q_{3-m} = (y_1^{3-m}, y_2^{3-m}) \in \mathcal{G}_h^{3-m,q} \cap \partial\omega$ such that $Q_{3-m} \in B(Q_m, h(1 + M^2)^{1/2})$. Let us select $\mathcal{G}_h^{i,j}$. If the segment $P_0 Q_0$, connecting the points P_0 and Q_0 and $\mathcal{G}_h^{m,p} \cap \omega$ has an acute angle, we put $(i, j) = (m, p)$; otherwise $P_0 Q_0$ and $\mathcal{G}_h^{3-m,q} \cap \omega$ form an acute angle and we take $(i, j) = (3 - m, q)$. Hence, for sufficiently large C and small h there exists such a point $A \in \omega$ with $\mathcal{G}_h^{i,j} \cap B(P_0, Ch) = H_h^{i,j,k}(A, A + \delta h)$, where $(i, j) = (m, p)$ or $(i, j) = (3 - m, q)$ and

$$\text{dist}|A Q_i| \leq \text{dist}|Q_i Q_0| + \text{dist}|Q_0 P_0| + \text{dist}|P_0 A| \leq (M + 2C)h := ch.$$

Since the set $\mathcal{U}$ does not depend on h , we take h sufficiently small in order to define the point A_- . $\square$

4.2 Korn's Inequality for $\Xi(h)$

Let $H_0^1(\Xi(h))^3$ be the closure of $C_0^\infty(\overline{\Xi(h)} \setminus (\Gamma_1(h) \cup \Gamma_2(h)))^3$ with respect to the Sobolev $H(\Xi(h))^3$ -norm.

Theorem 4.1 *For $u \in H_0^1(\Xi(h))^3$ the inequality*

$$|u; \Xi(h)|_0^2 \leq c\mathcal{E}(u; \Xi(h)), \quad (4.7)$$

is valid where c is a constant independent of u and $h \in (0, h_0]$.

Proof Let $\chi \in C^\infty(\mathbb{R})$ be a cut-off function, $\chi(t) = 0$ for $t < 1$ and $\chi(t) = 1$ for $t > 2$. Put $\chi_{Ch}(y) = \chi((Ch)^{-1} \text{dist}(y, \partial\omega))$ and $\mathcal{O}_\chi(Ch) := \text{supp}(\nabla \chi)$. According to (2.4) and the definition of the cut-off function χ we have

$$|\chi u; \Xi(h)|_0^2 \leq c\mathcal{E}(\chi u; \Xi(h)) \leq c(\mathcal{E}(u; \Xi(h)) + h^{-2}\|u; L_2(\mathcal{O}_\chi(Ch))\|^2), \quad (4.8)$$

where $\mathcal{O}_\chi(Ch) := \mathcal{O}_\chi(Ch) \times (-h/2, h/2)$.

In order to estimate the second term in the right-hand side of (4.8) we introduce the domains

$$L_h^{i,j,k}(A, B) = \{x = (y, z) \in \Pi_h^{i,j} : y \in \mathcal{H}_h^{i,j,k}(A, B)\},$$

and the cylinders $\mathbb{T}_h(P; r) = \{(y, z) : (y_1, y_2) \in B(P; hr), z \in (-h/2, h/2)\}$, where $B(P; hr)$ is a ball of radius rh with a center at a point P .

Let $P^{(1)}$ be a point in $\mathbb{O}_\chi(Ch)$ and $P_0^{(1)}$ be the projection of $P^{(1)}$ on the plane $z = 0$. By Lemma 4.2 we have $\mathcal{G}_h^{i,j}$ and points A_+, A_- such that the sets $H_h^{i,j,k}(A_+, A_+ + \delta h)$ and $H_h^{i,j,k}(A_-, A_- - \delta h)$ satisfy conditions (4.6). Let $L_h^{i,j,k}$ correspond to the domain $H_h^{i,j,k}$. By Lemma 4.2, for a sufficiently large C_0 , we have $L_h^{i,j,k}(A_+ + \delta h) \subset \mathbb{T}_h(P_0^{(1)}; C_0)$. Let us take a ball $\mathbb{B}_{lh}^{i,j,k} \subset L_h^{i,j,k}(A_+, A_+ + \delta h)$ of radius lh with sufficiently small $l > 0$. The cylinder $\mathbb{T}_h(P_0^{(1)}; C_0)$ is star-shaped with respect to the ball $\mathbb{B}_{lh}^{i,j,k}$. Hence, by (4.4), we have

$$\begin{aligned} \|u; \mathbb{T}_h(P_0^{(1)}; C_0)\|_h^2 &\leq c \left(\mathcal{E}(u; \mathbb{T}_h(P_0^{(1)}; C_0)) + h^{-2} \|u; L_2(\mathbb{B}_{lh}^{i,j,k})\|^2 \right) \\ &\leq c \left(\mathcal{E}(u; \mathbb{T}_h(P_0^{(1)}; C_0)) + h^{-2} \|u; L_2(L_h^{i,j,k}(A_+, A_+ + \delta h))\|^2 \right) \\ &\leq c \left(\mathcal{E}(u; \mathbb{T}_h(P_0^{(1)}; C_0)) + |u; L_h^{i,j,k}(A_-, A_+ + \delta h)|_i^2 \right). \end{aligned} \quad (4.9)$$

The last inequality follows from definition (2.6) of the norm $|u; \cdot|_i$. According to (4.9) and Theorem 2.2 we have the estimate

$$\|u; \mathbb{T}_h(P_0^{(1)}; C_0)\|_h^2 \leq c \left(\mathcal{E}(u; \mathbb{T}_h(P_0^{(1)}; C_0)) + \mathcal{E}(u; \Pi_h^{i,j}) \right). \quad (4.10)$$

We take a next point $P^{(2)} \subset \mathbb{O}_\chi(Ch) \setminus \mathbb{T}_h(P_0^{(1)}; C_0)$ and find a cylinder where u satisfies the estimate (4.10). Let us continue the process. We define points $P^{(3)}, P^{(4)}, \dots$ and obtain a system of cylinders which cover the domain $\mathbb{O}_\chi(Ch)$. Combining the estimates (4.10) for all those cylinders, we come to

$$\|u; \mathbb{O}_\chi(Ch)\|_h^2 \leq c \mathcal{E}(u; \Xi(h)). \quad (4.11)$$

By (4.8) and (4.11), we have

$$|\chi u; \Xi(h)|_0^2 \leq c \mathcal{E}(u; \Xi(h)). \quad (4.12)$$

To prove (4.7), it remains to find an estimate for the norm $|(1 - \chi)u; \Xi(h)|_0^2$. Set $\tilde{\mathbb{O}}_\chi(Ch) := (-h/2, h/2) \times \text{supp}(1 - \chi)$. In this domain we have $\rho_h(y) = O(h^1)$. Thus, the following inequality holds:

$$\begin{aligned} &|(1 - \chi)u; \Xi(h)|_0^2 \\ &\leq c \left(h^{-2} \|u; L_2(\mathbb{O}_\chi(Ch))\| \right. \\ &\quad \left. + \|u; \tilde{\mathbb{O}}_\chi(Ch) \cap (\Pi^1(h) \cup \Pi^2(h))\|_h + \|u; \tilde{\mathbb{O}}_\chi(Ch) \cap \Omega_h\|_h \right). \end{aligned} \quad (4.13)$$

The required estimate for the first two terms on the right-hand side of (4.13) follows from (4.11), (2.4). In order to estimate the third term we cover the domain $\mathcal{O}_\chi(Ch)$ by a system of circles $\mathcal{T}_h^{(j)}$, $j = 1, \dots, J$, with radii Rh where R is a sufficiently large number. We project those circles onto the curve $\partial\omega$. Let $\mathcal{S}_h^{(j)}$ be a union of a circle $\mathcal{T}_h^{(j)}$ and all points situated between $\mathcal{T}_h^{(j)}$ and the projection of $\mathcal{T}_h^{(j)}$ on $\partial\omega$.

We prove by contradiction that $\mathcal{S}_h^{(j)}$ is star-shaped with respect to the circle $\mathcal{T}_h^{(j)}$. Suppose that $\tau^- \in \mathcal{S}_h^{(j)} \setminus \mathcal{T}_h^{(j)}$, $\tau^+ \in \mathcal{T}_h^{(j)}$ and $(\tau^-, \tau^+) \not\subset \mathcal{T}_h^{(j)}$. Hence, we have a point τ^0 such that

$\tau^0 \in (\tau^-, \tau^+)$ and $\tau^0 \in \partial\omega$. For sufficiently small h , there exists a coordinate system (t_1, t_2) where the curves $\partial\omega \cap \mathcal{S}_h^{(j)}$ are defined by the equality $t_2 = \varphi(t_1)$ and φ satisfies (4.5). The Lipschitz condition (4.5) implies

$$|\tau_2^- - \tau_2^0| \leq M|\tau_1^- - \tau_1^0|, \quad (4.14)$$

where τ_i^-, τ_i^0 are coordinates of the points τ^-, τ^0 in the coordinate system (t_1, t_2) . According to the definition of the domain $\mathcal{S}_h^{(j)}$ and the points τ^-, τ^0 we have the geometrical relation

$$\frac{R}{C+R} \geq \frac{|\tau_1^+ - \tau_1^-|}{|\tau_2^+ - \tau_2^-|} \geq \frac{|\tau_1^0 - \tau_1^-|}{|\tau_2^0 - \tau_2^-|}. \quad (4.15)$$

By (4.14) and (4.15), $(R+C)R^{-1} \geq M^{-1}$ for sufficiently large C and, therefore, we have come to a contradiction. Thus, the domain $\mathcal{S}_h^{(j)}$ is star-shaped with respect to the circle $\mathcal{T}_h^{(j)}$ and consequently the domain $\mathbb{S}_h^{(j)} = \mathcal{S}_h^{(j)} \times (-h/2, h/2)$ is star-shaped with respect to the cylinder $\mathbb{T}_h^{(j)} = \mathcal{T}_h^{(j)} \times (-h/2, h/2)$. Let $\mathbb{B}_h^{(j)}$ be a ball of radius rh , where r is a number such that $\mathbb{B}_h^{(j)} \subset \mathbb{T}_h^{(j)}$. By (4.4), we have

$$\|u; \mathbb{S}_h^{(j)}\|_h^2 \leq C(\mathcal{E}(u; \mathbb{S}_h^{(j)}) + h^{-2}\|u; L_2(\mathbb{B}_h^{(j)})\|^2). \quad (4.16)$$

We sum the inequalities (4.16) over all $j = 1, \dots, J$ and obtain

$$\|u; \widetilde{\mathcal{O}}_\chi(\mathcal{C}h) \cap \Omega_h\|_h \leq C(\mathcal{E}(u; \mathcal{O}_\chi(\mathcal{C}h) \cap \Omega_h) + \|u; \mathcal{O}_\chi(\mathcal{C}h) \cap \Omega_h\|_h). \quad (4.17)$$

According to (4.13), (4.11) and (4.17) we find that

$$|(1-\chi)u; \Xi(h)|_0^2 \leq c\mathcal{E}(u; \Xi(h)). \quad (4.18)$$

Combining (4.12) and (4.18) completes the proof. $\square$

From the obvious inequality

$$\min\{1, \gamma\} \mathcal{E}(u; \Xi(h)) \leq \mathcal{E}_\gamma(u; \Xi(h)),$$

we obtain the following result.

Corollary 4.1 *For $u \in H_0^1(\Xi(h))^3$, the following estimate is valid*

$$\min\{1, \gamma\} |u; \Xi(h)|_0^2 \leq c\mathcal{E}_\gamma(u; \Xi(h)), \quad (4.19)$$

where the constant c is independent of the vector field u and the parameters $\gamma > 0$, $h \in (0, h_0]$.

4.3 The Estimate in (4.19) is Optimal

In order to verify that (4.19) is asymptotically sharp for $\gamma > 1$ we define the vector field

$$U(x) = (e_1 + e_2 + e_3)X_h(y) - z\frac{\partial X_h}{\partial y_1}(y)e_1 - z\frac{\partial X_h}{\partial y_2}(y)e_2 \quad (4.20)$$

where $X_h \in C^\infty(\mathbb{R}^2)$ is a cut-off function, $X_h(y) = 1$ for $y \in \Omega_h$ and $X_h(y) = 0$ for $y \in \Gamma(h)$.

The field in (4.20) satisfies the equalities $\varepsilon_{3j}(U) = \varepsilon_{j3}(U) = 0$, $j = 1, 2, 3$ and

$$\begin{aligned}\varepsilon_{11}(U) &= \frac{\partial X_h}{\partial y_1}(y) - z \frac{\partial^2 X_h}{\partial y_1^2}(y), & \varepsilon_{22}(U) &= \frac{\partial X_h}{\partial y_2}(y) - z \frac{\partial^2 X_h}{\partial y_2^2}(y), \\ \varepsilon_{12}(U) &= \varepsilon_{21}(U) = \frac{1}{2} \frac{\partial X_h}{\partial y_1}(y) + \frac{1}{2} \frac{\partial X_h}{\partial y_2}(y) + z \frac{\partial^2 X_h}{\partial y_1 \partial y_2}(y).\end{aligned}$$

Hence,

$$\mathcal{E}_\gamma(U; \Xi(h)) = O(\gamma^0 h^2 h^{-2}) = O(\gamma^0 h^0). \quad (4.21)$$

The factor h^2 appears in (4.21) due to the size of the 3d-Lebesgue measure

$$\text{mes}_3(\text{supp } |\nabla X_h|) = O(h^2).$$

The factor h^{-2} arises from the next relation for $k = 1$:

$$|\nabla^k X_h(y)| = O(h^{-k}). \quad (4.22)$$

All the terms in the expressions $|U; \Xi(h)|_0$ (see (2.3)) have the same order h^0 as $\mathcal{E}_\gamma(U; \Xi(h))$. The relations

$$\int_{\Xi(h)} \frac{h^2}{\rho_h^4} |U_3|^2 dy dz = h^2 O(h^1 h^{-3}) = O(h^0), \quad (4.23)$$

$$\int_{\Xi(h)} \frac{1}{\rho_h^2} |U_i|^2 dy dz = O(h^1 h^{-1}) = O(h^0) \quad (4.24)$$

follow from

$$\int_{\omega} \rho_h^{-k}(y) dy = O(h^{-k+1}) \quad (4.25)$$

with $k = 2$ and $k = 4$. The orders, analogous to (4.24), for the other terms of $|U; \Xi(h)|_0$ follow from (4.22) with $k = 1, 2$, (4.25) with $k = 2$, and the relation $|z| = O(h^1)$ for $z \in \Xi(h)$.

5 An Improved Weighted Korn Inequality for the Matrix

5.1 Korn's Inequality for $\Theta(h)$

In the previous section we have obtained the Korn inequality (4.19) and have proved that it is asymptotically sharp for $\gamma > 1$ on the junction $\Xi(h)$ and, in particular, on the matrix $\Theta(h)$. In this section we derive the Korn inequality for the matrix $\Theta(h)$ which is asymptotically sharp in the case $1 > \gamma > 0$.

Let $\Lambda(h)$ be an open subset of $\Xi(h)$ and let us introduce in the space $H^1(\Lambda(h))^3$ the weighted anisotropic norm

$$\begin{aligned}
|u; \Lambda(h)|_{\gamma, \bullet} := & \left(\int_{\Lambda(h)} \left\{ \min\{1, \gamma\} \left(\left| \frac{\partial u_1}{\partial y_1} \right|^2 + \left| \frac{\partial u_2}{\partial y_2} \right|^2 + \left| \frac{\partial u_3}{\partial z} \right|^2 \right) \right. \right. \\
& + h^2 \beta_1(h, \gamma, y) \left(\left| \frac{\partial u_1}{\partial y_2} \right|^2 + \left| \frac{\partial u_2}{\partial y_1} \right|^2 + \left| \frac{\partial u_1}{\partial z} \right|^2 + \left| \frac{\partial u_3}{\partial y_1} \right|^2 + \left| \frac{\partial u_2}{\partial z} \right|^2 + \left| \frac{\partial u_3}{\partial y_2} \right|^2 \right) \\
& \left. \left. + \beta_1(h, \gamma, y)(|u_1|^2 + |u_2|^2) + h^2 \beta_2(h, \gamma, y)|u_3|^2 \right\} dy dz \right)^{1/2} \quad (5.1)
\end{aligned}$$

where β_1 and β_2 are weight factors which depend on the geometrical parameter h , the physical parameter γ and the variable y .

Lemma 5.1 For $u \in H_0^1(\Xi(h))^3$ the following inequality holds:

$$|u; \Theta(h)|_{\gamma, \bullet}^2 \leq c \mathcal{E}_\gamma(u; \Xi(h)), \quad (5.2)$$

where β_1, β_2 defined in (3.2), c is a constant independent of u, γ and $h \in (0, h_0]$.

Proof Away from the lateral boundary the medium is periodic. The standard periodicity cell implies

$$S(h) = \left\{ (y, z); |y_i| < \frac{1}{2}h \text{ and } |z| < \frac{1}{2}h \right\}$$

We consider the cells

$$S^{(m)}(h) := \{(y, z); (y - mh, z) \in S(h)\}$$

for all $m \in \mathbb{Z}^2$ with $S^{(m)}(h) \subset \Omega_h$.

Let us choose circular cylinders $G_i(h) := \Pi^{i, q(m)}(h) \cap S^{(m)}(h)$, $i = 1, 2$, directed parallel to the y_{3-i} -axes, where $q(m) = \{j : \Pi^{i, j}(h) \cap S^{(m)} \neq \emptyset\}$ (see Fig. 3). We set the coordinate origin in the mass center of the cylinder $G_1(h)$. Let $Q(h)$ be the set

$$Q(h) = S^{(m)}(h) \setminus (G_1(h) \cup G_2(h)).$$

On the cell $S^{(m)}(h)$ the vector u admits the representation

$$u(x) = u^\perp(x) + d(x)a, \quad (5.3)$$

where $a \in \mathbb{R}^6$ and the vector $u^\perp$ satisfies the orthogonality condition

$$\int_{Q(h)} d(x)^\top u^\perp(x) dx = 0. \quad (5.4)$$

Let $s(h)$ denote the distance from the center of the parallelepiped $S(h)$ to the lateral surface $\Gamma(h)$ of $\Xi(h)$. In order to estimate the norm of the second term in (5.3) we introduce the number $\kappa_h = s(h) - h$. The inequality

$$c\rho_h(y) \leq \kappa_h \leq C\rho_h(y)$$

holds because of the definitions of the function ρ_h and the number κ_h .

By (5.1) and (1.4), we have

$$\begin{aligned} & |d(\cdot)a; Q(h)|_{\gamma, \bullet}^2 \\ & \leq ch^3 (\mu_1 ((|a_1|^2 + |a_2|^2) + h^2 (|a_4|^2 + |a_5|^2 + |a_6|^2)) \\ & \quad + h^2 \mu_2 (|a_3|^2 + h^2 (|a_4|^2 + |a_5|^2))) \\ & \leq ch^3 (\mu_1 ((|a_1|^2 + |a_2|^2) + h^2 (|a_4|^2 + |a_5|^2 + |a_6|^2)) + h^2 \mu_2 |a_3|^2) \end{aligned} \quad (5.5)$$

where

$$\mu_1(h, \gamma) = \min\{(\kappa_h)^{-2}, \gamma h^{-2}\}, \quad \mu_2(h, \gamma) = \min\{(\kappa_h)^{-4}, \gamma h^{-4}\}. \quad (5.6)$$

Taking the scalar product of (5.3) with the first unit vector e_1 and integrating it over $\Sigma_1(h) = \partial G_1(h) \setminus \partial S(h)$, we obtain

$$|\Sigma_1(h)|a_1 = \int_{\Sigma_1(h)} (u_1(x) - u_1^\perp(x)) ds_x. \quad (5.7)$$

We have used the symmetry of the circular cylinder $G_1(h)$ while calculating the left-hand side in (5.7).

Since $|\Sigma_1(h)| = O(h^2)$, we arrive by the Cauchy-Bunyakowsky-Schwarz inequality at the estimate

$$|a_1|^2 \leq ch^{-2} \left(\kappa_h^2 \int_{\Sigma_1(h)} \rho_h^{-2} u_1(x)^2 ds_x + \int_{\Sigma_1(h)} u_1^\perp(x)^2 ds_x \right). \quad (5.8)$$

To process the right-hand side, we use the well-known trace inequality

$$h \int_{\partial Q(h)} |U|^2 ds_x \leq c \int_{Q(h)} \{h^2 |\nabla U|^2 + |U|^2\} dx \quad (5.9)$$

for a vector-function $U \in H^1(Q(h))$ (see, e.g. [21], Proposition 1.2.8). From (5.9) with $U = u_1^\perp$ it follows that

$$h \int_{\Sigma_1(h)} |u_1^\perp|^2 ds_x \leq c \int_{Q(h)} \{h^2 |\nabla u_1^\perp|^2 + |u_1^\perp|^2\} dx = ch^2 \|u_1^\perp; Q(h)\|_h^2. \quad (5.10)$$

Equality (5.4) implies the estimate

$$\|u_1^\perp; Q(h)\|_h^2 \leq c \mathcal{E}(u_1^\perp; Q(h)) \quad (5.11)$$

(see, for instance [21, Theorem 2.3.3], [23, Proposition 2.1]). Hence, see [21, Proposition 3.4.12],

$$\int_{\Sigma_1(h)} |u_1^\perp|^2 ds_x \leq ch \mathcal{E}(u_1^\perp; Q(h)). \quad (5.12)$$

By the trace inequality (5.9) for $U = \rho_h^{-1} u_1$, we obtain

$$\kappa_h^2 \int_{\Sigma_1(h)} \rho_h^{-2} u_1(x)^2 ds_x \leq ch^{-1} \kappa_h^2 |u; G_1(h)|_1^2. \quad (5.13)$$

Thus, in accordance with (5.8), (5.12), (5.13) and the identity

$$\mathcal{E}_\gamma(u^\perp; Q(h)) = \mathcal{E}_\gamma(u; Q(h))$$

we find

$$|a_1|^2 \leq ch^{-3}(\kappa_h^2 |u; G_1(h)|_1^2 + h^2 \gamma^{-1} \mathcal{E}_\gamma(u^\perp; Q(h))), \quad (5.14)$$

where

$$\mathcal{E}_\gamma(u; Q(h)) := \gamma \mathcal{E}(u; Q(h)).$$

In order to estimate a_3 we take the scalar product of (5.3) with e_3 and arrive at the inequality

$$|a_3|^2 \leq ch^{-2} \left(\kappa_h^4 \int_{\Sigma_1(h)} \rho_h^{-4} u_3(x)^2 ds_x + \int_{\Sigma_1(h)} u_3^\perp(x)^2 ds_x \right)$$

(cf. (5.8)). By analogy to (5.14), from the definition of the norm $|\cdot|_1$ in (2.6) and (5.9) for $U = u_3^\top$ and $U = \rho_h^{-2} u_3$, one obtains the formula

$$|a_3|^2 \leq ch^{-5}(\kappa_h^4 |u; G_1(h)|_1^2 + h^4 \gamma^{-1} \mathcal{E}_\gamma(u; Q(h))). \quad (5.15)$$

We find an estimate for a_6 by taking the scalar product of (5.3) with $x_2 e_1$. In the same way as in (5.8) we have

$$|a_6|^2 \leq c \left(\int_{\Sigma_1(h)} x_2^2 ds_x \right)^{-1} \left(\kappa_h^2 \int_{\Sigma_1(h)} \rho_h^{-2} u_1(x)^2 dx + \int_{\Sigma_1(h)} u_1^\perp(x)^2 ds_x \right). \quad (5.16)$$

Hence,

$$|a_6|^2 \leq ch^{-5} \{ \kappa_h^2 |u; G_1(h)|_1^2 + h^2 \gamma^{-1} \mathcal{E}_\gamma(u; Q(h)) \}. \quad (5.17)$$

We have used the relation $ch^2 \leq x_2^2 \leq Ch^2$ for $x \in \Sigma_1(h)$.

Taking the scalar product of (5.3) with $x_2 e_1$, we obtain the inequality

$$|a_5|^2 \leq ch^{-5} \{ \kappa_h^2 |u; G_1(h)|_1^2 + h^2 \gamma^{-1} \mathcal{E}_\gamma(u; Q(h)) \}, \quad (5.18)$$

similarly to (5.17).

In order to estimate the components a_2 and a_4 we use the trace inequality for the cylinder G_2 . Let the center of the circular cylinder G_2 have the coordinates $\delta_1(h), \delta_2(h), \delta_3(h)$ and let us introduce the coordinates $(x'_1, x'_2, x'_3) = (x_1 + \delta_1(h), x_2 + \delta_2(h), x_3 + \delta_3(h))$. For the first term in the right-hand side of the representation

$$u(x') = u^\perp(x') + d(x')a. \quad (5.19)$$

we have the estimate

$$\int_{\Sigma_2(h)} |u_2^\perp|^2 ds_x \leq ch \mathcal{E}(u^\perp; Q(h))$$

(see the derivation of (5.12)).

Let us turn to the second term in (5.19). The vector $d(x)a$ in the new coordinates x' has the form

$$d(x')(a'_1, a'_2, a'_3, a_4, a_5, a_6)^\top,$$

where

$$a'_1 = a_1 + 2^{-1/2}\delta_3 a_5 - 2^{-1/2}\delta_2 a_6,$$

$$a'_2 = a_2 - 2^{-1/2}\delta_3 a_4 + 2^{-1/2}\delta_1 a_6,$$

$$a'_3 = a_3 + 2^{-1/2}\delta_2 a_4 - 2^{-1/2}\delta_1 a_5.$$

Take the scalar product (5.3) with e_2 and with $e_2 x'_3$, and then integrate the results over $\Sigma_2(h) = \partial\mathcal{G}_2(h) \setminus \partial S(h)$. We obtain the equalities

$$|\Sigma_2(h)| (a_2 - 2^{-1/2}\delta_3 a_4 + 2^{-1/2}\delta_1 a_6) = \int_{\Sigma_2(h)} (u_2(x') - u_2^\perp(x')) ds_{x'}, \quad (5.20)$$

$$a_4 \int_{\Sigma_2(h)} x_3'^2 ds_{x'} = -2^{1/2} \int_{\Sigma_2(h)} x'_3 (u_2(x') - u_2^\perp(x')) ds_{x'}. \quad (5.21)$$

First we consider (5.21). Analogously to (5.16) one verifies that

$$|a_4|^2 \leq c \left(\int_{\Sigma_2(h)} x_3'^2 ds_{x'} \right)^{-1} \kappa_h^2 \int_{\Sigma_2(h)} (\rho_h^{-2} u_2(x')^2 + u_2^\perp(x')^2) ds_{x'}.$$

Since $ch^2 \leq x_3'^2 \leq Ch^2$ for $x \in \Sigma_2(h)$, just as in (5.17) we have

$$|a_4|^2 \leq ch^{-5} (\kappa_h^2 |u; G_2(h)|_2^2 + h^2 \gamma^{-1} \mathcal{E}_\gamma(u; Q(h))). \quad (5.22)$$

In order to estimate a_2 , we use (5.17) and (5.22). Since $|\Sigma_1(h)| = O(h^2)$, it follows from (5.20) that

$$|a_2|^2 \leq c \left(h^{-2} \left(\kappa_h^2 \int_{\Sigma_2(h)} (\rho_h^{-2} u_2(x)^2 + u_2^\perp(x)^2) ds_x \right)^2 + |a_4|^2 \delta_3^2 + |a_6|^2 \delta_1^2 \right). \quad (5.23)$$

The first term has the same form as the right-hand side of (5.8) and therefore it allows the estimate

$$\begin{aligned} & h^{-2} \left(\kappa_h^2 \int_{\Sigma_2(h)} \rho_h^{-2} u_2(x)^2 ds_x + \int_{\Sigma_2(h)} u_2^\perp(x)^2 ds_x \right)^2 \\ & \leq ch^{-3} (\kappa_h^2 |u; G_1(h)|_2^2 + h^2 \gamma^{-1} \mathcal{E}_\gamma(u; Q(h))). \end{aligned} \quad (5.24)$$

By (5.22), (5.17) and the relations $|\delta_j(h)| \leq ch$ for $j = 1, 2, 3$ we have

$$|a_4|^2 \delta_3^2 + |a_6|^2 \delta_1^2 \leq ch^{-3} (\kappa_h^2 |u; G_1(h)|_1^2 + \kappa_h^2 |u; G_2(h)|_2^2 + h^2 \gamma^{-1} \mathcal{E}_\gamma(u; Q(h))). \quad (5.25)$$

The inequalities (5.23), (5.24), (5.25) imply

$$|a_2|^2 \leq ch^{-3} (\kappa_h^2 |u; G_1(h)|_1^2 + \kappa_h^2 |u; G_2(h)|_2^2 + h^2 \gamma^{-1} \mathcal{E}_\gamma(u; Q(h))). \quad (5.26)$$

We replace the terms $|a_k|$ in (5.5) by their bounds in (5.14), (5.15), (5.17), (5.18), (5.22), (5.26) and obtain the inequality

$$|da; \mathcal{Q}(h)|_{\gamma, \bullet}^2 \leq c \{ \mu_1(h, \gamma) (R_h^2(|u; G_1(h)|_1^2 + |u; G_2(h)|_2^2) + h^2 \gamma^{-1} \mathcal{E}_\gamma(u; \mathcal{Q}(h))) + \mu_2(h, \gamma) (R_h^4 |u; G_1(h)|_1^2 + h^4 \gamma^{-1} \mathcal{E}_\gamma(u; \mathcal{Q}(h))) \}. \quad (5.27)$$

Thus, we have processed the second term in the representation (5.3). According to the definitions in (4.3), (5.1) of the norms $\|\cdot; \mathcal{Q}(h)\|_h^2$, $|\cdot; \mathcal{Q}(h)|_{\gamma, \bullet}^2$ and the formula in (5.10) it holds that

$$|u^\perp; \mathcal{Q}(h)|_{\gamma, \bullet}^2 \leq c (\min\{1, \gamma\} \mathcal{E}(u; \mathcal{Q}(h)) + \gamma \|\cdot; \mathcal{Q}(h)\|_h^2) \leq c \mathcal{E}_\gamma(u; \mathcal{Q}(h)). \quad (5.28)$$

We sum (5.27), (5.28) over all $\mathcal{Q}(h) \in \Xi(h)$ and we arrive by (5.6) at the inequality

$$|u; \Theta(h)|_{\gamma, \bullet}^2 \leq c \left(\sum_q (|u; \Pi^{1,q}(h)|_1^2 + |u; \Pi^{2,q}(h)|_2^2) + \mathcal{E}_\gamma(u; \Omega(h)) \right). \quad (5.29)$$

Now the estimate in (5.2) follows from (2.7) and (5.29). $\square$

According to the evident relation $h^2 \rho_h^{-2} \leq 1$ we have

$$\begin{aligned} h^2 \min\{\rho_h^{-2}, \gamma h^{-2}\} &\leq \min\{1, \gamma\}, \\ \min\{\rho_h^{-2}, \gamma h^{-2}\} &\geq \rho_h^{-2} \min\{1, \gamma\}, \\ \min\{\rho_h^{-4}, \gamma h^{-2}\} &\geq \rho_h^{-4} \min\{1, \gamma\}. \end{aligned}$$

Hence, one can introduce the norm (3.1) and combine Corollary 4.1 and Lemma 5.1 into one Theorem 3.1. Note that the norms $|\cdot; \Theta(h)|_{\gamma, \bullet}$ and $|\cdot; \Theta(h)|_{\gamma, 0}$ differ only by factors on the terms $|\frac{\partial u_1}{\partial y_2}|^2$ and $|\frac{\partial u_2}{\partial y_1}|^2$.

5.2 The Weight Distribution for the Composite is Optimal: Example 1

Let us construct particular vector fields in order to verify that the estimate (3.3) is asymptotically sharp. In the case of $\gamma > 1$ we have the equality $\min\{1, \gamma\} |u; \Theta(h)|_{\gamma, 0} = |u; \Theta(h)|_0$ and a suitable vector field has the form (4.20).

If $\rho_h^{-4} \geq \gamma h^{-4}$, we use the field

$$U(x) = (1 - \chi(h^{-1} r_0 |x - x_0|))(e_1 + e_2 + e_3), \quad (5.30)$$

where χ is a cut-off function defined in the proof of Theorem 4.1. We take r_0 and x_0 so that $\text{supp}(x \mapsto 1 - \chi(h^{-1} r_0 |x - x_0|)) = \mathbb{B}_h$ belongs to one periodicity cell $S(h)$, $\mathbb{B}_h \subset \mathcal{Q}(h)$ and $\text{dist}(\mathbb{B}_h, \partial \Omega_h) \leq Ch$. On the one hand, we have $\mathcal{E}_\gamma(U; \Xi(h)) = O(\gamma h^{-2} h^3) = O(\gamma h^1)$. On the other hand, all terms in the right-hand side of (3.3) also are of order γh^1 . Note, that in (3.1) the weight factor $\rho_h(y)$ has an influence only on the factors $\beta_1(h, \gamma, y)$ and $\beta_2(h, \gamma, y)$. Therefore, the field (5.30) verifies the terms of (3.1) with the factor $\min\{1, \gamma\}$ for all $x \in \Xi(h)$.

5.3 The Weight Distribution for the Composite is Optimal: Example 2

In the case of $\rho_h^{-2} \leq \gamma h^{-2}$ we introduce the field

$$U(y) = h^{-1} w_3(y) e_3 + w_1(y) e_1 + w_2(y) e_2 - zh^{-1} \left(\frac{\partial w_3(y)}{\partial y_1} e_1 + \frac{\partial w_3(y)}{\partial y_2} e_2 \right), \quad (5.31)$$

where the smooth functions w_1, w_2, w_3 satisfy for $0 < \alpha < 1$ the following:

$$w_j(y) = 0 \quad \text{and} \quad \nabla w_3 = 0 \quad \text{for } y \in \{y \in \mathbb{R}^2; \text{dist}(y, \omega) < h^\alpha\}.$$

We have $\mathcal{E}_\gamma(U; \Xi(h)) = O(h^1)$. All integrals in the definition in (3.1) of the norm $|U; \Xi(h)|_{\gamma,0}^2$ have the same order h^1 .

5.4 The Weight Distribution for the Composite is Optimal: Example 3

Let us consider the case when both $\gamma h^{-4} < \rho_h^{-4}$ and $\gamma h^{-2} < \rho_h^{-2}$ hold. In order to verify that the estimate (3.3) is asymptotically sharp, we construct different fields for each term of the norm (3.1). We introduce the field

$$U(y) = (1 - \chi(r_0 h^{-1} \gamma^{-1/2} |y - y_0|)) e_3 \quad (5.32)$$

for the u_3 -terms in (3.1). The support $\mathbb{C}_{h,\gamma}$ of the cut-off function $1 - \chi(r_0 h^{-1} \gamma^{-1/2} |y - y_0|)$ is a cylinder having a base with diameter $4r_0^{-1} h^1 \gamma^{1/2}$ and height h . It follows from the inequality $\gamma h^{-2} < \rho_h^{-2}$ that $\gamma^{1/2} < h \rho_h < 1$. Hence we can take r_0 and y_0 in (5.32) such that $\mathbb{C}_{h,\gamma}$ belongs to one periodicity cell $S(h)$ and $\mathbb{C}_{h,\gamma} \subset Q(h)$. We obtain $E_\gamma(U, \Xi(h)) = O(\gamma^1 h^3 \gamma^1 h^{-1} \gamma^{-1}) = O(h\gamma)$. The terms with $|\frac{\partial U_3}{\partial y_j}|$ in the norm $|U; \Xi(h)|_{\gamma,0}$ have the same order $h\gamma$. To verify that the estimate (3.3) is asymptotically sharp for terms with $|U_3|$, we take r_0 and y_0 such that $\text{supp}(x \mapsto \chi(r_0 h^{-1} |y - y_0|))$ belongs to a cell $S(h)$ with $\text{dist}(S(h), \partial \Omega_h) \leq Ch$. Then we use the same relation $O(h^3 \gamma^1 h^{-4} h^2) = O(h\gamma)$ for those terms in the norm $|U; \Xi(h)|_{\gamma,0}$. Let us consider the field

$$U(x) = (1 - \chi(r_0 h^{-1} |y - y_0|))(1 - \chi(r_1 h^{-1} \gamma^{-1/2} |z - z_0|))(e_1 + e_2).$$

This field is equal to zero everywhere in $\Xi(h)$ except in a cylinder with the base diameter $4r_0^{-1} h^1$ and height $4h^1 \gamma^{1/2}$. We have

$$E_\gamma(U, \Xi(h)) = O(\gamma^1 h^3 \gamma^{1/2} \gamma^{-1} h^{-2}) = O(h\gamma^{1/2}).$$

The terms of (3.1) with $|\frac{\partial u_i}{\partial z}|$, $i = 1, 2$, have the same order $h^3 \gamma^{1/2} h^2 \gamma^1 h^{-2} \gamma^{-1} h^{-2} = h\gamma^{1/2}$. The other terms on $|U; \Xi(h)|_{\gamma,0}$ have the lower order $h\gamma^{3/2}$. For the field (5.30) all the terms of (3.1) with the factor $\min\{1, \gamma\}$ have the same order as the energy functional. This field also verifies the terms of (3.1) with $|U_1|$ and $|U_2|$.

6 Korn's Inequality for the Isolated and for the Welded Rods

6.1 An Improved Weighted Korn Inequality for the Systems of Rods

On one hand, it follows from (2.7) that

$$\sum_{j: \Pi^{i,j}(h) \in \Pi^i(h)} |u; \Pi^{i,j}(h)|_i^2 \leq c \mathcal{E}_\gamma(u; \Xi(h)). \quad (6.1)$$

On the other hand, we have according to (4.19)

$$\min\{1, \gamma\} |u; \Pi^i(h)|_0^2 \leq c \mathcal{E}_\gamma(u; \Xi(h)). \quad (6.2)$$

We introduce the norm

$$\begin{aligned}
 |u; \Pi^i(h)|_{i,\bullet} := & \left(\sum_{j: \Pi^{i,j}(h) \in \Pi_i(h)} \int_{\Pi^{i,j}(h)} \left\{ \left| \frac{\partial u_1}{\partial y_1} \right|^2 + \left| \frac{\partial u_2}{\partial y_2} \right|^2 + \left| \frac{\partial u_3}{\partial z} \right|^2 \right. \right. \\
 & + \frac{h^2}{\rho_h^2} \left(\left| \frac{\partial u_1}{\partial y_2} \right|^2 + \left| \frac{\partial u_2}{\partial y_1} \right|^2 + \left| \frac{\partial u_3}{\partial y_1} \right|^2 + \left| \frac{\partial u_3}{\partial y_2} \right|^2 + \left| \frac{\partial u_1}{\partial z} \right|^2 + \left| \frac{\partial u_2}{\partial z} \right|^2 \right) \\
 & \left. \left. + \frac{h^2}{\rho_h^4} |u_3|^2 + \frac{1}{\rho_h^2} |u_i|^2 + \beta_1(h, \gamma, y) |u_{3-i}|^2 \right\} dy dz \right)^{1/2} \quad (6.3)
 \end{aligned}$$

and again obtain another Korn inequality for the systems Π^i .

Lemma 6.1 For $u \in H_0^1(\Xi(h))^3$ the following inequality holds

$$|u; \Pi^i(h)|_{i,\bullet}^2 \leq c \mathcal{E}_\gamma(u; \Xi(h)), \quad (6.4)$$

where c is a constant independent of u and $h \in (0, h_0]$.

Proof Let $i = 2$ (for $i = 1$ we may give a similar proof). We put the origin of the coordinate system at the center of the cylinder $G_1(h)$. On the cell $S(h)$ we represent the vector u in the form

$$u(x) = u^\perp(x) + d(x)b \quad (6.5)$$

where $b \in \mathbb{R}^6$ and $u^\perp$ satisfies the condition

$$\int_{G_2(h)} d(x)^\top u^\perp(x) dx = 0. \quad (6.6)$$

Note that in (6.6) we integrate over $G_2(h)$ in contrast to the integral in (5.4).

At first we estimate a norm for the second term in (6.5). We put db in (6.3) and obtain the inequality

$$\begin{aligned}
 |db; G_2(h)|_{i,\bullet}^2 & \leq ch^3 (h^2 \kappa_h^{-2} (b_3^2 + b_4^2 + b_5^2) + h^2 \kappa_h^{-4} (b_3^2 + b_4^2 y_2^2 + b_5^2 y_1^2) \\
 & \quad + \kappa_h^{-2} (b_2^2 + b_4^2 z^2 + b_6^2 y_1^2) + \mu_1 (b_1^2 + b_5^2 z^2 + b_6^2 y_2^2)) \\
 & \leq ch^3 (h^2 \kappa_h^{-2} (b_3^2 + b_4^2 + b_5^2) + h^2 \kappa_h^{-4} b_3^2 + \kappa_h^{-2} b_2^2 + \mu_1 b_1^2). \quad (6.7)
 \end{aligned}$$

In order to estimate b_1 we multiply (6.5) by e_1 and integrate over $\Sigma_2(h) = \partial G_2(h) \setminus \partial S(h)$:

$$|\Sigma_2(h)| b_1 = \int_{\Sigma_2(h)} (u_1(x) - u_1^\perp(x)) ds_x. \quad (6.8)$$

By the Cauchy-Bunyakowsky-Schwarz inequality we have

$$|b_1|^2 \leq ch^{-2} \left(\mu_1^{-2} \int_{\Sigma_2(h)} \beta_1^2 u_1(x)^2 ds_x + \int_{\Sigma_2(h)} u_1^\perp(x)^2 ds_x \right).$$

From (6.8) and (5.9) with $U = u_1^\perp$ and $U = \rho_h^2 u_1$ it follows that

$$\begin{aligned}
|b_1|^2 &\leq c h^{-3} \left(\mu_1^{-2} \int_{Q(h)} \beta_1^2 (h^2 |\nabla_y u_1|^2 + |u_1|^2) dx + h^2 \|u_1\|; Q(h) \| \|_h \right) \\
&\leq c h^{-3} (\mu_1^{-2} |u; Q(h)|_{\gamma, \bullet}^2 + \gamma^{-1} h^2 \mathcal{E}_\gamma(u; Q(h))) \\
&\leq c h^{-3} \mu_1^{-2} (|u; Q(h)|_{\gamma, \bullet}^2 + \mathcal{E}_\gamma(u; Q(h))).
\end{aligned}$$

It is not necessary to apply the trace inequality for the estimate of the other terms $|b_k|$ of (6.7). Indeed, the required estimate follows immediately from (2.7). For instance, we multiply (6.5) by e_2 , integrate over $\Sigma_2(h)$ and derive the estimate

$$\begin{aligned}
|b_2|^2 &\leq ch^{-3} \kappa_h^2 \left(\int_{G_2(h)} \rho_h^{-2} u_2^2(x) dx + \int_{G_2(h)} \rho_h^{-2} u_2^\perp(x)^2 dx \right) \\
&\leq ch^{-3} \kappa_h^2 (|u; G_2(h)|_i^2 + \| \| u; G_2(h) \| \|_h^2) \\
&\leq ch^{-3} \kappa_h^2 (|u; G_2(h)|_i^2 + \mathcal{E}(u; G_2(h))). \tag{6.9}
\end{aligned}$$

In order to estimate $|b_3|$, we take the scalar product of (6.5) with e_3 and get

$$\begin{aligned}
|b_3|^2 &\leq ch^{-3} \kappa_h^4 h^{-2} \left(h^2 \int_{G_2(h)} \rho_h^{-4} u_3^2(x) dx + h^2 \int_{G_2(h)} \rho_h^{-4} u_3^\perp(x)^2 dx \right) \\
&\leq ch^{-5} \kappa_h^2 (|u; G_2(h)|_i^2 + \| \| u; G_2(h) \| \|_h^2) \\
&\leq ch^{-5} \kappa_h^2 (|u; G_2(h)|_i^2 + \mathcal{E}(u; G_2(h))). \tag{6.10}
\end{aligned}$$

Similarly one obtains estimates for $|b_{3+j}|$, $j = 1, 2, 3$, by taking scalar products of (6.5) with $x_m e_n$ and corresponding n and m . Hence, from (6.7) it follows that

$$|db; G_2(h)|_{i, \bullet}^2 \leq c (|u; G_2(h)|_i^2 + \mathcal{E}(u; G_2(h))). \tag{6.11}$$

By definition of the norm $|\cdot; G_2(h)|_i$ in (2.6) it holds that

$$|u^\perp; G_2(h)|_i^2 \leq c \| \| u^\perp; G_2(h) \| \|_h^2 \leq c \mathcal{E}(u; G_2(h)). \tag{6.12}$$

Indeed, the second estimate in (6.11) follows from Theorem 2.3.3 [17]. We sum (6.11), (6.12) over all cells of the junction $\Xi(h)$ and arrive at the inequality

$$\sum_q |u; \Pi_i(h)|_{i, \bullet} \leq c \left(|u; \Theta(h)|_\gamma^2 + |u; \Pi^i(h)|_i^2 + \sum_q \mathcal{E}(u; \Pi^i(h)) \right).$$

This estimate yields (6.4) by means of (2.7). $\square$

We combine (6.1), (6.2) and (6.4), introduce the norm (3.4) and arrive at the final Theorem 3.2 on the Korn inequality for the system of fibers. Note that due to the relation $h^2 \beta_1 \leq h^2 \rho_h^{-2}$ we cannot improve the coefficient $h^2 \rho_h^{-2}$ for the derivatives $\partial u_i / \partial z$ and $\partial z / \partial u_i$ with the help of the argument in the proof of Lemma 5.1.

6.2 The Weight Distribution for the Rods is Optimal

Let us verify that the estimate (3.5) is asymptotically sharp. We take a rod $\Pi^{i,j}(h) \in \Pi^i(h)$ with length $O(1)$. To simplify the notation, we suppose that the origin of the coordinates

(y, z) belongs to this rod. Next we construct some auxiliary vector fields using the cut-off functions

$$X_h^\Pi(y_i) = \chi(h^{-\alpha}(y_i - y_i^0))(1 - \chi(h^{-\alpha}(y_i - y_i^0))), \quad X_h^\Theta(x) = \chi(h^{-1}r_0 \text{dist}(x; \Pi^{i,j}(h))).$$

Here y_i^0 is the y_i -coordinate of the point $P^0 \in \Pi^{i,j}(h) \cap \partial\Omega_h$, χ is the cut-off function defined in the proof of Theorem 4.1 and $0 < \alpha < 1$. We take r_0 so that $\text{supp } X_h^\Theta(x) \cap \Pi^{3-i,j}(h) = \emptyset$ and introduce the vector field

$$U(x) = X_h^\Theta(x) \left(X_h^\Pi(y_i) e_{3-i} - y_{3-i} \frac{\partial}{\partial y_i} X_h^\Pi(y_i) e_i \right). \quad (6.13)$$

Note that $\mathcal{E}(U; \Xi(h)) = \mathcal{E}(U; \Pi^{i,j}(h)) + \gamma \mathcal{E}(U; \Theta(h))$ and

$$\mathcal{E}(U; \Pi^{i,j}(h)) = \|\varepsilon_{ii}(U); L_2(\Pi^{i,j}(h))\|^2.$$

According to the relation for the 3d-Lebesgue measure

$$\text{mes}_3 \left(\text{supp} \left\{ X_h^\Theta \frac{\partial}{\partial y_i} X_h^\Pi \right\} \right) = O(h^{2+\alpha}) \quad (6.14)$$

we have

$$\mathcal{E}(U; \Pi^{i,j}(h)) = O(h^2 h^{2+\alpha} h^{-4\alpha}). \quad (6.15)$$

The factors h^2 and $h^{-4\alpha}$ appear in (6.15) due to the relation $|y_{3-i}| = O(h)$ for $y_{3-i} \in \Pi^{i,j}(h)$ and the estimate

$$\left| \frac{\partial^2}{\partial y_i^2} X_h^\Pi(y_i) \right| = O(h^{-2\alpha}).$$

By (6.14) and the estimate

$$|\nabla X_h^\Theta(x)| = O(h^{-1})$$

we obtain

$$\gamma \mathcal{E}(U; \Theta(h)) = O(\gamma h^{2+\alpha} h^{-2}) = O(\gamma h^\alpha). \quad (6.16)$$

Let us consider the case

$$\gamma h^{-2} \leq h^2 \rho_h^{-4} \quad (6.17)$$

For $x \in \text{supp } X_h^\Theta(x)$ we have

$$ch^\alpha \leq \rho_h(x) \leq Ch^\alpha. \quad (6.18)$$

According to (6.17) and (6.18) the inequality $\gamma h^\alpha \leq h^{4-3\alpha}$ holds. Therefore,

$$\mathcal{E}(U; \Xi(h)) \simeq \mathcal{E}(U; \Pi^{i,j}(h)) = O(h^{4-3\alpha}).$$

Due to (4.25) with $k = 4$ the norm (3.4) has the same order $h^{4-3\alpha}$.

If $h^2 \rho_h^{-4} < \gamma h^{-2} < \rho_h^{-2}$, then by (6.17) and (6.18)

$$\mathcal{E}(U; \Xi(h)) \simeq \mathcal{E}(U; \Theta(h)) = O(\gamma h^\alpha)$$

and due to (6.14) the norm (3.4) has the same order γh^α .

Finally, in the case

$$\rho_h^{-2} \leq \gamma h^{-2}$$

we consider the vector field

$$U(x) = (e_1 + e_2 + e_3)\chi(h^{-1}r_0|x - x_0|).$$

We take r_0 and x_0 such that $\text{supp } \chi(h^{-1}r_0|x - x_0|) = \mathbb{B}_h$ belongs to $\Pi^{i,j}(h)$ and $\text{dist}(\text{supp } \chi(h^{-1}r_0|x - x_0|), \Gamma^{i,j}(h)) = O(h)$. Then all terms in (3.5) have order h^1 .

6.3 An Improved Weighted Korn Inequality for Plate with Welded Rods

Let us confirm (2.10). We prove this inequality by repeating the scheme of the proofs of Lemma 4.1 and Lemma 6.1 with some simplifications (we do not use a trace inequality).

On the cell $S(h)$ we represent vector u in the form of (6.5). From (5.11) it follows that

$$h \|u^\perp; Q(h)\|_h^2 \leq c \mathcal{E}_h(u^\perp; Q(h)). \quad (6.19)$$

In order to estimate $d(x)b$ we introduce the norm

$$\begin{aligned} |u; \Theta(h)|_{\times, \bullet} := & \left(\int_{\Theta(h)} \left\{ \sum_{j=1}^2 \left[h \left| \frac{\partial u_j}{\partial y_1} \right|^2 + h \left| \frac{\partial u_j}{\partial y_2} \right|^2 + \frac{h^2}{\rho_h^2} \left(\left| \frac{\partial u_j}{\partial z} \right|^2 + \left| \frac{\partial u_3}{\partial y_j} \right|^2 \right) \right. \right. \right. \\ & \left. \left. \left. + \frac{1}{\rho_h^2} |u_j|^2 \right] + \left| \frac{\partial u_3}{\partial z} \right|^2 + \frac{h^2}{\rho_h^4} |u_3|^2 \right\} dy dz \right)^{1/2}. \end{aligned} \quad (6.20)$$

We put db in (6.20) and similarly to (6.7) obtain the inequality

$$|u; \Theta(h)|_{\times, \bullet} \leq ch^3 (h^2 \kappa_h^{-2} (b_3^2 + b_4^2 + b_5^2) + h^2 \kappa_h^{-4} b_3^2 + \kappa_h^{-2} b_2^2 + \kappa_h^{-2} b_1^2). \quad (6.21)$$

Estimates of the terms $|b_k|$ of (6.7) follow from (2.8). For instance, similarly as in (6.9), (6.10), we multiply (6.21) by e_1 , integrate over $\Sigma_2(h)$ and derive the estimate

$$\begin{aligned} |b_1|^2 & \leq ch^{-3} \kappa_h^2 \left(\int_{G_2(h)} \rho_h^{-2} u_1^2(x) dx + \int_{G_2(h)} \rho_h^{-2} u_1^\perp(x)^2 dx \right) \\ & \leq ch^{-3} \kappa_h^2 (|u; G_2(h)|_{\times, i}^2 + \|u; G_2(h)\|_h^2) \\ & \leq ch^{-3} \kappa_h^2 (|u; G_2(h)|_{\times, i}^2 + \mathcal{E}(u; G_2(h))). \end{aligned}$$

In the same way we obtain the estimates for $|b_k|$, $k > 1$ by taking scalar products of (6.5) with e_j , $x_m e_n$ and corresponding j, n, m and come to the inequality

$$|db; \Theta(h)|_{\times, \bullet}^2 \leq c (|u; G_2(h)|_i^2 + \mathcal{E}(u; G_2(h))). \quad (6.22)$$

We sum (6.22), (6.19) over all cells of the junction $\Xi(h)$ and arrive at the inequality according to the definition (2.11) of the norm $|u; \Theta(h)|_{\times, 0}$ arrive at the (2.10).

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