2-D REPRESENTATION OF 3-D LIMIT SURFACES

By Jan G. M. van Mier¹ and René de Borst²

INTRODUCTION

Constitutive models for plain concrete and geomaterials (rocks and soils) are often based on general (isotropic) limit surfaces. For example, plasticity theories (e.g., Vermeer and de Borst 1984) rely largely on accurately formulated yield surfaces. The yield surface which contains all peak stress points is usually called a limit or failure surface, and is needed even in the simplest plasticity models. Several mathematical models which aim at describing the complex three-dimensional shape of the failure surface have been proposed in the past (e.g., Lade 1982; Ottosen 1977; Podgorski 1985; Schreyer 1984; Willam and Warnke 1974). The recent model by Podgorski contains many of the earlier formulations as a special case, and can be considered the most general formulation.

Visualization of failure surfaces is normally done in planar cross sections. The representation chosen is often closely related to the type of experiments used for tuning a certain formulation. In the case of plane stress, a representation in the plane $\sigma_3 = 0$ is sufficient. When standard triaxial tests are carried out (i.e., triaxial compression $\sigma_1 < \sigma_2 = \sigma_3 < 0$, or triaxial extension $\sigma_1 = \sigma_2 < \sigma_3 < 0$), a representation of results in the meridian plane seems straightforward. Results obtained from experiments along a hydrostatic-deviatoric stress path in a true triaxial apparatus may conveniently be represented in the meridian plane, or as contours ($\sigma_0 = \text{constant}$) in the deviatoric plane. When results are obtained from proportional stress path experiments (i.e., $\sigma_2 = \alpha\sigma_1$, $\sigma_3 = \beta\sigma_1$, with α and β constant), contours are usually plotted in planes $\sigma_3 = \beta\sigma_1$ (e.g., Aschl, et al. 1976; van Mier 1984).

The latter representation has an important property which influences directly the number of experiments to be carried out for establishing the shape of the envelope curves. In this technical note, attention is paid to this property of contours in the plane $\sigma_3 = \beta\sigma_1$, which seems to have gone unnoticed until now. For simplicity, the Drucker-Prager model is used for demonstrating the characteristics of such contours, but the conclusions pertain to other (isotropic) formulations as well (Lade 1982; Ottosen 1977; Podgorski 1985; Schreyer 1984; Willam and Warnke 1984). Furthermore, some recent experimental results on concrete obtained by the first writer (van Mier 1984) are compared with a suitable formulation for this material.

¹Research Engr., Inst. TNO for Building Materials and Structures, Concrete Structures Dept., P.O. Box 49, 2600 AA Delft, The Netherlands.

²Research Engr., Inst. TNO for Building Materials and Structures, Software Engineering Dept., P.O. Box 49, 2600 AA Delft, The Netherlands.

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DRUCKER-PRAGER FAILURE CRITERION

The Drucker-Prager failure criterion is an extension of the von Mises criterion and is given by

$$aI_1 + \sqrt{J_2} = k \dots\dots\dots (1)$$

where $I_1 = \sigma_1 + \sigma_2 + \sigma_3$ is the first invariant of the stress tensor; $J_2 = (1/6)[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$ is the second invariant of the deviatoric stress tensor; and k and a are material constants.

Consider a proportional triaxial stress path: $\sigma_2 = \alpha\sigma_1$ and $\sigma_3 = \beta\sigma_1$. For such a stress path, Eq. 1 can be rewritten as

$$a(1 + \alpha + \beta)\sigma_1 + \sqrt{\frac{1}{6}\sigma_1^2[(1 - \alpha)^2 + (\alpha - \beta)^2 + (\beta - 1)^2]} = k \dots\dots\dots (2)$$

If Eq. 2 were symmetric, a stress path $\sigma_2 = \sigma_1/\alpha$ would also yield Eq. 2. Instead, such a stress path results in

$$a(1 + \alpha + \beta)\sigma_1 + \sqrt{\frac{1}{6}\sigma_1^2[(1 - \alpha)^2 + (1 - \alpha\beta)^2 + (\alpha\beta - \alpha)^2]} = k \dots\dots\dots (3)$$

Comparing Eqs. 2 and 3, we observe that different values for the principal stress are calculated, unless $\beta = 0$.

For $\alpha = 1$, Eqs. 2 and 3 both yield

$$a(2 + \beta)\sigma_1 + \sqrt{\frac{1}{3}\sigma_1^2(1 - \beta)^2} = k \dots\dots\dots (4)$$

so that there is continuity in the failure envelope. The slope of the failure envelope is also continuous (smooth failure surface), but the slope only equals -45° for $\alpha = 1$ when $\beta = 0$. Hence, the yield contour only intersects the diagonal $\sigma_1 = \sigma_2$ perpendicularly when $\beta = 0$ as indicated in Fig. 1. This nonorthogonality can be proved rather simply for the Drucker-Prager Criterion, as application of the implicit function theorem to Eq. 2 yields for the slope f' at $\sigma_1 = \sigma_2$.

FIG. 1.—Shape of Contours on Planes $\sigma_3 = \beta\sigma_1$ for Various Values of β

FIG. 2.—Comparison of Experimental Data for Concrete by van Mier (1984) ($f_c = 46.6 \text{ N/mm}^2$, Cubes 100 mm, Loaded between Brushes) with General Failure Criterion Proposed by Podgorski (1985)

$$f' = -\frac{2a\sqrt{3} + \frac{\sigma_1}{|\sigma_1|}}{2a(1 + \beta)\sqrt{3} + (1 - 2\beta)\frac{\sigma_1}{|\sigma_1|}} \dots\dots\dots (5)$$

We observe that only for $\beta = 0$, Eq. 5 reduces to $f' = -1$, which proves the assertion.

The nonorthogonality and nonsymmetry, as shown here for the Drucker-Prager criterion, adhere to any (isotropic) failure criterion when the failure envelope is plotted on a plane $\sigma_3 = \beta\sigma_1$ with $\beta \neq 0$. Nevertheless, orthogonality and symmetry are frequently assumed (e.g., Aschl, et al. 1976; van Mier 1984), or even predicted as a result from the plastic fracturing model (Bažant and Kim 1979).

Furthermore, a comparison of recent triaxial strength data obtained by the first writer (van Mier 1984) with a recent failure criterion (Podgorski 1985) is shown in Fig. 2. For stress paths $\sigma_2 = \alpha\sigma_1$ with $0 \leq \alpha \leq 1$, the

agreement between the model and the experimental data is satisfactory. It is furthermore noted that the envelopes of the Podgorski model, which result from intersecting the three-dimensional failure surface with tilted planes, clearly display nonsymmetry and nonorthogonality when $\beta \neq 0$.

CONCLUSION

It has been shown that a two-dimensional representation of failure surfaces in planes $\sigma_3 = \beta\sigma_1$ yields nonsymmetric envelope curves which intersect the diagonal $\sigma_1 = \sigma_2$ nonorthogonally when $\beta \neq 0$. However, orthogonality and symmetry are assumed in several publications when this representation is used, also for $\beta \neq 0$. One should be careful in adopting assumptions regarding the shape of any arbitrary cross section of a failure surface.

APPENDIX.—REFERENCES

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