CHAPTER 14

ON FROUDE-CAUCHY SIMILITUDE

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ABSTRACT

It is common engineering practice in a hydraulic model study involving both gravity waves and a solid structure to measure the hydrodynamic forces on the model of the structure and then calculate the resulting internal structural stresses. Because of the large variety of available elastic material, and the latest development in solid state physics, it is now feasible to measure directly these structural stresses in a scale model study.

It is shown how the similitude of elastic forces in structures subjected to wave action can be made compatible with the Froude similitude valid for hydraulic motion. Several examples are presented to illustrate the method. These include the study of the elastic response of an ice floe, the motion of an underwater membrane-type oil storage tank, the behavior of the Mohole riser, and the motion of a Texas Tower type of structure under wave action. Results obtained in the NESCO wave tank are also presented.

INTRODUCTION

The rules of similitude can be obtained by three different approaches: dimensional analysis, inspectional analysis, or the most general method which consists of deducing the conditions of similitude from the Navier-Stokes equation and the equations of elasticity.

In hydraulic textbooks, similitude is usually presented as a natural consequence of dimensional analysis. Then the similitude of Froude, Reynolds, Mach, Weber, Cauchy, etc. are presented. But no scale model has ever been built according to an equality of Weber number, or even of Reynolds number. Practically, the Reynolds similitude does not exist in scale model technology, but the similitude of head loss, a function of the Reynolds number, is sometimes adjusted.

Because of the inherent inadequacies of dimensional analysis, the engineer will often rather deduce the laws of similitude by "inspectional analysis" (Le Méhauté, 1962). Knowledge and understanding of the phenomena under study is necessary for deducing the rules of similitude, for neglecting phenomena of secondary importance, and for deciding the relative importance of scale effects. Inspectional analysis is also required for the interpretation of scale model results. The problem needs to be mathematically formulated under a differential form, but does not need to be integrated. The law of similitude can be deduced from the law of
motion under a differential form. The scale model remains the best analogue computer. It is often the only way of taking into account nonlinear effects such as convective inertia forces.

In any case, the dynamics of the system obey the general vectorial law:

\[ I + G + P + F + E + C = 0 \]

where \( I \) is the inertial force, \( G \) the gravity force, \( P \) the pressure force, \( F \) the friction force, \( E \) the elastic force, and \( C \) the capillary force. On the scale model,

\[ I' + G' + P' + F' + E' + C' = 0 \]

and for similitude,

\[ \frac{I}{I'} = \frac{G}{G'} = \cdots = \frac{C}{C'} \]

Once all these forces are mathematically expressed, it is realized that a complete similitude is impossible. Then it is important to realize what can be neglected.

The scale is chosen as a compromise between economics on the one hand and the technical requirements for similitude on the other hand.

For most problems, elastic forces do not need to be scaled. Still, it can be shown in general that there is a relationship between elasticity moduli \( E \), such as \( E\text{_model} = E\text{field} \times \text{linear scale} \), if combined with the same relative density, permits a reproduction of similitude of elastic force compatible with the Froude similitude.

The purpose of this paper is to present four example scale model studies where the Cauchy similitude for elastic forces has been made compatible with the Froude similitude for hydrodynamics. They are: the study of the elastic response of an ice floe under wave action, the study of the motion of an underwater membrane-type oil tank, the behavior of a riser of the Mohole platform, and finally, the response of a Texas Tower type of structure under wave action.

THE RESPONSE OF A FLOATING ELASTIC SHEET TO WAVE ACTION

Ice floes in the Arctic have occasionally been split by \( \sim \) catastrophic fissures. These fissures seem to be caused by rapid variations of air temperature, sudden changes in barometric pressure, or sea swells generated by distant storms. The fact that these cracks app
during the winter seems to support the idea that distant storms are sometimes responsible for the fissures.

The purpose of this study was to obtain theoretical and experimental information on the dynamic interaction between floating ice sheets and gravity waves. Although specifically designed for the study of ice floes, it is evident that both theory and experiments will apply to the study of any kind of elastic material.

From the theoretical viewpoint, the problem consists of finding the three velocity potentials corresponding to the incident and reflected wave motion \( \phi_1 \), the motion under the ice floe \( \phi_2 \) and the transmitted motion \( \phi_3 \) \((\nabla^2 \phi_i = 0, i = 1, 2, 3)\), and to match these velocity potentials along a vertical at the junctions of the neighboring regions by equating pressure and horizontal velocity (Hendrickson, Webb, and Quigley, 1962) (see Figure 1).

The free surface condition for \( \phi_1 \) and \( \phi_2 \) is, of course, the Cauchy-Poisson condition: \( \phi_{tt} + g \phi_z = 0 \). For \( \phi_2 \), this condition has to be replaced by

\[ p \phi_t + p g w = - p(x,t) \]

where \( w \) is the plate deflection and \( p \) is the pressure exerted by the plate on the water. The equation governing the motion of the plate is

\[ D \frac{\partial^4 w}{\partial x^4} + \rho_s h \frac{\partial^2 w}{\partial t^2} = p(x,t) \]

where

\[ D = \frac{E h^3}{12 (1 - v^2)} \]

is the flexural rigidity of the elastic plate, \( E \) the elastic modulus of the plate, \( h \) the plate thickness, \( v \) the Poisson's ratio of the plate material, \( \rho_s \) the density of the ice, and \( \rho \) the fluid density.

The plate fiber stress is given by

\[ S_x = - \frac{E h}{2 (1 - v^2)} \left| \frac{\partial^2 w}{\partial x^2} \right| \]

According to the Froude similitude, the scale for the forcing function \( p(x,t) \) is the linear scale \( \lambda \). Consequently the scale for
FIGURE 1
must also be $\lambda$. More specifically, the scale for \( \frac{\partial^4 w}{\partial x^4} \) is $\lambda^{-3}$, so the scale for $D$ is $\lambda^4$.

This can be achieved by varying $E$, $h$, or $v$. The simplest method consists, of course, of choosing the scale for $h$ to be linear, keeping the same value for $v$, in which case one finds $E_{\text{model}} = E_{\text{field}} \times \lambda$. It is then easily verified that the scale for fiber stress is $\lambda$.

Considering now the term $\rho_s h \frac{\partial^2 w}{\partial t^2}$, it is seen that it is sufficient that $\rho_{s,\text{model}} = \frac{\rho_{s,\text{field}}}{\lambda}$ for achieving a complete Froude-Cauchy similitude.

Considering that the elasticity modulus for ice is around $10^6$ lb/in$^2$, it was found that a sheet of polyethylene with a relative density of 0.92, a modulus of elasticity of $2 \times 10^4$ lb/in$^2$, and a Poisson's ratio of 0.43 will fulfill the similitude condition at a scale of $\lambda = 1/50$.

Experiments were performed in the NESCO wave tank with a sheet of polyethylene 16 feet long, four feet wide and two inches thick, representing a two-dimensional ice floe 800 feet long and 8.3 feet thick (Le Méhauté and Szczesny, 1965) (see Figure 2). This sheet was monitored by strain gages, pressure transducers and displacement pick-up devices and subjected to waves with periods from one second to four seconds corresponding to ocean waves with periods ranging from seven to 28 seconds. The figures included show the installation (Figure 2) and some examples of the experimental results (Figure 3). It was concluded that the probability of ice cracks due to wave action alone is very remote.

UNDERWATER OIL STORAGE TANK

As oil drilling structures are installed farther and farther offshore it may become uneconomical to pipeline crude oil from wells for storage in tanks on shore. Hence underwater oil storage tanks and tankers are now considered as an alternative engineering solution.

A membrane-type oil storage tank anchored at the bottom of the ocean at depths of 100 to 200 feet is considered. At such depths the wave motion is considerably reduced, but still not negligible. Aside of many other advantages, a flexible rubber type structure will reduce the wave forces considerably. However, some resonance effects must be eliminated. Such an anchored bag will look like a pillow (see Figure 4).

From a theoretical viewpoint, the problem consists of determining the motion of two liquids of different densities separated by a membrane. It is a classical problem in hydrodynamics. In this particular case, the
FIGURE 2
FROUDE-CAUCHY SIMILITUDE

**Figure 3**

1. **W, Maximum Absolute Deflection, in**
   - Coordinate distance of sheet, ft

2. **S_k, Maximum Absolute Bending Stress, lbs/in^2**
   - Coordinate distance of sheet, ft
problem can be approximated as follows. Since the bag is fixed like a pillow on the bottom, we will apply the assumption of the linear long wave theory to the oil motion. If $D_o$ is the average oil depth and $\eta$ the variation of oil level around $D_o$, the equations governing the oil motion may be written

$$\eta_t + D_o (u_x + v_y) = 0$$

$$\rho u_t = -p^*$$

$$\rho v_t = -p^*$$

where

$$p^* = p_H + p_w(t) + \Delta p(t).$$

Here $p_H$ is the hydrostatic pressure:

$$p_H = \rho_o g z + \rho_w g d - (\rho_w - \rho_o) g (D_o + \eta)$$

where $\rho_w$ and $\rho_o$ are the sea water and crude oil densities respectively and $d$ the water depth. $p_w(t)$ is the pressure variation due to the water wave. It will be assumed that this wave remains undisturbed by the bag located at the bottom, i.e.,

$$p_w = \rho_w g \frac{H}{Z} \frac{\cosh kD}{\cosh kd} \cosh (\sigma t - kx) \approx \rho_w g \frac{H}{Z} \frac{\cos (\sigma t - kx)}{\cosh kd} = B \cos (\sigma t - kd).$$

$\Delta p$ is the pressure difference between the two sides of the membrane. If $T_m$ is the tension in the membrane,

$$\Delta p = T_m (\eta_{xx} + \eta_{yy}) = \frac{T_m}{R} = e \frac{N}{R}$$

where $(\eta_{xx} + \eta_{yy})$ is the total curvature $1/R$, $e$ the membrane thickness, and $N$ the stress per unit area. In the case of a bag in the shape of a cylinder,

$$\eta_{xx} = 0 \quad \text{and} \quad \eta_{yy} = \frac{1}{R},$$

$R$ being the radius of the bag. Moreover, $\frac{dR}{R} \approx \frac{d\eta}{D} = \frac{dN}{E}$. Then,
after some straightforward operations in which $\eta$ is eliminated and $v$ is supposed to be small, one obtains

$$u_{tt} - C_B^2 u_{xx} = B \cos(\sigma t - kx)$$

where

$$C_B = \left[ \frac{c^2}{\rho} - \frac{c_g^2}{\rho} \right]^{1/2} = \left[ \frac{cR}{\rho R} - \rho D \frac{\rho_w - \rho_o}{\rho_o} \right]^{1/2}$$

It is recognized that $C_p$ is the wave celerity in an elastic pipe and $C_g$ is the wave celerity of a density wave.

This equation has a number of solutions. In the case of an infinitely long bag, a progressive wave will be generated within the bag and resonance will occur when the surface wave celerity equals the wave celerity $C_B$ within the bag. In the case of a bag of finite length, a standing wave motion is superimposed to the progressive wave motion which still exists because the boundaries at the two ends of the tank are partly movable. The resonance condition for the standing wave occurs when $T = 2t/nC_B$ where $t$ is the length of the bag and $n$ is an integer.

A more complete three-dimensional theory in which $u$ and $v$ are eliminated instead of $\eta$ will give:

$$\eta_{tt} - g D_o \frac{\rho_w - \rho_o}{\rho_o} \nabla^2 \eta + \frac{T D_o}{\rho_o} \nabla^4 \eta = \frac{D_o}{\rho_o} \nabla^2 p_o$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

and $p_o$ is the pressure on the outside of the membrane.

Some interesting solutions can be deduced from this equation once the boundary conditions are specified. The similitude condition may also be deduced from this equation and will give the same results as the more approximate theory above.

The case of a partly full tank, the case of non-resonant motion
and the effect of non-linearity due to the law of tension-displacement of the membrane have also been investigated thoroughly (Le Méhauté, 1964). The most significant results are presented on the following schematic graphs (Figure 5). The maximum stress has been calculated theoretically in the case of resonant and non-resonant motion. These theoretical developments, although interesting, are not the purpose of this paper, which is the determination of the law of similitude.

Without any further development, it is seen that the similitude requirements are:

\[
\left[ gD \frac{\rho_w - \rho_o}{\rho_o} \right]_m^{\frac{1}{2}} = \lambda \left[ gD \frac{\rho_w - \rho_o}{\rho_o} \right]_f^{\frac{1}{2}}
\]

which gives \( D_m = \lambda D_f \) and \( \frac{\rho_w - \rho_o}{\rho_o} \bigg|_m = \frac{\rho_w - \rho_o}{\rho_o} \bigg|_f \), where subscripts \( m \) and \( f \) refer to model and field respectively. This condition may be fulfilled if the scale model tank is filled with kerosene, alcohol, or gasoline, in order to correct for the density of fresh water instead of sea water.

For elasticity,

\[
\left[ \frac{eE}{\rho_o R} \right]_m^{\frac{1}{2}} = \lambda \left[ \frac{eE}{\rho_o R} \right]_f^{\frac{1}{2}}
\]

which gives

\[
\frac{eE}{\rho_o} \bigg|_m = \lambda^2 \frac{eE}{\rho_o} \bigg|_f
\]

i.e., in practice,

\[
eE \bigg|_m = \lambda^2 eE \bigg|_f
\]

or more generally, in the case where the law of tension-displacement is not linear, one should have \( T x e \bigg|_m = \lambda^2 T x e \bigg|_f \). For the particular material under consideration (a Firestone product 1/4 inch thick) it was found that the similitude condition for elasticity will be respected provided one uses a very thin rubber (1/64 inch) for the scale model at a scale of 1/30.

It is interesting to note that (by chance) the Reynolds similitude
FIGURE 5
for the motion within the tank will also be satisfied, making the Froude-Cauchy—Reynolds similitude compatible. Indeed, it is easily verified that

\[ v_m = v_f \times \lambda^{3/2} \]

since the kinematic viscosity for oil is \(10^{-4}\), \(1.6 \times 10^{-5}\) for alcohol, and \(0.5 \times 10^{-6}\) for gasoline.

**MOHOLE RISER AND TEXAS TOWER**

The analysis of the dynamics of the Mohole riser has been carried out in great detail (Galef, et al, 1965). A scale model study of the riser and floating platform has been considered for measuring the forces on the riser near the platform where they are the greatest. The diagrams in Figure 6 are self explanatory.

The motion of the platform under wave action obeys the Froude similitude provided mass and volume are distributed accordingly. It is evident that the motion of the deep part of the riser will not be in too good similitude due to scale effects. But the coupling force between the riser and the platform, where actually the largest forces are exerted, can be studied in similitude of elasticity. Although these rules of similitude can be deduced from the general equations as established in the previously mentioned report, we will simply consider the equation for the moment at the top of the riser, i.e., at the connection between the riser and the platform. This moment is

\[ M = \theta \sqrt{\frac{T_R}{m}} EI + HF \]

where \(\theta\) is the angle of deviation of the riser

\(T_R\) is the tension

\(I\) is the inertia

\(HF\) is the moment due to the hydrodynamic forces (drag and inertial forces).

Inertial forces due to water will be in relatively good similitude provided the external riser diameters are related by the geometric scale. For practical reasons, the cross section of the riser, which is an annulus, has to be reproduced as a full disk in the model (see next page). The inertial force due to the mass of the riser itself will be in similitude provided the density of the model equals the average density of the riser, including the steel annulus and the water inside. Thus

\[ \rho_m A_m = \rho_w A_w + \rho_R A_R \]
FIGURE 6
where $\rho$ is the density and $A$ the cross section. Subscript $m$ refers to the model, $w$ to the fluid within the riser, and $R$ to the steel annulus of the riser. It is pointed out that a relatively large error in $\rho_m$ has only a secondary effect on the riser behavior.

The drag force will not be in good similitude, but this effect is also negligible near the platform. The Cauchy similitude is compatible with the Froude similitude provided: (Subscript $m$ refers to model, $R$ to riser.)

$$\frac{M_m}{M_R} = \lambda^4, \quad \frac{\theta_m}{\theta_R} = 1$$

i.e.,

$$\frac{T_{EI}^m}{T_{EI}^R} = \lambda^8$$

The scale model characteristics for a full disk are:

$$T = \pi R_m^2 E_m \frac{\Delta l_m}{l_m}$$

where $\frac{\Delta l_m}{l_m}$ is the relative elongation, and

$$I = \frac{\pi R_m^4}{4}$$

while for the annulus riser they are

$$T \equiv 2 \pi R_R \Delta R_R E \frac{\Delta l_R}{l_R}$$

and
Inserting these expressions within the similitude condition gives, with \( \frac{\Delta f_m}{f_m} = \frac{\Delta f_R}{f_R} \):

\[
E_m = E_R \lambda \frac{\Delta R_R}{R_R} 2\sqrt{A}
\]

It is interesting to mention that the similitude condition for tension requiring \( \frac{T_m}{T_R} = \lambda^3 \) gives:

\[
E_m = 2 E_R \lambda \frac{\Delta R_R}{R_R}
\]

The first condition for \( E_m \) prevails insofar as the reaction of the platform upon the riser is concerned. This condition is actually also a good approximation for reproducing the motion of the riser in similitude. Indeed, in this case it is seen that inertial forces, bending forces, and tension are almost in similitude from the platform to the bottom.

It has been found that a wire of polytetrafluoroethylene, readily available on the market, fulfills all the similitude requirements at a scale of approximately 1/200.

Indeed, its relative density is 2.2 and its modulus of elasticity is 0.58 \( \times 10^5 \) psi (the scale will be determined exactly after these variables have been measured on a sample).

At such a scale, the water depth at the well location under the wave tank will have to be 90 feet. If one might compromise by stopping the depth on the floor of the building, it would be equivalent to approximately 1000 feet in the ocean.

Another solution will consist of distorting the riser vertically, i.e., by making \( \Delta M_m = \Delta M_R \times \lambda \), and

\[
\frac{\Delta f_m}{f_m} = \frac{A \Delta f_R}{f_R}
\]

such that \( A f_m = \lambda f_R \). Then,

\[
E_m = E_R 2 \sqrt{A} \frac{\Delta R_R}{R_R} \sqrt{A}
\]

If the depth of the scale model is six feet, for example, \( A \) will be equal to 15.
Similar rules of similitude can be deduced as easily for studying the response of a Texas Tower type of structure to wave action. The limitation of this kind of study is, of course, determined by the relative importance of the drag force, which has to be small by comparison with inertial force. For this reason, this kind of study will be suitable for relatively large piles.

CONCLUSION

It has been seen through a minimum of inspectional analysis how it is possible to make the Cauchy similitude compatible with the Froude similitude. The few examples which have been treated are only representative cases from many problems. A whole family of problems of hydroelasticity can be successively studied by application of the Froude-Cauchy similitude: hydrodynamic impact, wave slamming, wave breaking on a vertical wall, hydroelastic vibrations, etc. Each problem needs to be analyzed independently in order to assess the value and the limit of a scale model study. In most cases the scale will have to be chosen as a function of the elastic properties of the material. However, the large variety of plastic or rubber type materials offers a wide range of possible scales.

The development of solid state physics now permits the monitoring of scale model structures by highly sensitive strain gages and pressure transducers. It has been seen that even polyethylene can be equipped with strain gages without fear of redundancy: only two strain gages out of 24 were lost during the tests.

The old method consists of measuring the pressure forces and calculating the structural forces. Very often, the best method in scale model technology should be the direct measurement of the stress within hydraulic structures by application of the Froude-Cauchy similitude.

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REFERENCES


APPENDIX I
NOTATION

\[ \begin{align*}
A_m & \quad \text{cross section of the model of the riser} \\
A_r & \quad \text{cross section of the material of the riser} \\
A_w & \quad \text{cross section of the fluid within the riser} \\
B & \quad \text{amplitude} \\
C_B & \quad \text{wave celerity within the oil storage tank} \\
D & \quad \text{flexural rigidity} \\
D_o & \quad \text{average oil depth} \\
e & \quad \text{thickness of the membrane} \\
E & \quad \text{elasticity modulus} \\
g & \quad \text{gravity acceleration} \\
h & \quad \text{thickness of the plate} \\
H & \quad \text{wave height} \\
HF & \quad \text{moment of the hydrodynamic force} \\
I & \quad \text{inertia of the riser} \\
k & \quad \frac{2\pi}{L} \\
l & \quad \text{length of the oil storage tank} \\
L & \quad \text{wave length} \\
M & \quad \text{moment of the riser on the platform} \\
n & \quad \text{integer} \\
p & \quad \text{pressure} \\
P_H & \quad \text{hydrostatic pressure} \\
P_o & \quad \text{pressure outside the membrane} \\
P_w & \quad \text{pressure due to water waves} \\
R & \quad \text{radius of curvature of the membrane}
\end{align*} \]
$S_x$ fiber stress
$t$ time variable
$T$ wave period
$T_m$ tension of the membrane
$T_R$ tension of the riser
$u$ horizontal component of velocity in the OX direction
$v$ horizontal component of velocity in the OY direction
$w$ plate deflection
$x$ horizontal coordinate
$y$ horizontal coordinate
$z$ vertical coordinate
$\Delta p$ pressure variation due to the tension of the membrane
$\eta$ variation of oil level around $D_o$
$\theta$ angle of deviation of the riser
$\lambda$ linear scale
$\nu$ Poisson's ratio
$\nu_f$ kinematic viscosity for the prototype
$\nu_m$ kinematic viscosity on the scale model
$\rho$ density of water
$\rho_m$ density of the model of the riser
$\rho_r$ density of the material of the riser
$\rho_w$ density of sea water; also density of the fluid within the riser
$\sigma \equiv \frac{2\pi}{L}$
$\phi$ potential function