illustrative examples of optimization techniques for quantitative and qualitative water management

report on investigation

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illustartive examples of optimization techniques for quantitative and qualitative water management

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VOORWOORD

Het voor u liggende rapport is het resultaat van onderzoek in het kader van TOW-H naar de toepassingsmogelijkheden van optimalisatietechnieken in het waterbeheer. Daarbij is gesteund op eerdere ervaringen in TOW-H en PAWN-kader alsmede op eigen speurwerkspanningen van het Waterloopkundig Laboratorium (project S 331). Dit rapport moet in samenhang gezien worden met de eerder verschenenrapporten R 999-1, -3, -4 en -5 en S 331; het vormde oorspronkelijk een onderdeel van R 999-5. Het rapport is in het Engels gesteld ten behoeve van een betere communicatie met personen en instanties buiten Nederland.

Het rapport beoogt een illustratie te geven van de in bovengenoemde studies behandelde optimalisatietechnieken en hun voor- en nadelen. Dit is gedaan aan de hand van vier probleemsituaties, die verschillen met betrekking tot de infrastructuur voor de watertoevoer. Een overzicht van de onderzochte schematisaties en gebruikte technieken is gegeven in hoofdstuk 2. Gekozen is voor een waterverdelingsprobleem over drie landbouwgebieden, waarbij schade kan optreden ten gevolge van zowel watertekort als hoge zoutgehalte.

De volgende technieken zijn behandeld:
- Lineair Programmeren;
- Dynamisch Programmeren;
- (Gemengd) Geheeltallig Programmeren;
- Lagrange Multipliers; en
- Niet-lineair Programmeren.

Deze, ter beperking van de omvang van het onderzoek, noodzakelijke keuze is voornamelijk gedaan vanwege de samenhang in de benaderingswijze. Een belangrijk gemeenschappelijk uitgangspunt is voorts geweest dat het vergelijkend onderzoek gebaseerd zou zijn op deterministische gegevens, wat wil zeggen, dat alle relevante gegevens van tevoren bekend zijn. De gebruikte technieken zijn afkomstig uit het vakgebied van de Besliskunde (in het Engels "Operations Research" genoemd) en vormen een redelijk overzicht van wat aan deterministische technieken beschikbaar is. Stochastische technieken, die eveneens zinvol kunnen worden toegepast op het behandelde type problemen, verdienen de voorkeur indien de onzekerheid met betrekking tot de te gebruiken data een belangrijke factor vormt. Vanzelfsprekend moet in een reëel geval de resulterende verfijning van het optimaliseringsalgorithmie afgewogen worden tegen grotere
rekenkosten en de noodzaak van beschikbaarheid van gegevens over kansverdelingen, welke over het algemeen moeilijker te verkrijgen zijn dan deterministische gegevens.

Het rapport is zodanig geschreven dat lezing van de hoofdstukken 1, 2 en 3 een algemeen overzicht verschaf van opzet en resultaten van de betreffende technieken. De hoofdstukken 4, 5, 6 en 7 en de appendices A, B en C geven aanvullende informatie en de uiteindelijke formulering van de diverse algorithmen. In appendix D wordt in het kort ingegaan op de ideeën achter de simplexmethode, die bij Lineair Programmeren wordt toegepast. Voor de precieze indeling van het rapport wordt verwezen naar hoofdstuk 1 van het rapport zelf.
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LIST OF SYMBOLS

- denotes multiplication
* - (used as a superscript) denotes a particularly chosen value for a variable (Section 7.2)

A
B - irrigation areas (not in Section 3.1.4)
C

t - time period: can be 1, 2 or 3: starting values are sometimes characterized by t=0; see also page 9

In the following list of variables, parameters $i$ and $t$ are used to indicate areas and time periods respectively. Variables not listed have only local ad hoc significance. Also variables of the annexes are not given. Indicated are Sections where variables are first used.

$a_{t1}$ - Section 7.3 slopes of the parts of the piecewise linearized reservoir salt balance functions
$a_{t2}$
$a_{t3}$

$b_{t1}$ - Section 7.3 maximum values of $Y_{t1}$ and $Y_{t2}$ in the linearization of the reservoir salt balance functions
$b_{t2}$

$C_{t1}$
$C_{t2}$ - Section 7.3 slopes of the parts of the piecewise linearized polder salt balance functions
$C_{t3}$

$C_t(i)$ - Chapter 2 salt concentration (salinity) of the water allocated to area $i$ in period $t$

$CAV_t$ - Section 3.5 average salt concentration of the reservoir releases in period $t$

$CL_t$ - Section 3.10 (objective) cumulative loss function in period $t$ for problem 4
LIST OF SYMBOLS (continued)

\( CLOSS_t \) - Section 3.5 part of \( CAV_t \) which causes losses

\( CNOT_t \) - Section 3.5 part of \( CAV_t \) which does not cause losses

\( CPAV_t \) - Section 3.9 average salt concentration in the polder reservoir during period \( t \)

\( CPLOSS_t \) - Section 3.9 part of \( CPAV_t \) which causes losses

\( CPNOT_t \) - Section 3.9 part of \( CPAV_t \) which does not cause losses

\( CPOL_t \) - Section 3.9 salt concentration in the polder reservoir after period \( t \)

\( CRES_t \) - Section 3.5 salt concentration in the reservoir after period \( t \)

\( CRIV_t \) - Section 3.5 salt concentration in the river during period \( t \)

\( CSEEP_t \) - Section 3.9 salt concentration of the saline seepage in the polder during period \( t \)

\( D_t \) - Section 4.2 total demand of the three irrigation areas during period \( t \)

\( d_{t1} \) - Section 7.3 maximum values of \( YPOL_{t1} \) and \( YPOL_{t2} \) in the linearization of the polder salt balance functions

\( d_{t2} \) - Section 7.3 maximum values of \( T_{t1}(i) \) and \( T_{t2}(i) \) in the linearization of the shortage loss function from area \( i \) during period \( t \)

\( d_{t1}(i) \) - Section 7.3 maximum values of \( T_{t1} \) and \( T_{t2} \) in the linearization of the shortage loss function from area \( i \) during period \( t \)

\( I_{t1} \) - Section 7.3 integer variables with value 0 or 1, a 1 indicating that \( Y_t \) is smaller than \( b_{t1} \) or bigger than \( b_{t1} \) or\( b_{t2} \) resp. so that \( Y_{t1}, Y_{t2} \) or \( Y_t \) must be filled
LIST OF SYMBOLS (continued)

K  - Section 7.3 very big number used by Mixed Integer Programming to simulate a not-binding inequality; must certainly be bigger than all $Y_t$

$\lambda_1$  - Section 6.2 Lagrange Multipliers

$\lambda_2$

$L$  - Section 6.2 Lagrangian (function) (to be minimized)

$L^*_t$  - Section 4.2 (shortage) loss during period $t$

$R_t$  - Section 3.1 total allocation of water (outflow) in period $t$

$R_t^{(i)}$  - Section 3.7 allocation of water to area $i$ in period $t$

$RFL_t$  - Section 3.5 volume of flushing of the polder in period $t$

$RIV_t$  - Section 4.2 river flow during period $t$

$S$  - Section 6.2 (square root of the) slack variables used to pass inequality constraints into equalities

$SEEP_t$  - Section 3.9 saline seepage volume into the polder in period $t$

$T_t$  - Section 3.1 total shortage in period $t$

$T_t^{(i)}$  - Chapter 2 shortage of area $i$ in period $t$

$T_{t1}$

$T_{t2}$  - Section 3.2 parts in which $T_t$ are divided by the linearization of the objective function

$T_{t3}$

$T_{t1}^{(i)}$

$T_{t2}^{(i)}$  - Section 7.3 parts in which $T_t^{(i)}$ are divided by the linearization of the shortage loss functions

$T_{t3}^{(i)}$
LIST OF SYMBOLS (continued)

\( V_t \)  
- Section 3.1  
reservoir storage after period \( t \); \( V_0 \) is the initial reservoir content

\( V_{POL} \)  
- Section 3.9  
(constant) volume of the polder reservoir

\( Y_t \)  
- Section 3.1  
inflow to the reservoir in period \( t \)

\( Y_{t1} \)  
- Section 7.3  
parts in which \( Y_t \) are divided by the linearization of the reservoir salt balance functions

\( Y_{t2} \)  
- Section 7.3  
parts in which \( Y_{POL} \) are divided by the linearization of the polder salt balance functions

\( Y_{POLt} \)  
- Section 3.9  
flow into the polder reservoir in time period \( t \)

\( Y_{POLt1} \)  
- Section 7.3  
parts in which \( Y_{POLt} \) are divided by the linearization of the polder salt balance functions

\( Y_{POLt2} \)  

\( Y_{POLt3} \)  

\( Z_t \)  
- Section 3.7  
flow remaining in the river downstream of the reservoir intake
ILLUSTRATIVE EXAMPLES OF OPTIMIZATION TECHNIQUES
FOR QUANTITATIVE AND QUALITATIVE WATERMANAGEMENT

1 Introduction

1.1 Aim of this report

The use of mathematical programming or optimization techniques\(^1\) in the analysis of Dutch watermanagement problems on a national scale was introduced with the PAWN\(^2\) study (1977-1980). This created the need for the Dutch engineers involved, to have more insight in the possibilities and limitations of this kind of techniques. As a consequence various research activities were undertaken as a part of TOW-H\(^3\) and DHL\(^4\)-research. The results of these research activities are described in TOW-H reports R999-1 [1]\(^5\), R999-3 [2], R999-4 [3] and R999-5 [4] and the DNL research report S331 [5]. In [1] and [2] the possibilities for Linear Programming (LP) methods were considered, in relation to quantity problems only. In [3] a survey is given of different aspects of applying mathematical programming techniques to the water allocation problem when considering quantity as well as quality (conservative substances). In this report [3] particular attention was paid to the non-linearities resulting from considering quality aspects. Computational procedures were outlined to overcome these non-linearities, taking into account the problem of dimensionality and the presence of uncertainties (e.g. hydrologic conditions). Using suggested optimization schemes from [3], a computational example was given in [5] on a simplified Dutch watermanagement network. In this study

\(^1\) It should be emphasized that (unless stated otherwise) the words optimization and optimal in the context of this report refer to the application of mathematical programming techniques. Optimal solutions as results of the application of these techniques are in terms of an often strongly schematized and mathematically described objective function and not in terms of an objective behind a real world decision making process (see also section 1.2).

\(^2\) PAWN = Policy Analysis of the Watermanagement for the Netherlands. Study carried out between 1977 and 1980 by Rand Corporation (USA), the Delft Hydraulics Laboratory (DHL) and Rijkswaterstaat (RWS: the Dutch Governmental Institute for Public Works), sponsored by RWS.


\(^4\) DHL = Delft Hydraulics Laboratory.

\(^5\) The numbers between square brackets refer to the list of references concluding this report.
combinations of Linear Programming with Dynamic Programming (DP), and of LP with an alternative solution method, based on elements of goal coordination, were used to overcome the dimensionality problem.

In a TOW-H meeting the wish was expressed to summarize the performed research and to illustrate theoretical considerations with sample calculations, possibly complemented with new approaches to and/or extensions of the basic ideas. As a result two reports were made. In [4] the main aspects of previously performed research are presented and a review is given of the possibilities and limitations of the application of mathematical programming techniques to water management problems.

In the present report different techniques are illustrated by application on sample problems. These sample problems present different schematizations of the same basic problem: allocation of water from a river to irrigation areas via reservoirs. Basically 4 techniques and combinations thereof are considered namely: Linear and Non-linear Programming, Dynamic Programming and Lagrange Multipliers.

1.2 Formulation of the problem

Allocation of water is subject to a decision making process where it is tried to find an optimal solution. Optimal in this sense means the best possible solution in an often extremely complex "decision-world" where many incompatible interests play a role, uncertain developments have to be taken into account and many alternative solutions are present.

In such a process of looking for an optimal or best decision, use can be made of scientifically oriented investigations that may use various kinds of techniques or mathematical models.

One of the available techniques is formal (analytical) optimization. In case of its use an objective function is formulated, expressing, for example, the losses due to shortage and/or due to bad water quality. This function is minimized by analytical differentiation and solved for the values of the (independent) decision variables that equate the derivatives of the objective function to zero. Constraints on the values of variables cannot be taken account of in a natural way.
The mathematical programming procedure enables an efficient comparison of many options, taking into account the specific characteristics of the system considered, such as the capacity of canals and reservoirs, the operating rules of weirs etc. Solution of the mathematical problem gives an indication of the best possible solution in terms of the objective function, but the calculations are usually based on a set of simplifying assumptions and criteria, which necessarily have to be very explicit.

The price one has to pay for use of the synthesizing capacity of optimization techniques, which enables the simultaneous comparison of many alternatives is a simplified schematization of the problem. As an alternative approach simulation can be used here, where on the one hand only one option at a time can be considered but on the other hand much more freedom exists in the selection of the level of detail in the schematization.

Formulation of an optimization problem requires:
a. a description of the system in the form of a set of mathematical equations or constraints; and
b. a mathematical formulation of the objectives and related constraints.

Both the constraints and the objective function depend on the kind of problem to be considered and play a decisive role in the kind of optimization technique to be used. Considering the water allocation problem with quantity and quality aspects, in the following an indication is given of what kind of problems can be handled using which type of technique.

In previous TOW-H reports ([1], [2] and [3]) it has been shown that, when considering quantity only, the optimal distribution of water through a network and the optimal allocation to users connected to this network can be modelled by Linear Programming (LP). The constraint set for such optimization problems is generally linear (some special features of the network may cause non-linearities, see e.g. [2]). Particular linearization methods can then be used if the objective function is non-linear. E.g. if the objective function is convex respectively concave then respectively Separable and Mixed Integer Programming can be used.
When considering quality as well as quantity aspects in the distribution/allocation problem, the continuity equations for quality parameters have to be considered. These equations are non-linear in the decision variables for quantity and quality (namely quadratic). Moreover when reservoirs are present decisions in one period affect the situation in coming periods and special techniques have to be used which allow this sequential allocation to be made. Other aspects such as uncertainty add further to the complexity of the sequential allocation problem, but are not treated here.

The use of formal mathematical optimization techniques to solve the above mentioned kind of problems will normally include the application of simplifying assumptions in order to make the optimization approach work and as a consequence will result in a dilemma between the amount of detail and the computational burden. Simplifications used in solving the problem will, however, make the interpretation of the numerical results more difficult.

Several methods have been suggested in [3] to solve the quantity/quality (conservative substances) optimization problem. In the search towards an applicable optimization technique the characteristics of the optimization problem and the influence they have on the use of a particular optimization method were discussed. The finally proposed solution made use of iterative application of Linear Programming, however, the convergence of the iteration remained to be tested. A first test has been performed in [5] concluding that convergence may be a problem due to a necessary piece-wise linearization procedure. Further investigation is required to reach a final decision about the suggested method. Rather than elaborating on the conditions and possible ways for convergence, in this report, however, the problem is approached from a different angle by solving the full non-linear problem more directly.

1.3 Basic assumptions and goals

In this report several solutions to the quantity/quality problem, where economic losses occur due to water shortages and bad water quality, are presented which address the following problem: "In which way can a given infrastructure be managed/operated in order to minimize damage in a certain period due to shortage or bad quality of water?".
In order to develop an optimal scheme for management and operation some assumptions have to be made:

a. A loss function is provided that gives the relation between shortage and bad quality of water and caused damage.
b. The supply of water is known in quantity and quality in advance for all the periods considered.
c. The demand is known in advance for all the periods considered.
d. All existing restrictions are quantified.

At least reasonable guesses have to be present concerning assumptions b and c.

The particular aim is then to illustrate, by means of applying selected optimization techniques to a set of problems, the relation between some basic characteristics of the decision making problem and an appropriate optimization technique; special emphasis is put on the dimensionality of the problem which is a major determinant in the choice of an optimization technique and was also the main concern in [3].

1.4 Overview of the report

Four allocation problems are considered which are solved with different optimization techniques. The solution of each of these four problems is illustrated and the choice of a particular technique, based on the characteristics of a particular problem is discussed. This provides insight in the use of optimization techniques, their potential and limitations.

The four sample problems are defined in Chapter 2. Chapter 3 gives an overview of the mathematical programming formulations, the general characteristics of the solution methods and the results of the computations, without going into detail concerning intricacies of the calculation methods; the latter are treated in the subsequent chapters.

In problem 1 only quantity considerations are made and this results in a quite simple problem which is solved respectively with Linear and Dynamic Programming in Chapter 4. A detailed comparison is made of both techniques. Problem 1 makes further a comparison possible between quantity and combined quantity plus quality considerations which are considered in problem 2. This problem is solved in Chapter 5 using Dynamic Programming. Problems 1 and 2 involve a DP-
solution procedure using single variable stages. In this case the decisions in a particular stage (corresponding to a particular time period) are uniquely determined once the state variables are fixed.

With problem 3, treated in Chapter 6, an allocation decision problem is considered in which there are still some degrees of freedom left for the decision variables within a particular stage. A combination of DP and the Lagrange Multiplier (LM) technique is used as an optimization technique. The DP-procedure is then used to optimize over the time intervals or stages while the LM technique optimizes the decisions within the stages.

The dimensionality of the problem is increased in problem 4 which is treated in Chapter 7. Basically the same solution strategy could be used as for problem 3, however, this would result in a very high computation time. Therefore another optimization technique is developed namely an iterative application of Non-linear Programming. This results in a relatively low computation time.

Consideration of quality aspects requires the determination of an average quality over a particular decision interval; this average quality can then be used in the decision making. This operation is needed because the salt concentration is a continuously varying function of flow and storage variables. In Annex A two methods are presented to calculate a representative average salt concentration assuming complete mixing.

In Annex B an overview is given of the Lagrange Multiplier technique using some examples, Annex C elaborates on the linearization of non-linear functions, while Annex D presents a graphical application of LP on a simplified version of problem 1.
2 Description of the sample problems

Four allocation problems, which differ in the supply infrastructure, will be used for the illustration of mathematical programming techniques. The following presents a general problem description, an explication on the chosen supply and demand situations and an overview of the four supply alternatives. At the end of this chapter a summary of the sample problems is given.

Problem description

The first three problems are constructed in such a way that in each case the same total demand and supply for water is considered. The physical infrastructure for the distribution of the water is different. Problem 4 knows the same demand but an extra supply of saline water. All problems form a part of the basic problem situation as given in Figure 2.1.

Water can be supplied to the irrigation areas A, B and C from a reservoir with a total storage capacity of 160 (m\(^3\)/s) and a dead storage of 40 (m\(^3\)/s\(^1\)). Irrigation area C can also be supplied with water out of a polder system (basically a reservoir with a constant level) or directly out of the river.

1) Note on the units

- Consider the reservoir in Figure 2.1 between the river and the irrigation areas. The water quantity continuity equation for period \( t \), with length \( T \), yields

\[
V_{t+1} = V_t + Y_t - R_t
\]

(2.1)

where \( V_{t+1} \) represents the stored water volume at time \( t+1 \) and \( V_t \) that on time \( t \), \( Y_t \) the total volume of inflow into the reservoir in period \( t \) and \( R_t \) the total volume of water released in period \( t \). One can measure all these water quantities in m\(^3\), but it is convenient to express all water quantities in m\(^3\)/s by dividing (2.1) on both sides by \( T \):

\[
V_{t+1} = V_t + \frac{Y_t}{T} - \frac{R_t}{T}
\]

(2.2)

Herein is \( V_t = \frac{V_t}{T} \) etc. All quantities are now in m\(^3\)/s and can therefore be compared with the river supply. \( V_t \) can be considered upon as the flow resulting if the reservoir is emptied at a constant rate in period \( t \). In the following all flow and storage variables are thus expressed in m\(^3\)/s.

- All salt concentrations are expressed in mg/l.
- Damages and losses are expressed in some monetary unit.
Figure 2.1 Schematic representation of the various water supply possibilities
The storage in the polder is kept at a constant value of 40 (m³/s). Upwards saline seepage occurs into the polder. The volume of this seepage in each interval amounts to 15 (m³/s) and the salt concentration is 1000 (mg/l). There is a possibility to flush the polder in order to reduce the salinity.

Losses occur due to salinity and/or water shortage. During the growing season the demand for water is varying as are the sensibilities to water shortage. Salt damage sensibilities are kept constant during the growing season.

The problem to solve is: how to operate the physical system in order to minimize the total losses in the irrigation areas A, B and C during the growing season.

For computational purposes the growing season is divided into three subsequent periods of equal duration. In the schematizations and the calculation schemes that follow in the subsequent chapters the following notation will be used concerning the time periods.

<table>
<thead>
<tr>
<th>Time period number</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actual time at beginning and end of the periods (expressed in some time unit)</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Both the time period number and the actual time are used to denote particular variables. It will be clear from the formulation which indication has been used.

Supply

The river is the primary supply of water; the quantity and salinity of the water in the river for the three time periods are as follows:

<table>
<thead>
<tr>
<th>period</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>flow (m³/s)</td>
<td>240</td>
<td>210</td>
<td>130</td>
</tr>
<tr>
<td>salt concentration (mg/l)</td>
<td>250</td>
<td>300</td>
<td>350</td>
</tr>
</tbody>
</table>

Table 2.1 Water supply and salinity of supplied water
The initial storage and salt concentration in the reservoir are respectively 80 (m³/s) and 200 (mg/l). The initial concentration in the polder is 250 (mg/l). Seepage into the polder area is 15 (m³/s).

**Demand**

The data on demand in the three periods for the three areas are summarized in table 2.2 together with shortage and salinity loss functions.

The following notation is used:

\[ T_t(A) : \text{shortage in the allocation to area A in period } t \ (= 1, 2 \text{ or } 3). \]
\[ C_t(B) : \text{salt concentration in the water allocated to area B in period } t. \]

<table>
<thead>
<tr>
<th>Name irrigation</th>
<th>Demand (m³/s) in period</th>
<th>Relative magnitude of the area</th>
</tr>
</thead>
<tbody>
<tr>
<td>area</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>A</td>
<td>105</td>
<td>150</td>
</tr>
<tr>
<td>B</td>
<td>35</td>
<td>50</td>
</tr>
<tr>
<td>C</td>
<td>70</td>
<td>100</td>
</tr>
</tbody>
</table>

**Shortage loss functions**

<table>
<thead>
<tr>
<th>Area</th>
<th>Loss in period</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>[0.5 T_{1-2}^2(A)]</td>
<td>0.6</td>
<td>0.4</td>
<td>[T_{3-2}^2(A)]</td>
</tr>
<tr>
<td>B</td>
<td>[1.6 T_{1-2}^2(B)]</td>
<td>2.1</td>
<td>1.4</td>
<td>[T_{3-2}^2(B)]</td>
</tr>
<tr>
<td>C</td>
<td>[0.75 T_{1-2}^2(C)]</td>
<td>0.9</td>
<td>0.6</td>
<td>[T_{3-2}^2(C)]</td>
</tr>
</tbody>
</table>

**Salinity loss functions (equal for each period)**

<table>
<thead>
<tr>
<th>Area</th>
<th>Damage</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>[0 \ (\text{if } C_t(A) &lt; 200); 9 \ (C_t(A) - 200) \ (\text{if } C_t(A) &gt; 200)]</td>
</tr>
<tr>
<td>B</td>
<td>[0 \ (\text{if } C_t(B) &lt; 200); 3 \ (C_t(B) - 200) \ (\text{if } C_t(B) &gt; 200)]</td>
</tr>
<tr>
<td>C</td>
<td>[0 \ (\text{if } C_t(C) &lt; 200); 6 \ (C_t(C) - 200) \ (\text{if } C_t(C) &gt; 200)]</td>
</tr>
</tbody>
</table>

Table 2.2 Demand and loss functions
Downstream of the diversions to the reservoir and the polder area a minimum flow of 20 (m³/s) is required in each period.

**Infrastructural alternatives**

The optimal distribution and allocation of water for different alternatives of the water supply infrastructure form the subject of calculation with different optimization techniques in the following chapters. The formulation of the alternatives is given below; they form the basis for the formulation of 4 optimization problems. In all cases complete mixing of water in the reservoirs is assumed.

- A first alternative which will be considered about the above allocation problem is that of providing the water for the three areas out of the reservoir and instituting, in case of shortage, an equal distribution of shortage per unit area. In this case the configuration of Figure 2.2 results.

![Diagram](image)

**Figure 2.2 Problems 1 and 2: all three irrigation areas supplied from the reservoir and an equal distribution per unit area of the shortages**

This allocation problem of distributing the water over the three time intervals in such a way, that total losses are minimized, will first be considered without quality aspects and solved by Linear and Dynamic Programming. This alternative will be referred to as problem 1.
• A second problem which will be considered uses the same water supply alternative as that of the above problem 1 but with quality considerations. This problem is solved by Dynamic Programming.

• A third problem is that of supplying the water for irrigation area C from the river and supplying areas A and B separately from the reservoir. For this problem the number of decisions in each time period is increased. This optimization problem will be solved by a combination of Dynamic Programming and the Lagrange Multiplier technique, though Linear Programming could also be used. Figure 2.3 gives a sketch of the considered infrastructure.

Figure 2.3 Problem 3: irrigation area C supplied from the river and irrigation areas A and B supplied from the reservoir

• The last alternative (problem 4) which will be considered is the supply of water for irrigation area C from the polder reservoir and for areas A and B separately from the reservoir (Figure 2.4).
Figure 2.4 Problem 4: irrigation area C supplied from the polder reservoir and irrigation areas A and B supplied from the reservoir

The consideration of one more reservoir means an increase in the dimensionality of the problem. Another optimization technique namely an iterative solution using Non-linear Programming will be used in this case.

Summary
Figure 2.5 presents a summary outline of the four water allocation problem situations, while Table 2.3 gives an overview of the associated data.
Figure 2.5 Summary outline of the alternative problem situations
<table>
<thead>
<tr>
<th></th>
<th>Time period</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>River flow</td>
<td></td>
</tr>
<tr>
<td>(m³/s)</td>
<td></td>
</tr>
<tr>
<td>Salt concentration river flow (mg/l)</td>
<td>240</td>
</tr>
<tr>
<td>Seepage into polder area (m³/s)</td>
<td>250</td>
</tr>
<tr>
<td>Salt concentration seepage (mg/l)</td>
<td>15</td>
</tr>
<tr>
<td>Downstream minimum requirement (m³/s)</td>
<td>1000</td>
</tr>
<tr>
<td>Water demand</td>
<td></td>
</tr>
<tr>
<td>- area A (m³/s)</td>
<td></td>
</tr>
<tr>
<td>- area B (m³/s)</td>
<td></td>
</tr>
<tr>
<td>- area C (m³/s)</td>
<td></td>
</tr>
<tr>
<td>Total (m³/s)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Shortage loss function</td>
<td></td>
</tr>
<tr>
<td>- area A</td>
<td>0.5 ( T^2_1(A) )</td>
</tr>
<tr>
<td>- area B</td>
<td>1.6 ( T^2_1(B) )</td>
</tr>
<tr>
<td>- area C</td>
<td>0.75 ( T^2_1(C) )</td>
</tr>
<tr>
<td>Salinity loss function</td>
<td></td>
</tr>
<tr>
<td>(equal for each period)</td>
<td></td>
</tr>
<tr>
<td>- area A</td>
<td>0 if ( C_t(A)&lt;200 ); 9*(( C_t(A)-200 )) if ( C_t(A)&gt;200 )</td>
</tr>
<tr>
<td>- area B</td>
<td>0 if ( C_t(B)&lt;200 ); 3*(( C_t(B)-200 )) if ( C_t(B)&gt;200 )</td>
</tr>
<tr>
<td>- area C</td>
<td>0 if ( C_t(C)&lt;200 ); 6*(( C_t(C)-200 )) if ( C_t(C)&gt;200 )</td>
</tr>
<tr>
<td>Reservoir capacity</td>
<td></td>
</tr>
<tr>
<td>minimum storage (m³/s)</td>
<td>160</td>
</tr>
<tr>
<td>initial storage (m³/s)</td>
<td>40</td>
</tr>
<tr>
<td>initial salt concentration (mg/l)</td>
<td>80</td>
</tr>
<tr>
<td>Polder constant volume</td>
<td></td>
</tr>
<tr>
<td>initial salt concentration (mg/l)</td>
<td>200</td>
</tr>
</tbody>
</table>

Relation between areas A, B and C: 3 : 2 : 1

Table 2.3 Overview of the data associated with the water allocation problems
3 Overview of solutions of the sample problems

This chapter presents in four subsequent sections (3.1 through 3.4): the mathematical programming formulation, the general characteristics of the solution methods and the results of the computations for each of the four problems of Chapter 2. In the following chapters 4, 5, 6 and 7, the solution methods are explained into more detail. The last section (3.5) of Chapter 3 includes a summary.

In all calculations complete mixing in the reservoirs is supposed.

3.1 Problem 1

The three irrigation areas are being considered as one irrigation area with one composite loss function for each of the three periods. No quality aspects are taken into account.

The total water shortage is given by \( T_t = T_t(A) + T_t(B) + T_t(C) \) for each period \( t \). The losses are calculated as follows:

Composite loss functions:

- loss in period 1 = \( 0.5\left(\frac{3}{6} T_1\right)^2 + 1.6\left(\frac{2}{6} T_1\right)^2 + 0.75\left(\frac{1}{6} T_1\right)^2 = 0.3236 \ T_1^2 \)

- loss in period 2 = \( 0.6\left(\frac{3}{6} T_2\right)^2 + 2.1\left(\frac{2}{6} T_2\right)^2 + 0.9\left(\frac{1}{6} T_2\right)^2 = 0.4083 \ T_2^2 \)

- loss in period 3 = \( 0.4\left(\frac{3}{6} T_3\right)^2 + 1.4\left(\frac{2}{6} T_3\right)^2 + 0.6\left(\frac{1}{6} T_3\right)^2 = 0.2722 \ T_3^2 \)

The objective function is composed of the loss functions for water shortage in the three periods. The constraint set contains the continuity relationships for the reservoir, upper and lower bounds on reservoir storage, upper bounds on inflows to the reservoir, and equations defining the water shortages.

3.1.1 Formulation of the mathematical programming model

Additional notation:
\( V_t \): reservoir storage at the end of period \( t \).
\( Y_t \): inflow to the reservoir in period \( t \).
\( R_t \): allocation (outflow) in period \( t \).
\( T_t \): shortage in period \( t \).

**Objective function:**

\[
\min [0.3236 \, T_1^2 + 0.4083 \, T_2^2 + 0.2722 \, T_3^2]
\] (3.1)

**Subject to the constraints:**

- \( V_1 = 80 + Y_1 - R_1 \)
  \( V_2 = V_1 + Y_2 - R_2 \)
  \( V_3 = V_2 + Y_3 - R_3 \)
  (reservoir content continuity equation for each time period; initial storage is 80) (3.2)

- \( 40 < V_1 < 160 \)
  \( 40 < V_2 < 160 \)
  \( 40 < V_3 < 160 \)
  (dead storage is 40 and maximum capacity is 160 for each time period) (3.3)

- \( 240 - Y_1 \geq 20 \) \( Y_1 < 220 \)
  \( 210 - Y_2 \geq 20 \) \( Y_2 < 190 \)
  \( 130 - Y_3 \geq 20 \) \( Y_3 < 110 \)
  (downstream restflow must be larger than downstream minimum flow requirement (= 20)) (3.4)

- \( T_1 = 210 - R_1 \)
  \( T_2 = 300 - R_2 \)
  \( T_3 = 180 - R_3 \)
  (definition of shortage in each period as the difference between demand and allocation) (3.5)

The mathematical programming model is linear except for the objective function.

### 3.1.2 Solution with Linear Programming (LP)

A solution to the above model can be obtained by linearizing the objective function and applying LP. The non-linear objective function in the model is convex and can be linearized straightaway by piecewise linearization. This is described in Chapter 4 and Annex C.

The solution for problem 1 is the following (in m\(^3\)/s)
\[
\begin{array}{cccc}
V_1 &=& 140 & Y_1 &=& 220 & T_1 &=& 50 \\
V_2 &=& 60 & Y_2 &=& 190 & T_2 &=& 30 \\
V_3 &=& 40 & Y_3 &=& 110 & T_3 &=& 50 \\
R_1 &=& 160 \\
R_2 &=& 270 \\
R_3 &=& 130 \\
\end{array}
\]

Total loss: 1898.

As easily can be seen by comparing the \(V_t\) and the river flows (240, 210 and 130 \(m^3/s\) for the three periods) the minimum of 20 \(m^3/s\) for the downstream flow is met. An error of about 2\% is introduced in the solution because of the approximation of the non-linear curves by linear pieces. In Section 4.2 more attention is given to this question.

3.1.3 Solution with Dynamic Programming (DP)

Dynamic Programming can be considered as an optimization procedure suitable for solving sequential allocation problems. To prepare an optimization problem for solution with DP one has to identify or recognize different stages in the problem in which decisions can be made which have consequences that are independent from decisions in other stages. The stages are connected to each other through variables called state variables which characterize the stages.

The general structure of DP is presented in R999-5 [4]. Briefly the solution procedure for the above allocation problem is as follows. The procedure is fully worked out in Section 4.3.

In a first computational procedure (generally in backward direction) the optimal solutions respectively are determined for period 3, periods 3+2 and periods 3+2+1 for different sets of state variables. This (recursive) procedure yields an overall optimal solution. In a second computational procedure in the other direction (generally forward) the state variables associated with the overall optimal solution are determined. For the problem at hand, the reservoir volumes \(V_t\) are regarded as state variables.

**Backward procedure**

**Stage 1** – Solve the optimization problem in period 3 for different values of \(V_2\). This yields the function \(CL_3(V_2)\), which represents the minimum
losses in period 3, if a quantity \( R_3 \) is available to be allocated. \( CL_3 \) is a function of \( V_2 \) only as \( R_3 = V_3 - V_2 \) and one can suppose that at the end of period 3 all available water is allocated and consequently \( V_3 = 40 \text{ m}^3/\text{s} \) (dead storage).

**Stage 2** - Solve the optimization problem in periods 3 + 2 for different values of \( V_1 \). This yields the function \( CL_2(V_1) \) which represents the minimum losses in periods 3 + 2; the minimum for each particular \( V_1 \) is obtained by choosing that value of \( V_2 \) for which the sum of the losses \( CL_3(V_2) \) in period 3 and the losses in period 2 with a quantity \( R_2 \) available is minimal; \( CL_2 \) can be expressed as a function of \( V_1 \) only (\( R_2 = V_2 - V_1 \)).

**Stage 3** - Solve the optimization problem in periods 3 + 2 + 1 for \( V_0 \) (the starting value, which is fixed). This yields the function \( CL_1(V_0) \), which represents the maximum benefit in periods 3 + 2 + 1 if a given quantity \( R_1 \) is available. \( CL_1 \) is fixed (\( R_1 = V_1 - V_0 \) and \( V_0 = 80 \)).

The usual procedure for the optimization, in view of the usual single variable stages is to discretize the state variables in some relevant values and to make the calculations for those values.

**Forward procedure**

After the stages 1 through 3 are performed, an overall minimal loss value has been found: \( CL_1(V_0) \). This is reached by choosing the optimal values \( R_1^* \) and \( V_1^* \) for \( R_1 \) and \( V_1 \). Given \( V_1^* \) the optimal solution for periods 2 and 3 is recalled: \( CL_2(V_1^*) \), which is reached by choosing \( R_2^* \) and \( V_2^* \) as optimal values for \( R_2 \) and \( V_2 \). Similarly the function \( CL_3 \) determined in stage 1 is reached with \( R_3^* \) as optimal value for \( R_3 \) (\( R_3^* = V_3 - V_2^* \) and \( V_3 \) is known beforehand).

It can be observed that for this case the optimization in each stage is simple because there is only one decision variable which is completely determined by the state variables at the beginning and end of the particular stage (or equivalently the state at the end of the stage is uniquely determined by the value of the state variable at the beginning of the stage and the decision in the stage). So there is no optimization left within a particular stage. As we will see problems 3 and 4 are cases where an optimization is still required.
within a stage. In far most of the cases DP is used with single variable stages or uniquely determined decisions within a stage.

The optimal decisions and reservoirs states as determined by DP are then (in m$^3$/s)

<table>
<thead>
<tr>
<th>$V_1$</th>
<th>$Y_1$</th>
<th>$T_1$</th>
<th>$R_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>140</td>
<td>220</td>
<td>50</td>
<td>160</td>
</tr>
<tr>
<td>$V_2$</td>
<td>$Y_2$</td>
<td>$T_2$</td>
<td>$R_2$</td>
</tr>
<tr>
<td>60</td>
<td>190</td>
<td>30</td>
<td>270</td>
</tr>
<tr>
<td>$V_3$</td>
<td>$Y_3$</td>
<td>$T_3$</td>
<td>$R_3$</td>
</tr>
<tr>
<td>40</td>
<td>110</td>
<td>50</td>
<td>130</td>
</tr>
</tbody>
</table>

(3.7)

Total loss: 1857.

3.1.4 Comparison of the LP versus DP solution procedure

The solution methods with LP and DP are quite different. Starting from the same problem formulation a completely different approach is taken. Basically in the LP approach, the problem is linearized (if non-linear) and all the variables are considered together by the solution procedure. An efficient solution is then possible because for linear problems the optimal solution occurs at a corner point of the feasible region. The simplex algorithm searches then systematically among those corner points.

In the DP approach a decomposition of the problem into preferably single variable problems is tried. Those decomposed problems are usually much simpler to solve and have the advantage that they do not have to be linear (specifically for a case with a single variable or uniquely determined stages). Problems are then, however, encountered if the number of interaction variables (state variables) between the decomposed problems becomes large because the amount of computation increases exponentially with the number of state variables. The implications of such an increase in the number of state variables is clearly illustrated in the solution approach to problem 4.

The constraints in the optimization problem play a different role in the LP and DP solution procedure. In the LP method additional constraints create additional corner points and as a crude rule of thumb the computation time is proportional with the third power of the number of constraints. In the DP procedure additional constraints limit the interval over which the variables
have to be searched and so they limit computation time. The number of discretized values by which the state variables are represented in the DP solution procedure have further a large impact on the computation time.

The decision whether to choose for a LP or DP solution method to a particular problem depends thus on:

a) the linearity and the form of possible non-linearities in the objective function and related constraints, which schematize the problem (LP);

b) the number of stages and corresponding state variables, resulting from a particular decomposition (DP);

c) the number of the constraints (LP); and

d) the number of discrete values of the state variables, required to reach a particular accuracy (DP).

Usually if the problem is linear or if the non-linearities have the right shape (convex in minimization problems) and if the constraints are not too numerous a LP solution approach will be preferable.

Another consideration in the choice between LP and DP is that for a LP solution the optimization problem has only to be shaped in the general model format (for a minimization problem)

\[
\begin{align*}
\min_{\mathbf{x}} & \quad [C \mathbf{x}] \\
\text{s.t.} & \quad \mathbf{A} \mathbf{x} \leq \mathbf{b} \\
& \quad \mathbf{x} \geq 0
\end{align*}
\] (3.8)

(\(\mathbf{x}\) is the vector of decision variables, \(C\) the vector of their coefficients in the objective function, \(B\) the vector of right members of the inequality constraints and \(A\) the matrix of coefficients of the decision variables therein) for which a standard computer program is available to solve the model. For the DP procedure the calculations and bookkeeping as performed in Table 4.1 has to be programmed for the computer separately for each case. Some general structure can be given to such a program but most of the programming has to be done for each problem separately.

The optimal decisions and reservoir states in the LP and DP solutions to problem 1 are identical: compare equations (3.6) and (3.7). The minimum total
loss is, however, different. In the DP algorithm no approximations to the original loss curve are used so there is no approximation error for the discrete points for which the calculations are made. However, due to this discretization of the variables the solution can only be identified at these points. It will only be accidental that the true optimum is located at such a point. In the LP procedure the decision variables are considered continuous, however, discontinuities are introduced when a piecewise linear representation of non-linear relationships is made. This has the effect that optimal solutions tend to be driven to the points of discontinuity and in that sense another kind of discretization is occurring in the solution procedure and again only accidentally will the true optimum be determined. The error thus introduced in both methods becomes smaller when smaller discretization intervals (for DP) and more linear pieces (for LP) are considered (obviously the LP procedure has no error when the problem is completely linear).

In the above solutions to problem 1 it is only accidental that the optimal solutions for the decision variables as determined by LP and DP are identical; they would only be expected to be close to each other (if a sufficient number of linear pieces and discretization intervals is used).

3.2 Problem 2

The same configuration of water supply is considered as for problem 1, however, damages due to salt concentrations in the supplied water are added to the trade-off for an optimal allocation of water over the three intervals. For this purpose the loss functions corresponding to water quality as defined in Chapter 2 are added to the objective function. More complex is the modelling of the salt concentration of the release from the reservoir. The concentration in the reservoir at each point in time is non-linearly related to the volume and flow variables in the salt balance equation. Moreover, the salt concentration of the water applied to the field is variable over the considered time interval, causing the necessity to determine an average concentration over the decision interval to which possible damage can be attributed.

Two alternative methods to model the salt balance equation are presented in Annex A. The second method is used in the formulation of the optimization model below.
3.2.1 Formulation of the mathematical programming model

Additional notation:

- $C_{\text{RES}}_t$: salt concentration in the reservoir at the end of period $t$.
- $C_{\text{RIV}}_t$: salt concentration in the river in period $t$.
- $C_{\text{AV}}_t$: average salt concentration of the releases from the reservoir in period $t$.
- $C_{\text{NOT}}_t$: variables which help define the salinity loss function; respectively they determine the portion of $C_t(A)$ which does and that which does not cause losses.

The salt concentration of the water applied to the fields A, B and C viz. $C_t(A)$, $C_t(B)$ and $C_t(C)$ are equal to the salt concentration of the releases from the reservoir. In the model formulation the possible damage to crops has therefore been directly related to $C_{\text{AV}}_t$. As in problem 1 the three irrigation areas are considered as one area so a composite salinity loss function (of $C_{\text{AV}}_t$) can be considered.

Objective function:

\[
\min \left[ 0.3236 T_1^2 + 0.4083 T_2^2 + 0.2722 T_3^2 + 18 \ C_{\text{LOSS}}_1 + 18 \ C_{\text{LOSS}}_2 + 18 \ C_{\text{LOSS}}_3 \right] \tag{3.9}
\]

Subject to the following constraints:

\[
\begin{align*}
V_1 &= 80 + Y_1 - R_1 \\
V_2 &= V_1 + Y_2 - R_2 \\
V_3 &= V_2 + Y_3 - R_3
\end{align*}
\]
(reservoir content continuity equation for each time period; initial storage is 80) \tag{3.11}

\[
40 < V_1 < 160 \\
40 < V_2 < 160 \\
40 < V_3 < 160
\]
(dead storage is 40 and maximum capacity is 160 for each time period) \tag{3.12}

\[
\begin{align*}
240 - Y_1 &> 20 \\
210 - Y_2 &> 20 \\
130 - Y_3 &> 20
\end{align*}
\]
(downstream restflow must be larger than downstream minimum flow requirement (= 20)) \tag{3.13}
\[ T_1 = 210 - R_1 \]  
\[ T_2 = 300 - R_2 \]  
\[ T_3 = 180 - R_3 \]  
(definition of shortage in each period as the difference between demand and allocation)  

\[ V_1 \cdot CRES_1 = 80 \cdot 200 + Y_1 \cdot CRIV_1 (= 250) - R_1 \cdot CAV_1 \]  
\[ V_2 \cdot CRES_2 = V_1 \cdot CRES_1 + Y_2 \cdot CRIV_2 (= 300) - R_2 \cdot CAV_2 \]  
\[ V_3 \cdot CRES_3 = V_2 \cdot CRES_2 + Y_3 \cdot CRIV_3 (= 350) - R_3 \cdot CAV_3 \]  
(salt balance equations for each period: second method of Annex A; see Equation (A.5))

\[ CAV_1 = \frac{1}{4} CRES_1 + \frac{1}{4} \cdot 200 \]  
\[ CAV_2 = \frac{1}{4} CRES_2 + \frac{1}{4} CRES_1 \]  
\[ CAV_3 = \frac{1}{4} CRES_3 + \frac{1}{4} CRES_2 \]  
(average salt concentration for each time period)

\[ CAV_1 = CNOT_1 + CLOSS_1 \& CNOT_1 < 200 \]  
\[ CAV_2 = CNOT_2 + CLOSS_2 \& CNOT_2 < 200 \]  
\[ CAV_3 = CNOT_3 + CLOSS_3 \& CNOT_3 < 200 \]  
(division of salt concentration in part without causing damage (CNOT_t) and part causing damage (CLOSS_t))

### 3.2.2 Solution with Dynamic Programming (DP)

The optimization model presented above can be put in the format of a multi-stage decision process in a similar way as has been presented in Section 3.1.3. Using the problem data given in Chapter 2, Figure 3.1 gives the schematization. For the decomposition into stages two state variables are to be considered now, namely the volume and salt concentration of storage in the reservoir at the end of a particular time period. For each stage, consideration is made of the loss resulting from shortage and high salt concentration. In each stage two decisions have to be made namely, on the inflow into the reservoir \((Y_t)\) and on the release from the reservoir \((R_t)\). The determination of the inflow is not so simple as in problem 1 because it is not clear that more is always better. However, because of the extra equation introduced by the salt balance the two decisions are uniquely determined when the state variables \(V_t\), \(CRES_t\), \(V_{t-1}\) and \(CRES_{t-1}\) are specified. The equations are further described in Chapter 5.
Figure 3.1: Schematization of problem 2 as a multistage decision process.

**Stage 1**
- **Given**: Reservoir content and salinity in period 1:
  - $V_0$
- **State Variables**:
  - $V_0$, $V_1$
  - CRES₂

**Decision**: $R_1$
- **Allocation**: $R_1$ to irrigation areas
- **Resulting Losses**:
  - Shortage: $0.3236 \cdot (210 - R_1)^2$
  - Salinity: $18 \cdot (CAV₁ - 200)$

**Stage 2**
- **Given**: Reservoir content and salinity in period 2:
  - $V_1$
- **State Variables**:
  - $V_1$, $V_2$
  - CRES₂

**Decision**: $R_2$
- **Allocation**: $R_2$ to irrigation areas
- **Resulting Losses**:
  - Shortage: $0.4083 \cdot (300 - R_2)^2$
  - Salinity: $18 \cdot (CAV₂ - 200)^2$

**Stage 3**
- **Given**: Reservoir content and salinity in period 3:
  - $V_2$
- **State Variables**:
  - $V_2$, $V_3$
  - CRES₃

**Decision**: $R_3$
- **Allocation**: $R_3$ to irrigation areas
- **Resulting Losses**:
  - Shortage: $0.2722 \cdot (180 - R_3)^2$
  - Salinity: $18 \cdot (CAV₃ - 200)^2$
The following optimal solution has been reached for problem 2 (in m³/s and mg/l)

\[
\begin{array}{cccccc}
V_1 &=& 160 & Y_1 &=& 217 & \text{CRES}_1 = 247\% & T_1 &=& 73 & R_1 &=& 137 \\
V_2 &=& 80 & Y_2 &=& 183 & \text{CRES}_2 = 293 & T_2 &=& 37 & R_2 &=& 263 & \text{(3.18)} \\
V_3 &=& 40 & Y_3 &=& 104 & \text{CRES}_3 = 346 & T_3 &=& 36 & R_3 &=& 144 \\
\end{array}
\]

Total loss: 6451.

3.3 Problem 3

The three irrigation areas are now considered separate for their water supply. The water supply to irrigation area C is considered directly from the river.

The objective function is composed of the individual loss functions (shortage and salinity losses) for each area for each time interval.

The salt concentration of water supplied to area C is fixed by the supply in the river and cannot be controlled: it constitutes a fixed loss to area C. It is indicated as a constant in the objective function and forms a part of the total loss incurred by this water supply system. It consequently can remain further outside of the optimization considerations in the model.

Because the water the areas A and B receive from the reservoir has the same salt concentration, their individual salt damage loss functions can be added together and put as a function of this concentration.

The salt balances are modelled in the same way as for problem 2 namely the second method of Annex A is used.

3.3.1 Formulation of the mathematical programming model

Additional notation:

\[
\begin{align*}
R_t(A), R_t(B), R_t(C) & : \text{respectively allocation of water to irrigation areas A, B and C in time period } t. \\
Z_t & : \text{remaining flow in the river downstream of the reservoir intake.}
\end{align*}
\]
Objective function:

\[
\begin{align*}
\text{min } & \left[ 0.5 T_1^2(A) + 0.6 T_2^2(A) + 0.4 T_3^2(A) + \\
& 1.6 T_1^2(B) + 2.1 T_2^2(B) + 1.4 T_3^2(B) + \\
& 0.7 T_1^2(C) + 0.9 T_2^2(C) + 0.6 T_3^2(C) + \\
& +(3+9)\text{LOSS}_1 + (3+9)\text{LOSS}_2 + (3+9)\text{LOSS}_3 + \\
& +6(250-200) + 6(300-200) + 6(350-200) \right] \\
\text{(shortage loss in area A)} \\
\text{(shortage loss in area B)} \\
\text{(shortage loss in area C)} \\
\text{(salinity losses areas A and B)} \\
\text{(salinity losses area C)}
\end{align*}
\]

Subject to the following constraints:

- \( V_1 = 80 + y_1 - R_1(A) - R_1(B) \) & \( 40 \leq V_1 \leq 160 \) (reservoir content continuity equation)
- \( V_2 = V_2 + y_2 - R_2(A) - R_2(B) \) & \( 40 \leq V_2 \leq 160 \) per period: initial storage is 80; \( 3.20 \)
- \( V_3 = V_3 + y_3 - R_3(A) - R_3(B) \) & \( 40 \leq V_3 \leq 160 \) dead and maximum storage are 40 and 160

- \( Z_1 = 240 - y_1 & z_1 - R_1(C) \geq 0 \) (definition remaining flow after reservoir intake; minimum downstream flow after area C intake per period is 20)
- \( Z_2 = 210 - y_2 & z_2 - R_2(C) \geq 0 \)
- \( Z_3 = 130 - y_3 & z_3 - R_3(C) \geq 0 \)

- \( T_1(A) = 105 - R_1(A) \) & \( T_2(A) = 150 - R_2(A) \) & \( T_3(A) = 90 - R_3(A) \) (definition of shortages)
- \( T_1(B) = 35 - R_1(B) \) & \( T_2(B) = 50 - R_2(B) \) & \( T_3(B) = 30 - R_3(B) \) per period

- \( V_1 \cdot \text{CRES}_1 = 80 \cdot 200 + y_1 \cdot 250 - R_1(A) \cdot \text{CAV}_1 - R_1(B) \cdot \text{CAV}_1 \) (reservoir salt continuity)
- \( V_2 \cdot \text{CRES}_2 = V_2 \cdot \text{CRES}_2 + y_2 \cdot 300 - R_2(A) \cdot \text{CAV}_2 - R_2(B) \cdot \text{CAV}_2 \) for each time period
- \( V_3 \cdot \text{CRES}_3 = V_3 \cdot \text{CRES}_3 + y_3 \cdot 350 - R_3(A) \cdot \text{CAV}_3 - R_3(B) \cdot \text{CAV}_3 \) (second method of Annex A; see Eq. (A.5))

- \( \text{CAV}_1 = \text{CRES}_1 \cdot \text{CAV} \cdot 200 \) (average salt concentration for each time period)
- \( \text{CAV}_2 = \text{CRES}_2 \cdot \text{CAV} \cdot 200 \)
- \( \text{CAV}_3 = \text{CRES}_3 \cdot \text{CAV} \cdot 200 \)

- \( \text{CAV}_1 = \text{CNOT}_1 + \text{LOSS}_1 \) & \( \text{CNOT}_1 < 200 \) (division of salt concentration in part without causing damage (\( \text{CNOT}_1 \))
- \( \text{CAV}_2 = \text{CNOT}_2 + \text{LOSS}_2 \) & \( \text{CNOT}_2 < 200 \) and part causing damage (\( \text{LOSS}_2 \))
- \( \text{CAV}_3 = \text{CNOT}_3 + \text{LOSS}_3 \) & \( \text{CNOT}_3 < 200 \)
3.3.2 Solution with Dynamic Programming (DP) and Lagrange Multipliers (LM)

Similar to the formulation for problem 2 the overall optimization can be separated into sub-optimizations over each period. The stages thus defined are again linked to each other by the volume and salt concentration of the storage in the reservoir.

Following the forward DP-approach (see end of Section 3.1.3) cumulative loss functions \( CL_t \) are defined, representing the minimum reachable loss by varying all variables of that particular and all foregoing stages, with exception of \( V_t, V_{t-1}, \text{CRES}_t \) and \( \text{CRES}_{t-1} \) (\( CL(V_o, \text{CRES}_o) = 0 \) because \( V_0 \) and \( \text{CRES}_0 \) are constant); for example for period 2 we get:

\[
CL_2(V_2, \text{CRES}_2) = \min \left[ 0.6T_2^2(A) + 2.1T_2^2(B) + 0.9T_2^2(C) + 12\text{LOSS}_2 + CL_1(V_1, \text{CRES}_1) \right] \tag{3.26}
\]

This objective function serves together with all constraints for the time period at hand as the optimization model per period.

The state variables for problem 3 are the same as for problem 2, however an important change occurs in the number of decisions in each stage.

- In the case of problem 2 the subproblems in the DP procedure are single variable problems or problems for which the decisions in the stage are uniquely determined with the specification of the state variables. In these cases the optimization in fact is performed only through the optimal choice of the state variables \( V_{t-1}, \text{CRES}_{t-1} \) for a particular combination of \( V_t \) and \( \text{CRES}_t \). The optimal string of reservoir states is then determined by the folding back procedure as has been illustrated in Figure 3.1.

- In the case of problem 3 there is still an optimization left within the stages. After the specification of the combination \( V_t \) and \( \text{CRES}_t \) for each choice of state variables \( V_{t-1}, \text{CRES}_{t-1} \) an optimization has to be performed in relation to the water distribution over the areas A, B and C; after this has been done an optimal choice can be made among the \( V_{t-1} \) and \( \text{CRES}_{t-1} \) that have been considered. There are thus two levels of optimization, one within a particular stage or time period and an optimization over the stages performed by the DP solution procedure.
A choice has to be made for the procedure which solves the optimization problem within a stage. In case of problem 3 it is convenient to choose the LM (Lagrange Multiplier) technique as the optimization procedure because it gives an analytical solution which can conveniently be used in the DP calculations. However, also LP could be used (with a linearized objective function).

The application of the LM technique is illustrated in detail in Annex B with some examples. See for the problem at hand Chapter 6.

The same DP calculation procedure can be followed as for problem 2 except that the equations which derive the stage decisions from the specified states are now extended by the equations derived from the LM technique.

The optimal solution then is the following (in m³/s and mg/l)

<table>
<thead>
<tr>
<th>( V_1 ) = 140</th>
<th>( Y_1 ) = 173</th>
<th>( \text{CRES}_1 ) = 244</th>
<th>( T_1(A) ) = 15</th>
<th>( T_1(B) ) = 12</th>
<th>( T_1(C) ) = 23</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_2 ) = 60</td>
<td>( Y_2 ) = 103</td>
<td>( \text{CRES}_2 ) = 282</td>
<td>( T_2(A) ) = 10</td>
<td>( T_2(B) ) = 7½</td>
<td>( T_2(C) ) = 13 (3.27)</td>
</tr>
<tr>
<td>( V_3 ) = 40</td>
<td>( Y_3 ) = 66</td>
<td>( \text{CRES}_3 ) = 336</td>
<td>( T_3(A) ) = 19</td>
<td>( T_3(B) ) = 15½</td>
<td>( T_3(C) ) = 16</td>
</tr>
</tbody>
</table>

Total loss: 4020.

The fixed quality loss in irrigation area C, not taken into account in the optimization is 6*(250-200) + 6*(300-200) + 6*(350-200) = 1800, so the total loss for this supply alternative is 5820.

3.4 Problem 4

The three irrigation areas are considered separate for their water supply as in problem 3. Area C however receives water not directly from the river but from a polder reservoir with a constant volume, subject to seepage and with a possibility for flushing in order to reduce the salt concentration in the polder area.

The objective function in the model is again composed of the individual loss terms for shortage and salinity for each area in each time period. Salt concentration losses for areas A and B are again represented by a composite function for each time period. The salt balances for this case are modelled
following the first method presented in Annex A, just to show the approach to problems caused by concave functions; the solution includes the use of Mixed Integer Programming which is explained in more detail in Chapter 7.

3.4.1 Formulation of the mathematical programming model

Additional notation:

- $\text{CPOL}_t$: salt concentration in the polder reservoir at time $t$.
- $\text{CPAV}_t$: average salt concentration in the polder reservoir in time period $t$; this concentration is used to calculate loss to crops in area C due to high salinity.
- $\text{CPNOT}_t$, $\text{CPLOSS}_t$: variables which help define the salinity loss function; they are associated with the average salt concentration of the polder water $\text{CPAV}_t$ and represent the portion of $\text{CPAV}_t$ which does and that which does not cause losses.
- $\text{VPOL}$: constant volume of the polder reservoir.
- $\text{YPOL}_t$: flow into the polder reservoir in time period $t$.
- $\text{RFL}_t$: volume of flushing of the polder in time period $t$.
- $\text{SEEPT}_t$: saline seepage into the polder in time period $t$.
- $\text{CSEEP}_t$: salt concentration of the saline seepage in the polder in time period $t$.

Objective function:

$$
\text{min} \left[ 0.5T_1^2(A) + 0.6T_2^2(A) + 0.4T_3^2(A) + 1.6T_1^2(B) + 2.1T_2^2(B) + 1.4T_3^2(B) + 0.75T_1^2(C) + 0.9T_2^2(C) + 0.6T_3^2(C) + 12\text{CLOSS}_1 + 12\text{CLOSS}_2 + 12\text{CLOSS}_3 + 6\text{CLOSS}_1 + 6\text{CLOSS}_2 + 6\text{CLOSS}_3 \right]$$

(shortage loss in area A) (shortage loss in area B) (shortage loss in area C) (salinity losses areas A and B) (salinity losses area C)

Subject to the constraints:

- $V_1 = 80 + Y_1 - R_1(A) - R_1(B) \leq 40 \text{ with } 1 \leq V_1 \leq 160$ (reservoir content continuity equation)
- $V_2 = V_1 + Y_2 - R_2(A) - R_2(B) \leq 40 \text{ with } 2 \leq V_2 \leq 160$ (per period: initial storage is 80; (3.29)
- $V_3 = V_2 + Y_3 - R_3(A) - R_3(B) \leq 40 \text{ with } 3 \leq V_3 \leq 160$ (dead and maximum storage are 40 and 160)
\( \text{Z}_1 = 240 - Y_{1}^1 \text{ & } \text{Z}_1 - \text{YPOL}_1 > 20 \) \\
\( \text{Z}_2 = 210 - Y_{2}^2 \text{ & } \text{Z}_2 - \text{YPOL}_2 > 20 \) \\
\( \text{Z}_3 = 130 - Y_{3}^3 \text{ & } \text{Z}_3 - \text{YPOL}_3 > 20 \)

(definition remaining flow after reservoir intake; minimum downstream flow after area C intake; per period)

\( \text{YPOL}_1 + \text{SEEP}_1^1 = R_1^1(C) + \text{RFL}_1 \) \\
\( \text{YPOL}_2 + \text{SEEP}_2^2 = R_2^2(C) + \text{RFL}_2 \) \\
\( \text{YPOL}_3 + \text{SEEP}_3^3 = R_3^3(C) + \text{RFL}_3 \)

(polder reservoir content continuity equation; for each time period)

\( \text{T}_1^1(A) = 105 - R_1^1(A) \text{ & } \text{T}_2^2(A) = 150 - R_2^2(A) \text{ & } \text{T}_3^3(A) = 90 - R_3^3(A) \) \\
\( \text{T}_1^1(B) = 35 - R_1^1(B) \text{ & } \text{T}_2^2(B) = 50 - R_2^2(B) \text{ & } \text{T}_3^3(B) = 30 - R_3^3(B) \)

(definition of shortages per period)

\( \text{T}_1^1(C) = 70 - R_1^1(C) \text{ & } \text{T}_2^2(C) = 100 - R_2^2(C) \text{ & } \text{T}_3^3(C) = 60 - R_3^3(C) \)

\( \text{CRES}_1 = 250 + (200 - 250) e^\frac{-Y_{1}^1}{V} \) \\
\( \text{CRES}_2 = 300 + (\text{CRES}_1 - 300) e^\frac{-Y_{2}^2}{V} \) \\
\( \text{CRES}_3 = 350 + (\text{CRES}_2 - 350) e^\frac{-Y_{3}^3}{V} \)

(reservoir salt continuity equations for each time period; first method of Annex A; see Equation (A.4))

\( \text{CAV}_1 = \frac{1}{2} \text{CRES}_1 + \frac{1}{2} \times 200 \) \\
\( \text{CAV}_2 = \frac{1}{2} \text{CRES}_2 + \frac{1}{2} \text{CRES}_1 \) \\
\( \text{CAV}_3 = \frac{1}{2} \text{CRES}_3 + \frac{1}{2} \text{CRES}_2 \)

(average salt concentration in reservoir; for each time period)

\( \text{CAV}_1 = \text{CNOT}_1 + \text{CLOSS}_1 \text{ & } \text{CNOT}_1 < 200 \) \\
\( \text{CAV}_2 = \text{CNOT}_2 + \text{CLOSS}_2 \text{ & } \text{CNOT}_2 < 200 \) \\
\( \text{CAV}_3 = \text{CNOT}_3 + \text{CLOSS}_3 \text{ & } \text{CNOT}_3 < 200 \)

(division of salt concentration in part without causing damage (\text{CNOT}_\ell) (3.35) and part causing damage (\text{CLOSS}_\ell))

\( \text{CPOL}_1 = \frac{250 \cdot \text{YPOL}_1 + \text{SEEP}_1 \cdot \text{CSEEP}_1}{\text{YPOL}_1 + \text{SEEP}_1} (1 - \frac{\text{YPOL}_1}{\text{VPOL}}) + 250 e^\frac{-\text{YPOL}_1}{\text{VPOL}} \) \\
\( \text{CPOL}_2 = \frac{300 \cdot \text{YPOL}_2 + \text{SEEP}_2 \cdot \text{CSEEP}_2}{\text{YPOL}_2 + \text{SEEP}_2} (1 - \frac{\text{YPOL}_2}{\text{VPOL}}) + 300 e^\frac{-\text{YPOL}_2}{\text{VPOL}} \) \\
\( \text{CPOL}_3 = \frac{350 \cdot \text{YPOL}_3 + \text{SEEP}_3 \cdot \text{CSEEP}_3}{\text{YPOL}_3 + \text{SEEP}_3} (1 - \frac{\text{YPOL}_3}{\text{VPOL}}) + 350 e^\frac{-\text{YPOL}_3}{\text{VPOL}} \)

(polder reservoir salt continuity equation for each time period; first method of Annex A; see Equation (A.6))
3.4.2 Solution with an iterative application of Non-linear Programming

Just as the models for problems 2 and 3 the above optimization model can be put in the format of a multistage decision process. The stages are defined in a similar way, however the linkage between stages is expanded with another state variable namely the salt concentration in the polder reservoir.

In Section 7.1 it will be shown that the computational effort due to the additional state variable is increased by a factor of order 50. When considering more state variables the required computation effort would quickly be prohibitive.

The DP solution procedure to decompose the problem into stages becomes then unpractical and even impossible. A much more efficient approach is presented by using Non-linear Programming. The major difficulty is then, however, linearization of the non-linear problem. In the approach to problem 4 the linearization cannot be accomplished in one step and therefore an iterative application is necessary.

Solution procedure:

"Shifting curves" are constructed for particular values of estimated storage and salt concentration in the reservoirs at the beginning of the various time periods. Based on a piece-wise linearized version of the thus derived curves an optimization run is made with the linearized model. Better estimates for storages and salt concentration become then available which are used as a basis for a next optimization run. This is continued until the chosen and newly calculated values are close enough according to a preset normative
value. Further explication is given in Section 7.2.

The (convex) non-linear terms in the objective function of the optimization model do not present any special difficulties. They can be linearized as has been shown in the LP solution to problem 1 (see Section 4.1). The major difficulties are caused by the (concave) non-linear constraints representing the salt balances of the reservoir and the polder. Because of the concavity integer variables have to be used. Further explication is given in Section 7.3. The way in which the non-linear constraints (3.33) and (3.36) can be linearized and more particular how the continuity between the different intervals is maintained is also explained there.

The above solution procedure was programmed using Linear Programming. The solutions converged very quickly: the final solution was already reached in the third iteration. The three cycles took about 20 sec. of total computation time (IBM 370).

The obtained solution for problem 4 is the following (in $m^3/s$ and mg/l)

<table>
<thead>
<tr>
<th>$V_1$</th>
<th>$V_2$</th>
<th>$V_3$</th>
<th>$CRES_1$</th>
<th>$CRES_2$</th>
<th>$CRES_3$</th>
<th>$CPOL_1$</th>
<th>$CPOL_2$</th>
<th>$CPOL_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>140</td>
<td>45</td>
<td>40</td>
<td>242</td>
<td>272</td>
<td>333</td>
<td>253</td>
<td>393</td>
<td>503</td>
</tr>
</tbody>
</table>

$RFL_1 = 5 \quad T_1(A) = 30 \quad T_1(B) = 10 \quad T_1(C) = 0$
$RFL_2 = 0 \quad T_2(A) = 0 \quad T_2(B) = 0 \quad T_2(C) = 0$
$RFL_3 = 0 \quad T_3(A) = 30 \quad T_3(B) = 10 \quad T_3(C) = 10$

Total loss: 5873.
3.5 Summary

3.5.1 Methods

In this report examples have been given to illustrate the application of mathematical programming algorithms in water-distribution problems including both quantity and quality aspects. Completeness is not strived at, so several other possible mathematical programming methods are left out, such as dynamic optimization (maximum principle of Pontryagin) and methods such as automatic control.

The examples include a water supply system for which several infrastructural schematizations are made with varying degree of reality. The selection of the method to be applied depends on the structure of the real world problem and on the schematization which is acceptable in view of this problem. A comparison between LP and DP approaches is presented in Section 3.1.4, while considerations about the accuracy of these methods are given in Section 4.2 and 4.4 respectively. Section 7.1 includes general remarks about the application of Non-linear Programming techniques.

3.5.2 Overview of results

The four problems, which are subject to the investigations in this report, represent three different schematizations of the one water distribution scheme in Figure 2.1: in problem 1 the same schematization is used as in problem 2 without considering salinity losses.

The three alternative water supply schemes can be arranged according to their potential to supply water to the irrigation areas causing a minimum of loss due to water shortages and salinity. The ranking is presented in Figure 3.2.
First

Problem 3 supply situation

Minimum total loss 5820 units

Second

Problem 4 supply situation

Minimum total loss 5873 units

Third

Problem 2 supply situation

Minimum total loss 6451 units

Figure 3.2 Overview and ranking of results
Observation: problem 4 includes seepage in polder area C (15 m³/s with a concentration of 1000 mg/l). The extra salinity losses resulting for area C are nearly outweighed by the benefits of an increased water availability and flexibility of the supply situation: compare the results for problems 3 and 4.

This means that only problems 2 and 3 are directly comparable as far as the influence of schematization of the infrastructure on the results of the calculation is concerned. The possibilities to regulate the water distribution are greater for problem 3 than for problem 2; the result coincides with expectations. Looking at the schematizations as alternative supply systems, a water manager should weigh the reduced losses against the investment and maintenance costs of a more flexible water distribution infrastructure.

3.5.3 State variables

Before we discuss the optimal settings of the state variables for all four problems the values of the most relevant variables are summarized below.

**problem 1**

\[
\begin{align*}
V_1 &= 140 & Y_1 &= 220 & R_1 &= 160 & T_1 &= 50 \\
V_2 &= 60 & Y_2 &= 190 & R_2 &= 270 & T_2 &= 30 \\
V_3 &= 40 & Y_3 &= 110 & R_3 &= 130 & T_3 &= 50
\end{align*}
\]

**problem 2**

\[
\begin{align*}
V_1 &= 160 & Y_1 &= 217 & R_1 &= 137 & T_1 &= 73 & CRES_1 &= 247.5 \\
V_2 &= 80 & Y_2 &= 183 & R_2 &= 263 & T_2 &= 37 & CRES_2 &= 293 \\
V_2 &= 40 & Y_3 &= 104 & R_3 &= 144 & T_3 &= 36 & CRES_3 &= 346
\end{align*}
\]

**problem 3**

\[
\begin{align*}
V_1 &= 140 & Y_1 &= 173 & R_1(A) &= 90 & T_1(A) &= 15 & CRES_1 &= 244 \\
V_2 &= 60 & Y_2 &= 103 & R_2(A) &= 140 & T_2(A) &= 10 & CRES_2 &= 282 \\
V_3 &= 40 & Y_3 &= 66 & R_3(A) &= 71 & T_3(A) &= 19 & CRES_3 &= 336
\end{align*}
\]

\[
\begin{align*}
Z_1 &= 67 & R_1(B) &= 23 & T_1(B) &= 12 & R_1(C) &= 47 & T_1(C) &= 23 \\
Z_2 &= 107 & R_2(B) &= 42 & T_2(B) &= 7.5 & R_2(C) &= 87 & T_2(C) &= 13 \\
Z_3 &= 64 & R_3(B) &= 14.5 & T_3(B) &= 15.5 & R_3(C) &= 44 & T_3(C) &= 16
\end{align*}
\]
problem 4

\[ \begin{align*}
V_1 &= 140 & Y_1 &= 160 & R_1(A) &= 75 & T_1(A) &= 30 & CRES_1 &= 242 & CPOL_1 &= 253 \\
V_2 &= 45 & Y_2 &= 105 & R_2(A) &= 150 & T_2(A) &= 0 & CRES_2 &= 272 & CPOL_2 &= 393 \\
V_3 &= 40 & Y_3 &= 75 & R_3(A) &= 60 & T_3(A) &= 30 & CRES_3 &= 333 & CPOL_3 &= 503 \\
RFL_1 &= 5 & Z_1 &= 80 & R_1(B) &= 25 & T_1(B) &= 10 & R_1(C) &= 70 & T_1(C) &= 0 \\
RFL_2 &= 0 & Z_2 &= 105 & R_2(B) &= 50 & T_2(B) &= 0 & R_2(C) &= 100 & T_2(C) &= 0 \\
RFL_3 &= 0 & Z_3 &= 55 & R_3(B) &= 20 & T_3(B) &= 10 & R_3(C) &= 50 & T_3(C) &= 10
\end{align*} \]

Volumes (\( V_L \))

In all four problems we observe that the reservoir volume falls to 40 at the end of period 3, which is precisely the dead storage of the reservoir; conditions are such that all available water has to be used. Furthermore, the biggest claim on the reservoir contents is made during period 2. This is a consequence of the fact, that shortages have their largest loss coefficients in period 2 (see the respective objective functions).

Reservoir inlets (\( Y_L \))

In problem 1 the maximum inlets to the reservoir are used, that are possible by the upstream river flow and the minimum required downstream flow. Although we have the same physical situation in problem 2 here the inlets are somewhat lower and not all the available riverwater is used due to quality considerations. Because the initial reservoir salt concentration lies below the river salt concentration, salinity losses can be lowered by keeping the reservoir salinity on a lower level; this is however accompanied with a lower water delivery (see the \( R_L \)-columns) which not fully outweighs the quality amelioration. Reservoir inlets in problems 3 and 4 are further complicated by the circumstance that the remaining river flow (\( Z_L \)) must not only effectuate a minimum downstream flow but a flow into the polder as well, without producing too great losses to irrigation areas A and B.

Shortages and allocated water (\( T_L, R_L \))

By the problem formulation the shortages (\( T_L \)) are determined directly, from which the allocated quantities (\( R_L \)), which are the real controls, can be
calculated. Although the water demand for all areas is biggest during period 2, shortages are - in general - lowest just then. This reflects the fact, mentioned above, that losses due to shortage are bigger then (per unit of shortage) than in other periods. In the problems 3 and 4 the reservoir's task is simplified by the decoupling of area C; that it is easier now to maintain the dead storage in the reservoir is reflected by the very low figures for the shortages. Seepage represents an extra flow that can be used in problem 4 to compensate for the shortages for area C completely.

Salt concentrations \((CRES_t, CPOL_t)\)

Reservoir concentrations in problem 3 are lower than in problem 2 because of the lower inlet: the salt concentration of the river is higher than the initial salinity of the reservoir, so it is advantageous not to let too much river water enter the reservoir. In problem 4 still more water is diverted to the polder in order to reduce the polder reservoir salinity which is menaced by the high salinity of the seepage; this gives a further reduction of the areas salinity of the reservoir (but a bigger shortage in areas A and B).

In all cases the salinity is increasing in time. This is a result of low starting values for (polder) reservoir salinity, increasing salt concentration in the river and (in problem 4) seepage of ground water with a high salinity level.

Seepage and flushing \((SEEP_t, RFL_t)\)

From the obtained optimal settings can be seen, that in problem 4 no flushing occurs except during period 1. This comes about because the polder salinity is lower (or better: not much higher) than the combined saline effect of river inlet and seepage. Looking at period 3 e.g. we see a seepage of 15 with concentration 1000 and inlet of 35 with concentration 350; together they supply 50 with a concentration of 555 which is higher than the initial concentration of the polder reservoir, 393. This occurs also during period 1, but in that period the reduction from 410.7 (without flushing) to 400 (with flushing of 5) in the salinity of the incoming water outweighs the losses in area A and B that could have been avoided by diverting the same quantity of 5 to areas A and B. A side-effect of this delivery to A and B would have been a higher
and B. A side-effect of this delivery to A and B would have been a higher salinity of the reservoir content and ultimately of the water inlets to areas A and B.

Sensitivity

The solutions show that the river downstream requirements are decisive for the results; so the model is likely to be sensitive for this minimum requirement. Another important barrier is the minimum storage of the reservoir. Requirements, not important if the data are chosen as they were, are:
- the maximum reservoir storage;
- the salt concentration up to which no damage occurs.
The constant volume of the polder reservoir only plays a background rôle.

Uncertainties

In this report complete knowledge of all relevant data is assumed. In real world situations a stochastic element should be introduced into the computations. These stochastic elements include the supply as well as the demand side (agriculture).
4 Solution to problem 1

The formulation of the mathematical programming model and an overview of the solutions to problem 1 with LP and DP is given in Section 3.1. In this chapter attention is paid to:

(i) the piece-wise linearization of the objective function as required for the solution by the LP approach (Section 4.1);
(ii) the influence of linearization errors in the LP solution (Section 4.2);
(iii) a more detailed description of the DP solution (Section 4.3); and
(iv) the influence of discretization on the DP solution (Section 4.4).

4.1 Linearization of the objective function

According to the Equation (3.1) the objective function has the following quadratic terms: 0.3236 $T_1^2$, 0.4083 $T_2^2$, and 0.2722 $T_3^2$. They represent the composite losses due to water shortages in periods 1, 2 and 3 respectively. Being non-linear, a linearization of these terms is needed; an example of such a linearization is given in Figure 4.1; the horizontal axis is divided in three pieces; the points on each graph corresponding with the divisions are connected by straight line segments; the resulting graph represents a piecewise linear function which is an approximation of the original quadratic function. Annex C gives some general considerations about the linearization procedure.

The linearized model is then as follows:

**Objective function**

\[
\begin{align*}
\min & \quad 8.09 T_{11} + 24.27 T_{12} + 45.30 T_{13} + \\
& + 10.21 T_{21} + 30.62 T_{22} + 57.16 T_{23} + \\
& + 6.80 T_{31} + 20.42 T_{32} + 38.09 T_{33}
\end{align*}
\]  

(4.1)

Subject to the constraints

of Equations (3.2) - (3.5) plus the constraints related to the linearization:
Figure 4.1 Example of piecewise linearization of the convex non-linear terms in the objective function

\[
T_1 = T_{11} + T_{12} + T_{13} \quad \& \quad T_{11} < 25 \quad \& \quad T_{12} < 25 \\
T_2 = T_{21} + T_{22} + T_{23} \quad \& \quad T_{21} < 25 \quad \& \quad T_{22} < 25 \\
T_3 = T_{31} + T_{32} + T_{33} \quad \& \quad T_{31} < 25 \quad \& \quad T_{32} < 25
\]  \hspace{1cm} (4.2)

This model can be solved using a standard LP solution procedure. The convexity of the functions insures that the variables \(T_{11}, \ldots, T_{33}\) for each linearization are considered in the right sequence by the solution procedure (see Annex C). An example of a graphical application of LP to a still simpler problem is given in Annex D.

4.2 The influence of linearization errors in the LP solution

In Sections 3.1.2 and 3.1.4 errors were mentioned made due to the linearization. In this case the error corresponding to the optimal decision set can easily be evaluated. The solutions in the first and third period, \(T_1 = 50\) and \(T_3 = 50\) (Equation (3.6)) are on a cornerpoint of the piecewise linear repre-
sentation (Figure 4.1) and are thus on the original non-linear curve. The solution for the second period is in between two cornerpoints. The error can then be illustrated as follows (Figure 4.2). The linearized loss function does not coincide with the quadratic loss function. In solving problem 1 with LP we used the linearized, in DP the original version. The calculated values differ by 41 as shown in Figure 4.2, being the difference in the total calculated losses between the LP and DP solutions during period 2 (408 and 367 respectively).

Figure 4.2 Effect of the linearization on the accuracy of the loss function

In this case this is the only error due to linearization, but that is merely coincidental. The LP algorithm tries to minimize the given objective function. If the original objective function is linearized or otherwise altered, the slopes of the actual function on particular points does not coincide with the ones coming from the original objective function. Taking the example of Figure 4.2, we see that the linearized objective function in the neighbourhood of \( T_2 = 30 \) turns out to be steeper than the true objective function, i.e. more penalty is given for higher values of \( T_2 \) in that neighbourhood. Therefore the algorithm tends to seek values for \( T_2 \) more in the direction of 25 to avoid this penalty. In the neighbourhood of 45 the solution would have been driven in the direction of 50. In general (with convex objective functions) the solution with the aid of the piecewise linearized objective function will be directed towards the discretized points. These effects have influence on all
other decision variables (in this example $T_1$ and $T_3$).

One way to get a better approximation while retaining the piecewise linearization is to partition the horizontal axis into more pieces; to linearize again within these pieces and to solve the (new) LP problem again. As mentioned in Section 3.1.4 the computation time of LP strongly depends on the number of discretization intervals.

Another way to enhance the accuracy of optimum searching is to linearize again only in the neighbourhood of the solution of the LP problem already solved. Then one gets locally, in that neighbourhood, a better approximation to the non-linear objective function and a new LP problem can be solved with that newly linearized objective function. Hereby one must assure that the LP procedure avoids picking a totally different solution from the one before only by the fact that out of that neighbourhood the new objective function may differ significantly from the former. This method has strong resemblance with the one used for precisizing the DP solution by taking more values of the state variables in the neighborhood of the former solution.

4.3 Solution with DP

The optimization model presented in Section 3.1 can be solved in an alternative way by using Dynamic Programming as was shown in Section 3.1.31). The model can be formulated as follows using the DP-schematization in a forward direction (let $CL_t$ stand for the cumulative loss function in period $t$):

Stage 1

$$CL_1(V_1) = \min \left[ 0.3236 \ T_1^2 \right]$$

$$\begin{align*}
V_1 & = 80 + Y_1 - R_1 \\
40 & < V_1 < 160 \\
Y_1 & < 220 \\
T_1 & = 210 - R_1
\end{align*}$$

1) Reference is also made to TOW-H report R999-5 [4].
Stage 2

\[ CL_2(V_2) = \min \left[ 0.4083 \; T_2^2 + CL_1(V_1) \right] \]
\[ Y_2, T_2 \]
\[ \text{s.t. } V_2 = V_1 + Y_2 - R_2 \]
\[ 40 < V_2 < 160 \]
\[ Y_2 < 190 \]
\[ T_2 = 300 - R_2 \]

Stage 3

\[ CL_3(V_2) = \min \left[ 0.2722 \; T_3^2 + CL_2(V_2) \right] \]
\[ Y_3, T_3 \]
\[ \text{s.t. } V_3 = V_2 + Y_3 - R_3 \]
\[ 40 < V_3 < 160 \]
\[ Y_3 < 110 \]
\[ T_3 = 180 - R_3 \]

The calculations to derive the optimal allocation policy can best be organized in a table which enables us to keep an overview over the calculations for each stage, the connection between stages and the retracing of the optimal solution. Table 4.1 presents such schematization applied to problem 1. In this case two decision variables are involved in each stage t namely the inflow \( Y_t \) and outflow \( R_t \) of the reservoir. However, the determination of the optimal \( Y_t \) is obvious: it is clear that the best strategy is to make the inflow as large as possible which means in most of the cases that the total available water is diverted to the reservoir.

The calculations in the table are organized as follows:
- following the dynamic programming recursive equation, for each chosen value for \( V_t \) an optimal solution has to be determined for the current stage plus previous stages (the last represented by the previous cumulative function value \( CL_{t-1}(V_{t-1}) \));
- this optimal solution gives then the function value \( CL_t(V_t) \); in the table this is accomplished by specifying a value for \( V_t \) in column 2;
all the relevant values of the state variable at the beginning of the stage are entered in column 3;
- with $V_t$ and $V_{t-1}$ specified the decisions $R_t$ and $Y_t$ are easily determined: namely, the quantity $V_{t-1} - V_t$ is available from storage and a particular amount ($RIV_t$) from the river; let $D_t$ denote the total demand in period $t$; the determination of $R_t$ and $Y_t$ can then be summarized as follows:

\[
\begin{align*}
    & \text{if } D_t > V_{t-1} - V_t + RIV_t \\
    & \quad \text{then } R_t = V_{t-1} - V_t + RIV_t \\
    & \quad Y_t = RIV_t \\
    & \text{if } D_t < V_{t-1} - V_t + RIV_t \\
    & \quad \text{then } R_t = D_t \\
    & \quad Y_t = D_t - (V_{t-1} - V_t)
\end{align*}
\]

- from the decision $R_t$ the shortage $T_t$ is determined and from this the loss;
- the cumulative loss corresponding to the chosen reservoir states $V_t$ and $V_{t-1}$ is then calculated and placed in column 8.

Starting from a particular $V_t$ these calculations are performed for all relevant $V_{t-1}$. The cumulative function value corresponding to $V_t$ is then chosen as the minimum value among all cumulative values for chosen $V_{t-1}$ in column 8 (this corresponds to the minimization at the right hand side of the recursive equations).

The typical numerical solution procedure with DP can be observed in the table: the calculations proceed through the stages until the last stage and then a backward procedure takes place to determine the string of overall optimal decisions. The dashed line in Table 4.1 illustrates this procedure.

Note that in the third stage only $V_3 = 40$ is considered to save on calculations because it is logical that in the optimization, if shortages occur, all the water in the reservoir will be used and no water will be left in the reservoir after the third interval.
<table>
<thead>
<tr>
<th>( t )</th>
<th>( V_t )</th>
<th>( V_{t-1} )</th>
<th>( R_t )</th>
<th>( Y_t )</th>
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*Line retracing the optimal states*

- **658.2** optimal cumulative loss values corresponding to the optimal states
- **110** optimal decisions in each stage
- **60** optimal states are underlined

**Table 4.1 Dynamic Programming solution to problem 1**
4.4 The influence of discretization on the DP solution

One important difference (see also Section 3.1.4) of the DP solution method from LP is the fact that calculations are made at discrete points (values of the state variables). Only at these points information is known about losses (see Table 4.1). The values of the state variables associated with the minimum loss are searched for only amongst these values for which the calculations are made. These values virtually never will coincide with the true optimum. The found values of the state variables give mostly a good idea of the region where the values associated with the true minimum loss can be found, but data for additional values of the state variables have to be included into the DP table. These additional values can be chosen in the neighbourhood of the best solution thusfar. In problem 1 we found a solution with $V_1 = 140$, $V_2 = 60$ and $V_3 = 40$; had we used other values of $V_L$ in the table we should have got another solution. We can now repeat calculations for example with (see also Chapter 5):

<table>
<thead>
<tr>
<th>$V_1$</th>
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<th>$V_3$</th>
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<tr>
<td>150</td>
<td>70</td>
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</tbody>
</table>

Table 4.2 Discretized values for the second computation round of the DP solution to problem 1

The best value we should have got then for the loss function corresponds to a new approximation to the optimal solution of the original problem; in this case we get the same optimum but $V_1 = 138$, $V_2 = 65$, $V_3 = 40$ could also have been the next, better, approximation to the optimal solution. To obtain a high accuracy we have to do large amounts of work by several iterations needed. On the other hand, by all schematizations made, accuracy as a goal makes no
sense, because the modelling errors are (much) greater. In Section 6.2 an example is presented that illustrates the iterations needed to enhance the accuracy.

Another problem must be pointed out here. Due to the choice of the values for which computations are to be performed, or in other words by the discretization, the true minimum of the loss function can lie far away from the calculation "grid" points and can therefore be overlooked. Further rounds may result in relative minima instead of the true minimum. This problem occurs especially when only few intervals are chosen for the discretization. In the present problems one can show that relative minima do not exist.
5 Solution to problem 2

The formulation of the mathematical programming model and an overview of the solutions to problem 2 with DP are given in Section 3.2. In the following the recursive equations are formulated and the DP solution is considered into more detail.

The optimization model can be represented by the DP recursive equations as follows. $CL_t$ represents again the cumulative loss function but in this case it is a function of the volume of water stored in the reservoir ($V_t$) and the salt concentration of the water in the reservoir ($CRES_t$). The DP schematization is effectuated in the forward direction.

Stage 1

$$\begin{align*}
CL_1(V_1,CRES_1) &= \min_{Y_1,R_1} \left[ 0.3236 \ T_1^2 + 18 \cdot CLOSS_1 \right] \\
& \text{s.t.} \quad V_1 = 80 + Y_1 - R_1 \\
&\quad 40 < V_1 < 160 \\
&\quad Y_1 < 220 \\
&\quad T_1 = 210 - R_1 \\
&\quad V_1 \cdot CRES_1 = 80 \cdot 200 + Y_1 \cdot 250 - R_1 \cdot CAV_1 \\
&\quad CAV_1 = \frac{1}{4} \cdot CRES_1 + \frac{1}{4} \cdot 200 \\
&\quad \text{CNOT}_1 < 200
\end{align*}$$

Stage 2

$$\begin{align*}
CL_2(V_2,CRES_2) &= \min_{Y_2,R_2} \left[ 0.4083 \ T_2^2 + 18 \cdot CLOSS_2 + CL_1(V_1,CRES_1) \right] \\
& \text{s.t.} \quad V_2 = V_1 + Y_2 - R_2 \\
&\quad 40 < V_2 < 160 \\
&\quad Y_2 < 190 \\
&\quad T_2 = 300 - R_2 \\
&\quad V_2 \cdot CRES_2 = V_1 \cdot CRES_1 + Y_2 \cdot 300 - R_2 \cdot CAV_2 \\
&\quad CAV_2 = \frac{1}{4} \cdot CRES_1 + \frac{1}{4} \cdot CRES_2 \\
&\quad \text{CNOT}_2 < 200
\end{align*}$$
Stage 3

\[
CL_3(V_3, \text{CRES}_3) = \min [0.2722 T_3^2 + 18 \cdot \text{CLOSS}_3 + CL_2(V_2, \text{CRES}_2)] \\
Y_3, R_3
\]

s.t. \[
V_3 = V_2 + Y_3 - R_3 \\
40 < V_1 < 160 \\
Y_3 < 110 \\
T_3 = 180 - R_3 \\
V_3 \cdot \text{CRES}_3 = V_2 \cdot \text{CRES}_2 + Y_3 \cdot 350 - R_3 \cdot \text{CAV}_3 \\
\text{CAV}_3 = \frac{1}{3} \cdot \text{CRES}_2 + \frac{1}{3} \cdot \text{CRES}_3 \\
\text{CAV}_3 = \text{CNOT}_3 + \text{CLOSS}_3 \\
\text{CNOT}_3 < 200
\]

The calculations can be organized similarly to the calculations for problem 1 presented in Table 4.1. A basic difference, however, lies in the fact that the number of state variables has doubled. The effect of this expansion is shown as follows.

In problem 1, for a particular \(V_t\), the right hand side of the recursive equation has been evaluated for relevant \(V_{t-1}\). For problem 2 with two state variables this becomes: for a particular combination of \(V_t\) and \(\text{CRES}_t\), the right hand side of the recursive equation is evaluated for all relevant combinations of \(V_{t-1}\) and \(\text{CRES}_{t-1}\). Thus several columns have to be added to the calculation table. To determine then the whole function \(CL_t(V_t, \text{CRES}_t)\) all relevant combinations of \(V_t\) and \(\text{CRES}_t\) have to be considered. In each line of the table the values for \(V_t\), \(\text{CRES}_t\), \(V_{t-1}\) and \(\text{CRES}_{t-1}\) are specified. The decisions (uniquely determined if the state variables are fixed) are then calculated, using (for stage 2):

\[
Y_2 = \frac{V_2 \cdot \text{CRES}_2 - V_1 \cdot \text{CRES}_1 + R_2 \cdot \text{CAV}_2}{300} \quad \text{(from the salt balance equation)}
\]

with \(\text{CAV}_2 = \frac{1}{3} \cdot \text{CRES}_1 + \frac{1}{3} \cdot \text{CRES}_2\); then

\[
R_2 = V_1 - V_2 + Y_2 \quad \text{(from the volume balance equation)}
\]
and
\[
T_2 = 300 - R_2 \quad (R_2 < 300) \\
= 0 \quad (R_2 > 300) \\
CLOSS_2 = CAV_2 - 200 \quad (CAV_2 > 200) \\
= 0 \quad (CAV_2 < 200)
\]

and the losses corresponding to these allocations can be calculated as:
shortage loss = 0.4083 \cdot T_2^2 \\
quality loss = 18 \cdot CLOSS_2

In the same manner as in Table 4.1 the cumulative loss function corresponding to \(V_{t-1}\) and \(CRES_{t-1}\) is added to the loss in stage \(t\). The function value \(CL_t(V_t, CRES_t)\) for a particular \(V_t\) and \(CRES_t\) is then chosen as the minimum cumulative loss resulting from all the considered combinations of \(V_{t-1}\) and \(CRES_{t-1}\). The table contains thus all columns of the table from problem 1 but two extra columns are added which contain \(CRES_{t-1}\) and \(CRES_t\).

The computations for problem 2 were carried out with discretization intervals which change for each stage in order to improve accuracy without increasing too much the computation time. Furthermore the discretization intervals indicated below were applied after a first optimization was performed using larger intervals. This optimization showed the approximate location of the optimum. Knowing this, a discretization with smaller intervals (considering the same number of intervals) could be constructed around the approximate location. This procedure drastically improves the accuracy with only double computation time (only some input data to the computer program have to be changed for this second run). See also Section 4.4.

Alternatively a larger number of smaller intervals could be considered to obtain a better accuracy; however, this is a less attractive alternative because computation time increases exponentially with the number of intervals. E.g., if in a particular stage 5 values are considered for each of the 4 variables then \(5^4\) lines (with the accompanying calculations) have to be considered in the DP calculation table, but if instead 10 values are considered for each variable then \(10^4\) lines have to be considered, which means a \(2^4 = 16\) fold computation time.
The discretized values for $V_1$, $CRES_1$, $V_2$, $CRES_2$, $V_3$ and $CRES_3$ in the second computation round are the following:

<table>
<thead>
<tr>
<th>$V_1$</th>
<th>$CRES_1$</th>
<th>$V_2$</th>
<th>$CRES_2$</th>
<th>$V_3$</th>
<th>$CRES_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>246</td>
<td>40</td>
<td>292</td>
<td>40</td>
<td>334</td>
</tr>
<tr>
<td>60</td>
<td>246½</td>
<td>60</td>
<td>293</td>
<td>60</td>
<td>336</td>
</tr>
<tr>
<td>80</td>
<td>247</td>
<td>80</td>
<td>294</td>
<td>80</td>
<td>338</td>
</tr>
<tr>
<td>100</td>
<td>247½</td>
<td>100</td>
<td>295</td>
<td>100</td>
<td>340</td>
</tr>
<tr>
<td>120</td>
<td>248</td>
<td>120</td>
<td>296</td>
<td>120</td>
<td>342</td>
</tr>
<tr>
<td>140</td>
<td>248½</td>
<td>140</td>
<td>297</td>
<td>140</td>
<td>344</td>
</tr>
<tr>
<td>160</td>
<td>160</td>
<td>298</td>
<td>160</td>
<td></td>
<td>346</td>
</tr>
</tbody>
</table>

Table 5.1 Discretized values for the second computation round of the DP solution to problem 2.

The intervals for the concentrations are quite small, because it takes very large flows to make a considerable change in the concentration by mixing.
6 Solution to problem 3

The formulation of the mathematical programming model and an overview of the solutions to problem 3 with a combination of DP and Lagrange Multiplier (LM) optimization are given in Section 3.3. In the following the problem is formulated as a multistage decision problem and the optimization within one stage with LM is explained. It is important to realize that the solution to problem 3 considers two levels of optimization:

(i) within a particular stage or time period (with LM optimization); and
(ii) over all stages (performed by DP).

6.1 Formulation as a multistage decision process

Following the setup from Section 3.3.2 we split the overall optimization process in sub-optimizations linked by salt concentrations and reservoir storage. The formulation is as follows:

stage 1

Objective function

\[
CL_1(V_1, CRES_1) = \min \left[ 0.5 \ T_1^2(A) + 1.6 \ T_1^2(B) + 0.75 \ T_1^2(C) + 12 \LOSS_1 \right]
\]

Subject to the constraints

- \( V_1 = 80 + Y_1 - R_1(A) - R_1(B) \) (reservoir content continuity equation) (6.2)
- \( 40 \leq V_1 \leq 160 \) (dead and maximum storage)
- \( Z_1 = 240 - Y_1 \) & \( Z_1 = R(C) \geq 20 \) (remaining flow; downstream requirement) (6.3)
- \( T_1(A) = 105 - R_1(A) \) (definition of shortages per irrigation area)
- \( T_1(B) = 35 - R_1(B) \)
- \( T_1(C) = 70 - R_1(C) \)

- \( V_1 \cdot CRES_1 = 80 \cdot 200 + Y_1 \cdot 250 - (R_1(A) + R_1(B)) \cdot CAV_1 \) (reservoir salt continuity equation) (6.4)

- \( CAV_1 = \frac{1}{2} \cdot CRES_1 + \frac{1}{2} \cdot 200 \) (average salt concentration)
- \( CAV_1 \cdot CNOT_1 + LOSS_1 \) & \( CNOT_1 < 200 \) (CNOT is not, LOSS is causing damage)
stage 2

Objective function

\[ CL_2(V_2, CRES_2) = \min \{ 0.6T_2(A) + 2.1T_2(B) + 0.9T_2(C) + 12CLoss_2 + CL_1(V_1, CRES_1) \} \]

\[ T_2(A), T_2(B), T_2(C), CLoss_2 \]

Subject to the constraints

- \[ V_2 = V_1 + Y_2 - R_2(A) - R_2(B) \] \hspace{1cm} (6.6)
- \[ 40 < V_2 < 160 \]
- \[ Z_2 = 210 - Y_2 \quad \& \quad Z_2 - R_2(C) > 20 \] \hspace{1cm} (6.7)

- \[ T_2(A) = 150 - R_2(A) \] \hspace{1cm} (6.8)
- \[ T_2(B) = 50 - R_2(B) \] \hspace{1cm} (6.9)
- \[ T_2(C) = 100 - R_2(C) \] \hspace{1cm} (6.10)

- \[ V_2 \cdot CRES_2 = V_1 \cdot CRES_1 + Y_2 \cdot 300 - (R_2(A) + R_2(B)) \cdot CAV_2 \] \hspace{1cm} (6.11)
- \[ CAV_2 = \frac{1}{4} \cdot CRES_2 + \frac{1}{4} \cdot CRES_1 \] \hspace{1cm} (6.12)
- \[ CAV_2 = CNOT_2 + CLoss_2 \quad \& \quad CNOT_2 < 200 \] \hspace{1cm} (6.13)

stage 3

Objective function

\[ CL_3(V_3, CRES_3) = \min \{ 0.4T_3(A) + 1.4T_3(B) + 0.6T_3(C) + 12CLoss_3 + CL_2(V_2, CRES_2) \} \]

\[ T_3(A), T_3(B), T_3(C), CLoss_3 \]

Subject to the constraints

- \[ V_3 = V_2 + V_3 - R_3(A) - R_3(B) \] \hspace{1cm} (6.15)
- \[ 40 < V_3 < 160 \]
- \[ Z_3 = 130 - Y_3 \quad \& \quad Z_3 - R_3(C) > 20 \]
\[ T_3(A) = 90 - R_3(A) \]
\[ T_3(B) = 30 - R_3(B) \]
\[ T_3(C) = 60 - R_3(C) \] (6.16)

\[ V_3 \cdot CRES_3 = V_2 \cdot CRES_2 + Y_3 \cdot 350 - (R_3(A) + R_3(B)) \cdot CAV_3 \]
\[ CAV_3 = \frac{1}{4} \cdot CRES_3 + \frac{1}{4} \cdot CRES_2 \]
\[ CAV_3 = CNOT_3 + CLOSS_3 \& CNOT_3 < 200 \] (6.17)

### 6.2 Optimization within a stage with LM

After specification of the values of the state variables, still an optimization within a stage must be performed. A choice was made to use LM (see Annex B). The following presents an example for stage 2 of the problem formulated in Section 6.1.

From the reservoir continuity equation (6.6) and the salt balance equation (6.11) \( Y_2 \) can be solved as

\[ Y_2 = \frac{V_1(CAV_2-CRES_1)}{300-CAV_2} + \frac{V_2(CRES_2-CAV_2)}{300-CAV_2} \] (6.18)

By the choice for \( V_1 \), \( CRES_1 \), \( V_2 \) and \( CRES_2 \) in the DP solution scheme for stage 2 therefore \( Y_2 \) is also determined (\( CAV_2 \) follows from (6.12)).

Equation (6.7) becomes

\[ 210 - Y_2 - R_2(C) > 20 \]

or using (6.10)

\[ 110 - Y_2 + T_2(C) > 20 \]

the reservoir continuity equation (6.6) gives, together with (6.8) and (6.9)

\[ T_2(A) + T_2(B) = -V_1 + V_2 - Y_2 + 200 \]

After these transformations the following optimization within stage 2 is left for a given set of \( V_1 \), \( CRES_1 \), \( V_2 \) and \( CRES_2 \).
Objective function

\[
\min \left[ 0.6 T_2^2(A) + 2.1 T_2^2(B) + 0.9 T_2^2(C) \right] \\
T_2(A), T_2(B), T_2(C)
\]  

Subject to the constraints

\[
90 - Y_2 + T_2(C) > 0 
\]  

\[
T_2(A) + T_2(B) = -V_1 + V_2 - Y_2 + 200 
\]  

These constraints do not adequately fix the variables \( T_2(A), T_2(B) \) and \( T_2(C) \). It can thus easily be seen that unlike the situation in problem 2 there are some degrees of freedom left for the water distribution between areas A, B and C (the quality situation remains fixed) after the specification of the state variables \( V_1 \), \( CRES_1 \), \( V_2 \) and \( CRES_2 \).

As was stated before the LM solution method was chosen to perform the remaining optimization step within a stage. Following the method in Annex B the Lagrangian function for problem (6.19) - (6.21) is as follows:

\[
L = 0.6 T_2^2(A) + 2.1 T_2^2(B) + 0.9 T_2^2(C) \\
+ \lambda_1 (90 - Y_2 + T_2(C) - S^2) \\
+ \lambda_2 (T_2(A) + T_2(B) + V_1 - V_2 + Y_2 - 200)
\]

\( S^2 \) : positive slack variable to make Equation (6.20) an equality constraint

\( \lambda_1, \lambda_2 \) : Lagrange Multipliers

Necessary conditions for an optimum are then

\[
\frac{\partial L}{\partial T_2(A)} = 1.2 T_2(A) + \lambda_2 = 0 
\]  

(6.22)
\[ \frac{\partial L}{\partial T_2(B)} = 4.2 \ T_2(B) + \lambda_2 = 0 \]  \hspace{1cm} (6.23)

\[ \frac{\partial L}{\partial T_2(C)} = 1.8 \ T_2(C) + \lambda_1 = 0 \]  \hspace{1cm} (6.24)

\[ \frac{\partial L}{\partial S} = -2S\lambda_1 = 0 \]  \hspace{1cm} (6.25)

\[ \frac{\partial L}{\partial \lambda_1} = 90 - Y_2 + T_2(C) - S^2 = 0 \]  \hspace{1cm} (6.26)

\[ \frac{\partial L}{\partial \lambda_2} = T_2(A) + T_2(B) + V_1 - V_2 + Y_2 - 200 = 0 \]  \hspace{1cm} (6.27)

We put (6.22) and (6.23) in (6.27) and get

\[ -\frac{\lambda_2}{4.2} - \frac{\lambda_2}{4.2} + V_1 - V_2 + Y_2 - 200 = 0 \]

\[ \lambda_2 = \frac{1.2 \times 4.2}{5.4} (V_1 - V_2 + Y_2 - 200) \]

and from this

\[ T_2(A) = -\frac{4.2}{5.4} (V_1 - V_2 + Y_2 - 200) \]  \hspace{1cm} (6.28)

\[ T_2(B) = -\frac{1.2}{5.4} (V_1 - V_2 + Y_2 - 200) \]  \hspace{1cm} (6.29)

Equation (6.25) offers two possibilities:

a) if \( S = 0 \) we have in (6.26):

\[ T_2(C) = Y_2 - 90 \]  \hspace{1cm} (6.30)

b) if \( \lambda_1 = 0 \) then from Equation (6.24)

\[ T_2(C) = 0 \]  \hspace{1cm} (6.31)

and from Equation (6.26)

\[ S^2 = 90 - Y_2 \]  \hspace{1cm} (6.32)
Situation a) corresponds with (6.20) as restricting factor, so the minimum downstream river flow is binding; as with b) the shortage of area C will be truly nullified. It is clear that solution b), if feasible, will always be preferred for an optimal solution.

As it turns out the solution to the optimization problem within a stage can easily be determined with equations (6.28) to (6.31). As in problem 2 two rounds of calculations were made, the first to determine an approximate solution and the second to determine a more accurate solution. In the calculations the following values were used for the state variables in the second round:

<table>
<thead>
<tr>
<th>$V_1$</th>
<th>CRES 1</th>
<th>$V_2$</th>
<th>CRES 2</th>
<th>$V_3$</th>
<th>CRES 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>238</td>
<td>40</td>
<td>270</td>
<td>40</td>
<td>306</td>
</tr>
<tr>
<td>60</td>
<td>240</td>
<td>60</td>
<td>274</td>
<td>40</td>
<td>312</td>
</tr>
<tr>
<td>80</td>
<td>242</td>
<td>80</td>
<td>278</td>
<td>40</td>
<td>318</td>
</tr>
<tr>
<td>100</td>
<td>244</td>
<td>100</td>
<td>282</td>
<td>40</td>
<td>324</td>
</tr>
<tr>
<td>120</td>
<td>246</td>
<td>120</td>
<td>286</td>
<td>40</td>
<td>330</td>
</tr>
<tr>
<td>140</td>
<td>248</td>
<td>140</td>
<td>290</td>
<td>40</td>
<td>336</td>
</tr>
<tr>
<td>160</td>
<td>160</td>
<td>160</td>
<td>294</td>
<td>40</td>
<td>342</td>
</tr>
</tbody>
</table>

Table 6.1 Discretized values for the second computation round of the DP solution to problem 3

As an illustration of the effect of discretization two further computation rounds were made. The following optimal solutions were reached:

**second**

- $V_1 = 140$
- $V_2 = 60$
- $V_3 = 40$
- CRES 1 = 244
- CRES 2 = 282
- CRES 3 = 336
- damage = 5820.4

**third**

- $V_1 = 140$
- $V_2 = 64$
- $V_3 = 40$
- CRES 1 = 243
- CRES 2 = 281
- CRES 3 = 335$
- damage = 5775.3

**fourth**

- $V_1 = 139$
- $V_2 = 64$
- $V_3 = 40$
- CRES 1 = 243.4
- CRES 2 = 281$
- CRES 3 = 334$
- or 335
- damage = 5766.2
It is clear that the third iteration gives a reasonable improvement, while the fourth does not make much sense. After the fourth round $V_1$ and $V_2$ have an accuracy of $1 \, \text{m}^3/\text{s}$, $CRES_1$ of $0.2 \, \text{mg/l}$, $CRES_2$ and $CRES_3$ of $\frac{1}{4} \, \text{mg/l}$, while $V_3$ is fixed. The deviation from the true optimal setting now seems to give less than 1 unit extra loss, so that further iterations are clearly redundant.
7 Solution to problem 4

The formulation of the mathematical programming model and an overview of the solutions to problem 4 with the use of Non-linear Programming are given in Section 3.4. In Section 3.4.2 a solution is formulated for the linearization of the non-linearities in the salt continuity equation by using "shifting curves". This chapter elaborates on the principle of this linearization procedure and formulates a Mixed Integer Linear Programming model.

7.1 Introduction

The nature of problem 4 corresponds to the quantity/quality problem, with multiple dimensions and decisions, which was considered in the report R 999-4 [3]. In R 999-4 a preliminary suggestion was made towards a solution, making use of special "value functions" attached to the state variables. These would enable to make marginality considerations between the intervals (solutions within the intervals are generated e.g. by LP) and to arrive iteratively at an overall optimal solution over the stages or intervals. The iterative procedure was considered because of the computational problem caused by the use of DP in case of the large dimensionality of the optimization problem.

This problem however could be solved with the same optimization in two levels as used for the solution of problem 3. The increased dimension of the problem would drastically increase the required computation time. A comparison of the solution effort with that of problem 3 will be shown in the following.

Computational effort for problem 3:

The amount of computation can be related to the number of "lines" considered in the dynamic programming table (see Table 4.1). The number of such lines in the solution to problem 3 is the following (for the number of discretization intervals specified for each of the variables see Table 6.1):

\[
\begin{array}{cccccc}
V_1 & CRES_1 & V_1 & CRES_1 & V_2 & CRES_2 & V_2 & CRES_2 & V_3 & CRES_3 \\
7 \times 6 + 7 \times 6 \times 7 \times 7 + 7 \times 7 \times 7 \times 8 &= 4844 \\
stage 1 & stage 2 & stage 3
\end{array}
\]
The total computation time for problem 3 (IBM 370) was 10 sec.

computational effort for the DP approach to problem 4

Considering the same number of intervals for the state variables as in problem 3 and for the new state variable CPOL, the same number of intervals as for the salt concentration in the reservoir, namely

\[ \text{CPOL}_1 = 6 \text{ intervals} \]
\[ \text{CPOL}_2 = 7 \text{ intervals} \]
\[ \text{CPOL}_3 = 8 \text{ intervals} \]

then the following number of "lines" have to be calculated:

\[ V_1 \text{ CRES}_1 \text{ CPOL}_1 V_2 \text{ CRES}_2 V_1 \text{ CRES}_1 \text{ CPOL}_1 \text{ CPOL}_2 V_2 \text{ CRES}_2 V_3 \text{ CRES}_3 \text{ CPOL}_2 \text{ CPOL}_3 \]

\[ 7 \times 6 \times 6 + 7 \times 7 \times 7 \times 6 \times 6 \times 7 + 7 \times 7 \times 7 \times 8 \times 7 \times 8 = \]

\[ \text{stage 1} \]
\[ \text{stage 2} \]
\[ \text{stage 3} \]
\[ 240352 \]

If the computational efforts to calculate a particular "line" for problems 3 and 4 are considered equal (the effort will actually be higher for problem 4 because of the larger number of decision variables) then the following (optimistic) computation time can be expected (IBM 370):

\[ \frac{240352}{4844} \times 10 \text{ sec} = 496 \text{ sec} \]

One additional state variable has increased the computational effort with a factor of order 50.

Some experiments were carried out using these above mentioned value functions but they showed some problems with convergence (see § 331 [5]). Instead of elaborating on the conditions and possible ways for convergence, in this chapter another approach is taken which avoids the above mentioned convergence problem and also is considered to be computationally more efficient.

The method consists again of an iterative solution, however in each iteration the total number of intervals is considered together. In each iteration a linearized problem is solved by Linear Programming.
Because of the explicit enumeration of the salt concentration at the beginning and end of each interval in the approach using DP, the salt concentration could be modelled using the salt balance equation (see Annex A). In the case where different time periods are considered in one optimization step, some form of decoupling between the periods is necessary in order to be able to linearize the optimization problem. This is possible using the first approach to the modelling of salt illustrated in Annex A. The particular linearization that is carried out for this case is illustrated in the following sections.

7.2 Continuity between intervals; shifting curves

In Section 3.4.2 we signalled the problem of maintaining the connection between the different intervals while linearizing the non-linear constraints (3.33) and (3.36). The way in which we achieve continuity is illustrated with the use of the three diagrams below (Figure 7.1) corresponding to the three time intervals. The reservoir salt concentration, at the end of the time period, is set out against the flow into the reservoir. Following the salt balance equations (3.33) the reservoir salt concentration depends on the reservoir volume and concentration at the beginning of the interval and the flow into the reservoir during the interval. The shapes of the curves shown in the diagrams are determined for particular values of the reservoir concentration ($C_{RES}^*$) and volume ($V^*$) at the beginning of the interval (represented by the full line).

The curves can be linearized and put in the model, however the constant assumed reservoir volume and concentration are variables which form part of the decision variable set of the optimization model. Those linearizations can however be used in consecutive optimizations where after each calculation the updated values are used to reconstruct the curves. Important in this process, in order to get reasonable results is, however, that the continuity relationship for salt and volume are respected in the model. For the reservoir volumes the continuity is explicitly guaranteed by the reservoir balance equation. This is not directly the case for the salt concentration in the reservoir. How continuity can be maintained is shown in the three diagrams. The considerations are not relevant for interval 1 because the initial values are given for this interval.
Figure 7.1  Representation of the salt continuity equations by means of shifting curves
The salt continuity equation for interval 2 is

\[ \frac{Y_2}{V_1} - \frac{Y^*}{V_1} = \frac{CRES_2}{300} + (CRES_1^* - 300) e^{-V_1} \]  

(7.1)

This curve starts with the concentration \( CRES_2 \) set equal to a value \( CRES_1^* \) for \( Y_2 = 0 \) and asymptotically approaches 300.

Chosen values for \( CRES_1^* \) (\( CRES_1^* \)) and \( V_1^* \) (\( V_1^* \)) determine the starting point and the curvature towards the 300 asymptote. If, however, the curve is completely fixed like this the influence of a changing \( CRES_1 \) resulting from considerations in interval 1 is not in any way represented in the following interval, so in fact no trade-offs in salinity can be made in this way between the intervals, though this is the aim of the optimization model.

To make such trade-off possible the following mechanism is proposed. Instead of fixing the starting point of the curve on the assumed value \( CRES_1^* \) this point is left variable by introducing the following equation:

\[ \frac{Y_2}{V_1} - \frac{Y^*}{V_1} = \frac{CRES_2}{300} + \Delta CRES_1 + (CRES_1^* - 300) e^{-V_1} \]  

(7.2)

The curvature is then still determined by the chosen value \( CRES_1^* \), however, this curve is allowed to move up and down over a distance \( \Delta CRES_1 \) as indicated in Figure 7.1.

In this method continuity is insured by the condition that for a particular interval the initial concentration in this interval is equal to the final concentration in the previous interval. The distribution of the concentration over the second interval follows then the curve determined by the values \( CRES_1^* \) and \( V_1^* \). These values are updated after each run. Note that in the initial iterations the concentration in the reservoir can go beyond the theoretical maximum of 300 (the inflow concentration). The method converges very quickly to a situation where the value \( CRES_1 \) determined in the present run becomes equal to the value \( CRES_1^* \) determined in the previous run.

How the method converges can be illustrated as follows (Figure 7.2).
Figure 7.2 Illustration of the convergence with the shifting curve method

Assume that in a particular previous iteration a value of $\text{CRES}_1^{[1]}$ is obtained. With this value curve A can be constructed. In the subsequent run where A is allowed to move up or down e.g. the position A' is reached as an optimal decision (given the previous constant values for $\text{CRES}_1$, $V_1$, etc. used in the curve) and a new value $\text{CRES}_1^{[2]}$ is determined. Using this value a new curve B can be constructed starting from $\text{CRES}_1^{[2]}$ towards the 300 asymptote. The curvature of B will be smaller or larger than the one of A as $\text{CRES}_1^{[2]}$ is larger or smaller than $\text{CRES}_1^{[1]}$. The whole procedure depends in fact on the adjustment of the curvature to the optimal values for $\text{CRES}_1$ and $V_1$.

For each value of $\text{CRES}_1$ and $V_1$ the corresponding curvature is determined in the subsequent step. If $\text{CRES}_1$ and $V_1$ move towards the final optimal solution then the curve moves automatically to its final "optimal" curvature.

Applying the above procedure, the reservoir salt balances for the three time intervals become:
\[ CRES_1 = 250 + (200 - 250) e^{-\frac{Y_1}{80}} \]

\[ CRES_2 = 300 + (CRES_1^* - 300) e^{-\frac{Y_2}{v_1}} \]

\[ CRES_3 = 350 + (CRES_2^* - 350) e^{-\frac{Y_3}{v_2}} \]

(7.3)

7.3 Linearization; Mixed Integer Programming

Reservoir

The salt balance equations (7.3) can be linearized using a piece-wise linear representation as shown in figure 7.3.

Figure 7.3 Linearization of the salt balance equation for the reservoir

This concave function has the wrong shape for a minimization problem: if \( Y_1 \) is partitioned in parts \( Y_{11} \), \( Y_{12} \) and \( Y_{13} \) as in the figure the slope of the linearized parts is smallest in the third interval: \( a_{13} < a_{12} < a_{11} \). This can be interpreted as \( Y_{13} \) being "cheaper" than \( Y_{12} \) and \( Y_{11} \). Without taking measures, the algorithm should fill \( Y_{13} \) before \( Y_{12} \) and \( Y_{11} \). Therefore, integer variables have been considered in the formulation of the linearized equations.

The method used here is Mixed Integer Programming. This method, developed in [7], is described in [3]. Its aim is to convert a problem containing a concave
function into a problem in which integers are introduced that secure the right treatment of that function despite its concaveness. Mixed Integer Programming is also apt for the conversion of other types of problems into a format suitable for application of LP. In the following $K$ represents a very large number introduced to essentially annul restrictions (e.g. if $I_{13} = 1$ the last restriction in period 1 becomes $0 < Y_{13} < K$ so that $Y_{13}$ is left free).

If $Y_1 = Y_{11} + Y_{12} + Y_{13}$ equation (3.33) for period 1 becomes

$$CRES_1 = 200 + a_{11} Y_{11} + a_{12} Y_{12} + a_{13} Y_{13}$$

(7.4)

with the additional integer equations

$$b_{11} (I_{12} + I_{13}) < Y_{11} < b_{11}$$

$$b_{12} I_{13} < Y_{12} < b_{12} (I_{12} + I_{13})$$

$$0 < Y_{13} < K I_{13}$$

$$I_{11} + I_{12} + I_{13} = 1$$

Similarly equations (3.3) for periods 2 and 3 become

$$CRES_2 = CRES_1 + a_{21} Y_{11} + a_{22} Y_{22} + a_{23} Y_{23}$$

(7.5)

$$b_{21} (I_{22} + I_{23}) < Y_{21} < b_{21}$$

$$b_{22} I_{23} < Y_{22} < b_{22} (I_{22} + I_{23})$$

$$0 < Y_{23} < K I_{23}$$

$$I_{21} + I_{22} + I_{23} = 1$$

and

$$CRES_3 = CRES_2 + a_{31} Y_{31} + a_{32} Y_{32} + a_{33} Y_{33}$$

(7.6)
\[ \begin{align*}
\frac{b_{31}}{b_{32}} \left( I_{32} + I_{33} \right) &< Y_{31} < \frac{b_{31}}{b_{32}} \left( I_{32} + I_{33} \right) \\
0 &< Y_{33} < k I_{33} \\
I_{32} + I_{32} + I_{33} & = 1
\end{align*} \]

Polder

The salt balance equations for the polder can be linearized in a way much similar to the linearization we used for the reservoir salt balances. The only difference is a second inflow to the polder because of saline seepage. The polder salt balance equation for interval 2 was the following (Equation (3.36)):

\[ CPOL_2 = \frac{300 \cdot YPOL_2 + 15 \cdot 1000}{YPOL_2 + 15} + (CPOL_1^* - \frac{300 \cdot YPOL_2 + 15 \cdot 1000}{YPOL_2 + 15}) e^{-\frac{YPOL_2 + 15}{40}} \] (7.7)

This relationship for constant \( CPOL_1^* \) is shown in figure 7.4 below.

![Figure 7.4 Linearization of the salt balance equation for the polder.](image)
The starting point of the curve (at YPOL$_2$ = 0) is now determined as follows:

Equation (7.7) for YPOL$_2$ = 0 gives

\[
\text{CPOL}_2 = 1000 + (\text{CPOL}_1 - 1000) e^{-\frac{15}{40}}
\]

\[
\approx 0.687 \times \text{CPOL}_1 + 313
\]

A linearization procedure can be followed as has been used for the reservoir, namely Eq. (7.7) can be replaced by the following linear equation:

\[
\text{CPOL}_2 = 0.687 \times \text{CPOL}_1 + 313 - c_{21} \times \text{YPOL}_{21} - c_{22} \times \text{YPOL}_{22} - c_{23} \times \text{YPOL}_{23} \tag{7.8}
\]

and conditions

\[
\text{YPOL}_{21} \leq d_{21} \\
\text{YPOL}_{22} \leq d_{22}
\]

This curve has the right, convex, shape (because of the saline seepage which in this case makes the polder always more saline than the incoming water) for a direct linearization without the use of integer variables.

Similarly the linearized equations for the other time periods 1 and 3 become then

\[
\text{CPOL}_1 = 485 \ (\approx 0.687 \times 250 + 313) - c_{11} \times \text{YPOL}_{11} - c_{12} \times \text{YPOL}_{12} - c_{13} \times \text{YPOL}_{13} \tag{7.9}
\]

\[
\text{YPOL}_{11} \leq d_{11} \\
\text{YPOL}_{12} \leq d_{12}
\]

\[
\text{CPOL}_3 = 0.687 \times \text{CPOL}_2 + 313 - c_{31} \times \text{YPOL}_{31} - c_{32} \times \text{YPOL}_{32} - c_{33} \times \text{YPOL}_{33} \tag{7.10}
\]

\[
\text{YPOL}_{31} \leq d_{31} \\
\text{YPOL}_{32} \leq d_{32}
\]

**Objective function**

The linearization of the objective function can be done by a piecewise linearization, identical to the one used in the LP solution to problem 1, see Figure 7.5.
Figure 7.5 Example linearization of $0.5 \, T_1^2(A)$

Here $0.5 \, T_1^2(A)$ is replaced by the linear expression

$$c_{11}(A) \cdot T_{11}(A) + c_{12}(A) \cdot T_{12}(A) + c_{13}(A) \cdot T_{13}(A) \quad (7.11)$$

and the associated additional constraints

$$T_1(A) = T_{11}(A) + T_{12}(A) + T_{13}(A)$$

$$T_{11}(A) \leq d_{11}(A)$$

$$T_{12}(A) \leq d_{12}(A)$$

7.4 Summary

The solution to problem 4 with the use of Non-linear Programming, as defined by Equations (3.28) to (3.38), is obtained by an iterative solution of the Mixed Integer Linear(ized) Programming problem presented below. To arrive at this linear optimization model, linearizations have been performed using particular values for $V_1$, $V_2$, $CRES_1$, $CRES_2$, $CPOL_1$, $CPOL_2$ (namely $V_1^*$, ..., $CPOL_2^*$). For the next iteration new linearizations are calculated using the values for
$V_1, \ldots, CPOL_2$ which are obtained in the current optimization round. This procedure is continued until two consecutive optimizations have converged on the same values for $V_1, \ldots, CPOL_2$.

Objective function

\[
\min \left[ c_{11}(A) \cdot T_{11}(A) + c_{12}(A) \cdot T_{12}(A) + \ldots + c_{33}(A) \cdot T_{33}(A) + \ldots + c_{33}(C) \cdot T_{33}(C) + 12 \text{LOSS}_1 + 12 \text{LOSS}_2 + 12 \text{LOSS}_3 + 6 \text{CPLOSS}_1 + 6 \text{CPLOSS}_2 + 6 \text{CPLOSS}_3 \right]
\]

Subject to the constraints

\[
\begin{align*}
V_1 & = 80 + Y_1 - R_1(A) - R_1(B) & 40 < V_1 < 160 & Z_1 = 240 - Y_1 & Z_1 - YPOL_1 > 20 & YPOL_1 + 15 = R_1(C) + RFL_1 \\
V_2 & = V_1 + Y_2 - R_2(A) - R_2(B) & 40 < V_2 < 160 & Z_2 = 210 - Y_2 & Z_2 - YPOL_2 > 20 & YPOL_2 + 15 = R_2(C) + RFL_2 \\
V_3 & = V_2 + Y_3 - R_3(A) - R_3(B) & 40 < V_3 < 160 & Z_3 = 130 - Y_3 & Z_3 - YPOL_3 > 20 & YPOL_3 + 15 = R_3(C) + RFL_3 \\
T_1(A) & = 105 - R_1(A) & T_1(B) = 35 - R_1(B) & T_1(C) = 70 - R_1(C) \\
T_2(A) & = 150 - R_2(A) & T_2(B) = 50 - R_2(B) & T_2(C) = 100 - R_2(C) \\
T_3(A) & = 90 - R_3(A) & T_3(B) = 30 - R_3(B) & T_3(C) = 60 - R_3(C)
\end{align*}
\]

\[
\begin{align*}
T_1(A) & = T_{11}(A) + T_{12}(A) + T_{13}(A) & T_1(A) \leq d_{11}(A) & T_{12}(A) \leq d_{12}(A) \\
\vdots & & \vdots & \vdots & \vdots \\
T_3(C) & = T_{31}(C) + T_{32}(C) + T_{33}(C) & T_3(C) \leq d_{31}(C) & T_{32}(C) \leq d_{32}(C)
\end{align*}
\]

reservoir salt balances

\[
CRES_t = CRES_{t-1} + a_t \cdot Y_{t-1} + a_t \cdot Y_t + a_t \cdot Y_{t+1} \quad (CRES_0 = 200)
\]

\[
Y_t = Y_{t-1} + Y_{t+1} + Y_t
\]

\[
b_t \cdot (T_{t2} + T_{t3}) < Y_t < b_t \cdot T_{t1}
\]

\[
b_t \cdot T_{t2} \cdot T_{t3} < Y_{t2} < b_t \cdot (T_{t2} + T_{t3})
\]

\[
0 \leq Y_{t3} \leq K \cdot T_{t3}
\]

\[
T_{t1} + T_{t2} + T_{t3} = 1 \quad (t = 1, 2, 3)
\]
CAV \_t = \frac{1}{4}CRES \_t + \frac{1}{4}CRES \_t\_1

CAV \_t = \text{CNOT} \_t + \text{CLOSS} \_t

\text{CNOT} \_t < 200

\text{polder salt balances}

\text{CPOL} \_t = 0.687 \text{CPOL} \_t\_1 + 313 - C\_t \_1 \cdot \text{YPOL} \_t \_1 - C\_t \_2 \cdot \text{YPOL} \_t \_2 - C\_t \_3 \cdot \text{YPOL} \_t \_3 \quad (\text{CPOL} \_0 = 250)

\text{YPOL} \_t \_1 = \text{YPOL} \_t \_1 + \text{YPOL} \_t \_2 + \text{YPOL} \_t \_3

\text{YPOL} \_t \_1 < d \_1 \quad \& \quad \text{YPOL} \_t \_2 < d \_2

\text{CPAV} \_t = \frac{1}{4}\text{CPOL} \_t + \frac{1}{4}\text{CPOL} \_t\_1

\text{CPAV} \_t = \text{CPOL} \_t + \text{CLOSS} \_t

\text{CPNOT} \_t < 200 \quad (t = 1, 2, 3)
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SALT CONCENTRATION IN A RESERVOIR DUE TO INFLOW AND RELEASE

A.1 One inflow and one release

Consider a reservoir with initial content $V_o$ (m$^3$/s) and salt concentration $C_o$ (mg/l). During a certain time period a constant inflow $Q_{in}$ (m$^3$/s) with concentration $C_{in}$ occurs into the reservoir and mixes completely with the reservoir content. A constant release $Q_{out}$ (m$^3$/s) takes place. The salt concentration of the outflow at a particular moment is then the salt concentration in the reservoir at that moment.

The salt concentration in the reservoir as a function of time assuming complete mixing can then be represented by the following equations:

$$\frac{d(VC)}{dt} = Q_{in} C_{in} - Q_{out} C$$

$$V = V_o + (Q_{in} - Q_{out})t$$

$V =$ volume at time $t$

$C =$ salt concentration at time $t$

Solution of these equations gives (in the case that $Q_{in} \neq Q_{out}$; the case that $Q_{in} = Q_{out}$ is given below):

$$C = C_{in} \left[ 1 - \left(1 + \frac{Q_{in}-Q_{out}}{V_o} \right) t \right] - \frac{Q_{in}}{Q_{in}-Q_{out}}$$

$$+ C_o \left(1 + \frac{Q_{in}-Q_{out}}{V_o} \right) t - \frac{Q_{in}}{Q_{in}-Q_{out}}$$

$$= C_{in} + (C_o-C_{in}) \left(1 + \frac{Q_{in}-Q_{out}}{V_o} \right) t$$

(A.1)

This equation is tiresome to handle; therefore two approximations to it are worked out below:
a) If \( Q_{\text{in}} = Q_{\text{out}} \) then equation (A.1) must be replaced by:

\[
C = C_{\text{in}} (1 - e^{-\frac{Q_{\text{in}}}{V o} t}) + C_o e^{-\frac{Q_{\text{in}}}{V o} t} = C_{\text{in}} + (C_o - C_{\text{in}}) e^{-\frac{Q_{\text{in}}}{V o} t} \tag{A.2}
\]

The concentration \( C_1 \) at the end of a period \( \Delta t \) is then

\[
C_1 = C_{\text{in}} (1 - e^{-\frac{Q_{\text{in}}}{V o} \Delta t}) + C_o e^{-\frac{Q_{\text{in}}}{V o} \Delta t} = C_{\text{in}} + (C_o - C_{\text{in}}) e^{-\frac{Q_{\text{in}}}{V o} \Delta t} \tag{A.3}
\]

If further \( Q_{\text{in}} \) is considered as the total inflow in period \( \Delta t \) then we get

\[
C_1 = C_{\text{in}} (1 - e^{-\frac{Q_{\text{in}}}{V o}}) + C_o e^{-\frac{Q_{\text{in}}}{V o}} \tag{A.4}
\]

The approximation in using Equation (A.4) instead of (A.1) when \( Q_{\text{in}} \neq Q_{\text{out}} \) is illustrated by the following sample comparisons:

consider \( V_o = 100 \)
\( C_o = 200 \)
\( C_{\text{in}} = 250 \)

then for:

<table>
<thead>
<tr>
<th>( Q_{\text{in}} )</th>
<th>( Q_{\text{out}} )</th>
<th>( C_1 ) exact</th>
<th>( C_1 ) approx</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>20</td>
<td>219.82</td>
<td>222.56</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>222.56</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>40</td>
<td>226.76</td>
<td></td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>227.16</td>
<td></td>
</tr>
<tr>
<td>160</td>
<td>80</td>
<td>239.14</td>
<td></td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>239.14</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>140</td>
<td>244.11</td>
<td></td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>244.11</td>
<td></td>
</tr>
</tbody>
</table>

Equation (A.4) can be considered as a good approximation to Equation (A.1);
b) Another approximation to the concentration at the end of period $\Delta t$ is to calculate the concentration with the following salt balance equation:

$$V_1 \cdot C_1 = V_o \cdot C_o + Q_{in} \cdot C_{in} - Q_{out} \cdot C_{\text{average}}$$  \hspace{1cm} (A.5)

with $C_{\text{average}} = \frac{1}{2} C_o + \frac{1}{2} C_1$.

This salt balance equation would be correct if the time distribution of the concentration would be linear. Using again the sample comparisons introduced above, the concentrations calculated by applying the approximate salt balance compare to the earlier values as follows:

<table>
<thead>
<tr>
<th>$Q_{in}$</th>
<th>$Q_{out}$</th>
<th>$C_1$</th>
<th>$\text{Equation (A.1)}$</th>
<th>$\text{Equation (A.4)}$</th>
<th>$\text{Equation (A.5)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>20</td>
<td>219.32</td>
<td>222.56</td>
<td>222.56</td>
<td>220.00</td>
</tr>
<tr>
<td>60</td>
<td>60</td>
<td>226.76</td>
<td>223.00</td>
<td>227.27</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>40</td>
<td>227.16</td>
<td>227.78</td>
<td>231.60</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>231.60</td>
<td>233.33</td>
<td>239.14</td>
<td></td>
</tr>
<tr>
<td>160</td>
<td>80</td>
<td>239.14</td>
<td>241.67</td>
<td>237.67</td>
<td></td>
</tr>
<tr>
<td>140</td>
<td>140</td>
<td>237.67</td>
<td>241.18</td>
<td>237.67</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>200</td>
<td>244.11</td>
<td>250.00</td>
<td>241.18</td>
<td></td>
</tr>
</tbody>
</table>

The approximation with Equation (A.5) is better because the $Q_{out}$ is taken into account.

To determine the damage to agricultural production, one is interested in the salt concentration during a particular period. This concentration is not a constant but its effect can very well be represented by an average concentration because of the supposed linearity between salt concentration and damage.

It is proposed to use the average between the concentration at the beginning and the end of a period as an indicator for damage to the crops in the period. Two alternative methods can then be used to represent quality considerations in the models:
a) using Equation (A.4)

\[ C_1 = C_{in}(1 - e^{-\frac{Q_{in}}{V_0}}) + C_o e^{-\frac{Q_{in}}{V_0}} \]

and an equation expressing the average concentration namely:

\[ C_{average} = \frac{1}{2} C_o + \frac{1}{2} C_1 \]

b) using the approximate salt balance

\[ V_1 C_1 = V_o C_o + Q_{in} C_{in} - Q_{out} C_{average} \]

with \( C_{average} = \frac{1}{2} C_o + \frac{1}{2} C_1 \).

Method a) is applied in the Non-linear Programming application in Chapter 7. Method b) is used in the Dynamic Programming application in Chapters 5 and 6.

A.2 Two inflows with different salt concentrations and one release

Consider a reservoir with initial content \( V_o \) and salt concentration \( C_o \). During a time period \( \Delta t \) two constant flows \( Q_{in}^1 \) and \( Q_{in}^2 \) with respective concentrations \( C_{in}^1 \) and \( C_{in}^2 \) enter the reservoir and mix completely with the reservoir content. A constant release \( Q_{out} \) takes place during period \( \Delta t \).

The salt concentration in function of time, assuming instant mixing, can be represented by the following equations

\[ \frac{d(CV)}{dt} = Q_{in}^1 C_{in}^1 + Q_{in}^2 C_{in}^2 - Q_{out} C_{out} \]

\[ V = V_o + (Q_{in}^1 + Q_{in}^2 - Q_{out})t \]

Solution of these equations gives
\[
\begin{align*}
C &= \frac{Q'_1 C'_{in} + Q''_{in} C''_{in}}{Q'_1 + Q''_{in}} [1 - (1 + \frac{Q'_{in} + Q''_{in} - Q_{out}}{V_0} t) - \frac{Q''_{in} + Q''_{in} - Q_{out}}{Q'_1 + Q''_{in} - Q_{out}} ] \\
&\quad + C_0 (1 + \frac{Q'_{in} + Q''_{in} - Q_{out}}{V_0} t)
\end{align*}
\]

The approximation for the concentration \( C_1 \) at the end of a period \( \Delta t \) is calculated in a similar way as in the previous section for \( Q'_1 + Q''_{in} = Q_{in} \):

\[
C_1 = \frac{Q'_1 C'_{in} + Q''_{in} C''_{in}}{Q'_1 + Q''_{in}} \left( 1 - \frac{Q'_{in} + Q''_{in}}{V_0} \right) + C_0 e^{-\frac{Q'_{in} + Q''_{in}}{V_0}} \]  \hspace{1cm} (A.6)

This equation is used in Chapter 7 for the polder reservoir.
APPLICATION OF LAGRANGE MULTIPLIERS TO AN OPTIMIZATION PROBLEM WITH
EQUALITY AND INEQUALITY CONSTRAINTS

B.1 Application to an optimization problem with equality constraints

Consider the problem

$$\min [f(X_1, X_2)]$$

subject to the equality constraints

$$g_1(X_1, X_2) = b_1$$
$$g_2(X_1, X_2) = b_2$$

A function $L$, called the Lagrangian, can be defined as follows

$$L (X_1, X_2, \lambda_1, \lambda_2) = f(X_1, X_2) + \lambda_1 [g_1(X_1, X_2) - b_1] + \lambda_2 [g_2(X_1, X_2) - b_2]$$

where $\lambda_1$ and $\lambda_2$ are Lagrange Multipliers.

It can then be shown that a necessary condition for an optimal solution to the above problem is indicated by

$$\frac{\delta L}{\delta X_1} = \frac{\delta L}{\delta X_2} = \frac{\delta L}{\delta \lambda_1} = \frac{\delta L}{\delta \lambda_2} = 0$$

If the functions $f$, $g_1$ and $g_2$ have the right shape such that no local optima exist, this condition is also sufficient.

For a more detailed explanation on the Lagrangian function and Lagrange Multipliers see e.g. Haimes [6].

To illustrate the use of Lagrange Multipliers an example is given. Consider the following problem (taken from Haimes [6]): find the radius $r$ and the height $h$ of a water reservoir with a shape of a closed cylinder and of a given volume $V_0$ having a minimum surface area.
Objective function

\[ \min [2\pi rh + 2\pi^2] \]

Subject to

\[ \pi r^2 h = V_o \]

Lagrangian

\[ L (r, h, \lambda) = 2\pi rh + 2\pi^2 + \lambda(\pi r^2 h - V_o) \]

Necessary conditions

\[ \frac{\delta L}{\delta r} = 2\pi(h+2r) + 2\pi r^2 h = 0 \]
\[ \frac{\delta L}{\delta h} = 2\pi r + \lambda\pi r^2 = 0 \]
\[ \frac{\delta L}{\delta \lambda} = \pi r^2 h - V_o = 0 \]

We get the following simultaneous equations

\[ h + 2r + \lambda rh = 0 \]
\[ 2r + \lambda r^2 = 0 \]
\[ \pi r^2 h - V_o = 0 \]

Solving these gives as optimal values for \( \lambda, r \) and \( h \)

\[ \lambda^* = -\frac{2}{r} \]
\[ r^* = \sqrt[3]{\frac{V_o}{2\pi}} \]
\[ h^* = \frac{3}{4} \frac{V_o}{\pi} \]

The Lagrange Multiplier \( \lambda \) has a specific meaning namely it is the value by which the total area can be reduced if the volume \( V_o \) is taken one unit less.
B.2 Application to an optimization problem with inequality constraints

A classical method for solving a problem with inequality constraints is by transforming those equations into equality constraints by introducing slack variables.

Consider the problem

\[ \min \{ f(x_1, x_2) \} \]

Subject to the constraints

\[ g_1(x_1, x_2) = b_1 \]
\[ g_2(x_1, x_2) > b_2 \]

The second constraint can be converted to an equality by introducing the real variable \( S \) satisfying

\[ S^2 = g_2(x_1, x_2) - b_2 \]

Because \( S \) is required to be real, \( S^2 \) will always be positive which is necessary to represent the constraint adequately.

The two constraints can now be written as

\[ g_1(x_1, x_2) = b_1 \]
\[ g_2(x_1, x_2) - S^2 = b_2 \]

The Lagrangian \( L \) becomes then

\[ L(x_1, x_2, S, \lambda_1, \lambda_2) = f(x_1, x_2) + \lambda_1[g_1(x_1, x_2) - b_1] + \lambda_2[g_2(x_1, x_2) - S^2 - b_2] \]

The necessary conditions are then

\[ \frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial g_1}{\partial x_1} + \lambda_2 \frac{\partial g_2}{\partial x_1} = 0 \]
\[
\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} + \lambda_1 \frac{\partial g_1}{\partial x_2} + \lambda_2 \frac{\partial g_2}{\partial x_2} = 0
\]
\[
\frac{\partial L}{\partial \lambda_1} = g_1 - b_1 = 0
\]
\[
\frac{\partial L}{\partial \lambda_2} = g_2 - S^2 - b_2 = 0
\]
\[
\frac{\partial L}{\partial S} = -2\lambda_2 S = 0
\]

The difference with the previous case (equality constraints) is located in the last equation. We look at two cases for \(-2\lambda_2 S = 0\)

a) \(S = 0\); then \(g_2 = b_2\)
   The second constraint is binding.

b) \(\lambda_2 = 0\)
   As \(S^2\) may be greater than zero the second constraint is not binding.

The equations resulting from the necessary conditions are solved for the two cases and the solution resulting in the smallest value for the objective function is chosen as the optimal solution.

An example further illustrates the above procedure:

\[
\min [f (x_1, x_2)] = x_1^2 + x_2^2 + 3x_1
\]

Subject to

\[
x_1 + x_2 = 30
\]
\[
x_1 + 2x_2 \geq 40
\]

Transformation of the inequality constraint

\[
x_1 + 2x_2 - S^2 = 40
\]

Lagrangian function

\[
L(x_1, x_2, \lambda_1, \lambda_2, S) = x_1^2 + x_2^2 + 3x_1 + \lambda_1(x_1+x_2-30) + \lambda_2(x_1+2x_2-S^2-40)
\]
\[ \frac{\partial L}{\partial x_1} = 2x_1 + 3 + \lambda_1 + \lambda_2 = 0 \]
\[ \frac{\partial L}{\partial x_2} = 2x_2 + \lambda_1 + 2\lambda_2 = 0 \]
\[ \frac{\partial L}{\partial \lambda_1} = x_1 + x_2 - 30 = 0 \]
\[ \frac{\partial L}{\partial \lambda_2} = x_1 + 2x_2 - s^2 - 40 = 0 \]
\[ \frac{\partial L}{\partial s} = -2\lambda_2 s = 0 \]

Solution of the set of simultaneous equations:

a) if \( s = 0 \)
then solution of the equations gives:

\[ x_1 = 20 \]
\[ x_2 = 10 \]

objective value = 560

b) if \( \lambda_2 = 0 \)
then solution of the equations gives:

\[ x_1 = \frac{57}{4} = 14.25 \]

objective value = 493.87

\[ x_2 = \frac{63}{4} = 15.75 \]

Thus optimal solution \( x_1 = 14.25 \)
\[ x_2 = 15.75 \] objective value = 493.87
LINEARIZATION OF NON-LINEAR FUNCTIONS; CONVEXITY

Linearization

As the mathematical theory of linear functions is simpler and much further developed than that of non-linear functions, the latter are often linearized in order to employ methods for linear functions that otherwise could not be used.

When linearizing a non-linear function, as in the case of the function \( f \) in figure C.1 below, one searches for a linear (or better: affine) function, in the figure a straight line, that deviates minimally from the non-linear function.

\[ Y = f(x) \]
\[ Y = g_1(x) \]
\[ Y = g_2(x) \]

\[ Y = f(x) \]
\[ Y = g_1(x) \]
\[ Y = g_2(x) \]

Figure C.1 Example of linearization of a non-linear function

It can be seen, that determining which straight line is best apt for use as linearization depends on the interval to be considered. In the interval from 0 to A a linear approximation \( Y = g_1(x) \) is chosen; it has the form \( Y = a + bx \) with \( b = (c-a)/(A-0) \). In the interval from B to C: \( Y = g_2(x) \) is chosen as linearization and in this interval a much better approximation results than with \( g_1(x) \). In general it can be observed that narrowing the interval in which one linearizes gives a smaller error in the corresponding intervals in nearly all practical cases.
Piecwise linearization

In case of a big interval, or, which amounts to the same, a highly non-linear function one can make use of piecewise linearization, where the interval at issue is partitioned in several subintervals; in each of these intervals a linearization is made with the restriction that head and tail must join. The procedure is illustrated in a Figure C.2 where the objective function for period 1 in problem 1 is shown.

Figure C.2 Linearization of the shortage loss function $0.3236T_1^2$

The horizontal axis is partitioned into four parts by the more or less arbitrarily chosen points 25, 50, and 90; within each interval the function is linearized by taking the function values, in this case 202.25 for $T_1 = 25$ and 809.00 for $T_1 = 50$, and joining them. This is clearly a simple but not the best way in doing this piecewise linearization, but the results for problem 1 show that the discrepancies between the true (non-linear) function and this piecewise linearized version are relatively small.
In Figure C.2 also the slopes are given: the linearized function can be defined as:

\[ g(T_1) = 8.09T_1 \quad (0 < T_1 < 25) \]
\[ g(T_1) = 202.25 + 24.27(T_1 - 25) = 8.09 \times 25 + 24.27(T_1 - 25) \quad (25 < T_1 < 50) \]
\[ g(T_1) = 809.00 + 45.30(T_1 - 50) = 8.09 \times 25 + 24.27 \times 25 + 45.30(T_1 - 50) \quad (50 < T_1 < 90) \]

It is not possible to linearize the whole part of the function beyond 50, because the function is unbounded. However, the use of the third expression for \( g \) can be extended to that part of the function. That is the reason that no restriction is laid on \( T_{13} \). Far beyond 90 the error grows rapidly, but knowledge of the problem at hand says, that the maximum possible shortage is 150 (the demand in period 2). From the solution it can be seen, that the third interval is not needed, because the maximum shortage turns out to be 50.

**Convexity**

The loss function of the last figure is an example of a convex function, i.e. a function where the line segment between two points on the graph lies always above it. In the same manner a function is called concave if such a line segment remains under the graph. The importance of convexity when minimizing (and of concavity when maximizing) lies in the fact that in mathematical programming procedures one is assured of the first part of the variable being filled first. This comes about because of the lower losses (or the higher benefits) per unit of shortage. We illustrate this with Figure C.2 for the first loss term of problem 1.

The variable \( T_1 \) has been divided into three parts: \( T_{11}, T_{12}, \text{ and } T_{13} \) with \( T_1 = T_{11} + T_{12} + T_{13} \) and \( T_{11} < 25 \) and \( T_{12} < 25 \). For a particular choice of \( T_1 \) an infinity of combinations of \( T_{11}, T_{12}, \text{ and } T_{13} \) exists, all together giving that value of \( T_1 \); e.g. if \( T_1 = 60 \) example combinations are: 10, 20, 30; 20, 25, 15; 0, 0, 60; and 25, 25, 10. Of these the last one was meant: \( T_{11} \) (and \( T_{12} \)) must be filled (to 25) before \( T_{12} \) (and \( T_{13} \)) comes up. This means that we must append extra conditions appertaining to \( T_{11}, T_{12}, \text{ and } T_{13} \).

By the optimization however these extra restrictions are automatically fulfilled in the case of convexity: the losses in the first part of the \( T_1 \)-range
(the part for $T_{11}$) are lower than those in the latter parts, so it is cheapest to have $T_{11}$ as big as possible. The same can be said about $T_{12}$. This comes all about by the monotone growing of the slope of the (linearized) functions, in other words by the convexity.

Problem 4 includes a reservoir salt balance function which is concave. Due to this concavity another method must be used to ensure the right sequence in filling the contributions to the running variable, in this case $Y_1$. The method used is Mixed Integer Programming (see Section 7.3 and [3]).
APPLICATION OF LP ON A SIMPLIFIED VERSION OF PROBLEM 1

The LP technique will be illustrated with the following simplified and slightly modified version of the linearized model for problem 1. Suppose that in an extremely dry year (or in a region where no river exists) the river flow is less than 20. Then no water intake is possible. The situation then represents an area, where water for irrigation comes only from a lake, that will not be refilled. Suppose that the initial contents of the lake was 160 (the maximum) and look solely at periods 1 and 2. The optimization problem becomes after linearizing (compare Section 4.1):

Objective function

\[
\min \left[ 8.09 \ T_{11} + 24.27 \ T_{12} + 45.30 \ T_{13} + 10.21 \ T_{21} + 30.62 \ T_{22} + 57.16 \ T_{23} \right] \tag{D-1}
\]

Subject to the constraints

\[
\begin{align*}
V_1 &= 160 - R_1 \quad (Y_1 = 0) \quad \text{reservoir content continuity equation for both time periods} \\
V_2 &= V_1 - R_2 \quad (Y_2 = 0) \\
40 &\leq V_1 \leq 160 \\
40 &\leq V_2 \leq 160 \\
T_1 &= 210 - R_1 \quad \text{definition of shortages} \quad \text{per period} \\
T_2 &= 300 - R_2 \\
T_{11} &+ T_{12} + T_{13} \leq 25 \\
T_{21} &+ T_{22} + T_{23} \leq 25 \\
0 &\leq T_{13} \\
0 &\leq T_{23} \\
T_{1} &= T_{11} + T_{12} + T_{13} \\
T_{2} &= T_{21} + T_{22} + T_{23}
\end{align*}
\]

The only variables that can be influenced are \( R_1 \) and \( R_2 \); but as the objective function is formulated in terms of \( T_1 \) and \( T_2 \) we shall express all constraints in terms of \( T_1 \) and \( T_2 \), which is possible because of the shortage definitions (Equations D-4) from which also results that \( T_{11}, T_{12}, T_{21} \) and \( T_{22} \) are having their maximum value of 25. Equations (D-5) give then:

\[
T_{13} = T_1 - 50 \quad \text{and} \quad T_{23} = T_2 - 50.
\]

Thus the objective function becomes:

\[
\min \left[ 8.09 \cdot 25 + 24.27 \cdot 25 + 45.30 \cdot (T_1 - 50) + 10.21 \cdot 25 + 30.62 \cdot 25 + 57.16 \cdot (T_2 - 50) \right]
\]
Combining Equations (D-4) and (D-2) one gets:
\[ V_1 = T_1 - 50 \] and \[ V_2 = T_1 - 50 + T_2 - 300. \]
This together with (D-3) results in the inequalities \( 90 < T_1 < 210 \) and \( 390 < T_1 + T_2 < 510 \). After some reshuffling the following formulation is obtained:

Objective function

\[
\min \left[ -3293.25 + 45.30 \ T_1 + 57.16 \ T_2 \right]
\]

Subject to the constraints

\[
\begin{align*}
90 &< T_1 < 210 \\
390 &< T_1 + T_2 < 510 \\
T_2 &< 300 \ (\text{from } R_2 = 0) \\
T_1 &> 0 \\
T_2 &> 0
\end{align*}
\]

Figure D-1 Graphical LP-solution.
In Figure D-1 the constraints are presented in graphical form. On the "permitted" side of the boundaries a part of the figure is shaded. As we see the restriction $T_1 + T_2 < 510$ is not restrictive because the reservoir content cannot grow. Also $T_1 > 0$ and $T_2 > 0$ are not restrictive. The shaded region is the part of the plane where solutions to the problem have to be searched; these are feasible combinations of $T_1$ and $T_2$.

Then the loss function $-3293.25 + 45.30 T_1 + 57.16 T_2$ for different values of the loss is drawn. A loss of 10000 is not feasible because the loss function has no point in common with the feasible region. Also 15000 is not possible; its loss function is nearer to the feasible region and parallel to the preceding one. The first loss function reaching the region is the one with $T_1 = 210$ and $T_2 = 390 - 210 = 180$. This represents the minimum loss of 16508.55 and gives us the optimum solution reached for.

It is seen that the optimal solution lies in a corner of the feasible region. This is no coincidence, since optima always lie on corners (or at least at borders) of the admitted regions. This is exactly the reason why LP is such an efficient algorithm: it only has to search amongst the corners. The number of corners does however depend strongly on the number of constraints. See for this point the discussion in Section 3.1.4.