A CONSERVATIVE MURD SCHEME ON MOVING DOMAINS. APPLICATION TO THREE-DIMENSIONAL FREE SURFACE FLOWS

A. Decoene∗, J.-F. Gerbeau†

∗INRIA Rocquencourt,
z.B.P. 105 78153 Le Chesnay Cedex France
e-mail: Astrid.Decoene@inria.fr
†INRIA Rocquencourt,
e-mail:Jean-Frederic.Gerbeau@inria.fr

Key words: linear advection problem, moving domains, MURD schemes, Geometric Conservation Laws, three-dimensional Free Surface Flows

Abstract. We introduce a monotonic and conservative numerical scheme for the resolution of the linear advection problem set on a moving domain. The scheme is based on an extension to moving domains of the Multidimensional Upwind Residual Distributive (MURD) approach. The properties ensured by the residual distribution schemes set on fixed domains are not altered by the domain movement, except for the conservation of the advected quantity. We introduce in this paper an additional condition which ensures the conservation properties of the MURD scheme on a moving domain. Several numerical tests have been performed, providing satisfying results in terms of conservation of the advected quantity.

1 INTRODUCTION

We introduce in this paper a numerical scheme for the resolution of the linear advection problem set on a moving domain, based on the Multidimensional Upwind Residual Distributive (MURD) approach – see for instance [1, 2, 10]. The schemes based on this approach present a number of attractive features – such as lower cross-diffusion due to multidimensional upwinding – but their main advantage is that they can be made monotonic by construction.

However, they have been developed for problems defined on fixed domains, and their extension to moving domains is not yet well-established. Now the domain movement can affect numerical schemes in a negative way. Indeed, it has been proven by several authors – see for instance [4, 5, 6] – that some schemes suffer from a loss of their accuracy, stability or conservation properties when applied to a problem set on a moving domain. In certain cases, the satisfaction of a particular condition – the so-called Geometric Conservation
Law (GCL) [4, 5, 6] – ensures the preservation of these properties. Therefore, particular care must be taken when applying a MURD scheme to a problem set on a moving domain.

We derive a generic form of MURD schemes for the linear advection problem with moving boundaries formulated in the ALE frame, to which we refer as ALE-MURD schemes. We then show that a particular constraint must be satisfied to ensure the accurate conservation of the advected quantity when the domain moves. This constraint is strongly related to the concept of the GCL. Then we describe how the conservation constraint can be satisfied in the particular framework of three-dimensional domains moving in the vertical direction only. This result has been applied in the framework of the three-dimensional free surface flow problem: we have extended the MURD schemes solving the advection of a scalar in the Telemac-3D system [7] to the use of a general type of ALE mappings. In particular, the conservation property of these schemes has been improved, reaching a very satisfactory level, as illustrated by numerical results at the end of this paper.

2 PROBLEM SETTINGS

Let $I = [0,T]$ be an open interval and, for each $t \in I$, let $\Omega_t \subset \mathbb{R}^d$ be a time dependent domain in $\mathbb{R}^d$ with a smooth enough boundary. Throughout we will denote the set $\{(x,t) \mid x \in \Omega_t, \ t \in I\}$ by $\Omega_t \times I$. We consider the non-conservative formulation of the linear advection equation of a scalar $u$ with divergence free velocity:

$$\frac{\partial u}{\partial t} + \beta \cdot \nabla u = f \quad \text{in} \quad \Omega_t \times I,$$

$$u = u_0 \quad \text{in} \quad \Omega_t \quad \text{at} \quad t = 0,$$

where $\beta$ is the advection velocity, assumed to be sufficiently regular, and $f$ is a source term. The system is closed by a suitable boundary condition: $u$ is prescribed at the inflow boundaries $\Gamma_{in,t}$ of the domain, i.e.

$$u(x,t) = u_{in}(x,t) \quad \forall \ x \in \Gamma_{in,t}, \ t \in I,$$

where $\Gamma_{in,t}$ is defined by $\Gamma_{in,t} = \{x \in \partial \Omega_t / \beta \cdot n < 0\}$.

The ALE formulation is adopted in order to deal with the moving boundaries. We define a fixed reference configuration $\hat{\Omega} \subset \mathbb{R}^d$ and a mapping $\hat{A}_t$ which at each time $t \in I$ associates to a point $\hat{x}$ in $\hat{\Omega}$ a point $x$ in $\Omega_t$:

$$\hat{A}_t : \hat{\Omega} \longrightarrow \Omega_t, \quad x(\hat{x},t) = \hat{A}_t(\hat{x}).$$

We assume that

$$\hat{A}_t \in \left(W^{1,\infty}(\hat{\Omega})\right)^d \quad \text{and} \quad \hat{A}_t^{-1} \in \left(W^{1,\infty}(\Omega_t)\right)^d.$$

The ALE mapping defines an instantaneous domain velocity $c(\hat{A}_t(\hat{x}),t) = \frac{\partial \hat{A}_t}{\partial t}(\hat{x})$. It has to be conforming to the evolution of the domain boundaries, that is $c \cdot n = \beta \cdot n$.
A. Decoene, J.-F. Gerbeau

at the physical boundaries. Let \( \psi : \Omega \times I \rightarrow \mathbb{R} \) be a function defined on the Eulerian frame. The corresponding function on the ALE frame is then \( \hat{\psi} : \hat{\Omega} \times I \rightarrow \mathbb{R} \), defined as

\[
\hat{\psi}(\hat{x}, t) = \psi(x, t) \quad \text{where} \quad x = \hat{A}_t(\hat{x}).
\]

We denote by \( \hat{J}_t = \left[ \frac{\partial \hat{A}_t}{\partial \hat{x}_j} \right] \) the Jacobian matrix of \( \hat{A}_t \), and by \( \hat{J}_t \) its determinant.

We can now write the non-conservative ALE formulation of problem (1):

\[
\frac{\partial u}{\partial t} \bigg|_{\hat{x}} + (\beta - c) \cdot \nabla u = f \quad \text{in} \quad \Omega \times I, \\
u = u_0 \quad \text{in} \quad \Omega_{t_0} \quad \text{at} \quad t = 0.
\]

The time interval \( I \) is divided in \( N_t \) time steps of equal length \( \Delta t \), and the problem is discretized in space using the finite element method. At each time \( t^n = n\Delta t \), let \( \hat{A}_{h,n} \) denote the discrete ALE mapping and let \( T_h^n \) be the mesh of the real domain \( \Omega_t \).

The solution \( u \) is approximated in a Lagrangian finite element space \( X^n_h \) whose degrees of freedom are situated on the vertices of the mesh. Throughout we will make the following abuse of notation : \( i \in K_0 \) means that node \( i \) belongs to the element \( K_0 \) of the mesh. In addition we denote by \( N_h \) the dimension of \( X^n_h \) – which is the same at each time \( t^n \) – and we introduce \( \{ \psi^n_i \}_{i=1, \ldots, N_h} \), a set of basis functions for this space.

Given the initial condition \( u_0 \) we consider the following explicit Euler time-advancing scheme:

for each \( n = 0, \ldots, N_t - 1 \) find \( u^{n+1}_h \in X^{n+1}_h \) such that

\[
\int_{\Omega^{n+1}_h} \left( u^{n+1}_h - u^n_h \circ A_{n+1,n} \right) \, dx + \Delta t \int_{\Omega^n_h} (\beta_h - c_h)^c \cdot \nabla (u^n_h \circ A_{c,n}) \, dx = 0,
\]

where \( \Omega^n_h \) and \( (\beta_h - c_h)^c \) denote respectively the approximations of the real domain and the ALE advection velocity at some arbitrary time \( t^c \) within \( t^n \) and \( t^{n+1} \), and where

\[
X_{h,0}(\Omega_{h,t}, t) = \{ \psi_h \in X^n_h \mid \psi = u^n_m \text{ in } \Gamma^n_m \}.
\]

Note that the discrete advection velocity is assumed to be divergence free in the sense of

\[
\int_{\Omega^n_h} \psi^c_i \, \text{div} \beta^c_h \, dx = 0 \quad \forall \ i = 1, \ldots, N_h, \quad \forall \ t^c \in I.
\]
3 A GENERIC FORM OF ALE-MURD SCHEMES FOR THE LINEAR
ADVECTION PROBLEM IN NON-CONSERVATIVE FORM.

3.1 Derivation of the scheme

The residual distribution approach consists in computing, for each element of the mesh,
the residual (or fluctuation) within each time step – i.e. the source term for the local in-
crement of the unknown. It is then optimally distributed over the vertices of the element
with weighting coefficients summing up to unity for consistency. These nodal local resid-
uals are finally assembled in order to retrieve a global increment at each node. There
are several ways to perform the optimal distribution of the residual inside an element. A
significant advantage of the approach lies in the fact that the different properties required
for the numerical scheme can easily be expressed. The distribution function is then chosen
such that the conditions imposed by these properties are satisfied. We refer to [1, 2, 10]
for more details on these techniques.

In the particular case studied here, the global increment \( \delta M^{n+1}_u \) of the advected
variable \( u_h \) is due to the different advective fluxes considered within \([t^n, t^{n+1}]\) as expressed by
(7). Equation (7) can be expressed locally as follows: for \( n = 0, \ldots, N_t - 1 \),

\[
\forall K_0 \in \hat{T}_h, \quad \delta M^{n+1}_{u,K_0} = - \Delta t \Phi^{n+1}_{K_0},
\]

where \( \Delta t \Phi^{n+1}_{K_0} \) is the local residual in element \( K_0 \in \hat{T}_h \) within time step \([t^n, t^{n+1}]\),

\[
\Delta t \Phi^{n+1}_{K_0} = \Delta t \sum_{i \in K_0} u^n_i \int_{K_c} \left( (\beta_h - c_h)^c \cdot \nabla \psi^c_i \right) dx,
\]

and \( \delta M^{n+1}_{u,K_0} \) is the local increment of \( u \) in \( K_0 \), that is

\[
\delta M^{n+1}_{u,K_0} = \sum_{i \in K_0} \left( u^{n+1}_{i,K_0} - u^n_{i,K_0} \right) \int_{\Omega^{n+1}} \psi^{n+1}_i dx.
\]

Note that each nodal increment \( \delta u^{n+1}_i = u^{n+1}_i - u^n_i \) is assumed to result from the sum of
the contributions of each element in the mesh \( \hat{T}_h \) containing the \( i \)-th node, that is:

\[
\delta u^{n+1}_i = \sum_{K_0 \in \hat{T}_h, i \in K_0} \delta u^{n+1}_{i,K_0}.
\]

For more details on the derivation of the scheme we refer to [3].

The residual distribution principle can now be applied, leading to a first definition
characterizing a more general class of ALE schemes, which we will refer to as ALE Resid-
ual Distributive (ALE-RD) schemes for the linear advection problem in non-conservative
form.
Definition 3.1 The ALE-Residual Distributive schemes for the linear advection problem in non-conservative form consist in distributing each local residual $\Delta t \Phi_{K_0}^{n+1}$, defined by (11), to the nodes of its corresponding element, contributing to the increment of the advected variable $u_h$ as follows:

$$\forall K_0 \in T_h, \forall i \in K_0, \quad \delta u_i^{n+1} = -\frac{\Delta t}{V_i^{n+1}} \beta_{i,K_0}^{n+1} \Phi_{K_0}^{n+1}, \quad (14)$$

where $V_i^n = \int_{\Omega_{h}^{n+1}} \psi_i^n \, d\mathbf{x}$. The distribution coefficients $\beta_{i,K_0}^{n+1}$ are chosen such that

$$\forall K_0 \in T_h, \sum_{i \in K_0} \beta_{i,K_0}^{n+1} = 1 \quad (15)$$

in order to ensure consistency. The increment $\delta u_i^{n+1}$ is retrieved by assembling contributions from all the elements using (13). Each nodal value of $u_h$ is thus updated as

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{V_i^{n+1}} \sum_{K_0 \in T_h, i \in K_0} \beta_{i,K_0}^{n+1} \Phi_{K_0}^{n+1} \quad \forall i = 1, \ldots, N_h. \quad (16)$$

Definition 3.2 A residual distributive scheme is multidimensionally upwind if the local residuals are distributed to downstream nodes only; that is if the distribution coefficients are chosen such that, for any mesh element $K_0$, if $i$ is an upstream node of $K_0$ then $\beta_{i,K_0} = 0$.

We call ALE-MURD scheme a multidimensionally upwind ALE-RD scheme.

Different schemes can now be designed on this basis by choosing a particular set of distribution coefficients – making sure that the consistency condition (15) as well as the multidimensional upwinding condition are satisfied. The optimal distribution can be chosen such that particular properties are satisfied by the scheme.

Referring to [10], four essential design criteria can be imposed on a residual distributive scheme, ensuring particular properties. Consistency guarantees to a residual distributive scheme raised on a fixed domain to be conservative. Multidimensional upwinding prevents the numerical solution from unphysical oscillations and minimizes the creation of cross-diffusion. Positivity ensures a maximum principle which prohibits the creation of new extrema. Finally, the linearity preservation criteria guarantees that the scheme preserves the exact steady state solution when this is a piecewise linear polynomial function on the considered triangulation of the domain. The reader may refer to [10] for a complete description of the criteria and the properties they guarantee to residual distributive schemes set on a fixed domain, as well as an extended list and description of the existing MURD
schemes.

Clearly, the properties ensured by the multidimensional upwinding, the positivity and the linearity preservation criteria are preserved on a moving domain. On the contrary, the conservation property is not automatically preserved, since the expression of the required discrete conservation changes when the domain moves. We will next formulate the additional constraint the numerical scheme must satisfy to be conservative.

3.2 Conservation property of the scheme

Consistency is only a necessary condition for a distributive scheme to be conservative when set on a moving domain. Indeed, further conditions must be satisfied to make sure that the advected quantity is really preserved.

According to the time and space discretization (7) of the linear advection problem, the required conservation property is:

\[
\int_{\Omega_{n+1}^h} u_{h}^{n+1} d\mathbf{x} - \int_{\Omega_{n}^h} u_{h}^{n} d\mathbf{x} = - \Delta t \int_{\Gamma_{\text{liq}}^c} (u_{h}^c \circ \mathcal{A}_{c,n}) (\beta - c_{h})^c \cdot n_h d\Gamma, \tag{17}
\]

where \(\Gamma_{\text{liq},t}\) denotes the fictitious liquid boundaries. The following proposition can be established.

**Proposition 3.1** If the local fluxes (11) are computed exactly and if the discrete advection velocity is divergence-free in the sense of (9), then the numerical scheme (7) is conservative in the sense of (17) if the following relation holds:

\[
\forall \ n = 1, \ldots, N_t, \quad \forall \ i = 1, \ldots, N_h,
\]

\[
\int_{\Omega_{n+1}^h} \psi_{h}^{n+1} d\mathbf{x} - \int_{\Omega_{n}^h} \psi_{h}^{n} d\mathbf{x} - \Delta t \int_{\Omega_{h}^c} \psi_{h}^c \ div c_{h}^c d\mathbf{x} = 0, \tag{18}
\]

where \(t^c\) is the time at which the advective fluxes are considered within each time step.

This conservation constraint only involves terms related to the mesh movement. Clearly, (18) is automatically satisfied in the case where the domain doesn’t move. In fact, this traduces a loss of the advected quantity arising from the domain movement when a particular condition is not satisfied. Therefore, it is strongly related to the Geometric Conservation Laws (GCL) (see [4, 5, 6]).

In the next section we describe the application of the ALE-MURD schemes proposed in this paper to a particular framework: the linear advection problems set on three-dimensional domains moving in the vertical direction only. Especially, we will show how the conservation constraint of these schemes can be satisfied.
3.3 Conservation in the framework of three-dimensional domains moving in the vertical direction only

In this section we consider a linear advection problem set on a three-dimensional domain moving only in the vertical direction. In this case, the whole problem can be formulated in the ALE approach using a mapping of the form

$$\hat{A}_t : \hat{\Omega} \rightarrow \Omega_t, \quad x(\hat{x}, t) = \hat{A}_t(\hat{x})$$

with $x = \hat{x}$, $y = \hat{y}$ and $z = Z(x, y, \hat{z}, t)$, \(\text{(19)}\)

where $Z(x, y, \hat{z}, t)$ is an arbitrary continuous and monotonic ($\partial Z/\partial \hat{z} \geq 0$) function such that the velocity of the domain $c$ satisfies

$$c \cdot n = \beta \cdot n \quad \text{on} \quad \Gamma_{imp,t} \quad \text{and} \quad c \cdot n = 0 \quad \text{on} \quad \Gamma_{liq,t},$$

where $\Gamma_{imp,t}$ denotes the impervious boundaries of the domain and $\Gamma_{liq,t}$ denotes its fictitious liquid boundaries – which only move in the vertical direction. Figure 1 shows an example of such a mapping. Note that $c = (0, 0, c)^T$.

![Figure 1: Example of a mapping describing a 3D domain moving in the vertical direction only.](image)

The problem is discretized in space and time as described previously. We point out that the mesh velocity $c_h$ is considered constant within each time step. The motion of the mesh can thus be described by:

$$x = \hat{A}_{h,n+1}(\hat{x}) = \hat{A}_{h,n}(\hat{x}) + \Delta t \, \hat{c}^{n,n+1}_h \quad \text{for} \quad \hat{x} \in \hat{\Omega},$$

\(\text{(20)}\)

for each $n = 1, \ldots, N_t - 1$.

In this particular framework it can be shown that the conservation constraint (18) is automatically verified. Indeed, we can state the following proposition.
Proposition 3.2 If the local fluxes (11) are computed exactly and the advection velocity is divergence-free in the sense of (9), and if the mesh moves in the vertical direction only and its velocity is considered constant within each time step, then the advection ALE-MURD scheme (7) is conservative in the sense of (17).

For the proof of these proposition we refer to [3]. From this proposition we can conclude that the ALE-MURD schemes introduced in this paper are conservative in the particular framework of the linear advection problem set on a three-dimensional domain moving in the vertical direction only, provided the advection velocity is divergence-free at least in the sense of (9) and the mesh velocity is considered constant within each time step.

4 Numerical illustration: application to 3D free surface flows.

In this section we aim to evaluate the conservation property of the ALE-MURD schemes introduced in this paper when used in the particular framework described in section 3.3. For this purpose, we make use of the PSI scheme implemented in the Telemac-3D system [7] for the linear advection of a scalar in the framework of three-dimensional free surface flows.

Note that the MURD schemes in the Telemac-3D system – see the work by Janin [8] – were initially implemented for the use of a particular and simple mapping called the classical sigma transformation (see for instance [3]). Now an important lack of conservation was revealed when using a different mapping. We have therefore updated these advection schemes taking profit of the ALE-MURD approach introduced in this paper. They are now compatible with any ALE mapping for a three-dimensional domain moving in the vertical direction only. In particular, they are conservative up to a very satisfactory level with any vertical discretization of the domain.

We illustrate this statement through a numerical test simulating the effect of a tracer source in the middle of a closed basin. We consider a square closed basin of side $L = 4000$ m, with a constant bottom at $z = -10$ m. At the initial time the water height is flat ($h = 10$ m) and no motion is taken. After one time step, water is discharged with an excess tracer value $T = 333$ from a point source located in the middle of the water column at the centre of the model. The coordinates of this source point are $x = 2000$ m, $y = 2000$ m and $z = -5$ m. The water discharge of the source is $Q = 20 \text{m}^3/\text{s}$. A constant horizontal viscosity coefficient of $\nu_h = 10^{-4}$ and no vertical diffusion are chosen for the fluid velocities. No diffusion at all is considered on the tracer, so that only the advection scheme is evaluated. The advection is performed using successively a characteristic method type scheme and the PSI ALE-MURD scheme. The simulations are performed with a time step length of 5 seconds for a time interval of 30 minutes. Note that the tracer considered is active, that means a state equation relates the fluid density to the tracer.

Figure 2 gives the evolution of the relative loss of tracer quantity during the simulation. It reveals that the characteristic scheme has a serious problem with conservation. On the
contrary, the conservation property of the MURD scheme is excellent: the relative loss of tracer quantity during the simulation is zero. The tracer profiles obtained with both

![Relative loss of the advected quantity using the characteristic method (solid line with crosses) and the PSI ALE-MURD scheme (solid line with plane circles).](image)

**Figure 2:** Relative loss of the advected quantity using the characteristic method (solid line with crosses) and the PSI ALE-MURD scheme (solid line with plane circles).

![Tracer profiles obtained at t = 1800 s using (a) the characteristic method and (b) the PSI ALE-MURD scheme.](image)

**Figure 3:** Tracer profiles obtained at $t = 1800$ s using (a) the characteristic method and (b) the PSI ALE-MURD scheme.

schemes at $t = 1800$ seconds are shown in vertical cross-sections at $y = 2000$ m in Figure
3. The effect of the tracer on the flow has been taken into account since the velocities are mainly vertical and oriented in the direction of the free surface: the heat goes upwards. We observe that the tracer does not reach the free surface when advected with the characteristic scheme, which reveals a further problem of this method.

In order to show that the scheme is conservative for any vertical discretization of the domain, we have considered a different configuration. A cone-shaped obstacle is placed on the bottom at the middle of the domain, and we compare the results obtained on two different meshes, whose initial vertical cross-sections are shown in Figure 4. Mesh (a) has been obtained using the classical sigma transformation whereas mesh (b) has been obtained using another mapping, included in the recently implemented generalized sigma coordinate system (see [3]). Note that mesh (b) has a fixed horizontal plane at \( z = -4 \), so that the layers containing the tracer source do not move during the entire simulation.

![Figure 4: Initial vertical cross-sections of meshes (a) and (b).](image)

Figure 5 gives the evolution of the relative loss of tracer quantity during the simulation. On both meshes, the ALE-MURD scheme is perfectly conservative. Figure 6 shows the tracer profiles obtained after 360 time steps using both meshes, as vertical-cross sections at \( y = 2 \) m. Note that the result obtained using mesh (b) is more satisfactory, since the tracer only goes upwards. This does not hold when using mesh (a): the tracer also spreads out in the bottom direction. We point out that this result confirms the advantage of the general sigma coordinate introduced in [3] and implemented in Telemac-3D: the mesh can be adapted to the particular needs of the applications.
Figure 5: Relative loss of advected quantity using mesh a) (solid line with crosses) and mesh b) (solid line with plane circles).

Figure 6: Tracer profiles at $t = 1800$ s using mesh (a) and mesh (b).

5 CONCLUSION

We have derived a generic form of MURD schemes in the ALE frame, in order to apply them to advection problems posed on a moving domain. This has lead us to introduce a particular constraint the scheme must verify to preserve its conservation property when the domain moves. On a three-dimensional domain moving only in the vertical direction,
this constraint is easy to verify: it is enough to consider a mesh velocity which is constant within each time step. This result has allowed us to correct the lack of conservation in the MURD schemes implemented in the telemac-3D system, by extending them to the general ALE frame and ensuring the conservation constraint.

As a next step we aim to find a way to ensure the conservation property using a more accurate approximation of the domain velocity. It would also be interesting to extend our results to other explicit advection schemes.

REFERENCES


