NOTES ON THE UNSTEADY RECTILINEAR MOTION
OF A PERFECT GAS

IV. Some studies on centered rarefaction waves

by

J.A. Steketee

DELTFT - THE NETHERLANDS

October 1979
NOTES ON THE UNSTEADY RECTILINEAR MOTION OF A PERFECT GAS

IV. Some studies on centered rarefaction waves

by

J.A. Steketee

DELT - THE NETHERLANDS

October 1979
SUMMARY

In this report a class of solutions for the equations of the unsteady, rectilinear motion of a perfect gas is constructed. The velocity is prescribed to be of the form

$$u = A \frac{x}{t} + C,$$

and the homentropic, centered simple waves of the classical theory are included, together with the homogeneous solutions obtained in [1]. Also several more general flows are obtained, amongst others the so-called Von Mises solutions mentioned in [3].

The generalizations admitted, enable us to eliminate a difficulty found in [1] when two different homogeneous flows were matched along a common particle path. Jumps then appeared in temperature, density and entropy. By matching a homogeneous flow with a generalized flow no discontinuities in the parameters appear.

Several details of the special cases included in the family are discussed, and it is pointed out that the family of solutions constructed appears already in a paper of H. Ardavan-Rhad [6].
<table>
<thead>
<tr>
<th>CONTENT</th>
<th>page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2. The Equations of Motion</td>
<td>4</td>
</tr>
<tr>
<td>3. The velocity field $u = A \frac{x}{t} + C$ and the general solution</td>
<td>7</td>
</tr>
<tr>
<td>4. The Homentropic solutions</td>
<td>13</td>
</tr>
<tr>
<td>5. The compatibility condition</td>
<td>25</td>
</tr>
<tr>
<td>6. The solutions homogeneous in $h$, $t$</td>
<td>27</td>
</tr>
<tr>
<td>7. The generalized flows</td>
<td>35</td>
</tr>
<tr>
<td>8. Some related investigations</td>
<td>41</td>
</tr>
<tr>
<td>9. Concluding Remarks</td>
<td>44</td>
</tr>
<tr>
<td>References</td>
<td>45</td>
</tr>
<tr>
<td>Appendix</td>
<td>A-1</td>
</tr>
<tr>
<td>Figures</td>
<td></td>
</tr>
</tbody>
</table>
1. INTRODUCTION

In a recent report some solutions of the Lagrangian equations of motion were considered, which were prescribed to be homogeneous functions of the Lagrangian mass coordinate \( h \) and the time \( t \) [1]. These in general non-homentropic solutions were distinguished one from another by a different degree of homogeneity \( n \), with \( n \) appearing in the Poisson relation

\[
pV^\gamma = \exp \left( \frac{S}{c_v} \right) = B(h) = \bar{B} h^{n(\gamma+1)}
\]  

(1.1)

In the relation (1.1) all the symbols have their usual meaning, \( p \) is the pressure, \( V \) the specific volume, \( S \) the entropy per unit mass, \( c_v \) the constant specific heat at constant volume, \( c_p \) the constant specific heat at constant pressure, \( \gamma = c_p/c_v \), while \( \bar{B} \) is a constant.

All these distinct flows were found to possess the same velocity distribution and the same particle paths in the physical \( x,t \)-plane. The velocity distribution assumed the form

\[
u(x,t) = (1 - \kappa) \left( \frac{x - x_0}{t} - U \right) + U
\]  

(1.2)

with \( x_0, U \) and \( \kappa = \frac{\gamma - 1}{\gamma + 1} \) all constants.

It was also found that two different flows of this family could be joined together along a common particle path, in such a way that along the common path continuity of pressure and velocity was maintained.

The discontinuities in density and temperature however, which had to be accepted along this common path made the appearance of these composite flows in practical situations seem remote.

In the present report some slight alterations and generalizations of the flows found in [1] are considered.
In particular it is found that the undesirable discontinuities in density and temperature, mentioned above, can in several cases be removed, when two different flows of the new family are matched along a common particle path.

Before presenting an outline of the report some other remarks may be in order.

Speaking generally it seems natural in fluid dynamics that a certain flow, characterized by its velocity field, will require a definite pressure distribution for its realization. In steady incompressible inviscid flow the pressure distribution will, apart from constant velocities usually determine the velocity field uniquely, and conversely. In gases the density is an additional variable and when also variations in entropy are admitted there is clearly more scope. Also it would seem that the geometrical and kinematical complexity of the flow field will have some influence; if the geometrical and kinematical constraints are becoming less, there may be greater freedom left for other parameters.

As a consequence the simple velocity distribution (1.1) and the associated particle paths can apparently be realized by many distributions of pressure, density, and entropy. This report may be considered as an exercise to explore in this simple case the available freedom, when the velocity distribution and therefore also the particle paths are given.

Physically speaking the problems to be studied are simple and represent the conversion of random molecular motion i.e. heat, measured by the temperature $T$, into "directed motion", observed as the velocity $u$ and into work performed during expansion upon the moving boundaries.

The report is divided into 9 sections. In section 2 the equations of motion are derived and in addition to the variables $u$, $a^2$ and $S$ some other relations with pressure density, enthalpy etc. are collected.
In section 3 the velocity field is prescribed and the general solutions of the system are constructed yielding the relations (3.14) and (3.16). In the solutions two arbitrary functions $S(\psi)$ and $f(\psi)$ appear which are related by a compatibility condition obtained from the momentum equation.

In the following sections special cases included in the family of flows are discussed. In section 4 the homentropic flows are considered. These are the homentropic centered simple waves of the classical theory but also slightly more general types, which are indicated as Von Mises' flows, since they appear in the book of Von Mises [3]. Some details are worked out.

In section 5 the general solutions of the compatibility condition are considered. In section 6 a summary of the homogeneous flows constructed in [1] is presented and the relation with the solutions of this report is constructed. Also the matching of two homogeneous flows along a common particle path is discussed and the discontinuities in temperature, density and entropy are indicated.

In section 7 the homogeneous flows are generalized and it becomes clear that a homogeneous flow and a generalized flow can be matched along a common particle path without introduction of discontinuities. Also some details of these generalized flows are considered.

In section 8 some discussion is given of other literature, related to our work. In particular a paper by H. Ardavan-Rhad [6] is discussed, where essentially the solutions of this report already appear.

Some final remarks are made in section 9 and this is followed by a list of references, an Appendix and some figures.
2. THE EQUATIONS OF MOTION

The starting point is formed by the three conservation laws of mass, momentum and energy in the physical $x, t$-plane. Only expansion waves will be considered and compressions leading to shock-waves are excluded. Therefore effects of viscosity and heat conduction will be neglected throughout. The three conservation laws then read

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = 0 ,
\]  
(2.1)

\[
\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u^2 + p) = 0 ,
\]  
(2.2)

\[
\frac{\partial}{\partial t} \left\{ \rho (U + \frac{1}{2} u^2) \right\} + \frac{\partial}{\partial x} \left\{ \rho u (H + \frac{1}{2} u^2) \right\} = 0 ,
\]  
(2.3)

with $x$ the Cartesian coordinate, $t$ the time, $\rho$ the density, $p$ the pressure, $u$ the velocity, $U$ the internal energy per unit mass and $H$ the enthalpy per unit mass.

The gas to be considered is an ideal gas, satisfying the law of Boyle-Gay Lussac

\[
\frac{p}{\rho} = RT ,
\]  
(2.4)

with $R$ the gas constant per unit mass and $T$ the temperature. This implies that upon taking $\rho$ and $T$ as independent thermodynamic variables the internal energy $U$ is independent of $\rho$ and a function of $T$ only (Joule effect). Similarly the enthalpy $H$ is independent of the pressure $p$ and a function of the temperature $T$ only if $p$ and $T$ are chosen as independent thermodynamic variables. Assuming further that the specific heats at constant volume $c_v$ and at constant pressure $c_p$ are constant one may write

\[ U = c_v T , \quad H = c_p T . \]  
(2.5)
Since the gas is allowed to perform work by expansion only the second law of thermodynamics can be written in the form

$$ T \frac{DS}{Dt} = \frac{DU}{Dt} + p \frac{D}{Dt} \left( \frac{1}{\rho} \right) , $$

(2.6)

with $S$ denoting the entropy per unit mass and

$$ \frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} . $$

(2.7)

By simple manipulations, employing (2.6) the equations (2.1)-(2.3) can be reduced to a set of three equations for the velocity $u$, the square of the speed of sound $a^2 = \gamma RT$ (with $\gamma = c_\rho/c_v$) and the entropy $S$, which are taken as the preferred variables. The equations so obtained are

$$ \frac{\partial a^2}{\partial t} + u \frac{\partial a^2}{\partial x} + (\gamma - 1) a^2 \frac{\partial u}{\partial x} = 0 , $$

(2.8)

$$ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\gamma - 1} \frac{\partial a^2}{\partial x} = \frac{a^2}{\gamma R} \frac{\partial S}{\partial x} , $$

(2.9)

$$ \frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} = 0 . $$

(2.10)

The equation (2.10) is the familiar statement that a fluid element will retain its entropy during its lifetime. If all fluid elements have the same entropy the flow is homentropic and $S$ is a constant. The equation (2.10) can then be left out altogether and (2.8), (2.9) may be written in the familiar form

$$ \frac{\partial}{\partial t} \left( 2a \frac{a}{\gamma - 1} \right) + u \frac{\partial}{\partial x} \left( 2a \frac{a}{\gamma - 1} \right) + a \frac{\partial u}{\partial x} = 0 , $$

(2.11)

$$ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + a \frac{\partial}{\partial x} \left( \frac{2a}{\gamma - 1} \right) = 0 , $$

(2.12)

leading to the characteristic form, the Riemann-invariants, the classical theory of simple waves etc.
If the flow is non-homentropic different fluid elements will have different entropy values and the eq. (2.10) has to be retained. This equation then introduces the particle paths as the third set of characteristics of the system (2.8)-(2.10).

It may be noticed that

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial H}{\partial x} - T \frac{\partial S}{\partial x}, \quad (2.13)$$

a relation between derivatives of pressure, enthalpy and entropy, familiar from Crocco's Theorem, and also easily deduced from (2.6). It transforms the familiar pressure gradient term in the momentum equation to the form appearing in (2.9).

Finally since the speed of sound, the temperature, the density and other parameters are related by

$$a^2 = \gamma RT = \gamma \frac{p}{\rho} = \gamma \cdot \rho^{\gamma - 1} \cdot \exp \left( \frac{S}{c_v} \right) = (\gamma - 1) H = \gamma (\gamma - 1) U, \quad (2.14)$$

the equation (2.8) can be considered as the continuity equation, but if desired also as a part of the energy equation i.e. for the internal energy

$$\frac{\partial U}{\partial t} + u \frac{\partial U}{\partial x} + \frac{p}{\rho} \frac{\partial u}{\partial x} = 0. \quad (2.15)$$

Once the three parameters $u$, $S$ and $a^2$ are found the temperature $T$ follows from (2.14), while the pressure $p$ and density $\rho$ are deduced from

$$(\gamma - 1) \ln \rho = \ln a^2 - \frac{1}{c_v} S - \ln \gamma, \quad (2.16)$$

$$\frac{\gamma - 1}{\gamma} \ln p = \ln a^2 - \frac{1}{c_p} S - \ln \gamma, \quad (2.17)$$

where the logarithms can be easily removed if one so desires.
3. THE VELOCITY FIELD $u = A \frac{x}{t} + C$ AND THE GENERAL SOLUTION

The flows to be considered are prescribed to have the velocity distribution

$$ u = A \frac{x}{t} + C , $$ (3.1)

with $A$ and $C$ constants. This form includes the velocity distribution (1.2) of the homogeneous flows studied in [1]. It is clear from (3.1) and from the continuity equation (2.1) that

$$ \frac{\partial u}{\partial x} = \frac{A}{t} = - \frac{1}{\rho} \frac{Dp}{Dt} , $$ (3.2)

indicating that the specific expansion rate depends on $t$ only and is constant throughout the flow at each instant $t$. This simple property holds true for all velocity distributions linear in $x$ as for example

$$ u = a(t) x + b(t) , $$ (3.3)

mentioned by Gundersen [2].

The origin $x = 0$, $t = 0$ is a singularity in the field where the specific expansion rate is infinite.

To construct the particle paths the equation

$$ \frac{dx}{dt} = u = A \frac{x}{t} + C , $$ (3.4)

has to be integrated yielding

$$ t^{1-A} \left( \frac{x}{t} + C^* \right) = \psi , \quad A \neq 1 . $$ (3.5)

In (3.5) $C^*$ is a constant related to $C$ and $A$ by $C = C^*(A - 1)$, while $\psi$ is constant along a particle path.
Also the condition \( A \neq 1 \) has to be satisfied. For \( A = 1 \) a special case results to be dealt with in an Appendix.

Since \( \psi \) changes its value from one particle path to another we may take \( \psi \) as a Lagrangian particle coordinate. It is easily verified that

\[
\frac{D\psi}{Dt} = \frac{\partial \psi}{\partial t} + u \frac{\partial \psi}{\partial x} = 0 ,
\]

and since the entropy \( S \) is constant for a particle, the entropy may be considered as a function \( S(\psi) \) of \( \psi \).

Substituting the Lagrangian variables \((\psi, t)\) in (3.1) one obtains

\[
u = A \psi t^{A-1} - C^* ,
\]

which shows the difference between flows with \( A > 1 \) and \( A < 1 \). For \( A > 1 \) one has \( t^{A-1} = 0 \) for \( t = 0 \) and the constant \(-C^*\) represents the initial velocity for the entire flow. The speed of the fluid elements increases with increasing \( t \), reaching infinite values for \( t \to +\infty \).

Since these infinite velocities, with the accompanying expansions have to be obtained from conversion of random molecular motions, initially present, it will require infinite temperatures at \( t = 0 \). It will be found that the equations of motion cannot admit this situation.

For \( A < 1 \) one has \( t^{A-1} = \infty \) for \( t = 0 \) and the initial speed of all the fluid elements is infinite. For increasing \( t \) the fluid slows down while expanding and reaching the constant terminal velocity \(-C^*\) at \( t \to \infty \). Only this case will be found to satisfy the equations.

To proceed the velocity distribution (3.1) is substituted into eq. (2.8) yielding for \( a^2 \) the linear partial differential equation

\[
\frac{\partial a^2}{\partial t} + \left( A \frac{x}{t} + C \right) \frac{\partial a^2}{\partial x} + (\gamma - 1) \frac{A}{t} a^2 = 0 .
\]
The characteristic equations of (3.8) are

\[
\frac{dt}{t} = \frac{dx}{Ax + Ct} = -\frac{da^2}{(\gamma - 1) Aa^2}.
\]  

(3.9)

Two independent solutions of the set (3.9) are found to be the expression (3.5) for the particle paths and

\[
a^2 t (\gamma - 1) A = \text{const} = C_2.
\]  

(3.10)

A general solution of (3.8) can then be written down. Since in eq. (2.9) the explicit form of \(a^2\) is required the general integral is taken in the form

\[
a^2 = t^{-(\gamma - 1) A} \cdot f(\psi),
\]  

(3.11)

with \(f(\psi)\) an arbitrary function of \(\psi\).

Substituting \(u\) and \(a^2\) from (3.1) and (3.11) together with \(S(\psi)\) into the momentum equation (2.9) one finds that this equation can be satisfied only provided

\[
A = 1 - \kappa = \frac{2}{\gamma + 1},
\]  

(3.12)

in agreement with (1.2) and with \(A < 1\), while furthermore

\[
f'(\psi) - \frac{1}{\rho} S'(\psi) f(\psi) = 2\kappa^2 \psi,
\]  

(3.13)

has to be satisfied with the dashes denoting derivatives to \(\psi\). The equation (3.13) is a compatibility condition to be satisfied by the two arbitrary functions \(f(\psi)\) and \(S(\psi)\). Collecting the results, we have found

\[
u = (1 - \kappa) \frac{x}{t} + C = (1 - \kappa) t^{-\kappa} \cdot \psi - C^*,
\]
\[ a^2 = t^{-2\kappa} f(\psi), \]
\[ S = S(\psi), \]
\[ \psi = t^\kappa \left( \frac{x}{t} + C^* \right), \]

with \( C = -\kappa C^* \), and \(-C^*\) the terminal velocity of all the fluid elements. The arbitrary functions \( f(\psi) \) and \( S(\psi) \) moreover have to satisfy the compatibility condition (3.13).

To keep the further analysis as simple as possible it is advantageous to study the flows from a reference system moving with the uniform terminal velocity \(-C^*\) with respect to the \(x,t\)-system used so far. If the origins of the new \(x',t'\)-system and the old \(x,t\)-system coincide the required Galilei transformations are

\[ x' = x + C^* t, \quad x = x' - C^* t', \]
\[ u' = u + C^*, \quad u = u' - C^*, \]
\[ t' = t, \quad t = t'. \] (3.15)

The equations of motion are invariant under these transformations, while the solutions (3.14) assume the simple form

\[ u = (1 - \kappa) \frac{x}{t} = (1 - \kappa) t^{-\kappa} \psi, \]
\[ a^2 = t^{-2\kappa} f(\psi), \]
\[ S = S(\psi), \]
\[ \psi = x t^{-(1-\kappa)}, \]

(3.16)
where the dashes of $x'$ and $t'$ were omitted after the substitution of (3.15) into (3.14) was completed. In this new frame of reference the terminal velocity is clearly zero for all the fluid elements.

From the solutions for $u$, $a^2$ and $S$ one deduces the temperature $T$, the density $\rho$ and the pressure $p$ by employing (2.14), (2.16) respectively (2.17).

Eliminating the time $t$ between $u$ and $a^2$ in (3.16) one finds the relation

\[ (1 - \kappa)^2 \psi^2 a^2 - u^2 f(\psi) = 0 , \tag{3.17} \]

indicating that a simple relation between $u$, $a$ and $\psi$ exists in the flows considered. This relation will reappear in the discussion of special cases.

The characteristic directions associated with the equations (2.8) and (2.9) are given by

\[ \frac{dx}{dt} = u + a , \quad \frac{dx}{dt} = u - a , \tag{3.18} \]

while the equation (2.10) introduces the direction of the particle paths as the third characteristic direction i.e.

\[ \frac{dx}{dt} = u . \tag{3.19} \]

It is convenient to write the equations (3.18) and (3.19) also in terms of the Lagrangian variables $\psi$ and $\tau$. Employing (3.16) and the plus sign for the speed of sound $a$ the equations (3.18) and (3.19) become

\[ t \frac{d\psi}{dt} = \sqrt{f(\psi)} , \quad t \frac{d\psi}{dt} = -\sqrt{f(\psi)} , \quad \frac{d\psi}{dt} = 0 . \tag{3.20} \]

These Lagrangian equations are usually simpler to deal with than the
Eulerian equations (3.18), (3.19).

In the sections which follow the homentropic flows will be considered first. Then some non-homentropic flows will be considered, in particular the homogeneous solutions of [1], with some slight extensions.
4. THE HOMENTROPIC SOLUTIONS

In this section the analytical details of the homentropic solutions in the system (3.16) will be considered.

Since the entropy distribution simplifies to a constant

\[ S(\psi) = S_0 = \text{const} , \]  

(4.1)

the compatibility condition (3.13) also simplifies and yields

\[ f(\psi) = \kappa^2 \psi^2 + f_o , \]  

(4.2)

with \( f_o \) an integration constant. In the \( \psi, f(\psi) \)-plane (4.2) is a parabola, symmetric with respect to \( \psi = 0 \) and with a minimum \( f = f_o \) for \( \psi = 0 \). For \( f_o > 0 \) the function \( f(\psi) \) assumes only positive values, for \( f_o = 0 \) the value \( f(\psi) = 0 \) is reached for \( \psi = 0 \), and for \( f_o < 0 \) the values of \( f(\psi) \) are both positive and negative.

Since \( f(\psi) \) apart from a power of \( t \) represents the square of the speed of sound, which cannot be negative, it follows that three distinct cases have to be considered.

Case I. \( f_o = \kappa^2 \alpha^2 \), \( f(\psi) = \kappa^2 (\psi^2 + \alpha^2) \)

Case II. \( f_o = 0 \), \( f(\psi) = \kappa^2 \psi^2 \)  

(4.3)

Case III. \( f_o = -\kappa^2 \alpha^2 \), \( f(\psi) = \kappa^2 (\psi^2 - \alpha^2) \), \( |\psi| \geq \alpha \)

The flows with the integration constant \( f_o \neq 0 \) do not appear in the homogeneous solutions studied in [1]. They are mentioned however in the book of R. von Mises [3] and it seems appropriate to call them Von Mises' solutions.

The discussion of the homentropic flows begins with Case II and proceeds
then to the Cases I and III.

**Case II**, \(f(\psi) = \kappa^2 \psi^2\). Substitution of \(f(\psi)\) into (3.16) yields

\[
 u = (1 - \kappa) \frac{x}{t} = (1 - \kappa) \psi t^{-\kappa}, \quad a^2 = \kappa^2 \psi^2 t^{-2\kappa} = \kappa^2 \left( \frac{x}{t} \right)^2.
\]  

(4.4)

The expressions (4.4) together with (4.1) represent the two centered simple waves of the classical theory taken together. The positive half plane \(x > 0\) represents the wave with

\[
 s = u - \frac{2a}{\gamma - 1} = \text{const} ,
\]  

(4.5)

while the negative half plane \(x < 0\) represents the wave with

\[
 r = u + \frac{2a}{\gamma - 1} = \text{const} .
\]  

(4.6)

The two waves are joined along the vacuumline \(x = 0\), where due to the choice of our reference frame the velocity vanishes.

To verify this we have from (4.4)

\[
 u > 0 , \quad a = \kappa \psi t^{-\kappa} = \kappa \frac{x}{t} ,
\]  

(4.7)

valid for \(x > 0\) and \(\psi > 0\), and

\[
 u < 0 , \quad a = -\kappa \psi t^{-\kappa} = -\kappa \frac{x}{t} ,
\]  

(4.8)

valid for \(x < 0\) and \(\psi < 0\).

The expression (3.17) valid throughout the flow takes the form

\[
 u^2 - \left( \frac{1 - \kappa}{\kappa} \right)^2 a^2 = r s = 0
\]  

(4.9)

by using (4.5) and (4.6). From (4.5) and (4.6) together with (4.7) and
(4.8) one observes that for $x > 0$, $\psi > 0$ also $r > 0$ and hence from (4.9)

$$s = 0.$$  \hfill (4.10)

This can also be verified from $u$ in (4.4) and (4.7). It shows that the half plane $x > 0$ is the simple wave $s = 0$.

In the same way one verifies that the half plane $x < 0$ represents the simple wave $r = 0$.

The Eulerian characteristic equations (3.18) yield

$$\frac{dx}{dt} = u + a = \frac{x}{t}, \quad \frac{dx}{dt} = u - a = (1 - 2\kappa) \frac{x}{t},$$  \hfill (4.11)

valid in the positive half plane $x > 0$, and

$$\frac{dx}{dt} = u + a = (1 - 2\kappa) \frac{x}{t}, \quad \frac{dx}{dt} = u - a = \frac{x}{t},$$  \hfill (4.12)

valid in the negative half plane $x < 0$. Integration yields for $x > 0$

$$x = r^* t, \quad x = s^* t^{1-2\kappa},$$  \hfill (4.13)

and for $x < 0$ one obtains in similar fashion

$$x = r^* t^{1-2\kappa}, \quad x = s^* t.$$  \hfill (4.14)

The relations (4.13) and (4.14) verify the well-known property of simple waves, that the $r$-characteristics are straight lines for the simple wave with $s = \text{const}$ and its converse.

Simple experiments, where parts of the centered simple waves discussed can be produced are well known. We mention the ordinary shock-tube flow, the flow in a tube where originally a section filled with a uniform gas at rest and a completely evacuated section are separated
by a membrane, which is then instantaneously removed and the flow generated when a piston, initially moving with uniform speed, while pushing a uniform gas column is suddenly brought to rest.

Finally, some attention may be paid to the Lagrangian expressions in the $\psi, t$-plane.

Calculating the characteristics from the Lagrangian equations (3.20) one finds

$$t \frac{d\psi}{dt} = \kappa \psi , \quad t \frac{d\psi}{dt} = -\kappa \psi , \quad (4.15)$$

yielding upon integration

$$\psi = r^* t^\kappa , \quad \psi = s^* t^{-\kappa} , \quad (4.16)$$

for the $r$- respectively the $s$-characteristics in the positive half plane. Substituting $\psi$ from (3.16) the equations (4.13) are recovered.

By interchanging the + and − signs in the equations (4.15) and their integrals (4.16) the characteristics in the half plane $\psi < 0$ are obtained. Substituting $\psi$ from (3.16) again the equations (4.14) are recovered.

In earlier work on the Lagrangian equations of motion the Lagrangian mass coordinate $h$ was used, while here a Lagrangian particle coordinate $\psi$ is employed. The relation between the two Lagrangian coordinates is obtained by comparing the expressions (see [4])

$$dx = V \, dh + u \, dt , \quad (4.17)$$

and the differential of $x$ obtained from $\psi$ in (3.16) i.e.

$$dx = t^{1-\kappa} \, d\psi + u \, dt . \quad (4.18)$$
It follows that

\[ dh = t^{1-\kappa} \rho \, d\psi, \tag{4.19} \]

and after some simple manipulation, employing (2.16) this yields

\[ dh = \left( \frac{\kappa}{\sqrt{Y}} \right)^{1-\kappa} \exp \left( -\frac{S_0}{R} \right) \frac{1-\kappa}{\psi^\kappa} \, d\psi. \tag{4.20} \]

Comparison of (4.20) and its integral, which is easily obtained then agrees with results obtained elsewhere, which show that the characteristics of homentropic centered simple waves in the \( h, t \)-plane are of the form

\[ \frac{h}{t} = \text{const}, \quad ht = \text{const}. \tag{4.21} \]

i.e. a pencil of straight lines passing through the origin and a set of equilateral hyperbolae.

In Figures 1 and 2 the flows have been drawn in the physical \( x, t \)-plane and the Lagrangian \( \psi, t \)-plane.

**Case 1,** \( f(\psi) = \kappa^2(\psi^2 + \alpha^2) \). The Von Mises' flow so obtained is represented in the \( \psi, t \)-plane by

\[ u = (1 - \kappa) \psi \, t^{-\kappa}, \quad a^2 = \kappa^2 \, t^{-2\kappa}(\psi^2 + \alpha^2), \quad S = S_0, \tag{4.22} \]

and in the \( x, t \)-plane by

\[ u = (1 - \kappa) \frac{x}{t}, \quad a^2 = \kappa^2 \left( \frac{x}{t} \right)^2 + \kappa^2 \alpha^2 \, t^{-2\kappa}, \quad S = S_0. \tag{4.23} \]

In the form (4.23) the solution appears in the book of Von Mises ([3], p. 78).
The velocity distribution and the particle paths are identical with those of the simple waves considered before, but \( a^2 \) and the associated parameters have changed. Also the simple wave properties are lost.

The possibility to extend \( a^2 \) with a term depending on the time only is due to the simplicity of the velocity distribution. Since \( \frac{\partial u}{\partial x} \) and the specific expansion rate depend on the time only the coefficients of \( a^2 \) in the third term of (2.8) and (3.8) no longer contain \( x \) and a solution for \( a^2 \) depending on \( t \) can be superposed without difficulty. The momentum equation (2.9) is not affected by this extension since only the \( x \) derivative of \( a^2 \) plays a part.

It should be noted that the preceding argument remains valid if eq. (2.8) is reduced to an equation for \( a \) instead of \( a^2 \). In the momentum equation (2.9) however the nonlinear term

\[
\frac{2a}{\gamma - 1} \frac{\partial a}{\partial x},
\]

then appears and at this point the argument fails. It suggests that the point is subtle and not trivial. It also explains why \( a^2 \) was chosen as a variable and not the speed of sound itself.

Compared to the solution of Case II the flow takes place at a higher temperature level so that the possibility to reach \( T = 0 \) and the associated vacuum situation appears only when \( t \to \infty \). There is however no vacuum line \( x = 0, \psi = 0 \) and the flows in the two half planes form one whole.

It is of interest that also for this flow the properties can be worked out exactly as we now proceed to show.

To obtain the temperature, density and pressure in the \( x,t \)-plane and \( \psi,t \)-plane the appropriate substitutions of (4.22) and (4.23) into (2.14), (2.16) and (2.17) can be made.
The Lagrangian characteristic equations (3.20) take the form

\[ t \frac{d\psi}{dt} = \kappa \sqrt{\psi^2 + \alpha^2}, \quad t \frac{d\psi}{dt} = -\kappa \sqrt{\psi^2 + \alpha^2}, \quad \text{(4.26)} \]

and yield upon integration

\[ \sqrt{\psi^2 + \alpha^2} + \psi = r^+ t^\kappa, \quad \sqrt{\psi^2 + \alpha^2} + \psi = s^+ t^{-\kappa}, \quad \text{(4.27)} \]

with \( r^+ \) and \( s^+ \) constant along the \( r^- \) respectively \( s^- \)-characteristics.

Alternative forms can be obtained by employing the relation

\[ (\sqrt{\psi^2 + \alpha^2} + \psi) (\sqrt{\psi^2 + \alpha^2} - \psi) = \alpha^2, \quad \text{(4.28)} \]

and by introducing other characteristic constants \( r^- \) and \( s^- \) related to \( r^+ \) and \( s^+ \) by

\[ r^+ r^- = s^+ s^- = \alpha^2. \quad \text{(4.29)} \]

In this way the alternative forms

\[ \sqrt{\psi^2 + \alpha^2} - \psi = \frac{\alpha^2}{r^+} t^{-\kappa} = r^- t^{-\kappa}, \quad \text{(4.30)} \]

\[ \sqrt{\psi^2 + \alpha^2} - \psi = \frac{\alpha^2}{s^+} t^\kappa = s^- t^\kappa, \]

are obtained for the \( r^- \) respectively the \( s^- \)-characteristics.

To verify that along an \( r^- \)-characteristic the Riemann-invariant \( r \), defined by (4.6) is a constant \( u \) and \( a \) from (4.22) are substituted in the first expression of (4.27).

This yields

\[ r^+ = \frac{a}{\kappa} + \frac{u}{1 - \kappa} = \frac{1}{1 - \kappa} \left( u + \frac{2a}{\gamma - 1} \right) = \frac{r}{1 - \kappa} = \frac{\alpha^2}{r^-}, \quad \text{(4.31)} \]
and since \( r^+ \) is constant along the characteristic the same applies to \( r \). In the same way one obtains for \( s \) defined by (4.5) and the second relation in (4.30)

\[
\frac{s^-}{s^+} = a \frac{u}{1 - \kappa} = -\frac{1}{1 - \kappa} \left( u - \frac{2a}{\gamma - 1} \right) = -\frac{s^-}{s^+} = \frac{a^2}{s^+} .
\]  

(4.32)

Subtraction of the first respectively the second relations in (4.27) and (4.30) yields upon substitution for \( \psi \) the expression in \( x \) and \( t \) from (3.16)

\[
\frac{x^+}{t} = \frac{r^+}{2} - \frac{a^2}{2r^+} t^{-2\kappa} = \frac{a^2}{2} - \frac{r^-}{2} t^{-2\kappa} ,
\]

(4.33)

\[
\frac{x^-}{t} = -\frac{a^2}{2s^-} + \frac{s^+}{2} t^{-2\kappa} = -\frac{s^-}{2} + \frac{a^2}{2s^-} t^{-2\kappa} ,
\]

(4.34)

representing the characteristics in the \( x,t \)-plane.

Taking the limit \( \alpha \to 0 \) in the formulae with \( r^+ \) and \( s^+ \), the simple wave characteristics (4.13) in the positive half plane are recovered. The same limit in the formulae with \( r^- \) and \( s^- \) yields the simple wave characteristics (4.14) in the negative half plane.

The flow considered here is not a simple wave, as was mentioned, but a general homentropic flow. To show that some reminders of the simple wave are still present we consider the relation (3.17) which may be written for this case in the form

\[
\left( u \sqrt{1 + \frac{a^2}{\psi}} + \frac{2a}{\gamma - 1} \right) \left( u \sqrt{1 + \frac{a^2}{\psi}} - \frac{2a}{\gamma - 1} \right) = 0 .
\]

(4.35)

Since \( u > 0 \) in the positive half plane \( x > 0 \), and \( \psi > 0 \) and since \( a \) is always positive, we have in the positive half plane

\[
u \sqrt{1 + \frac{a^2}{\psi}} - \frac{2a}{\gamma - 1} = 0 .
\]

(4.36)

In the same way we find, that in the negative half plane
\[ u \sqrt{1 + \frac{\alpha^2}{\psi^2} + \frac{2a}{\gamma - 1}} = 0. \] (4.37)

To study general homentropic flows it is often convenient to take the Riemann invariants \( r \) and \( s \) defined by (4.5) and (4.6) as independent variables and to consider \( t \) and \( x \) as the dependent variables. The equation for \( t \) then becomes the Euler-Poisson-Darboux equation

\[ \frac{\partial^2 t}{\partial r \partial s} + \frac{1}{2\kappa} \cdot \frac{1}{r - s} \cdot \left( \frac{\partial t}{\partial s} - \frac{\partial t}{\partial r} \right) = 0. \] (4.38)

From (4.22), (4.23), (4.5) and (4.6) it is easy to solve \( t \) in the form

\[ t^{2\kappa} = -\left( \frac{2a}{\gamma + 1} \right)^2 \cdot \frac{1}{rs}, \] (4.39)

and to verify that this expression is a solution of (4.38). From

\[ u = \frac{1}{2} (r + s), \] (4.40)

one also deduces \( x \) in terms of \( r \) and \( s \) by

\[ x = \frac{\gamma + 1}{4} \cdot t(r + s). \] (4.41)

Also \( \psi \) can now easily be deduced in terms of \( r \) and \( s \).

The relation between the Lagrangian mass coordinate \( h \) and the Lagrangian particle coordinate \( \psi \) is also in this case given by (4.19) as one easily verifies and substituting \( a^2 \) from (4.22) into (2.16) one finds

\[ dh = \left\{ \frac{\kappa^2}{\gamma} \left( \psi^2 + \alpha^2 \right) \right\}^{\frac{1-\kappa}{2\kappa}} \exp \left( -\frac{S_0}{R} \right) d\psi. \] (4.42)

For a monatomic and diatomic case the integral (4.42) can be evaluated explicitly resulting in lengthy forms that cannot be inverted.

To generate the flow considered here in an experiment appropriate
initial- and boundary-conditions, deduced from the formulae, have to be imposed. They have seemed neither simple nor obvious so far. This also applies to the assigned value of $\alpha^2$.

Case III, $f(\psi) = \kappa^2(\psi^2 - \alpha^2)$, $|\psi| > \alpha$. The Von Mises' flow in this case has physical significance only provided $|\psi| > \alpha$. The flow takes place at a lower temperature level than the simple waves of Case II. The temperature $T = 0$, resulting in a vacuum situation, therefore appears already at $\psi = \pm \alpha$. The vacuum line along the axis $\psi = 0$ in Case II now appears at $\psi = +\alpha$ and $\psi = -\alpha$. In the strip $-\alpha < \psi < +\alpha$ there is no physically acceptable flow. As a result we have essentially two distinct flows but taken together.

The analytical aspects are similar to Case I and it seems unnecessary to go through it with the same amount of details.

The flow is represented in the $\psi,t$-plane by

$$u = (1 - \kappa) t^{-\kappa} \psi, \quad a^2 = \kappa^2 t^{-2\kappa} (\psi^2 - \alpha^2), \quad S = S_0,$$

(4.43)

and in the $x,t$-plane by

$$u = (1 - \kappa) \frac{x}{t}, \quad a^2 = \kappa^2 \left(\frac{x}{t}\right)^2 - \kappa^2 \alpha^2 t^{-2\kappa}, \quad S = S_0,$$

(4.44)

The Lagrangian characteristic equations (3.20) now become

$$t \frac{d\psi}{dt} = \kappa \sqrt{\psi^2 - \alpha^2}, \quad t \frac{d\psi}{dt} = -\kappa \sqrt{\psi^2 - \alpha^2},$$

(4.45)

and are tangent to the lines $\psi = \pm \alpha$ in the $\psi,t$-plane. Integration of (4.45) yields

$$\psi + \sqrt{\psi^2 - \alpha^2} = r^+ \ t^\kappa = \frac{\alpha}{r} t^\kappa, \quad \psi + \sqrt{\psi^2 - \alpha^2} = s^+ \ t^{-\kappa} = \frac{\alpha}{s} t^{-\kappa},$$

(4.46)

$$\psi - \sqrt{\psi^2 - \alpha^2} = r^- \ t^{-\kappa} = \frac{\alpha}{r} t^{-\kappa}, \quad \psi - \sqrt{\psi^2 - \alpha^2} = s^- \ t^\kappa = \frac{\alpha}{s} t^\kappa,$$

(4.47)
where an identity similar to (4.28) has been used together with (4.29) to relate \( r^+ \), \( r^- \) and \( s^+ \), \( s^- \).

Substituting \( u \) and \( a \) from (4.43) into the appropriate expressions in (4.46) and (4.47) one finds

\[
\frac{u}{1 - \kappa} + \frac{a}{\kappa} = r^+ = \frac{1}{1 - \kappa} \left( u + \frac{2a}{\gamma - 1} \right) = \frac{r}{1 - \kappa}, \tag{4.48}
\]

and so \( r = \text{const} \) along a \( r \) characteristics, while

\[
\frac{u}{1 - \kappa} - \frac{a}{\kappa} = s^- = \frac{1}{1 - \kappa} \left( u - \frac{2a}{\gamma - 1} \right) = \frac{s}{1 - \kappa}, \tag{4.49}
\]

indicating that along an \( s \)-characteristic the Riemann invariant \( s \) is constant.

Subtracting the corresponding equations in (4.46) and (4.47) and substitution of \( \psi \) in terms of \( x, t \) from (3.16) yields

\[
\frac{x}{t} = \frac{r^+}{2} - \frac{\alpha^2}{2r^+} t - 2\kappa = \frac{\alpha^2}{2r^+} - \frac{r^-}{2} t - 2\kappa, \tag{4.50}
\]

\[
\frac{x}{t} = \frac{s^+}{2} t - 2\kappa - \frac{\alpha^2}{2s^+} = \frac{\alpha^2}{2s^+} - \frac{s^-}{2} - s^- \tag{4.51}
\]

which represent the characteristics in the \( x, t \)-plane. The relation (3.17) now takes the form

\[
(u \sqrt{1 - \frac{\alpha^2}{\psi^2} + \frac{2a}{\gamma - 1}}) (u \sqrt{1 - \frac{\alpha^2}{\psi^2} - \frac{2a}{\gamma - 1}}) = 0, \tag{4.52}
\]

yielding for \( \psi > +\alpha \), where \( u > 0 \)

\[
u \sqrt{1 - \frac{\alpha^2}{\psi^2} - \frac{2a}{\gamma - 1}} = 0, \tag{4.53}
\]

and for \( \psi < -\alpha \), with \( u < 0 \)

\[
u \sqrt{1 - \frac{\alpha^2}{\psi^2} + \frac{2a}{\gamma - 1}} = 0. \tag{4.54}
\]
Expressing $t$ in terms of the Riemann invariants one finds

$$t^{2\kappa} = \left( \frac{2\alpha}{Y+1} \right)^2 \frac{1}{r s},$$

(4.54)

while (4.19) relating the Lagrangian mass coordinate $h$ with $\psi$ now takes the form

$$dh = \left\{ \frac{k^2}{Y} (\psi^2 - \alpha^2) \right\}^{1-\kappa} 2\kappa \exp \left( -\frac{S\varphi}{R} \right) d\psi.$$  

(4.55)

For the experimental realization of this flow the same remarks apply as for Case I.

This completes the discussion of the homentropic flows and we turn now to non-homentropic cases.
5. THE COMPATIBILITY CONDITION

In the general solution (3.16) two arbitrary functions $f(\psi)$ and $S(\psi)$ appear. They cannot be chosen independently since they have to satisfy the compatibility condition

$$f'(\psi) - \frac{1}{c_p} S'(\psi) f(\psi) = 2\kappa^2 \psi$$

(3.13)

This condition may be considered as a non-homogeneous differential equation for the determination of $f(\psi)$.

The general solution of this differential equation is composed of the general solution of the homogeneous equation plus a particular solution of the non-homogeneous equation and it is not difficult to construct this general solution. The process may however be simplified considerably if the entropy $S(\psi)$ is replaced by other expressions $B(\psi)$ and $b(\psi)$ defined by

$$p \nu^\gamma = \exp \left( \frac{S(\psi)}{c_v} \right) = B(\psi) = b(\psi)^\gamma,$$

(5.1)

and yielding for $S(\psi)$

$$S(\psi) = c_v \ln B(\psi) = c_p \ln b(\psi).$$

(5.2)

Employing $b(\psi)$ instead of $S(\psi)$ the eq. (3.13) can be brought in the form

$$\frac{d}{d\psi} \left( \frac{f(\psi)}{b(\psi)} \right) = 2\kappa^2 \frac{\psi}{b(\psi)}$$

(5.3)

The general solution of $f(\psi)$ now requires only a quadrature and yields

$$f(\psi) = 2\kappa^2 b(\psi) \left( \int \frac{\psi}{b(\psi)} d\psi + C \right).$$

(5.4)

with $C$ an integration constant.

To express $b(\psi)$ in terms of $f(\psi)$ we write (5.3) in the form
\[
\frac{d}{d\psi} \left( \frac{f(\psi)}{b(\psi)} \right) = 2\kappa^2 \frac{\psi}{f(\psi)} \cdot \frac{f(\psi)}{b(\psi)},
\]

which yields upon integration

\[
b(\psi) = f(\psi) \exp \left( -2\kappa^2 \int \frac{\psi}{f(\psi)} d\psi + C \right),
\]

and employing (5.2)

\[
\frac{1}{c_p} S(\psi) = \ln f(\psi) - 2\kappa^2 \int \frac{\psi}{f(\psi)} d\psi + C.
\]

where C again is an integration constant.

Before leaving the compatibility condition it should be remarked that in the homentropic case the general solution of the homogeneous equation reduces to a constant. It was this constant which distinguished the Von Mises' solutions from the simple waves of Case II. It will be seen that the homogeneous solutions of [1] appear when in the general solution of \( f(\psi) \), only the particular solution of the non-homogeneous equation is considered.
6. THE SOLUTIONS HOMOGENEOUS IN h, t

In Report LR-258 [1] solutions of the Lagrangian equations of motion are considered, which are prescribed to be homogeneous functions of the Lagrangian mass coordinate h and the time t. It is to begin with assumed that h and t are positive \((0 \leq h \leq +\infty, 0 \leq t \leq +\infty)\). For the velocity \(u\), the speed of sound \(a\), and the Cartesian coordinate \(x\) the following expressions were obtained

\[
\begin{align*}
\begin{aligned}
u(h,t) &= u_0 h^{n+\kappa} t^{-\kappa} + U_\infty, \\
a(h,t) &= a_0 h^{n+\kappa} t^{-\kappa}, \\
x(t) &= \frac{u_0}{1 - \kappa} h^{n+\kappa}.
\end{aligned}
\end{align*}
\]  

(6.1)

In these expressions \(u_0\), \(a_0\), \(U_\infty\), \(x_0\) and \(n\) are constants. The constant \(U_\infty\) is the terminal velocity of all the fluid elements when \(t \to \infty\), \(x_0\) is the point where the entire mass is concentrated at \(t = 0\) and \(n\) is the degree of homogeneity. Transferring to a reference frame moving with the constant speed \(U_\infty\), and origin in \(x_0\), the formulae (6.1) simplify after the required Galilei transformation to

\[
\begin{align*}
\begin{aligned}
u &= u_0 h^{n+\kappa} t^{-\kappa}, \\
a &= a_0 h^{n+\kappa} t^{-\kappa}, \\
x t^{-1 - \kappa} &= \frac{u_0}{1 - \kappa} h^{n+\kappa}.
\end{aligned}
\end{align*}
\]  

(6.2)

The constants \(u_0\) and \(a_0\) finally are related by

\[
a_0 = \pm \frac{1}{1 - \kappa} \sqrt{\frac{\kappa(1 + \kappa) (n + \kappa)}{n + 1 + \kappa} u_0,}
\]

(6.3)

where the constant \(\lambda\) is defined by (see LR-258, p. 17)
\[ \lambda = \sqrt{\frac{\kappa (1 + \kappa) (n + \kappa)}{n + 1 + \kappa}}. \tag{6.4} \]

It was found that not all values of \( n \) were admissible and that

\[ -(1 + \kappa) \leq n \leq -\kappa, \tag{6.5} \]

was a "forbidden interval". For values of \( n \) in this interval the pressure \( p \) and the specific volume \( V \) obtained opposite signs, which is physically unacceptable. As a consequence the solutions separate naturally in two classes, those with \( n > -\kappa \), and those with \( n < -(1 + \kappa) \).

It was found that for \( n > -\kappa \) the constant \( u_o \) had to be taken positive and the + sign in (6.3) has to be selected. For \( n < -(1 + \kappa) \) one has to take \( u_o < 0 \) and the – sign in (6.3). Also it was found that the interval \( 0 < h < +\infty \) corresponds with \( 0 < x < +\infty \) for \( n > -\kappa \) and with \( -\infty < x < 0 \) for \( n < -(1 + \kappa) \).

It was possible to extend the solutions to negative values of \( h \), in such a way that the interval \( -\infty < h < 0 \) corresponds with flows obtained from the previous one's i.e. (6.2) by reflection with respect to the line \( x = 0 \) in the \( x,t \)-plane and reversing the direction of the velocity at the same time. Introducing \( h^* = -h \) \(( -\infty < h < 0)\) these solutions may be written

\[ u = -u_o \left( h^* \right)^{n+\kappa} t^{-\kappa}, \]

\[ a = a_o \left( h^* \right)^{n+\kappa} t^{-\kappa}, \tag{6.6} \]

\[ -x t^{-(1-\kappa)} = \frac{u_o}{1 - \kappa} \left( h^* \right)^{n+\kappa}. \]

To relate the expressions of the present report with those of Report LR-258 we have from (3.6)

\[ u = (1 - \kappa) \frac{x}{t} = (1 - \kappa) t^{-\kappa} \psi, \]
\[ a = \pm t^{-\kappa} \sqrt{f(\psi)} , \]  
\[ \psi = x t^{-(1-\kappa)} , \]  
while (6.2) may be rewritten
\[ u = u_o h^{n+\kappa} t^{-\kappa} = (1 - \kappa) \frac{x}{t} , \]  
\[ a = (1 - \kappa) \frac{a_o}{u_o} \frac{x}{t} = \pm \lambda \frac{x}{t} . \]  
Similar expressions can be obtained from (6.6).
The expressions for \( u \) in terms of \( x, t \) agree, while one also obtains
\[ (1 - \kappa) \psi = u_o h^{n+\kappa} . \]  
Comparing the expressions for the speed of sound it follows that
\[ f(\psi) = (1 - \kappa)^2 \left( \frac{a_o}{u_o} \right)^2 \psi^2 = \lambda^2 \psi^2 . \]  
Other useful relations valid for the homogeneous solutions of LR-258 are
\[ u - u_o \frac{a}{a_o} a = 0 \text{ if } 0 \leq h \leq +\infty , \]  
\[ u + u_o \frac{a}{a_o} a = 0 \text{ if } -\infty \leq h < 0 . \]  
These relations appearing on p. 10, respectively p. 34, of Report LR-258 can be obtained here from (6.2) respectively (6.6) and also by substitution of (6.10) into (3.17) yielding
\[ u^2 - \left( \frac{u_o}{a_o} \right)^2 a^2 = \left( u + u_o \frac{a}{a_o} a \right) \left( u - u_o \frac{a}{a_o} a \right) = 0 . \]  
Substitution of (6.10) into the compatibility condition (3.13) yields
\[
\frac{1}{c_p} S'(\psi) = \frac{2(\lambda^2 - \kappa^2)}{\lambda^2} \frac{1}{\psi} = \frac{2N}{\psi}, \tag{6.13}
\]

where a new constant \( N \) has been introduced by

\[
N = \frac{\lambda^2 - \kappa^2}{\lambda^2} = \frac{n}{(1 + \kappa)(n + \kappa)}. \tag{6.14}
\]

One may verify that the relation \( N, n \) is such that

\[
-\infty < n < -(1 + \kappa), \quad \frac{1}{1 + \kappa} < N < 1, \\
-\kappa < n < 0, \quad -\infty < N < 0, \tag{6.15}
\]
\[
0 < n < +\infty, \quad 0 < N < \frac{1}{1 + \kappa},
\]

and so \( N \) never exceeds +1 for all admissible values of \( n \).

Integration of (6.13) now yields

\[
\frac{1}{c_p} S(\psi) = 2N \ln \psi + \frac{1}{c_p} S(1), \tag{6.16}
\]

with \( S(1) \) an integration constant. Employing (5.2) this may be re-written

\[
b(\psi) = b(1) \psi^{2N}. \tag{6.17}
\]

The characteristics of the homogeneous flows have been considered in Report LR-258. Here they may be deduced in terms of \( \psi, t \) from (3.20) yielding

\[
t \frac{d\psi}{dt} = \lambda \psi, \quad t \frac{d\psi}{dt} = -\lambda \psi, \tag{6.18}
\]

and upon integration
\[ \psi = r \cdot t^\lambda, \quad \psi = s \cdot t^{-\lambda}. \quad (6.19) \]

To conclude the discussion of the homogeneous solutions we consider the matching of two flows of different degree of homogeneity. It was shown in LR-258 that for flows with \( 0 < h < +\infty \) matching could be achieved provided both flows have either \( n < -\kappa \) or \( n < -(1 + \kappa) \).

Considering the correspondence with the \( x,t \)-plane this should cause no surprise since a flow with \( n > -\kappa \) is mapped in the half plane \( x > 0 \), while a flow with \( n < -(1 + \kappa) \) is mapped in the half plane \( x < 0 \).

Also the velocities have opposite directions. It was the problem of matching a flow with \( n > -\kappa \) and a flow with \( n < -(1 + \kappa) \) which led to the extension of the homogeneous flows to negative values of \( h \) (section 11 of [1]).

We consider here the matching of a flow (1) with \( h \) negative, and \( n_1 < -(1 + \kappa) \) to a flow (2) with \( h \) positive and \( n_2 > -\kappa \). From the conditions stated about signs and corresponding domains it follows that \( u_{(1)}^{(1)} < 0, u_{(2)}^{(2)} > 0 \) and \( x > 0, \psi > 0 \).

Written out in detail we have for flow (1)

\[ u_{(1)}^{(1)} = -u_{(1)}^{(1)} (h^*)^{n_1+\kappa} t^{-\kappa} = (1 - \kappa) \frac{x}{t} = (1 - \kappa) \psi t^{-\kappa}, \]

\[ a_{(1)}^{(1)} = a_{(1)}^{(1)} (h^*)^{n_1+\kappa} t^{-\kappa} = -\frac{a_{(1)}^{(1)}}{u_{(1)}^{(1)}} u = -(1 - \kappa) \frac{a_{(1)}^{(1)}}{u_{(1)}^{(1)}} \frac{x}{t} = \]

\[ = \lambda^{(1)} \frac{x}{t} = \lambda^{(1)} \psi t^{-\kappa}, \quad (6.20) \]

\[ -x \frac{t^{-(1-\kappa)}}{t^{(1-\kappa)}} = \frac{u_{(1)}^{(1)}}{1-\kappa} (h^*)^{n_1+\kappa} = -\psi, \]

\[ \frac{1}{c_p} S_{(1)}^{(1)}(\psi) = 2N_{(1)} \ln \psi + \frac{1}{c_p} S_{(1)}^{(1)}(1). \]

and
\[ \lambda (1) = \sqrt{\frac{\kappa (1 + \kappa) (n_1 + \kappa)}{n_1 + 1 + \kappa}}, \quad N (1) = \frac{n_1}{(1 + \kappa) (n_1 + \kappa)}. \]  \hspace{1cm} (6.21)

In the same way we have for flow (2)

\[ u (2) = u_o (2) h^{n_2 + \kappa} t^{-\kappa} = (1 - \kappa) \frac{x}{t} = (1 - \kappa) \psi t^{-\kappa}, \]

\[ a (2) = a_o (2) h^{n_2 + \kappa} t^{-\kappa} = \frac{a_o (2)}{u_o (2)} u (2) = (1 - \kappa) \frac{a_o (2)}{u_o (2)} \frac{x}{t} = \lambda (2) \frac{x}{t} = \lambda (2) \psi t^{-\kappa}, \]  \hspace{1cm} (6.22)

\[ \frac{1}{c_p} S (2) (\psi) = 2N (2) \ln \psi + \frac{1}{c_p} S (2) (1), \]

and

\[ \lambda (2) = \sqrt{\frac{\kappa (1 + \kappa) (n_2 + \kappa)}{n_2 + 1 + \kappa}}, \quad N (2) = \frac{n_2}{(1 + \kappa) (n_2 + \kappa)}. \]  \hspace{1cm} (6.23)

The two flows are joined along a common particle path \( \psi = \psi_o \) and the continuity of velocity and pressure requires

\[ u (1) (\psi_o, t) = u (2) (\psi_o, t), \]  \hspace{1cm} (6.24)

\[ p (1) (\psi_o, t) = p (2) (\psi_o, t). \]

Since the two flows have the same velocity distributions and particle paths the first condition in (6.24) is satisfied for arbitrary \( \psi = \psi_o \), without further restrictions. One may deduce

\[ (1 - \kappa) \psi_o = -u_o (1) (n^*) n_1^{1 + \kappa} = u_o (2) h^{n_2 + \kappa}, \]  \hspace{1cm} (6.25)

which relates the Lagrangian mass coordinates at the common particle path.
Employing the relation (2.17) for the pressure one may write the second condition of (6.24) in the form

\[ 2(N^{(2)} - N^{(1)}) \ln \psi_o + 2 \ln \frac{\lambda^{(1)}}{\lambda^{(2)}} + \frac{1}{c_p} \left\{ S^{(2)}(1) - S^{(1)}(1) \right\} = 0. \quad (6.26) \]

If the two flows are completely determined, including \( S^{(2)}(1) \) and \( S^{(1)}(1) \), this equation determines the path \( \psi_o \) where matching may occur. If \( \psi_o \) is prescribed together with flow (1), a relation for the parameters results, which determines flow (2).

Employing now (2.16) for \( \rho \) we deduce

\[ \frac{\gamma - 1}{\gamma} \ln \frac{\rho^{(1)}(\psi_o,t)}{\rho^{(2)}(\psi_o,t)} = \frac{2}{\gamma} \ln \frac{\lambda^{(1)}}{\lambda^{(2)}} + 2(N_2 - N_1) \ln \psi_o + \]

\[ + \frac{1}{c_p} \left\{ S^{(2)}(1) - S^{(1)}(1) \right\}, \quad (6.27) \]

and with (6.26) and the definitions of \( \lambda^{(1)} \), \( \lambda^{(2)} \) this may be written

\[ \ln \frac{\rho^{(2)}(\psi_o,t)}{\rho^{(2)}(\psi_o,t)} = 2 \ln \frac{\lambda^{(1)}}{\lambda^{(2)}} = \ln \left( \frac{n_1 + \kappa}{n_1 + 1 + \kappa} \cdot \frac{n_2 + 1 + \kappa}{n_2 + \kappa} \right), \quad (6.28) \]

indicating that there is a jump in the density across the path \( \psi = \psi_o \). The same holds true for the entropy and the temperature. One obtains by means of (6.26)

\[ \frac{1}{c_p} \left\{ S^{(2)}(\psi_o) - S^{(1)}(\psi_o) \right\} = -2 \ln \frac{\lambda^{(1)}}{\lambda^{(2)}}, \quad (6.29) \]

\[ \frac{T^{(2)}(\psi_o,t)}{T^{(1)}(\psi_o,t)} = \left( \frac{a^{(2)}(\psi_o,t)}{a^{(1)}(\psi_o,t)} \right)^2 = \left( \frac{\lambda^{(2)}}{\lambda^{(1)}} \right)^2 \quad (6.30) \]

The discontinuities in density, entropy and temperature make it seem unlikely that these composite flows will appear as a consequence of some spontaneous process. Special efforts will be required to realize them.
In the following sections slight extensions of the homogeneous flows are considered which allow composite flows where all the parameters are continuous at the common path $\psi = \psi_0$. 
7. THE GENERALIZED FLOWS

The homogeneous flows of the preceding section have the entropy distribution

\[ \frac{1}{c_p} S(\psi) = 2N \ln \psi + \frac{1}{c_p} S(1). \]  

(6.16)

This expression was obtained from the compatibility condition (3.13) after \( f(\psi) \) had been found.

Considering the compatibility condition (3.13) as a differential equation for \( f(\psi) \) we may reverse the process and taking the entropy distribution (6.16) we determine the most general form of \( f(\psi) \).

Writing (6.16) in the alternative form

\[ b(\psi) = b(1) \psi^{2N}, \]  

(7.1)

by using (5.1) one obtains from (5.4)

\[ f(\psi) = \frac{\kappa^2}{1 - N} \psi^2 + c \psi^{2N} = \lambda^2 \psi^2 + c \psi^{2N}, \]  

(7.2)

where \( C \) is an arbitrary constant. For \( C = 0 \) one returns to the homogeneous flows of the preceding section. The term with \( C \) represents the general solution of the differential equation (3.13) when the R.H.S. is put equal to zero.

The solutions (7.2) are related to the homogeneous solutions of section 6 in the same way as the Von Mises' solutions of section 4 to the isentropic simple waves.

To verify directly, that indeed the expression for \( \alpha^2 \) with \( f(\psi) \) given by (7.2) will yield a proper solution, it is convenient to return to the equations of motion (2.8)-(2.10). Assuming that a solution of the system
has been found in the form

\[ u = u_1(x,t), \quad a^2 = a_1^2(x,t), \quad S = S_1(x,t), \tag{7.3} \]

one may ask whether a solution can be found with the same velocity- and entropy-distribution but a different form for the square of the sound speed (which is practically the temperature). Written out we ask for the conditions on \( a_2^2(x,t) \) in order that

\[ u = u_1(x,t), \quad a^2 = a_1^2(x,t) + a_2^2(x,t), \quad S = S_1(x,t), \tag{7.4} \]

be again a solution of the system (2.8)-(2.10).

The conditions are easily found to be

\[ \frac{\partial}{\partial t} (a_2^2) + u_1 \frac{\partial}{\partial x} (a_2^2) + (\gamma - 1) a_2 \frac{\partial u_1}{\partial x} = 0, \tag{7.5} \]

\[ \frac{1}{\gamma - 1} \frac{\partial}{\partial x} (a_2^2) - \frac{a_2^2}{\gamma R} \frac{\partial S_1}{\partial x} = 0. \tag{7.6} \]

The equation (7.6) can be reduced to

\[ \frac{\partial}{\partial x} \left\{ \ln a_2^2 - \frac{S_1}{c_p} \right\} = 0, \tag{7.7} \]

and so the expression in braces can be only a function of the time yielding

\[ \ln a_2^2 - \frac{S_1}{c_p} = \ln f(t), \quad a_2^2 = f(t) \exp \left( \frac{S_1}{c_p} \right). \tag{7.8} \]

Comparing (6.16) and the second term on the R.H.S. in (7.2) in relation to (7.8) one observes that things are in order. Rewriting (7.5) in the form

\[ \frac{\partial}{\partial t} (\ln a_2^2) + u_1 \frac{\partial}{\partial x} (\ln a_2^2) + (\gamma - 1) \frac{\partial u_1}{\partial x} = 0, \tag{7.9} \]

using (7.8) and equation (2.10) for the entropy it is found that
\[
\frac{\partial}{\partial t} \left\{ \ln f(t) \right\} + (\gamma - 1) \frac{\partial u}{\partial x} = 0
\]  
(7.10)

yielding

\[
f(t) = \text{const.} \cdot t^{-2\kappa}
\]  
(7.11)

which agrees with the form of \(a^2\) obtained from (7.2).

In the remainder of this section two things will be considered. First a flow of the form (7.2) will be matched to a homentropic centered simple wave, which represents a homogeneous flow of degree \(n = 0\).

For the resulting generalized flow we shall as second point work out some details of the flow. In particular the characteristics will be calculated. The homentropic centered simple wave is determined by

\[
u = (1 - \kappa) \frac{x}{t} = (1 - \kappa) t^{-\kappa} \psi,
\]

\[
a^2 = t^{-2\kappa} f(\psi), \quad f(\psi) = \kappa^2 \psi^2,
\]  
(7.12)

\[
S(\psi) = S_h = \text{const.}
\]

The generalized flow is determined by

\[
u = (1 - \kappa) \frac{x}{t} = (1 - \kappa) t^{-\kappa} \psi,
\]

\[
a^2 = t^{-2\kappa} f(\psi), \quad f(\psi) = \lambda^2 \psi^2 + C \psi^{2N},
\]  
(7.13)

\[
\frac{1}{c_p} S(\psi) = 2N \ln \psi + \frac{1}{c_p} S(1).
\]

It is assumed that the homentropic centred simple wave is present for the interval \(0 < \psi < \psi_o\), while at \(\psi = \psi_o\) the flow passes into the the generalized non-homentropic flow of (7.13), which is present for
\[ \psi > \psi_0. \]

Since the two flows have the same velocity distributions and particle paths the continuity of the velocity along \( \psi = \psi_0 \) is assured without further constraints. Considering the expressions (2.16) and (2.17) it will be clear that continuity of pressure and density along \( \psi = \psi_0 \) is assured if the entropy \( S \) and the square of the speed of sound \( a^2 \) are continuous. Due to the form of \( a^2 \) in (7.12) and (7.13) this requires continuity of \( f(\psi) \) at \( \psi = \psi_0 \). Written out the requirements are

\[
\frac{1}{c_p} S_h = 2N \ln \psi_o + \frac{1}{c_p} S(1), \quad (7.14)
\]

\[
k^2 \psi_o^2 = \lambda^2 \psi_o^2 + C \psi_o^{2N}. \quad (7.15)
\]

The first relation determines \( \psi_o \) if \( S_h, S(1) \) and \( N \) are given. If \( \psi_o \) is prescribed some other combination of the parameters \( S_h, S(1) \) and \( N(n) \) can be deduced.

The condition (7.15) determines \( C \) in terms of \( k, \lambda \) and \( \psi_o \). One finds

\[
C = (k^2 - \lambda^2) \psi_o^{2(1-N)} = -\lambda^2 N \psi_o^{2(1-N)}. \quad (7.15)
\]

Substitution of (7.16) into \( f(\psi) \) in (7.2) or (7.13) then yields

\[
f(\psi) = \lambda^2 \psi_o^2 \left\{ \left(\frac{\psi}{\psi_o}\right)^2 - N \left(\frac{\psi}{\psi_o}\right)^{2N} \right\}. \quad (7.17)
\]

Selecting the constants in the way just discussed, so that (7.14) and (7.15) are satisfied the continuity at \( \psi = \psi_o \) of pressure, density, temperature and entropy is now assured.

To complete the discussion the characteristics in the generalized flow with \( \psi > \psi_o \) will be computed. The equations for the characteristics in the \( \psi,t \)-plane are
\[ t \frac{d\psi}{dt} = + \sqrt{f(\psi)}, \quad t \frac{d\psi}{dt} = - \sqrt{f(\psi)}, \quad (3.20) \]

Substitution of (7.17) yields

\[ t \frac{d\psi}{dt} = + \lambda \frac{\psi}{\psi_0} \sqrt{1 - N \left( \frac{\psi_0}{\psi} \right)^2 (1 - N)} \quad (7.18) \]

Let

\[ q = \frac{\psi}{\psi_0}, \quad s = q^{1 - N} \quad (7.19) \]

and substituting \( s \) into (7.18) one obtains

\[ \sqrt{s^2 - N} \frac{ds}{\sqrt{s^2 - N}} = \pm \frac{\kappa^2}{\lambda} \cdot \frac{dt}{t} \quad (7.20) \]

and integration yields

\[ s + \sqrt{s^2 - N} = r^+ t^\kappa^2 \lambda^{-1} \]

\[ s + \sqrt{s^2 - N} = s^+ t^{-\kappa^2 \lambda^{-1}} \quad (7.21) \]

with \( r^+ \) and \( s^+ \) constant along the respective characteristics. Since

\[ (s + \sqrt{s^2 - N})(s - \sqrt{s^2 - N}) = N \quad (7.23) \]

one also deduces

\[ s - \sqrt{s^2 - N} = \frac{N}{r^+} t^{-\kappa^2 \lambda^{-1}} = r^- t^{-\kappa^2 \lambda^{-1}} \quad (7.24) \]

\[ s - \sqrt{s^2 - N} = \frac{N}{s^+} t^{\kappa^2 \lambda^{-1}} = s^- t^{\kappa^2 \lambda^{-1}} \quad (7.25) \]

where
\[ r^+ r^- = s^+ s^- = N. \] (7.26)

Addition of the corresponding equations in (7.21), (7.22), (7.24) and (7.25) yields

\[ 2s = 2 \left( \frac{\psi}{\psi_0} \right)^{1-N} = r^+ t^{\kappa_\lambda - 1} + \frac{N}{r^+} t^{-\kappa_\lambda - 1}, \] (7.27)

\[ 2s = 2 \left( \frac{\psi}{\psi_0} \right)^{1-N} = s^+ t^{-\kappa_\lambda - 1} + \frac{N}{s^+} t^{\kappa_\lambda - 1}, \] (7.28)

Since \( N, \lambda \) and \( \kappa \) are related by (6.14) one may study in detail the difference between the characteristics (7.27) and (7.28) obtained here and those obtained for the homogeneous flow in (6.19). Substitution of \( \psi \) in terms of \( x \) and \( t \) into (7.27) and (7.28) the characteristics may be written in \( x \) and \( t \).

It will be clear that other matching problems can also be considered. For example one may consider matching a homentropic Von Mises flow to a non-homentropic homogeneous flow etc.

This will not now be pursued.
8. SOME RELATED INVESTIGATIONS

The investigation reported here is closely connected to the study of homogeneous solutions reported in [1] and summarized in [5].

Since investigations on non-homentropic flows are not very numerous it was possible and seemed worth while in [1] to single out some related investigations (see section 12 of [1]). The papers discussed there are equally relevant for the present investigation. In particular the Von Mises' solutions appearing in reference [3] may be recalled.

Another investigation to be added here is by Ardavan-Rhad [6]. His investigation is concerned with the overtaking of a shock-wave by a rarefaction wave as originally studied by Friedrichs [7]. To start with he constructs a non-homentropic flow, which has the following form. If \( \sigma \) denotes

\[
\sigma = -\frac{1}{\gamma R} (S - S_0),
\]  

(8.1)

with \( S \) the entropy it reads

\[
x = [u + a h(\sigma)] t ,
\]

(8.2)

\[
t = a \frac{1}{S} f(\sigma) ,
\]

(8.3)

\[
u = \frac{2}{\gamma - 1} a h(\sigma) + \text{const} ,
\]

(8.4)

\[
f = (h^2 - 1) - \frac{1}{2 \kappa} \exp \left[ \frac{\gamma + 1}{2} \left( \frac{d\sigma}{h^2 - 1} \right) \right] ,
\]

(8.5)

with \( h \) and \( f \) representing arbitrary functions of \( \sigma \), and obtained by generalizing the simple wave of the homentropic flow to the form (8.4).

We verify that the solution (8.2)-(8.5) practically coincides with the solution (3.14), obtained here. From (8.3) one obtains
\begin{equation}
a = t^{-\kappa} f^{\kappa}(\sigma), \quad a^2 = t^{-2\kappa} f^{2\kappa}(\sigma), \tag{8.6}
\end{equation}

and comparison with (3.14) yields

\begin{equation}
f(\psi) = f^{2\kappa}(\sigma). \tag{8.7}
\end{equation}

For the flows in this report the relation (3.17) applies throughout the flow and considering the half plane with \( u > 0 \) one obtains from (3.17)

\begin{equation}
u - \frac{2}{\gamma - 1} a \frac{\kappa \psi}{\sqrt{f(\psi)}} = 0. \tag{8.8}
\end{equation}

Assuming that the constant in (8.4) can be removed by an appropriate Galilei transformation we have on comparison of (8.4) and (8.8)

\begin{equation}
h(\sigma) = \frac{\kappa \psi}{\sqrt{f(\psi)}}. \tag{8.9}
\end{equation}

To check this relation we rewrite (8.2) in the form

\begin{equation}
u = \frac{x}{t} - a h(\sigma), \tag{8.10}
\end{equation}

and deduce from (3.16)

\begin{equation}
u = \frac{x}{t} - \kappa \cdot \frac{x}{t} = \frac{x}{t} - \kappa \psi t^{-\kappa} = \frac{x}{t} - \kappa \psi \frac{a}{\sqrt{f(\psi)}}. \tag{8.11}
\end{equation}

Finally we rewrite (8.5) in the form

\begin{equation}
d \left[ \ln \left( f^{2\kappa}(\sigma) (h^2 - 1) \right) \right] = (\gamma - 1) \frac{d\sigma}{h^2 - 1}. \tag{8.12}
\end{equation}

Considering \( S \) and \( \sigma \) as functions of our particle coordinate \( \psi \), so that from (8.1) it follows

\begin{equation}
\frac{d\sigma}{d\psi} = - \frac{1}{\gamma R} S'(\psi), \tag{8.13}
\end{equation}
one easily converts (8.12) to the form

\[ \frac{h^2 - 1}{f(\psi)} \frac{df(\psi)}{d\psi} + \frac{d}{d\psi} (h^2 - 1) = - \frac{1}{c_p} S'(\psi). \]  

(8.14)

Upon substitution of (8.9) this relation then takes the form of the compatibility condition (3.13) completing the verification.
9. CONCLUDING REMARKS

The present investigation was undertaken to consider, whether the discontinuities, which appeared when two homogeneous flows, studied in [1] were matched along a common particle path, could be removed. It turns out that this is indeed possible without affecting the velocity field and the particle paths. The generalization of the homogeneous flows needed to achieve this is analogous to the solutions presented by Von Mises in [3], which generalize homentropic centered simple waves. Since these Von Mises' flows have hardly ever been referred to this seems of some interest.

The analysis depends crucially on the form of the velocity distribution \( u \) being linear in \( x \), and on using \( a^2 \) as a variable in preference of the speed of sound itself. Also employing a reference frame moving with the constant terminal velocity provides some definite simplification.

The solutions constructed are essentially the same as those presented by Ardavan-Rhad [6] who studied the problem of Friedrichs when a shock wave is overtaken by a rarefraction wave.

The solutions presented in this report appear not yet to have the generality needed to describe fully the simplest wave interactions, as for example the Friedrichs problem, and some further extension seems necessary.
REFERENCES


APPENDIX. THE VELOCITY FIELD WITH A = 1

Consider a rectilinear motion of an incompressible fluid, specified by the constraint that each fluid element perform a uniform motion. The only proper flow possible with this constraint is a flow where all fluid elements have the same velocity, since otherwise the continuity of the mass cannot be maintained.

If the fluid is a perfect gas and the same constraint is imposed, the same velocity distribution \( u = \text{const.} \) represents a possible solution. In particular if the gas is homentropic this flow is a special case of the motion, where the two Riemann invariants are constant throughout the entire flow.

However since a gas can expand, the flow \( u = \text{const.} \) is not the only possible flow which satisfies the constraint that each fluid element move uniformly. There is another possibility with the velocity distribution

\[
u = \frac{x}{t} = \eta.
\]

This is the special case of (3.1) with \( A = 1 \) and the non-essential constant \( C \) taken zero (non-essential since the equations of motion are Galilean invariant and a velocity \( u = C \) can always be superposed).

Since each fluid element moves with constant speed the particle acceleration has to vanish and so

\[
\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0,
\]

yielding

\[
u = \text{const.} \text{ for } \frac{dx}{dt} = u.
\]
It follows that lines $u = \text{const}$ are straight lines in the $x, t$-plane and the slope $\frac{dx}{dt}$ represents the constant velocity. Integration of (A1.3) yields

$$u = \frac{x - x_0(u)}{t - t_0(u)}, \quad (A1.4)$$

and selecting the centered solution with $x_0(u) = t_0(u) = 0$ one has obtained (A1.1). Since the differential equation (A1.3) is the equation for the particle paths the lines $\eta = \text{const}$ are at the same time the particle paths, and so one may take the Lagrangian particle coordinate $\psi$ equal to $\eta$.

Similar to the flows in section 3, the specific expansion rate is constant throughout the gas at each instant. Also from the momentum equation it follows in view of (A1.2) that the pressure $p$ is independent of $x$ and a function of $t$ only.

Substituting (A1.1) into the various equations of motion yields

$$t \frac{\partial \rho}{\partial t} + x \frac{\partial \rho}{\partial x} = -\rho,$$

$$t \frac{\partial T}{\partial t} + x \frac{\partial T}{\partial x} = -\gamma \frac{1}{T},$$

$$t \frac{\partial S}{\partial t} + x \frac{\partial S}{\partial x} = 0,$$  

indicating that $\rho, T$ and $S$ are functions homogeneous in $x$ and $t$ of degree $-1$, $-(\gamma - 1)$ respectively zero.

It follows that

$$\rho = t^{-1} \rho^*(\eta), \quad T = t^{-(\gamma-1)} T^*(\eta), \quad S = S(\eta), \quad (A1.6)$$

and since the pressure $p$ depends on $t$ only, one concludes from the equation of state (2.4) that
\( \rho^*(\eta) T^*(\eta) = D = \text{const}, \quad \rho = R \, D \, t^{-\gamma} \), \hspace{1cm} (A1.7)

Also the isentropic relation yields

\[ R \, D = \rho^*(\eta)^\gamma \exp \left\{ \frac{1}{c_v} \, S^*(\eta) \right\} . \] \hspace{1cm} (A1.8)

In particular for the homentropic case (A1.7) and (A1.8) reduce to

\[ S(\eta) = S_o = \text{const}, \quad \rho^*(\eta) = \rho_o = \text{const}, \quad T^*(\eta) = T_o = \text{const} . \] \hspace{1cm} (A1.9)

Since the internal energy \( U \), the enthalpy \( H \), and the square of the speed of sound \( a^2 \) are proportional to \( T \), it follows that these parameters decrease at a rate proportional to \( t^{-(\gamma-1)} \).

Since each fluid element retains its speed, there is no conversion of internal energy into kinetic energy. The only process taking place is the conversion of internal energy into work done at the front and the back of the expanding gas column. Denoting the right- and left-end of the gas column by subscripts \( r \) and \( l \) respectively, one easily finds from the energy equation, that at each instant

\[ (up)_r - (up)_l = - \int_{x_l}^{x_r} dx \, \rho \, \frac{ DU }{ Dt } . \] \hspace{1cm} (A1.10)

Substituting new variables \((\eta, t)\) in the integral, and keeping \( t \) fixed the integral is easily found to yield

\[ R \, D \, t^{-\gamma}(\eta_r - \eta_l) , \] \hspace{1cm} (A1.11)

corresponding with the L.H.S. in (A1.10) since \( \rho \) is independent of \( x \), and so \( \rho_l = \rho_r \).
Fig. 1. Physical x,t-plane ($x \geq 0, t \geq 0$).

Lines $\psi = \text{const}$ - particle paths
Lines $s^* = \text{const}$ - s-characteristics.
Straight lines $r^* = \text{const} = r$-characteristics.
$\gamma = \frac{5}{3}, \kappa = \frac{1}{4}$. 
Fig. 2. Lagrangian $\psi, t$-plane ($\psi \geq 0$, $t \geq 0$).

- Lines $\psi = \text{const}$ - particle paths.
- Lines $r^* = \text{const}$ - $r$-characteristics
- Lines $s^* = \text{const}$ - $s$-characteristics.

$\gamma = \frac{5}{3}$, $\kappa = \frac{1}{4}$. 