Bearing Capacity of Spatially Random Rock Masses Obeying Hoek-Brown Failure Criterion

Tamara AL-BITTAR[a], Michael MICHAEL[b] and Abdul-Hamid SOUBRA[b]
[a]*Lebanese University, Kobe, North Lebanon*
[b]*University of Nantes, France*

Abstract. In this paper, a probabilistic analysis is presented to compute the ultimate bearing capacity of a strip footing resting on a spatially varying rock mass. The rock is assumed to follow the generalized Hoek-Brown failure criterion. The uniaxial compressive strength of the intact rock (σ_c) was considered as a random field and the Geological Strength Index (*GSI*) was modeled as a random variable. The deterministic model was based on numerical simulations. The uncertainty propagation methodology employed in the analysis makes use of a non-intrusive approach to build up a sparse polynomial chaos expansion for the system response. The probabilistic numerical results were presented in the case of a weightless rock mass. The variability of the ultimate bearing capacity was found to decrease with the decrease in the autocorrelation distance. Sobol indices have shown that for the very large values of the autocorrelation distance, the variability of the ultimate bearing capacity is mainly due to σ_c; however, in the case of very small values of the autocorrelation distance, *GSI* is the most weighed variable.

Keywords. Rock mechanics, Hoek-Brown failure criterion, bearing capacity, probabilistic analysis, spatial variability, sparse polynomial chaos expansion.

1. Introduction

The analysis and design of a strip footing resting on a rock mass obeying Hoek-Brown (HB) failure criterion are generally based on deterministic approaches. In this paper, the behavior of a strip footing resting on a HB rock mass is studied using a probabilistic approach. Our aim is to determine the probabilistic ultimate bearing capacity of a strip footing resting on a spatially varying rock mass and subjected to a vertical load. The rock mass is assumed to follow the generalized HB failure criterion (Hoek and Brown, 1980; Hoek et al., 2002). This criterion is characterized by four parameters (i) the geological strength index (*GSI*), (ii) the uniaxial compressive strength of the intact rock (σ_c), (iii) the intact rock material constant (m_i) and (iv) the disturbance coefficient (*D*). Mao et al. (2012) have modeled these four parameters as random variables and have performed a probabilistic analysis of the ultimate bearing capacity of foundations. These authors have shown that the variability of the ultimate bearing capacity is mainly due to the uniaxial compressive strength of the intact rock (σ_c) and the geological strength index (*GSI*). Based on this study, only these two

parameters were considered herein as uncertain. The rock uniaxial compressive strength of the intact rock (σ_c) was considered as a log-normal random field characterized by a square exponential autocorrelation function. The Expansion Optimal Linear Estimation (EOLE) method proposed by Li and Der Kiureghian (1993) was used to dicretize this random field. As for *GSI*, this parameter characterizes the overall rock mass condition and it does not represent a precise physical parameter varying in space. Thus, it cannot be modeled as a random field and will be treated herein as a random variable.

As for the probabilistic method of analysis, the classical Monte Carlo Simulation (MCS) methodology is generally used when dealing with random fields (Ching et al., 2011). In spite of being a rigorous method, MCS requires a large number of calls to the deterministic model. This is not convenient in the case where a computationally-expensive (finite element or finite difference) deterministic model is used. This paper makes use of an efficient probabilistic approach which significantly reduces the number of calls of the deterministic model. The Sparse

Polynomial Chaos Expansion (SPCE) methodology is proposed in this regard.

The paper is organized as follows: The next section aims at presenting the deterministic model used for the computation of the ultimate bearing capacity. It is followed by a presentation of the probabilistic method. Finally, the probabilistic numerical results are presented and discussed. The paper ends with a conclusion.

2. Deterministic Model

The Hoek-Brown failure criterion only deals with intact rocks or heavily jointed rock masses. A heavily jointed rock mass involves sufficiently dense and randomly distributed joints so that in the scale of the problem, it can be regarded as an isotropic assembly of interlocking particles. Consequently, rocks with few discontinuities cannot be considered in this framework. The HB failure criterion can be described by the following equation (Hoek et al., 2002):

$$\sigma_1 - \sigma_3 = \sigma_c \left[m(\sigma_3 / \sigma_c) + s \right]^n \qquad (1)$$

where σ_1 and σ_3 are respectively the major and minor principal stresses at failure and σ_c is the uniaxial compressive stress of the rock at failure. The parameters m, s and n are given by the following equations:

$$m = m_i . e^{\left(\frac{GSI-100}{28-14D} \right)}, \quad s = e^{\left(\frac{GSI-100}{9-3D} \right)},$$
$$n = (1/2) + (1/6) \left[e^{-GSI/15} - e^{-20/3} \right] \qquad (2)$$

In these equations, the geological strength index (*GSI*) characterizes the quality of rock. On the other hand, the parameter m_i is the value of parameter m for intact rock and can be obtained from experimental tests. It varies from 4 for very fine weak rock like claystone to 33 for coarse igneous light-colored rock like granite. Finally, D is the disturbance coefficient. It varies from 0.0 for undisturbed in situ rock masses to 1.0 for very disturbed rock masses.

The deterministic model used to calculate the ultimate bearing capacity of a strip footing resting on a HB rock mass and subjected to a vertical load was based on the commercial numerical code FLAC[3D]. A footing of breadth B=1m is considered in the analysis. For this calculation, a rock mass of 20m wide by 6m deep was considered. The rock behavior was modeled by an elastic perfectly plastic model obeying the generalized HB failure criterion. The rigid strip footing was modeled by a prescribed uniform velocity for all the rock nodes in contact with the footing. A value of 10^{-6} m/time step was chosen for this velocity since a smaller value was proven to negligibly decrease the value of the ultimate bearing capacity. Notice that the ultimate bearing capacity was computed by integrating the reaction forces at the different nodes of the footing when reaching the steady state of plastic flow.

3. Probabilistic Analysis

First, the discretisation of the log-normal random field of σ_c is briefly presented. It is followed by a brief presentation of the SPCE methodology used for the probabilistic analysis.

3.1. Discretization of the Random Field

Consider a 2D non-isotropic log-normal random field Z^{LN} described by: (i) a log-normal marginal cumulative distribution function F_G, and (ii) a square exponential autocorrelation function ρ_Z^{LN} [(x, y), (x', y')] which gives the values of the correlation between two arbitrary points (x, y) and (x', y'). Notice that this function is given as follows:

$$\rho_z^{LN}[(x, y), (x', y')] = \exp \left(- \left(\frac{|x-x'|}{a_x} \right)^2 - \left(\frac{|y-y'|}{a_y} \right)^2 \right) \qquad (3)$$

where a_x and a_y are the autocorrelation distances along x and y respectively. The EOLE method proposed by Li and Der Kiureghian (1993) to discretize a random field is used herein. In this method, one should first define a stochastic grid composed of s grid points (or nodes) and determine the log-normal autocorrelation matrix $\sum_{\chi:\chi}^{LN}$ which gives the correlation between each grid point of the stochastic mesh and the other grid points of this mesh using Equation (3). The

log-normal autocorrelation matrix $\Sigma^{LN}_{\chi:\chi}$ should then be transformed into the Gaussian space using the Nataf transformation (Nataf, 1962). As a result, one obtains a Gaussian autocorrelation matrix $\Sigma^{G}_{\chi:\chi}$ that can be used to discretize the Gaussian random field Z as follows:

$$\tilde{Z}(x,\ y) \cong \mu_{\ln Z} + \sigma_{\ln Z} \sum_{j=1}^{N} \frac{\xi_j}{\sqrt{\lambda_j}}.\phi_j.\Omega \qquad (4)$$

where μ_{lnZ} and σ_{lnZ} are the mean and standard deviation values of the underlying normal distribution (i.e. ln(Z)); (λ_j, ϕ_j) are the eigenvalues and eigenvectors of the Gaussian autocorrelation matrix $\Sigma^{G}_{\chi:\chi}$; Σ^{G} ; Ω is the correlation vector between the value of the field at an arbitrary point (x, y) and its values at the different grid points; ξ_j (j=1, ..., N) is a vector of standard normal random variables; and N is the number of terms (expansion order) retained in EOLE method. This number N is obtained (i) by sorting the eigenvalues λ_j (j=1, …, s) in a descending order and (ii) by choosing the number N of eigenmodes that leads to a variance of the error which is smaller than a prescribed tolerance ε ($\varepsilon \approx 10\%$ in this paper). Notice that the variance of the error for EOLE is given by Li and Der Kiureghian (1993) as follows:

$$Var\left[\,Z(x,\ y) - \tilde{Z}(x,\ y)\,\right] = \sigma^2 - \sum_{j=1}^{N} \frac{1}{\lambda_j}\left(\left(\phi_j\right)^T \Omega\right)^2 \quad (5)$$

where $Z(x,\ y)$ and $\tilde{Z}(x,\ y)$ are respectively the exact and the approximate values of the random fields at a given point (x, y) and $\left(\phi_j\right)^T$ is the transpose of the eigenvector ϕ_j . Once the Gaussian random field is obtained, it should be transformed into the log-normal space by exponentiating the approximated Gaussian random field $\tilde{Z}(x,\ y)$ given by Equation (4).

3.2. Sparse Polynomial Chaos Expansion SPCE for the System Response

In this section, one first presents the polynomial chaos expansion (PCE) and then its extension, the sparse polynomial chaos expansion (SPCE). The polynomial chaos expansion methodology allows one to replace a complex deterministic model whose uncertain input parameters are random variables by a meta-model. Thus, the random system response may be easily calculated (when performing the probabilistic analysis by MCS) using a simple analytical equation. Within the PCE methodology, the system response Γ of a given model with M random variables can be expressed by a PCE as follows:

$$\Gamma_{PCE}(\xi) = \sum_{\beta=0}^{\infty} a_\beta \Psi_\beta(\xi) \cong \sum_{\beta=0}^{P-1} a_\beta \Psi_\beta(\xi) \qquad (6)$$

where P is the number of terms retained in the truncation scheme, $\xi = \left\{\xi_i\right\}_{i=1,\dots,M}$ is a vector of M independent standard random variables that represent the M random variables, a_β are unknown coefficients to be computed and Ψ_β are multivariate Hermite polynomials. These multivariate Hermite polynomials can be obtained from the product of one-dimensional Hermite polynomials as follows:

$$\Psi_\beta = \prod_{i=1}^{M} H_{\alpha_i}(\xi_i) \qquad (7)$$

where α_i (i=1, …, M) are a sequence of M non-negative integers and $H_{\alpha_i}(.)$ is the α_i^{th} one-dimensional Hermite polynomial. The coefficients a_β of the PCE are computed in this paper using a non-intrusive technique where the deterministic calculations are done using the finite difference software FLAC[3D] treated as a black box. The most used non-intrusive method is the regression approach (Blatman and Sudret, 2010). This method is used in the present work. In fact, for a PCE of order p, only the multivariate polynomials Ψ_β of degree less than or equal to p should be retained. This leads to a number P of the unknown PCE coefficients (see

Equation (6)) equal to $\dfrac{(M+p)!}{M!\,p!}$. It should be noticed that the number of the PCE coefficients to be computed grows dramatically with the size M of the input random vector and the PCE order p. When dealing with a random field as is the case in the present paper (and especially when considering small values of the autocorrelation distance), the discretization of the random field may lead to a significant number of random variables which makes the determination of the PCE coefficients unfeasible because of the significant increase in the number of calls of the deterministic model. To address such problem, the sparse polynomial chaos expansion developed by Blatman and Sudret 2010 is used herein. Indeed, Blatman and Sudret 2010 have shown that the number of significant terms in a PCE is relatively small since the multivariate polynomials Ψ_β corresponding to high-order interaction (i.e. those resulting from the multiplication of the H_{α_i} with increasing α_i values) are associated with very small values for the coefficients a_β. Based on this observation, these authors have proposed a so-called hyperbolic truncation scheme to determine the significant Ψ_β terms. This scheme suggests that the q-norm $\|\alpha\|_q$ of the retained Ψ_β term should be less than or equal to the order p of the PCE. The q-norm is given by:

$$\|\alpha\|_q = \left(\sum_{i=1}^{M}(\alpha_i)^q\right)^{1/q} \tag{8}$$

where q is a coefficient (0<q<1). In this formula, q can be chosen arbitrarily. Blatman and Sudret 2010 have shown that sufficient accuracy is obtained for $q \geq 0.5$.

The proposed SPCE methodology leads to a sparse polynomial chaos expansion that contains a small number of unknown coefficients which can be calculated from a reduced number of calls of the deterministic model with respect to the classical PCE methodology. Once the SPCE coefficients are determined, a global sensitivity analysis (GSA) based on Sobol indices can be easily performed. Notice that the first order Sobol index of a given random variable ξ_i (i=1,…, M) gives the contribution of this variable to the variability of the system response. The first order Sobol index is given by Salteli et al. 2000 as follows:

$$S(\xi_i) = \frac{Var\left[E\left(Y\mid\xi_i\right)\right]}{Var\left(Y\right)} \tag{9}$$

where Y is the system response, $E\left(Y\mid\xi_i\right)$ is the expectation of Y conditional on a fixed value of ξ_i, and *Var* denotes the variance. In the present paper, the system response is represented by a SPCE. Thus, by replacing Y in Equation (9) with the SPCE expression, one obtains the Sobol index as a function of the different terms of the SPCE (Sudret, 2008) as follows:

$$S(\xi_i) = \frac{\sum_{\beta\in I_i}(a_\beta)^2 E\left[(\Psi_\beta)^2\right]}{\sum_{\beta\in[I_1,I_2,…,I_M]}(a_\beta)^2 E\left[(\Psi_\beta)^2\right]} \tag{10}$$

where a_β are the obtained SPCE coefficients, Ψ_β are the multivariate Hermite polynomials, E[.] is the expectation operator, and $E\left[(\Psi_\beta)^2\right]$ is given by Sudret (2008) as follows:

$$E\left(\Psi_\beta^2\right) = \prod_{i=1}^{M}\alpha_i! \tag{11}$$

In this equation, α_i are the same sequence of M non-negative integers used in Eq (7). Notice finally that I_i which appears in the numerator of Eq (10) denotes the set of indices β for which the corresponding Ψ_β terms are only functions of the random variable ξ_i (i.e. they only contain the variable ξ_i), and I_i (i=1, …, M) in the denominator of the same equation regroup all the indices β for which the corresponding Ψ_β terms are functions of all the random variables ξ_i (i=1, …, M). In the present paper where both a random variable (*GSI*) and a random field (σ_c) are involved, the Sobol index of the random field (σ_c) is computed as the sum of the Sobol indices of the different variables that represent this field.

4. Numerical Results

The aim of this section is to present the probabilistic numerical results. The mean value and coefficient of variation of σ_c (referred to in this paper as reference values) were taken as follows: $\mu_{\sigma_c} = 10 MPa$, $COV_{\sigma_c} = 25\%$. On the other hand, the mean value and coefficient of variation of *GSI* are given as follows: $\mu_{GSI} = 25$, $COV_{GSI} = 10\%$. As for the autocorrelation distance (*a*) of the random field σ_c, the reference value adopted is 2m. Notice that the intact rock material constant (m_i) and the disturbance coefficient (*D*) were assumed to be deterministic. Their corresponding values were respectively $m_i = 8$ and D=0.3.

4.1. Sobol Indices

Figure 1 depicts the values of Sobol indices (for the reference case) as given by the obtained SPCE for (i) the random variable *GSI* and (ii) the 35 random variables representing the random field σ_c. The first random variable ξ_1 corresponds to *GSI* and its Sobol index was found to be equal to 0.66. However, the last 35 random variables [i.e. ξ_i for i=2, …, 36] are those corresponding to the σ_c random field. The sum of their Sobol indices gives the weight of the random field σ_c in the variability of the ultimate bearing capacity. This sum was found to be equal to 0.34.
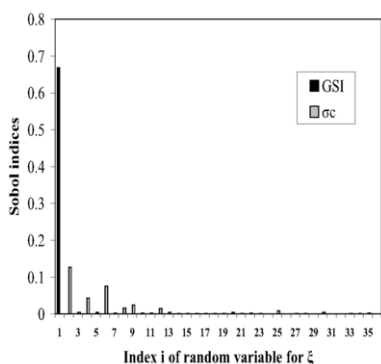

Figure 1. *Sobol indices of the random variable GSI [i.e. ξi(i=1)] and the random field σc [i.e. ξi(i=2, ..., 36)]*

Figure 1 shows that only six random variables (ξ_2, ξ_4, ξ_6, ξ_8, ξ_9, ξ_{12}) of the σ_c random field are the most influential (they involve 89% of the variance of σ_c). This can be explained by the fact that the ultimate bearing capacity is a quantity that depends on the average distribution of the spatially-varying rock property over the entire domain and it is therefore quite insensitive to small-scale fluctuations of σ_c. Notice that the first eigenmodes provide the average distribution of σ_c over the rock domain; however, the remaining eigenmodes give the small scale fluctuations around this average distribution.

4.2. Effect of the Autocorrelation Distance

Table 1 presents the first two statistical moments of the ultimate bearing capacity for different values of the autocorrelation distance of σ_c. This table shows that the variability of the bearing capacity decreases when the autocorrelation distance decreases. For the very large values of the autocorrelation distance (i.e. *a* =100m), the coefficient of variation of the ultimate bearing capacity tends to a constant maximal value which is the value corresponding to the case of a random variable. This is because in this case the value of σ_c (which is uniform within a given realization) varies within a large range corresponding to the values obtained from the PDF of σ_c. The decrease in the autocorrelation distance of the random field σ_c from infinity to a finite value (moderate or small where $a \leq 10m$) limits the correlation between the values of this random field (in a given simulation) to a finite zone which leads to several zones with different values of σ_c over the entire rock mass. This means that in a single simulation, one obtains a set of weak and strong zones. The position of these zones may change from simulation to another one. This leads to a decrease in the variability of the ultimate bearing capacity because the presence of the rock mass heterogeneity will produce a somewhat close global behavior of the footing because of the averaging phenomenon over the zone of possible failure mechanism.

Table 1. Effect of the autocorrelation distance a on the statistical moments (μ, σ,) of the ultimate bearing capacity

Autocorrelation distance a[m]	$\mu \, [MPa]$	$\sigma \, [MPa]$	COV%
0.5	1.486	0.288	19.35
1	1.459	0.301	20.88
2	1.462	0.342	23.40
5	1.484	0.408	27.49
10	1.512	0.450	29.78
50	1.557	0.486	31.41
100	1.560	0.488	31.41
Random variable	1.560	0.488	31.41

Table 1 shows that the probabilistic mean value of the ultimate bearing capacity presents a minimum when the autocorrelation distance is nearly equal to the footing breadth B (i.e. when a =1m). Notice that the minimal probabilistic mean was also observed by Fenton and Griffiths (2003) when considering the bearing capacity of foundations resting on a soil mass.

Finally, Figure 2 shows the effect of the autocorrelation distance on the Sobol indices of the random field σ_c and the random variable *GSI*. The results show that for very large values of the autocorrelation distance, the variability of the ultimate bearing capacity is mainly due to σ_c. Similar results were obtained by Mao et al. (2012) where the uncertain parameters were modeled by random variables. On the other hand, Figure 2 shows that the decrease in the autocorrelation distance of σ_c reduces its weight in the variability of the ultimate bearing capacity and increases the weight of *GSI*. This result can be explained by the fact that the small values of the autocorrelation distance increase the rock mass heterogeneity (i.e. one obtains a set of weak and strong zones) which will produce a somewhat close global behavior of the footing from simulation to another one because of the averaging phenomenon over the zone of possible failure mechanism. The expected decrease in the variability of the ultimate bearing capacity with the decrease in the autocorrelation distance of σ_c is reflected herein by a decrease in the weight of σ_c in the variability of this response.
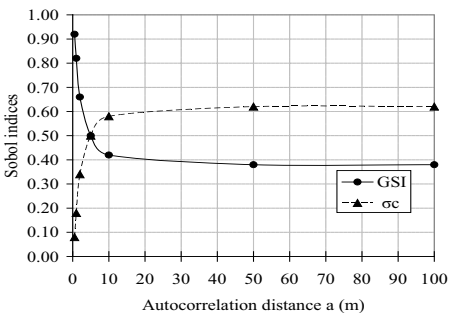

Figure 2. Influence of the autocorrelation distance a on the Sobol indices of *GSI* and σ_c

5. Conclusions

A probabilistic analysis of a vertically loaded strip footing resting on a spatially varying rock mass has been performed to compute the ultimate bearing capacity. The rock was assumed to follow the generalized Hoek-Brown failure criterion. The uniaxial compressive strength of the intact rock (σ_c) was considered as a random field and the Geological Strength Index (*GSI*) was modeled as a random variable. The uncertainty propagation methodology employed in the analysis makes use of a non-intrusive approach to build up a sparse polynomial chaos expansion for the system response. Also, a global sensitivity analysis based on Sobol indices was performed. The deterministic model was based on numerical simulations using FLAC^{3D} software. The probabilistic numerical results were presented in the case of a weightless rock mass. These results have shown that for the very large values of the autocorrelation distance, the variability of the ultimate bearing capacity is mainly due to σ_c; however, in the case of very small values of the autocorrelation distance, *GSI* is the most weighed variable. With a decrease in the autocorrelation distance of the uniaxial compressive strength of the intact rock (σ_c), a smaller variability of the ultimate bearing capacity was obtained; however, the probabilistic mean value of the ultimate bearing capacity presents a minimum. This minimum was obtained when the autocorrelation distance is equal to the footing breadth B.

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