Numerical Pricing of Equity Barrier Options with Local Volatility

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MASTER OF SCIENCE in APPLIED MATHEMATICS

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MEREL ISABEL STOUT

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“Numerical Pricing of Equity Barrier Options with Local Volatility.”

Merel Isabel Stout

Delft University of Technology

Responsible Professor
Prof.dr.ir. C.W. Oosterlee

Other members of the thesis committee
Dr. C. Kraaiikamp
Dr. V. Leijdekker
Dr. R. Pietersz

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Abstract

This thesis is about the pricing of equity barrier options under local volatility. We study Dupire’s non-parametric local volatility model, which can be defined in terms of call option prices or in terms of implied volatilities. No-arbitrage conditions are derived for the call option surface, and equivalent conditions for the total variance surface. Dupire’s model is implemented based on a Stochastic Volatility Inspired parameterization of the implied volatility surface. The Dupire-SVI model can accurately reproduce the implied volatility smile. Furthermore, we show how to incorporate dividends into our local volatility model. Lastly, we discuss option pricing by solving forward-backward stochastic differential equations with the BCOS method, a Fourier cosine expansion method. We propose a novel pricing method for barrier options by applying the BCOS method to reflected forward-backward stochastic differential equations. We compare the BCOS results to those of the Crank-Nicolson scheme.

Keywords: equity derivatives, barrier options, local volatility, Dupire, no-arbitrage conditions, Stochastic Volatility Inspired model, dividends, Fourier cosine expansion method, (reflected) forward-backward stochastic differential equations.
# Contents

1 Introduction ................................. 5
   1.1 Structure of this thesis ...................... 6

2 Preliminaries ............................... 8
   2.1 Martingale pricing approach .................. 9
   2.2 Pricing PDEs and the Black-Scholes model ..... 10
   2.3 Feynman-Kac theorem .......................... 11
   2.4 Barrier options ............................... 12
      2.4.1 Product description ........................... 12
      2.4.2 Analytical solutions ....................... 13
      2.4.3 Barrier options and market prices ........... 14

3 Local Volatility Models .................... 15
   3.1 Dupire’s non-parametric local volatility model .. 15
   3.2 Local volatility in terms of implied volatilities .. 17
   3.3 Barrier options and local volatility ............... 18
   3.4 Constant Elasticity of Variance model .......... 19
   3.5 Conditions for an arbitrage-free volatility surface ... 20
      3.5.1 Calendar spread arbitrage .................... 21
      3.5.2 Call spread arbitrage and butterfly arbitrage .... 22

4 Dividends Modeling ......................... 24
   4.1 Basic concepts in dividends modeling .......... 24
      4.1.1 Volatility and dividends .................... 25
   4.2 Stock price dynamics with cash dividends .......... 26
      4.2.1 Stock price dynamics in the clean space ......... 28
   4.3 Pricing in the clean space ..................... 28
      4.3.1 No-arbitrage conditions ..................... 29
      4.3.2 Implied volatility and local volatility in the clean space .. 30
   4.4 Proportional dividends ....................... 30

5 Building the Volatility Surface .......... 32
   5.1 Heston model ............................... 32
      5.1.1 Implied volatilities and Heston parameters ....... 32
5.1.2 Heston model and arbitrage ................................................. 33
5.1.3 Dupire local volatility on Heston call option price surface ............... 34
5.2 Numerical results of the Dupire-Heston model .................................. 35
5.3 Stochastic Volatility Inspired model ........................................... 36
5.3.1 Implied volatilities and SVI parameters ...................................... 36
5.3.2 SVI model and arbitrage ....................................................... 37
5.3.3 Dupire local volatility on SVI volatility surface .............................. 40
5.4 Numerical results of the Dupire-SVI model ..................................... 41
5.4.1 Constructing the volatility surface .......................................... 42
5.4.2 Reproducing the market prices of vanillas .................................. 44
5.4.3 Pricing vanillas in the presence of dividends .................................. 46

6 Numerical Methods for Pricing Barrier Options .................................. 48
6.1 Finite difference method for option pricing ..................................... 48
6.1.1 Crank-Nicolson scheme ......................................................... 49
6.1.2 Time-dependent barrier options .............................................. 52
6.2 Numerical Fourier method for option pricing .................................... 53
6.2.1 Cosine Series Expansion (COS) method .................................... 53
6.2.2 COS method for barrier options ............................................. 58
6.3 Forward-backward SDEs and option pricing ..................................... 60
6.3.1 Forward-backward SDEs ......................................................... 60
6.3.2 FBSDEs and vanilla options .................................................... 62
6.3.3 Reflected FBSDEs and barrier options ....................................... 63
6.3.4 Discretization of FBSDEs ......................................................... 64
6.4 Numerical Fourier method for FBSDEs ......................................... 66
6.4.1 Backward stochastic differential equation COS (BCOS) method ............ 66
6.4.2 BCOS method for barrier options ........................................... 68
6.5 Numerical results of the BCOS method with local volatility ................. 71
6.5.1 BCOS method for Constant Elasticity of Variance .......................... 71
6.5.2 BCOS method for Dupire local volatility .................................... 72

7 Conclusion ................................................................. 75

A Derivations and proofs ......................................................... 80
A.1 Alternative representation Dupire’s local volatility ............................. 80
A.2 Partial derivatives SVI ............................................................. 83
A.3 Corollary 5.1 SVI parameter restrictions ....................................... 85
A.4 Lemma 6.8 Discrete characteristic function ..................................... 86

B Calibrated SVI parameters ....................................................... 87
Chapter 1

Introduction

This research is about the pricing of equity barrier options with local volatility, where we focus on three main subjects: the construction of a local volatility model, a dividends model, and numerical pricing techniques for barrier options. Before we define our research objectives in more detail, we introduce some financial terminology that will be used throughout this thesis.

An equity derivative is a financial product whose price depends on the value of a stock of a company. Having a stock is a synonym for owning a share of the company. The stock price path \( S_t \) is the underlying process for the pricing of equity derivatives. The volatility is a measure for the variation of the underlying stock price over time. Furthermore, we need to take dividends into account. A dividend is a share of the profit of a company which is paid out to the shareholders, which depends on the performance of the company. The dividend amount and timing may differ, but are announced in advance.

A common derivative is a call option, which gives the buyer the right, but not the obligation, to purchase the underlying stock at a certain expiration time \( T \) for a certain strike price \( K \). A put option, on the other hand, gives the buyer the right to sell the underlying stock for such a prefixed strike price \( K \). Hence, the payoff value of both a call, as well as a put option at expiration are given by:

\[
\text{Call}(S_T, T) = \max\{S_T - K, 0\}; \\
\text{Put}(S_T, T) = \max\{K - S_T, 0\},
\]

respectively, where \( S_T \) is the value of the stock at time \( T \). Because exercising options is not mandatory, their value is always non-negative. If the strike value equals the stock value at time \( t \) we say the option is at-the-money (ATM). Derivatives which can only be exercised at time \( T \) and depend solely on the final payoff are often called European options, or in the case of the call and put options, vanillas.

Barrier options are path-dependent options; their value depends on the path of the underlying stock price. The payoff depends on whether or not a certain barrier has been triggered. There are various barrier options, which we discuss in more detail in Section 2.4. Due to the path-dependency of barrier options, it is important to correctly model the volatility of the underlying stock over the entire lifetime of the option.

The value of an option at the time of expiry is given by the final payoff. But we would like to know the option value at an earlier time as well. We define the fair price of a derivative at time \( 0 \leq t \leq T \) as the expected discounted payoff under the risk-neutral measure. A risk-neutral probability can be interpreted as the risk-adjusted probability of future outcomes. The Feynman-Kac Theorem [23] describes the link between the solution of a partial differential equations (PDEs) and the computation of expected values.

In 1973, Black and Scholes [1] introduced the famous Black-Scholes model for derivative pricing. They derived a pricing PDE that defines the evolution of the option price over time. In the traditional Black-Scholes framework a constant volatility value is assumed. The implied volatility of an option is the volatility value that reproduces the market price in the Black-Scholes model. Analytical solutions for the price of barrier options exist under constant volatility.

If we compute the implied volatilities from market data, we find different volatilities for different maturities and strike values. The Black-Scholes model is not able to reproduce the skewness of the implied volatility curve that is present in the market data. We refer to the volatility term structure as the
dependency of the volatility on the time to maturity for a given strike. The dependency of the volatility on the strike for a fixed time to maturity is often called the volatility smile. The volatilities at various strikes and maturities constitute the (implied) volatility surface.

In this thesis, we study the local volatility model, where the volatility is defined as a deterministic function of the stock price and time. In 1994 Dupire [11] introduced a non-parametric local volatility model, where the volatility is fully implied by market data. The model can be calibrated exactly to any given set of arbitrage-free implied volatilities. However, building this arbitrage-free implied volatility surface turns out to be a challenging task. Furthermore, under a local volatility model, numerical pricing methods are required for the computation of barrier options values.

1.1 Structure of this thesis

The structure of this thesis is as follows: we start out in Chapter 2 with a short overview of preliminary mathematical knowledge. We present mathematical definitions and results of, among others, arbitrage, risk-neutral measure, pricing PDEs and the Fundamental Theorem of Asset Pricing. The notations that will be used throughout this thesis are introduced and we discuss barrier options in more detail. Chapter 2 is especially recommended for people who are not familiar with the mathematical framework of (equity) derivative pricing.

Chapter 3 to Chapter 6 constitute the main part of the thesis, in which we focus on our research objectives:

- Incorporate dividend payments into the local volatility model.
- Price barrier options under the local volatility model.
- Test whether or not the Backward Stochastic Differential Equation Cosine Series Expansion method [25] is applicable to our local volatility barrier model.

In Chapter 3 we discuss the non-parametric local volatility model of Dupire [11]. We derive an expression for the local volatility model in terms of European call option prices and in terms of implied volatilities. We give a short note on barrier options and the use of local volatility models. We also discuss the Constant Elasticity of Variance model, a parametric local volatility model. In Section 3.5, we show that arbitrage can be defined in terms of calendar spread arbitrage in the expiry direction, and in butterfly and call spread arbitrage in the strike direction. We derive conditions to guarantee the absence of arbitrage in the call option price surface, and equivalent conditions for the total variance surface.

The second research question is how to incorporate dividend payments into the local volatility model. We discuss how to model dividend payments in Chapter 4. We start by discussing basic concepts in dividends modeling. Buchler [2] introduced a model which is consistent with the assumption of cash dividends. The net present value of the future dividends is taken out of the stock price process $S_t$, such that a martingale process $\tilde{S}_t$ remains. Pricing under $\tilde{S}_t$, instead of under $S_t$, allows us to model the local volatility in a ‘clean space’ without dividends. We show that we can price barrier options in the clean space as well, but the barrier level becomes time-dependent. An extension of the model to proportional dividends is given in Section 4.4.

The local volatility model should be in line with the market view on European option prices. To define the local volatility term as derived in Chapter 3, we need the full implied volatility surface, but market prices are not available for all strikes and expiries. We discuss in Chapter 5 how we can parameterize the implied volatility surface by the SVI model. However, the SVI model might not be arbitrage-free. Hence, we study the Heston model [23] as a theoretical case, since the Heston model is arbitrage-free by definition.

For both the Heston model and the SVI model we give a model description and study how the model parameters impact the implied volatility smile. We discuss no-arbitrage conditions and how suitable the models are as input to Dupire’s local volatility model. By reproducing the implied volatility surface,
we can test whether or not our local volatility model is consistent with the vanilla prices. The local volatility model is tested for a stock without dividend payments, and for a dividend paying stock.

The third research objective is to price barrier options under the local volatility model. For this, numerical pricing methods are required. We start Chapter 6 with the Crank-Nicolson scheme, a finite difference method for solving the pricing PDE and present numerical examples under constant volatility. Next, we discuss a Fourier method for derivative pricing, the Cosine Series Expansions (COS) method. Introduced by Oosterlee and Fang in 2008 [13], the COS method is a Fourier pricing technique for European options based on the cosine series expansion of the underlying density function. We discuss the COS extension to path-dependent options such as (discrete) barriers [14]. Numerical examples of pricing with the COS method are presented as well.

In Section 6.3, we discuss forward-backward stochastic differential equations and their use in option pricing. We propose to price barrier options by using reflected forward-backward stochastic differential equations. The BCOS method [24] is used to solve the forward-backward stochastic differential equations. We illustrate the BCOS method with several numerical examples under geometric Brownian motion.

Lastly, for our final research objective, we want to test whether or not we can use the BCOS method with local volatility for the pricing of barrier options. In Chapter 6.5 we present the numerical results of the BCOS method for pricing equity vanillas and barrier options with local volatility. We compare the BCOS results under Dupire local volatility to the results of the Crank-Nicolson scheme.
Chapter 2

Preliminaries

This chapter gives a short overview of definitions and theorems in stochastic calculus and financial mathematics. We define the mathematical framework and notations that will be used throughout this thesis. Also some financial interpretations of these mathematical concepts are given. For a more elaborate insight into these, and other, concepts the reader is referred to the book of Shreve [27].

Let \( W_t \), for \( 0 \leq t \leq T \), be a Brownian motion on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \( \mathbb{P} \) is the real world probability measure, and let \( \mathcal{F}_t \), for \( 0 \leq t \leq T \), be a filtration for this Brownian motion. Here \( T \) is a fixed final time. We consider a stock price process whose differential is modelled by a diffusion process:

\[
dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t, \quad 0 \leq t \leq T.
\]

We need to introduce some definitions and theorems from stochastic calculus to be able to define the price of a financial derivative based on this underlying stock price process.

**Definition 2.1** (Conditional expectation). If \( X \) is an integrable random variable in the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), and if \( \mathcal{G} \subset \mathcal{F} \), we say that \( Y \) is a conditional expectation of \( X \) with respect to \( \mathcal{G} \) if:

- \( Y \) is measurable w.r.t. \( \mathcal{G} \),
- \( \mathbb{E}[|Y|] < \infty \),
- \( \int_{\mathcal{G}} Yd\mathbb{P} = \int_{\mathcal{G}}Xd\mathbb{P}, \quad \forall G \in \mathcal{G} \).

We write: \( Y = \mathbb{E}[X|\mathcal{G}] \).

**Lemma 2.1.** Some useful properties of conditional expectations:

- **Conditional Jensen’s inequality:** if \( \phi : \mathbb{R} \to \mathbb{R} \) is a convex function, and \( \phi(X) \in L^1 \), then:
  \[
  \phi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\phi(X)|\mathcal{G}].
  \]

- **Tower property:** if \( \mathcal{H} \subset \mathcal{G} \) is a sub-\( \sigma \)-algebra, then:
  \[
  \mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}].
  \]

- **Taking out what is known:** if \( Z \) is \( \mathcal{G} \)-measurable and \( XZ \in L^1 \), then:
  \[
  \mathbb{E}[XZ|\mathcal{G}] = Z\mathbb{E}[X|\mathcal{G}] \quad \text{a.s.}
  \]

**Proof.** We refer to [29], Chapter 9, for a proof of the properties.

**Definition 2.2** (Adapted). If the random variables \( \{X_t : 0 \leq t \leq T\} \) are such that each \( X_t \) is \( \mathcal{F}_t \)-measurable, then we say that \( X_t \) is adapted to the filtration.

**Definition 2.3** (Predictable). A continuous-time stochastic process \( \{X_t\}_{t \geq 0} \) is predictable if \( X_t \), considered as a mapping from \( \Omega \times \mathbb{R}_+ \), is measurable with respect to the \( \sigma \)-algebra generated by all left-continuous adapted processes.
2.1 Martingale Pricing Approach

A predictable process, also called previsible, is a stochastic process whose values are known, in a sense, just in advance of time, whereas adapted processes, are not known in advance. But they do not depend on future events either, if $X_t$ is adapted to $\mathcal{F}_t$, all information that is required to define the value of $X_t$ is available at time $t$:

$$E[X_t|\mathcal{F}_t] = X_t \quad \text{for } X_t \text{ measurable.}$$

**Definition 2.4** (Discount process). We assume the interest rate curve $\{r_t : 0 \leq t \leq T\}$ to be a deterministic process. We define the discount process as,

$$P(t,T) = e^{-\int_t^T r_s ds},$$

throughout this thesis, we often use a constant interest rate $r$.

2.1 Martingale pricing approach

**Definition 2.5** (Martingale). A stochastic process $\{X_t : 0 \leq t \leq T\}$ is a martingale w.r.t. the filtration $\mathcal{F}$, if $X$ is a $\mathcal{F}$-adapted process such that $X_t$ is integrable for all $t \in [0,T]$ and such that:

$$E[X_t|\mathcal{F}_s] = X_s \quad \text{a.s., } 0 \leq s \leq t \leq T.$$

Now, we can give a proper definition of the risk-neutral measure and the way it is often used in derivative pricing.

**Definition 2.6** (Risk-neutral measure). A probability measure $Q$ is said to be risk-neutral if:

- $Q$ and $P$ are equivalent (i.e., for every $A \in \mathcal{F}$, $Q(A) = 0$ if and only if $P(A) = 0$), where $P$ is the real world probability measure.
- under $Q$ the discounted stock price $P(t,T)S_t$ is a martingale.

An important concept in finance is replicating your portfolio by taking positions in other financial products. If the replicating portfolio has the same payoff as the original portfolio, the possibility of making a loss can be eliminated, this technique is called hedging. A market model is complete when each derivative security can be hedged.

An arbitrage is a trading strategy, so that one starts with zero capital and at some later time $T$ is sure not to have lost money and furthermore has a positive probability of to have made money.

**Definition 2.7** (Arbitrage). An arbitrage opportunity emerges, when a portfolio value process $X_t$ satisfying $X_0 = 0$, also satisfies:

$$P(X_T \geq 0) = 1, \quad P(X_T > 0) > 0.$$

Such an opportunity exists if and only if there is a way to start with positive capital $X_0 > 0$ and to beat the money market account:

$$P \left( X_T \geq \frac{X_0}{P(0,T)} \right) = 1, \quad P \left( X_T > \frac{X_0}{P(0,T)} \right) > 0.$$

**Theorem 2.2** (Fundamental Theorems of Asset Pricing).

- If a market model has a risk-neutral probability measure, then it does not admit arbitrage.
- A market model that has a risk-neutral probability measure is complete if and only if the risk-neutral measure is unique.

**Proof.** We refer to [27] for a proof of the Fundamental Theorems of Asset Pricing.
Lemma 2.3 (Risk-neutral pricing formula). From the Fundamental Theorem of Asset Pricing it follows that the price at time $t$ of any contingent claim with payoff $V(S_t, T)$ at time $T$ is given by:

$$V(S_t, T) = \mathbb{E}_Q [P(t, T)V(S_T, T)|\mathcal{F}(t)], \quad P(T, T) = 1.$$  

where the expectation is taken under the risk neutral measure $\mathbb{Q}$.

Proof. We refer to [27] for a proof.

2.2 Pricing PDEs and the Black-Scholes model

Another way of pricing derivatives is by taking a closer look at the dynamics of the underlying stock process.

Theorem 2.4 (Itô’s formula). Let $f(x, t)$ be a function for which the partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ are defined and continuous. Define the diffusion process:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \quad 0 \leq t \leq T.$$  

Then,

$$f(X_T, T) = f(X_0, 0) + \int_0^T \frac{\partial f(X_t, t)}{\partial t} dt + \int_0^T \frac{\partial f(X_t, t)}{\partial x} dX_t + \frac{1}{2} \int_0^T \frac{\partial^2 f(X_t, t)}{\partial x^2} dX_t dX_t$$

$$+ \int_0^T \sigma(X_t, t) \frac{\partial f(X_t, t)}{\partial x} dW_t.$$  

In differential notation we have:

$$df = \left( \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2(X_t, t) \frac{\partial^2 f}{\partial x^2} \right) dt + \mu(X_t, t) \frac{\partial f}{\partial x} dW_t.$$  

Proof. For a proof of Itô’s formula we refer to Shreve [27].

A common way of modeling the dynamics of $S_t$ under the risk-neutral measure is by a geometric Brownian motion:

$$dS_t = rS_t dt + \sigma(S_t, t)S_t dW_t,$$

where $r$ is the interest rate, and $\sigma(S_t, t)$ the volatility term. The differential of a derivative security $V(S_t, t)$ whose value depends on $S_t$ and $t$, can be derived using Itô’s formula (Theorem 2.4):

$$dV = \left( \frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2(S_t, t)S_t^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma(S_t, t)S_t \frac{\partial V}{\partial S} dW_t.$$  

In finance, the sensitivities of an option price with respect to its variables are called Greeks. For example, the first and second partial derivative of $V$ with respect to the stock price $S$ are called the option’s delta and gamma, respectively. The partial derivative with respect to the time is called theta and vega is the derivative with respect to the volatility. By so-called delta-hedging, and using the no-arbitrage assumption, the well-known Black-Scholes pricing PDE can be derived:

Lemma 2.5 (Black-Scholes PDE). The Black-Scholes pricing partial differential equation describes the value $V$ of the option over time:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2(S_t, t)S_t^2 \frac{\partial^2 V}{\partial S^2} + rS_t \frac{\partial V}{\partial S} - rV = 0.$$  

Proof. For a proof we refer to [27].
Once the Black-Scholes PDE, with boundary and terminal conditions, is derived for a derivative, the PDE can be solved to find the option value. For constant volatility term $\sigma$ and interest rate $r$ the Black-Scholes PDE has an analytic solution for European options. The terminal and boundary conditions for a call option are given by:

$$C(S_t, T) = \max\{S_T - K, 0\}; \quad C(0, t) = 0, \quad \text{for } 0 \leq t \leq T; \quad C(S_t, t) \approx S_t \quad \text{for large } S_t,$$

where $C(\cdot, \cdot)$ denotes the call option price. The solution of the Black-Scholes pricing PDE for a call option price at time $t = 0$ is given by [19]:

$$C(S_t, t) = S_t N(d_1) - Ke^{-r(T-t)}N(d_2), \quad (2.1)$$

with

$$d_1 = \frac{\log \left( \frac{S_t}{K} \right) + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}; \quad d_2 = d_1 - \sigma \sqrt{T-t}.$$

The implied volatility for a given strike and maturity $\sigma_{BS}(K, T)$ is the value for $\sigma$ that when inserted into the Black-Scholes formula (2.1) returns the market price of a European option for that strike and maturity. The observed implied volatility of the market prices is not a constant, as it varies for different strike values and expiries. In many financial markets we see an implied volatility smile, meaning higher implied volatilities for extreme strike values and low implied volatilities for strike values close to the spot price. The Black-Scholes formula does not have a closed form solution of its inverse. Thus the implied volatility needs to be computed using a root-finding technique.

**Definition 2.8** (Total variance). The total variance is the implied volatility squared, times the time to maturity, and is denoted by

$$\omega(K, T) := \sigma_{BS}^2(K, T)T.$$

**Definition 2.9** (log-moneyness). The log-moneyness of an option on a stock $S_t$, with strike $K$ and expiry $T$, is defined as:

$$x(K, T) := \log \left( \frac{K}{F(0, T)} \right),$$

where $F(t, T)$ is the price of a forward contract:

$$F(t, T) := \frac{S_t}{P(t, T)}; \quad F(0, T) := \frac{S_0}{P(0, T)}$$

which is the price agreed on at time $t$, for buying the stock at time $T$. We often abbreviate: $F(0, t) := F_t$.

**Remark 2.1.** The volatility surface is often parameterized by its total variance. The SVI model, that we discuss in Section 5.3, is a parameterization of $\omega$ as a function of the log-moneyness $x$.

### 2.3 Feynman-Kac theorem

We have seen derivative pricing by two different methods, by solving a pricing PDE and by computing an expectation. The Feynman-Kac formula establishes a link between parabolic partial differential equations and stochastic processes.

**Theorem 2.6** (Feynman-Kac). Given a constant interest rate $r$, let $V(S_t, t)$ be a sufficiently differentiable function of time $t$ and stock price $S_t$. Suppose that $V(S_t, t)$ satisfies the following partial differential equation, with general drift $\mu(S_t, t)$, and volatility term $\sigma(S_t, t)$:

$$\frac{\partial V}{\partial t} + \mu(S_t, t) \frac{\partial V}{\partial S} + \frac{1}{2} \sigma(S_t, t)^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - rV = 0,$$

with final condition:

$$V(S_T, T) = g(S_T, T).$$
Then, the solution for $V(S_t, t)$ at time $0 \leq t < T$ is given by:

$$V(S_t, t) = \mathbb{E}[P(t, T)g(S_T, T)|\mathcal{F}_t],$$

where the expectation is taken under risk-neutral measure $Q$, with respect to a process $S_t$, defined by:

$$dS_t = rS_t dt + \sigma(S_t, t)S_t dW_t.$$

Proof. We refer to [23] for a proof of Feynman-Kac. \qed

2.4 Barrier options

This research focuses on the pricing of barrier options, which are (weakly) path dependent options; their value depends on the path of the underlying stock price. The dependence is weak, since we only need to know whether or not a certain barrier has been triggered. There are various barrier options, they can be specified by an ‘in’ or ‘out’ barrier. If the stock price triggers the barrier in an ‘out’ option, the option can not be exercised anymore, and becomes worthless. For an ‘in’ option, it is the other way around, if the barrier is not reached the option has no payoff. Wilmott discusses barrier options in detail in his book Derivatives [30].

2.4.1 Product description

Just as for vanillas we define the values of barrier options by their expected payoffs. For example, the price of a down-and-out call option is given by:

$$\mathbb{E}\left[(S_T - K)^+ 1_{\{\min_{0 \leq t \leq T} S_t > B\}}\right],$$

with barrier $B$. The value of a barrier option, here denoted by $V(S, t)$, also needs to satisfy the Black-Scholes pricing PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2(S_t, t)S_t^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0,$$

with appropriate boundary conditions:

- An ‘out’ barrier option is knocked out if the underlying asset path reaches the barrier. Hence, at the barrier $B$ the option value should satisfy the boundary condition:

$$V(B, t) = 0 \quad \text{for } t < T. \quad (2.2)$$

Therefore, the Black-Scholes PDE is solved over $S \in [0, B]$. Furthermore, a terminal condition is required, this condition is specified by the payoff received if the barrier is not triggered. For an down-and-out call option this is:

$$V(S, T) = \max\{S - K, 0\} 1_{\{S > B\}}. \quad (2.3)$$

- For ‘in’ options there is no payoff if the barrier is not reached before expiry:

$$V(S, T) = 0.$$

If the barrier is triggered, the option must have the same value as the corresponding vanilla option. For example, for a barrier option with call payoff, the value at the barrier should satisfy:

$$V(B, t) = C(B, t),$$

where $C(B, t)$ is the Black-Scholes price of an European call option.
The relation between the price of vanillas and barriers is illustrated in Figure 2.1. The figure on the left side shows the asset-path of the standard option, the one in the middle corresponds to the ‘up-and-out’, and ‘up-and-in’ options, and the right side figure illustrates the ‘down-and-out’, and ‘down-and-in’ options.

Consider the ‘up’ barrier, the asset path is split into two parts, the up-and-out option follows the same dynamics of the standard option, as long as the barrier is not triggered. Once the barrier is hit, this option becomes worthless, but the ‘up-and-in’ is switched on, and follows from there on the value of the standard option. Hence, the value of ‘in’ and ‘out’ barrier options add up to the vanilla option value:

\[ \text{in} + \text{out} = \text{vanilla}. \] (2.4)

Hence, barrier options are less expensive than their vanilla counterparts. As an example, suppose we purchase an up-and-out call option. In the case of moderate increases of the stock price (below the barrier), the payoff will be the same as for the ordinary call option. But we do not have to pay for all the potential of stock prices above the barrier.

**Remark 2.2.** In derivative pricing, instead of computing the value of an ‘in’ option explicitly, it is common to price these options as the difference between the vanilla option and the ‘out’ option (2.4).

### 2.4.2 Analytical solutions

For constant interest rate \( r \) and volatility \( \bar{\sigma} \) the Black-Scholes pricing PDE can be solved analytically for barrier options. The solutions for several types of barrier options can be found in Wilmott [30]. For example, for \( B < K \), the value of an down-and-out call option is given by:

\[ V(S,t) = C(S,t) - C(B^2, t) \left( \frac{S}{B} \right)^{1 - 2r/\bar{\sigma}^2}. \] (2.5)

One can confirm that this is indeed the solution by checking that \( V(S,t) \) satisfies the Black-Scholes equation and boundary conditions (2.2) and (2.3). From the relationship between in and out barrier options (2.4), we immediately obtain the value of the down-and-in call as well:

\[ V(S,t) = C(B^2, t) \left( \frac{S}{B} \right)^{1 - 2r/\bar{\sigma}^2}. \] (2.6)

However, under local volatility, finite difference methods are required to numerically compute the solutions of the pricing PDE. We could also compute the expected payoff of barrier options by recovering their probability density function using a Fourier-cosine expansion. We discuss numerical pricing methods in Chapter 6.
2.4.3 Barrier options and market prices

The numerically derived barrier option values should be in line with the available market prices on the same underlying stock price process. If we have a sufficient amount of listed market prices of barrier options we could use a parameterization or 3D interpolation (over the strike, expiry and barrier directions) to derive the value of barrier options with different variables. However, market prices of barrier options usually do not exist, barriers are mostly traded over-the-counter (OTC), which means directly between two parties.

We will model the volatility as a deterministic function of the stock price and time, i.e. a so-called local volatility model is chosen, which is in line with the market view. However, we can not calibrate the volatility term to the market, since there are no listed barrier option prices. But the underlying process should not depend on the derivative we want to price. Hence, we can use the listed market prices of call options to calibrate our local volatility term. Therefore, our first research objective is to reproduce vanilla prices with our local volatility model. If the model is in line with the market view, option pricing under the local volatility model should result in the same call option prices as seen in the market.

Due to the path dependency of barrier options it is important to have an accurate local volatility term for the entire life time of the option. In the next chapter we discuss how to derive the local volatility term from call option prices.
Chapter 3

Local Volatility Models

In this chapter, we derive Dupire’s non-parametric local volatility model in terms of European call option prices such that the prices of call options under the local volatility model are in line with the market view. Small errors in numerical approximations of option prices can lead to significant relative errors in the local volatility model. Therefore, we reformulate the local volatility in terms of implied volatilities. We also discuss the Constant Elasticity of Variance model, which is a parametric local volatility model.

Consistent pricing requires the volatility surface to be free of arbitrage. Absence of arbitrage can be characterized in terms of calendar spreads in the maturity direction and butterfly arbitrage in the strike direction. Some necessary conditions for an arbitrage-free surface are derived.

We consider non-dividend paying stocks in this chapter, while we discuss dividends modeling in Chapter 4.

3.1 Dupire’s non-parametric local volatility model

In 1994, Dupire [11] introduced a non-parametric local volatility model which allows the local volatility surface to be extracted from available call prices. We assume the call prices $C(K, T)$ for a given spot $S_0$ to be functions of the strike $K$ and time to maturity $T$.

**Definition 3.1 (Local volatility).** Given a call price surface $C(K, T)$, a local volatility model defines a deterministic function $\sigma(S_t, t)$ of the stock price and time, such that the diffusion process:

$$dS_t = rS_t dt + \sigma(S_t, t)S_t dW_t,$$

reproduces the given call prices as defined by the risk-neutral pricing formula (Lemma 2.3):

$$P(t, T)\mathbb{E}[(S_T - K)^+], = C(K, T).$$

According to Dupire [11], there is a unique risk-neutral diffusion process for the spot price, which is compatible with European option prices. Hence, if the spot price is a one-dimensional diffusion process, the model will be complete.

Let $f(S_t, t)$ denote the probability density function of the underlying stock process $S_t$ at time $t$. So we can rewrite equation (3.2):

$$C(K, T) = P(t, T)\mathbb{E}[(S_T - K)^+]$$

$$= P(t, T)\mathbb{E}[(S_T - K)1_{S_T> K}]$$

$$= P(t, T) \int_{K}^{\infty} (s - K) f(s, T) ds.$$
The partial derivatives of $C(K,T)$ with respect to time to maturity $T$ and strike $K$ are required, i.e.,

$$\frac{\partial C}{\partial K} = -P(t,T) \int_{K}^{\infty} f(s,T) ds,$$

$$\frac{\partial^2 C}{\partial K^2} = P(t,T) f(K,T),$$

where we use $\lim_{s \to \infty} f(s,T) = 0$. And for the time derivative:

$$\frac{\partial C}{\partial T} = \frac{\partial P(t,T)}{\partial T} \int_{K}^{\infty} (s-K) f(s,T) ds + P(t,T) \int_{K}^{\infty} (s-K) \frac{\partial f(s,T)}{\partial T} ds$$

$$= -rC(K,T) + P(t,T) \int_{K}^{\infty} (s-K) \frac{\partial f(s,T)}{\partial T} ds. \quad (3.4)$$

Furthermore, the probability density function $f(S_t,t)$ satisfies the Fokker-Planck partial differential equation (Kolmogorov forward equation).

**Theorem 3.1** (Fokker-Planck equation). The risk-neutral transition density $f(S_t,t)$ associated to the differential $(3.1)$ for $S_t$ satisfies the Fokker-Planck PDE, for $0 \leq t \leq T$:

$$\frac{\partial}{\partial t} f(S_t,t) + r \frac{\partial}{\partial S} (f(S_t,t)) - \frac{1}{2} \frac{\partial^2}{\partial S^2} \sigma^2(S_t,t) f(S_t,t) = 0,$$  

$$f(S_0,t_0) = \delta(S_0). \quad (3.5)$$

**Proof.** We refer to [27] for a proof. \ 

Now, substitute the Fokker-Planck expression $(3.5)$ at $t = T$ in equation $(3.4)$, which gives

$$\frac{\partial C}{\partial T} = -rC(K,T) + P(t,T) \int_{K}^{\infty} (s-K) \frac{\partial f(s,T)}{\partial T} ds$$

From the above equation we can derive the expression for local volatility by evaluating the integral. First, we split the integral in two parts:

$$I_1 := P(t,T) \int_{K}^{\infty} (s-K) \left( -r \frac{\partial}{\partial S} f(s,T) + \frac{1}{2} \frac{\partial^2}{\partial S^2} \sigma^2(s,T) s^2 f(s,T) \right) ds \quad (3.6)$$

These integrals can be solved by integration by parts, starting with $I_1$:

$$I_1 = -rP(t,T) \left\{ [(s-K) s f(s,T)]_{K}^{\infty} - \int_{K}^{\infty} sf(s,T) ds \right\}$$

$$= rP(t,T) \int_{K}^{\infty} sf(s,T) ds,$$

where we use $\lim_{s \to \infty} f(s,T) = 0$. Note that we can rewrite expression $(3.3)$:

$$C(K,T) = P(t,T) \int_{K}^{\infty} sf(s,T) ds - P(t,T) \int_{K}^{\infty} K f(s,T) ds$$

$$= P(t,T) \int_{K}^{\infty} sf(s,T) ds + K \frac{\partial C}{\partial K},$$

such that $I_1$ can be written as:

$$I_1 = r \left( C(K,T) - K \frac{\partial C}{\partial K} \right).$$
The second integral $I_2$ can be solved by integration by parts as well:

\[
I_2 = \frac{1}{2} P(t, T) \left\{ \left[ (s - K) \frac{\partial}{\partial s} \sigma^2(s, T)s^2 f(s, T) \right]_K^\infty - \int_K^\infty \frac{\partial}{\partial s} \sigma^2(s, T)s^2 f(s, T) ds \right\}
\]

\[
= -\frac{1}{2} P(t, T) \int_K^\infty \frac{\partial}{\partial s} \sigma^2(s, T)s^2 f(s, T) \, ds
\]

\[
= -\frac{1}{2} P(t, T) \left[ \sigma^2(s, T)s^2 f(s, T) \right]_K^\infty
\]

\[
= \frac{1}{2} P(t, T) \sigma^2(K, T) K^2 f(K, T).
\]

Use \( \frac{\partial^2 C}{\partial K^2} = P(t, T) f(K, T) \) to derive:

\[
I_2 = \frac{1}{2} \sigma^2(K, T) K^2 \frac{\partial^2 C}{\partial K^2}.
\]

Substituting $I_1$ and $I_2$ into equation (3.6) results in the so-called Dupire-equation:

\[
\frac{\partial C}{\partial T} = \frac{1}{2} \sigma^2(K, T) K^2 \frac{\partial^2 C}{\partial K^2} + r \left( C(K, T) - K \frac{\partial C}{\partial K} \right) - rC(K, T).
\]

Solving the Dupire equation for the volatility term \( \sigma(K, T) \) results in the following expression for Dupire’s local volatility:

\[
\sigma_{LV}^2(K, T) = \frac{\frac{\partial C}{\partial T} + rK \frac{\partial C}{\partial K}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}.
\]  \hspace{1cm} (3.7)

From here on we denote the local volatility term by \( \sigma_{LV} \). An alternative way of deriving Equation (3.7), is by writing the local volatility as a conditional expectation, both derivations are discussed in detail in [21].

### 3.2 Local volatility in terms of implied volatilities

Analytical solutions for (3.7) are usually not available. Given a grid of call prices, finite difference methods can be used to approximate the derivatives. However, the second partial derivative with respect to the strike in the denominator of (3.7) can cause difficulties for extreme strike values and for small expiries. For these \( K \) and \( T \) values it is harder to accurately price the option and to compute its partial derivatives. Since we divide by the second partial derivative, small errors in its numerical approximation can lead to substantial relative errors in the local volatility model.

If the call option price surface is parameterized, the derivatives may be computed analytically, however, we choose to reformulate the model in terms of the corresponding Black-Scholes implied volatilities, as explained in [23]. Recall the Black-Scholes price for a call option at \( t = 0 \) corresponding to the market price:

\[
C(K, T) = S_0 N(d_1) - Ke^{-rT} N(d_2),
\]  \hspace{1cm} (3.8)

with

\[
d_1 = \log \left( \frac{S_0}{K} \right) + \left( r + \frac{1}{2} \sigma_{BS}^2(K, T) \right) T \sigma_{BS}(K, T) \sqrt{T}, \quad d_2 = d_1 - \sigma_{BS}(K, T) \sqrt{T}.
\]

Given a spot price \( S_0 \), time to maturity \( T \) and implied volatility \( \sigma_{BS}(K, T) \), recall the definitions of the forward price \( F(t, T) \), log-moneyness \( x \) and the total (implied) variance \( \omega \):

\[
F(t, T) = \frac{S_t}{P(t, T)}, \quad x(K, T) = \log \left( \frac{K}{F(0, T)} \right), \quad \omega(K, T) = \sigma_{BS}^2(K, T) T.
\]
We propose a change of variables and define the call price in terms of the log-moneyness $x$ and total variance $\omega$:

$$c(x, \omega) = S_0 N(d_1) - S_0 e^x N(d_2), \quad (3.9)$$

where

$$d_1 = -\frac{x}{\sqrt{\omega}} + \frac{1}{2} \sqrt{\omega}, \quad d_2 = d_1 - \sqrt{\omega}.$$ 

In Dupire’s original formula (3.7) partial derivatives of $C(K, T)$ with respect to strike and expiry are used as input. We apply the chain rule for functions of two variables to express these derivatives in terms of $c(x, \omega)$:

$$\frac{\partial C}{\partial T} = -r \frac{\partial c}{\partial x} + \frac{\partial c}{\partial \omega} \frac{\partial \omega}{\partial T};$$

$$\frac{\partial C}{\partial K} = \frac{1}{K} \frac{\partial c}{\partial x} + \frac{\partial c}{\partial \omega} \frac{\partial \omega}{\partial K};$$

$$\frac{\partial^2 C}{\partial K^2} = 2K^2 \frac{\partial c}{\partial \omega} \left\{ \left( 1 - \frac{1}{2} K x \frac{\partial \omega}{\partial K} \right)^2 + \frac{1}{2} K \left( \frac{\partial \omega}{\partial K} + K \frac{\partial^2 \omega}{\partial K^2} \right) - \frac{1}{4} K^2 \left( \frac{1}{4} + \frac{1}{\omega} \right) \right\}. \quad (3.10)$$

Substituting these partial derivatives into (3.7) results in a local volatility in terms of total variance:

$$\sigma^2_{LV}(K, \omega) = \frac{\partial \omega}{\partial T} + rK \frac{\partial \omega}{\partial K} \left( 1 - \frac{1}{2} K x \frac{\partial \omega}{\partial K} \right)^2 + \frac{1}{2} K \left( \frac{\partial \omega}{\partial K} + K \frac{\partial^2 \omega}{\partial K^2} \right) - \frac{1}{4} K^2 \left( \frac{1}{4} + \frac{1}{\omega} \right). \quad (3.11)$$

We can also rewrite the local volatility in terms of the implied volatilities:

$$\sigma^2_{LV}(K, T) = \frac{\sigma_{BS}^2 + 2\sigma_{BS}T \left( \frac{\sigma_{BS}}{\sigma_{BS}T} + rK \frac{\sigma_{BS}}{\sigma_{BS}T} \right)}{\left( 1 - \frac{1}{2} K x \sigma_{BS} \frac{\sigma_{BS}}{\sigma_{BS}T} \right)^2 + K \sigma_{BS}T \left( \frac{\sigma_{BS}}{\sigma_{BS}T} - \frac{1}{4} K \sigma_{BS} \frac{\sigma_{BS}}{\sigma_{BS}T} \right)^2 + K \frac{\sigma_{BS}}{\sigma_{BS}T}}. \quad (3.11)$$

The complete derivation of equation (3.11) is given in Appendix A.1, the derivation of (3.10) is done in a similar way.

The local volatility term is now defined as a function of the implied volatility surface. Note that we only have a limited number of quoted market prices and their implied volatilities, so whenever a volatility for a different strike and maturity is required, some sort of interpolation has to be used in order to compute the new volatility. An alternative approach is to use a parametric form; build a parameterized volatility surface such that implied volatilities and corresponding market option prices match as closely as possible. The Heston model (Chapter 5.1) and Stochastic Volatility Inspired model (Chapter 5.3) are examples of such parameterizations.

**Remark 3.1.** In the case the implied volatility does not exhibit any dependence on the strike, we have:

$$\sigma^2_{LV}(T) = \sigma^2_{BS}(K, T)^2 + 2\sigma_{BS}(K, T)T \frac{\partial \sigma_{BS}}{\partial T}.$$ 

So, in the case of no strike-dependence, we can find the following expression for the Black-Scholes implied volatility in terms of the local volatility:

$$\sigma^2_{BS}(K, T) = \frac{1}{T} \int_0^T \sigma^2_{LV}(u) du,$$

Hence, when there is no strike dependence, the implied variance can be seen as the time average of the local variance.

### 3.3 Barrier options and local volatility

Local volatility models are often used in equity derivative pricing, since they are consistent with the implied volatility smile of vanillas. However, when it comes to barriers and other exotics, critical quants
use the different behavior of the Greeks, if compared to vanillas, as an argument to not use local volatility models.

Recall that the vega of an option indicates the option’s sensitivity to a change in the volatility. In the vicinity of the barrier, the vega of barrier options behaves differently than the vega of their vanilla counterparts. For example, for a European call option an increase in the volatility always results in an increase in option value, whereas for a knock-out barrier option an increase of the volatility may result in a higher probability of hitting the barrier. Figure 3.1 shows the (Black-Scholes) option price of a European call option and the price of an up-and-out call option plotted against the implied volatility. We have used the following parameter values:

\[ S_0 = 100, \quad K = 100, \quad r = 0.01, \quad T = 1 \text{ and } B = 130. \]

Figure 3.1: Relation between volatility and the vanilla call option and up-and-out call option price

The graph shows a near linear relation between the option price of the regular call option and the volatility. The option price of the up-and-out call option behaves in a non-linear way; for an increasing volatility the option exhibits a flat vega.

A local volatility model that is consistent with the implied volatility smile of vanilla options is therefore not necessarily consistent with the implied volatility of the barrier option. But using different volatility models for different options would result in a logical flaw, since the dynamics of the underlying stock price process should not depend on which option is priced. Therefore, we only require the local volatility model to be consistent with the volatility smile of vanilla options.

Another downside is that forward volatilities, implied by local volatility models, tend to become too flat, which leads to a mis-pricing of forward-starting barrier options [23]. The so-called stochastic volatility models, in which the volatility is modeled as a diffusion process, do not exhibit this problem.

Nonetheless, many market practitioners often prefer local volatility models over stochastic volatility for equity derivative pricing. First of all, the model is an easy derivation from the Black-Scholes framework, and secondly, local volatility can be fitted exactly to an arbitrage-free surface of listed market prices. Moreover, since the local volatility calibration is implicit, it comes almost for free in terms of computational time. For a stochastic volatility model, however, the parameters of the underlying process need to be calibrated to the data, which can be a time consuming, and potentially inaccurate exercise.

3.4 Constant Elasticity of Variance model

Dupire’s model is a non-parametric local volatility model, but many parametric models for the local volatility exist as well. One of these models is the Constant Elasticity of Variance (CEV) model [23]. The CEV model dynamics, under the risk-neutral measure, are defined by the following SDE:

\[ dS_t = r S_t dt + \sigma S_t^\lambda dW_t, \quad (3.12) \]

with \( \lambda \) a positive constant. Note, for \( \lambda = 1 \), the underlying process follows a geometric Brownian motion, for \( \lambda = 0 \) the stock price is normally distributed and for \( \lambda = 0.5 \) the process equals a Cox-Ingersoll-Ross
CHAPTER 3. LOCAL VOLATILITY MODELS

type process [20]. For the first two cases, the volatility is constant. But for \( \lambda \not\in \{0, 1\} \) the volatility of the process equals: \( \sigma(S, t) = \bar{\sigma}S^{\lambda - 1} \). In this case, the local volatility is given explicitly in terms of the stock price \( S_t \), but does not depend on time. Parametric models are calibrated to market data so that we find the values \( \bar{\sigma} \) and \( \lambda \) that match the market prices best. An advantage of Dupire’s model over parametric models is that no calibration step is required. We will use the CEV model in our numerical examples in Section 6.5.

Lemma 3.2. In the CEV model with constant interest rate \( r \), and volatility parameter \( \lambda \in \mathbb{R}^+ \), the prices of European call options at time \( t_0 = 0 \) are given by:

\[
C(K, T) = \begin{cases} 
S_0 \left[ 1 - F_{\chi^2(b+2,c)}(a) \right] - Ke^{-rT}F_{\chi^2(b,a)}(c), & \text{for } \lambda \in (0, 1); \\
S_0 \left[ 1 - F_{\chi^2(-b,a)}(c) \right] - Ke^{-rT}F_{\chi^2(-b,a)}(c), & \text{for } \lambda > 1,
\end{cases}
\]

(3.13)

where \( F_{\chi^2(a,b)}(c) = \mathbb{P}(\chi^2(a,b) < c) \) is the non-central chi squared cumulative distribution function with degrees of freedom parameter \( a \), non-centrality parameter \( b \), which is evaluated at value \( c \). The parameters here are given by:

\[
a = \frac{K^{2(1-\lambda)}}{(1-\lambda)^2 \bar{\sigma}^2 T}, \quad b = \frac{1}{1-\lambda}, \quad c = \left( S_0 e^{rT} \right)^2 \frac{2(1-\lambda)}{(1-\lambda)^2 \bar{\sigma}^2 T}.
\]

Proof. For a proof we refer to [23].

An extension of the CEV model, in which the volatility is modeled as a stochastic process is the Stochastic Alpha-Beta-Rho (SABR) model [23].

3.5 Conditions for an arbitrage-free volatility surface

In Chapter 2, we defined the arbitrage opportunity as a trading strategy so that one starts with zero capital, and at some later time \( T \) is sure not to have lost money, and furthermore, has a positive probability of to have made money (Definition 2.7). In this section we discuss how absence of arbitrage can be characterized in terms of calendar spreads in the maturity direction and butterfly arbitrage in the strike direction.

Following the Fundamental Theorem of Asset Pricing (Theorem 2.2), absence of arbitrage corresponds to the existence of a risk-neutral probability measure. But Carr, Géman, Madan and Yor [6] introduce the phrase static arbitrage to describe an arbitrage opportunity. The ‘static’ part of the phrase signifies that the position taken in the underlying stock at a particular time can only depend on the time and on the stock price \( S_t \).

In the paper by Carr and Madan [8] it is shown that for a grid of call prices the absence of call spread, butterfly spread and calendar spread arbitrages is sufficient to exclude all static arbitrages from a set of option price quotes across strikes and maturities. We will give definitions of these spreads in Subsections 3.5.1 and 3.5.2.

Theorem 3.3 (Static arbitrage). A volatility surface is free of static arbitrage if and only if the following conditions are satisfied:

- the volatility surface is free of calendar spread arbitrage;
- each volatility smile is free of call spread arbitrage and butterfly arbitrage.

Proof. For a proof we refer to Carr and Madan [8].
3.5. CONDITIONS FOR AN ARBITRAGE-FREE VOLATILITY SURFACE

3.5.1 Calendar spread arbitrage

The calendar spread is a spread of options of the same underlying stock, same fixed strike price $K$, but different expiries.

**Definition 3.2.** A call calendar spread involves a short position in a call with expiry $T_1$ and a long position in a call with expiry $T_2$:

$$C(K, T_2) - C(K, T_1), \quad 0 \leq T_1 < T_2.$$  

The idea behind the calendar spread is to ‘sell time’, which is why calendar spreads are also known as time spreads. To guarantee the absence of arbitrage in the maturity direction, the price of a call calendar spread should always be non-negative.

We can derive the reason why a calendar spread should be non-negative by a no-arbitrage argument. Consider call options with strike $K$ and expiries $T_1$ and $T_2$ such that $T_1 < T_2$. The holder of a call option has the right to buy a share of stock for the strike price $K$ at the option’s expiry. The call option holder is able to earn an interest on the strike price throughout the life of an option, where

$$P(t, T_2)K < P(t, T_1)K.$$  

**Note.** If we switch to the forward measure we do not encounter the discount factor.

**Lemma 3.4.** The price of a call calendar spread should always be non-negative.

**Proof.** Consider a function $\phi : \mathbb{R} \to \mathbb{R}$ by: $\phi(y) = (y - L)^+$, where $L \in \mathbb{R}$ is a constant. Then $\phi$ is convex in $y$. Let $X_t$ be a martingale. By Jensen’s inequality for conditional expectations (Lemma 2.1), $\phi(X_t)$ is a sub-martingale:

$$\phi(X_s) = \phi(E[X_t|\mathcal{F}_s]) \leq E[\phi(X_t)|\mathcal{F}_s], \quad 0 \leq s \leq t.$$  

Taking the expectation and using the monotonicity of the expectation and the tower property (Lemma 2.1) we get:

$$E[\phi(X_s)] \leq E[E[\phi(X_t)|\mathcal{F}_s]] = E[\phi(X_t)], \quad 0 \leq s \leq t.$$  

Hence, for fixed $T_1$ and $T_2$ such that $0 \leq T_1 \leq T_2 \leq T$:

$$E \left[(X_{T_1} - L)^+\right] \leq E \left[(X_{T_2} - L)^+\right]. \quad (3.14)$$  

Since $S_t$ is a martingale under the forward measure, pricing the call options under the forward measure results in the following inequality:

$$C(K, T_1) = E \left[(S_{T_1} - K)^+\right] \leq E \left[(S_{T_2} - K)^+\right] = C(K, T_2).$$  

Hence, given a fixed strike value $K$, call options should be non-decreasing over time:

$$C(K, T_2) - C(K, T_1) \geq 0 \text{ for all } 0 \leq T_1 < T_2.$$

Calendar spread arbitrage is usually expressed as the monotonicity of European call option prices with respect to the maturity:

$$\frac{\partial C}{\partial T} \geq 0.$$  

Since the main focus of this thesis is on volatility modeling, we would rather define calendar spread arbitrage on the volatility surface explicitly. For this, we follow the paper by Gatheral et al [17]. If we again apply a change of variables to total variance $\omega$ (Definition 2.8) and log-moneyness $x$ (Definition 2.9), we are able to establish a sufficient condition in terms of $\omega$.  

21
Lemma 3.5. The volatility surface is free of calendar spread if and only if
\[ \frac{\partial \omega(x,T)}{\partial T} \geq 0, \quad \text{for all } x \in \mathbb{R} \text{ and } T > 0. \]

Proof. We prove the lemma in a similar way as Lemma 3.4. We define \( X_t := S_t/F_t \), where \( F_t := F(0,t) \) is the forward price. Note that under the forward measure, \( X_t \) is a martingale. Consider two call options with the same moneyness:
\[ K_1/F_{T_1} = K_2/F_{T_2} =: e^x, \]
Compute the call option price under the forward measure:
\[ C(K_1,T_1) = E\left[ (S_{T_1} - K_1)^+ \right] = E\left[ F_{T_1} \left( \frac{S_{T_1}}{F_{T_1}} - \frac{K_1}{F_{T_1}} \right)^+ \right] = E\left[ F_{T_1} (X_{T_1} - e^x)^+ \right] \]
Hence, again by using equation (3.14) we derive the following inequality for \( C(K_1,T_1) \) and \( C(K_2,T_2) \):
\[ \frac{C(K_1,T_1)}{K_1} = e^{-x}E\left[ (X_{T_1} - e^x)^+ \right] \leq e^{-x}E\left[ (X_{T_2} - e^x)^+ \right] = \frac{C(K_2,T_2)}{K_2}. \]
Keeping the moneyness constant, call option prices are non-decreasing in time to expiration. Recall the Black-Scholes formula (3.15) in terms of \( x \) and \( \omega \):
\[ c(x,\omega) = S_0 N(d_1) - S_0 e^x N(d_2), \quad (3.15) \]
where
\[ d_1 = -\frac{x}{\sqrt{\omega}} + \frac{1}{2} \sqrt{\omega}, \quad d_2 = d_1 - \sqrt{\omega}. \]
Note that \( c(x,\omega) \) is strictly increasing in \( \omega \):
\[ \frac{\partial c}{\partial \omega} = \frac{1}{2} S_0 e^x N'(d_2) \frac{1}{\sqrt{\omega}} > 0. \]
For fixed \( x \) the total variance should be a non-decreasing function of \( T \):
\[ \frac{\partial c}{\partial T} \geq 0 \quad \text{iff} \quad \frac{\partial c}{\partial \omega} \frac{\partial \omega}{\partial T} \geq 0 \quad \text{iff} \quad \frac{\partial \omega}{\partial T} \geq 0. \]

By Lemma 3.5, for a fixed \( x \), the total variance is increasing in \( T \). Hence, there is no calendar arbitrage spread if for any two dates \( T_1 \) and \( T_2 \) the corresponding slices of the term structure do not intersect:
\[ \omega(x,T_1) \neq \omega(x,T_2) \quad \text{for all } x \in \mathbb{R}. \]
If the total variance surface is parameterized we can find the roots analytically. Otherwise, a root-finding technique such as the bisection method can be used to verify this condition. We can define calendar spread in terms of implied volatilities by the following condition:
\[ \frac{\partial \omega}{\partial T} = \sigma^2_{BS} + 2\sigma_{BS} \frac{\partial \sigma_{BS}}{\partial T} T \geq 0. \]

3.5.2 Call spread arbitrage and butterfly arbitrage

Arbitrage in the strike direction can be characterized by call spread and butterfly arbitrage. For these kinds of arbitrages we consider only one smile of the volatility surface for a given fixed maturity \( T \).
We define the call spread, butterfly option and corresponding arbitrage conditions by:
3.5. CONDITIONS FOR AN ARBITRAGE-FREE VOLATILITY SURFACE

Definition 3.3. To guarantee the absence of call spread arbitrage in a volatility smile, the price of a call spread should always be non-positive:

\[ C(K_2, T) - C(K_1, T) \leq 0 \text{ for all } 0 \leq K_1 < K_2. \]

Definition 3.4. A smile is free of butterfly arbitrage if a butterfly option always has a non-negative value:

\[ C(K_1, T) - 2C((K_1 + K_2)/2, T) + C(K_2, T) \geq 0. \]

The call option holder has the option to buy a share of stock for a strike value \( K \) at the time of expiry. Hence, it is clear that the value of a call option should be a decreasing function in the strike direction:

\[ \frac{\partial C}{\partial K} \leq 0. \]

According to Definition 3.4 a call option price is a convex function of the strike:

\[ \frac{\partial^2 C}{\partial K^2} \geq 0, \]

We will show that this condition for the absence of arbitrage in the strike direction corresponds to a non-negative density function of the underlying asset process. Recall from Section 3.1 the formulas for the partial derivatives of \( C(K, T) \) with respect to the strike value:

\[
\begin{align*}
\frac{\partial C}{\partial K} &= -P(t, T) \int_{\infty}^{K} f(s, T) ds, \\
\frac{\partial^2 C}{\partial K^2} &= P(t, T) f(K, T),
\end{align*}
\]

where \( f(K, T) \) is the probability density function. It follows from these expressions that if \( f(K, T) \) is non-negative, both partial derivatives have the desirable sign and the absence of arbitrage is guaranteed. In the paper of Gatheral et al [17] a lemma is derived, in which the no-arbitrage conditions in the strike direction are defined in terms of the total variance.

Lemma 3.6. A volatility smile for a fixed expiry \( T \) is free of call spread and butterfly arbitrage if and only if \( g(x) \geq 0 \) for all \( x \in \mathbb{R} \) and \( \lim_{x \to +\infty} d_1(x) = -\infty \), where,

\[
g(x) := \left(1 - \frac{\omega'(x)^2}{2\omega(x)}\right)^2 - \frac{\omega'(x)^2}{4} \left(\frac{1}{\omega(x)} + \frac{1}{4}\right) + \frac{\omega''(x)}{2}.\]

Note that we have dropped the \( T \) dependence of the total variance \( \omega(x, T) \) and abbreviated the notation to \( \omega(x) \) instead.

Proof. By explicit differentiation of the Black-Scholes call option price for fixed \( T \) we can derive the following expression for the probability density function:

\[
f(x) = P(t, T) \left. \frac{\partial^2 C}{\partial K^2} \right|_{K=F.e^x} = \frac{g(x)}{K \sqrt{2\pi} \omega} e^{-\frac{1}{2}d_2^2}.\]

It follows that \( f(x) \) is non-negative if and only if \( g(x) \geq 0 \) for all \( x \in \mathbb{R} \).

We conclude that in order to have an arbitrage-free volatility surface the partial derivatives of \( C(K, T) \) and \( \omega \) with respect to the strike and time to maturity need to have the correct sign. Furthermore, since we use these derivatives as input to Dupire’s local volatility model (3.7) or (3.10), butterfly arbitrage and calendar spread arbitrage can result in negative local variance values.
Chapter 4

Dividends Modeling

In Chapter 3, we assumed an underlying stock without dividend payments. In this chapter we discuss how to incorporate dividends into our local volatility model. We start with some basic considerations regarding dividends modeling, we study the behavior of option values and implied volatilities across the dividend date.

We show that the stock price can be modeled in an arbitrage-free way, that is consistent with the assumption of cash dividends. We derive a martingale process $\tilde{S}_t$ by taking the net present value of the dividends out of the stock price process $S_t$. Under $\tilde{S}_t$, pricing is the same as in the no dividend case. Hence, we are able to define a local volatility term $\tilde{\sigma}_{LV}$ for $\tilde{S}_t$. Following the paper of Buehler [2], our model can be expanded to proportional dividends as well.

4.1 Basic concepts in dividends modeling

A dividend is a payment by a company to its shareholders, following a no-arbitrage argument, the dividend payment will cause a decrease in the stock price. A straightforward method to model dividend payouts in the Black-Scholes framework is by considering a continuous dividend yield. If the underlying receives a dividend of $qS_t dt$ in each time step $dt$ then the pricing PDE is of the form:

$$
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2(S_t, t) S_t^2 \frac{\partial^2 V}{\partial S^2} + (r - q) S_t \frac{\partial V}{\partial S} - r V = 0.
$$

However, only for an index where many different stocks are taken into account a continuous dividend yield might be suitable, but even then the dividend payments will be clustered over the year and do not behave like a continuously-paid yield. For derivatives on one stock, dividends need to be modeled as discrete drops in the stock price process, since there usually are one or two payments a year.

We assume that the ex-dividend dates $0 < \tau_1 < \tau_2 < \ldots$, as well as the size of payments, are known in advance. Let denote $\tau_j^-$ and $\tau_j^+$ the moments just before and after a dividend payment, respectively. The person who buys the stock on or before $\tau_j^-$ will also get the rights to the dividend. The person who buys the stock at $\tau_j^+$ or later will not receive the dividend. So the stock price needs to be balanced by a drop in its price by the amount of dividend:

$$
S_{\tau_j^+} = S_{\tau_j^-} - \alpha_j,
$$

where $\alpha_j$ is the cash dividend payment at time $\tau_j$. We can not observe this drop in stock price in practice. Dividends go ex overnight while there is no trading. Since the forward contract is an agreement to buy a share of stock at a future time $T$, the forward price encounters the same drop:

$$
F(t, T) = \frac{S_t - \sum_{j:t<\tau_j<T} P(t, \tau_j) \alpha_j}{P(t, T)}. \quad (4.1)
$$

Since the amount of dividend that is being paid is dependent on the performance of the company, i.e. the performance of the stock price, an alternative way of modeling dividends is by taking a proportional
value:

\[ S_{\tau_j^+} = (1 - \beta_j)S_{\tau_j^-}. \]

This method is often used, since we can easily incorporate dividends into the forward price:

\[ F(t, T) = \prod_{j : t < \tau_j < T} (1 - \beta_j)S_t, \]

and it is also more accurate for longer term options. But the estimated dividend payments are usually absolute values, so we need a conversion from the estimated cash dividends to proportional dividends. This needs to be done in such a way that the forward price remains the same. The conversion is done iteratively starting with the first dividend at time \( \tau_1 \). If the forward prices are matched at all dividend dates \( \tau_j \), then the forward prices are equal at the time intervals between the dividends as well. The relative dividend at time \( \tau_j \) is determined by:

\[ \beta_j = \frac{\alpha_j}{F(0, \tau_j^-)}, \]

where \( F(0, \tau_j^-) \) denotes the forward just before the dividend date. Hence, we can switch from cash dividends to proportional dividends. However, the dividends are announced early in advance, hence, won’t be affected by a certain increase or decrease in the stock price close to the dividend date, therefore we will use only cash dividends \( \alpha_j \) in our volatility models.

We use a no-arbitrage argument to study how the drop in the underlying stock price path affects the option price. Note that the holder of the option does not profit from the dividend payment. Furthermore, if the dividend date and amount are known in advance, then there is no surprise in the drop in the stock price either. Hence, the value of the option does not change in value across the dividend date:

\[ V(S_{\tau_j^-}, \tau_j^-) = V(S_{\tau_j^+}, \tau_j^+) = V(S_{\tau_j^-} - \alpha_j, \tau_j^+). \]

In Section 2.4 we discussed the pricing of barrier options for stocks with no dividends. Across the dividend date, for the barrier option value a similar no-arbitrage argument as for the vanilla option holds true. Take for example a down-and-out barrier option. We assume that we know the timing and amount of the dividend payment in advance. We already expect the stock price to drop below the barrier so the option value becomes zero even before the actual dividend is paid. In the next subsection, we discuss how the drop in the stock price affects the volatilities.

### 4.1.1 Volatility and dividends

In this subsection, we investigate how the volatility surface behaves in the presence of dividends. For this, we assume that the implied volatility surface \( \sigma_{BS}(K, T) \) corresponding to the stock price process \( S_t \) with dividends, is fully known. Consider call options with expiries \( \tau_j^- \) and \( \tau_j^+ \), just before and after a dividend payment \( \alpha_j \) at time \( \tau_j \) and strike values \( K_1 \) and \( K_2 \) where the latter is defined by:

\[ K_2 = K_1 - \alpha_j. \]

According to the no-arbitrage argument both call options have the same price, since the value of the second option should not change across the dividend date, and at time \( \tau_j^- \) the following equality holds:

\[ S_{\tau_j^-} - K_1 = S_{\tau_j^+} + \alpha_j - K_1 = S_{\tau_j^+} - K_2. \]

Hence,

\[ C(S_t, t, K_1, \tau_j^-) = C(S_t, t, K_2, \tau_j^-) = C(S_t, t, K_2, \tau_j^+), \]

where \( C(S,t,K,T) \) denotes the Black-Scholes call price. For the corresponding implied volatilities it holds that:

\[ \sigma_{BS}(K_1, \tau_j^-) = \sigma_{BS}(K_2, \tau_j^+). \]

At the dividend date \( \tau_j \) a jump occurs in the term structure of the implied volatilities. But if there were known jumps in volatility, this would create a calendar spread arbitrage opportunity within the Black-Scholes framework. In the presence of dividends a call spread actually can have a negative value, as we discuss in Example 4.1.
CHAPTER 4. DIVIDENDS MODELING

Example 4.1. Consider again the call option with expiry on time \( \tau_j^- \):

\[
C(S_t, t, K_1, \tau_j^-) = C(S_t, t, K_2, \tau_j^+) \geq C(S_t, t, K_1, \tau_j^+)
\]

where the latter inequality follows from the increase in strike value: \( K_1 > K_2 \). So if we would build a calendar spread with fixed strike \( K_1 \) and \( T_1, T_2 \), respectively, before and after the dividend date, the value might get negative.

We could apply a dividend adjustment to all strike values in a grid, and use a smooth time interpolation to avoid the jumps in the volatility surface. However, the model would not be consistent anymore. To define a consistent model, we have to switch to a different underlying process, which we will discuss in the upcoming sections.

4.2 Stock price dynamics with cash dividends

Many dividend models assume the stock price process follows a geometric Brownian motion in between dividend dates and will drop across the dividend dates:

\[
dS_t = rS_t dt + \sigma(S_t - dW_t - \sum_j \alpha_j 1_{\{t = \tau_j^-\}}).
\]  

(4.3)

As explained in the book of Shreve [27] we can still apply Itô calculus to stock price processes with jumps. Furthermore, most PDE solvers can be generalized in such a way that they can handle these drops as well. But the question is how to determine the local volatility since we are clearly not in the Dupire framework anymore: the dynamics of the stock price process are not represented by one-dimensional diffusion process, due to the ex-dividend jumps in the stock price.

It is important to notice that the stock price dynamics need to be floored to ensure that the stock price never falls below any future dividend amount and causes negative stock prices, i.e.,

\[
S_{\tau_j^-} > \alpha_j \quad \forall j.
\]

Intuitively, the restriction means that the stock price at any time needs to be above the net present value of the dividends, the discounted value of all future dividends. Instead of taking the time interval \([0, \infty]\), we consider dividends up to a fixed time horizon \( T^* \) in the future. We floor the stock price process by:

\[
S_t > \sum_{j:t<\tau_j<T^*} P(t, \tau_j) \alpha_j.
\]

We can extract dividend information from market data using the put-call parity. The put-call parity describes the relationship between the value of a call and a put option with same time to maturity and the same strike value. Consider a portfolio \( \Pi_t \) of a long position in a European call and a short position in a European put, both with expiry \( T \) and strike \( K \). Our portfolio at the time of expiry is worth:

\[
\Pi_T = \max\{S_T - K, 0\} - \max\{K - S_T, 0\} = S_T - K.
\]

For time \( t : 0 \leq t \leq T \), the portfolio equals:

\[
\Pi_t = S_t - \sum_{j:t<\tau_j<T} P(t, \tau_j) \alpha_j - P(t, T) K.
\]

Hence, we can imply the net present value of the dividend payments during the life time of an option by listed market option prices of European calls and puts. We denote the net present value of all future dividends up to the fixed time horizon \( T^* \) by:

\[
D_t := \sum_{j:t<\tau_j<T^*} P(t, \tau_j) \alpha_j,
\]

such that,

\[
S_t > D_t.
\]

(4.4)
4.2. STOCK PRICE DYNAMICS WITH CASH DIVIDENDS

We assume the timing and size of the dividends to be known in advance, such that $D_t$ is a deterministic process. Instead of focusing on the stock price process we use the ‘clean’ asset dynamics, where the net present value of the future dividends is removed from the stock price process:

$$\tilde{S}_t = S_t - D_t,$$

with initial value:

$$\tilde{S}_0 = S_0 - D_0 = S_0 - \sum_{j:0<\tau_j<T} P(t,\tau_j)\alpha_j.$$

It may seem strange at first to let $\tilde{S}_t$ depend on dividend dates beyond the life time of the option. Many other models, often only depend on dividend payments up to the time of expiry of the option. As an example, we consider the escrowed dividend model [18], a dividend model that is often used in equity derivative pricing. In this model the initial stock value is adjusted:

$$S^*_0 = S_0 - \sum_{j:0<\tau_j<T} P(0,\tau_j)\alpha_j,$$

and $S^*_t$ is modeled as an ordinary geometric Brownian motion. However, this definition of the underlying dynamics is derivative-dependent, since options with different expiries result in different underlying dynamics, which leads to a logical flaw in the model. Furthermore, if one will only take the dividends up to expiry into account an arbitrage opportunity may occur by taking positions in options whose expiries are just before and after a dividend payment. That is why we prefer our definition of $\tilde{S}_t$.

Another useful property of the definition of $\tilde{S}_t$ is that the discounted process $P(0,t)\tilde{S}_t$ is a martingale.

**Lemma 4.1.** The discounted process $P(0,t)\tilde{S}_t$ is a martingale.

**Proof.** First we consider $\tilde{S}_t$ in a time interval between two dividend payments: $\tau_{k-1} \leq s < t < \tau_k$. Note that for times $s$ and $t$ the number of future dividends $\alpha_j$ is the same:

$$P(s,t)D_t = P(s,t) \sum_{j:t<\tau_j<T^*} P(t,\tau_j)\alpha_j = \sum_{j:s<\tau_j<T^*} P(s,\tau_j)\alpha_j = D_s,$$

Between dividend payments the stock price process $S_t$ is modeled as an ordinary diffusion process (4.3). Hence, $P(0,t)S_t$ is a martingale between dividend dates, and so is $P(0,t)\tilde{S}_t$:

$$\mathbb{E} \left[ P(s,t)\tilde{S}_t \mid \mathcal{F}_s \right] = \mathbb{E} \left[ P(s,t)(S_t - D_t) \mid \mathcal{F}_s \right] = \mathbb{E} \left[ P(s,t)S_t \mid \mathcal{F}_s \right] - P(s,t)D_t = S_s - D_s = \tilde{S}_s.$$

Next, we investigate what happens across dividend date $\tau_k$. Note that for the net present value of the dividends the following equality holds across the dividend date:

$$D_{\tau_k} = \sum_{j:\tau_k<\tau_j<T^*} P(\tau_j^-,\tau_j)\alpha_j = \sum_{j:\tau_k<\tau_j<T^*} P(\tau_k,\tau_j)\alpha_j + \alpha_k = D_{\tau_k} + \alpha_k.$$

Therefore, $\tilde{S}_t$ is continuous at time $\tau_k$:

$$\tilde{S}_{\tau_k} = (S_{\tau_k^-} - D_{\tau_k^-}) = (S_{\tau_k^-} - D_{\tau_k} - \alpha_k) = (S_{\tau_k} - D_{\tau_k}) = \tilde{S}_{\tau_k}.$$

Now we can show that $P(0,t)\tilde{S}_t$ is a martingale for all $t$. Let $\tau_{k-1} \leq s < t < \tau_{k+1}$, then we have:

$$\tilde{S}_s = \mathbb{E} \left[ P(s,\tau_k^-)\tilde{S}_{\tau_k^-} \mid \mathcal{F}_s \right] = \mathbb{E} \left[ P(s,\tau_k^-)\tilde{S}_{\tau_k} \mid \mathcal{F}_s \right] = \mathbb{E} \left[ P(s,\tau_k)\tilde{S}_{\tau_k} \mid \mathcal{F}_s \right].$$

These steps can be used to recursively ‘jump’ over all dividend dates. Hence, $P(0,t)\tilde{S}_t$ is a martingale.

**Corollary 4.2.** The process $\tilde{S}_t$ is non-negative which leads to the deterministic process $D_t$ being an absorbing state for $S_t$: if $\tilde{S}_t = 0$ at a certain time $t = \tilde{t}$, then $\tilde{S}_t$ will remain at zero:

$$\tilde{S}_t = 0, \text{ for all } t > \tilde{t},$$

and, hence,

$$S_t = D_t \text{ for all } t > \tilde{t}.$$
CHAPTER 4. DIVIDENDS MODELING

The stock price process $S_t$ follows, from time $\hat{t}$ on, the deterministic process $D_t$ and never exceeds the net present value of the future dividends anymore. In this setting the stock price process $S_t$ will never become negative. This is an advantage over other cash dividends models, for example the model by Haug et al [18], which needs to cap the dividends to make sure the dividend-drops do not cause $S_t$ to become negative. We assumed the dividend amount to be known in advance, the need of resizing the dividend payment would result in a logical flaw.

4.2.1 Stock price dynamics in the clean space

The dynamics of $\tilde{S}_t$ in the clean space follow a one-dimensional diffusion process, and are given by:

$$d\tilde{S}_t = r\tilde{S}_t dt + \tilde{\sigma}_{LV}(\tilde{S}_t, t) \tilde{S}_t dW_t,$$

where $\tilde{\sigma}(S_t, t)$ is the local volatility term corresponding to $\tilde{S}_t$. We can now find the dynamics of $S_t$ as well in terms of this local volatility:

$$dS_t = d\tilde{S}_t + dD_t = r\tilde{S}_t dt + rD_t - dt - \sum_j \alpha_j 1\{t = \tau_j\}.$$

With these dynamics we are able to price financial derivatives based on $S_t$, but it will be easier to just price in the ‘clean’ space with underlying process $\tilde{S}_t$ where we don not encounter dividend drops.

4.3 Pricing in the clean space

We can treat $\tilde{S}_t$ as a stock without dividends. So, we can define the local volatility and corresponding no arbitrage conditions in a similar way as we have done in Chapter 3 and start pricing with $\tilde{S}_t$ as the underlying process. Consider an option based on $S_t$, with value $V(S_t, t)$ and an option with the same time to maturity $T$, based on the underlying process $\tilde{S}_t$, denoted by $\tilde{V}(\tilde{S}_t, t)$. Following a no-arbitrage argument, two portfolios that guarantee the same payoff at time $T$, should have the same value at all times $0 \leq t \leq T$. Thus, if the payoff of an option in the clean space equals the payoff of an option under $S_t$, these options have the same value over $[0, T]$. Hence, for a call option, let us define an adjusted strike value:

$$\tilde{K} = K - D_T,$$

such that,

$$\mathbb{E}\left[(S_T - K)^+ | \mathcal{F}_t\right] = \mathbb{E}\left[(\tilde{S}_T + D_T - K)^+ | \mathcal{F}_t\right] = \mathbb{E}\left[(\tilde{S}_T - \tilde{K})^+ | \mathcal{F}_t\right].$$

Therefore, the call option prices should always be equal:

$$C(K, T) = \tilde{C}(\tilde{K}, T),$$

where we denote $\tilde{C}(\tilde{K}, T)$ for the call option priced under $\tilde{S}$ with adjusted strike value $\tilde{K}$. We discuss a simple test case in Example 4.2.

Example 4.2 (Pricing in the clean space). We consider a call option on a dividend-paying stock $S_t$, with two future dividend payments:

$$\alpha_1 = 1, \tau_1 = 0.5 \text{ and } \alpha_2 = 1, \tau_2 = 1.5,$$

we model $S_t$ as a geometric Brownian motion with discrete dividend drops of size $\alpha_1$ and $\alpha_2$ at times $\tau_1$ and $\tau_2$, respectively. For this example we use the following parameter values:

$$S_0 = 100, \quad K = 80, \quad r = 0, \quad T = 1, \quad \bar{\sigma} = 0.1,$$

28
such that we have $\hat{S}_0 = 98$ and $\hat{K} = 79$.

For the pricing under $S_t$ we use a Monte Carlo simulation. For $M = 10^5$ simulated paths, we obtain a confidence interval $[18.9646, 19.0867]$ for the option value. The corresponding Black-Scholes price under $\hat{S}_t$ is 19.0488.

Lastly, we consider the pricing of barrier options in the clean space. For path-dependent options we need to make sure that the option value based on $\hat{S}_t$ equals the option value under $S_t$ at each time step $t$ of the life time of the option. This can be accomplished by introducing a time-dependent barrier level.

Take for example the down-and-out call, with constant barrier $B$ level:

$$E \left[ (S_T - K)^+ 1_{\{\forall t \leq T: S_t > B\}} \right] = E \left[ (\hat{S}_T - \hat{K})^+ 1_{\{\forall t \leq T: \hat{S}_t > \hat{B}_t\}} \right],$$

where $\hat{B}_t$ is the time-dependent adjusted barrier level:

$$\hat{B}_t = B - D_t.$$

Therefore, even though we can consider the underlying process $\hat{S}_t$ as a no-dividend paying stock, pricing barrier options with dividends becomes more complicated than in the no-dividend case. In Chapter 6 we will discuss numerical pricing methods for barrier options.

### 4.3.1 No-arbitrage conditions

The potential negative value of a calendar spread (Example 4.1) illustrates the fact that the introduction of cash dividends changes the no-arbitrage conditions. However, these conditions will hold for pricing under $\hat{S}_t$:

$$\frac{\partial \hat{C}}{\partial \hat{K}} \leq 0, \quad \frac{\partial^2 \hat{C}}{\partial \hat{K}^2} \geq 0, \quad \frac{\partial \hat{C}}{\partial \hat{T}} \geq 0.$$

Furthermore, a common arbitrage problem in dividends models with adjusted volatilities is that American call options may become cheaper than European call options. The difference between an American and a European option is that the American call option can be exercised at any time during the life time of the option. Hence the option value is given by:

$$C^A(S_t, t, K, T) = \max \{g(S_t), E[P(t, T)g(S_T)|\mathcal{F}_t]\},$$

where $g(\cdot)$ is the payoff function. In case of no dividends, the holder of an American call option will never choose to exercise early (following the same reasoning as for the non-negativeness of the calendar spread), so we have,

$$C^A = C^E.$$

However, in the presence of dividends, it may be more attractive to exercise the American call option just before a relative large dividend payment takes place. Now, the European and American call prices do not have to be equal anymore, but due to the extra optionality of the American option, the American price should always be greater than or equal to the ordinary European option:

$$C^A \geq C^E,$$

with an adjusted volatility, this inequality can be violated. If we consider the pricing of American options in the clean space $\hat{C}^A$, we need to make sure that at any time $0 \leq t \leq T$ the clean price equals the option value under $S_t$. For this, we introduce a time-dependent strike value,

$$\hat{K}_t = K - D_t,$$

such that for each time step $0 \leq t \leq T$, the American option under $\hat{S}_t$ with time-depending $\hat{K}_t$ has the same value as the American option under $S_t$ with fixed strike $K$:

$$\max \{S_t - K, E[P(t, T)(S_T - K)|\mathcal{F}_t]\} = \max \{\hat{S}_t - \hat{K}_t, E[P(t, T)(\hat{S}_T - \hat{K}_T)|\mathcal{F}_t]\},$$

with this formula we can price American options in the clean space.
4.3.2 Implied volatility and local volatility in the clean space

For pricing financial derivatives in the clean space, the implied volatility and local volatility terms corresponding to \( \tilde{S}_t \) are required. Note that we can think of \( \tilde{S}_t \) as the volatile part of the stock price process. Hence, it makes sense to look at the implied volatilities of \( \tilde{S}_t \) directly, instead of those of \( S_t \). If a call option price surface \( C(K, T) \) is fitted to market data under \( S_t \), the option values \( \tilde{C}(\tilde{K}, T) \) under \( \tilde{S}_t \) should equal these prices. Hence, we can extract the implied volatilities \( \tilde{\sigma}_{BS} \) from the call surface \( C(K, T) \) and translate this surface to a surface in the clean space \( \tilde{C}(\tilde{K}, T) \).

In case of a parameterized volatility surface, for example the SVI model, which we discuss in Chapter 5.3, we may find the implied variance directly from the parameterization. However, it would be even better if we would calibrate the volatility model directly in the clean space. From the fitted implied volatility surface \( \tilde{\sigma}_{BS}(\tilde{K}, T) \) we can again derive the Dupire local volatility term.

We know that \( \tilde{S}_t \) is a diffusion process and, as discussed in Subsection 4.3.1, follows the common arbitrage conditions as defined in Section 3.5. Hence, once we have constructed the arbitrage-free implied volatility surface we can derive the Dupire local volatility term for the dynamics of \( \tilde{S}_t \):

\[
\tilde{\sigma}_{LV}^2(K, T) = \frac{\tilde{\sigma}_{BS}^2 + 2\tilde{\sigma}_{BS}T \left( \frac{\partial \tilde{\sigma}_{BS}}{\partial T} + rK \frac{\partial \tilde{\sigma}_{BS}}{K} \right)}{\left( 1 - Kx \frac{\partial \tilde{\sigma}_{BS}}{\partial T} \right)^2 + K \tilde{\sigma}_{BS}T \left( \frac{\partial \tilde{\sigma}_{BS}}{\partial K} - \frac{1}{4} K \frac{\partial^2 \tilde{\sigma}_{BS}}{K} \left( \frac{\partial \tilde{\sigma}_{BS}}{\partial K} \right)^2 + K \frac{\partial^2 \tilde{\sigma}_{BS}}{\partial K^2} \right)}.
\]

Once the local volatility term is derived from listed market prices we can use the dynamics of \( \tilde{S}_t \) for pricing barrier options. This makes life much easier because we now do not encounter dividend payments, and, hence, do not have to deal with discrete jumps in our numerical option pricing methods anymore.

4.4 Proportional dividends

In Section 4.2, we discussed how to model cash dividends \( \alpha_j \), but the dividend model proposed by Buehler [2] is consistent for proportional dividends as well. In this section we discuss the proportional dividend extension of our model to show that it still works, but we only take cash dividends into account in our numerical tests. We follow the paper of Buehler [2] and use the same notations.

We assume that at each dividend date \( \tau_j \) a proportional dividend is paid first, followed by a cash dividend payment:

\[
S_{\tau_j} = (1 - \beta_j) S_{\tau_j^-} - \alpha_j.
\]

Buehler defines a “proportional growth factor”,

\[
R(t, T) = \Pi_{j:t<\tau_j<T} \left( 1 - \beta_j \right) \frac{P(t, T)}{P(t, \tau_j)},
\]

such that the price of a forward contract in the presence of proportional and cash dividends is given by:

\[
F(t, T) = R(t, T)S_t - \sum_{j:t<\tau_j<T} R(\tau_j, T)\alpha_j,
\]

which is a combination of the cash dividend forward price (4.1) and the proportional dividend forward price (4.2). We abbreviate the notations:

\[
F_t := F(0, t), \quad R_t := R(0, t).
\]

We again define a deterministic process that floors the stock price process:

\[
S_t \geq D_t, \quad \text{where} \ D_t := \sum_{j:t<\tau_j<T} \frac{\alpha_j}{R(t, \tau_j)}.
\]

In this case we can see \( D_t \) as the “growth rate-discounted” value of all remaining future cash dividends.
4.4. PROPORTIONAL DIVIDENDS

Theorem 4.3 (Theorem 2.1 of Buehler [2]). All non-defaulting arbitrage-free stock price processes for \( S_t \) can be written as:

\[
S_t = (F_t - D_t)X_t + D_t, \tag{4.5}
\]

where \( X_t \) is a (local) martingale with \( X_0 = 1 \).

Proof. We define

\[ X_t := \frac{S_t - D_t}{F_t - D_t}. \]

By construction, \( X_t \) is non-negative with \( X_0 = 1 \). We can show that \( X_t \) is a martingale in a similar way as we have done for \( P(0,t)\tilde{S}_t \) (Lemma 4.1). Note that in between dividend dates \( S_t/R_t \) is a martingale and across the dividend date we have:

\[
S_{\tau_k} - D_{\tau_k} = S_{\tau_k} - (D_{\tau_k} + \alpha_k) = S_{\tau_k} - D_{\tau_k}.
\]

For the complete proof of the theorem we refer to Buehler [2].

Corollary 4.4. Note that we can derive an alternative representation of (4.5):

\[
S_t = \bar{S}_0 R_t X_t + D_t,
\]

where the adjusted initial value is given by:

\[ \bar{S}_0 := S_0 - \sum_{0 < \tau_j < T} \frac{\alpha_j}{R_{\tau_j}}. \]

If \( \beta_j = 0 \) for all \( j \), we are back in the clean setting defined by \( \tilde{S}_t \).

Pricing can be done under \( X_t \), just as we have seen in Section 4.3. We adjust the strike value:

\[ \bar{K} := \frac{K - D_T}{F_T - D_T}, \]

such that

\[
C(K,T) = P(0,T)E \left[ (S_T - K)^+ \right] = P(0,T)E \left[ (X_T(F_T - D_T) + D_T - K)^+ \right] = P(0,T)E \left[ \left( \frac{X_T - \bar{K}}{F_T - D_T} \right)^+ \right] = P(0,T)(F_T - D_T)E \left[ (X_T - \bar{K})^+ \right].
\]

Hence, we derive the following relation between pricing under \( S_t \) and pricing under \( X_t \):

\[
C(K,T) = P(0,T)(F_T - D_T)\bar{C}(X, \bar{K}).
\]

In this section we have shown how our dividend model can be extended to proportional dividends as well. But, as explained before, for our numerical tests we focus solely on cash dividends.
Chapter 5
Building the Volatility Surface

In Chapter 3, it was shown how the local volatility model can be defined in terms of the implied volatility surface. In this chapter we discuss how to build such a volatility surface. We want to study the Stochastic Volatility Inspired model as a parameterization of the implied volatility surface. However, one of the drawbacks of the SVI model is that it may not be arbitrage-free. Hence, we discuss the Heston model as a theoretical test case, because the Heston model is arbitrage-free by definition.

We study both models, starting with the Heston model. We give a short model description and investigate how the model parameters affect the implied volatility smile. We derive no-arbitrage conditions and discuss the suitability of the parameterization for the local volatility model.

5.1 Heston model

The Heston model [23] is a stochastic volatility model; the volatility term is no longer defined by a deterministic function, but depends on a second Brownian motion. We have two stochastic differential equations, one for the underlying asset price, \( S_t \), and one for the variance process, \( v_t \). The Heston dynamics under the risk-neutral measure are given by:

\[
\begin{align*}
    dS_t &= rS_t dt + \sqrt{v_t} S_t dW^S_t, \\
    dv_t &= \kappa(\theta - v_t) dt + \gamma \sqrt{v_t} dW^v_v.
\end{align*}
\]

We define parameter \( \rho_{x,v} \) to be the correlation between the underlying Brownian motions,

\[ dW^S_t dW^v_v = \rho_{x,v} dt. \]

Note that, for a positive volatility of the volatility, denoted by \( \gamma > 0 \), the Heston model is mean reverting, it converges from an initial variance value \( v_0 > 0 \) to a long-term mean \( \theta \geq 0 \). The parameter \( \kappa \geq 0 \) is called the speed of mean reversion, and determines how fast the volatility term converges to the long-term mean. To guarantee the variance \( v_t \) stays positive, the model parameters need to satisfy the Feller condition [23]:

\[ 2\kappa\theta \geq \gamma^2. \]

If the Feller condition is not satisfied, the variance process may reach zero. In practice, the parameters calibrated to market data often do not satisfy the Feller condition.

A closed-form solution for the European call option price for the Heston model exists [22], however, it contains an integral that cannot be solved exactly. We can solve this integral by numerical integration, however, Fourier methods, such as the Carr-Madan method [7] and the COS method [13], are much faster for a strip of strike values.

5.1.1 Implied volatilities and Heston parameters

Following the example of Oosterlee et al [23], we study the effect of the Heston parameters on the implied volatility smile. We change each parameter while keeping the others fixed and discuss the influence of
the parameter on the volatility smile. There is no closed-form solution for the implied volatility term in the Heston model, hence, we need a root-finding technique such as the Newton-Raphson algorithm to determine the implied volatilities. We set $S_0 = 100, T = 1.0$ and $r = 0.0$, and take the following reference parameter values:

$$\rho_{x,v} = 0\%, \; \kappa = 1, \; \gamma = 0.1, \; \nu_0 = 0.05, \; \theta = 0.05. \tag{5.1}$$

In all figures, the red line corresponds to the above parameters. The obtained results for different values of, respectively, the speed of mean-reversing $\kappa$, and the long-term variance $\theta$, are shown in Figure 5.1. The parameter $\kappa$ has only a limited effect on the shape of the volatility smile. Increasing the value of $\theta$ results in a vertical translation of the smile.

![Figure 5.1: The influence of parameters $\kappa$ and $\theta$ on the volatility surface.](image)

The other two parameters of which we study the effect on the volatility smile are the correlation, $\rho_{x,v}$, and the volatility of the volatility, $\gamma$. These parameters have a more significant effect on the shape of the volatility smile. The obtained results are shown in Figure 5.2. A higher $\gamma$ value increases the curvature in the volatility smile, whereas an increasing negative correlation affects the skew of the smile.

![Figure 5.2: The influence of parameters $\gamma$ and $\rho_{x,v}$ on the volatility surface.](image)

### 5.1.2 Heston model and arbitrage

Recall the two statements of the Fundamental Theorem of Asset Pricing (Theorem 2.2):

- If a market model has a risk-neutral probability measure, then it does not admit arbitrage;
- A market model that has a risk-neutral probability measure is complete if and only if the risk-neutral measure is unique.

The Heston model is defined under the risk-neutral measure, hence, by definition, the Heston model does not admit arbitrage. Figure 5.3 shows the recovered density function of the Heston model for
$X_t = \log S_t$, with $S_0 = 1$ and with the same values for the parameters as in the previous section. For the recovery of the density we have used the COS formula (6.6).

No-arbitrage in the strike direction corresponds to a non-negative density function, as discussed in Section 3.5.2. However, even though the Heston model does not admit arbitrage in theory, in practice we can still encounter problems with computing the local volatility term, due to numerical errors in the finite difference approximations of the partial derivatives.

Note that the second statement of the Fundamental Theorem of Asset Pricing does not apply to the Heston model. Since there are two stochastic process, $S_t$ and $v_t$, both defined by a Brownian motion: $W^x_t$ and $W^v_t$, there is an extra degree of uncertainty and there is no unique risk-neutral measure. Hence, the Heston model is not complete.

5.1.3 Dupire local volatility on Heston call option price surface

In this section we discuss how to use the Heston model as input to Dupire’s local volatility model. We use the Dupire local volatility model in terms of the call option prices:

$$
\sigma_{LV}^2(K, T) = \frac{\partial C}{\partial T} + r K \frac{\partial C}{\partial K} + \frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}. 
$$

The Heston model is considered as a theoretical test case here, therefore we will not use real market data, but artificial data generated by a fixed set of Heston parameters. Recall that the volatility smile is arbitrage-free for any set of parameters. Pricing is done with the COS method. We discuss the COS method in Section 6.2.1. The Heston model is not a parameterization of the volatility surface, but describes the dynamics of the underlying process. So, for each local volatility term $\sigma_{LV}(S_t, t)$ we need to compute four call option values for the finite difference approximation of the partial derivatives. Due to the property in Lemma 6.5, we can compute the call option prices for all strike values for a fixed time $t$ at once. A central difference approximation is used for the partial derivatives in the strike direction:

$$
\frac{\partial C}{\partial K} \approx \frac{C(K + h, T) - C(K - h, T)}{2h},
$$

$$
\frac{\partial^2 C}{\partial K^2} \approx \frac{C(K + h, T) - 2C(K, T) + C(K - h, T)}{h^2},
$$

and a forward difference for the time direction:

$$
\frac{\partial C}{\partial T} \approx \frac{C(K, T + h) - C(K, T)}{h}.
$$

In the remainder of this chapter the application of the Dupire local volatility formula on a call option price surface under the Heston model will be referred to as the Dupire-Heston model. In the next section we present numerical results of the Dupire-Heston model.
5.2 Numerical results of the Dupire-Heston model

We study the Dupire-Heston model for a simple test case, as is done in [28]. For this, we construct a call option price surface for fixed Heston parameters. The call price surface is used to compute Dupire’s local volatility term. With the local volatility model we can price call options, using, for example, a finite difference method, where we evaluate the local volatility at each time step \( T = t \) in \( K = S_t \). From these call option prices we extract the Black-Scholes implied volatility, which should be consistent with the original implied Heston volatilities.

Summarizing:

**Heston test case:**
- Construct the Heston call price surface with the COS method.
- Evaluate the local volatility term (3.7) at each time step \( T = t \) in \( K = S_t \).
- Compute call option value under local volatility.
- Derive implied volatilities from computed option prices.
- Compare to the initial Heston implied volatilities.

For this test, we take parameter values:

\[
\rho_{x,v} = 0\%, \quad \kappa = 1, \quad \gamma = 0.1, \quad v_0 = 0.05, \quad \theta = 0.05, \quad (5.2)
\]

and set \( S_0 = 100, T = 1.0 \) and \( r = 0.0 \). The results of the Heston test-case are plotted in Figure 5.4. The blue dots are the implied volatilities of the underlying Heston model, whereas the red ones are the implied volatilities extracted from the call option prices under local volatility.

For strike values close to the spot value the Dupire-Heston model performs well, but for options that lie far in-the-money, and in particular, far out-of-the-money the implied volatilities are less accurate. These out-of-the-money options have values close to zero, therefore numerical errors can occur in the computation of the second derivative with respect to the strike \( \frac{\partial^2 C}{\partial K^2} \). Since we divide by this term, small errors in its numerical approximation can lead to substantial relative errors in the local volatility model.

We could switch to the local volatility model in terms of the implied volatility, however, there is no closed form solution for the implied volatilities. They need to be approximated as well and for small expiries the vega of the call is small, so a whole range of implied volatilities could result in the same option price. Another solution is to interpolate the call option prices or volatility surface to obtain the local volatility for all expiries and strike values. The thin plate splines (TPS) method is often used in the literature [21].

Figure 5.4: Results for the local volatility model with Heston (T = 1)
5.3 Stochastic Volatility Inspired model

In this section we discuss the Stochastic Volatility Inspired (SVI) model, that was introduced by Gatheral [15], as a parameterization of the implied volatility surface. One of the research objectives is to use the SVI model as input to Dupire's local volatility model. We start out with a study of the volatility model and in Section 5.4 we present numerical results.

The SVI parameterization depends on five parameters, \(a, b, \rho, l, \tilde{\sigma}^2\). For a given parameter set \(\{a, b, \rho, l, \tilde{\sigma}^2\}\) the implied variance is given by:

\[
\sigma_{SVI}^2(x) = a + b \left( \rho(x - l) + \sqrt{(x - l)^2 + \tilde{\sigma}^2} \right), \tag{5.3}
\]

where \(x\) is the log-moneyness (Definition 2.9). Furthermore, we set parameter restrictions: \(a \in \mathbb{R}, b \geq 0, |\rho| < 1, l \in \mathbb{R}, \tilde{\sigma} > 0\) and \(a + b\tilde{\sigma}\sqrt{1 - \rho^2} \geq 0\).

**Corollary 5.1.** As a direct result of the parameter restrictions, the SVI model has the following properties:

- \(\sigma_{SVI}^2(x)\) is a convex function in \(x\).
- \(\sigma_{SVI}^2(x) \geq 0\) for all \(x\).

**Proof.** For a proof we refer to Appendix A.3.

**Remark 5.1.** Note that our SVI model (5.3) is different from the method proposed in Gatheral [17], where the total implied variance is parameterized instead of the implied variance. By adjusting parameters \(a\) and \(b\) one can easily switch from one parameterization to another, we denote \(\hat{a} := aT\) and \(\hat{b} := bT\).

The SVI model is closely related to stochastic volatility models. In [16] it is shown how (5.3) is the limit-case of the implied volatility smile produced by the Heston model as the maturity goes to infinity.

5.3.1 Implied volatilities and SVI parameters

Each of the parameters has an impact on the implied volatility smile. A general understanding of the influence of the parameters is important, for the calibration process. We study their effect on the smile by varying each parameter separately, and restrictions on time-dependency will be put in place. Since the SVI model parameterizes the variance, there is no need for numerical approximations of the implied volatility, since we can compute the volatility surface directly from the SVI formula (5.3).

In Figures 5.5, 5.6 and 5.7 we see how the volatility surface is affected by a change in a specific parameter. In all figures the red line corresponds to the parameters:

\[a = -0.1, \; b = 0.6, \; \rho = 0.35, \; l = -0.02, \; \tilde{\sigma}^2 = 0.1.\]

By increasing \(a\) the general level of variance increases, so it leads to a vertical translation of the smile. Parameter \(a\) can be used to fit the at-the-money volatilities. For small expiries, it usually follows from the market data that \(a\) is negative. An increase in \(b\) results in increasing the slopes of the wings, tightening the smile. As the smile usually decreases over time, a natural restriction on \(b\) is that it decreases over time. Clearly \(b\) should be positive to have a realistic convex smile (assuming a ‘frown’ will not occur in the market).

The parameter \(\rho\) determines the angle of the smile. Decreasing \(\rho\) will tilt the smile to the right, so the left tail will rise, and the right tail will drop. In general, the right tail of a volatility smile becomes flatter with increasing expiry. Therefore, \(\rho\) is restricted to be decreasing. The parameter \(\tilde{\sigma}^2\) controls the curvature and smoothness of the smile. A higher value of \(\tilde{\sigma}^2\) results in a smoother smile pattern. The only restriction to be applied to \(\tilde{\sigma}\) is that is should remain positive. Lastly, the parameter \(l\) shifts the smile along the x-axis. Increasing \(l\) translates the smile to the right.
5.3. STOCHASTIC VOLATILITY INSPIRED MODEL

Figure 5.5: The influence of parameters $a$ and $b$ on the volatility surface.

Figure 5.6: The influence of parameters $\rho$ and $\tilde{\sigma}^2$ on the volatility surface.

Figure 5.7: The influence of parameter $l$ on the volatility surface.

5.3.2 SVI model and arbitrage

The Stochastic Volatility Inspired model does not guarantee the absence of arbitrage for all parameter combinations, therefore, it is important to check whether the SVI surface is arbitrage-free before we actually use it as input to Dupire's model. The following conditions need to be satisfied:

- \[ a + b\tilde{\sigma}\sqrt{1 - \rho^2} \geq 0 \quad \iff \quad \sigma_{SVI} \geq 0; \]
- \[ \frac{\partial\omega(x,T)}{\partial T} \geq 0 \quad \iff \quad \text{no calendar spread arbitrage}; \]
- \[ g(x) \geq 0 \quad \iff \quad \text{no butterfly spread arbitrage}. \]
We start with the calendar spread condition, in the SVI case this property results in searching for real-valued roots of a quartic polynomial.

**Lemma 5.2.** The SVI surface is free of calendar spread arbitrage if a certain quartic polynomial has no real root:

\[
\delta_4 x^4 + \delta_3 x^3 + \delta_2 x^2 + \delta_1 x + \delta_0 = 0,
\]

where the \( \delta_i \) depend on the SVI parameter sets corresponding to maturities \( T_1 \) and \( T_2 \).

**Proof.** As a direct result of Lemma 3.5, it is sufficient to check whether there are no intersection points: for any two dates \( T_1 \) and \( T_2 \) the corresponding slices of the term structure:

\[
\omega(x, T_1) \neq \omega(x, T_2)
\]

for all \( x \in \mathbb{R} \).

In the SVI case these slices are parameterized by expression (5.3), so that for parameter sets \( \{a_1, b_1, \rho_1, l_1, \sigma_1^2\} \) and \( \{a_2, b_2, \rho_2, l_2, \sigma_2^2\} \), corresponding to expiries \( T_1 \) and \( T_2 \), respectively, the following equation should have no real-valued roots:

\[
\hat{a}_1 + \hat{b}_1 \left( \rho_1(x - l_1) + \sqrt{(x - l_1)^2 + \sigma_1^2} \right) = \hat{a}_2 + \hat{b}_2 \left( \rho_2(x - l_2) + \sqrt{(x - l_2)^2 + \sigma_2^2} \right)
\]

(5.4)

where \( \hat{a}_1 = a_1 T_1, \hat{a}_2 = a_2 T_2, \hat{b}_1 = b_1 T_1 \), and \( \hat{b}_2 = b_2 T_2 \). Rewriting Eq. (5.4) gives us:

\[
\sqrt{(x - l_1)^2 + \sigma_1^2} = \xi_1 + \xi_2 x + \sqrt{(x - l_2)^2 + \sigma_2^2},
\]

(5.5)

where we set

\[
\xi_1 := \hat{a}_2 - \hat{a}_1 + \hat{b}_1 \rho_1 l_1 - \hat{b}_2 \rho_2 l_2
\]

and

\[
\xi_2 := \hat{b}_2 \rho_2 - \hat{b}_1 \rho_1.
\]

Squaring (5.5) leads to

\[
(x - l_1)^2 + \sigma_1^2 = \xi_1^2 + 2\xi_1 \xi_2 x + \xi_2^2 x^2 + 2(\xi_1 + \xi_2 x) \sqrt{(x - l_2)^2 + \sigma_2^2} + (x - l_2)^2 + \sigma_2^2,
\]

which we can rearrange such that we have the square-root term on one side, squaring both sides leads to a quartic polynomial in \( x \):

\[
\delta_4 x^4 + \delta_3 x^3 + \delta_2 x^2 + \delta_1 x + \delta_0 = 0,
\]

where each coefficients can expressed in terms of parameter \( \{a_1, b_1, \rho_1, l_1, \sigma_1^2\} \) and \( \{a_2, b_2, \rho_2, l_2, \sigma_2^2\} \). We refer to [32] for explicit expressions.

Note that the absence of real-valued roots of the quartic polynomial is a sufficient but not necessary condition. In the case of one or more real-valued roots, we need to check whether these are solutions of the original problem (5.4). As we will see in Section 5.4, even though the SVI volatility is calibrated in a way that satisfies Lemma 3.5, once we interpolate between the given expiries, calendar spread arbitrage can occur.

According to Lemma 3.6 the existence of a non-negative function \( g(x) \) is a sufficient condition to guarantee the absence of butterfly arbitrage.

**Lemma 5.3.** The SVI surface for fixed expiry \( T \) is free of arbitrage in the strike direction if \( g(x) \geq 0 \) for all \( x \in \mathbb{R} \) and \( \lim_{x \to +\infty} d_4(x) = -\infty \), where,

\[
g(x) = \left( 1 - \frac{x \sigma'_{SVI}(x)}{\sigma_{SVI}(x)} \right)^2 - \frac{1}{4} \sigma_{SVI}(x)^2 \sigma'_{SVI}(x)^2 T^2 + \sigma_{SVI}(x) \sigma''_{SVI}(x) T,
\]

with

\[
\sigma'_{SVI}(x) = \frac{\dot{b}}{2 \sigma_{SVI}(x)} \left( \rho + \frac{x - l}{\sqrt{(x - l)^2 + \sigma^2}} \right);
\]

\[
\sigma''_{SVI}(x) = \frac{-b \sigma'_{SVI}(x)}{2 \sigma_{SVI}(x)^2} \left( \rho + \frac{x - l}{\sqrt{(x - l)^2 + \sigma^2}} \right) + \frac{b}{2 \sigma_{SVI}(x)} \frac{\sqrt{(x - l)^2 + \sigma^2} - \frac{(x - l)^2}{\sqrt{(x - l)^2 + \sigma^2}}}{(x - l)^2 + \sigma^2}.
\]
Proof. As shown in Lemma 3.6, \( g(x) \) is defined by:

\[
g(x) = \left(1 - \frac{x\omega'(x)}{2\omega(x)}\right)^2 - \frac{\omega'(x)^2}{4} \left(\frac{1}{\omega(x)} + \frac{1}{4}\right) + \frac{\omega''(x)}{2}.
\]

(5.6)

In the SVI model, the total variance surface is parameterized as:

\[
\omega(x) = \sigma_{\text{SVI}}^2(x)T,
\]

with \( \sigma_{\text{SVI}}(x) \) as in (5.3). Note that we have dropped the \( T \) dependence of the total variance \( \omega(x, T) \) and abbreviated the notation to \( \omega(x) \) instead. We can rewrite (5.6) in terms of \( \sigma_{\text{SVI}}(x) \), so that we have

\[
g(x) = \left(1 - \frac{x\sigma_{\text{SVI}}'(x)}{\sigma_{\text{SVI}}(x)}\right)^2 - \frac{1}{4}\sigma_{\text{SVI}}(x)^2\sigma_{\text{SVI}}'(x)^2T^2 + \sigma_{\text{SVI}}(x)\sigma_{\text{SVI}}''(x)T.
\]

For the SVI model we have analytical expressions for the first and second partial derivatives with respect to \( x \). The computation of the partial derivatives of the SVI model can be found in Appendix A.2. Substituting these derivatives in \( g(x) \) proves the lemma.

Gatheral [15], claims that the SVI model is arbitrage-free in the strike direction if the parameter restrictions are satisfied. However, examples of parameter sets have been given [33] that satisfy the parameter restrictions, but still admit arbitrage.

**Example 5.1** (SVI smile with arbitrage). We take the following parameter set,

\[
\{a, b, \rho, l, \tilde{\sigma}\} = \{-0.0410, 0.1331, 0.3586, 0.3060, 0.4153\},
\]

for \( T = 1, [33] \). The corresponding function \( g(x) \) becomes negative for certain \( x \) values, as is shown in Figure 5.8 from [17]. Hence, the butterfly spread arbitrage condition is not satisfied, and the probability density function can become negative.

![Figure 5.8: Plots of the total variance \( \omega \) (left) and the function \( g(x) \), [17].](image)

In order to correctly define a local volatility model in terms of an SVI volatility surface, the arbitrage-free construction of the SVI surface is of the utmost importance. Because of the non-linear behavior of \( g \), there are no general conditions on the parameter values that would eliminate butterfly arbitrage, hence, during the calibration, we explicitly need to check whether \( g(x) \) is non-negative for all \( x \) values. For this, the bisection method can be used.

**Remark 5.2.** In [17], the SVI formulation in (5.3) is called the ‘raw SVI parameterization’, and the authors present alternative, but equivalent ways to parameterize the surface, for which the no-arbitrage conditions are easier to evaluate.
CHAPTER 5. BUILDING THE VOLATILITY SURFACE

5.3.3 Dupire local volatility on SVI volatility surface

In the remainder of this chapter the application of the Dupire local volatility formula on an SVI volatility surface under the Heston model will be referred to as the Dupire-SVI model. Recall that the local volatility in terms of implied volatilities is given by:

\[ \sigma_{LV}^2(K,T) = \frac{\sigma_{SVI}^2 + 2\sigma_{SVI} T \left( \frac{\partial \sigma_{SVI}}{\partial T} + r K \frac{\partial \sigma_{SVI}}{\partial K} \right)}{\left(1 - \frac{K}{\sigma_{SVI}} \frac{\partial \sigma_{SVI}}{\partial K} \right)^2 + K \sigma_{SVI} T \left( \frac{\partial \sigma_{SVI}}{\partial K} - \frac{1}{4} K \sigma_{SVI} T \left( \frac{\partial \sigma_{SVI}}{\partial K} \right)^2 + K \frac{\partial^2 \sigma_{SVI}}{\partial K^2} \right)}. \]

In the SVI case, the local volatility can be seen as a function of \( \sigma_{SVI}(x) \) with respect to strike and maturity. For a given time to maturity \( T \) the parameter set \( \{a, b, \rho, l, \tilde{\sigma}\} \) is the same for all values of strike \( K \). Hence, we can compute the partial derivatives of \( \sigma_{SVI}(x) \) with respect to strike and maturity. First, we compute the first and second derivative of the SVI volatility with respect to the log-moneyness \( x \):

\[ \sigma'_{SVI}(x) = \frac{b}{2\sigma_{SVI}(x)} \left( \rho + \frac{x-l}{\sqrt{(x-l)^2 + \tilde{\sigma}^2}} \right); \]

\[ \sigma''_{SVI}(x) = -\frac{b\sigma'_{SVI}(x)}{2\sigma_{SVI}(x)^2} \left( \rho + \frac{x-l}{\sqrt{(x-l)^2 + \tilde{\sigma}^2}} \right) + \frac{b}{2\sigma_{SVI}(x)} \frac{\sqrt{(x-l)^2 + \tilde{\sigma}^2} - \frac{(x-l)^2}{\sqrt{(x-l)^2 + \tilde{\sigma}^2}}}{(x-l)^2 + \tilde{\sigma}^2}. \]

The derivations can be found in Appendix A.2. Secondly, we need the partial derivatives of \( x \) with respect to the strike \( K \):

\[ \frac{\partial x}{\partial K} = \frac{1}{K}, \quad \frac{\partial^2 x}{\partial K^2} = -\frac{1}{K^2}, \quad \frac{\partial x}{\partial T} = -r. \]

Now we can apply the chain rule:

\[ \frac{\partial \sigma_{SVI}}{\partial K} = \sigma'_{SVI}(x) \frac{\partial x}{\partial K}; \]

\[ \frac{\partial^2 \sigma_{SVI}}{\partial K^2} = \sigma''_{SVI}(x) \left( \frac{\partial x}{\partial K} \right)^2 + \sigma'_{SVI}(x) \frac{\partial^2 x}{\partial K^2}. \]

Using these expressions we can define the local variance term in the Dupire-SVI model as

\[ \sigma_{LV}^2(x,T) = \frac{\sigma_{SVI}^2 + 2\sigma_{SVI} \frac{\partial \sigma_{SVI}}{\partial T} T + b r \left( \rho + \frac{x-l}{\zeta} \right) T}{\left(1 - \frac{K}{\sigma_{SVI}} \frac{\partial \sigma_{SVI}}{\partial K} \right)^2 - \frac{b^2}{4} \left( \rho + \frac{x-l}{\zeta} \right)^2 \left( \frac{1}{\sigma_{SVI}} + \frac{T}{4} \right) T + \frac{b \tilde{\sigma}^2}{2 \zeta^2} T}, \]

where

\[ \zeta = \sqrt{(x-l)^2 + \tilde{\sigma}^2}. \]

Since the parameters are time dependent, we can not simply take \( \frac{\partial \sigma_{SVI}}{\partial T} = \sigma'_{SVI}(x) \frac{\partial x}{\partial T} \). Furthermore, we have a fitted parameter set for the given expiries in the market only. Time-interpolation is required to obtain \( \frac{\partial \sigma_{SVI}}{\partial T} \).

Time-interpolation

Different types of parameter interpolations are possible. Linear interpolation can lead to jumps and discontinuities in the local volatility surface, hence, we require a sufficiently smooth interpolation technique. We can interpolate the parameters \( a(T), b(T), \rho(T), l(T) \) and \( \tilde{\sigma}(T) \) separately and derive analytical time-derivatives in line with the interpolation method. Denote \( a := a(T) \) and \( \dot{a} := \frac{da}{dT} \).
5.4. NUMERICAL RESULTS OF THE DUPIRE-SVI MODEL

Using these interpolated parameter values we can construct the whole SVI volatility surface for all strike and maturity values. The partial derivative of $\sigma_{\text{SVI}}(K, T)$ with respect to $T$ is given by

$$
\frac{\partial \sigma_{\text{SVI}}}{\partial T} = \frac{1}{2\sigma_{\text{SVI}}(x)} \left[ \hat{a} + b \left\{ \rho(x - \hat{l}) + \sqrt{(x - \hat{l})^2 + \hat{\sigma}^2} \right\} \right].
$$

(5.12)

The partial derivatives as given in (5.10), (5.11) and (5.12), together with the SVI formulation (5.3), can be used in Dupire’s formula for local volatility $\sigma_{\text{LV}}(K, T)$ as defined in (3.11). No further interpolation or calibration is needed. When more complicated interpolation schemes are introduced, the time derivative can become quite involved. In that case, we prefer a finite difference approximation:

$$
\frac{\partial \sigma_{\text{SVI}}}{\partial T} \approx \frac{\sigma_{\text{SVI}}(K, T + \Delta T) - \sigma_{\text{SVI}}(K, T)}{\Delta T}.
$$

(5.13)

It is important to check that for all $T$ the interpolated parameter sets satisfy the parameter restrictions and no-arbitrage requirements as well. In practice, it is is a challenging task to find correctly interpolated parameter values for all expiries.

A different approach is to interpolate the total variance $\omega$, instead of the parameters. For the given expiries, we can analytically compute the value of $\omega$ for all strike values $K$. Next, we interpolate $\omega$ over $T$ for every value of $K$.

The main advantage of total variance interpolation is that by using a monotonic cubic interpolation we can easily model the total variance as non-negative and increasing in $T$. Hence, only the butterfly arbitrage condition remains. A disadvantage is that by interpolating the total variance, we do not have the specific parameter values for the interpolated expiries. So we cannot use the analytical derivatives of the SVI model anymore. Instead, we use central difference methods for approximating the derivatives in strike direction:

$$
\frac{\partial \omega}{\partial K} \approx \frac{\omega(K + \Delta K, T) - \omega(K - \Delta K, T)}{2\Delta K};
$$

$$
\frac{\partial^2 \omega}{\partial K^2} \approx \frac{\omega(K + \Delta K, T) - 2\omega(K, T) + \omega(K - \Delta K, T)}{(\Delta K)^2}.
$$

(5.14)

And again a forward difference for the time direction:

$$
\frac{\partial \omega}{\partial T} \approx \frac{\omega(K, T + \Delta T) - \omega(K, T)}{\Delta T}.
$$

(5.15)

Once we have derived the partial derivatives of $\omega$ with respect to strike and expiry, we can use the expression of local volatilities in terms of the total variance, as stated in (3.10):

$$
\sigma_{\text{LV}}^2(K, \omega) = \frac{\frac{\partial \omega}{\partial T} + rK \frac{\partial \omega}{\partial K}}{(1 - \frac{1}{2} K \frac{\partial \omega}{\partial K})^2} + \frac{K \frac{\partial^2 \omega}{\partial K^2}}{(1 - \frac{1}{2} K \frac{\partial \omega}{\partial K})^2} - \frac{1}{4} K^2 \left( \frac{\partial \omega}{\partial K} \right)^2 \left( \frac{1}{4} + \frac{K}{\omega} \right).
$$

Note, at time to maturity $T = 0$, the total variance is zero for all $K$. We define the local volatility in $T = 0$ by: $\sigma_{\text{LV}}^2 = \frac{\partial \omega}{\partial T}$.

5.4 Numerical results of the Dupire-SVI model

In this section we present the results of the local volatility surface based on an SVI parameterization. An SVI model on market data of a non-dividend paying stock is considered. Different time interpolation methods are studied to construct an implied volatility surface from which a local volatility model is derived. The local volatility model is tested with a similar test case as in Section 5.1. Next we incorporate dividends and analyze the effects on the volatility surface. We test whether we can also reproduce the prices of dividend-paying stocks.
5.4.1 Constructing the volatility surface

The SVI model is fitted to option prices for Ebay. This dataset was chosen since no dividends are expected yet for Ebay, and thus the Ebay stock is a non-dividend paying stock. The model is calibrated to the data at a given set of expiries \( \{T_1, T_2, \ldots, T_N\} \), so that for each of these expiries we have a corresponding parameter set \( \{a_n, b_n, \rho_n, l_n, \tilde{\sigma}_n^2\} \). The fitted parameter values can be found in Appendix B. The SVI model is fitted in such a way that the variance is increasing over time. To obtain the term structure for all \( T \), we interpolate over time, and study the following types of interpolation:

- Method 1: Linear interpolation of SVI parameters, analytical computation of the time derivative (5.12) and local volatility model in terms of \( \sigma_{SVI} \) (3.11).
- Method 2: Monotonic cubic interpolation of SVI parameters, finite difference approximation of the time derivative (5.13) and local volatility model in terms of \( \sigma_{SVI} \) (3.11).
- Method 3: Monotonic cubic interpolation of total variance \( \omega \), numerical approximation of strike and time derivatives, (5.14), and (5.15), respectively, and local volatility model in terms of \( \omega \) (3.10).

Method 1

The volatility surfaces corresponding to Method 1 can be found in Figure 5.9. The plot of the implied volatility surface shows the characteristic smile. The total variance surface is smooth as well, but if we take a closer look at the surface we see that the total variance is not always increasing over time, which results in calendar spread arbitrage (Lemma 3.5), and also a negative local variance term. This causes complex valued local volatility (complex values are set to zero), which makes the surface not suitable for pricing. The discontinuities in the local variance and local volatility surface are caused by the discontinuities in the partial time derivative due to the linear interpolation of the SVI parameters.

![Figure 5.9: Volatility surfaces (Method 1)](image)

Method 2

We try a more advanced, shape-preserving, interpolation technique to interpolate the parameters. Furthermore, we use the finite difference approximation for the partial derivative with respect to time. The results are in Figure 5.10, and the surfaces for the implied volatility and total variance look similar to
those of Method 1. Clearly, these surfaces are, again, not arbitrage-free, since the local variance surface contains negative terms.

Figure 5.10: Volatility surfaces (Method 2).

The local variance value for small expiries are so extreme that the surface at higher expiries almost seems flat. Therefore, we plot the local variance and local volatility again, but now for $T \in [0, 1]$, the negative local variance values only occur for small expiries (Figure 5.11).

Figure 5.11: Volatility surfaces for $T \in [0, 1]$ (Method 2).

Method 3

To make sure the total variance is increasing for all expiries, we switch to interpolating the total variance itself. For this, we use a shape-preserving monotonic cubic interpolation. A downside of this method, is that we do not explicitly know the parameter values at the interpolated times, therefore we use numerical approximations of all required partial derivatives. The resulting volatility surfaces are presented in Figure 5.12. We see the total variance surface, which is now always increasing over time, and the corresponding, smooth, implied volatility surface. Method 3 results in slightly lower total variance values, compared to the first two methods, and thus in higher implied volatilities. The constructed local variance is now strictly positive, which gives us a well-defined local volatility surface, which is sufficiently smooth for pricing, as we will show in the next section. From here on, we focus on the third method.
5.4.2 Reproducing the market prices of vanillas

The local volatility model (Method 3) is analyzed with a simple test case. The constructed local volatility surface should be in line with the original SVI implied volatility surface, meaning that option pricing under the local volatility model should result in the same vanilla option prices as the Black-Scholes prices implied by the SVI volatility surface. Hence, to test this, we price an at-the-money call option, compute the implied volatility and compare it to the SVI volatility. We take parameter values $S = 57.12$, $K = 57.12$, $r = 0.01$, $T = 0.5$, which gives us a implied volatility value $0.266157$, and corresponding Black Scholes call option price: 4.41562. The call option under local volatility is priced with the Crank-Nicolson scheme, and the results are in Table 5.1.

<table>
<thead>
<tr>
<th>$N_x$</th>
<th>400</th>
<th>450</th>
<th>500</th>
<th>550</th>
<th>600</th>
<th>650</th>
<th>700</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implied vol</td>
<td>0.265162</td>
<td>0.265191</td>
<td>0.265127</td>
<td>0.265159</td>
<td>0.265097</td>
<td>0.265138</td>
<td>0.265121</td>
</tr>
<tr>
<td>Time (s)</td>
<td>0.030</td>
<td>0.033</td>
<td>0.037</td>
<td>0.045</td>
<td>0.045</td>
<td>0.049</td>
<td>0.052</td>
</tr>
</tbody>
</table>

Table 5.1: Numerical results of the Dupire-SVI model, target values are 4.41562 and 0.266157.

The implied volatility can be reproduced up to an absolute error of 1e-3, which is sufficiently close to the true SVI value. Next, we consider the implied volatility smile for different expiries. The obtained results for $T = 0.5$, $T = 1.0$, and $T = 1.5$ are shown in Figures 5.13, 5.14, and 5.15, respectively. The smiles are plotted against a relative strike value; $K$ is expressed as a percentage of the stock price. The computed implied volatilities closely match the initial SVI volatilities even for extreme strike values. We conclude that pricing under the local volatility model accurately reproduces the vanillas.
SVI compared to Heston

Heston has the advantage that interpolating over the parameters always results in an arbitrage-free model, but the Heston model describes the underlying dynamics, and is not a parameterization of the surface itself. Hence, to obtain the surfaces, numerical methods are required for the computation of call option prices and implied volatilities.

Although, the SVI model does not guarantee the absence of arbitrage, once a arbitrage-free surface is constructed, a parameterization of the whole surface is available, which makes it possible to accurately
compute the local volatility term, even for extreme strike values and small expiries, whereas the Heston model encounters difficulties. We observe that for these extreme strike values, local volatility based on the SVI surface performs better than the Dupire-Heston model.

5.4.3 Pricing vanillas in the presence of dividends

In this section, we consider the SVI model that is fitted to market data for a dividend-paying stock. We choose Microsoft, which usually pays four dividends per year. We take $T^* = 10$, thus we take the net present value of the expected dividend payments over the next ten years into account.

The SVI model is calibrated under $S_t$, but for the construction of the local volatility model, we need to switch to a volatility model in the clean space:

$$\sigma^2_{SVI}(\tilde{K},T) = a + b \left( \rho (\tilde{x} - l) + \sqrt{(\tilde{x} - l)^2 + \tilde{\sigma}^2} \right),$$

where the adjusted log-moneyness is given by:

$$\tilde{x} = \log \left( \frac{\tilde{K}}{\tilde{F}} \right) = \log \left( \frac{P(0,T)(K + D_T)}{S_0 - D_0} \right),$$

with $D_0$ and $D_T$, the NPV of the future dividends in $[0,T^*]$ and $[T,T^*]$ respectively, and the parameters $\{a,b,\rho,l,\tilde{\sigma}^2\}$ fitted in the $S_t$ setting. The calibrated parameters are given in Appendix B. We use Method 3 from Subsection 5.4.1 for the construction of the volatility surfaces. The obtained surfaces are shown in Figure 5.16.

![Figure 5.16: Volatility surfaces in the clean space.](image)

The surfaces in the clean space look similar to those in Subsection 5.4.1 for a non-dividend paying stock. The total variance is always increasing, and the local variance is non-negative. Next, we test whether we can accurately reproduce the implied volatility smile with the local volatility model for Microsoft. The obtained results for $T = 0.5$, $T = 1.0$, and $T = 1.5$ are shown in Figures 5.17, 5.18, and 5.19, respectively. The absolute errors are slightly larger than those for the non-dividend paying stock, but the resulting implied volatility values are still sufficiently close to the original SVI implied volatilities. Note that we plotted the smiles against $K$, and not in the clean space ($\tilde{K}$). If we take a look at the smiles for different expiries, we see the call option prices for fixed $K$ are not always increasing over time, which is due to the expected dividend payments between the expiries.
5.4. NUMERICAL RESULTS OF THE DUPIRE-SVI MODEL

Figure 5.17: Results for the local volatility model (T = 0.5)

Figure 5.18: Results for the local volatility model (T = 1)

Figure 5.19: Results for the local volatility model (T = 1.5)
Chapter 6

Numerical Methods for Pricing Barrier Options

In this chapter we discuss different numerical pricing methods for the computation of the option value. Different pricing approaches exist, as we can compute the expected discounted payoff or solve the corresponding pricing partial differential equation. We start with the finite difference approach. There are no closed-form solutions for the pricing partial differential equation of time dependent barrier options under non-constant volatility. Therefore, we approximate the partial differential equation by a difference equation using the Crank-Nicolson scheme [30].

A different approach is by computing the expected payoff. Usually the probability density function of the underlying process is not known, but for some models we know its characteristic function. The probability density function and the characteristic function form a Fourier pair, which give rise to a whole new class of pricing methods with Fourier techniques. Oosterlee and Fang [13] introduced a pricing method based on Fourier cosine series expansions, called the COS method. We discuss the COS method and its extension to barrier options in Section 6.2.

Next, we discuss forward-backward stochastic differential equations and their link with option pricing. We propose to price barrier options by solving reflected forward-backward stochastic differential equations. In Section 6.4, we discuss how the COS method can be used for solving forward-backward stochastic differential equations. This Backward SDE COS (BCOS) method has recently been presented by Ruijter and Oosterlee [24], [25]. We extend the BCOS method to Dupire’s local volatility and apply the BCOS method to reflected forward-backward SDEs, so that we can compute barrier option values. This is a new area of application of the BCOS method.

In Section 6.5, we present results of the BCOS method with local volatility. The BCOS method is applied to option pricing problems under Constant Elasticity of Variance and Dupire processes. We compare option prices under Dupire’s local volatility to those obtained by the Crank-Nicolson scheme. Pricing is done in the clean space, so we do not take dividends into account.

6.1 Finite difference method for option pricing

As discussed in Section 2.4, the value of a barrier option, here denoted by \( V(S, t) \), needs to satisfy the following pricing PDE:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 V \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0,
\]

with appropriate boundary conditions. With finite difference methods we approximate the pricing partial differential equation by a difference equation valid on a grid. We follow Wilmott [30]. Let \( \Delta t \) be the time step and \( \Delta S \) the asset step, such that the grid contains asset values \( S = i\Delta S \), and times \( t = T - m\Delta t \), where \( 0 \leq i \leq N_x \) and \( 0 \leq m \leq M \). Hence, we solve for the asset value going from \( S = 0 \) up to the asset value \( S := N_x \Delta S \). Since the Black-Scholes equation is valid in principle for
0 ≤ S < ∞, the domain needs to be chosen sufficiently large. Typically the boundaries of the domain should be around five times the standard deviation of the spot price. However, we often use a larger computational domain. For barrier options we do not need to solve for all values of S; there is no need to make the grid extend beyond the barrier. Before we discuss pricing barrier options we start with finite difference pricing methods in general. We discretize the partial differential equation. The option value at each grid point is denoted by:

\[ V_k^i = V(i\Delta S, T - m\Delta t), \]

so that the superscript is the time variable and the subscript the asset variable. Note that we work backwards in time, an increasing \( m \) value corresponds to a shorter time to expiry. We write the pricing PDE in a more general form:

\[
\frac{\partial V}{\partial t} + a(S, t) \frac{\partial^2 V}{\partial S^2} + b(S, t) \frac{\partial V}{\partial S} + c(S, t) V = 0
\]

We denote

\[
a_{m}^i = a(i\Delta S, T - m\Delta t), \quad b_{m}^i = b(i\Delta S, T - m\Delta t) \quad \text{and} \quad c_{m}^i = c(i\Delta S, T - m\Delta t).
\]

and approximate the partial differentials by finite difference equations.

### 6.1.1 Crank-Nicolson scheme

We use the Crank-Nicolson method for the pricing of equity derivatives. The Crank-Nicolson scheme is given by:

\[
\frac{V_i^m - V_i^{m+1}}{\Delta t} + \frac{a_{m}^i}{2} (V_{i+1}^{m+1} - 2V_i^{m+1} + V_{i-1}^{m+1}) \Delta S^2 + \frac{b_{m}^i}{2} (V_{i+1}^{m+1} - V_{i-1}^{m+1}) \Delta S + \frac{c_{m}^i}{2} V_i^{m+1} = 0.
\]

The Crank-Nicolson scheme is a second-order method in time. We rewrite the difference equation such that we have the \( m \) terms and the \( m + 1 \) terms on different sides:

\[-A_{i}^{m+1}V_{i-1}^{m+1} + (1 - B_{i}^{m+1}) V_{i}^{m+1} - C_{i+1}^{m+1}V_{i+1}^{m+1} = A_{i}^{m}V_{i-1}^{m} + (1 - B_{i}^{m}) V_{i}^{m} + C_{i}^{m}V_{i+1}^{m}.
\]

where

\[
A_{i}^{m} = \frac{1}{2}v_1a_{i}^{m} - \frac{1}{4}v_2b_{i}^{m}, \quad B_{i}^{m} = -v_1a_{i}^{m} + \frac{1}{2}\Delta t c_{i}^{m} \quad \text{and} \quad C_{i}^{m} = \frac{1}{2}v_1a_{i}^{m} + \frac{1}{4}v_2b_{i}^{m},
\]

with

\[
v_1 = \frac{\Delta t}{\Delta S^2}, \quad v_2 = \frac{\Delta t}{\Delta S}.
\]

These equations only hold for \( 1 \leq i \leq N_x - 1 \), since the boundaries need to be addressed separately, such that we have \( N_x - 1 \) equations with \( N_x + 1 \) unknowns. The Crank-Nicolson scheme can be written
The boundary conditions supply the two missing equations. For a call option we have boundary conditions:

\[ V^m_0 = 0 \quad \text{and} \quad V^m_{N_x} = N_x \Delta S - Ke^{-r m \Delta t}. \]  

(6.1)

But we also can take more advanced boundary conditions, which are more generally applicable to different options. Since \( S_t = 0 \) is an absorbing state for the underlying stock price process, for most options the payoff is fixed once \( S_t = 0 \), thus:

\[ \frac{\partial V}{\partial t}(0, t) - r V(0, t) = 0. \]

Numerically, this is represented by

\[ V^m_0 = (1 - r \Delta t) V^{m-1}_0. \]  

(6.2)

Furthermore, when the payoff is at most linear in the underlying stock for large values of \( S_t \) then one can use the upper boundary condition

\[ \frac{\partial^2 V}{\partial S^2}(S_t, t) \to 0 \quad \text{for} \quad S_t \to \infty. \]

The finite difference representation is

\[ V^m_{N_x} = 2 V^m_{N_x-1} - V^m_{N_x-2}. \]  

(6.3)

We prefer boundary conditions (6.2) and (6.3) over the boundary conditions in (6.1). The final conditions at time \( T \) are given by the payoff of the option and from there on we work backwards in time. For a call option we have:

\[ V^0_i = \max\{i \Delta S - K, 0\}. \]

Next, we illustrate the application of the Crank-Nicolson scheme by numerical examples. We start with examples under geometric Brownian motion (GBM).

**Example 6.1** (Pricing a European call option under GBM). We test the speed and accuracy of the Crank-Nicolson scheme by pricing a European call option. We set:

\[ S_0 = 100, \quad r = 0.01, \quad T = 1, \quad K = 100, \quad \bar{\sigma} = 0.3, \]

such that the Black-Scholes reference value is 12.3683. We take \( M = 200 \) time steps and vary the size of \( N_x \) for the domain \([0, 400]\). The observed errors are presented in Table 6.1.
### Table 6.1: Error when a European call option is priced with the CN method.

<table>
<thead>
<tr>
<th>$N_x$</th>
<th>Error</th>
<th>Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>250</td>
<td>9.97e-05</td>
<td>0.003</td>
</tr>
<tr>
<td>300</td>
<td>2.837e-03</td>
<td>0.005</td>
</tr>
<tr>
<td>350</td>
<td>5.35e-05</td>
<td>0.006</td>
</tr>
<tr>
<td>400</td>
<td>1.593e-03</td>
<td>0.007</td>
</tr>
<tr>
<td>450</td>
<td>3.47e-05</td>
<td>0.007</td>
</tr>
<tr>
<td>500</td>
<td>1.017e-03</td>
<td>0.008</td>
</tr>
<tr>
<td>550</td>
<td>2.52e-05</td>
<td>0.009</td>
</tr>
</tbody>
</table>

Figure 6.1 shows the convergence of the computed call option price towards the exact Black-Scholes price. The computed option value is oscillating. The oscillating behavior may be caused by the size of the uniform grid $[0, 400]$. Furthermore, we could improve the convergence by applying an implicit method for the first time steps, since the 'non-smoothness' at the strike can also cause the oscillations.

#### Example 6.2

(Pricing an up-and-out call option under GBM) In the case of an up-and-out call option we build our grid in such a way that the barrier level is the upper bound of the domain. Hence, we obtain boundary conditions:

$$V_0^m = 0 \quad \text{and} \quad V_N^m = 0,$$

with payoff function

$$V_i^m = \max\{i\Delta S - K, 0\}, \quad i = 0, 1, \ldots, N_x - 1,$$

if the barrier is not triggered. Note that we can only monitor whether the barrier is hit or not discretely, instead of continuously monitoring. Therefore, more time steps are required than in Example 6.1. We take $N_x = 700$ and vary $M$. In this example, we take the same parameter values as in Example 6.1 and consider a barrier at $S_t = 120$, so that we have parameter values:

$$S_0 = 100, \quad r = 0.01, \quad T = 1, \quad K = 100, \quad \bar{\sigma} = 0.3, \quad B = 120.$$

The corresponding Black-Scholes reference value of the up-and-out call option in this example is 0.42868. The observed errors and CPU times are given in Table 6.2.

### Table 6.2: Error when an up-and-out call option is priced with the CN method.

<table>
<thead>
<tr>
<th>$M$</th>
<th>Error</th>
<th>Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>3.1716e-02</td>
<td>0.003</td>
</tr>
<tr>
<td>100</td>
<td>6.615e-03</td>
<td>0.006</td>
</tr>
<tr>
<td>150</td>
<td>1.325e-03</td>
<td>0.008</td>
</tr>
<tr>
<td>200</td>
<td>3.489e-04</td>
<td>0.011</td>
</tr>
<tr>
<td>250</td>
<td>6.47e-05</td>
<td>0.014</td>
</tr>
<tr>
<td>300</td>
<td>1.65e-05</td>
<td>0.017</td>
</tr>
<tr>
<td>350</td>
<td>9.61e-06</td>
<td>0.02</td>
</tr>
</tbody>
</table>

The results of the numerical test are shown in Table 6.2. The Crank-Nicolson scheme can accurately reproduce the Black-Scholes price of an up-and-out call option, in small CPU times.
In Chapter 5, we have constructed an SVI volatility surface from which we can extract Dupire’s local volatility term. The constructed local volatility model should be in line with the original SVI implied volatility surface, meaning that option pricing under the local volatility model should result in the same vanilla option prices as the Black-Scholes prices implied by the SVI volatility surface.

As we explained in Section 3.3, the local volatility model is not consistent with the Black-Scholes prices for barrier options. We illustrate the pricing of barrier options under local volatility with an example.

**Example 6.3** (Option pricing under local volatility). In this example, we wish to price an up-and-out call option under Dupire local volatility. We use the Dupire-SVI volatility model based on market data from Ebay (Section 5.4.1), and use parameter values

\[ S_0 = 57.12, \quad K = 57.12, \quad r = 0.01, \quad T = 1.0, \quad B = 100. \]

We present the numerical results of the regular call option, and the option value of the up-and-out barrier call option in Table 6.3. We compare the option values to the Black-Scholes prices implied by the SVI volatility surface, 6.3775, and 5.0711, respectively.

<table>
<thead>
<tr>
<th>( M )</th>
<th>call price</th>
<th>UO call price</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>6.31517</td>
<td>4.15633</td>
</tr>
<tr>
<td>100</td>
<td>6.35505</td>
<td>5.18746</td>
</tr>
<tr>
<td>150</td>
<td>6.35855</td>
<td>5.25242</td>
</tr>
<tr>
<td>200</td>
<td>6.36145</td>
<td>5.25394</td>
</tr>
<tr>
<td>500</td>
<td>6.36447</td>
<td>5.25513</td>
</tr>
<tr>
<td>1000</td>
<td>6.36669</td>
<td>5.25469</td>
</tr>
</tbody>
</table>

Table 6.3: Pricing under Dupire-SVI.

Like we expected, it looks as if the local volatility price of the regular call option converges to the implied Black-Scholes price, whereas the price the up-and-out call option does not.

### 6.1.2 Time-dependent barrier options

We discussed in Chapter 4 how to incorporate dividends into our local volatility model; instead of pricing under the stock price \( S_t \), we price under the clean process \( \tilde{S}_t \), so that we do not encounter any dividends. However, when pricing barrier options in the clean space, the barrier level needs to be adjusted, hence, the clean space equivalent of an ordinary barrier options has a time-dependent barrier level.

In this subsection we discuss how to price time-dependent barrier options with the Crank-Nicolson scheme. The pricing PDE of the up-and-out call option is now given by:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 (S_t, t) S_t^2 \frac{\partial^2 V}{\partial S^2} + r S_t \frac{\partial V}{\partial S} - r V = 0,
\]

with boundary conditions:

\[ V(0, t) = 0, \quad V(B_t, t) = 0 \quad \text{for} \quad t < T, \quad \text{and} \quad V(S_T, T) = \max \{ S_T - K, 0 \}, \]

where

\[ B_t = B - \text{NPV Div}(t, T^*). \]

We approximate \( \tilde{B}_t \) by a step-function, for this we define a time grid \( 0 =: t_0 < t_1 < ... t_{\tilde{M}} := T^* \), such that we have

\[ \tilde{B}_t = \sum_{m=1}^{\tilde{M}} \frac{\tilde{B}_{t_m} - \tilde{B}_{t_{m-1}}}{2} 1_{\{t_{m-1} \leq t < t_m\}}. \]

**Example 6.4** (Time-dependent barrier level). We consider an up-and-out call option with parameter values

\[ S_0 = 100, \quad r = 0.01, \quad T = 0.5, \quad K = 100, \quad \sigma = 0.3, \]
and time-dependent barrier level

\[ B_t = \begin{cases} 
130 & t \in [0.25, 0.5]; \\
120, & t \in [0, 0.25]. 
\end{cases} \]

We take \( N_x = 700 \) and vary \( M \), the results are shown in Table 6.4. Pricing is done in two steps, firstly, we compute the option value on the time interval \([0, 0.5]\) for a stock price domain \([0, 130]\). Next, we resized our grid and computed the option value on time interval \([0, 0.25]\). This leads to an increase in the computational time.

<table>
<thead>
<tr>
<th>( M )</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>400</th>
<th>500</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>0.954184</td>
<td>0.952866</td>
<td>0.952855</td>
<td>0.952857</td>
<td>0.952858</td>
</tr>
<tr>
<td>Time (s)</td>
<td>0.298</td>
<td>0.551</td>
<td>0.802</td>
<td>1.053</td>
<td>1.315</td>
</tr>
</tbody>
</table>

Table 6.4: Results under time-dependent barrier level \( B_t \).

6.2 Numerical Fourier method for option pricing

Finite difference methods are used to solve the pricing PDE, whereas Fourier methods are based on the characteristic function of the underlying process for the computation of the expected payoff. In 2008, Fang and Oosterlee [13] introduced a numerical Fourier method for pricing financial derivatives based on the Fourier cosine series expansion of the underlying process. We discuss this Cosine Series Expansion (COS) method in Section 6.2.1. In 2009, the COS method was extended to pricing early-exercise and discretely monitored barrier options [14]. In Section 6.2.2, we apply the COS method to discretely and continuously monitored barrier options under constant volatility.

6.2.1 Cosine Series Expansion (COS) method

In this subsection we discuss the COS method, we follow the steps and notations from the book by Oosterlee et al [23]. From the Fundamental Theorem of Asset Pricing (Theorem 2.2) it follows that the price at time \( t \) of any contingent claim on \( S_t \) with payoff \( g(S_T) \) at time \( T \) is given by:

\[ V(S_t, t) = P(t,T)E^Q [g(S_T)|\mathcal{F}_t], \]

in integral form we can write this expectation as:

\[ V(S_t, t) = P(t,T) \int g(S_T) f(y, T|S_t, t) dy, \]  

(6.4)

where \( f(\cdot|\cdot) \) is the conditional density function. The density function of the underlying stochastic process is usually not known. However, for many models, e.g., the Heston stochastic volatility dynamics or the geometric Brownian motion with constant volatility, the characteristic function is known.

**Definition 6.1 (Characteristic function).** Let \( X \) be a random variable with density function \( f(y) \). Then the characteristic function of \( X \) is defined to be

\[ \phi(u) := E [e^{iuX}] = \int_R e^{iu}f(y)dy. \]

The characteristic functions of affine processes have explicit forms. Here, we assume the characteristic function of the underlying process is given. For the COS method, we start by explaining how a probability density function can be recovered from its characteristic function. First of all, the density \( f(y) \) and its characteristic function \( \phi(u) \) form a Fourier pair.

**Lemma 6.1 (Inverse Fourier Transform).** Let \( \phi(u) \) be the characteristic function corresponding to random variable \( X \). Then we can recover the probability density function \( f(y) \) of \( X \) as the inverse Fourier transform of \( \phi(u) \):

\[ f(y) = \frac{1}{2\pi} \int_R e^{-iyu} \phi(u)du. \]
CHAPTER 6. NUMERICAL METHODS FOR PRICING BARRIER OPTIONS

Proof. For a proof we refer to [29], Chapter 16.

Furthermore, for a function supported on a finite interval $[a, b] \in \mathbb{R}$, the Fourier cosine series expansion can be obtained.

Note. We follow the notation in [23]. Therefore, in this chapter, the symbols $a$ and $b$ do not denote the SVI parameters, but are used to define a finite interval $[a, b]$.

Lemma 6.2 (Fourier cosine series expansion). Let $f(y)$ be a real-valued function, supported on a finite interval $[a, b] \in \mathbb{R}$. Then we can compute $f(y)$ by its Fourier cosine series expansion:

$$f(y) = \sum_{k=0}^{\infty} A_k \cos \left( k\pi \frac{y-a}{b-a} \right),$$

where

$$A_k := \frac{2}{b-a} \int_{a}^{b} f(y) \cos \left( k\pi \frac{y-a}{b-a} \right) dy.$$  \hfill (6.5)

Proof. For a proof we refer to [31].

Note. We use $\sum'$ to denote that the first Fourier cosine coefficient is weighted by $0.5$.

We use Lemma 6.1 and Lemma 6.2 to recover the conditional probability density function from its characteristic function.

Theorem 6.3 (COS density recovering). Let $f(x|y)$ be the conditional probability density function as in 6.4. Then we can approximate $f(y|x)$ by

$$\hat{f}(y|x) = \sum_{k=0}^{N-1} \tilde{A}_k \cos \left( k\pi \frac{y-a}{b-a} \right),$$

where

$$\tilde{A}_k = \frac{2}{b-a} \Re \left\{ \phi \left( \frac{k\pi}{b-a} \right| x \right\} \exp \left\{ -ik\pi \frac{a}{b-a} \right\}. \hfill (6.6)$$

Proof. Since any real function with finite support has a cosine series expansion (Lemma 6.2), we truncate the infinite integration range of the density function. Suppose $[a, b] \in \mathbb{R}$ is chosen so that the truncated integral approximates the infinite counterpart. Then we define the characteristic function corresponding to the truncated integral as:

$$\hat{\phi}(u|x) := \int_{a}^{b} e^{iyu} f(y|x)dy \approx \int_{\mathbb{R}} e^{iyu} f(y|x)dy = \phi(u|x).$$

We substitute the Fourier argument $u = \frac{k\pi}{b-a}$, and multiply by a factor $\exp\{-ik\pi \frac{a}{b-a}\}$:

$$\hat{\phi} \left( \frac{k\pi}{b-a} \right| x \right) \exp \left\{ -ik\pi \frac{a}{b-a} \right\} = \int_{a}^{b} \exp \left\{ iy \frac{k\pi}{b-a} - ik\pi \frac{a}{b-a} \right\} f(y|x)dy.$$

Take the real parts of both sides, and use $\Re\{e^{iu}\} = \cos u$:

$$\Re \left\{ \hat{\phi} \left( \frac{k\pi}{b-a} \right| x \right\} \exp \left\{ -ik\pi \frac{a}{b-a} \right\} = \int_{a}^{b} \cos \left( k\pi \frac{y-a}{b-a} \right) f(y|x)dy.$$

The right-hand side equals the definition of Fourier cosine coefficients $A_k$ in (6.5), so:

$$A_k = \frac{2}{b-a} \Re \left\{ \hat{\phi} \left( \frac{k\pi}{b-a} \right| x \right\} \exp \left\{ -ik\pi \frac{a}{b-a} \right\}.$$
The coefficients $A_k$ are based on $\hat{\phi}(u)$, corresponding to the truncated integration range. But we want to use the original characteristic function $\phi(u)$ of the underlying process. Hence, we replace $A_k$ by $\bar{A}_k$:

$$\bar{A}_k = \frac{2}{b-a} \text{Re} \left\{ \phi \left( \frac{k\pi}{b-a} \right| x \right\} \exp \left\{ -ik\pi \frac{a}{b-a} \right\}.$$ 

The last step is to truncate the series summation, such that we have

$$f(y|x) \approx N^{-1} \sum_{k=0}^{N-1} \bar{A}_k \cos \left( k\pi y - a \frac{y-a}{b-a} \right).$$

Hence, if the characteristic function is known, we can use the COS formula (6.6) to recover the density function. An error analysis is given in [13]. We illustrate this with an example.

**Example 6.5 (Density recovering).** To test the approximated density function (6.6) we analyze the results of the recovered density function of the standard normal distribution, for which the probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$ 

In Table 6.5 we see that the COS method is highly efficient for density recovering. Only a small numbers of terms $N$ is needed for a small error. The results are computed over an integration range $[a, b] = [-10, 10]$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Error</th>
<th>Time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.2538</td>
<td>0.004703</td>
</tr>
<tr>
<td>8</td>
<td>0.1075</td>
<td>0.004777</td>
</tr>
<tr>
<td>16</td>
<td>0.0072</td>
<td>0.004929</td>
</tr>
<tr>
<td>32</td>
<td>4.0376e-07</td>
<td>0.004942</td>
</tr>
<tr>
<td>64</td>
<td>2.7756e-17</td>
<td>0.005230</td>
</tr>
</tbody>
</table>

Table 6.5: Maximum error when $f(x)$ is recovered from $\phi(u)$ by a Fourier cosine expansion.

To price financial derivatives with European payoff, we substitute the approximated (recovered) density function $\hat{f}(x)$ in the pricing formula (6.4).

**Theorem 6.4 (COS pricing formula).** Let $g(y)$ be the payoff function of a European option under a process $X_t$, with corresponding conditional characteristic function $\phi(u|x)$. Then the option value $V(x, t)$ at time $t$, for $X_t = x$, can be approximated by

$$\hat{V}(x, t) = P(t, T) \sum_{k=0}^{N-1} \text{Re} \left\{ \phi \left( \frac{k\pi}{b-a} \right| x \right\} \exp \left\{ -i k \frac{a \pi}{b-a} \right\} \mathcal{V}_k(T),$$

where $\mathcal{V}_k(T)$ is the $k$th Fourier cosine coefficient of the payoff at the terminal time:

$$\mathcal{V}_k(T) = \frac{2}{b-a} \int_a^b g(y) \cos \left( k\pi \frac{y-a}{a-b} \right) dy.$$ 

**Proof.** We insert the COS approximation of the conditional probability density function $\hat{f}(y|x)$ in the pricing formula (6.4).

$$\hat{V}(x, t) = P(t, T) \int_a^b g(y) \sum_{k=0}^{N-1} \bar{A}_k \cos \left( k\pi \frac{y-a}{b-a} \right) dy$$

$$= P(t, T) \int_a^b g(y) \sum_{k=0}^{N-1} \frac{2}{b-a} \text{Re} \left\{ \phi \left( \frac{k\pi}{b-a} \right| x \right\} e^{-ik\pi \frac{a}{b-a}} \cos \left( k\pi \frac{y-a}{b-a} \right) dy.$$
By interchanging the integral and summation (Fubini’s Theorem) in Equation (6.9), we derive:

$$
\hat{V}(x, t) = P(t, T) \sum_{k=0}^{N-1} \text{Re} \left\{ \phi \left( \frac{k \pi}{b-a} \bigg| x \right) e^{-ik \pi \frac{x}{a+b}} \right\} \frac{2}{b-a} \int_{a}^{b} g(y) \cos \left( k \pi \frac{y-a}{a-b} \right) dy,
$$

we define $V_k(T)$:

$$
V_k(T) := \frac{2}{b-a} \int_{a}^{b} g(y) \cos \left( k \pi \frac{y-a}{a-b} \right) dy.
$$

With the COS pricing formula (6.8) we can compute the price of European options. We demonstrate the COS method for the European call option under geometric Brownian motion. For this, we switch to the adjusted log-stock process $X_t = \log S_t/K$, so that we have

$$
g(y) = K (e^y - 1)^+, \tag{6.13}
$$

for the payoff. Hence, the Fourier cosine coefficients are given by

$$
V_k(T) = \frac{2}{b-a} \int_{a}^{b} K (e^y - 1)^+ \cos \left( k \pi \frac{y-a}{a-b} \right) dy.
$$

In this case, the coefficients can be computed analytically, so for a call option we denote the terminal coefficient by:

$$
G_k(0, b, a, b) := \frac{2}{b-a} K [\chi_k(0, b, a, b) - \psi_k(0, b, a, b)], \tag{6.10}
$$

where the function $\chi_k$ and $\psi_k$ are given by the integrals:

$$
\chi_k(z_1, z_2, a, b) = \int_{z_1}^{z_2} e^y \cos \left( k \pi \frac{y-a}{b-a} \right) dy \quad \text{and} \quad \psi_k(z_1, z_2, a, b) = \int_{z_1}^{z_2} \cos \left( k \pi \frac{y-a}{b-a} \right) dy,
$$

with analytic solutions (see [26]):

$$
\chi_k(z_1, z_2, a, b) = \frac{1}{1 + \left( \frac{k \pi}{b-a} \right)^2} \left[ \cos \left( k \pi \frac{z_2-a}{b-a} \right) e^{z_1} - \cos \left( k \pi \frac{z_1-a}{b-a} \right) e^{z_2} \right] + \frac{k \pi}{b-a} \sin \left( k \pi \frac{z_2-a}{b-a} \right) e^{z_1} - \frac{k \pi}{b-a} \sin \left( k \pi \frac{z_1-a}{b-a} \right) e^{z_2}, \tag{6.11}
$$

and,

$$
\psi_k(z_1, z_2, a, b) = \begin{cases} 
\left[ \sin \left( k \pi \frac{z_2-a}{b-a} \right) - \sin \left( k \pi \frac{z_1-a}{b-a} \right) \right] \frac{b-a}{k \pi} & \text{if } k \neq 0; \\
\frac{z_2-z_1}{z_2-z_1} & \text{if } k = 0.
\end{cases} \tag{6.12}
$$

Before we can start the numerical computation of option prices with formula (6.8), we first need to define the integration range $[a, b]$. The size of the integration range is important for accurate pricing. A small interval will lead to a larger integration-range truncation error, whereas a larger interval requires many Fourier cosine coefficients $N$ to achieve a certain level of accuracy. We follow the choice of integration range as given in [23], where we center the domain at $x = \log(S/K)$ and use the cumulants of the underlying stochastic process to derive the size, i.e.

$$
[a, b] := \left[ (\zeta_1 + x) - L \sqrt{\zeta_2 + \sqrt{\zeta_4}}, (\zeta_1 + x) + L \sqrt{\zeta_2 + \sqrt{\zeta_4}} \right], \tag{6.13}
$$

where $L \in [6, 12]$ is a constant, we take $L = 10$, and $\zeta_1, \ldots, \zeta_4$ being the cumulants of the underlying stochastic process. Note that the fourth cumulant of the underlying stochastic process, for example in the Heston model, is quite involved, which makes it harder to accurately derive the integration range.

Now we have all the information to start option pricing by the COS method. In Example 6.6 we use the COS method for the pricing of a European call option.
Example 6.6 (Pricing with the COS method). We test the COS method by pricing a European call option under a geometric Brownian motion. For this we switch to the log-stock process \( X_t = \log S_t \), such that we have

\[
dX_t = \left( r - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t.
\]

The characteristic function of the geometric Brownian motion reads

\[
\phi(u | x, t) = \mathbb{E} \left[ e^{iuX_t} | X_t = x \right] = \exp \left\{ iux + iu \left( r - \frac{1}{2} \sigma^2 \right) (T - t) - \frac{1}{2} \sigma^2 u^2 (T - t) \right\},
\]

where we have used that the increments of the Brownian motion \( W_t \) are normally distributed:

\[
(W_t + \Delta t - W_t) \sim N(0, \Delta t).
\]

The cumulants are

\[
\zeta_1 = \left( r - \frac{1}{2} \sigma^2 \right) (T - t), \quad \zeta_2 = \sigma^2 (T - t), \quad \text{and} \quad \zeta_4 = 0.
\]

We test the COS method for different strike values. In this example, we choose \( S_0 = 100, \ r = 0.1, \ T = 1, \ K \in \{80, 100, 120\}, \ \text{and} \ \sigma = 0.3 \).

The Black-Scholes reference values are \([29.4317, 16.7341, 8.6063]\). We vary the number of Fourier coefficients \( N \) and present the observed errors for the three strike values in Table 6.6. It shows that we can accurately reproduce the Black-Scholes value of the call option by the COS method.

<table>
<thead>
<tr>
<th>( N )</th>
<th>16</th>
<th>32</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error K = 80</td>
<td>2.8289e-01</td>
<td>6.6685e-06</td>
<td>1.7764e-14</td>
</tr>
<tr>
<td>Error K = 100</td>
<td>2.9773e-01</td>
<td>4.9562e-06</td>
<td>3.2330e-13</td>
</tr>
<tr>
<td>Error K = 120</td>
<td>3.2394e-01</td>
<td>6.5925e-06</td>
<td>2.5935e-13</td>
</tr>
<tr>
<td>Time (msec)</td>
<td>7.282</td>
<td>0.106</td>
<td>0.126</td>
</tr>
</tbody>
</table>

Table 6.6: Error when pricing an European call option with the COS method.

In Example 6.6, we have performed the computations for different strike values simultaneously. This is possible due to a useful property of Lévy processes.

Lemma 6.5 (COS Pricing formula for Lévy processes). If the underlying process is a Lévy process, we can simplify the COS pricing formula to

\[
\hat{V}(x, t) = P(t, T) \sum_{k=0}^{N-1} \text{Re} \left\{ \phi \left( \frac{k\pi}{b-a} \right) \exp \left\{ i k \pi \frac{x-a}{b-a} \right\} \right\} V_k(T) \tag{6.14}
\]

Proof. For Lévy processes the conditional characteristic function can be represented by

\[
\phi(u | x) = \phi(u) \cdot e^{iu x} \quad \text{with} \quad \phi(u) := \phi(u | x = 0).
\]

This property gives us

\[
\hat{V}(x, t) = P(t, T) \sum_{k=0}^{N-1} \text{Re} \left\{ \phi \left( \frac{k\pi}{b-a} \right) \exp \left\{ -i \frac{k a \pi}{b-a} \right\} \right\} V_k(T)
\]

\[
= P(t, T) \sum_{k=0}^{N-1} \text{Re} \left\{ \phi \left( \frac{k\pi}{b-a} \right) \exp \left\{ i k \pi \frac{x-a}{b-a} \right\} \right\} V_k(T).
\]

We refer to [23], for a complete proof. □

Examples of exponential Lévy processes are the Heston model and geometric Brownian motion, so for both underlying models we can write the conditional characteristic function as the product of the unconditional characteristic function and an exponent. Therefore, Lemma 6.5 enables us to compute \( V(x, t) \) for a vector of strike values \( K \) simultaneously, which is important for calibration.
Corollary 6.6. For a given column vector of strikes, \( K = [K_1, \ldots, K_n]^T \), we consider the vector
\[
\mathbf{x} = \left[ \log \left( \frac{S_0}{K_1} \right) , \ldots , \log \left( \frac{S_0}{K_n} \right) \right]^T
\]
such that, for Lévy processes, we compute European call options price for different strike values simultaneously with:
\[
V(x, t) = P(t, T) K \sum_{k=0}^{N-1} \Re \left\{ \phi \left( \frac{k\pi}{b-a} \right) \exp \left\{ i k \pi \frac{x-a}{b-a} \right\} \right\} \hat{G}_k',
\]
where \( \hat{G}_k' \) is given by
\[
\hat{G}_k' := \frac{2}{b-a} \left[ \chi_k(0, b, a, b) - \psi_k(0, b, a, b) \right].
\]

Remark 6.1 (The Greeks). We can easily compute the Greeks by the COS method as well [14], e.g., we can compute the delta and gamma of an option by
\[
\hat{\Delta}_t = P(t, T) \frac{2}{b-a} \sum_{k=0}^{N-1} \Re \left\{ \phi \left( \frac{k\pi}{b-a} \right) e^{i \pi \frac{a-x}{b-a}} \right\} \frac{\hat{V}_k(T)}{S_t};
\]
\[
\hat{\Gamma}_t = P(t, T) \frac{2}{b-a} \sum_{k=0}^{N-1} \Re \left\{ \phi \left( \frac{k\pi}{b-a} \right) e^{i \pi \frac{a-x}{b-a}} \left[ -i k \pi \frac{x-a}{b-a} + \left( \frac{i k \pi}{b-a} \right)^2 \right] \right\} \frac{\hat{V}_k(T)}{S_t^2}.
\]

Above, we have discussed option pricing with the COS method for European options. In the next subsection we discuss how to price path-dependent options with the COS method under constant volatility. In case of a local volatility term the use of the COS method is not straightforward. We discuss the local volatility extension of the COS method in Section 6.4.

6.2.2 COS method for barrier options

In Section 6.2.1 we discussed how to price European options with the COS method. Fang and Oosterlee [14] extended the COS method to early-exercise and (discrete) barrier options. In this section we show how the COS method indeed can be applied to pricing barrier options and discuss an efficient algorithm proposed in [14], for the computation of the corresponding Fourier coefficients.

For the pricing of the barrier options, we choose a set of observation dates \( t_1 < \cdots < t_M = T \), and at each time step we monitor whether the barrier has been hit or not. Consider an up-and-out call option that is monitored \( M \) times. We denote the barrier with \( B \) and \( \xi := \log(B/K) \). Then the option price satisfies the following recursive formula:
\[
V(x, t_{m-1}) = \begin{cases} 
0, & x \geq \xi, \\
e^{-r(t_m-t_{m-1})} \int_{\xi}^x V(y, t_m) f(y) dy, & x < \xi,
\end{cases}
\]
for \( m = M, M-1, \ldots, 2 \). We approximate the option value \( V(x, t_{m-1}) \) for \( x < \xi \) by the COS formula (6.14):
\[
\hat{V}(x, t_{m-1}) = e^{-r(t_m-t_{m-1})} \sum_{k=0}^{N-1} \Re \left\{ \phi \left( \frac{k\pi}{b-a} \right) e^{i k \pi \frac{x-a}{b-a}} \right\} V_k(t_m),
\]
where \( V_k(t_m) \) are the Fourier-cosine series coefficients of \( V(x, t_m) \) on \([a, b] \), i.e.
\[
V_k(t_m) = \frac{2}{b-a} \int_a^b V(y, t_m) \cos \left( k \pi \frac{y-a}{b-a} \right) dy.
\]
The up-and-out option has no value on \([\xi, b] \), this gives us
\[
V_k(t_m) = \frac{2}{b-a} \int_{\xi}^b V(y, t_m) \cos \left( k \pi \frac{y-a}{b-a} \right) dy.
\]
We assume in this section that the underlying follows a geometric Brownian motion with constant volatility. Therefore, we can use the unconditional characteristic function (Lemma 6.5). At the time of expiry, we can compute the Fourier cosine coefficients analytically

\[ \mathcal{V}_k(t_M) = \begin{cases} G_k(0, \xi, a, b), & \xi \geq 0, \\ 0, & \xi < 0, \end{cases} \]

where the \( G_k \) are again the coefficients corresponding to the payoff of a call option (6.10) over an interval \([0, \xi]\). We can interpret the definition of \( \mathcal{V}_k(t_M) \) as follows, a negative value for \( \xi \) corresponds to a barrier level below the strike value. Hence, for \( \xi < 0 \), the option value is zero: \( V(y, T) = 0 \). In case of a barrier level above the strike value (\( \xi > 0 \)), the payoff of an up-and-out barrier equals the payoff of a regular call option on \([a, \xi]\).

Given the terminal Fourier cosine coefficients, we work backwards in time to compute the coefficients at earlier times, \( t_1, t_2, \ldots, t_{M-1} \).

\[
\hat{V}_k(t_{m-1}) = \frac{2}{b-a} \int_a^\xi \hat{V}(y, t_{m-1}) \cos \left( \frac{k\pi y}{a-b} \right) dy \\
= e^{-r(t_{m-1}-t_m)} \frac{2}{b-a} \int_a^\xi \sum_{j=0}^{N-1} \text{Re} \left\{ \phi \left( \frac{j\pi}{b-a} \right) e^{-i\frac{k\pi}{b-a}} \hat{V}_j(t_m) \right\} \cos \left( \frac{k\pi y}{a-b} \right) dy \\
= e^{-r(t_{m-1}-t_m)} \sum_{j=0}^{N-1} \text{Re} \left\{ \phi \left( \frac{j\pi}{b-a} \right) \hat{V}_j(t_m) M_{k,j}(a, \xi) \right\},
\]

where the matrix elements \( M_{k,j} \) are given by:

\[
M_{k,j}(a, \xi) := \frac{2}{b-a} \int_a^\xi \exp \left\{ ij\pi \frac{y}{b-a} \right\} \cos \left( \frac{k\pi y}{a-b} \right) dy. \tag{6.15}
\]

For barrier options with a time-dependent barrier level \( B(t) \) the size of the interval \([a, \xi]\) varies over time. Since the values \( \hat{V}_k \) are independent of \( x \), we can calculate the coefficients for different values of \( x \) simultaneously. For this we use a matrix consisting of the elements \( M_{k,j} \). Fang and Oosterlee [14] proposed an efficient algorithm based on the Fast Fourier Transform, which is used for the computation of \( \hat{V}_k \). If the underlying process is not a Lévy process the efficient FFT based algorithm can not be applied in a straightforward way, in that case we can use a discrete cosine transform, which we will discuss in Section 6.4.1.

The option value at \( t_0 \) is approximated by

\[
\hat{V}(x, t_0) = e^{-r(t_1-t_0)} \sum_{k=0}^{N-1} \text{Re} \left\{ \phi \left( \frac{k\pi}{b-a} \right) e^{ik\pi (x-a)/b-a} \right\} \hat{V}_k(t_1). \tag{6.16}
\]

We illustrate the barrier extension of the COS method with two examples. Firstly, we price a discretely monitored barrier option. Next, we consider the pricing of an option with a continuously monitored barrier.

**Example 6.7** (Pricing a discrete down-and-out call option with the COS method). We test the COS method by pricing a down-and-out call option under a geometric Brownian motion.

\[
S_0 = 100, \quad r = 0.1, \quad T = 0.2, \quad K = 100, \quad B = 95, \quad \text{and} \quad \bar{\sigma} = 0.6.
\]

The barrier is monitored \( M = 4 \) times, thus at times \( t = 0.05, 0.1, 0.15 \) and \( t = 0.2 \). The COS results are compared to the reference value 9.49052. We vary the number of Fourier coefficients \( N \) and present the observed errors in Table 6.7. We observe that the COS method can accurately reproduce the reference value of the down-and-out call option.
CHAPTER 6. NUMERICAL METHODS FOR PRICING BARRIER OPTIONS

<table>
<thead>
<tr>
<th>$N$</th>
<th>16</th>
<th>32</th>
<th>64</th>
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<tr>
<td>Error</td>
<td>6.7398e-02</td>
<td>1.3041e-03</td>
<td>1.4708e-05</td>
</tr>
<tr>
<td>Time (msec)</td>
<td>0.27831</td>
<td>0.25144</td>
<td>0.30729</td>
</tr>
</tbody>
</table>

Table 6.7: Error when a discrete down-and-out call option is priced with the COS method.

Remark 6.2. In case of continuously monitored barriers, small time steps are required. Broadie et al [5] proposed a continuity correction for the discretely monitored barrier option, so that the discretely monitor option converges faster to the continuously monitored barrier.

\[
\hat{B} = B \exp\{\pm \beta \bar{\sigma} \sqrt{T/m}\},
\]

(6.17)

with $-$ for an up-and-out option and $+$ for a down-and-out option, where the constant $\beta$ is given by $\beta = \zeta(1/2)/\sqrt{2\pi} \approx 0.5826$, where $\zeta$ is the Riemann zeta function. We use this continuity correction in all our numerical examples.

Example 6.8 (Pricing a continuously monitored up-and-out option with the COS method). We test whether the COS method can accurately reproduce the analytical value of a continuous up-and-out call option as well. For this, we choose a fixed number of Fourier coefficients $N = 1024$ and vary the number of monitoring times $M$. We use parameter values:

$S_0 = 100, \ r = 0.01, \ T = 0.2, \ K = 100, \ B = 125$, and $\bar{\sigma} = 0.25$.

The corresponding Black-Scholes up-and-out call option price $3.0914$ is used as a reference value. The option value of a regular call option is $3.2017$. The observed errors and CPU times are presented in Table 6.8. We conclude that we can accurately price barrier options with the COS method. However, a large number of time steps $M$ is required.

<table>
<thead>
<tr>
<th>$M$</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>10000</th>
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<tr>
<td>Error</td>
<td>1.2844e-02</td>
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<td>1.2193e-04</td>
<td>2.7295e-05</td>
</tr>
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<td>Time (sec)</td>
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<td>1.7490e-02</td>
<td>1.7157e-01</td>
<td>1.7407</td>
</tr>
</tbody>
</table>

Table 6.8: Error when a continuously monitored up-and-out call option is priced by the COS method.

In this section we discussed how to price European options and barrier options under constant volatility. For local volatility models, the corresponding characteristic function of the underlying process is often not known. In Section 6.4 we discuss how to extend the COS method to local volatility models.

6.3 Forward-backward SDEs and option pricing

In this section we discuss how we can price barrier options under local volatility by solving a forward-backward SDE. We start out with a definition of forward-backward SDEs. Next, we show how FBSDEs can be used in derivative pricing, and propose to price the barrier option by solving a reflected forward-backward SDE. To solve the forward-backward SDE we derive a discretization scheme, following the paper by Ruijter and Oosterlee [25].

6.3.1 Forward-backward SDEs

In financial mathematics there are many examples of forward stochastic differential equations (FSDE), which have to satisfy an initial condition.

\[
X_t = x, \quad dX_s = \mu(X_s, s)ds + \sigma(X_s, s)dW_s, \quad t \leq s \leq T.
\]

Take for example the stock price process (3.1), whose dynamics follow a geometric Brownian motion with a given spot price $S_0$. A backward stochastic differential (BSDE), on the other hand, is an SDE with a terminal condition,

\[
dY_s = -f(Y_s, Z_s, s)ds + Z_s dW_s, \quad Y_T = \xi.
\]
The function \( f \) is often called the \textit{driver function}, and is predictable. Furthermore, \( f \) should be uniformly \textit{Lipschitz}, i.e., there exists a constant \( \Gamma > 0 \) such that
\[
|f(y_1, z_1, t) - f(y_2, z_2, t)| \leq \Gamma \left( |y_1 - y_2| + |z_1 - z_2| \right), \quad \forall (y_1, z_1), (y_2, z_2).
\]
The solution to the BSDE is given by a pair of processes \((Y, Z)\) with \( Y \) a continuous real-valued adapted process and \( Z \) a real-valued predictable process, satisfying:
\[
Y_t = \xi + \int_t^T f(Y_s, Z_s, s) \, ds - \int_t^T Z_s \, dW_s, \quad 0 \leq t \leq T.
\]
Unlike an FSDE, the solution of a BSDE is thus defined as a pair of adapted processes \((Y, Z)\). Note that BSDEs cannot be considered as time-reversed FSDEs, because at time \( t \) the pair \((Y_t, Z_t)\) is \( \mathcal{F}_t \)-measurable and the process does not yet ‘know’ the terminal condition. We define a pair consisting of an FSDE and a BSDE as a \textit{forward-backward stochastic differential equation}.

**Definition 6.2** (Decoupled Forward-Backward Stochastic Differential Equation). We consider the following FSDE
\[
X_t = x, \quad dX_s = \mu(X_s, s) \, ds + \sigma(X_s, s) \, dW_s, \quad t \leq s \leq T, \tag{6.18}
\]
and the associated BSDE
\[
dY_s = -f(X_s, Y_s, Z_s, s) \, ds + Z_s \, dW_s, \quad Y_T = g(X_T), \tag{6.19}
\]
where the terminal condition of the BSDE is determined by the terminal value of the FSDE. The system (6.18) and (6.19) is said to be a forward-backward stochastic differential equation (FBSDE).

The Black-Scholes formula, for example, can be presented by a system of \textit{decoupled forward-backward SDEs}. We will discuss option pricing by solving forward-backward SDEs in Section 6.3.2. The phrase \textit{decoupled FBSDE} means that the dynamics of the FSDE at time \( t \) are not affected by the current value of the BSDE \( Y_s \). The solution of the forward-backward SDE contains the forward stochastic process
\[
X_t = X_0 + \int_0^t \mu(X_s, s) \, ds - \int_0^t \sigma(X_s, s) \, dW_s, \quad X_0 = x, \tag{6.20}
\]
and the backward stochastic process:
\[
Y_t = g(X_T) + \int_t^T f(X_s, Y_s, Z_s, s) \, ds - \int_t^T Z_s \, dW_s, \quad 0 \leq t \leq T. \tag{6.21}
\]
In Chapter 2, the Feynman-Kac (Theorem 2.6) was given, which establishes a link between FSDEs and linear parabolic PDEs. Similar as in the FSDE case, the FBSDE has also an corresponding PDE representation.

**Theorem 6.7** (Generalization of the Feynman-Kac formula). Let \( V \) be a \( C^{1,2} \) function and suppose that there exists a constant \( \Gamma \) such that, for each \((x, t)\),
\[
|V(x, t)| + |\sigma(x, t)D_x V(x, t)| \leq |\Gamma(1 + x)|.
\]
Suppose \( V \) is the solution of the following system of quasilinear parabolic partial differential equation:
\[
\begin{cases}
\frac{\partial V(x, t)}{\partial t} + LV(x, t) + f(t, x, V(x, t), D_x V(x, t) \sigma(x, t)) = 0, \\
V(x, T) = g(x),
\end{cases}
\]
where \( D_x V \) is the gradient of \( V \) and \( L \) denotes the second order differential operator
\[
LV(x, t) = \mu(x, t)D_x V(x, t) + \frac{1}{2} \sigma^2(x, t)D_x^2 V(x, t).
\]
Then the pair \((Y, Z)\), defined by
\[
Y_s = V(X_s, s) \quad \text{and} \quad Z_s = D_x V(X_s, s) \sigma(X_s, s) \quad \text{for} \quad t \leq s \leq T, \tag{6.22}
\]
is the unique solution of BSDE (6.19).
Proof. For a proof we refer to [12], Proposition 3.5.

Just as in the FSDE case, we can solve FBSDE in its probabilistic representation or through its corresponding PDE. We choose to solve the FBSDE in its probabilistic representation. In Section 6.2.1 we have seen how the COS method can be applied to FSDEs for which the corresponding characteristic function is known. However, there is usually no analytical solution of the characteristic function for an underlying process with non-constant volatility and drift term. Therefore, Oosterlee and Ruijter [24] proposed to discretize the FBSDE, and apply the BCOS method.

A discretization scheme is given in Section 6.3.4, and we will discuss the BCOS method in Section 6.4. But first, we will discuss the link between solving FBSDEs and option pricing.

### 6.3.2 FBSDEs and vanilla options

In this section, we discuss how to price options by solving a FBSDE and follow the papers by Oosterlee and Ruijter [24], [25]. The BSDE $Y_t$ (6.21) can be considered as the option that needs to be priced. We assume it replicates the call option exactly, that is at the time of expiry, the value of the replicating portfolio equals the option’s payoff,

$$Y_T = (S_T - K)^+.$$  

The terminal condition of $Y_t$ depends on the value of the underlying stock price process. We study the pricing of a call option in the Black-Scholes framework, thus the dynamics of the stock price process are given by:

$$dS_t = \mu S_t dt + \sigma S_t dW^P_t.$$  

Following [25], we defined the dynamics of $S_t$ under the real-world measure. The Black-Scholes price of a call option has an analytical solution, which can be used as a reference value. To obtain the price of the call option, we set up a self-financing portfolio $Y_t$, where we take a position of $a_t$ in stocks $S_t$ and the rest of the portfolio value is invested in the money market with risk-free return rate $r$. The option value at initial time should be equal to the initial value of the portfolio as well. The portfolio evolves according to the SDE

$$dY_t = r(Y_t - a_t S_t) dt + a_t dS_t = \left( rY_t + \frac{\mu - r}{\sigma} a_t S_t \right) dt + \sigma a_t S_t dW_t.$$  

If we set $Z_t = \sigma a_t S_t$, then $(Y, Z)$ solves the BSDE (Theorem 6.7),

$$dY_t = -f(Y_t, Z_t, t) dt + Z_t dW_t,$$

$$f(Y_t, Z_t, t) = -r Y_t - \frac{\mu - r}{\sigma} Z_t,$$

$$Y_T = (S_T - K)^+,$$

The value of the call option is given by $C(S_t, t) = Y_t$ and the hedging strategy corresponds to $\sigma S_t \frac{\partial C}{\partial S_t}(S_t, t) = Z_t$. The position $a_t$ in stocks equals the delta of the option; $a_t = \frac{\partial C}{\partial S_t}(S_t, t)$, this hedging strategy is called delta-hedging.

In the above setting, we can price options under the real-word measure $\mathbb{P}$. However, in the the Black-Scholes framework, the value of a call option does not depend on $\mu$. We prefer to price under the risk neutral measure $\mathbb{Q}$. The underlying dynamics are given by the FSDE

$$dS_t = r S_t dt + \sigma S_t dW^Q_t.$$  

We obtain the following BSDE:

$$dY_t = -f(Y_t, Z_t, t) dt + Z_t dW_t,$$

$$f(Y_t, Z_t, t) = -r Y_t,$$

$$Y_T = (S_T - K)^+.$$  

The driver function $f(Y_t, Z_t, t)$ is under the risk-neutral measure independent of $Z_t$. In the upcoming sections, where we discuss the pricing under local volatility and the pricing of barrier options, we always price under the risk-neutral measure $\mathbb{Q}$. Hence, we do not have to approximate $Z_t$, which will make life easier in Section 6.4.1, where we discuss the BCOS method.
In Section 6.3.2, we discussed option pricing of European options by solving FBSDEs; the value of
the replicating portfolio is given by \( Y_t \), whereas \( Z_t \) corresponds to the delta-hedging strategy. In this
subsection we discuss how to price barrier options in this setting.

There is no straightforward way of delta-hedging a barrier option. Consider, for example, a knock-
out option. When the stock price path hits the barrier level, the option is turned off, resulting in a
discontinuity in the option value and the delta. A common hedge strategy in the Black-Scholes case, is
to use a static hedge, where the barrier option value is approximated by a portfolio of vanilla options,
but these replicating portfolios do not hold under local volatility.

The Forward-Backward SDE setting as defined in previous subsections is therefore not sufficient for
the pricing of barrier options. No portfolio strategy exists that replicates the barrier option value
exactly, because the path-dependence of the option value on \( X_t \) is not taken into account. Therefore,
we introduce an additional process \( K_t \), so that a replicating portfolio can be set up. For this, we define
another type of FBSDEs, the reflected forward-backward SDEs.

**Definition 6.3 (Reflected forward-backward stochastic differential equation).** We define the following
set of equations

\[
X_t = X_t + \int_t^T \mu(X_s, s) ds + \int_t^T \sigma(X_s, s) dW_s,
\]

\[
Y_t = g(X_T) + \int_t^T f(X_s, Y_s, Z_s, s) ds + K_T - K_t - \int_t^T Z_s dW_s,
\]

\[
Y_t \leq U_t,
\]

\[
0 = \int_0^T (Y_t - U_t) dK_t,
\]

to be a Reflected Forward-Backward SDE (RFBSDE) [12].

The process \( U_t \) is called the obstacle and the process \( K_t \) serves to 'push \( Y_t \) below the obstacle'. To solve
this RFBSDE we start with a partition \( 0 = t_0 < t_1 < t_2 < \ldots < t_m < \ldots < t_M = T \). At each time
step we wish to monitor the barrier option. For notational convenience we write \( X_{t_m} = X_{t_m}, Y_{t_m} = Y_{t_m} \),
etcetera.

We first solve the unreflected BSDE on a time interval \( [t_{m-1}, t_m] \):

\[
\tilde{Y}_{m-1} = Y_{m-1} + \int_{t_{m-1}}^{t_m} f(X_s, Y_s, Z_s, s) ds - \int_{t_{m-1}}^{t_m} Z_s dW_s,
\]

(6.23)

and subsequently, we check whether the solution does not exceed the obstacle. For an up-and-out call
option we compute the obstacle as a function of the FSDE: \( U_m = u(X_m, t_m) \), where

\[
u(X_m, t_m) = \begin{cases}
0 & X_m \geq B_m, \\
\tilde{Y}_m & X_m < B_m,
\end{cases}
\]

with \( \tilde{Y}_m \) the replicating portfolio of the ordinary call option. Note that we immediately incorporate
time-dependent barriers \( B_m \). We set \( Y_m \) to be

\[
Y_m = \min\{\tilde{Y}_m, U_m\},
\]

(6.24)

to make sure \( Y_m \) never exceeds \( U_m \). We define \( K_m \) as

\[
K_m = \sum_{j=1}^{m} \tilde{Y}_{j-1} - \tilde{Y}_{j-1},
\]

such that we have

\[
K_m - K_{m-1} = Y_{m-1} - \tilde{Y}_{m-1} = Y_{m-1} - Y_m - \int_{t_{m-1}}^{t_m} f(X_s, Y_s, Z_s, s) ds + \int_{t_{m-1}}^{t_m} Z_s dW_s.
\]

63
This implies
\[ Y_{m-1} = Y_m + \int_{t_{m-1}}^{t_m} f(X_s, Y_s, Z_s, s)ds + K_m - K_{m-1} - \int_{t_{m-1}}^{t_m} Z_s dW_s, \]
and by the definition of \(Y_m\), we have \(Y_m \leq U_m\) for all \(m = 0, \ldots, M\). The value of the barrier option is given by \(Y_T\). To solve (reflected) FBSDEs, we define a discretization scheme in the next section, and solve \(Y_T\) backwards in time with the BCOS method (Section 6.4.1).

### 6.3.4 Discretization of FBSDEs

In this section, we discuss a discretization of the FBSDE. First, we derive a discretization scheme for the FSDE \(X_t\), and discuss its discrete characteristic function. Then, we discuss a discretization scheme for the BSDE \(Y_t\).

**Discretization schemes for FSDEs**

We discuss two different discretization schemes, the Euler scheme and the Milstein scheme. We denote \(X_{m}^\Delta\) for the discretized FSDE and define \(\Delta W_m = W_t^{m+1} - W_t^m\). With \(W_t\) a Brownian motion, the increments \(\Delta W_m \sim \mathcal{N}(0, \sqrt{\Delta t})\) are normally distributed.

The Euler discretization converges strongly with order \(1/2\), and is given by:
\[ X_0^\Delta = x, \quad X_{m+1}^\Delta = X_m^\Delta + \mu (X_m^\Delta, t_m) \Delta t + \sigma (X_m^\Delta, t_m) \Delta W_{m+1}, \quad m = 0, \ldots, M - 1. \]

The Milstein approximation converges strongly with order 1, and has the form:
\[ X_{m+1}^\Delta = X_m^\Delta + \mu (X_m^\Delta, t_m) \Delta t + \sigma (X_m^\Delta, t_m) \Delta W_{m+1} + \frac{1}{2} \sigma (X_m^\Delta, t_m) \sigma_x (X_m^\Delta, t_m) (\Delta W_{m+1}^2 - \Delta t), \]

We can write the Euler and Milstein in the following general form:
\[ X_{m+1}^\Delta = x + m(x, t) \Delta t + s(x, t) \Delta W_{m+1} + \kappa(x, t) (\Delta W_{m+1})^2, \quad X_0^\Delta = x. \quad (6.25) \]

Note that the above discretization scheme consists of a drift term and volatility term that are dependent on \(x\) but are not yet time-dependent. For the Euler scheme we have
\[ m(x, t) = \mu(x, t), \quad s(x, t) = \sigma(x, t), \quad \kappa(x, t) = 0, \]
and for the Milstein scheme
\[ m(x, t) = \mu(x, t) - \frac{1}{2} \sigma(x, t) \sigma_x(x, t), \quad s(x, t) = \sigma(x, t), \quad \kappa(x, t) = \frac{1}{2} \sigma(x, t) \sigma_x(x, t), \]

For the general form (6.25) we can derive the characteristic function.

**Lemma 6.8** ([25]). The characteristic function of \(X_{m+1}^\Delta\), given \(X_m^\Delta = x\), for \(\kappa(x, t) \neq 0\) is given by
\[ \phi_{X_{m+1}^\Delta}(u | X_m^\Delta = x) = \mathbb{E} \left[ e^{iuX_{m+1}^\Delta} \big| X_m^\Delta = x \right] = \exp \left\{ iux + iu m(x, t) \Delta t - \frac{1}{2} u^2 s^2(x, t) \Delta t, \frac{1}{1 - 2iu \kappa(x, t) \Delta t} \right\} (1 - 2iu \kappa(x, t) \Delta t)^{-1/2}, \]
and for \(\kappa(x, t) = 0\) we have
\[ \phi_{X_{m+1}^\Delta}(u | X_m^\Delta = x) = \exp \left\{ iux + iu m(x, t) \Delta t - \frac{1}{2} u^2 s^2(x, t) \Delta t, \right\}. \]

**Proof.** A proof of Lemma 6.8 can be found in Appendix A.4.

There usually does not exist an analytical solution of the characteristic function for local volatility models. By using a discretization scheme for the underlying stock price process, we are able to define the discrete characteristic function. In Section 6.4.1 we will use the COS method based on the discrete characteristic function recursively to solve the discretized FBSDEs.
Discretization scheme for BSDEs

To solve forward-backward SDEs, not only a discretization scheme for the FSDE is required, but a scheme for the BSDE is required as well. Recall that the BSDE (6.21) is solved by a pair of stochastic processes \((Y, Z)\). We start with

\[
Y_m = Y_{m+1} + \int_{t_m}^{t_{m+1}} f(X_s, Y_s, s) \, ds - \int_{t_m}^{t_{m+1}} Z_s \, dW_s,
\]

(6.26)

We take conditional expectations at both sides of equation (6.26):

\[
Y_m = \mathbb{E}_m [Y_{m+1}] + \int_{t_m}^{t_{m+1}} \mathbb{E}_m [f(X_s, Y_s, s)] \, ds
\approx \mathbb{E}_m [Y_{m+1}] + \Delta t \theta f(\Lambda_m, t_m) + \Delta t (1 - \theta) \mathbb{E}_m [f(\Lambda_{m+1}, t_{m+1})], \quad 0 \leq \theta \leq 1,
\]

(6.27)

where \(\Lambda_m = (X_m, Y_m)\) and the integral is approximated by the \textit{theta-time discretization method}, taking \(\theta = 0.5\) results in the trapezoidal rule. In a similar way we can obtain an approximation for \(Z_m\). Since we wish to price under the risk-neutral measure \(\mathbb{Q}\), the driver function does not depend on \(Z_m\), therefore, we do not consider the approximation of \(Z_m\) in this report. From here on, we ignore the \(Z_m\) term.

Given a discretization scheme for \(X^\Delta_m\) we can derive schemes for \(Y^\Delta_m\), based on equations (6.27) and (6.22). We have for \(m = M - 1, \ldots, 0:\)

\[
Y^\Delta_m = \mathbb{E}_m [Y^\Delta_{m+1}] + \Delta t \theta f(\Lambda^\Delta_m, t_m^\Delta) + \Delta t (1 - \theta) \mathbb{E}_m \left[ f(\Lambda^\Delta_{m+1}, t^\Delta_{m+1}) \right],
\]

and terminal condition:

\[
Y^\Delta_M = g \left( X^\Delta_M \right).
\]

The terminal condition is a deterministic function of \(X^\Delta_M\), which is a Markov process. Then, using an induction argument, there are deterministic functions \(y(x, t_m)\) so that

\[
Y^\Delta_m = y \left( X^\Delta_m, t_m \right).
\]

So, the random variable \(Y^\Delta_m\) is a function of \(X^\Delta_m\), for each \(m = 0, \ldots, M\). The function \(y(x, t_m)\) is obtained in a backward manner by means of the following time iteration

\[
y(x, t_M) = g(x),
\]

and for \(m = M - 1, \ldots, 0:\)

\[
y(x, t_m) = \mathbb{E}_m^x \left[ y \left( X^\Delta_{m+1}, t^\Delta_{m+1} \right) + \Delta t \theta f \left( X^\Delta_m, y \left( X^\Delta_m, t^\Delta_m \right), t_m \right) + \Delta t (1 - \theta) \mathbb{E}_m^x \left[ f \left( X^\Delta_{m+1}, y \left( X^\Delta_{m+1}, t^\Delta_{m+1} \right), t^\Delta_{m+1} \right) \right] \right].
\]

(6.28)

where the conditional expectation \(\mathbb{E}_m^x [\cdot] \) represents \(\mathbb{E}_m [\cdot | X^\Delta_m = x] \). Note that the function \(y\) depends on the discretization partition \(\Delta\).

In the case \(\theta > 0\), the function \(y(x, t_m)\) has an implicit part: \(f \left( X^\Delta_m, y \left( X^\Delta_m, t^\Delta_m \right), t_m \right)\). In order to determine this term we perform \(P\) Picard iterations. The Picard method can be written as

\[
y^1(x, t_m) = \Delta t \theta f \left( x, y^0 \left( x, t_m \right), t_m \right) + h(x, t_m),
\]

\[
y^2(x, t_m) = \Delta t \theta f \left( x, y^1 \left( x, t_m \right), t_m \right) + h(x, t_m),
\]

\[
\vdots
\]

\[
y^P(x, t_m) = \Delta t \theta f \left( x, y^{P-1} \left( x, t_m \right), t_m \right) + h(x, t_m),
\]

where we take \(y^0(x, t_m) := \mathbb{E}_m^x [Y^\Delta_{m+1}] \) as an initial guess, and with \(h(x, t_m)\) the explicit part of (6.28):

\[
h(x, t_m) := \mathbb{E}_m^x \left[ y \left( X^\Delta_{m+1}, t^\Delta_{m+1} \right) + \Delta t (1 - \theta) \mathbb{E}_m^x \left[ f \left( X^\Delta_{m+1}, y \left( X^\Delta_{m+1}, t^\Delta_{m+1} \right), t^\Delta_{m+1} \right) \right] \right].
\]

(6.29)

Now, (6.28) provides us with a scheme to solve the BSDE backwards in time, starting at terminal time \(T\). There are several numerical methods to compute these conditional expectations backwards in time. For example, a binomial tree model is a method that is widely used in mathematical finance. In [24] the COS method is proposed for solving the BSDE, since it proves to be accurate and fast.
6.4 Numerical Fourier method for FBSDEs

Due to the local volatility, the characteristic function of the underlying stochastic process is usually not known. Hence, we can not use the COS method in a straightforward way. Recently, Ruijter and Oosterlee presented the Backward Stochastic Differential Equation COS (BCOS) method [24], [25], which can be used to solve forward-backward SDEs under local volatility.

In the BCOS method the FBSDEs are first discretized, and a discrete characteristic function is used for the computation of the conditional expectations. We use the discretization schemes for $X_t$ and $Y_t$ from Section 6.3.4. The BCOS method is illustrated by numerical examples under constant volatility.

Finally, we will discuss the pricing of barrier options as a new extension of the BCOS method.

6.4.1 Backward stochastic differential equation COS (BCOS) method

In the BCOS method we use the COS method recursively for the computation of the conditional expectations in the discretization scheme of the BSDE (6.28). As we have seen in Section 6.2.2 the COS method is illustrated by numerical examples under constant volatility.

We need to recover the Fourier coefficients where we let

$$Y_{k}(t_{m+1}) = \left\{ \phi \left( \frac{k\pi}{b-a} \right) x \right\} \exp \left\{ -ik\pi \frac{a}{b-a} \right\}.$$ 

With the help of the COS formula we derive the following approximations of these expectations:

$$\mathbb{E}_m [y (X_{m+1}^\Delta, t_{m+1})] = \sum_{k=0}^{N-1} \mathcal{Y}_k(t_{m+1}) \text{Re} \left\{ \phi \left( \frac{k\pi}{b-a} \right) x \right\} \exp \left\{ -ik\pi \frac{a}{b-a} \right\},$$

$$\mathbb{E}_m [f (A_{m+1}^\Delta, t_{m+1})] = \sum_{k=0}^{N-1} \mathcal{F}_k(t_{m+1}) \text{Re} \left\{ \phi \left( \frac{k\pi}{b-a} \right) x \right\} \exp \left\{ -ik\pi \frac{a}{b-a} \right\}.$$

We need to recover the Fourier coefficients where we let $\mathcal{Y}_k(t_{m+1})$ be the Fourier-cosine coefficients of $y(x, t_m)$, i.e.,

$$\mathcal{Y}_k(t_m) = \frac{2}{b-a} \int_a^b y(x, t_m) \cos \left( \frac{k\pi}{b-a} x \right) dx,$$

$$\mathcal{Y}_k(t_M) = \frac{2}{b-a} \int_a^b y(x, t_M) \cos \left( \frac{k\pi}{b-a} x \right) dx, \text{ for } m = M - 1, \ldots, 1,$$

and let $\mathcal{F}_k(t_m)$ be the Fourier-cosine coefficients of driver function $f(x, y(x, t_m), t_m)$, i.e.,

$$\mathcal{F}_k(t_M) = \frac{2}{b-a} \int_a^b f(x, y(x, t_m), t_m) \cos \left( \frac{k\pi}{b-a} x \right) dx,$$

$$\mathcal{F}_k(t_m) = \frac{2}{b-a} \int_a^b f(x, y(x, t_m), t_m) \cos \left( \frac{k\pi}{b-a} x \right) dx, \text{ for } m = M - 1, \ldots, 0.$$
For the computation of the implicit part, the coefficients of the driver function, \( F_k(t_m) \), a sufficient amount of Picard iterations is required. These coefficients are given by

\[
F_k^P(t_m) := \frac{2}{b-a} \int_a^b f(x,y^P(x,t_m),z(x,t_m),t_m) \cos\left(\frac{k\pi x-a}{b-a}\right) dx,
\]

such that we have for the Fourier-cosine coefficient of \( y \):

\[
Y_k(t_m) = \Delta t \theta F_k^P(t_m) + \mathcal{H}_k(t_m).
\]

In some cases, we can compute the Fourier-cosine coefficients analytically, for example, we have seen the analytical solution for the terminal coefficients of the European call option (6.10) in Section 6.2.1. **Example 6.9.** We discuss a specific case for the coefficients \( \mathcal{H}_k(t_m) \), to illustrate the coefficient recovery procedure of the BCOS method. We take the Euler scheme for a geometric Brownian motion, thus with constant drift and volatility term. In this case, the FSDE \( X_t \) is a Lévy process, hence, we can simplify the formula for \( \mathcal{H}_k(t_m) \) to

\[
\mathcal{H}_k(t_m) = \text{Re} \left\{ \sum_{j=0}^{N-1} [Y_j(t_{m+1}) + \Delta t(1 - \theta)F_j(t_{m+1})] \phi \left( \frac{k\pi}{b-a} \right) M_{k,j} \right\},
\]

where \( M_{k,j} \) denote the matrix elements from Section 6.2.2, Equation (6.15):

\[
M_{k,j} := \frac{2}{b-a} \int_{z_1}^{z_2} \exp \left\{ i j \pi \frac{y-a}{b-a} \right\} \cos \left( \frac{k\pi y-a}{a-b} \right) dy.
\]

Hence, we can compute the coefficients \( \mathcal{H}_k(t_m) \), under constant drift and volatility, by the efficient FFT algorithm of Fang and Oosterlee [14].

**Discrete Fourier-cosine transform**

For general drift and volatility, we use the *discrete Fourier-cosine transform* (DCT) to approximate the Fourier-cosine coefficients. We take \( Q \) grid-points and define a \( x \)-grid

\[
x_n := a + \left( n + \frac{1}{2} \right) \frac{b-a}{Q} \quad \text{and} \quad \Delta x := \frac{b-a}{Q}.
\]

We apply the following lemma for the approximation of the coefficients.

**Lemma 6.9.** Let \( \mathcal{H}_k(t_m) \) be the Fourier-cosine coefficients of a function \( h(x,t_m) \) and consider a \( x \)-grid as defined in (6.30). Then, we can approximate \( \mathcal{H}_k(t_m) \) by a Type II discrete Fourier-cosine transform

\[
\mathcal{H}_k(t_m) = \frac{2}{b-a} \sum_{n=0}^{Q-1} h(x_n,t_m) \cos \left( k\pi \frac{2n+1}{2Q} \right).
\]

**Proof.** The Fourier-cosine coefficient of \( h(t_m,x) \) is defined as

\[
\mathcal{H}_k(t_m) = \frac{2}{b-a} \int_a^b h(x,t_m) \cos \left( k\pi \frac{x-a}{b-a} \right) dx,
\]

on the \( x \)-grid we determine the value of \( h(x_n,t_m) \) and use the midpoint-rule for the integration

\[
\mathcal{H}_k(t_m) = \frac{2}{b-a} \sum_{n=0}^{Q-1} h(x_n,t_m) \cos \left( k\pi \frac{x_n-a}{b-a} \right) \Delta x,
\]

rewriting gives us

\[
\mathcal{H}_k(t_m) = \frac{2}{b-a} \sum_{n=0}^{Q-1} h(x_n,t_m) \cos \left( k\pi \frac{2n+1}{2Q} \right).
\]

**Remark 6.3.** This is the Type II discrete Fourier-cosine transform [25], which can be computed efficiently by the MATLAB function \texttt{dct}.
BCOS algorithm and a numerical example

The BCOS method of Ruijter and Oosterlee [25] can be summarized as:

<table>
<thead>
<tr>
<th>Algorithm BCOS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Initialization:</strong> Compute, or approximate, the terminal coefficients ( Y_{\Delta k}(t_M) ) and ( F_{\Delta k}(t_M) ).</td>
</tr>
<tr>
<td><strong>Main loop</strong> for ( m = M - 1 ) to 1:</td>
</tr>
<tr>
<td>• Approximate the functions ( y(x, t_m) ) and ( f(x, y(x, t_m), t_m) ) with the COS formula.</td>
</tr>
<tr>
<td>• Determine the Fourier cosine coefficients ( Y_{\Delta k}(t_m) ) and ( F_{\Delta k}(t_m) ), by using a DCT.</td>
</tr>
<tr>
<td><strong>Final step:</strong> Compute ( Y_{\Delta 0}(X_0) ).</td>
</tr>
</tbody>
</table>

We illustrate the BCOS method with an example under constant volatility.

**Example 6.10** (Pricing a European call option with the BCOS method). We wish to compute the Black-Scholes price of a European call option. We set \( \theta = 0.5 \), and take 5 Picard iterations for the computation of the implicit part of \( Y_{\Delta m} \). For the numerical approximation, we use \( N = 1024 \) and switch to log-stock price \( X_t = \log S_t \), with

\[
dX_t = rdt + \sigma dW_t.\]

The following parameter values are used

\[X_0 = \log 100, \quad K = 100, \quad r = 0.01, \quad \sigma = 0.25, \quad T = 0.1,\]

with the exact solution \( Y_0 = V(X_0, t_0) = 10.4035 \). Table 6.9 contains the observed errors and corresponding CPU times for a different number of time steps \( M \).

<table>
<thead>
<tr>
<th>( M )</th>
<th>10</th>
<th>50</th>
<th>100</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>8.66974e-09</td>
<td>3.47530e-10</td>
<td>8.94715e-11</td>
<td>2.24354e-11</td>
<td>2.35207e-11</td>
</tr>
<tr>
<td>Time (sec)</td>
<td>0.169382</td>
<td>0.403154</td>
<td>0.775071</td>
<td>2.234883</td>
<td>5.066060</td>
</tr>
</tbody>
</table>

Table 6.9: BCOS results for a European call option under GBM.

The BCOS method can accurately and quickly reproduce the price of the call option. We compare the absolute error per time step of the BCOS method to the Crank-Nicolson scheme in Figure 6.2. For the Crank-Nicolson scheme \( N_x = 700 \) asset steps were taken. The BCOS method prices regular call options more accurate than the Crank-Nicolson scheme.

### 6.4.2 BCOS method for barrier options

In Section 6.2.2, we have seen the pricing of barrier options under constant volatility with the COS method. In this section we study the pricing of barrier options as a reflected forward-backward SDE problem, as was proposed in Section 6.3.3. The BCOS method is used to solve the reflected FBSDE. This approach enables us to price barrier options under local volatility. This is a new field of application of the BCOS method.

We define the characteristic function \( \phi_{X^\Delta_m} \) under the general form (6.25) of the discretization scheme for FSDE \( X^\Delta_m \). The barrier option is monitored at each time step \( t_m \):

\[0 = t_0 < t_1 < t_2 < \ldots < t_m < \ldots < t_M = T,\]

and we apply the continuity correction from (6.17). We denote with \( Y_m \) (6.24), the replicating portfolio of the barrier option, and with \( \tilde{Y}_m \) (6.23) the unreflected BSDE. For the discretization of \( \tilde{Y}_m \) we use (6.28) from Section 6.3.4, and approximate \( Y_m \) with the COS formula.
6.4. NUMERICAL FOURIER METHOD FOR FBSDES

Figure 6.2: Absolute error when pricing a European call under GBM.

We consider an up-and-out call option. The Fourier-cosine coefficients at the terminal time are the same as for the barrier option under constant volatility (Section 6.2.2).

\[ Y_k(t_M) = \begin{cases} G_k(0, \xi, a, b), & \xi \geq 0, \\ 0, & \xi < 0, \end{cases} \]

where \( G_k(0, \xi, a, b) \) is the payoff of a call option, as in Equation (6.10) and \( \xi = \log(B/K) \). The coefficients of the driver function are given by

\[ Y_k(t_M) = -r F_k(t_M). \]

We use the BCOS method (Section 6.4.1) to compute \( \tilde{Y}_{M-1} \) and compute the value of the reflected BSDE \( Y_{M-1}^\Delta \) by (6.24). We repeat this procedure backward in time up to \( m = 1 \). Note that we can easily incorporate a time-dependent barrier level. The barrier option value is computed in the final step and is given by \( Y_0^\Delta(X_0) \). We summarize the BCOS method for barrier options by

Algorithm BCOS for barrier options

**Initialization:** Compute, or approximate, the terminal coefficients \( Y_k(t_M) \) and \( F_k(t_M) \).

**Main loop for** \( m = M - 1 \) to 1:

- Approximate \( \tilde{Y}_m^\Delta \) and \( f \left( X_m^\Delta, \tilde{Y}_m^\Delta, t_m \right) \). with the COS formula.
- Compute \( U_m \) and \( Y_m^\Delta = \min\{\tilde{Y}_m^\Delta, U_m\} \) and \( f \left( X_m^\Delta, Y_m^\Delta, t_m \right) \).
- Determine the Fourier cosine coefficients \( Y_k^\Delta(t_m) \) and \( F_k^\Delta(t_m) \), by using a DCT.

**Final step:** Compute \( Y_0^\Delta(X_0) \).

We illustrate the BCOS method for barrier options with an example under constant volatility.

**Example 6.11** (BCOS method for up-and-out call under GBM). In this example, we apply the BCOS method to solve the reflected FBSDE corresponding to an up-and-out call option. For this, we choose a fixed number of Fourier coefficients \( N = 1024 \) and vary the number of monitoring times \( M \). We set
\( \theta = 0.5 \), and take 5 Picard iterations for the computation of the implicit part of \( \bar{Y}_m^\Delta \). We take the following parameter values

\[
S_0 = 100, \quad r = 0.01, \quad T = 0.1, \quad K = 100, \quad B = 125, \quad \text{and} \quad \bar{\sigma} = 0.25.
\]

The corresponding Black-Scholes price, 3.0914, is used as a reference value, the option price of a regular call option is 3.2017. The observed errors and CPU time are presented in Table 6.10.

<table>
<thead>
<tr>
<th>( M )</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>0.15508</td>
<td>1.39904e-02</td>
<td>7.32391e-04</td>
<td>4.43199e-04</td>
</tr>
<tr>
<td>Time (sec)</td>
<td>0.051111</td>
<td>0.072369</td>
<td>0.118306</td>
<td>0.627376</td>
</tr>
</tbody>
</table>

Table 6.10: Convergence of the up-and-out call option price.

**Example 6.12** (BCOS method for different barrier levels). Consider the up-and-out call option from example 6.11 for different barrier levels \( B \in \{110, 120, 130, 140, 150\} \). The number of Fourier coefficients that is used is \( N = 2048 \). The analytical Black-Scholes prices and the BCOS option value can be found in Table 6.11.

<table>
<thead>
<tr>
<th>( B )</th>
<th>110</th>
<th>120</th>
<th>130</th>
<th>140</th>
<th>150</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS price</td>
<td>0.96673</td>
<td>2.80512</td>
<td>3.17695</td>
<td>3.20100</td>
<td>3.201733</td>
</tr>
<tr>
<td>BCOS price</td>
<td>0.96215</td>
<td>2.81333</td>
<td>3.17646</td>
<td>3.20097</td>
<td>3.201739</td>
</tr>
</tbody>
</table>

Table 6.11: BCOS method for different barrier levels (\( M = 1000 \)).

Note that for the higher barrier levels the option value for the up-and-out call option approaches the regular call option price (3.201730). The convergence of the BCOS method and the Crank-Nicolson scheme for different barrier levels is plotted in Figure 6.3. For the Crank-Nicolson scheme \( N_x = 700 \) asset steps were taken.

It looks as if the BCOS method converges, up to a constant, with the same rate for the different barrier levels. The absolute error decreases for higher barrier levels. Furthermore, we observe oscillations towards \( M = 10^3 \). We believe this could be caused by numerical errors in the Fourier cosine coefficients. If we increase the number of coefficients \( N \), the oscillations damp out. In the right side plot, we observe that the barrier level has little influence on the absolute error of the Crank-Nicolson scheme.

![Figure 6.3: Absolute error for different barrier levels B.](image)
Remark 6.4. We would like to emphasize that because of our choice of dividends model, we can price equity barrier options (in the clean space) with this approach. Otherwise we would encounter jumps at the dividend dates. In [24], it is shown that the BCOS method can also be extended to jump-diffusion processes.

6.5 Numerical results of the BCOS method with local volatility

In Chapter 5, we have constructed a volatility surface from which we can extract Dupire’s local volatility term. In this section we present results of the BCOS method with local volatility. The method is applied to option pricing problems under Constant Elasticity of Variance and Dupire processes. We compare option prices under Dupire’s local volatility to those obtained by the Crank-Nicolson scheme.

6.5.1 BCOS method for Constant Elasticity of Variance

In this section we price options under the CEV model, that was discussed in Section 3.4. Recall the CEV model dynamics under the risk-neutral measure
\[
dX_t = rX_t dt + \tilde{\sigma}X_t^\lambda dW_t,
\]
where we have a drift term \(\mu(x) = rx\), and volatility term \(\sigma(x) = \tilde{\sigma}x^\lambda\). There only exists closed form solutions for the characteristic function for \(\lambda \in \{0, 0.5, 1\}\), but we can define the discrete characteristic function \(\phi_{X,T}^m\) for any \(\lambda \geq 0\). In order to compute the discrete characteristic function \(m(x,t), s(x,t),\) and \(\kappa(x,t)\) are required. For the Milstein scheme we have
\[
m(x,t) = rx - \frac{1}{2}\lambda \tilde{\sigma}^2 x^{2\lambda - 1}, \\
s(x,t) = \tilde{\sigma} x^\lambda, \\
\kappa(x,t) = \frac{1}{2}\lambda \tilde{\sigma}^2 x^{2\lambda - 1}.
\]

For the CEV model we compare our results to reference values as given in (3.13), where we use a numerical approximation of the non-central chi squared cumulative distribution function.

Example 6.13 (Pricing a European call option under CEV). In this example, the following parameter values are used:
\[
X_0 = 100, \quad K = 100, \quad r = 0.01, \quad \lambda = 0.2, \quad \tilde{\sigma} = 0.25, \quad T = 0.2,
\]
with corresponding reference value \(Y_0 = 0.2390\). We have used \(N = 1024\) Fourier-cosine coefficients. The numerical results are shown in Table 6.12. We observe that the BCOS method is fast and accurate in this example.

\[
\begin{array}{cccccc}
M & 10 & 50 & 100 & 500 & 1000 \\
\hline
\text{Error} & 1.93303e-02 & 3.76133e-03 & 1.81005e-03 & 2.59454e-04 & 5.38083e-05 \\
\text{Time (sec)} & 0.961053 & 0.968232 & 0.976666 & 1.027220 & 1.094697
\end{array}
\]

Table 6.12: Convergence of the CEV call option price.

Example 6.14 (BCOS method for different lambda values). We wish to test the convergence of the BCOS method for different \(\lambda\) values. In this example, the following parameter values are used:
\[
X_0 = 100, \quad K = 100, \quad r = 0.01, \quad \lambda \in \{0.2, 0.5, 0.8\}, \quad \tilde{\sigma} = 0.25, \quad T = 0.5.
\]
The reference values (3.13) are given in Table 6.13. In Figure 6.4 we see the convergence of the BCOS method for different \(\lambda\) values. It looks as if the value of \(\lambda\) has little influence on the rate of convergence (up to a constant).
Lastly, we wish to price a barrier option by the BCOS method.

**Example 6.15** (Up-and-out call option under CEV). In this example, we price an up-and-out call option by the BCOS method. For the discretization of the FSDE we use again the Milstein scheme. No closed form solutions exist for the up-and-out call option under the CEV model. Therefore, we compare our results to a numerical reference value obtained in the paper [4]. The following parameter values are used:

\[ X_0 = 100, \quad K = 100, \quad r = 0.1, \quad \lambda = 0.5, \quad \bar{\sigma} = 0.25X_0^{1-\lambda}, \quad T = 0.5, \quad B = 140. \]

We choose \( \bar{\sigma} \) so that \( \sigma(X_0) = 25 \) and \( N = 2048 \) Fourier cosine coefficients are used. The observed BCOS prices and CPU time are presented in Table 6.14, we compare them to barrier option prices obtained by Boyle et al. The value of the regular European call option is 9.5845. We observe that the BCOS prices are in line with the values in [4].

\[
\begin{array}{c|c|c|c|c}
M & 10 & 50 & 100 & 500 \\
\hline
Boyle price & 6.8932 & 7.0206 & 7.0295 & 7.0363 \\
BCOS price & 6.8193 & 7.0145 & 7.0062 & 7.0365 \\
BCOS CPU time (sec) & 0.053061 & 0.092151 & 0.116469 & 0.182798 \\
\end{array}
\]

Table 6.14: Convergence of the up-and-out call option price under CEV.

### 6.5.2 BCOS method for Dupire local volatility

In Chapter 5, we have constructed an SVI volatility surface from which we can extract Dupire’s local volatility model. In this section we present results of the BCOS method with Dupire local volatility. The BCOS prices are compared to those obtained by the Crank-Nicolson scheme. For the Crank-Nicolson scheme \( N_x = 750 \) asset steps were taken.

In the Dupire model, the stock price dynamics are modeled by a one-dimensional diffusion process

\[ dS_t = rS_t dt + \sigma_{LV}(S_t, t)S_t dW_t. \]

In order to compute the discrete characteristic function, the functions \( m(x, t), s(x, t), \) and \( \kappa(x, t) \) are required. We use the Euler scheme

\[ m(x, t) = rx, \quad s(x, t) = \sigma_{LV}(x, t), \quad \kappa(x, t) = 0. \]
The function \( s(x, t) \) is time-dependent, hence, the discrete characteristic function will change with each time step.

**Remark 6.5.** The models are partly implemented in C++, and partly in Matlab; the volatility surface is constructed in C++, whereas the BCOS method is implemented in Matlab. This leads to difficulties in the BCOS application. We started out with programming in C++, since it is the financial industry standard language, and switched to Matlab from an ease of implementation perspective. Matlab can, for example, easily compute the fast Fourier transform and discrete cosine transform.

For each BCOS time step \( m \), a local volatility smile, based on the market data from Ebay, is uploaded from C++. For the required points on the \( x \)-grid we use a basic interpolation method. Interpolation of the local volatility surface could cause arbitrage opportunities. Therefore, we keep the number of steps in the \( x \)-grid as small as possible. We take only \( N = 128 \) Fourier cosine coefficients, such that call options under GBM can be reproduced up to an absolute error of order 1e-05.

**Example 6.16 (Pricing a European call option under Dupire).** We price a European call option by the BCOS method for different expiries. We use the Dupire-SVI volatility model based on market data from Ebay (Section 5.4.1), and use parameter values \( S_0 = 57.12, K = 57.12, r = 0.01, T \in \{0.2, 0.5, 1.0\} \).

For the BCOS method, we choose a fixed number of Fourier coefficients \( N = 128 \). We set \( \theta = 0.5 \), and take 5 Picard iterations for the computation of the implicit part of \( \tilde{Y}_{m} \).

The constructed local volatility model should be in line with the original SVI implied volatility surface, meaning that option pricing under the local volatility model should result in the same vanilla option prices as the Black-Scholes prices implied by the SVI volatility surface. Therefore, we compare the BCOS prices and the prices obtained by the Crank-Nicolson scheme to the call option price in the Black-Scholes framework.

The prices, together with the SVI implied volatility, are given in Table 6.15 for \( M = 1000 \).

<table>
<thead>
<tr>
<th>( T )</th>
<th>0.2</th>
<th>0.5</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>SVI volatility</td>
<td>0.268883</td>
<td>0.266157</td>
<td>0.269362</td>
</tr>
<tr>
<td>BS price</td>
<td>2.79322</td>
<td>4.41562</td>
<td>6.37750</td>
</tr>
<tr>
<td>BCOS price</td>
<td>2.77850</td>
<td>4.41854</td>
<td>6.37955</td>
</tr>
<tr>
<td>CN price</td>
<td>2.77965</td>
<td>4.40418</td>
<td>6.36669</td>
</tr>
</tbody>
</table>

Table 6.15: European call option prices.

We compare the absolute error per time step of the BCOS metod to the Crank-Nicolson scheme in Figure 6.5. The corresponding CPU times of the BCOS method can be found in Table 6.16. The computation time has increased, compared to the CEV process (Table 6.14), because the characteristic function changes with each time step in the Dupire setting.

<table>
<thead>
<tr>
<th>( M )</th>
<th>10</th>
<th>50</th>
<th>100</th>
<th>500</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPU time (sec)</td>
<td>0.036526</td>
<td>0.125005</td>
<td>0.224783</td>
<td>0.869812</td>
</tr>
</tbody>
</table>

Table 6.16: CPU time BCOS method (\( N = 128 \)).

We observe in Figure 6.5 that, for all expiries, the BCOS method prices regular call options more accurate than the Crank-Nicolson scheme. We believe that the oscillations of the BCOS method are caused by the small number of Fourier cosine coefficients \( N \). We conclude that the BCOS method is suitable for option pricing under Dupire’s local volatility.
In Example 6.11 and 6.15, barrier options were successfully priced under geometric Brownian motion and CEV, respectively. However, in order to accurately price barrier options under Dupire local volatility by BCOS, we need a more refined $x$-grid. We believe an arbitrage-free interpolation method for the local volatility surface is required, such that we can take a sufficient amount of Fourier cosine coefficients $N$ and time steps $M$. 

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**Figure 6.5: BCOS Dupire results**

In Example 6.11 and 6.15, barrier options were successfully priced under geometric Brownian motion and CEV, respectively. However, in order to accurately price barrier options under Dupire local volatility by BCOS, we need a more refined $x$-grid. We believe an arbitrage-free interpolation method for the local volatility surface is required, such that we can take a sufficient amount of Fourier cosine coefficients $N$ and time steps $M$. 

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**Figure 6.5: BCOS Dupire results**
Chapter 7

Conclusion

In order to accurately price path-dependent equity derivatives, we wish to correctly model the volatility of the underlying stock over the entire lifetime of the option. In this thesis we studied the local volatility model, where the volatility is defined as a deterministic function of the stock price and time. We considered Dupire’s non-parametric local volatility model, and successfully showed how the local volatility term can be derived from a call option price surface or implied volatility surface. The Dupire model has the advantage that no calibration step is required.

A challenging task in volatility modeling is to guarantee the absence of arbitrage. We discussed how arbitrage can be defined in terms of calendar spread arbitrage in the expiry direction, and in butterfly and call spread arbitrage in the strike direction. No-arbitrage conditions are derived for the call option surface, and equivalent conditions for the total variance surface.

We implemented Dupire’s model based on a Stochastic Volatility Inspired parameterization of the implied volatility surface. The SVI model does not guarantee the absence of arbitrage. Hence, in the calibration of the SVI parameters the no-arbitrage conditions and, therefore, the suitability of the total variance surface as input to Dupire’s local volatility model should be taken into account. Different interpolation techniques for building the term structure were analyzed, and we succeeded in constructing an implied volatility surface that was suitable for derivative pricing. The implied volatility smile was accurately reproduced for different expiries.

We considered Dupire’s model based on a call option surface under the Heston dynamics as a test case, and compared the Heston results to those obtained with the SVI parameterization. Heston has the advantage that interpolating over the parameters always results in an arbitrage-free model, however, the Heston model describes the underlying dynamics, and is not a parameterization of the surface itself. Hence, to obtain the full surface, numerical methods were required for the computation of call option prices and implied volatilities. Although, it is harder to find an arbitrage-free surface for the SVI model, we prefer the SVI model over the Heston model, since a parameterization of the whole surface is available, which makes it possible to accurately compute the local volatility term, even for extreme strike values and small expiries.

It was shown how to incorporate dividends into our local volatility model. The ex-dividend dates, as well as the size of dividend payments, are assumed to be known in advance. After a dividend payment the stock price needs to be balanced by a drop in its price by the amount of dividend, but the option price stays continuous across the dividend date. For a consistent model we switched to a clean space, in which the net present value of the future dividends is taken out of the stock price process $S_t$, such that a martingale process $\tilde{S}_t$ remains. Pricing under $\tilde{S}_t$, instead of under $S_t$, allows us to consider the underlying stock price as a non-dividend paying stock.

We have seen that we can price barrier options in the clean space, but for this, the barrier level becomes time-dependent. We preferred to model the dividend payments as cash dividends, but showed that the proposed dividend model was suitable for proportional dividends as well. It was shown that the Dupire-SVI local volatility model is able to reproduce the implied volatility smile of a dividend paying-stock.

Several numerical pricing methods were discussed, starting with the Crank-Nicolson scheme, a finite difference method for solving the pricing PDE. Furthermore, we discussed the COS and BCOS methods,
which are Fourier pricing methods based on cosine series expansions. It was shown how the BCOS method can be used for solving forward-backward SDEs. We proposed a novel pricing method for barrier options by applying the BCOS method to reflected forward-backward SDEs. This is a new area of application for the BCOS method. Barrier options were successfully priced under geometric Brownian motion and CEV processes. We extended the BCOS method to Dupire’s local volatility as well. We concluded that the BCOS method is suitable for option pricing under local volatility.
Future research

We recommend the following issues for future research

- In order to correctly define a local volatility model in terms of an implied volatility surface, the arbitrage-free construction of this underlying surface is of the utmost importance. We propose to do future research on the calibration of the SVI parameterization, and set explicit parameter restrictions in order to construct a volatility surface that is suitable for Dupire’s local volatility model. Moreover, we propose to investigate the calibration of the SVI model in the clean space, and test the local volatility model on different sets of market data.

- We have seen the BCOS valuation of barrier options under local volatility for small expiries. In order to price long-term barrier options, we propose to do future research on a continuity correction for the monitoring of the barrier level between time steps. Furthermore, in future research, we recommend to do a more detailed error analysis of the BCOS method for barrier options.

- We concluded that the BCOS method was suitable for option pricing under Dupire local volatility. We propose to investigate arbitrage-free interpolation of the local volatility surface, such that we can accurately compute the local volatility term for small time steps and small asset steps.

- The models are partly implemented in C++, and partly in Matlab; the volatility surface, based on market data, is constructed in C++, whereas the BCOS method is implemented in Matlab. It is recommended to perform all computations in one programming language. Furthermore, we propose to do future research on more efficient implementations of the models.
Bibliography


Appendix A

Derivations and proofs

A.1 Alternative representation Dupire’s local volatility

Dupire’s non-parametric local volatility is in terms of a call option price defined by:

\[ \sigma_{LV}(K,T)^2 = \frac{\partial C}{\partial T} + rK \frac{\partial C}{\partial K} \frac{1}{2K^2 \frac{\partial^2 C}{\partial K^2}}. \tag{A.1} \]

However, it is possible to reformulate the model in terms of the corresponding Black Scholes implied volatilities \( \sigma_{BS}(t,K) \), as explained in [23]. The Black-Scholes price for a call option is given by:

\[ C(K,T) = S_0 N(d_1) - Ke^{r(T-t)} N(d_2), \]

where \( N(\cdot) \) is the standard normal distribution function, and:

\[ d_1 = \frac{\log \left( \frac{S_0}{K} \right) + \left( r + \frac{1}{2} \sigma_{BS}^2(t,K) \right) (T-t)}{\sigma_{BS}(t,K) \sqrt{T-t}}; \]
\[ d_2 = d_1 - \sigma_{BS}(t,K) \sqrt{T-t}. \]

We propose a change of variables and define the call price in terms of the log-moneyness \( x \) and total implied variance \( \omega \):

\[ c(x,\omega) = S_0 N(d_1) - S_0 e^x N(d_2), \tag{A.2} \]

where

\[ d_1 = -\frac{x}{\sqrt{\omega}} + \frac{1}{2} \sqrt{\omega}; \quad d_2 = d_1 - \sqrt{\omega}. \]

Various (partial) derivatives need to be computed. The derivatives of the standard normal distribution function are:

\[ N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}; \quad N''(x) = -x N'(x). \]

And the partial derivatives of \( d_1 \) and \( d_2 \) with respect to \( x \) and \( \omega \) are:

\[ \frac{\partial d_1}{\partial x} = -\frac{1}{\sqrt{\omega}}; \quad \frac{\partial d_2}{\partial x} = \frac{\partial d_1}{\partial x} = -\frac{1}{\sqrt{\omega}}; \]
\[ \frac{\partial d_1}{\partial \omega} = \frac{1}{2\omega \sqrt{\omega}} + \frac{1}{4 \sqrt{\omega}}; \quad \frac{\partial d_2}{\partial \omega} = \frac{\partial d_1}{\partial \omega} - \frac{1}{2} \frac{1}{\sqrt{\omega}} = \frac{1}{2 \omega \sqrt{\omega}} - \frac{1}{4 \sqrt{\omega}}. \]

In Dupire’s original formula (A.1) partial derivatives of \( C(K,T) \) with respect to strike and expiry are used as input. We apply the chain rule for functions of two variables to express these derivatives in
terms of \( c(x, \omega) \):

\[
\frac{\partial C}{\partial K} = \frac{\partial c}{\partial x} \frac{\partial x}{\partial K} + \frac{\partial c}{\partial \omega} \frac{\partial \omega}{\partial K} = \frac{1}{K} \frac{\partial c}{\partial x} + \frac{\partial c}{\partial \omega} \frac{\partial \omega}{\partial K}; \tag{A.3}
\]

\[
\frac{\partial^2 C}{\partial K^2} = \frac{\partial c}{\partial x} \frac{\partial^2 x}{\partial K^2} + \frac{\partial c}{\partial x} \left( \frac{\partial x}{\partial K} \right)^2 + \frac{\partial^2 c}{\partial x \partial K} \frac{\partial x}{\partial K} + \frac{\partial c}{\partial \omega} \frac{\partial^2 \omega}{\partial K^2} + \frac{\partial c}{\partial \omega} \frac{\partial^2 \omega}{\partial K^2} + \frac{\partial^2 c}{\partial x^2} \left( \frac{\partial x}{\partial K} \right)^2;
\]

\[
= \frac{1}{K^2} \left( \frac{\partial c}{\partial x} + \frac{\partial^2 x}{\partial x^2} \right) + 2 \frac{\partial^2 c}{\partial x \partial K} \frac{\partial x}{\partial K} + \frac{\partial c}{\partial \omega} \frac{\partial^2 \omega}{\partial K^2} + \frac{\partial c}{\partial \omega} \frac{\partial^2 \omega}{\partial K^2} + \frac{\partial^2 c}{\partial x^2} \left( \frac{\partial x}{\partial K} \right)^2; \tag{A.4}
\]

\[
\frac{\partial C}{\partial T} = \frac{\partial c}{\partial x} \frac{\partial x}{\partial T} + \frac{\partial c}{\partial \omega} \frac{\partial \omega}{\partial T} = -r \frac{\partial c}{\partial x} + \frac{\partial c}{\partial \omega} \frac{\partial \omega}{\partial T}. \tag{A.5}
\]

For the computation of the partial derivatives of \( c(x, \omega) \), we start with deriving an useful identity:

\[
S_0 N'(d_1) = S_0 e^{\frac{\omega}{2}} N'(d_2). \tag{A.6}
\]

We can derive identity (A.6) from:

\[
\frac{S_0 N'(d_1)}{S_0 e^{\frac{\omega}{2}} N'(d_2)} = \frac{e^{-\frac{1}{2} d_1^2}}{e^x e^{-\frac{1}{2} d_2^2}} = e^{-\frac{1}{2} (d_1^2 - d_2^2) - x}. \tag{A.7}
\]

Rewriting \( d_1^2 - d_2^2 = -2x \) into equation (A.7):

\[
\frac{S_0 N'(d_1)}{S_0 e^{\frac{\omega}{2}} N'(d_2)} = e^{-\frac{1}{2} (-2x) - x} = e^0 = 1,
\]

which proves the identity (A.6). Now we can compute the partial derivatives of \( c(x, \omega) \). The first partial derivative of the call price with respect to \( x \) is:

\[
\frac{\partial c}{\partial x} = S_0 N'(d_1) \frac{\partial d_1}{\partial x} - S_0 e^{\frac{\omega}{2}} N'(d_2) \frac{\partial d_2}{\partial x} - S_0 e^{\frac{\omega}{2}} N(d_2)
\]

\[
= (S_0 N'(d_1) - S_0 e^{\frac{\omega}{2}} N'(d_2)) \frac{\partial d_1}{\partial x} - S_0 e^{\frac{\omega}{2}} N(d_2)
\]

\[
= -S_0 e^{\frac{\omega}{2}} N(d_2). \tag{A.8}
\]

The first derivative with respect to the total variance is given by:

\[
\frac{\partial c}{\partial \omega} = S_0 N'(d_1) \frac{\partial d_1}{\partial \omega} - S_0 e^{\frac{\omega}{2}} N'(d_2) \frac{\partial d_2}{\partial \omega}
\]

\[
= (S_0 N'(d_1) - S_0 e^{\frac{\omega}{2}} N'(d_2)) \frac{\partial d_1}{\partial \omega} + \frac{1}{2} S_0 e^{\frac{\omega}{2}} N'(d_2) \frac{1}{\sqrt{\omega}}
\]

\[
= \frac{1}{2} S_0 e^{\frac{\omega}{2}} N'(d_2) \frac{1}{\sqrt{\omega}}. \tag{A.9}
\]

The partial derivative of \( c(x, \omega) \) with respect to \( x \) and \( \omega \) is:

\[
\frac{\partial^2 c}{\partial x \partial \omega} = -S_0 e^{\frac{\omega}{2}} N'(d_2) \frac{\partial d_2}{\partial \omega}
\]

\[
= -S_0 e^{\frac{\omega}{2}} N'(d_2) \left( \frac{1}{2} x \frac{1}{\omega} \frac{1}{\sqrt{\omega}} - \frac{1}{4} \frac{1}{\sqrt{\omega}} \right)
\]

\[
= \frac{\partial c}{\partial \omega} \left( -\frac{x}{\omega} + \frac{1}{2} \right). \tag{A.10}
\]
The partial derivatives of the total implied variance with respect to the strike and maturity are:
\[ \frac{\partial^2 c}{\partial x^2} = -S_0 e^x N(d_2) - S_0 e^x N'(d_2) \frac{\partial d_2}{\partial x}, \]
\[ = \frac{\partial c}{\partial x} + 2 \frac{\partial c}{\partial \omega}. \quad \text{(A.11)} \]
and second derivative with respect to \( \omega \):
\[ \frac{\partial^2 c}{\partial \omega^2} = \frac{1}{2} S_0 e^x N''(d_2) \frac{\partial d_2}{\partial \omega} \frac{1}{\sqrt{\omega}} - \frac{1}{4} S_0 e^x N'(d_2) \frac{1}{\omega^{3/2}} \]
\[ = \frac{\partial c}{\partial \omega} \left( \frac{d_2}{\sqrt{\omega}} - \frac{1}{2\omega} \right) \]
\[ = \frac{\partial c}{\partial \omega} \left( \frac{x^2}{2\omega^2} - \frac{1}{2\omega} - \frac{1}{8} \right). \quad \text{(A.12)} \]
Note that these partial derivatives all have a \( \frac{\partial c}{\partial \omega} \) term. Substitute (A.10), (A.11) and (A.12) into expression (A.4) for \( \frac{\partial^2 c}{\partial x^2} \):
\[ \frac{\partial^2 C}{\partial K^2} = \frac{1}{K^2} \left( \frac{\partial c}{\partial x} + \frac{\partial c}{\partial \omega} + 2 \frac{\partial c}{\partial x} \right) + \frac{2}{K} \frac{\partial c}{\partial K} \frac{\partial c}{\partial \omega} \left( \frac{x}{\omega} + \frac{1}{2} \right) + \frac{\partial c}{\partial x} \frac{\partial^2 c}{\partial x^2} \]
\[ + \frac{\partial c}{\partial \omega} \left( \frac{x^2}{2\omega^2} - \frac{1}{2\omega} - \frac{1}{8} \right) \left( \frac{\partial c}{\partial K} \right)^2 \]
\[ = \frac{\partial c}{\partial \omega} \left\{ 2 - \frac{2}{K} \frac{\partial c}{\partial K} \left( \frac{x}{\omega} + \frac{1}{2} \right) + \frac{\partial^2 c}{\partial K^2} \left( \frac{x^2}{2\omega^2} - \frac{1}{2\omega} - \frac{1}{8} \right) \left( \frac{\partial c}{\partial K} \right)^2 \right\}. \quad \text{(A.13)} \]
The partial derivatives of the total implied variance with respect to the strike and maturity are:
\[ \frac{\partial \omega}{\partial K} = 2\sigma \frac{\partial \sigma}{\partial K} T; \]
\[ \frac{\partial^2 \omega}{\partial K^2} = 2\sigma \frac{\partial^2 \sigma}{\partial K^2} T + 2 \left( \frac{\partial \sigma}{\partial K} \right)^2 T; \]
\[ \frac{\partial \omega}{\partial T} = \sigma^2 + 2\sigma \frac{\partial \sigma}{\partial T}. \]
These can be used to rewrite the term between brackets in (A.13):
\[ \frac{\partial^2 C}{\partial K^2} = \frac{\partial c}{\partial \omega} \frac{2}{K^2} \left\{ 1 + K \sigma \frac{\partial \sigma}{\partial K} T - 2K \frac{\partial \sigma}{\partial K} \frac{x}{\sigma} + K^2 \left( \frac{\partial \sigma}{\partial K} \right)^2 T + K^2 \frac{\partial^2 \sigma}{\partial K^2} T \right\} \]
\[ + \left( \frac{\partial \sigma}{\partial K} \right)^2 \frac{x^2}{\sigma^2} - K^2 \left( \frac{\partial \sigma}{\partial K} \right)^2 T - \frac{1}{4} K^2 \sigma^2 \left( \frac{\partial \sigma}{\partial K} \right)^2 T^2 \}
\[ = \frac{\partial c}{\partial \omega} \frac{2}{K^2} \left\{ \left( 1 - \frac{Kx}{\sigma} \frac{\partial \sigma}{\partial K} \right)^2 + \sigma T \left( \frac{\partial \sigma}{\partial K} - \frac{1}{4} K \sigma T \left( \frac{\partial \sigma}{\partial K} \right)^2 + K \frac{\partial^2 \sigma}{\partial K^2} \right) \right\}. \quad \text{(A.14)} \]
The numerator of the local variance (A.2) can be rewritten as:
\[ \frac{\partial C}{\partial T} + rK \frac{\partial T}{\partial K} = \frac{\partial c}{\partial \omega} + \frac{\partial c}{\partial \omega} \frac{\partial \omega}{\partial T} + rK \left( \frac{1}{K} \frac{\partial c}{\partial K} + \frac{\partial c}{\partial \omega} \frac{\partial \omega}{\partial K} \right) \]
\[ = \frac{\partial c}{\partial \omega} + \frac{\partial c}{\partial \omega} \frac{\partial \omega}{\partial T} + rK \frac{\partial c}{\partial \omega} \frac{\partial \omega}{\partial K} \]
\[ = \frac{\partial c}{\partial \omega} \left\{ \sigma^2 + 2\sigma T \left( \frac{\partial \sigma}{\partial K} + rK \frac{\partial \sigma}{\partial \omega} \right) \right\}. \quad \text{(A.15)} \]
Substitute (A.14) and (A.15) into (A.2) and divide out the \( \frac{\partial c}{\partial \omega} \) term:
\[ \sigma_{LV}(K,T)^2 = \frac{\sigma^2 + 2\sigma T \left( \frac{\partial \sigma}{\partial K} + rK \frac{\partial \sigma}{\partial \omega} \right)}{\left( 1 - \frac{Kx}{\sigma} \frac{\partial \sigma}{\partial K} \right)^2 + K \sigma T \left( \frac{\partial \sigma}{\partial K} - \frac{1}{4} K \sigma T \left( \frac{\partial \sigma}{\partial K} \right)^2 + K \frac{\partial^2 \sigma}{\partial K^2} \right)}. \quad \text{(A.16)} \]
Hence, we have an alternative representation of Dupire’s local volatility model in terms of the implied Black-Scholes volatilities \( \sigma = \sigma_{BS}(K,T) \).
A.2 Partial derivatives SVI

Recall the SVI parameterization of the volatility:

$$\sigma_{SVI}(x) = \sqrt{a + b \left( \rho(x - l) + \sqrt{(x - l)^2 + \sigma^2} \right)},$$  \hspace{1cm} (A.17)

where $x$ is the log-moneyness:

$$x := \log \left( \frac{K}{F} \right) = \log \left( \frac{K}{S_0} \right) - r(T - t).$$  \hspace{1cm} (A.18)

We need to compute the partial derivatives of $\sigma_{SVI}(x)$ with respect to strike and maturity. For a given time to maturity $T$ the parameter set $\lambda = \{a, b, \rho, l, \sigma\}$ is the same for all values of strike $K$. Hence, we can compute the partial derivatives in the strike direction analytically.

First we compute derivatives of the SVI volatility with respect to the log-moneyness $x$. The first derivative is straightforward:

$$\sigma'_{SVI}(x) = \frac{1}{2} \left( a + b \left( \rho(x - l) + \sqrt{(x - l)^2 + \sigma^2} \right) \right)^{-\frac{1}{2}} b \left( \rho + \frac{1}{2} \frac{2(x - l)}{\sqrt{(x - l)^2 + \sigma^2}} \right),$$

$$= \frac{b}{2\sigma_{SVI}(x)} \left( \rho + \frac{x - l}{\sqrt{(x - l)^2 + \sigma^2}} \right).$$  \hspace{1cm} (A.19)

And for the second derivative we apply the product rule:

$$\sigma''_{SVI}(x) = \frac{-b\sigma'_{SVI}(x)}{2\sigma_{SVI}(x)^2} \left( \rho + \frac{x - l}{\sqrt{(x - l)^2 + \sigma^2}} \right) + \frac{b}{2\sigma_{SVI}(x)} \frac{d}{dx} \left( \rho + \frac{x - l}{\sqrt{(x - l)^2 + \sigma^2}} \right).$$

For the latter term holds:

$$\frac{d}{dx} \left( \rho + \frac{x - l}{\sqrt{(x - l)^2 + \sigma^2}} \right) = \frac{d}{dx} \frac{x - l}{\sqrt{(x - l)^2 + \sigma^2}} = \frac{d}{dx} g(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2},$$

where $f'(x) = 1$ and $g'(x) = \frac{1}{2} \frac{2(x - l)}{\sqrt{(x - l)^2 + \sigma^2}} = \frac{f(x)}{g(x)}$. Hence,

$$\frac{d}{dx} \rho + \frac{x - l}{\sqrt{(x - l)^2 + \sigma^2}} = g(x) - f(x)^2 / g(x).$$

Substitute into equation (A.20):

$$\sigma''_{SVI}(x) = \frac{-b\sigma'_{SVI}(x)}{2\sigma_{SVI}(x)^2} \left( \rho + \frac{x - l}{\sqrt{(x - l)^2 + \sigma^2}} \right) + \frac{b}{2\sigma_{SVI}(x)} \frac{\sqrt{(x - l)^2 + \sigma^2} - \frac{(x - l)^2}{\sqrt{(x - l)^2 + \sigma^2}}}{(x - l)^2 + \sigma^2}.$$  \hspace{1cm} (A.21)

Note that this expression can be simplified using (A.19):

$$\sigma''_{SVI}(x) = \frac{-b^2}{4\sigma_{SVI}(x)^3} \left( \rho + \frac{x - l}{\sqrt{(x - l)^2 + \sigma^2}} \right)^2 + \frac{b}{2\sigma_{SVI}(x)} \frac{\sqrt{(x - l)^2 + \sigma^2} - \frac{(x - l)^2}{\sqrt{(x - l)^2 + \sigma^2}}}{(x - l)^2 + \sigma^2},$$

$$= \frac{b}{2\sigma_{SVI}(x)} \left\{ \frac{-b}{2\sigma_{SVI}(x)} \left( \rho + \frac{x - l}{\zeta} \right)^2 + \frac{\zeta - \frac{(x - l)^2}{\zeta^2}}{\zeta^2} \right\},$$

where,

$$\zeta = \sqrt{(x - l)^2 + \sigma^2}.$$
Note
\[ \zeta = \frac{(x-l)^2}{\zeta^2} = \frac{\zeta^2 - (x-l)^2}{\zeta^3} = \tilde{\sigma}^2 \]

Given the partial derivatives of \( \sigma_{SVI} \), the local variance expression:
\begin{align*}
\sigma_{LV}(x,T) &= \frac{\sigma_{SVI}^2 + 2\sigma_{SVI} T \left( \frac{\partial \sigma_{SVI}}{\partial T} + rK \frac{\partial \sigma_{SVI}}{\partial K} \right) + \left( 1 - \frac{Kx}{\sigma_{SVI}} \right)^2 + K \sigma_{SVI}^2 \left( \frac{\partial \sigma_{SVI}}{\partial K} \right)^2 + K^{\prime} \frac{\sigma_{SVI}}{\partial K^2} }{1 - \frac{Kx}{\sigma_{BS}} \frac{\partial \sigma_{BS}}{\partial K}},
\end{align*}
(A.22)
can be simplified, starting with the numerator:
\[ \sigma_{SVI}^2 + 2\sigma_{SVI} \left( \frac{\partial \sigma_{SVI}}{\partial T} + rK \frac{\partial \sigma_{SVI}}{\partial K} \right) T = \sigma_{SVI}^2 + 2\sigma_{SVI} T \frac{\partial \sigma_{SVI}}{\partial T} + bT \left( \rho + \frac{x-l}{\zeta} \right). \]

The terms in the denominator:
\[ 1 - \frac{Kx}{\sigma_{BS}} \frac{\partial \sigma_{BS}}{\partial K} = 1 - \frac{Kx}{\sigma_{SVI}} \sigma_{SVI} \left( \frac{1}{K} \right) = 1 - \frac{bx}{2\sigma_{BS}} \left( \rho + \frac{x-l}{\zeta} \right), \]
and
\begin{align*}
K \frac{\partial \sigma_{SVI}}{\partial K} &\left( \frac{\partial \sigma_{SVI}}{\partial K} - \frac{1}{4} K \frac{\partial \sigma_{SVI}}{\partial K}^2 + K \frac{\partial^2 \sigma_{SVI}}{\partial K^2} \right) \\
= &\sigma_{SVI}(x) \sigma_{SVI}(x) T - \frac{1}{4} \sigma_{SVI}(x)^2 \sigma_{SVI}(x) T^2 + \sigma_{SVI}(x) \sigma_{SVI}(x) T - \sigma_{SVI}(x) \sigma_{SVI}(x) T \\
= &- \frac{b^2}{16} \left( \rho + \frac{x-l}{\zeta} \right)^2 T^2 + \frac{-b^2}{4\sigma_{SVI}} \left( \rho + \frac{x-l}{\zeta} \right)^2 T + \frac{b}{2} \left( \zeta - \frac{(x-l)^2}{\zeta} \right) T \\
= &- \frac{b^2}{4} \left( \rho + \frac{x-l}{\zeta} \right)^2 \left( \frac{1}{\sigma_{SVI}^2} + \frac{T^2}{4} \right) T + \frac{b}{2} \left( \frac{\zeta - \frac{(x-l)^2}{\zeta} T}{\zeta^2} \right) T.
\end{align*}

Combining everything and substitute back into (A.22):
\[ \sigma_{LV}(x,T) = \frac{\sigma_{SVI}^2 + 2\sigma_{SVI} \frac{\partial \sigma_{SVI}}{\partial T} T + bT \left( \rho + \frac{x-l}{\zeta} \right) T}{\left( 1 - \frac{bx}{2\sigma_{SVI}} \left( \rho + \frac{x-l}{\zeta} \right) \right)^2 + \frac{b^2}{4} \left( \rho + \frac{x-l}{\zeta} \right)^2 \left( \frac{1}{\sigma_{SVI}^2} + \frac{T^2}{4} \right) + \frac{b}{2} \tilde{\sigma}^2 T}. \]
(A.23)

Equation (A.23) is used in the implementation of the Dupire-SVI model.
Corollary A.1. As a direct result of the parameter restrictions, the SVI model has the following properties:

- $\sigma^2_{SVI}(x)$ is a convex function in $x$.
- $\sigma^2_{SVI}(x) \geq 0$ for all $x$.

Proof. We proof the above statements for $l = 0$. Consider a function:

$$h(x) = a + b\left(\rho x + \sqrt{x^2 + \tilde{\sigma}^2}\right).$$

The first and second derivatives are given by

$$h'(x) = b\left(\rho + \frac{x}{\sqrt{x^2 + \tilde{\sigma}^2}}\right); \quad h''(x) = \frac{b\tilde{\sigma}^2}{(x^2 + \tilde{\sigma}^2)^{3/2}}.$$

Always taking the positive value of the square root and $b \geq 0$, $\tilde{\sigma} > 0$ results in:

$$h''(x) \geq 0, \forall x \in \mathbb{R}.$$

Hence, $h(x)$ is convex in $x$. Since $h(x)$ is a convex function, finding a root of $h'(x)$ would indicate a minimum, or a boundary. It is sufficient to check whether this minimum is greater than zero to make sure that $h(x)$ is always positive.

$$h'(x) = b\left(\rho + \frac{x}{\sqrt{x^2 + \tilde{\sigma}^2}}\right) = 0 \iff \rho = \frac{-x}{\sqrt{x^2 + \tilde{\sigma}^2}} \iff \hat{x} := \frac{-\rho\tilde{\sigma}}{\sqrt{1 - \rho^2}}.$$

Evaluate $h(x)$ in $\hat{x}$:

$$h(\hat{x}) = a + b\left(\frac{-\rho^2\tilde{\sigma}}{\sqrt{1 - \rho^2}} + \sqrt{\frac{\rho^2\tilde{\sigma}^2}{1 - \rho^2} + \tilde{\sigma}^2}\right) = a + b\left(\frac{-\rho^2\tilde{\sigma}}{\sqrt{1 - \rho^2}} + \sqrt{\frac{\rho^2\tilde{\sigma}^2 + (1 - \rho^2)\tilde{\sigma}^2}{1 - \rho^2}}\right) = a + b\left(\frac{-\rho^2\tilde{\sigma}}{\sqrt{1 - \rho^2}} + \sqrt{\frac{\tilde{\sigma}}{1 - \rho^2}}\right) = a + b\tilde{\sigma}\sqrt{1 - \rho^2} \geq 0,$$

where the inequality is satisfied through the parameter restriction. Without loss of generality we can prove these statements for all $l \in \mathbb{R}$. Hence, the parameter restrictions ensure that $\sigma_{SVI}(x)$ is convex and $\sigma_{SVI}(x) \geq 0$ for all $x \in \mathbb{R}$. 

\qed
A.4 Lemma 6.8 Discrete characteristic function

Lemma A.2. The characteristic function of $X_{m+1}^\Delta$, given $X_m^\Delta = x$, for $\kappa(x,t) \neq 0$ is given by

$$
\phi_{X_{m+1}^\Delta} (u \mid X_m^\Delta = x) = \mathbb{E} \left[ e^{iuX_{m+1}^\Delta} \mid X_m^\Delta = x \right] \\
= \exp \left\{ iux + ium(x,t)\Delta t - \frac{1}{2} u^2 s^2(x,t)\Delta t \right\} (1 - 2i\kappa(x,t)\Delta t)^{-1/2},
$$

and for $\kappa(x,t) = 0$ we have

$$
\phi_{X_{m+1}^\Delta} (u \mid X_m^\Delta = x) = \exp \left\{ iux + ium(x,t)\Delta t - \frac{1}{2} u^2 s^2(x,t)\Delta t \right\}.
$$

Proof. For the general form of $X_{m+1}^\Delta$ we have a discrete characteristic function

$$
\phi_{X_{m+1}^\Delta} (u \mid X_m^\Delta = x) = \mathbb{E} \left[ e^{iuX_{m+1}^\Delta} \mid X_m^\Delta = x \right] \\
= \mathbb{E} \left[ \exp \left\{ iux + ium(x,t)\Delta t + ius(x,t)\Delta W_{m+1} + i\kappa(x,t)(\Delta W_{m+1})^2 \right\} \mid X_m^\Delta = x \right] \\
= \exp \left\{ iux + ium(x,t)\Delta t \right\} \mathbb{E} \left[ \exp \left\{ ius(x,t)\Delta W_{m+1} + i\kappa(x,t)(\Delta W_{m+1})^2 \right\} \mid X_m^\Delta = x \right].
$$

(A.24)

We consider two different cases, for $\kappa(x,t) = 0$ we have

$$
\phi_{X_{m+1}^\Delta} (u \mid X_m^\Delta = x) = \exp \left\{ iux + ium(x,t)\Delta t - \frac{1}{2} u^2 s^2(x,t)\Delta t \right\},
$$

where we have used that $\Delta W_{m+1} \sim \mathcal{N}(0,\sqrt{\Delta t})$. For $\kappa(x,t) \neq 0$ we can apply a polynomial factorization

$$
ius(x)\Delta W_{m+1} + i\kappa(x,t)(\Delta W_{m+1})^2 = -\frac{1}{2} ius^2(x,t) \kappa(x) + i\kappa(x,t) \left( \Delta W_{m+1} + \frac{1}{2} \frac{s(x,t)}{\kappa(x)} \right)^2.
$$

Substituting the above equation into (A.24) gives us:

$$
\phi_{X_{m+1}^\Delta} (u \mid X_m^\Delta = x) = \exp \left\{ iux + ium(x,t)\Delta t - \frac{1}{2} \frac{us^2(x,t)}{\kappa(x)} \right\} \\
\cdot \mathbb{E} \left[ \exp \left\{ ius(x) \left( \Delta W_{m+1} + \frac{1}{2} \frac{s(x,t)}{\kappa(x,t)} \right)^2 \right\} \mid X_m^\Delta = x \right].
$$

(A.25)

where $\Delta W_{m+1} + \frac{1}{2} \frac{s(x,t)}{\kappa(x,t)} \sim \mathcal{N} \left( \frac{1}{2} \frac{s(x,t)}{\kappa(x,t)}, \Delta t \right)$ such that we have a non-central chi-squared distribution with one degree of freedom and non-centrality parameter $\frac{1}{4} \frac{s^2(x,t)}{\kappa^2(x,t)}$:

$$
\frac{1}{\Delta t} \left( \Delta W_{m+1} + \frac{1}{2} \frac{s(x,t)}{\kappa(x,t)} \right)^2 \sim \chi^2 \left( \frac{1}{4} \frac{s^2(x,t)}{\kappa^2(x,t)} \right).
$$

Hence, we can rewrite (A.25):

$$
\phi_{X_{m+1}^\Delta} (u \mid X_m^\Delta = x) = \exp \left\{ iux + ium(x,t)\Delta t - \frac{1}{2} \frac{us^2(x,t)}{\kappa(x,t)} \right\} \\
\cdot \exp \left\{ \frac{1}{4} \frac{us^2(x,t)}{\kappa(x,t)} \frac{iu}{\left( 1 - 2i\kappa(x,t)\Delta t \right)^{1/2}} \right\} (1 - 2i\kappa(x,t)\Delta t)^{-\frac{1}{2}}
$$

$$
= \exp \left\{ iux + ium(x,t)\Delta t - \frac{1}{2} \frac{us^2(x,t)}{\kappa(x,t)} \Delta t \right\} (1 - 2i\kappa(x,t)\Delta t)^{-\frac{1}{2}},
$$

\[\square\]
Appendix B

Calibrated SVI parameters

The following SVI parameter sets are used in Section 5.4.1 and Section 5.4.2:

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The following SVI parameter sets are used in Section 5.4.3:

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