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A multigrid method
for a discretization of the
incompressible Navier-Stokes equations
in general coordinates

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SUMMARY

A discretization for the incompressible Navier-Stokes equations in general coordinates is derived. A staggered grid arrangement is used for the unknowns: contravariant flux-components and pressure.

The performance of the nonlinear multigrid algorithm, in particular the performance of two smoothing methods SCGS and SCGS/LS is investigated .

INTRODUCTION

For the computation of flows in complex geometries boundary-fitted coordinates are popular nowadays. An arbitrarily shaped physical domain is mapped onto a computational rectangular block, causing a transformation of the equations. The equations considered are the incompressible Navier-Stokes equations. In developing numerical methods for the incompressible Navier-Stokes equations in curvilinear coordinates some choices need to be made:

- (i) a staggered or non-staggered grid
- (ii) the type of velocity unknowns: Cartesian, contravariant or other
- (iii) the iterative solver

In this work a staggered grid is used with contravariant flux components as unknowns. The solution method chosen is a multigrid method. Tensor notation proves indispensable for formulating physical conservation laws in general coordinates. It is found that discretization accuracy depends strongly on the way the geometric quantities (base vectors, metric tensor, Christoffel symbols) are discretized. It is assumed that the coordinate mapping is given only in terms of the physical coordinates of the points in the computational grid. The geometric quantities require differentiation of the mapping once or twice. The resulting work and inaccuracy which occurs if certain rules are not followed has led many authors to consider non-coordinate-invariant discretizations. However, this usually compromises stability and generality of the discretization. It seems attractive to use invariant discretizations of invariant physical laws, provided discretization accuracy can be maintained, and this is the approach that we follow. Desirable properties are: constant solutions are exact solutions of the discrete equations, and no loss of accuracy in the presence of grid discontinuities. These aims can be met by discretizing the geometric quantities in a special way, and by using fluxes rather than velocities as unknowns. If a coordinate invariant discretization of the equations is formulated in Gibbs' vector notation as in [7] explicit occurrence of Christoffel symbols is avoided. Instead other geometric quantities appear implicitly defining approximations of Christoffel symbols.

A nonlinear multigrid method is used to solve the equations. The behaviour of two smoothing methods, Symmetric Coupled Gauss-Seidel (SCGS) proposed in [11] and a variant (SCGS/LS (Line Solver)) [6], [8] is compared for discretizations in curvilinear coordinates.

THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS IN GENERAL COORDINATES

The notation used will be tensor notation ([1], [9]). In general curvilinear coordinates the incompressible Navier-Stokes equations are given by:

$$U_{,\alpha}^\alpha = 0 \quad (1)$$

$$T_{,\beta}^{\alpha\beta} = (\rho U^\alpha U^\beta)_{,\beta} + (g^{\alpha\beta} p)_{,\beta} - \tau_{,\beta}^{\alpha\beta} = \rho F^\alpha \quad (2)$$

where ρ is the fluid density and $\tau^{\alpha\beta}$ represents the deviatoric stress tensor given by

$$\tau^{\alpha\beta} = \mu(g^{\alpha\gamma} U_{,\gamma}^\beta + g^{\gamma\beta} U_{,\gamma}^\alpha) \quad (3)$$

with μ the viscosity coefficient. The contravariant metric tensor $g^{\alpha\beta}$ is defined as:

$$g^{\alpha\beta} = \mathbf{a}^{(\alpha)} \cdot \mathbf{a}^{(\beta)}$$

with $\mathbf{a}^{(\alpha)} = \frac{\partial \xi^\alpha}{\partial \mathbf{x}}$ the contravariant base vectors.

The equations are transformed, because the arbitrarily shaped domain Ω is mapped onto a rectangular block G , resulting in a boundary fitted grid as shown in Figure 1; $\mathbf{x}$ are Cartesian coordinates, ξ boundary conforming curvilinear coordinates.

Figure 1: The mapping of a physical domain Ω onto a rectangular block G

In order to get accurate discretizations on non-smooth grids, some requirements should be met [10]:

- (i) The geometric identity $\oint_S \mathbf{a}_\beta^{(\alpha)} dS_{(\alpha)} = 0$ (coming from the application of the divergence theorem to a constant vector field) should be satisfied numerically. This requirement imposes rules on the covariant and contravariant base vectors.
- (ii) Uniform flow fields should satisfy the discrete equations exactly. From this requirement the use of $V^\alpha = \sqrt{g} U^\alpha$, a relative contravariant tensor of weight one is found to be preferred as primary unknown, instead of U^α , although this by itself is not sufficient to meet this requirement; we will not go further into this here. Details can be found in [10], [4].

The covariant derivative of a relative contravariant tensor of weight one is defined by

$$V_{,\beta}^\alpha = \frac{\partial V^\alpha}{\partial \xi^\beta} + \left\{ \begin{matrix} \alpha \\ \beta \gamma \end{matrix} \right\} V^\gamma - \left\{ \begin{matrix} \gamma \\ \gamma \beta \end{matrix} \right\} V^\alpha$$

The equations are discretized with a finite volume method on a staggered grid arrangement, i.e. pressure unknowns are placed at the cell centers, the flux unknowns V^1 are located at the centers of ξ^2 faces, the V^2 unknowns at the centers of ξ^1 -faces. The total number of variables linked together in a momentum equation is 19 (13 flux unknowns, 6 pressure unknowns).

THE MULTIGRID SOLUTION ALGORITHM

The discretized equations are solved with the standard nonlinear multigrid algorithm [3], [2], [12]. Details of this algorithm are presented in [5]. Two smoothing methods are investigated for the discretization in curvilinear coordinates; the Symmetric Coupled Gauss-Seidel scheme (SCGS), proposed in [11], where the Navier-Stokes equations on staggered Cartesian grids were solved, and the line variant SCGS/LS proposed in [6], [8], which solves pressure unknowns along a line simultaneously. SCGS/LS is not well known yet and will be explained briefly: SCGS/LS is based on the SCGS smoother. For this smoother the five discretized equations per cell ($2 \times V^1 + 2 \times V^2$ momentum equations + continuity) are:

$$\begin{aligned}(A_c^1)_{i-1/2,j} V_{i-1/2,j}^1 &= F_{i-1/2,j}^1 \\ (A_c^1)_{i+1/2,j} V_{i+1/2,j}^1 &= F_{i+1/2,j}^1 \\ (A_c^2)_{i,j-1/2} V_{i,j-1/2}^2 &= F_{i,j-1/2}^2 \\ (A_c^2)_{i,j+1/2} V_{i,j+1/2}^2 &= F_{i,j+1/2}^2\end{aligned}$$

and

$$(V_{i+1/2,j}^1 - V_{i-1/2,j}^1)/\delta\xi^1 + (V_{i,j+1/2}^2 - V_{i,j-1/2}^2)/\delta\xi^2 = 0$$

where for example

$$\begin{aligned}F_{i+1/2,j}^1 &= A_{sw}^1 V_{i-1/2,j-1}^1 + A_s^1 V_{i+1/2,j-1}^1 + A_{se}^1 V_{i+3/2,j-1}^1 + \\ &A_w^1 V_{i-1/2,j}^1 + A_e^1 V_{i+3/2,j}^1 + \\ &A_{nw}^1 V_{i-1/2,j+1}^1 + A_n^1 V_{i+1/2,j+1}^1 + A_{ne}^1 V_{i+3/2,j+1}^1 + \\ &A_{sw}^2 V_{i,j-1/2}^2 + A_{se}^2 V_{i+1,j-1/2}^2 + \\ &A_{nw}^2 V_{i,j+1/2}^2 + A_{ne}^2 V_{i+1,j+1/2}^2 + \\ &A_{sw}^3 p_{i,j-1} + A_{se}^3 p_{i+1,j-1} + A_w^3 p_{ij} + \\ &A_e^3 p_{i+1,j} + A_{nw}^3 p_{i,j+1} + A_{ne}^3 p_{i+1,j+1}\end{aligned}$$

These equations are rewritten for solving them along a ξ^1 -line in terms of corrections, indicated by primes and residuals, as follows:

$$\begin{aligned}(A_c^1)_{i-1/2,j} (V^1)'_{i-1/2,j} - (A_w^3)_{i-1/2,j} p'_{i-1,j} - (A_e^3)_{i-1/2,j} p'_{i,j} &= R_{i-1/2,j}^1 \quad (4) \\ &= F_{i-1/2,j}^1 - (A_c^1)_{i-1/2,j} V_{i-1/2,j}^1 \\ (A_c^1)_{i+1/2,j} (V^1)'_{i+1/2,j} - (A_w^3)_{i+1/2,j} p'_{i,j} - (A_e^3)_{i+1/2,j} p'_{i+1,j} &= R_{i+1/2,j}^1 \\ &= F_{i+1/2,j}^1 - (A_c^1)_{i+1/2,j} V_{i+1/2,j}^1 \\ (A_c^2)_{i,j-1/2} (V^2)'_{i,j-1/2} - (A_n^3)_{i,j-1/2} p'_{i,j} &= R_{i,j-1/2}^2 \\ &= F_{i,j-1/2}^2 - (A_c^2)_{i,j-1/2} V_{i,j-1/2}^2 \\ (A_c^2)_{i,j+1/2} (V^2)'_{i,j+1/2} - (A_n^3)_{i,j+1/2} p'_{i,j} &= R_{i,j+1/2}^2 \\ &= F_{i,j+1/2}^2 - (A_c^2)_{i,j+1/2} V_{i,j+1/2}^2 \\ (V_{i+1/2,j}^1 - V_{i-1/2,j}^1)/\delta\xi^1 + (V_{i,j+1/2}^2 - V_{i,j-1/2}^2)/\delta\xi^2 &= R_{i,j}^3 \\ &= (V_{i-1/2,j}^1 - V_{i+1/2,j}^1)/\delta\xi^1 + \\ &\quad (V_{i,j-1/2}^2 - V_{i,j+1/2}^2)/\delta\xi^2\end{aligned}$$

(The equations for solving along a ξ^2 -line are found in a similar way).

Solving the set of equations (4) gives us a tri-diagonal system for pressures along the line:

$$(P_w)_{i-1,j} p'_{i-1,j} + (P_c)_{i,j} p'_{i,j} + (P_e)_{i+1,j} p'_{i+1,j} = R^p_{i,j}$$

where

$$\begin{aligned} (P_w)_{i-1,j} &= \left(-\frac{A_w^3}{A_c^1}\right)_{i-1/2,j} \cdot \frac{1}{\delta\xi^1} \\ (P_e)_{i+1,j} &= \left(\frac{A_e^3}{A_c^1}\right)_{i+1/2,j} \cdot \frac{1}{\delta\xi^1} \\ (P_c)_{i,j} &= \left(-\frac{A_c^3}{A_c^1}\right)_{i-1/2,j} \cdot \frac{1}{\delta\xi^1} + \left(\frac{A_w^3}{A_c^1}\right)_{i+1/2,j} \cdot \frac{1}{\delta\xi^1} - \left(\frac{A_n^3}{A_c^2}\right)_{i,j-1/2} \cdot \frac{1}{\delta\xi^2} + \left(\frac{A_s^3}{A_c^2}\right)_{i,j+1/2} \cdot \frac{1}{\delta\xi^2} \\ R^p_{i,j} &= R^3_{i,j} + \left(\frac{R^1}{A_c^1}\right)_{i-1/2,j} \cdot \frac{1}{\delta\xi^1} - \left(\frac{R^1}{A_c^1}\right)_{i+1/2,j} \cdot \frac{1}{\delta\xi^1} + \left(\frac{R^2}{A_c^2}\right)_{i,j-1/2} \cdot \frac{1}{\delta\xi^2} - \left(\frac{R^2}{A_c^2}\right)_{i,j+1/2} \cdot \frac{1}{\delta\xi^2} \end{aligned}$$

After solving the tridiagonal system the corrections for p are added to the current solution. The corrections for V^1 and V^2 are found from (4):

$$\begin{aligned} V^1_{i-1/2,j} &= [R^1_{i-1/2,j} + (A_w^3)_{i-1/2,j} p'_{i-1,j} + (A_e^3)_{i-1/2,j} p'_{i,j}] / (A_c^1)_{i-1/2,j} \\ V^1_{i+1/2,j} &= [R^1_{i+1/2,j} + (A_w^3)_{i+1/2,j} p'_{i,j} + (A_e^3)_{i+1/2,j} p'_{i+1,j}] / (A_c^1)_{i+1/2,j} \\ V^2_{i,j-1/2} &= [R^2_{i,j-1/2} + (A_n^3)_{i,j-1/2} p'_{i,j}] / (A_c^2)_{i,j-1/2} \\ V^2_{i,j+1/2} &= [R^2_{i,j+1/2} + (A_s^3)_{i,j+1/2} p'_{i,j}] / (A_c^2)_{i,j+1/2} \end{aligned}$$

We can move line by line in a forward or backward direction. The velocities are updated twice, while the pressures are updated once.

Underrelaxation is implemented as in [11] by modifying the main diagonal coefficients:

$$\begin{aligned} (A_c^1)_{i\pm 1/2,j} &:= (A_c^1)_{i\pm 1/2,j} / \alpha_1 \\ (A_c^2)_{i,j\pm 1/2} &:= (A_c^2)_{i,j\pm 1/2} / \alpha_2 \end{aligned}$$

TEST PROBLEMS, SOME RESULTS

To investigate the behaviour of SCGS and SCGS/LS in the multigrid method some driven cavity problems are calculated. The first problem is the classical problem: An equidistant grid in a square cavity. Then: a non-equidistant grid in a square cavity, an equidistant grid in a skewed cavity (skew angle 63°), and a non-equidistant grid in an L-shaped cavity. Figures 2a to 2d show the last three cavities. Boundary conditions for the first three cavities are: $u = 0$ on boundary I, II and IV, $u = -1, v = 0$ on III. The L-shaped cavity causes extra problems, namely boundary II consists of two parts as does IV. On only half of II ($y = 1$) the tangential velocity $u = -1$ must be prescribed.

All problems are computed with $Re = 100$ and $Re = 1000$. Average reduction factors r are compared, defined as

$$r = \left(\frac{\|res\|_{nit}}{\|res\|_0} \right)^{1/nit}$$

i.e. the 2-norm of the residual after nit iterations divided by the 2-norm of the starting residual. The algorithm starts with nested iteration; the starting vector is the zero solution on the coarsest ($= 2 \times 2$) grid.

The number of pre-smoothing iterations is 1 as is the number of post-smoothing iterations. The

Figure 2: The different cavities, 16×16 grid

results given are the reduction factors for the W-cycle, which was more efficient then the V-cycle closely followed by the F-cycle.

The smoothers: For different Reynolds numbers different values of the underrelaxation factors α_k are needed. However for both smoothing methods the optimal α_k 's are the same:

$$\text{For } \begin{array}{ll} Re = 100 & : \alpha_k = 0.7 \\ Re = 1000 & : \alpha_k = 0.3 \end{array}$$

For both Reynolds numbers the smoothing iterations are performed in alternating directions, a sweep in the ξ^1 -direction is followed by a sweep in the ξ^2 -direction.

Table 1 presents the reduction factors r for the first three problems for $Re = 100$, $nit = 10$. Table 2 shows r for $Re = 1000$, $nit = 20$. Finally table 3 gives r for the L-shaped cavity, $nit = 20$. Figure 3 shows streamlines for the skew cavity for $Re = 100$ and $Re = 1000$ (64×64 grid). In figure 4 the streamlines for the L-shaped cavity are presented for the 64×64 grid.

Table 1: Average reduction factors for the different cavity problems, $Re = 100$.

number of levels	grid	sq. cav. eq. grid		sq. cav. neq. grid		skew. cav. eq. grid	
		SCGS	SCGS/LS	SCGS	SCGS/LS	SCGS	SCGS/LS
4	16×16	.208	.220	.155	.209	.234	.185
5	32×32	.153	.200	.227	.270	.176	.174
6	64×64	.147	.153	.242	.336	.152	.130
7	128×128	.142	.169	.241	.520	.150	.146

For the square cavity with the non-equidistant grid r_{SCGS} seems better than $r_{SCGS/LS}$. This is due to very good reduction in the first iterations. The convergence factor however is the same for both smoothing methods:

$$\rho \approx 0.55$$

Some ideas to overcome problems with stretched cells are currently under investigation.

Figure 3: Streamlines for the skew cavity, a) $Re = 100$, b) $Re = 1000$, 64×64 grid.

Figure 4: Streamlines for the L-shaped cavity, a) $Re = 100$, b) $Re = 1000$, 64×64 grid.

Table 2: Average reduction factors for the different cavity problems, $Re = 1000$.

number of levels	grid	sq. cav. eq. grid		sq. cav. neq. grid		skew. cav. eq. grid	
		SCGS	SCGS/LS	SCGS	SCGS/LS	SCGS	SCGS/LS
4	16 x 16	.516	.466	.405	.459	.495	.565
5	32 x 32	.565	.495	.445	.459	.540	.540
6	64 x 64	.564	.569	.530	.499	.601	.585
7	128 x 128	.517	.558	.456	.519	.602	.607

For $Re = 1000$ there is not much difference for the three test problems. All reduction factors are level-independent. For the non-equidistant grid of problem 2 r is not worse than r of problem 1. This is due to upwind discretization in large parts of the domain.

Table 3: Average reduction factors for the L-shaped cavity, $Re = 100$ and $Re = 1000$.

number of levels	grid	$Re = 100$		$Re = 1000$	
		SCGS	SCGS/LS	SCGS	SCGS/LS
4	16 x 16	.334	.311	.608	.637
5	32 x 32	.402	.370	.686	.687
6	64 x 64	.411	.410	.767	.688

CONCLUSION

An invariant formulation of the incompressible Navier-Stokes equations in general coordinates has been presented in which Christoffel symbols occur. The discretization of the invariant formulation shows good results for many geometries and fairly non-uniform grids. A level independent convergence rate has been found for the test problems solving with a multigrid solution algorithm. Reduction factors for rectangular and some more complex geometries do not seem to differ much. Reduction factors for domains in which stretched cells occur are worse than for domains with square cells. This is a well known fact when a Gauss-Seidel type of smoothing method is being used. At the moment we are attempting to improve these reduction factors. The behaviour of the two smoothing methods SCGS and SCGS/LS does not differ much. It seems that SCGS is more robust in the variation of underrelaxation factors, especially for low Reynolds flows and in the choice of smoothing directions.

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