stratified flows

internal lee waves in a turbulent two-layer flow

5 internal lee waves in a turbulent two-layer flow

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Notation

A_n	area of layer cross-section
b_1, b_2	coefficients
B	width of flume
c	wave speed
f_n, f_i, f_b	friction factors
F, F_2, F_o	Froude numbers
g, g'	acceleration and reduced acceleration of gravity
h	water depth
h_b	height of bottom
h_n	layer depths
k	dimensionless wavenumber, turbulent kinetic energy
l	half the length of the obstacle
l_e	length scale of energy containing eddies
m	dimensionless damping rate
p_n	ratio of turbulence productions
P_n	turbulence production
q	total flow rate
q_n	layer flow rate
r	friction parameter
R	friction parameter
S_n	profile coefficient
t	time
u_n	layer velocity
u'	turbulence intensity of horizontal velocity
v_n	horizontal velocity disturbance
x	horizontal coordinate
β_n	coefficient
Δu	$= u_1 - u_2$, velocity difference between layers
$\Delta \rho$	$= \rho_2 - \rho_1$, density difference between layers
ϵ	dissipation rate

η	vertical displacement of interface
κ	wavenumber
ρ	reference density
ρ_n	layer density
τ_n, τ_i, τ_b	shear stresses
ϕ	function
ω	frequency

subscripts

b	bottom
i	interface
n	layer number, n=1 for the surface layer n=2 for the bottom layer
o	no friction, no lee wave
-	undisturbed quantity
·	amplitude

1 INTRODUCTION

The study which is described in this report forms part of an investigation toward assessing the relative importance of internal wave induced turbulence and mixing in relation to that produced by other mechanisms in the Rotterdam Waterway. It was prompted by exploratory field measurements which revealed significant internal wave activity in the Rotterdam Waterway.

Internal wave energy can be transferred into turbulent kinetic energy through distortion of the flow field and wave breaking. An increase in turbulence levels caused by internal waves may enhance mass and momentum transfer, and as a consequence influence the distribution of salinity and other solutes.

In general internal waves are damped by the turbulent exchange of momentum. The horizontal extent of an internal wave field generated at some location is determined by the damping rate of the waves. Both wave amplitudes and wave damping need to be considered when addressing the total transfer of wave energy to turbulent kinetic energy and its effect on the resulting turbulence levels.

The objective of the present work is to establish a relationship between internal wave characteristics and an increase in turbulence production; for internal lee waves generated over small scale bottom topography as observed in the Rotterdam Waterway. When the flow is near critical this small scale bottom topography generates large amplitude internal waves. A second objective therefore is to be able to predict the internal wave response to tidal flow in a stratified estuary over observed bottom topography.

The study consists of two parts, in which the observations in the Rotterdam Waterway were simulated by carrying out an exploratory two-layer laboratory experiment, and a mathematical model of internal wave generation by topography, which is based on the Boussinesq equations, was developed.

Stationary lee waves were selected as the object of the present study because of their significance in the Rotterdam Waterway field surveys. In addition they were selected for the laboratory experiments because of their experimental convenience, which included stationary conditions and the ease with which they could be generated.

The laboratory experiments were restricted to two-layer flow because of the limitations imposed by the available experimental installation. Consequently the interface was more stable, that is the Richardson numbers were larger, than found in the Rotterdam Waterway.

This report describes the study of internal lee wave damping which has been made. Section 2 explains the basic lee wave mechanism and develops a Boussinesq type theory of internal waves. Section 3 describes the two-layer steady state experiments which were performed, while Section 4 describes the results of the experiments and makes a comparison of the results with the theoretical developments.

In summary, see Section 5, experiments were performed in which internal lee waves were generated by the flow of a stratified fluid over a small topographic feature. Lee wave damping was observed, the magnitude of which was greater than predicted using linear theory, although non-linear effects may have influenced the results as critical conditions were approached. Observed increases in the levels of turbulent intensities were higher than theoretical estimates, however, there may also have been an increase in the turbulent intensities due to a wake effect in the lee of the obstacle.

The results of this investigation may also have a bearing on current numerical schemes, in which as a rule only the large wavelengths of both topography and flow are resolved, thus neglecting contributions from smaller scales, such as the internal waves observed in this study.

In a subsequent report on the field surveys the findings from the theory and from the experiments will be applied to the acoustic images obtained in these surveys. In this way quantitative information on the contribution of internal waves to turbulence production can be derived by an indirect method.

The study described in this report was initiated by the "Werkgroep Inhomogene Stromingen" at its meeting of April 28, 1987, when it decided to have the field surveys made. The laboratory study was authorized by the "Stuurgroep Inhomogene Stromingen" in its meeting of June 18, 1987.

The theoretical developments included in the study were made by Dr. Ir.

C. Kranenburg, Delft University of Technology. The laboratory experiments were performed at Delft Hydraulics by Dr. J.D. Pietrzak and Dr. Ir. G. Abraham.

2 Mathematical model of stationary lee waves in two-layer flow

2.1 Introduction

Consider the steady flow of a two-layer stratified fluid over a two-dimensional obstacle of small height that is placed on the bed. A fluid particle is displaced vertically when passing over the obstacle. The restoring force caused by the density difference induces a motion in the opposite direction. Some overshoot may occur at a sufficiently low water velocity, which results in an internal wave that distorts the interface. This wave is stationary: it propagates against the flow, but the wave (or phase) speed relative to the bed is zero. The group velocity, which is the speed at which the wave energy travels, generally is less than the phase speed (e.g., Turner, 1973). As a result wave energy travels in the downstream direction, and the wave forms in the lee of the obstacle. Small-amplitude lee waves in a two-layer fluid are of a sinusoidal nature.

If the water velocity is slowly increased, the phase speed relative to the water velocity must also increase for the lee wave to remain stationary. This causes the wavelength to increase as the phase speed (relative to the water velocity) increases with increasing wavelength. The maximum water velocity at which a lee wave can remain stationary and hence be formed is reached when the wavelength becomes infinite. The flow is then said to be (internally) critical, meaning that the long wave speed is zero with respect to the bed. Lee waves can form in subcritical flow, i.e. when the velocity of flow is smaller than that when the flow is critical, but not in supercritical flow when the velocity is larger than when it is critical.

In general the group velocity of the stationary lee wave decreases as the water velocity (and the wavelength) increase. In the long-wave limit, i.e., as critical conditions are approached, the group velocity approaches the phase velocity of the internal wave, and the group velocity relative to the bed becomes zero. Since the group velocity is the velocity at which wave energy is transported, the wave energy therefore tends to accumulate near the obstacle and a large-amplitude lee wave develops. This wave may break to form (in a two-layer fluid) an internal jump, which marks the transition from supercritical flow just downstream of the obstacle to subcritical flow (e.g., Lawrence, 1985). The flow

over the obstacle and the lee wave influence the flow upstream, but this effect is minor in the case of two-layer flow over a low obstacle.

A prerequisite when selecting a mathematical model to describe these lee waves is, that phase and group velocities are different. This rules out the often made assumption of a hydrostatic pressure distribution for long waves in a two-layer flow (e.g., Lawrence, 1985), since this assumption makes the phase velocity equal to the group velocity. Consequently no wave energy would be transported in the downstream direction, and no lee wave would be found even when the flow is subcritical. To obtain different group and phase speeds, Boussinesq type equations are adopted (e.g. Whitham, 1974, and Appendix B). The Boussinesq equations in an approximate way account for nonhydrostatic effects, a formal requirement still being that the horizontal length scale of the flow is large with respect to the total depth. These equations therefore are particularly suited to model near-critical flows over topography, which have relatively long lee waves. However, internal jumps which are short wave effects are represented as undular jumps.

Recently, near-critical stratified flow over an obstacle has been modelled using equations of Korteweg-de Vries type (Smyth, 1987; Melville and Helfrich, 1987). These equations apply to wave propagation in the upstream direction only. The Boussinesq equations presented herein allow for wave propagation in both upstream and downstream directions, and therefore may be preferable to describe flows in more general, unsteady conditions. Naghdi and Vongsarnpigoon (1986), Wu (1987) and others, applied the Boussinesq equations to flow of a single layer over an obstacle.

The observed damping of the lee waves with distance from the obstacle requires that friction at bottom, sidewalls and interface be taken into account. Since the flow is spatially varying, internal boundary layers at the bottom, sidewalls and interface could develop. However, the course pursued here is to ignore possible boundary layer formation and to examine whether simple friction formulations that apply to gradually varying flow are sufficient to explain the observed damping.

The same friction formulae are used in Section 2.5 to calculate the shear production of turbulent kinetic energy in the lee wave. However, allowance is

made for the effect of the increased turbulence production on the friction factors.

2.2 Equations

Neglecting exchange of mass between the layers and possible spanwise variations in the flow, the continuity equations for each layer become

$$\frac{\partial h_1}{\partial t} + \frac{\partial}{\partial x}(h_1 u_1) = 0 \quad (2.1)$$

$$\frac{\partial h_2}{\partial t} + \frac{\partial}{\partial x}(h_2 u_2) = 0 \quad (2.2)$$

where x is the horizontal co-ordinate, t is time, h_1 and h_2 are depths of surface and bottom layers, and u_1 and u_2 are mean (i.e. cross-section averaged) layer velocities, see figure 2.1.

Momentum equations that are derived for each layer, in principle describe external (or surface) as well as internal modes. In the case of small density differences the internal mode, which is of interest here, can be separated out by introducing the rigid-lid approximation. Adding the Boussinesq extension (e.g., Whitham, 1974; Wu 1987) to the hydrostatic long-wave equations (Schijf and Schönfeld, 1953; Appendix A) then yields

$$\begin{aligned} \frac{\partial u_1}{\partial t} - (S_1 - 1) \frac{u_1}{h_1} \frac{\partial h_1}{\partial t} + S_1 u_1 \frac{\partial u_1}{\partial x} + \frac{1}{\rho_1} \frac{\partial p_i}{\partial x} - g \frac{\partial h_1}{\partial x} - \frac{1}{3} h_1^2 \left(\frac{\partial^3 u_1}{\partial x^2 \partial t} + u_1 \frac{\partial^3 u_1}{\partial x^3} \right) + \\ + \frac{2}{B} \frac{\tau_1}{\rho} + \frac{1}{h_1} \frac{\tau_i}{\rho} = 0 \end{aligned} \quad (2.3)$$

$$\begin{aligned} \frac{\partial u_2}{\partial t} - (S_2 - 1) \frac{u_2}{h_2} \frac{\partial h_2}{\partial t} + S_2 u_2 \frac{\partial u_2}{\partial x} + \frac{1}{\rho_2} \frac{\partial p_i}{\partial x} + g \frac{\partial}{\partial x}(h_2 + h_b) + \\ - \frac{1}{3} h_2^2 \left(\frac{\partial^3 u_2}{\partial x^2 \partial t} + u_2 \frac{\partial^3 u_2}{\partial x^3} \right) + \frac{1}{2} h_2 u_2^2 \frac{d^3 h_b}{dx^3} + \frac{2}{B} \frac{\tau_2}{\rho} + \frac{1}{h_2} \frac{\tau_b}{\rho} - \frac{1}{h_2} \frac{\tau_i}{\rho} = 0 \end{aligned} \quad (2.4)$$

where g is the acceleration of gravity, ρ_1 and ρ_2 are the densities of the surface and bottom layers, ρ is a reference density, p_i the pressure at the interface, h_b the height of the bottom above a datum-line, B the (constant) width of the channel, and $\tau_1(\tau_2)$, τ_b and τ_i are shear stresses at the sidewalls of the surface (bottom) layer, the bottom and the interface, respectively. The rigid-lid approximation implies that $h_1 + h_2 + h_b = \text{constant}$. The profile coefficients S_n ($n = 1$ or 2), assumed constant, are given by $S_n = \langle u^2 \rangle_n / u_n^2$, where u is the horizontal velocity, and the caret brackets indicate averaging over a layer cross-section.

The third-order derivative terms in (2.3) and (2.4) represent non-hydrostatic contributions to the more common long-wave equations. Slightly more general expressions for these contributions could be introduced in (2.3) and (2.4), but some linearization consistent with the various approximations made has already been carried out here. The terms $u_n \partial^3 u_n / \partial x^3$, where $n = 1$ or 2 , are usually disregarded in the literature, which is justified if wave propagation in an otherwise quiescent fluid is considered. However, in the case under consideration the wave is stationary while the flow is near-critical. Under these conditions these terms are the leading ones. In the case of large-amplitude bottom topography additional terms have to be included in (2.3) and (2.4) (Peregrine, 1967; Wu, 1987).

The shear stresses in (2.3) and (2.4) are parameterized assuming quadratic friction laws,

$$\tau_n = \rho f_n u_n |u_n|$$

$$\tau_b = \rho f_b u_2' u_2' \quad (2.5)$$

$$\tau_i = \rho f_i \Delta u' \Delta u'$$

where $\Delta u = u_1 - u_2$ is the velocity difference between the layers, and f_n , f_b and f_i are constant friction factors. The coefficient f_n and f_b may be different. Tracy's (1965) experiments indicate $f_n \approx 0.85 f_b$ for a rectangular channel of aspect ratio 1/6.4. Knight, Demetriou and Hamed (1984) collected data on f_n/f_b as a function of the aspect ratio.

Strictly speaking, the derivation of the Boussinesq equations applies to an inviscid fluid. Although friction in real fluids will presumably alter the Boussinesq terms, this effect is neglected here since the influence of friction is taken as relatively weak.

A single momentum equation can be obtained by eliminating the interfacial pressure gradient between (2.3) and (2.4). The (small) density difference is retained in the gravity term only. The resulting equations for small-amplitude bottom topography become

$$h_1 + h_2 = h \quad (2.6)$$

$$u_1 h_1 + u_2 h_2 = q(t) \quad (2.7)$$

$$\begin{aligned} & \frac{\partial}{\partial t}(u_1 - u_2) - (S_1 - 1) \frac{u_1}{h_1} \frac{\partial h_1}{\partial t} + (S_2 - 1) \frac{u_2}{h_2} \frac{\partial h_2}{\partial t} + S_1 u_1 \frac{\partial u_1}{\partial x} - S_2 u_2 \frac{\partial u_2}{\partial x} + \\ & + g' \frac{\partial h_1}{\partial x} - \frac{1}{3} h_1^2 \left(\frac{\partial^3 u_1}{\partial x^2 \partial t} + u_1 \frac{\partial^3 u_1}{\partial x^3} \right) + \frac{1}{3} h_2^2 \left(\frac{\partial^3 u_2}{\partial x^2 \partial t} + u_2 \frac{\partial^3 u_2}{\partial x^3} \right) + \\ & - \frac{1}{2} h_2 u_2^2 \frac{d^3 h_b}{dx^3} + \frac{2}{B} \left(\frac{\tau_1}{\rho} - \frac{\tau_2}{\rho} \right) + \frac{h}{h_1 h_2} \frac{\tau_i}{\rho} - \frac{1}{h_2} \frac{\tau_b}{\rho} = 0 \end{aligned} \quad (2.8)$$

where h is the total water depth, q is the total flow rate per unit width, and g' , the reduced acceleration of gravity.

$$g' = \frac{\Delta\rho}{\rho_2} g \quad (2.9)$$

Here $\Delta\rho = \rho_2 - \rho_1$ is the density difference between the layers. In addition to (2.6) - (2.8) one of the continuity equations, (2.1) or (2.2), is needed to ensure conservation of mass for each layer.

2.3 Inviscid steady flow over small-amplitude bottom topography

In the experiments the flow was steady, the height and length of the obstacle on the bottom were relatively small (see Section 3.3), and the bottom of the flume was horizontal upstream as well as downstream of the obstacle. To simulate these experiments a small-amplitude steady state analysis is set up, in which the weak influence of friction over the small distance where the obstacle is present is neglected. The damping of the lee wave is calculated in Section 2.4.

Equations (2.6) - (2.8) are linearized by substituting

$$\begin{aligned} h_1 &= \bar{h}_1 - \eta \\ h_2 &= \bar{h}_2 + \eta - h_b \end{aligned} \quad (2.10)$$

$$u_1 = \bar{u}_1 + v_1$$

$$u_2 = \bar{u}_2 + v_2$$

where an overbar refers to the situation where the obstacle (the height of which is $h_b = h_b(x)$, see figure 2.2) is absent, η is the vertical displacement of the interface, and v_1 and v_2 are velocity disturbances. Both η and h_b are assumed to be small with respect to $\bar{h}_1$ and $\bar{h}_2$. Neglecting higher order contributions one thus obtains

$$\bar{h}_1 + \bar{h}_2 = h \quad (2.11)$$

$$\bar{u}_1 \bar{h}_1 = q_1, \quad \bar{u}_2 \bar{h}_2 = q_2$$

and

$$-\bar{u}_1 \eta + \bar{h}_1 v_1 \approx 0, \quad \bar{u}_2 (\eta - h_b) + \bar{h}_2 v_2 \approx 0$$

$$S_1 \bar{u}_1 \frac{dv_1}{dx} - S_2 \bar{u}_2 \frac{dv_2}{dx} - g' \frac{d\eta}{dx} - \frac{1}{3} \bar{u}_1 \bar{h}_1^2 \frac{d^3 v_1}{dx^3} + \frac{1}{3} \bar{u}_2 \bar{h}_2^2 \frac{d^3 v_2}{dx^3} +$$

$$- \frac{1}{2} \bar{h}_2 \bar{u}_2^2 \frac{d^3 h_b}{dx^3} \approx 0 \quad (2.12)$$

where q_1 and q_2 are the (constant) flow rates in surface and bottom layers ($q_1 + q_2 = q$). The undisturbed layer depths $\bar{h}_1$ and $\bar{h}_2$ and layer velocities $\bar{u}_1$ and $\bar{u}_2$ are also constant.

Eliminating v_1 and v_2 , a single equation with η as the dependent variable can be derived. Integrating this equation once with respect to x gives

$$bh^2 \frac{d^2 \eta}{dx^2} + (1-F^2)\eta = -F_2^2 h_b - b_2 h^2 \frac{d^2 h_b}{dx^2} + C \quad (2.13)$$

where C is a constant of integration. The internal Froude numbers F and F_2 are given by (the overbars are dropped)

$$F^2 = S_1 \frac{u_1^2}{g' h_1} + S_2 \frac{u_2^2}{g' h_2}, \quad F_2^2 = S_2 \frac{u_2^2}{g' h_2} \quad (2.14)$$

The coefficients b and b_2 derive from the Boussinesq terms in (2.8),

$$b = \frac{u_1^2 h_1 + u_2^2 h_2}{3g' h^2}, \quad b_2 = \frac{u_2^2 h_2}{6g' h^2} \quad (2.15)$$

The undisturbed layer depths h_1 and h_2 are defined so as to make $\eta = 0$ upstream of the obstacle. In this way a possible upstream influence, which would be independent of time (as in the experiments made), is avoided. Equation (2.13) with $h_b = \eta = 0$ then shows that the constant C is zero. The solutions of this equation are of an oscillatory nature, and hence a lee wave will develop, only if $F^2 < 1$, that is, if the flow is internally subcritical.

The two remaining terms on the RHS of (2.13) represent the forcing of the interface owing to the bottom topography. The assumption of large horizontal length scales that underlies the derivation of the Boussinesq equations, implies that the term containing the second derivative of h_b is small with respect to the one proportional to h_b . Therefore this term is neglected here.

It is assumed that the obstacle occupies an interval given by $-1 \leq x \leq 1$, see figure 2.2. The solution for $x > -1$ of the reduced version of (2.13) then can be written for subcritical flow as

$$\eta = \frac{F^2}{k b h} \left[\cos k \frac{x}{h} \int_{-1}^x h_b(\xi) \sin k \frac{\xi}{h} d\xi + \right. \\ \left. - \sin k \frac{x}{h} \int_{-1}^x h_b(\xi) \cos k \frac{\xi}{h} d\xi \right] \quad (2.16)$$

where ξ is a dummy variable and k is a dimensionless wavenumber given by

$$k = \left(\frac{1 - F^2}{b} \right)^{1/2} \quad (2.17)$$

The solution for the lee waves ($x > 1$) becomes

$$\eta = \frac{F_2^2}{k b h} \left[\cos k \frac{x}{h} \int_{-1}^1 h_b(\xi) \sin k \frac{\xi}{h} d\xi + \right. \\ \left. - \sin k \frac{x}{h} \int_{-1}^1 h_b(\xi) \cos k \frac{\xi}{h} d\xi \right] \quad (2.18)$$

which represents a simple harmonic wave.

In the experiments the obstacle had a cosine profile given by

$$h_b = \hat{h}_b \cos \left(\frac{\pi}{2} \frac{x}{1} \right) \quad (-1 \leq x \leq 1)$$

where $\hat{h}_b$ is the maximal height of the obstacle. Equation (2.18) then gives for the lee wave

$$\eta = \frac{\pi}{b} \frac{F_2^2}{k^2} \phi \left(k \frac{1}{h} \right) \hat{h}_b \sin k \frac{x}{h} \quad (2.19)$$

where the function ϕ is given by

$$\phi \left(k \frac{1}{h} \right) = \frac{k \frac{1}{h} \cos k \frac{1}{h}}{\left(k \frac{1}{h} \right)^2 - \frac{\pi^2}{4}} \quad (2.20)$$

In near-critical flow ($F^2 \approx 1$) the wave number k becomes small and, provided $F_2 > 0$, the lee wave amplitude becomes large. Obviously, the linear analysis must break down in this near-resonance situation.

Figure 2.3 shows the effect of the length of the obstacle as represented by the function ϕ on the lee-wave amplitude. It is note-worthy that the same form of the function ϕ is found for each vertical wave mode and arbitrary wavelength in

the case of linear density stratification and uniform velocity considered by Miles and Huppert (1969).

2.4 Damping of stationary lee wave by friction

In the experiments the layer velocities were such that $u_1 > u_2 > 0$. A wave disturbance according to (2.10), in which $h_b = 0$, is now introduced in (2.5)-(2.8) for this case. Linearizing gives for the undisturbed flow (2.11) and

$$\begin{aligned} S_1 \bar{u}_1 \frac{d\bar{u}_1}{dx} - S_2 \bar{u}_2 \frac{d\bar{u}_2}{dx} + g' \frac{d\bar{h}_1}{dx} + \frac{2}{B} (f_1 \bar{u}_1^2 - f_2 \bar{u}_2^2) + \\ + f_i \frac{h}{\bar{h}_1 \bar{h}_2} \Delta \bar{u}^2 - f_b \frac{\bar{u}_2^2}{\bar{h}_2} \approx 0 \end{aligned} \quad (2.21)$$

The equations for the wave disturbance become, see (2.12) for comparison,

$$- \bar{u}_1 \eta + \bar{h}_1 v_1 \approx 0, \quad \bar{u}_2 \eta + \bar{h}_2 v_2 \approx 0$$

$$\begin{aligned} S_1 \bar{u}_1 \frac{dv_1}{dx} - S_2 \bar{u}_2 \frac{dv_2}{dx} - g' \frac{d\eta}{dx} - \frac{1}{3} \bar{u}_1 \bar{h}_1^2 \frac{d^3 v_1}{dx^3} + \frac{1}{3} \bar{u}_2 \bar{h}_2^2 \frac{d^3 v_2}{dx^3} + \\ + \frac{4}{B} (f_1 \bar{u}_1 v_1 - f_2 \bar{u}_2 v_2) + f_i \frac{h (\bar{h}_2 - \bar{h}_1)}{(\bar{h}_1 \bar{h}_2)^2} \Delta \bar{u}^2 \eta + \\ + 2 f_i \frac{h}{\bar{h}_1 \bar{h}_2} \Delta \bar{u} (v_1 - v_2) + f_b \frac{\bar{u}_2^2}{\bar{h}_2^2} \eta - 2 f_b \frac{\bar{u}_2}{\bar{h}_2} v_2 \approx 0 \end{aligned} \quad (2.22)$$

Several terms containing derivatives of the undisturbed-flow variables have been neglected in (2.21) and (2.22). These derivatives are small because of the assumption of weak friction. More precisely, the wavelength should be small

with respect to a horizontal length scale of the flow (over a horizontal bottom) for the neglects to be justified. As a further result, the undisturbed flow variables in (2.22) may then, in first approximation, be treated as constant.

Equations (2.22) are solved by substituting the expression for a damped harmonic wave,

$$\eta \sim \text{Re} \left\{ \exp \left[-(m+ik) \frac{x}{h} \right] \right\} \quad (2.23)$$

where m is a dimensionless damping rate (h/m is an e-folding length of the wave amplitude). After some rearrangement and excluding a solution with $k = 0$ and $m < 0$ (which would represent a frictional upstream influence), the solution can be written as

$$8m^3 + 2 \frac{1 - F^2}{b} m - \frac{r}{b} = 0 \quad (2.24)$$

$$k^2 = 3m^2 + \frac{1 - F^2}{b} \quad (2.25)$$

where (the overbars are dropped)

$$\begin{aligned} r = 3f_b \frac{h}{h_2} \frac{u_2^2}{g'_1 h_2} + 4 \frac{h}{B} \left(f_1 \frac{u_1^2}{g'_1 h_1} + f_2 \frac{u_2^2}{g'_1 h_2} \right) + \\ + f_i \frac{h^2}{h_1 h_2} \Delta u \frac{3(u_1 h_2 + u_2 h_1) - u_1 h_1 - u_2 h_2}{g'_1 h_1 h_2} \end{aligned} \quad (2.26)$$

The solution of (2.24) and (2.25) is shown in figure 2.4. As expected the wavenumber k decreases as the Froude number increases. Friction augments the wavenumber in near-critical conditions so that k remains finite when the flow is critical. This result is doubtful, however, since the linear analysis becomes inaccurate as the critical flow-condition is approached. If $(1-F^2)/b$ is not small, an approximate solution of (2.24) and (2.25) gives (2.17) for the wavenumber in the case without friction, and

$$m \approx \frac{1}{2} \frac{r}{1-F^2} \quad (2.27)$$

Note that in this approximation m no longer depends on the parameter b representing the Boussinesq terms. The condition for this solution to hold good is

$$\frac{r^2 b}{(1-F^2)^3} \ll 1 \quad (2.28)$$

The wave damping is seen to be largest under near-critical conditions, but it decreases markedly for larger values of the parameter $(1-F^2)/b$. This behaviour is due to the fact that, as the Froude number increases, the wavenumber and hence the group velocity decrease. A smaller group velocity results in a smaller downstream transport of wave energy so that wave dissipation caused by friction effects, reduces the lee-wave amplitude more rapidly.

Under certain circumstances the parameter $(1-F^2)/b$ is sensitive to the relative layer depths. This is the case particularly for the present experiments, where equal flow rates q_1 and q_2 were prescribed. Figure 2.5 shows the sensitivity of $(1-F^2)/b$ to h_2/h for various values of a Froude number F_0 defined by ($q_1=q_2=q_0$, it is assumed here that $S_1 = S_2 = 1$)

$$F_0^2 = \frac{q_0^2}{g' h^3} \quad (2.29)$$

In these conditions the damping rate, and to a lesser extent the wavenumber, are also sensitive to factors influencing h_2/h , such as the shape of the bottom topography, the flow rates and a possible (weak) unsteadiness of the flow.

2.5 Estimate of increase in turbulence production caused by the lee waves

The velocity distribution in the lee-wave region differs from that in the absence of the obstacle. The horizontal velocity in the surface layer increases over the crests and decreases over the troughs. The reverse holds for the bottom layer. The different velocity distribution affects the production of turbulent kinetic energy (TKE), and as a result, turbulence parameters such as the dissipation rate, intensities and shear stresses will also be affected. It is therefore meaningful to estimate the changes in production of TKE caused by the lee wave.

Multiplying the equations of motion (2.3) and (2.4) by $\rho_1 B h_1 u_1$ and $\rho_2 B h_2 u_2$ respectively, yields transport equations for the hydrodynamical energy of the mean flow (the definition of the profile coefficients S_n becomes somewhat different). The loss terms, owing to friction, in these equations represent the turbulence production in each layer. Substituting from (2.5) gives the turbulence productions P_n , where $n = 1$ or 2 , as

$$P_1 = \rho_1 (2h_1 f_1 u_1^3 + B f_i u_1 \Delta u^2) \quad (2.30)$$

$$P_2 = \rho_2 (2h_2 f_2 u_2^3 + B f_b u_2^3 - B f_i u_2 \Delta u^2) \quad (2.31)$$

where $u_1 > u_2 > 0$.

The interfacial friction terms in the expressions are of secondary importance for the experimental conditions. In view of the approximate nature of the analysis to follow, and in order to avoid unnecessary complications, these terms are disregarded.

In a flow that is in local equilibrium the production of TKE (approximately) equals the dissipation rate. Local-equilibrium conditions require a horizontal length scale of the flow under consideration that is at least ten or twenty times the layer depth. The observed lee waves were shorter than this. However, it can be shown, by averaging the transport equation for TKE (e.g., Launder and Spalding, 1972) over the wavelength of a periodic wave, that the averaged dissipation equals the averaged production,

$$\rho_n \langle A_n^f \epsilon \, dA \rangle = \langle P_n \rangle \quad (2.32)$$

where A_n is the area of cross-section of the n^{th} layer, ϵ is the rate of dissipation per unit mass, and the caret brackets denote averaging over the wavelength. This approach implies that the damping of the wave over one wavelength is neglected, which is justified provided (2.28) holds. This condition was satisfied in Tests 1-3 of the present experiments.

The dissipation rate is usually modelled as (e.g. Launder and Spalding, 1972)

$$\epsilon \sim \frac{k^{3/2}}{l_e} \quad (2.33)$$

where k is the TKE, and l_e is a length scale of the energy containing eddies. Equations (2.30) - (2.33) in principle constitute a relationship between mean-flow variables and TKE. In shear flow a relationship with the shear stresses can be assumed (e.g., Bradshaw, 1978),

$$\frac{\tau}{\rho} \sim k \quad (2.34)$$

To examine the increase in turbulence production caused by the lee wave consider the ratio

$$\frac{\langle P_n \rangle}{P_{no}} = P_n \quad (2.35)$$

Here the subscript o refers to the flow in the absence of the lee wave. Assuming that in first approximation the length scale l_e is not affected by the lee wave, equations (2.32), (2.33) and (2.35) indicate that

$$\langle A_n \int k^{3/2} dA \rangle \cong P_n (A_n \int k^{3/2} dA)_o \quad (2.36)$$

If the distribution of the TKE across the layers were not influenced by the lee wave, equations (2.34), (2.36) and (2.5) defining the friction factors f would yield

$$\frac{f_1}{f_{10}} \cong P_1^{2/3} \quad (2.37)$$

$$\frac{f_2}{f_{20}} \cong \frac{f_b}{f_{bo}} \cong P_2^{2/3} \quad (2.38)$$

Equations (2.30), (2.31), (2.35), (2.37) and (2.38) then would give

$$p_1 \cong \left[\frac{\langle 2h_1 f_{1o} u_1^3 \rangle}{(2h_1 f_{1o} u_1^3)_o} \right]^3 \quad (2.39)$$

$$p_2 \cong \left[\frac{\langle 2h_2 f_{2o} u_2^3 + B f_{bo} u_2^3 \rangle}{(2h_2 f_{2o} u_2^3 + B f_{bo} u_2^3)_o} \right]^3 \quad (2.40)$$

However, the distribution of TKE is likely to be altered by the development of internal boundary layers at bottom and sidewalls in the rapidly varying flow observed. Therefore (2.39) and (2.40) are only indicative of the increase in turbulence production induced by the lee wave.

Evaluating (2.39) and (2.40) for a sinusoidal wave ($\eta = \hat{\eta} \sin \kappa x$) gives in the case where, as in the experiments, the layer flow rates are equal ($q_1 = q_2$).

$$p_1 \cong \left[1 - \left(\frac{\hat{\eta}}{h_1} \right)^2 \right]^{-9/2} \quad (2.41)$$

$$p_2 \cong \left[1 + \frac{1}{2} \frac{f_b B - 4f_2 h_2}{f_b B + 2f_2 h_2} \left(\frac{\hat{\eta}}{h_2} \right)^2 \right]^3 \left[1 - \left(\frac{\hat{\eta}}{h_2} \right)^2 \right]^{-15/2} \quad (2.42)$$

where the subscript o has been dropped, and h_1 and h_2 are undisturbed layer depths.

The numerical values: $h_1 = 0.16$ m, $h_2 = 0.24$ m, $B = 1$ m, $f_2/f_b = 0.85$, $\hat{\eta} = 0.03$ m (0.04 m), which approximately coincide with the experimental values, give $p_1 \cong 1.17$ (1.34) and $p_2 \cong 1.13$ (1.24). This example and the expressions for p_1 and p_2 indicate that the overall (that is, cross-sectionally averaged) increase in turbulence production can be nonnegligible for moderate amplitudes and quite substantial for larger ones. The lee-wave effect on the turbulence could be even more pronounced in the boundary layers mentioned.

3 EXPERIMENTAL METHODS

3.1 Introduction

The experiments are designed to study the generation and dissipation of near-critical internal lee waves by the flow of a turbulent two layer fluid over a topographic feature. Most previous studies of stratified flow over topography have been performed in towing tanks, in which the flow in most cases is laminar, and have concentrated on the upstream effects generated as the flow becomes critical. The working section in the present facility is 64 m long and 1 m wide. Layer Reynolds numbers of 20,000 are achieved which allow the study of background turbulence on the decay of internal waves. In addition the facility allows a fixed obstacle to be used so that the flow can achieve a steady state as critical conditions are approached.

In the following description the lower layer was always deeper than the upper layer and the height of the topography was constant at 1/10 of the total depth. Small amplitude topography was selected so that the internal lee wave phenomena could be studied. This is because large scale topography leads to an internal hydraulic jump, Lawrence (1985), which is not of primary interest in the present study. In addition a small obstacle reduces the effects of upstream influence which increase with obstacle height as critical conditions are approached (Baines 1984). The theoretical developments in Section 2.3 assume the upstream water level to be given and therefore neglect the effects of upstream influence.

3.2 Experimental apparatus

The experiments were performed in the tidal flume facility of Delft Hydraulics. A 64 m section of the 130 m tidal flume was adapted for two-layer steady state experiments. Details of the inlet and outlet design, and control systems are described by Uittenbogaard (1987).

The experimental facility is illustrated in Fig.3.1. A steady, recirculating, two-layer flow of salt water with a specified density difference is shown. Internal waves are generated by the flow of this two layer system, from left to

right, over a fixed obstacle. A density difference of about 25 kg m^{-3} was selected for all test conditions.

The computerised control system of the tidal flume monitors the background conductivity, temperature and flow rate of each layer. This is achieved by measuring the flow in the recirculation system (Fig. 3.1) with an A-D convertor sampling at a frequency of 3 Hz. This control system operates the brine injection system and thereby maintains a constant density difference. To compensate for the volume of brine injected an equal volume of fluid is removed. The average density of both layers increased during the experiments due to the recirculation system. The experimental facility allowed the steady flow of a two-layer system with a constant density difference to be maintained for more than 10 hours. The specified density difference is maintained to within $\pm 0.2 \text{ kg m}^{-3}$ and the flow rates to within 0.17 l s^{-1} .

In addition the background flow conditions were monitored (at the same location) by sampling the flow rate, conductivity, and temperature of both the upper and lower layers with an A/D convertor at a rate of 5 Hz. These data were stored on an Olivetti(M24SP) PC and transferred to the Vax 750 for further analysis. In order to check the background flow conditions the mean layer densities and flow rates were computed for each test.

The design of the splitter plate is discussed in Uittenbogaard (1987). A series of filters at the inflow section minimises the turbulence levels at the inlet. The splitter plate was hinged to allow variations in the height of the interface. In its equilibrium position the height of the lower layer is 0.30 m. Through elastic bending of the splitter plate the initial depth of the lower layer was adjusted to 0.20 m. An upper layer depth of 0.20 m was also prescribed, so that the total water depth was 0.40 m. The width of the flume was 1 m.

The splitter plate minimised mixing between the two layers yielding a two layer flow with an interfacial thickness upstream of the topography of the order 0.01m. Fig 4.4(a) is a typical density profile taken 1 m upstream of the topography. Recirculation takes place by the removal of both layers at the outlet section with another splitter plate, Fig. 3.1, which is also adjusted to give a lower layer depth of 0.20 m.

The vertical position of the splitter plate at the inlet was found to influence the flow conditions in the flume. At the inlet internal waves were generated whose wave characteristics were a function of the internal Froude number, as defined by Eqn. (2.14). As a Froude number of 1 was approached a hydraulic jump was seen at the inlet, as the Froude number decreased below 1 then a series of lee waves developed downstream. In all cases the waves were seen to decay within 20 m of the inlet and their effect on the experimental conditions was negligible except for their influence on the level of the interface. Their influence on the depth of the lower layer 1 m upstream of the topography is seen in Table 3.1. The conditions at the inlet were such that equal layer depths and velocities were prescribed. However, the layer depths were controlled by conditions at the outlet, which results in an adaptation near the inlet because of friction effects.

Velocity and turbulent intensity profiles were measured with four WSM micro-propeller velocity meters with a propeller diameter of 15 mm. These meters have a threshold velocity of 2.5 cms-1 and can be used to study velocity fluctuations of up to 15 Hz. Each profile consisted of 16 points in the vertical, which were located at 0.02 m intervals from 0.03 m to 0.33 m above the bottom. The emphasis of the measurements was on the lower layer dynamics.

Four WSM current meters were positioned across the centre of the flume and were separated at intervals of 0.10 m. The two meters located the farthest from the centre were 0.35 m from either wall see Fig 3.1. The 16 points in each vertical profile consisted of four consecutive sets of four simultaneous WSM current meter records.

Density profiles were measured in the centre line of the flume using a Bezo density profiler. This profiler recorded conductivity as a function of vertical position (between 0.5 cm and 39.5 cm above the bottom) as it moved up and down at a rate of 1cm/s. The temperatures of the upper and lower layers were also measured and used for the conversion of the conductivity profiles to density profiles. Visual observations of the stationarity of the test conditions were made by photographing shadowgraph images of the internal waves during a test. The waterlevels near the inlet and outlet were recorded.

The data collection system consisted of a ten channel A/D convertor which sampled at a frequency of 50 Hz. The analogue data were filtered with a 4th order 30 Hz low pass filter. A sampling period of 320 seconds was selected. The resolution of the convertor is 2048 bits / 10 volts. The channels were sampled cyclically and the data were collected in 4 blocks each containing 80 seconds of data. The convertor operates at one frequency for all the channels. The tidal flume is equipped with its own Vax-730. This system is used to store the sampled signals on disk (121 MB). The data were transferred to the Vax-750 for processing.

3.3 Experimental procedure

The objectives of the experimental work in the flume were to measure both the decay of internal lee waves and the associated increase in turbulence levels with distance along the flume in the presence of a turbulent background fluid. The internal lee waves were generated by installing a topographic feature with a cosine shape of amplitude, h_b , 0.04 m and length 0.20 m on the bottom of the flume, so that $l = 0.10$ m (section 2.3). The topography was located about 34.0 m downstream of the inlet, with its mid-point at 35.87 m as measured with respect to the tidal flume position markings. This ensured that the boundary layer at the obstacle was fully developed as the upstream length of the flume was many times larger than either the width or the depth of the flow.

Velocity and turbulence intensity profiles were measured both upstream and downstream of the topography. Interfacial displacements were measured from Bezo profiles. These profiles were either taken concurrently with the velocity data, but at a downstream location, in which case they consisted of 320 seconds of data or they were collected as supplementary data downstream of the topography using an Olivetti(M24SP) PC, while sampling for 120 seconds. The Bezo data were accurate to plus or minus 0.5 cm when locating the position of the interface.

Five experiments (Tests 1-3 and 5-6) were conducted which included velocity and turbulent intensity data, in addition five subsidiary experiments were conducted one of which (Test 4) included Bezo profiles while the others involved photographing the lee waves, Tests P1-P3. A summary of the conditions is given in Table 3.1.

Table 3.1 Summary of experimental conditions

Test number	$q_{1,2}$ ($\text{m}^3 \text{s}^{-1} \times 10^{-3}$)	h_2 (cm)	$\bar{\rho}_2$ (kg m^{-3})	$\bar{\rho}_1$	$\Delta\rho$ kgm^{-3}	F	Condition
1	24.5	24.0	1031.1	1006.2	24.9	0.90	Lee waves
2	24.0	24.0	1028.0	1003.1	24.9	0.89	Lee waves
3	23.0	23.8	1033.3	1008.3	25.0	0.84	Lee waves
4*	26.0	24.2	1029.8	1005.9	23.9	0.99	Lee waves
5	24.5	24.0	1037.2	1012.3	24.9	0.90	Flat bed
6	24.0	24.0	1037.5	1012.6	24.9	0.89	Flat bed
P1	20.0	22.0	1034.6	1009.6	25.0	0.68	Lee waves
P2	21.0	23.0	1034.6	1009.6	25.0	0.74	Lee waves
P3	25.0	24.5	1034.6	1009.6	25.0	0.92	Lee waves

* Brine injection problems

Test 4 was the condition closest to critical flow condition studied and as such it was not feasible to obtain a steady state. Tests 5 and 6 were performed in the absence of the topography for the same experimental conditions as Tests 1 and 2. These flat bed tests allowed the bed, side-wall and interfacial friction coefficients to be determined. They were also used to assess the increase in turbulence levels produced by the internal lee waves.

3.4 Data processing

Mean velocities were computed from

$$\bar{u} = \frac{1}{N} \sum_{i=1}^N u_i$$

where N = the number of samples
 u_i = the i^{th} velocity measurement
 $\bar{u}$ = the mean velocity

Turbulent intensities were computed from

$$u'^2 = \frac{1}{N-1} \sum_{i=1}^N (u_i - \bar{u})^2$$

where u'^2 = the variance and
 u' = the turbulent intensity

The statistical confidence limits can be computed according to standard procedures given in Bendat and Piersol(1971). In order to apply these procedures the assumption is made that the data are a sample from a stationary ergodic data set. Assuming a gaussian distribution the student t-test can be used to assign statistical confidence limits to the mean. In a similar manner the Chi-squared test can be used to assign statistical confidence limits to the variance (or u'^2) data. A 95 % level of significance was adopted in the present work. At the 95 % level of significance the mean velocity data is accurate to

$$(\bar{u} - 0.015 u') \leq \bar{u} < (\bar{u} + 0.015 u')$$

while the variance (or turbulent intensity squared) is accurate to

$$0.978 u'^2 \leq u'^2 < 1.022 u'^2$$

The current meter data were checked for wild points due to voltage fluctuations and instrument malfunction. In some cases the mean and variance were computed with less than 320 second records of data.

4 RESULTS

4.1 The determination of the friction coefficients

The friction coefficients were determined both from literature reviews and from the measurements made in the absence of the obstacle, that is, during Tests 5 and 6. The bottom friction coefficient was found to be 0.0032 ± 0.0002 . This value was obtained from the profile method,

$$u(z) = \frac{u^*}{k} \ln \frac{z}{z_0}$$

where u = the velocity of the fluid as a function of position (z) above the bed

u^* = the friction velocity

k = the Von Kármán constant (0.4)

z_0 = the roughness length

on the assumption that a logarithmic velocity profile applies to the lowest 20 % of the flow.

Open channel flow data can be used to establish a relationship between the bottom and wall friction coefficients as a function of the aspect ratio of each layer. The measurements of Tracy (1965) and the review of Knight, Demetriou and Hamed (1984) suggest that for a layer aspect ratio of 1:6 $fw \approx 0.85 fb$. This relationship was adopted in the present study.

Based on the reviews by Abraham et. al. (1979) and Arita and Jirka (1987) a value of $fi = 0.0013$ was chosen for the interfacial friction coefficient. The latter review provides information on salt wedge interfacial friction coefficients as a function of upper layer Reynolds number and Froude number. While the former review gives values of these coefficients as a function of Reynolds number but includes temperature wedge and exchange flows.

The interfacial slopes, measured in the absence of the obstacle in Tests 5 and 6 (see Fig 4.1 (a) and (b)), were used to verify the estimates of the friction coefficients. An equation for the friction coefficients was determined from equation (2.21) together with equation (2.11) to eliminate $\bar{u}_n$.

Substitution of the experimental values, together with the assumption that $S_1 = S_2 (= 1.04)$ and $f_1 = f_2 (= f_n)$ gave

$$640 f_b - 300 f_n - 270 f_i \approx 0.88$$

The above friction coefficients satisfy this equation.

4.2 The lee waves

Shadowgraphs, Fig 4.2, show a series of internal lee waves as a function of internal Froude number, and therefore implicitly as a function of lower layer depth upstream of the topography. The shadowgraphs cover the first 2.7 m of the flow downstream of the topography. Table 4.1 summarises the Froude number conditions associated with each shadowgraph and gives theoretical wavelengths and decay coefficients calculated using equations (2.19), (2.20) and (2.23) to (2.26) .

Sidewall effects seem to be the main cause of the apparent three-dimensional nature of the internal waves. The wavelengths, which are measured from the shadowgraphs, are also shown in Table 4.1. The wave amplitudes were not measured from the shadowgraphs because of the three-dimensional effects.

The flow was observed to separate from the crest of the obstacle, which is in line with the predictions of Huppert and Britter(1982, 1984). Flow visualization during Tests 1-3 using dye indicated that reattachment took place at 0.22 - 0.27 m downstream of the crest, slightly less than in the unstratified case. In these tests an effective length $2l$ of the obstacle including the separation zone was about 0.34 m. This value of $2l$ was used to calculate the theoretical lee-wave amplitudes from (2.19) and (2.20). Substituting the length of the obstacle proper ($2l = 0.20$ m) yielded predictions for Tests 1-3 that were too low by about 40 %.

Table 4.1 Predicted and observed wavelengths for the conditions of Fig. 4.2

Test number	Froude number	Wavelength (m)		m/h (m ⁻¹)	50% attenuation (m)
		Predicted	Measured		
P1	0.68	0.64	0.6	0.013	54.4
P2	0.74	0.73	0.7	0.017	39.6
3	0.84	1.01	1.0	0.035	19.9
2	0.89	1.23	1.2	0.053	13.0
1	0.90	1.41	1.4	0.066	10.6
P3	0.92	1.53	1.6	0.082	8.5
4	0.99	3.74	hydraulic jump	0.480	1.4

Quantitative information on the interfacial displacements downstream were obtained for Tests 1-4, as discussed in section 3.2. Fig. 4.3 shows a sequence of interfacial displacements as functions of distance downstream, where the distance scale is with respect to the centre of the obstacle. The discrete points are the measured displacements. Two theoretical curves are superimposed upon the data, in the first case the friction coefficients discussed in Section 4.1 are used in the calculations and in the second case the friction coefficients have been doubled. The theoretical curves were calculated in two steps. First of all the inviscid wave amplitude was calculated from equation 2.19 using the appropriate test conditions as input and then the exponential decay coefficients were determined using equations 2.23 - 2.26. The wave decay downstream from the topography was calculated by combining both solutions and adding them to the mean interfacial position observed 1 meter upstream of the topography as given in Table 3.1.

The calculated wavelengths, which are insensitive to the friction factors for Froude numbers below 0.92, compare favourably with the observed values. Apart from the first trough and crest the theoretical wave amplitudes for Tests 1-3 show reasonable agreement with the observed values. In fact the agreement is somewhat surprising, since formally the Boussinesq equations apply to long waves and the effective length of the obstacle was only 0.34 m. Also, the theoretical analysis was a linear one, while the observed amplitudes were substantial.

The calculations using the original friction factors apparently underestimate the observed wave damping, although the decrease in wave amplitude related to the first two or three measuring points in Tests 1 and 2 may be due to non-linear effects. In Test 1 the first wave amplitude is 4 cm in Test 2 it is 3.5 cm. In this case the wave amplitudes are not a small fraction of the layer depths and non-linear effects are likely to be important. Tests 1 and 2 show better agreement with the theoretical calculations if the friction coefficients are doubled. The larger friction coefficients in Test 3 have a small effect on the damping. It should be noted that the various friction factors obtained in Sections 2.5 and 4.1 do not support these larger values.

With an increase in Froude number the damping is seen to increase markedly, see Table 4.1 and Fig. 4.2. However as the Froude number is increased the subcritical flow changes to one which is usually termed critical. The interfacial displacement downstream is no longer a train of lee waves but there is an abrupt jump of the interface. The oscillatory lee wave train predicted by the theory for Test 4 is in disagreement with the observed response which is more like a hydraulic jump. This effectively limits the validity of the theory to Froude numbers up to about 0.9.

Above a Froude number of about 0.9 the flow became markedly unsteady. It was only possible to make comparisons between the observed and predicted wavelengths by photographing the flow after it had apparently adjusted to the flow conditions. Due to the transients the flow was continually adjusting itself so that after about an hour a hydraulic jump would appear. Given this limitation the wavelengths measured from the photographs appear to be accurately represented by the theory, even for a Froude number of 0.92.

4.3 Transient effects

For conditions where the Froude number was above 0.9 it was not experimentally feasible to achieve a steady state. Generally, as critical conditions are approached the obstacle acts to block the flow of the lower layer fluid upstream and the upstream depth of the lower layer increases. In the present experiments it is difficult to resolve whether this is the influence of the blocking action alone or also of conditions at the inlet. In either case as critical conditions are approached the upstream water level is influenced (Table 3.1).

From the point of view of blocking this increase in elevation is caused by a gravity wave of elevation which moves further and further upstream continually changing the lower layer depth upstream of the topography and therefore preventing the system from achieving a steady state.

Recent work by Melville and Helfrich (1987) and Smyth (1987) shows that the salient feature as resonance (critical conditions) is approached is that no steady state forms upstream of the obstacle. Solitons are periodically generated at the obstacle and sent upstream.

4.4 Mixing across the interface

Fig 4.4 shows density profiles for Test 1, 1 m upstream and 5 m downstream of the topography. These plots which are representative of the other conditions studied suggest that the internal lee waves have no effect on interfacial mixing. This could reflect the small velocity differences between the two layers and the large upstream interfacial Richardson numbers, of the order 100. Kranenburg (1987) discusses two-layer entrainment models and shows that mixing is a function of the velocity difference between the layers. In the present experiments this difference is small, even at the location of the troughs and crests the interfacial Richardson number is of the order 10. However, internal waves in the presence of a sheared two-layer flow may significantly affect mixing as observed by Geyer and Smith (1987) in a salt wedge estuary. This reflects the limitations of the two-layer inflow conditions. It should be noted that although there was no noticeable effect on interfacial mixing there was an effect on the production of TKE, so that mixing within each layer has increased.

4.5 The turbulent intensities

Fig. 4.5 shows the effect of the decaying lee waves on the velocity and the turbulence intensity distributions. The velocity profiles, both upstream of the topography and over the flat bed, increase approximately linearly outside a logarithmic bottom layer. Downstream of the topography, the lee waves distort the background flow. Under the troughs the lower layer fluid increases in velocity while the velocity is seen to decrease above the interface. The reverse is seen to happen over the crests. This effect is most noticeable in the first trough and crest.

Since the production of TKE is given by the product of the turbulent shear stress and the vertical gradient of horizontal velocity, an increase is expected under the troughs near the bed and across the interface near the crests. This effect is most noticeable near the bed where the local gradients are most strongly affected. If one compares the local increases in TKE near the bed then under the troughs there is an increase of 40 - 50 %. These effects are noticeable in Fig. 4.6, in which the vertical distributions of velocity and intensity, as functions of the upstream and downstream position, are shown.

As discussed in Section 2.5 the distribution of TKE is likely to be altered by the development of internal boundary layers at the bottom and side-walls. This is because of the rapid variations of the flow in the downstream direction and because of the three-dimensional effects which have been neglected in the theoretical developments. In the downstream direction the lee waves cause local acceleration of the fluid towards the troughs and deceleration towards the crests.

The layer averaged turbulent intensities are presented in Table 4.2. Although the local increases in turbulent intensity, for example near the bed, are larger under the troughs the depth averaged values are larger under the crests. Similarly the layer averaged values are smaller over the crests in the upper layer and larger over the troughs.

In order to make the assumption of local equilibrium the theoretical considerations of Section 2.5 are based on the production of TKE averaged over a wavelength. For the experimental conditions theory suggests that there should be an increase in the level of the cross-sectionally averaged turbulence production by a factor of 1.17-1.34 for the lower layer and 1.13-1.24 for the upper layer. Equation (2.36) indicates that $u' \sim k^{1/2} \sim p^{1/3}$, therefore the numerical values imply that the cross-sectionally averaged turbulence intensity increases by 5 - 10% for the lower layer and 4-7% for the upper layer.

Estimates of the experimental increases in each layer were made by averaging the values in each layer over the lee wave train, and comparing them to the layer averaged values found both upstream and in the flat bed tests. The results are given in Table 4.2. and suggest that in the experiments the increases are larger, in the lower layer being of the order 10 - 20 %. In the upper layer they range between 2-10 %, however, only limited data were collected in the upper layer.

There are two possible sources for the increases in the turbulence levels. The first, as discussed above, is based on the distortion of the fluid due to the lee waves. In the second case, the flow separated from the crest of the topography, giving rise to a region of recirculating flow in the lee. Further downstream the flow reattaches but in the wake zone the turbulence intensities are locally larger than those in the undisturbed flow. Nakagawa and Nezu (1987)

observed that the region with larger intensities typically extends over a downstream distance that is ten to twenty times the obstacle height. The length of the separation zone was only slightly reduced by the proximity of the first trough of the lee wave. In order to distinguish between the two sources of TKE the experiments would have to be repeated with a homogenous fluid, thereby eliminating the influence of the lee waves.

If the turbulent intensity data are non-dimensionalised with an appropriate background friction velocity, a comparison with open channel flow data can be made. The lower layer data of the flat bed experiments then show strong similarities with open channel flow data, see for example Nezu and Rodi (1986). In particular there is a reduction in intensities as the interface is approached which compares with the reduction seen at the surface in open channel flow experiments. It may, however, be a function of the WSM measuring volume (15 mm) in which only the large eddies are resolved. In order to examine the damping of u' near the interface further measurements are necessary.

Table 4.2(a) Layer-averaged streamwise turbulent intensities

x,m	Position with respect to wave	Mean turbulent intensity, cm/s	
		bottom layer	surface layer
(1)	(2)	(3)	(4)
(a) Test 1			
-1.10	upstream	0.76	0.48
-0.10	upstream	0.72	0.50
0.35	trough	0.78	0.55
1.02	crest	0.84	0.49
1.41	between crest and trough	0.76	0.55
1.80	trough	0.84	0.49
(b) Test 2			
-1.10	upstream	0.70	0.46
-0.10	upstream	0.68	0.50
0.31	trough	0.83	0.51
0.92	crest	0.97	0.47
1.56	trough	0.71	0.51
2.04	crest	0.79	0.54
(c) Test 3			
-1.10	upstream	0.75	0.39
-0.10	upstream	0.75	0.40
0.39	trough	0.78	0.45
0.87	crest	0.99	0.42
(d) Test 5			
-1.10	flat bed	0.67	0.49
1.02	flat bed	0.73	0.51
8.90	flat bed	0.71	0.53
(e) Test 6			
-1.10	flat bed	0.69	0.45
1.02	flat bed	0.72	0.47
8.90	flat bed	0.76	0.48

Table 4.2(b) Wavelength averaged streamwise turbulent intensities

Test number	Layer	Mean turbulent intensity (cms^{-1})			% Increases in the lee wave train compared with values	
		Lee wave train	Upstream	Flat Bed	(a) upstream	(b) Flat bed
1	Lower	0.81	0.74	0.70	9	16
1	Upper	0.52	0.49	0.51	6	2
2	Lower	0.83	0.69	0.72	20	15
2	Upper	0.51	0.48	0.47	6	8
3	Lower	0.89	0.75	-	19	-
3	Upper	0.44	0.40	-	10	-

5. SUMMARY

A Boussinesq type theory of internal waves in two-layer flow has been developed. Two-layer internal lee wave experiments have been performed, the results of which compare favourably with the theoretical results provided an effective length of the obstacle including the separation zone is used in the calculations. In order to simulate both the Rotterdam Waterway topography and to avoid significant upstream effects a small topographic feature was used in the study, with a height of $1/10$ the total water depth. A short obstacle with length less than the water depth was selected in order to retain the non-hydrostatic lee wave phenomena.

The theory correctly predicts the wavelengths but appears to underestimate the wave damping suggesting that the friction coefficients may be larger than estimated, although non-linear effects may be partly responsible for the disagreement. Increased friction caused by bottom topography would be relevant to salt intrusion studies. Further experiments should be performed to measure the shear stresses in order to accurately determine the friction coefficients. There are two effects which may lead to an increase in the production of turbulent energy, the first is associated with the presence of the lee waves while the second is associated with flow separation over the crest of the topography. Further experiments are required to separate the influence of the latter effect.

Theoretical estimates of the production of TKE were made in section 2.5 which gave a lower limit to the intensities measured downstream from the obstacle. The lee wave effect alone, i.e. excluding possible wake effects, may increase the turbulent intensities by 5 - 10 % for Froude numbers from 0.84 - 0.90. The experimental results suggest that separation and lee waves may cause an increase of the order 10 - 20 %.

The lee wave effect could be even more pronounced in the boundary layers near the bed and the sidewalls. Fig. 4.6 shows that near the bed the lee waves can increase the turbulent intensities and therefore the TKE, which may influence sediment transport.

The wave damping increases with the Froude number until at critical conditions a hydraulic jump forms. Significant dissipation of energy of the mean flow is likely to be associated with this condition, Lawrence(1985). The flow becomes unsteady however, as critical conditions are approached so that it was not practical to make turbulent intensity measurements above a Froude number of 0.9. In addition Fig.4.3 (d) demonstrates the limitations of the linear theory as critical conditions are approached. The theoretical curve suggests that there should be a damped lee wave, while the observations show a hydraulic jump.

The internal lee waves were seen to have no effect on the interfacial mixing despite their effect on the turbulence levels. This may be attributed in part to the small velocity differences between the layers, and the large Richardson numbers. Although the increase in turbulent intensities within each layer should increase the mixing within the layer.

Applications of the present theory to RWW conditions will be made in a subsequent report. The present theoretical and experimental observations leads one to expect a substantial internal wave induced increase in the overall turbulence levels around tidal slack when the flow is near critical. In order to assess their importance for the RWW, further progress must be made in determining the extent of the internal wave fields in the RWW.

If the experiments are repeated it is recommended to measure the shear stresses in the lee waves, and therefore to perform the experiments in conjunction with experiments on the stratified shear layer which have been planned in a different context.

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Fig. 2.1 Definition sketch of two-layer flow

Fig. 2.2 Definition sketch of two-layer flow over an obstacle

		A ₄
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The function ϕ defined by equation 2.20

Dimensionless wave number and associated damping rate
of lee waves

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Fig. 2.4

A₄

Sensitivity of the parameter $(1-F^2)/b$ to variations
in the depth of the bottom layer

The experimental facility

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Fig. 3.1

A₄

Interfacial slopes measured in the absence of topography during (a) Test 5 and (b) Test 6

Shadowgraphs of internal lee waves as a function of internal Froude number

Observed and theoretical lee-wave trains for (a) Test 1 and (b) Test 2. [+ observations, — using experimental friction factors, ---- using doubled friction factors]

Observed and theoretical lee-wave trains for (c) Test 3 and (d) Test 4. [+ observations, — using experimental friction factors, ---- using doubled friction factors]

Density profiles for Test 1 (a) 1 meter upstream and (b) 5 meters downstream

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Test 1

Fig. 4.4

Vertical profiles upstream
of the topography

Position -1.10 m

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Test 1

Fig. 4.5

Vertical profiles upstream
of the topography

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Position -0.10 m

Test 1

Fig. 4.5

Vertical profiles at the first
wave trough

Position 0.35 m

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Test 1

Fig. 4.5

Vertical profiles at the first
wave crest

Position 1.02 m

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Test 1

Fig. 4.5

Vertical profiles between a wave
trough and crest

Position 1.41 m

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Test 1

Fig. 4.5

Vertical profiles at the second
wave trough

Position 1.80 m

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Test 1

Fig. 4.5

Vertical profiles upstream
of the topography

Position -1.10 m

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Test 2

Fig. 4.5

Vertical profiles at the topography

Position -0.10 m

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Test 2

Fig. 4.5

Vertical profiles at the first
wave trough

Position 0.31 m

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Test 2

Fig. 4.5

Vertical profiles at the first
wave crest

Position 0.92 m

DELFT HYDRAULICS

Test 2

Fig. 4.5

Vertical profiles at the second
wave trough

Position 1.56 m

DELFT HYDRAULICS

Test 2

Fig. 4.5

Vertical profiles at the second
wave crest

Position 2.04 m

DELFT HYDRAULICS

Test 2

Fig. 4.5

Vertical profiles upstream
of the topography

Position -1.10 m

DELFT HYDRAULICS

Test 3

Fig. 4.5

Vertical profiles at the topography

Position -0.10 m

DELFT HYDRAULICS

Test 3

Fig. 4.5

Vertical profiles at the first
wave trough

Position 0.39 m

DELFT HYDRAULICS

Test 3

Fig. 4.5

Vertical profiles at the first
wave crest

Position 0.87 m

DELFT HYDRAULICS

Test 3

Fig. 4.5

Vertical profiles over the flat bed

Position -1.10 m

DELFT HYDRAULICS

Test 5

Fig. 4.5

Vertical profiles over the flat bed

Position 1.02 m

DELFT HYDRAULICS

Test 5

Fig. 4.5

Vertical profiles over the flat bed

Position 8.90 m

DELFT HYDRAULICS

Test 5

Fig. 4.5

Vertical profiles over the flat bed

Position -1.10 m

DELFT HYDRAULICS

Test 6

Fig. 4.5

Vertical profiles over the flat bed

Position 1.02 m

DELFT HYDRAULICS

Test 6

Fig. 4.5

Vertical profiles over the flat bed

Position 8.90 m

DELFT HYDRAULICS

Test 6

Fig. 4.5

Fig. 4.6 Distribution of (i) velocity (cm/s) and (ii) turbulent intensity (cm/s) for Test 1

Fig. 4.6 Distribution of (i) velocity (cm/s) and (ii) turbulent intensity (cm/s) for Test 2

Fig. 4.6 Distribution of (i) velocity (cm/s) and (ii) turbulent intensity (cm/s) for Test 3

Fig. 4.6 Distribution of (i) velocity (cm/s) and (ii) turbulent intensity (cm/s) for Test 5

Fig. 4.6 Distribution of (i) velocity (cm/s) and (ii) turbulent intensity (cm/s) for Test 6

Appendix A - Heuristic derivation of the Boussinesq terms in equations
(2.3) and (2.4)

The Boussinesq terms in the equations of motion are derived from a method given in Section 13.11 of Whitham (1974). It consists of introducing such higher-order terms in the shallow water equations that the dispersion relation obtained is identical with an expansion of the exact dispersion relation for small wavenumbers. This method can be shown to be equivalent to a formal expansion technique using the ratio of water depth to wavelength squared as a small parameter. It is applied here in two steps to the flow of a single layer of inviscid, homogeneous liquid. First the propagation of surface waves in a flow of uniform depth is considered, and second steady flow over an undulating bed is analysed. Only linearized terms are examined and therefore the Boussinesq terms which result from these special cases, may be added to describe the more general case of unsteady flow over topography. The two-layer equations of Section 2.2 are found by applying the equation of motion for a single layer to both the bottom and surface layers separately, thereby accounting for the coupling between the layers by introducing normal and shear stresses at the interface. In doing so, it must be kept in mind that the surface layer, so to speak, hangs on the rigid lid introduced, and consequently the gravity term in (2.3) has a minus sign.

The dispersion relation for surface-wave propagation in a layer of uniform depth h and velocity u is (e.g., Lamb, 1932)

$$(\omega - \kappa u)^2 = g \kappa \tanh \kappa h$$

where ω is the wave frequency and κ is the wavenumber. An approximate form of this expression for long waves (small κh) is

$$(\omega - \kappa u)^2 \approx gh \kappa^2 \left[1 - \frac{1}{3} (\kappa h)^2 \right] \quad (A1)$$

Following Whitham (1974), an equation of motion is sought that yields the same dispersion relation. In attempting this the following linearized continuity and momentum equations are chosen

$$\frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} + h \frac{\partial v}{\partial x} = 0$$

(A2)

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + g \frac{\partial \eta}{\partial x} - C_1 \frac{\partial^3 v}{\partial x^2 \partial t} - C_2 \frac{\partial^3 v}{\partial x^3} = 0$$

where η is the surface elevation, v is the horizontal velocity disturbance, and C_1 and C_2 are coefficients to be determined. Substituting the expression for a single, travelling harmonic wave,

$$(\eta, v) = (\hat{\eta}, \hat{v}) \cos(\omega t - \kappa x)$$

where $\hat{a}$ denotes an amplitude, the dispersion relation becomes

$$\begin{vmatrix} (\omega - \kappa u) & \kappa h \\ g\kappa & (\omega - \kappa u + C_1 \kappa^2 \omega - C_2 \kappa^3) \end{vmatrix} = 0$$

or

$$(\omega - \kappa u)^2 + \kappa^2 (\omega - \kappa u) (C_1 \omega - C_2 \kappa) - gh \kappa^2 = 0 \quad (A3)$$

Apart from higher-order terms in κh , (A1) and (A3) should be identical. In the first place this gives $C_2 = C_1 u$.

Equation (A3) then becomes

$$(\omega - \kappa u)^2 = gh \kappa^2 / (1 + C_1 \kappa^2) \approx gh \kappa^2 (1 - C_1 \kappa^2)$$

Comparison with (A1) shows that

$$C_1 = \frac{1}{3} h^2 \quad (A5)$$

Both C_1 and C_2 are independent of ω and κ , therefore (A2) is a suitable choice, although equivalent forms exist.

Both C_1 and C_2 are independent of ω and κ , therefore (A2) is a suitable choice, although equivalent forms exist.

Next consider steady flow over an undulating bed. Lamb (1932, p. 409) gives the amplitude of the surface elevation as

$$\hat{\eta} = \frac{\hat{h}_b}{\cosh \kappa h - (g/\kappa u^2) \sinh \kappa h}$$

where bottom and surface elevations are given by $h_b = \hat{h}_b \cos \kappa x$ and $\eta = \hat{\eta} \cos \kappa x$ respectively. For long-wave bottom corrugations (κh small) this expression becomes

$$\hat{h}_b/\hat{\eta} \approx 1 - \frac{gh}{u^2} + \left(\frac{1}{2} - \frac{1}{6} \frac{gh}{u^2}\right) (\kappa h)^2 \quad (A6)$$

The equation of motion which also yields this relationship should contain the last term of (A2), since it does not vanish in a steady-state situation, and a term representing the undulating bottom. The form chosen is ($\partial/\partial t=0$)

$$u \frac{dv}{dx} + g \frac{d\eta}{dx} - \frac{1}{3} h^2 u \frac{d^3 v}{dx^3} + C_3 \frac{d^3 h_b}{dx^3} = 0 \quad (A7)$$

where C_3 is another coefficient to be determined. The continuity equation is (also see Section 2.4)

$$hv + u (\eta - h_b) = 0 \quad (A8)$$

Substituting

$$(h_b, \eta, v) = (\hat{h}_b, \hat{\eta}, \hat{v}) \cos \kappa x$$

Comparing (A9) with (A6) shows that

$$C_3 = \frac{1}{2} h u^2$$

which is independent of ω and κ , as required.

Summarizing, the linearized Boussinesq terms on the LHS of the equation of motion for a single layer are

$$- \frac{1}{3} h^2 \left(\frac{\partial^3 u}{\partial x^2 \partial t} + u \frac{\partial^3 u}{\partial x^3} \right) + \frac{1}{2} h u^2 \frac{d^3 h_b}{dx^3}$$

Appendix B - Dispersion relation of two-layer Boussinesq equations

Substituting from (2.10), linearizing (2.2) and (2.5)-(2.8) yields (2.11) for the undisturbed flow, and, assuming the profile coefficients S_1 and S_2 to be unity, for the wave disturbance

$$-\bar{u}_1 \eta + \bar{u}_2 \eta + \bar{h}_1 v_1 + \bar{h}_2 v_2 \approx 0$$

$$\frac{\partial \eta}{\partial t} + \bar{u}_2 \frac{\partial \eta}{\partial x} + \bar{h}_2 \frac{\partial v_2}{\partial x} \approx 0 \quad (B1)$$

$$\frac{\partial}{\partial t} (v_1 - v_2) + \bar{u}_1 \frac{\partial v_1}{\partial x} - \bar{u}_2 \frac{\partial v_2}{\partial x} - g' \frac{\partial \eta}{\partial x} - \frac{1}{3} \bar{h}_1^2 \left(\frac{\partial^3 v_1}{\partial x^2 \partial t} + \bar{u}_1 \frac{\partial^3 v_1}{\partial x^3} \right) +$$

$$+ \frac{1}{3} \bar{h}_2^2 \left(\frac{\partial^3 v_2}{\partial x^2 \partial t} + \bar{u}_2 \frac{\partial^3 v_2}{\partial x^3} \right) + \frac{4}{B} (f_1 \bar{u}_1 v_1 - f_2 \bar{u}_2 v_2) +$$

$$+ f_i \frac{h}{\bar{h}_1 \bar{h}_2} \left[\Delta \bar{u}^2 \frac{\bar{h}_2 - \bar{h}_1}{\bar{h}_1 \bar{h}_2} \eta + 2 \Delta \bar{u} (v_1 - v_2) \right] + f_b \frac{\bar{u}_2}{\bar{h}_2} \left(\frac{\bar{u}_2}{\bar{h}_2} \eta - 2 v_2 \right) \approx 0$$

As in Section 2.3 it has been assumed here that $u_1 \geq u_2 \geq 0$, and that the friction is weak so that in first approximation the undisturbed flow variables in (B1) may be treated as constant.

The dispersion relation is obtained by substituting the expression for a single, travelling harmonic wave,

$$(\eta, v_1, v_2) = (\hat{\eta}, \hat{v}_1, \hat{v}_2) \operatorname{Re} \{ \exp [i(\omega t - \kappa x)] \} \quad (B2)$$

where ω is the (real) angular frequency, κ is the complex wavenumber, and a caret denotes an amplitude. The solution presented in section 2.3 is a special case of (B2) with $\omega = 0$. Equation (2.13) gives

$$\kappa = \frac{k - im}{h} \quad (B3)$$

On substitution from (B2), equation (B1) yield three homogeneous, linear equations for the unknown $\hat{\eta}$, $\hat{v}_1$ and $\hat{v}_2$. Setting the coefficient determinant equal to zero gives (the overbars are dropped)

$$\frac{\beta_1}{h_1} \left(\frac{\omega}{\kappa} - u_1 \right)^2 + \frac{\beta_2}{h_2} \left(\frac{\omega}{\kappa} - u_2 \right)^2 - g' - i \frac{R}{\kappa} = 0 \quad (B4)$$

where

$$\beta_n = 1 + \frac{1}{3} \kappa^2 h_n^2 \quad (n = 1 \text{ or } 2) \quad (B5)$$

and

$$R = \frac{4}{B} \left[f_1 \frac{u_1}{h_1} \left(\frac{\omega}{\kappa} - u_1 \right) + f_2 \frac{u_2}{h_2} \left(\frac{\omega}{\kappa} - u_2 \right) \right] + f_i \frac{h}{h_1 h_2} \left\{ 2 \Delta u \left[\frac{1}{h_1} \left(\frac{\omega}{\kappa} - u_1 \right) + \frac{1}{h_2} \left(\frac{\omega}{\kappa} - u_2 \right) \right] - \Delta u^2 \frac{h_2 - h_1}{h_1 h_2} \right\} + f_b \frac{u_2}{h_2} \left(2 \frac{\omega}{\kappa} - 3u_2 \right) \quad (B6)$$

Note that κ becomes complex because of the friction term iR/κ in (B4). The coefficients β_n derive from the Boussinesq terms.

If friction is ignored ($R=0$), κ will be real ($\kappa = k/h$) and (B4) will give

$$\frac{\omega}{\kappa} = \frac{\beta_1 u_1 h_2 + \beta_2 u_2 h_1 \pm \{h_1 h_2 [(\beta_1 h_2 + \beta_2 h_1) g' - \beta_1 \beta_2 \Delta u^2]\}^{1/2}}{\beta_1 h_2 + \beta_2 h_1} \equiv c_0 \quad (B7)$$

where c_0 is the wave speed in the absence of friction. It may be verified that (B7) is an approximation for small κ (the error is $O(\kappa^4 h^4)$) of a result that can be derived for arbitrary values of κ using potential theory (e.g., Lamb, 1932). Figure B1 illustrates this for the special case where $h_1 = h_2 = \frac{1}{2}h$ and $u_1 = u_2 = 0$.

Equations (B7) and (B5) show that due to the Boussinesq terms the wave speed decreases as the wave number increases. As a result the group velocity ($= d\omega/d\kappa$) is less than the wave speed which, as explained in section 2.1 is a necessary condition to obtain a stationary lee wave.

A first approximation of the damping rate m of a wave can be derived by substituting $\omega/\kappa \approx c_0$ in (B6) and recalculating ω/κ from (B4). Substituting from (B3) and approximating assuming $m \ll k$, this gives

$$m \approx \frac{h R_0}{2 c_0} \left\{ \frac{h_1 h_2}{(\beta_1 h_2 + \beta_2 h_1) g' - \beta_1 \beta_2 \Delta u^2} \right\}^{1/2} \quad (B8)$$

where

$$R_0 = (R)_{\omega/\kappa = c_0}$$

The damping rate also depends on the wave number $\kappa = k/h$. For example, a long wave travelling in the positive x-direction in a two-layer fluid with equal layer depths ($h_1 = h_2 = \frac{1}{2}h$) and layer velocities ($u_1 = u_2 > 0$) damps more rapidly than a shorter one. Furthermore, under certain circumstances R_0 may become negative for waves travelling in the positive x-direction, which points to unstable behaviour of the system. However, it is beyond the scope of this investigation to analyze (B8) in greater detail.

Wave speed c_0 as a function of wavenumber in a two layer fluid at rest. The interface is at mid-depth. Solid line: exact result: dashed line: equation (B7)