One-dimensional models for mountain-river morphology

A. Sieben
ONE-DIMENSIONAL MODELS
FOR MOUNTAIN-RIVER MORPHOLOGY.

by

A. Sieben
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Abstract.

In this report, some classical and new simplifications in mathematical and numerical models for river morphology are compared for conditions representing rivers in mountainous areas (high values of Froude numbers and relatively large values of sediment transport rates).

Options for simplification are summarized based on time scale of hydrographs and length scales of river geometry. This results in concepts based on quasi-steady and/or quasi-uniform flow assumptions. Additionally, the behaviour of frictionless, critical flow with a mobile bed is considered.

The non-linear interaction between changing flow and morphology is investigated for different values of the Froude number. Neglecting this interaction in numerical solution procedures appears to affect the solution. Also, mass and momentum contributions of sediment in transport on the mixture of water and sediment are analyzed.

It is shown that errors due to simplification in numerical models for river morphology vary with the different up- or downstream propagating waves that are part of the solution.

Conclusions further refer to the importance of wave non-uniformity (wave length, dominance of friction), Froude number and bed mobility on the error made when using simplified modelling concepts. Application of simplified modelling concepts based on subcritical low-land rivers in the modelling of transcritical and supercritical flows can result in significant errors.

Acknowledgements

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Chapter one.

Introduction.

1.1. General.

In mountain rivers, hydraulic and morphological behaviour deviates from that in low-land rivers. Effects of different time and length scales, input conditions, non-uniform widths, steep slopes and sediment grain-size distributions should be taken into account. Despite the multiple time and length scales of processes that can be observed in mountain rivers (Crosato, 1995), one-dimensional models can be applied to a certain extend. The complex conditions, however, emphasize the importance of schematization.

There is a major gap between complex conditions in proto-type rivers and one-dimensional representations in mathematical models. Hence, with respect to the non-uniform, unsteady features of a mountain river, schematization fundamentally affects the solution that is obtained. To perform a proper schematization of hydrograph, sediment characteristics and river geometry, and subsequent interpretation of the solution, one should know the model responses that are introduced due to simplifications.

Although research on river morphology includes impressive progress on formulation, simplification and analysis of mathematical models, applications were mainly focussed on low-land rivers. Because hydraulic and morphological processes in mountain rivers can be different from that in common low-land rivers, a number of classic analyses are repeated in this report, for conditions resembling mountain rivers. The effect of grain-size distributions, however, is not analyzed in this report.

Despite enormous expansion of computational capacity during the last decades, reduction of computational effort by using simplified models is attractive. In many cases, major events determine the morphology of mountain rivers. Then, flooding hazards are controlled by single, extreme events. This complicates any long-term prediction of morphology. Hence, to indicate future flooding probabilities for different areas, many extreme-event scenarios must be used to construct a representative distribution of flooding probability (FRIMAR, 1995).

The objective of this study is to analyse the behaviour of flood-wave solutions of mathematical models of rivers with mobile beds. Length and time scales of processes involved are indicated and characterised by key parameters. These
parameters enable the classification of model-responses to different river-configurations and hydrographs.

Consequences of different simplified modelling concepts are reviewed in an analytical way. This, however, restricts all results to convenient, hypothetical uniform rivers, and stresses the need for at least numerical verification.

In this report, analyses are extended with the help of packages for symbolic algebraic manipulations (Wolfram, 1991). Analytical expressions are analyzed with numerical procedures (e.g. Press et al. 1992)

1.2. State of the art: fixed-bed flood routing.

In this section, a very limited review is given of the numerous literature on mathematical models for flood-wave routing. A clear and elegant mathematical analysis of flood waves at fixed beds can be found in publications such as Vreugdenhil (1972); Grijzen and Vreugdenhil (1976); Ponce and Simons (1977) and Ponce et al. (1978).

The contributions of inertia and friction to harmonic solutions of the linearized mathematical model have been compared for different wave periods and flow regimes. This enabled reviewing wave types that can be present in a river. These types range from very short waves with negligible friction losses (surges) to extremely long waves dominated by friction.

As reviewed in Jansen et al. (1979), different simplifications can be justified for different wave types, :

- simple-wave model for inertia-dominated waves without friction
- diffusion analogy model for quasi-steady waves with non-uniform flow
- kinematic wave model for quasi-steady waves with quasi-uniform flow (Henderson, 1963)

The wave models mentioned represent solutions with limit behaviour. For intermediate waves as for example tidal waves, the complete (dynamic) model must be used. These results have been used extensively to apply simplified models in flood routing models for rivers and overland flow (e.g. Grijzen and Ogink, 1973; Jansen et al., 1979; Vreugdenhil, 1994).
1.3. State of the art: morphological models.

With the introduction of a mobile boundary at the level of the river bed, the model changes significantly with respect to fixed bed models. The increased complexity has challenged an ongoing line of research for simplifications in mathematical models for morphological changes in rivers.

A major step for simplifying and understanding morphological models has been the introduction of the quasi-steady flow assumption (De Vries, 1966). This implies an "instantaneous" adaption of flow variables with respect to "slow" changes in bed level. Assuming steady flow with respect to morphology enables adjustment of numerical models to large morphological time steps. Therefore, this approach has been applied widely since.

The quasi-steady flow assumption enables deriving morphological models with a mixed hyperbolic / parabolic character (Vreugdenhil and De Vries, 1973). The resulting equations are reviewed in Section 2.4. Vreugdenhil (1982) illustrates this mixed character by means of a Péclet number: quasi-steady models approximate a pure "diffusion" equation for long waves and a pure "convection" (or simple-wave) equation for short waves.

Ribberink and Van der Sande (1985) categorize the applicability of analytical solutions with respect to time and length scales. Simplified morphological models were compared successfully with over-load experiments.

Barneveld (1988) applied the analysis by Vreugdenhil (1972) to flood waves at mobile beds and compared different simplified models (kinematic model, diffusion analogy).

In this report, most of the results of the studies mentioned are reviewed. Apart from numerical tools, main differences are the range of key parameters (values of $Fr$) and the type of simplified models.
1.4. Model definition.

The mathematical model of a river flow with a mobile bed consists of water-mass and sediment-mass balances, a water-momentum equation and a sediment-transport predictor. The equations for the flow read

\[
\frac{\partial a}{\partial t} + u \frac{\partial a}{\partial x} + a \frac{\partial u}{\partial x} = S_1
\]

\[
u \frac{\partial a}{\partial t} + (u^2 - g_z a) \frac{\partial a}{\partial x} + \frac{a \partial u}{\partial t} + 2a \frac{\partial u}{\partial x} - g_x a \frac{\partial z_b}{\partial x} = S_2
\]

The sediment-mass balance is

\[
\frac{\partial z_b}{\partial t} + a \frac{\partial u}{\partial x} = S_3 \quad ; \quad \psi = \frac{1}{a} \frac{ds_b}{du}
\]

The source terms \(S_1\) and \(S_3\) represent lateral exchange of water mass and sediment mass respectively. The source term \(S_2\) contains the effects of gravity and friction and is defined as

\[
S_2 = g_x a + \frac{g_z u^2}{C^2}
\]

In Figure 1.1, the defined z- and x-axes are shown relative to \(Z_b\).

Fig. 1.1
To account for slope effects, the local bed level $z_b$ is defined relative to $Z_b$. The latter represents the spatially-averaged, constant equilibrium slope of the bed. The components of gravity forces are

$$
\begin{bmatrix}
g_x \\
g_z
\end{bmatrix} = -g \cos \theta_x \begin{bmatrix}
\frac{\partial Z_b}{\partial x} \\
\frac{\partial Z_b}{\partial x}
\end{bmatrix} = \begin{bmatrix}
g_i \\
-g
\end{bmatrix} ; \quad \tan \theta_x = \frac{\partial Z_b}{\partial x} = -i_b
$$

Due to its hyperbolic character, the set of three partial-differential equations can be transformed into a set of three ordinary-differential equations identified as the compatibility equations (e.g. Lai, 1986; Hirsch, 1988 and 1990). The latter are

$$
g_z n_i \frac{da}{d\tau} + n_i (n_i + u) \frac{du}{d\tau} + g_z (n_i + u) \frac{dz_b}{d\tau} =
$$

$$
= n_i (g_z a - u(u + n_i)) \frac{S_1}{a} + n_i (n_i + u) \frac{S_2}{a} + g_z (n_i + u) S_3
$$

($i = 1, 2$ and $3$) with

$$
\frac{d.}{d\tau} = \frac{\partial.}{\partial \tau} - n_i \frac{\partial.}{\partial x}
$$

(e.g. Lin and Shen, 1984; Sieben, 1994).

The characteristic equation that yields the characteristics of this system is

$$
n_i^3 + 2un_i^2 + (g_z a(1 + \psi + u^2)) n_i + g_z au\psi = 0
$$

(De Vries, 1966). It is noted that in this report, $n_i = -c_i$, where $c_i$ is the characteristic celerity [m/s].

1.5. Key parameters.

The mathematical exercise performed here is entirely similar to the one applied by Vreugdenhil (1972) and Barneveld (1988) to flood waves with mobile beds. For convenience, lateral exchanges of flow and sediment mass are left out ($S_1 = 0$ and $S_3 = 0$).
The mathematical model for flows at mobile beds reads

\[
\frac{\partial a}{\partial t} + u \frac{\partial a}{\partial x} + a \frac{\partial u}{\partial x} = 0 \tag{8}
\]

\[
u \frac{\partial a}{\partial t} + a \frac{\partial u}{\partial t} + (u^2 - g_x a) \frac{\partial a}{\partial x} + 2u a \frac{\partial u}{\partial x} - g_x a \frac{\partial z_b}{\partial x} = g_x a + \frac{g_x u^2}{C^2} \tag{9}
\]

\[
\frac{\partial z_b}{\partial t} + a \psi \frac{\partial u}{\partial x} = 0 \quad ; \quad \psi = \frac{1}{a} \frac{ds_b}{\partial u} \tag{10}
\]

The system of partial-differential equations can be linearized and written in non-dimensional form.

The zero-order terms of the flow are assumed to be uniform and steady. This yields

\[
g_x a + \frac{g_x u^2}{C^2} = 0 \tag{11}
\]

The derivatives of first-order terms can be made dimensionless by introducing the following transformations

\[
a' = \frac{a}{a_0} \quad ; \quad u' = \frac{u}{u_0} \quad ; \quad z_b' = \frac{z_b}{a_0} \quad \tag{12}
\]

\[
x' = \frac{\alpha x}{u_0 T} \quad ; \quad t' = \frac{t}{T}
\]

according to Vreugdenhil (1972, 1989). The time period \(T\) represents the period of the wave considered. For short-hand writing, the primes are left out.

The definition of the celerity \(u_0/\alpha\) is still optional. Vreugdenhil (1989) uses \(\alpha = Fr\) whereas Ponce and Simons (1977) use \(\alpha = 1\). In annex A it is concluded that the selection of \(\alpha\) has no effect on the analysis of harmonic solutions. It does, however, appear in the (quantitative) interpretation of the attenuation length of a wave.
The resulting system of linearized equations in non-dimensional form is

\[ \frac{\partial a}{\partial t} + \alpha \frac{\partial a}{\partial x} + \alpha \frac{\partial u}{\partial x} = 0 \]
\[ \frac{\partial a}{\partial t} + \alpha (1 + Fr^{-2}) \frac{\partial a}{\partial x} + \frac{\partial u}{\partial t} + 2\alpha \frac{\partial u}{\partial x} + \frac{\alpha}{Fr^{-2}} \frac{\partial z_b}{\partial x} + E \left( u - \frac{a}{2} \right) = 0 \]  \( (13) \)
\[ \alpha \psi \frac{\partial u}{\partial x} + \frac{\partial z_b}{\partial t} = 0 \]

with the linearized friction parameter written as

\[ E = 2 \frac{-g_z}{C^2} \frac{u_0 T}{a_0} = 2 \frac{-g_z z_b^{2/3} B^{1/3} T}{C^{2/3} Q^{1/3}} \]  \( (14) \)

Based on this model in non-dimensional form, three key parameters can be found that determine the solution of a flood wave at mobile beds (e.g. De Vries, 1992).

\[ E = 2 \frac{-g_z}{C^2} \frac{u_0 T}{a_0} ; \quad Fr = \frac{u_0}{\sqrt{-g_z a_0}} ; \quad \psi = \frac{1}{a_0} \frac{d s_b}{d u} \]  \( (15) \)

It is noted that the model is valid for values of \( \psi \ll 1 \) (Chapter five).

The parameter \( E \) represents the combined effect of friction and unsteadiness. It is noted that for mountain rivers with steep hydrographs a small value of \( E \) can be expected. A rather extreme example is the Kali Termas Lama at the Kelud Volcano in Indonesia (Slooff, 1993b), where steep slopes are combined with rather smooth river beds due to excessive supply of fine sediment.

However, this is not always the case; if the upper part of the Mallero is considered, a large value of \( E \) is found due to a small value of \( C \) (rough bed), despite the steep slope and short wave period. Analogously, values of \( Fr \) are not necessarily large in steep rivers. They can still be low when the bed is very rough.

The Johila river (De Vries, 1994) is an ephemeral river and combines a short flood-wave period with a relatively large average, "zero-order" discharge. The
characteristics of the Rio Apure (Delft Hydraulics, 1971) represent a low-land river.

Due to relatively smooth river beds, low values $E$ for the Oued Sebou result. Consequently, waves apparently have a dynamic character (Vreugdenhil, 1976)

<table>
<thead>
<tr>
<th>river</th>
<th>$i_b$</th>
<th>$C$</th>
<th>$Q_0/B$</th>
<th>$T/2$</th>
<th>$E$</th>
<th>$Fr$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Johila (India)</td>
<td>2.2</td>
<td>40</td>
<td>26.7</td>
<td>24</td>
<td>1.6</td>
<td>0.6</td>
</tr>
<tr>
<td>Mallero (Italy) Sondrio</td>
<td>17</td>
<td>35</td>
<td>2</td>
<td>12</td>
<td>9</td>
<td>0.9</td>
</tr>
<tr>
<td>Mallero (Italy) Laterna</td>
<td>36</td>
<td>20</td>
<td>2</td>
<td>12</td>
<td>20</td>
<td>1.2</td>
</tr>
<tr>
<td>K.Termas Lama (Indonesia)</td>
<td>5-8</td>
<td>50</td>
<td>2.1</td>
<td>12</td>
<td>3-4</td>
<td>1.1-1.4</td>
</tr>
<tr>
<td>Rio Apure (Venezuela)</td>
<td>0.01</td>
<td>40</td>
<td>6.7</td>
<td>4380</td>
<td>13</td>
<td>0.04</td>
</tr>
<tr>
<td>Oued Sebou (Marocco)</td>
<td>0.1</td>
<td>60-80</td>
<td>7.5</td>
<td>10</td>
<td>0.05</td>
<td>0.2</td>
</tr>
</tbody>
</table>
Chapter two.

Review of simplified morphological models.

2.1. Introduction.

As reviewed in Chapter one, time and length scales can be used to identify different options for simplification of mathematical or numerical models. Simplifications by analysis of time and length scales can be applied for solutions of mathematical model for the internal computation domain, as well as for consistent boundary conditions. For one-dimensional models it has already been assumed that effects of changes in transverse direction can be neglected.

The use of scales is described in Section 2.2. In Sections 2.3 and 2.4 different scales of solutions are analyzed with the help of harmonic solutions. In Section 2.5 to 2.8, the following simplifications are reviewed:

- quasi-steady flow (hyperbolic model)
- quasi-uniform flow (telegraph model)
- quasi-steady and quasi-uniform flow (parabolic model)
- critical flow (wave model)

2.2 Relevance of time and length scales.

Due to the hyperbolic character, hydraulic and morphological processes occur in a time-space domain. In general, the initial and/or boundary conditions prescribe an external length scale. The mathematical formulation of mass conservation and momentum transfer produces the internal time scale of a problem. The relation of length and time scales can be assigned to a representative propagation rate.

Obviously this implies that simplifications in mathematical models with respect to time, affect solutions with respect to space as well. The benefit of extreme scale differences is simplifying wave-type solutions to steady or uniform type solutions. This reduces the number of wave-type solutions involved, which enables more simple numerical (or even analytical) solution techniques.
To define absolute time (or length) scales, an absolute length (or time) scale is required. De Vries (1975) uses the analytical solution of the parabolic model to define a morphological time scale for different rivers, with the help of an arbitrary reference length. Because absolute scales should be specified from proto-type conditions, some qualitative remarks are made here with the help of relative scales.

To indicate different options for simplification, it is assumed here that separate time and length scales can be distinguished for hydraulic and morphological variables; $T_h$, $T_m$, $L_h$ and $L_m$. For wave type phenomena, time and length scales are related by a finite propagation rate. Then, obviously $T_h \to \infty$ (quasi-steady flow) would correspond with $L_h \to \infty$ (quasi-uniform flow). Likewise, $T_m \to \infty$ (quasi-fixed bed) corresponds with $L_m \to \infty$ (quasi-uniform bed). In the case of an infinite propagation rate, however, the relation between time and length scale is undetermined.

Generally, the typical length scales of flow and morphology coincide ($L_h = L_m$). In limited cases of local non-uniform (or even discontinuous) flow, different length scales can be observed. In this report, one length scale is applied.

If scales differ an order of magnitude, models can be simplified by neglecting changes in one scale with respect to the other. Neglecting a time or length scale implies that a solution with a wave-character can be approximated as a steady or uniform solution. Concerning time scales, the most important possibility is:

$$T_h/T_m \to 0; \text{ fixation of bed level during a number of (hydraulic) time steps (quasi-fixed bed).}$$

In the opposite situation ($T_m/T_h \to 0$) hydraulic conditions are fixed during a number of (morphological) time steps (constant flow conditions).

Both options are often used in numerical models to apply different time steps for hydraulics and morphology. This assumption of decoupled hydraulic and morphological changes is essential for simplifying morphological models in numerical procedures. Therefore, the non-linear interaction between changing hydraulics and morphology will be analyzed in more detail in Chapter three.
2.3. Characteristics-based time scales.

In Section 2.2, scales of morphology and hydraulics have been introduced without definition. It was also noted that time and length scales are related by means of a representative propagation rate. These rates can be obtained by substitution of harmonic solutions. With the linearized model, the relation of time and length scales can be found to be a function of Fr, E and ψ (Eq.15, Chapter one).

In this section, the effect of flow uniformity is neglected (E → 0). In Section 2.4, the relation of time and length scales is considered for larger values of E.

For infinitely small disturbances in model variables (E → 0), propagation rates approach the characteristics. By comparing characteristics of fixed bed and mobile bed models, De Vries (1965) distinguished "hydraulics-related" and "morphology-related" characteristics. These are approximated with

\[
\frac{n_{1,3}}{u} = -1 \pm \frac{1}{Fr} \quad ; \quad \frac{n_{12}}{u} = -\frac{\psi}{1-Fr^2} \quad ; \quad \psi = \frac{1}{a} \frac{ds_b}{du}
\]  

(1)

This is valid for Fr < 0.8 and Fr > 1.2. For transcritical flows (0.8 < Fr < 1.2), single characteristics can no longer be identified with either a hydraulic or morphological scale. It can be shown by using the compatibility equations that hydraulic as well as morphological variables change along each characteristic (Sieben, 1994).

If characteristics are assumed to be representative propagation rates for time and length scales, effects of non-uniformity and friction (characterized by E) are not taken into account.

\[
\frac{L}{T_h} = \left| 1 \pm \frac{1}{Fr} \right| \quad ; \quad \frac{L}{T_m} = \left| \frac{\psi}{1-Fr^2} \right|
\]  

(2)

This can be combined as

\[
\frac{T_m}{T_h} = \frac{(Fr \pm 1) (1 - Fr^2)}{Fr \psi}
\]  

(3)

The quasi-fixed bed approach (adaptation of hydraulic conditions without changes in bed level) would be valid if \(T_h/T_m \rightarrow 0\); \(\psi \rightarrow 0\) or \(Fr \rightarrow 0\).
The option $T_s/T_m \to 0$ can be interpreted as an "instantaneous" adaption of water depth and velocity, relative to slow changes in bed level (e.g. De Vries, 1971). This enables quasi-steady flow modelling as described by De Vries (1965) and Vreugdenhil and De Vries (1973).

It is noted that in the quasi-steady approach, the propagation rate for "hydraulic waves" is assumed to be infinite with respect to that of "morphological waves". Hence, $T_h \to \infty$ does not affect any length scale.

In numerical models based on this quasi-steady approach, two simplifications are applied. The first step concerns the bed levels, which is assumed constant or fixed during the quick adaption of hydraulic conditions. This may involve multiple (hydraulic) time steps. The resulting free surface levels are backwater curves. During the second step, morphological changes are computed assuming constant hydraulic conditions.

From Eq.3 it can be concluded that $T_s/T_h \to 0$ (constant hydraulic conditions with respect to morphology) cannot be observed for $Fr < 0.8$ and $Fr > 1.2$ and realistic values of $\psi$. Nevertheless, hydraulic conditions can often be assumed constant with respect to small changes in morphological conditions. Then, hydraulic conditions and subsequent sediment-transport gradients can be easily integrated over a period.

An example is the computation of season-integrated morphological changes, with hydraulic conditions that are assumed constant over a characteristic part of the season. A second example concerns morphology in tidal reaches, where morphological changes, integrated over a number of tidal periods, are often computed with periodic tidal conditions.

Analogously, Lyn (1987) derives a morphological time scale with the help of characteristics.

2.4. Time scales based on harmonic solutions.

In Section 2.3, time and length scales were related by means of characteristics. Because characteristics represent propagation of infinitely small disturbances, time and length scales, and subsequently representative propagation rates, can be different for longer flood wave solutions ($E \neq 0$).

Therefore, in this section, scales are distinguished by using the characteristic equations (characteristics and compatibility equations). Because higher values of
Fr are considered, the distinction of separate morphological and hydraulic scales is necessarily left out.

Along a characteristic, changes in a variable are defined as

$$\frac{d..}{d\tau} = \frac{\partial..}{\partial t} - n^a \frac{\partial..}{\partial x} = -n^a \frac{\partial..}{\partial x} \left( \frac{\partial../\partial t}{-n^a \partial../\partial x} + 1 \right) = -\zeta_i n^a \frac{\partial..}{\partial x}$$  \hspace{1cm} (4)

Now, changes in hydraulic and morphological variables along a specific characteristic are considered. If values of $\zeta$ approach infinity or unity, time and length scales along a characteristic can be decoupled. Then, obviously the wave character of the solution can be neglected. The possibilities are threefold.

The first option is

$$|\zeta_i| = \left| \frac{1}{-n^a} \frac{\partial../\partial t}{\partial../\partial x} + 1 \right| \gg 1 \rightarrow \frac{d..}{d\tau} \approx \frac{\partial..}{\partial t}$$  \hspace{1cm} (5)

with changes in time dominant over changes in space. This refers to an instantaneous adaption of variables in a quasi-uniform condition; the wave character (or relation between time and length scale) can be neglected.

![Fig. 2.1](image-url)

Fig.2.1
The second option is

$$
| \zeta_i | = \left| 1 - \frac{1}{-n_{ti}} \frac{\partial \zeta}{\partial t} + 1 \right| \approx 1 - \frac{d_{..}}{d\tau} = \frac{n_{ti}}{\partial x} \frac{\partial \zeta}{\partial x} \tag{6}
$$

with spatial changes dominant over changes in time. This refers to a quasi-steady situation; again the wave character of the solution can be neglected. Options one and two are illustrated qualitatively in Figure 2.1.

The third option is

$$
| \zeta_i | = \left| 1 - \frac{1}{-n_{ti}} \frac{\partial \zeta}{\partial t} + 1 \right| \approx 0 - \frac{d_{..}}{d\tau} \approx 0 \tag{7}
$$

with no changes along a characteristic. It is noted that in contrast with the first two options ($\xi \gg 1$ and $\xi \approx 1$), now the wave character is preserved. This option leads to a set of simple-wave equations along a specific characteristic (Riemann invariants).

To analyze $\xi$ values for flood wave conditions, a sinusoidal solution is used (Appendix A). Substitution yields

$$
\zeta_i = \frac{u}{-n_{ti}} \frac{1}{\alpha} \frac{\partial \zeta}{\partial t} + 1 = \frac{u}{-n_t} \frac{\omega}{\alpha} \frac{1}{k} + 1 = \frac{u}{-n_t} \frac{\omega}{\alpha |k|^2} + 1 \tag{8}
$$

The complex wave number is defined as $k = k_1 + i k_2$. This yields

$$
\zeta_i = \left( \frac{u}{-n_{ti}} \frac{\omega'}{\alpha |k'|^2} + 1 \right) - i \left( \frac{u}{-n_{ti}} \frac{\omega'}{\alpha |k'|^2} \right)
$$

Hence, the ratio of time and length-scales $\xi_i$ can be interpreted as wave-propagation rate relative to characteristic celerity. In addition, a phase shift is present between maxima of time and spatial derivatives.

Now, for the slow and fast downstream-travelling waves (indices II and III) and for the upstream-travelling wave I, the modulus of this ratio $| \zeta_i |$ can be constructed. The arrows indicate the effect of increasing $E$; ($E = 1$, $E = 50$, $E = 100$ and $E = 500$).
Indices refer to the characteristics (increasing with celerity); 1 identifies the upstream-directed characteristic $n_{i1}$, 2 refers to the small downstream-directed characteristic $n_{i2}$ and 3 refers to the large downstream-directed characteristic $n_{i3}$.

The results are presented in Figures 2.2, 2.3 and 2.4.
As can be concluded from the figures, there are different options. For small values of $E$, $\zeta \rightarrow 0$. It is noted that for $E \rightarrow 0$, the characteristic equation of wave $j$ can be reduced to

$$\frac{Da}{D\tau_j} = 0 \ ; \ \frac{Du}{D\tau_j} = 0 \ ; \ \frac{Dz_{b}}{D\tau_j} = 0$$ (10)

Because for the other two it can be assumed $\zeta_i \rightarrow \infty$ relative to $\zeta_j$, (with $i \neq j$) the only wave present will be a simple-wave, propagating with a rate $n_r$.

This implies that the total solution for $E \rightarrow 0$ (no friction or infinitely small wave length) consists of three simple-waves that propagate without deformation with a rate equal to a characteristic. The propagation and stability of such waves is analyzed in Sieben (1995).

$Fr < 0.4$.

For both waves I and III for $Fr < 0.4$, $\zeta_2 \rightarrow \infty$ for the smallest characteristic root, yielding

$$\frac{Da}{D\tau_2} = \frac{\partial a}{\partial t} \ ; \ \frac{Du}{D\tau_2} = \frac{\partial u}{\partial t} \ ; \ \frac{Dz_{b}}{D\tau_2} = \frac{\partial z_{b}}{\partial t}$$ (11)

Along the small, downstream-directed characteristic $n_{\alpha}$, time changes in variables are dominant over spatial gradients.
This implies that information propagating with \( n_{i2} \) refers to (zero-order) uniform-flow conditions and does not add to the solution of the wave. Neglecting this equation results in a set of equations that resembles the quasi-fixed bed approach. Hence, wave I and III correspond with solutions for the fixed bed model for \( Fr < 0.4 \).

For both waves I and III at \( Fr < 0.4 \), the contribution of spatial changes along the small \( n_{i2} \) can be neglected, whereas for wave II, changes along \( n_{i1} \) and \( n_{i3} \) are mainly spatial. With respect to wave II it can be stated \( \zeta_1 \to 1 \) and \( \zeta_3 \to 1 \) relative to \( \zeta_2 \).

Substitution of this approximation in the characteristic equations for the celerities \( n_{i1} \) and \( n_{i3} \) yields

\[
\frac{\partial u}{\partial t} = \frac{-g_z au}{(n_1+u)(n_3+u)} \frac{\partial a}{\partial t} ; \quad \frac{\partial z_b}{\partial t} = \frac{-n_1 n_3}{(n_1+u)(n_3+u)} \frac{\partial a}{\partial t} \tag{12}
\]

\[
\frac{\partial a}{\partial x} = -\left( \frac{g_z}{n_{i1} n_{i3}} + S_2 \right) \frac{(n_{i1}+u)(n_{i3}+u)}{g_z n_{i1} n_{i3}} \quad ; \quad \frac{\partial u}{\partial x} = \frac{ua \left( \frac{g_z}{n_{i1} n_{i3}} + S_2 \right)}{n_{i1} n_{i3}} \tag{13}
\]

Substitution into the characteristic equation for \( n_{i2} \) yields indeed the quasi-steady model (Section 2.6)

\[
\frac{\partial z_b}{\partial t} - n_{i2} \frac{\partial z_b}{\partial x} = n_{i2} \frac{S_2}{g_z} ; \quad \frac{\partial a}{\partial t} - n_{i2} \frac{\partial a}{\partial x} = 0 ; \quad \frac{\partial u}{\partial t} - n_{i2} \frac{\partial u}{\partial x} = 0 \tag{14}
\]

This confirms application of the quasi-steady flow concept for morphological computations for \( Fr < 0.4 \) and all values of \( E \).

Hence, for \( Fr < 0.4 \), waves in bed level propagate in downstream direction with a rate \(-n_{i2}\). The deformation of the bed-level wave is related to the source term \( S_2 \). Because \( | n_{i2} | << 1 \), the rate of deformation is relatively low (e.g. De Vries, 1994).

**Fr > 0.4.**

For higher values of \( Fr \), it is noted that spatial gradients become dominant in the solution. Since this is valid for all characteristics, decoupling of scales is
complicated. It is noted that no decoupling is possible if celerities are of similar magnitude. For large values of $E$, $\zeta$ approaches a constant value, implying that hydraulic conditions no longer affect the ratio of time and length scales of a wave. Therefore, time or length scales cannot be neglected but, due to the constant ratio, a scale might be eliminated.

For $0.4 < Fr < 1.0$ and smaller values of $E$, it can be assumed to a certain extent $\zeta_1 \to 1$ and $\zeta_2 \to 1$ relative to $\zeta_3$ with respect to wave III (Figure 2.5). This implies that wave III could be approximated by a convection-dominated simple-wave

$$\frac{\partial z_b}{\partial t} - n_3 \frac{\partial z_b}{\partial x} = n_3 \frac{S_2}{g_z} \quad ; \quad \frac{\partial a}{\partial t} - n_3 \frac{\partial a}{\partial x} = 0 \quad ; \quad \frac{\partial u}{\partial t} - n_3 \frac{\partial u}{\partial x} = 0 \quad (15)$$

This corresponds to the increased attenuation length for higher values of $Fr$ (Appendix A).

2.5. Distinction of temporal and spatial changes.

To illustrate the coupling between time and length scales in the mathematical model, the following analysis is carried out. Starting point for the analysis are the PDEs with variables written in dimensional form.

The system of mixed time and spatial changes is separated into two subsystems: one with changes in time and one with changes in space. Now, changes in time (space) can be stated to be zero and the corresponding effect on gradients in space (time) can be found.
The subsystems can be written as

\[
\begin{bmatrix}
\frac{\partial u}{\partial t} \\
\frac{\partial u}{\partial t} \\
\frac{\partial z_j}{\partial t}
\end{bmatrix} = [A]
\begin{bmatrix}
\delta_1 \\
\delta_2 \\
\delta_3
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial x} \\
\frac{\partial z_j}{\partial x}
\end{bmatrix} = [B]
\begin{bmatrix}
\delta_1 \\
\delta_2 \\
\delta_3
\end{bmatrix} + \left[ \begin{array}{c}
\frac{S_1-S_3/\psi}{u} \\
\frac{S_3/(a\psi)}{u} \\
\frac{-\frac{S_1}{u}(1+Fr^2) - \frac{S_2}{g_\alpha} - \left( \frac{1}{n_{r_1}} + \frac{1}{n_{r_2}} + \frac{1}{n_{r_3}} \right)S_3}{u}
\end{array} \right]
\]

(16)

with \(\delta_i\) a dummy correlation variable. In the following, the source terms \(S_1\) and \(S_3\) will be neglected. The matrices \(A\) and \(B\) are

\[
[A] = \begin{bmatrix}
\frac{(n_{r_2}+u)(n_{r_3}+u)}{g_\alpha u(n_{r_1}-n_{r_2})(n_{r_3}-n_{r_1})} & \frac{(n_{r_1}+u)(n_{r_3}+u)}{g_\alpha u(n_{r_2}-n_{r_1})(n_{r_3}-n_{r_2})} & \frac{(n_{r_1}+u)(n_{r_2}+u)}{g_\alpha u(n_{r_1}-n_{r_2})(n_{r_3}-n_{r_2})}
\end{bmatrix}
\]

(17)

\[
[B] = \begin{bmatrix}
\frac{(n_{r_2}+u)(n_{r_3}+u)}{g_\alpha u(n_{r_1}-n_{r_2})(n_{r_3}-n_{r_1})} & \frac{(n_{r_1}+u)(n_{r_3}+u)}{g_\alpha u(n_{r_2}-n_{r_1})(n_{r_3}-n_{r_2})} & \frac{(n_{r_1}+u)(n_{r_2}+u)}{g_\alpha u(n_{r_1}-n_{r_2})(n_{r_3}-n_{r_2})}
\end{bmatrix}
\]

(18)

The values of corresponding determinants are

\[
|A| = -\frac{1}{g_\alpha^2 u(n_{r_1}-n_{r_2})(n_{r_1}-n_{r_3})(n_{r_2}-n_{r_3})} ; \quad |B| = \frac{1}{g_\alpha^2 u(n_{r_1}-n_{r_2})(n_{r_1}-n_{r_3})(n_{r_2}-n_{r_3})}
\]

(19)
Eq. 19 shows that matrices $A$ and $B$ are non-singular, which implies that time and spatial scales are indeed coupled for every realistic condition, in correspondence with the hyperbolic character of the model.

2.6. Quasi-steady flow.

Hydraulic time scales can be neglected (instantaneous adaption) if

$$ \delta_1 = \frac{n_{t_1}+u}{n_{t_3}+u} \delta_3 ; \quad \delta_2 = \frac{n_{t_1}+u}{n_{t_3}+u} \delta_3 $$

(20)

which implies

$$ \frac{\partial z_b}{\partial t} = \frac{\delta_3}{g_z(n_3+u)} $$

$$ \frac{\partial a}{\partial x} = -\frac{(n_{t_1}+u)(n_{t_2}+u)(n_{t_3}+u)}{n_{t_1}n_{t_2}n_{t_3}u} \frac{\partial z_b}{\partial t} $$

$$ \frac{\partial u}{\partial x} = \frac{g_z u}{n_{t_1}n_{t_2}n_{t_3}} \frac{\partial z_b}{\partial t} $$

$$ \frac{\partial z_b}{\partial x} = \left( \frac{1}{n_{t_1}} + \frac{1}{n_{t_2}} + \frac{1}{n_{t_3}} + \frac{1}{u} \right) \frac{\partial z_b}{\partial t} - \frac{S_2}{g_z a} $$

This equation can be written as

$$ \frac{\partial z_b}{\partial t} + \frac{u \psi}{1-Fr^2} \frac{\partial z_b}{\partial x} = -\frac{u S_2 \psi}{g_z a (1-Fr^2)} $$

(22)

For $S_2 = 0$ (no friction), the simple-wave (convection) model is found (De Vries, 1973). It is noted that this simple-wave model becomes unstable for values of $Fr$ near unity.

If $S_2 = 0$ (uniform, steady flow), gradients in source term $S_2$ can be written as

$$ \frac{\partial S_2}{\partial x} = g_z \frac{\partial a}{\partial x} + 2 \frac{g_z u}{C^2} \frac{\partial u}{\partial x} = -3 \frac{g_z}{u \psi} \left( \frac{u^2}{C^2 a} \right) \frac{\partial z_b}{\partial t} $$

(23)
Linearizing Eq.22 in combination with Eq.23 yields the hyperbolic model (e.g. Vreugdenhil and De Vries, 1976).

\[
\frac{\partial z_b}{\partial t} + a(Fr^2 - 1) \frac{\partial^2 z_b}{\partial x \partial t} - \frac{ua\psi}{3} \frac{\partial^2 z_b}{\partial x^2} = 0
\]  \hspace{1cm} (24)

For flows with \( Fr \approx 1 \), this hyperbolic equation transforms into a parabolic equation.

Compared to the simple-wave model it can be seen that including friction stabilizes the model.

2.7. Quasi-uniform flow.

An analogous analysis can be applied for quasi-uniform flows. To obtain this it is stated

\[
\delta_1 = \delta_3 \frac{n_{t1}(n_{t1} + u)}{n_{t3}(n_{t3} + u)} ; \quad \delta_2 = \delta_3 \frac{n_{t2}(n_{t2} + u)}{n_{t3}(n_{t3} + u)}
\]  \hspace{1cm} (25)

Substitution yields

\[
\frac{\partial a}{\partial t} = 0 ; \quad \frac{\partial z_b}{\partial t} = 0 ; \quad \frac{\partial u}{\partial t} = \frac{\delta_3}{n_{t3}(n_{t3} + u)} ; \quad \frac{\partial z_b}{\partial x} = \frac{a\delta_3 - n_{t3}S_2(n_{t3} + u)}{g\alpha n_{t3}(n_{t3} + u)}
\]  \hspace{1cm} (26)

Elimination of \( \delta_3 \) gives the reduced momentum equation

\[
\frac{\partial u}{\partial t} - g\frac{\partial z_b}{\partial x} = S_2 \frac{a}{a}
\]  \hspace{1cm} (27)

Since velocity has been assumed constant (steady, uniform flow), it follows from the sediment-mass balance that bed levels are fixed.
If, however, gradients in velocity at a larger morphological length scale are assumed, combination of Eq.14 with the sediment-mass balance enables writing
\[
\frac{\partial^2 z_b}{\partial t^2} + g_z \psi a \frac{\partial^2 z_b}{\partial x^2} = -\psi a \frac{\partial S_2}{\partial x} \tag{28}
\]

Without friction effects \((S_2 = 0)\), this would equal the wave or d’Alembert equation with \(n_1 = \pm (-g_z a \psi)^{1/2}\). However, friction must be considered dominant for large length-scales. Therefore, elimination of \(S_2\) results in
\[
\frac{\partial z_b}{\partial t} + \frac{C^2 a}{-3g_z u} \frac{\partial^2 z_b}{\partial t^2} - \psi \frac{C^2 a^2}{3u} \frac{\partial^2 z_b}{\partial x^2} = 0 \tag{29}
\]

This is the telegraph equation.

2.8. Quasi-steady and quasi-uniform flow.

The assumptions of quasi-steady and quasi-uniform flows can be combined, if
\[
\delta_3 \left( \frac{n_{\text{fl}}}{n_{\text{t3}}} \right) \left( \frac{n_{\text{fl}+u}}{n_{\text{t3}+u}} \right) = \delta_3 \left( \frac{n_{\text{fl}+u}}{n_{\text{t3}+u}} \right)
\]
\[
\wedge
\]
\[
\delta_3 \left( \frac{n_{\text{f2}}}{n_{\text{t3}}} \right) \left( \frac{n_{\text{f2}+u}}{n_{\text{t3}+u}} \right) = \delta_3 \left( \frac{n_{\text{f2}+u}}{n_{\text{t3}+u}} \right)
\tag{30}
\]

These conditions can be satisfied if \(\delta_3 = 0\), yielding
\[
\frac{\partial z_b}{\partial t} = 0 \quad ; \quad \frac{\partial z_b}{\partial x} = -\frac{S_2}{g_z a} \tag{31}
\]

If, again, changes in morphology and hydraulics at the scale of the second-order term \(S_2\) are considered, friction can be eliminated. Then, the parabolic model as derived by De Vries (1973) is obtained.
\[
\frac{\partial z_b}{\partial t} - \frac{C^2 a^2 \psi}{3u} \frac{\partial^2 z_b}{\partial x^2} = 0 \tag{32}
\]

A review of the different simplified models can be found in Chapter four.
2.8. Critical flow.

Because quasi-steady flow models fail at values of \( Fr \) near unity (Eq.22), the following analysis is added. For values of \( Fr \) near unity, absolute values of two characteristics approach. With the similarity conditions (Appendix C), the exact solution of the characteristic equation can be found.

\[
n_{r1} = -2u \quad ; \quad n_{r2,3} = \pm u \frac{\sqrt{\psi/2}}{Fr} \quad ; \quad Fr = \sqrt{1+\psi/2}
\] (33)

This equals the solution presented by De Vries (1992).

Substitution into the compatibility equations yields

\[
\frac{\partial a}{\partial x} = - \frac{\partial z_b}{\partial x} + \frac{1}{g_z} \frac{\partial u}{\partial t} - \frac{S_2}{g_z} - u \frac{\partial u}{g_z} \frac{\partial z_b}{\partial x} + \frac{u}{n_{r2}^2} \frac{\partial z_b}{\partial t}
\]

\[
\frac{\partial a}{\partial t} = u \frac{\partial z_b}{\partial x} - \frac{u}{g_z} \frac{\partial u}{\partial t} + \frac{uS_2}{g_z} + \frac{n_{r2}^2}{g_z} \frac{\partial u}{\partial x} - \frac{\partial z_b}{\partial t}
\]

(34)

Differentiation to \( x \) and \( t \) and combination enables writing

\[
\frac{\partial^2 u}{\partial t^2} - n_{r2}^2 \frac{\partial^2 u}{\partial x^2} - g_z \frac{\partial^2 z_b}{\partial x^2} - \frac{g_z u}{n_{r2}^2} \frac{\partial^2 z_b}{\partial t^2} = \frac{\partial S_2}{\partial t} + u \frac{\partial S_2}{\partial x}
\]

(35)

Again differentiation to \( x \) enables elimination of \( u \) with the sediment mass balance.

Without friction, the following equation is found

\[
\frac{\partial^3 z_b}{\partial t^3} + \frac{g_z a \psi}{2} \frac{\partial^3 z_b}{\partial x^3} + g_z a \psi \frac{\partial^3 z_b}{\partial x^2 \partial t} + 2u \frac{\partial^3 z_b}{\partial t^2 \partial x} = 0
\]

(36)

This can be rewritten as

\[
\frac{\partial}{\partial t} \left( \frac{\partial^2 z_b}{\partial t^2} + \frac{g_z a \psi}{2} \frac{\partial^2 z_b}{\partial x^2} \right) + 2u \frac{\partial}{\partial x} \left( \frac{g_z a \psi}{2} \frac{\partial^2 z_b}{\partial x^2} + \frac{\partial^2 z_b}{\partial t^2} \right) = 0
\]

(37)
As can be expected, the solution consists of three simple waves. The fast wave (|n_1| = 2u) is not considered here.

The resulting model is

$$\frac{\partial^2 z_b}{\partial t^2} + \frac{g \alpha \psi}{2} \frac{\partial^2 z_b}{\partial x^2} = 0$$

It is noted here that propagation rates of the wave model derived earlier (Eq.16) are overpredicted with a factor $\sqrt{2}$ relative to the exact model.
Chapter three.

Decoupling hydraulics and morphology in numerical models.

3.1. Introduction.

One of the complicating features of morphological models is the non-linear interaction between bed and flow: changes in flow induce changes in bed and vice versa. As a result, mathematical models for hydrodynamics on fixed and mobile beds are fundamentally different. However, under certain conditions, the resemblance between both models enables a quasi-fixed bed approach for computing hydraulics in morphological models (De Vries, 1965).

Holly and Rahuel (1990) state decoupled models to be applicable in case of bed load due to relatively slow motions of bed load. Subsequently, development and usage of coupled models is motivated in case of suspended load, which moves at rates comparable to flow. It should be noted, however, that questions on decoupling should focus on non-linear interactions between bed level and hydraulic variables.

Because the decoupling of changes in hydraulics and morphology enables efficient solution procedures in numerical models of river morphology, the concept has been widely applied. At present, however, most of the applications are in low-land rivers. Therefore, this chapter includes the es of decoupling effects in numerical models with mountain-river conditions.

As reviewed in Chapter two, two steps can be distinguished in decoupled-solution techniques;

1) depth and velocity are calculated at a fixed bed

2) changes in bed level are computed with the adapted depth and velocity

The first step neglects changes in $z_b$ with respect to changes in $u$ and $a$, whereas the second step neglects changes in $u$ and $a$ with respect to changes in $z_b$.

In this chapter, the nonlinear interaction between morphological and hydraulic variables is analyzed by comparison mobile-bed and fixed-bed model formulations. A transformation of PDEs to a set of normal-differential equations is used (e.g., Lin, 1984).
The response of both sets of characteristic equations to a solution is compared. In Section 3.2, this solution has spatial gradients of first-order only. Then, the complete and decoupled version appear to be similar. Comparison, however, yields a set of similarity conditions, equivalent to the characteristic equation presented by De Vries (1965).

In Sections 3.3 and 3.4, a solution with spatial gradients of second-order is considered. Now, different responses can be found. This difference is quantified, and appears to be a function of Fr and t. In Section 3.5, the effect of linearizing the implicit formulation is analysed numerically.

Sections 3.2, 3.3 and 3.4 regards decoupling effects at the short time and length scale of characteristics. In Section 3.6, the decoupling effect is scaled with respect to higher-order terms, which enables analyzing the hydraulic-morphological interaction with the help of harmonic solutions (Chapter four).

3.2. Effects of first-order gradients.

In the compatibility equations (Section 1.3), changes in variables are defined along characteristic paths of information (Fig.3.1). Hence, these compatibility equations should be integrated along characteristics. This yields three equations (one for every root) of the form

\[ g_z n \Delta a_i + n(n_u + u) \Delta u_i + g_z (n_u + u) \Delta z_{bi} = \Delta t O_i \]  \hspace{1cm} (1)

The changes in variable \( U \) along \( n \) are defined as

\[ \Delta U_i = U^{n-1} - U_i^n \]  \hspace{1cm} (2)
To enable computation of variables $U^{n+1}$ at the new time level $n+1$, variables $U^n_i$ at the time level $n$, at different locations (1,2,3) must be used as initial conditions.

These initial conditions in the characteristic equations can be approximated with the help of a Taylor-series expansion. In a first-order approach, the variables at the point of arrival of the characteristic are

$$U^n_i = U^n_0 + \Delta t \left. n_i \frac{\partial U}{\partial x} \right|_n$$ (3)

Hence, changes in $U$ along $n_i$ are conform the definition (Eq.5, Chapter one)

$$\Delta U_i = U^{n+1} - \left( U^n_0 + \Delta t \left. n_i \frac{\partial U}{\partial x} \right|_n \right) = \Delta U - \Delta t \left. n_i \frac{\partial U}{\partial x} \right|_n$$ (4)

Substitution into the compatibility equations yields

$$g^2 n_\alpha \Delta a + n_i (n_i + u) \Delta u + g^2 (n_i + u) \Delta z_b =$$

$$\Delta \tau_i O_i = g^2 n^2 \Delta t \frac{\partial a}{\partial x} - n_i^2 (n_i + u) \Delta t \frac{\partial u}{\partial x} - g^2 (n_i + u) n_i \frac{\partial z_b}{\partial x}$$
Now, the system of PDE’s has been obtained again. Time-gradients in $a$, $u$ and $z_b$ can be solved by

\[
\begin{pmatrix}
\frac{\partial a}{\partial t} \\
\frac{\partial u}{\partial t} \\
\frac{\partial z_b}{\partial t}
\end{pmatrix} =
\begin{pmatrix}
-u \\
g_c \\
0
\end{pmatrix}
\frac{\partial a}{\partial x} + \begin{pmatrix}
-(n_{1a}+u)(n_{13}+u)(n_{3a}+u) \\
g_c (n_{1a}+n_{3a}+u) \\
n_{1a}n_{2a}n_{3a}
\end{pmatrix}
\frac{\partial u}{\partial x} + \begin{pmatrix}
0 \\
g_c \\
0
\end{pmatrix}
\frac{\partial z_b}{\partial x} + \begin{pmatrix}
S_1 \\
S_2 - uS_1 \\
a
\end{pmatrix} \frac{\partial S_3}{\partial x}
\]

(6)

Where $S_1$, $S_2$ and $S_3$ are source terms in flow-mass, flow-momentum and sediment-mass balance.

Because coefficients in the compatibility equations are determined by the unknown variables $U^{n+1}$, the solution is an implicit one. In the following, coefficients are computed with $U_0$ to enable an explicit solution. The effect of this assumption on the is is considered in Section 3.4.

For the decoupled or quasi-fixed bed model, a fixed bed model is combined with a sediment-mass balance. Solution of these models takes place in two steps. The first step concerns the hydraulic variables that can be determined from

\[
\frac{\partial a}{\partial t} + u \frac{\partial a}{\partial x} + a \frac{\partial u}{\partial x} = S_1
\]

(7)

\[
u \frac{\partial a}{\partial t} + (u^2 - g_c a) \frac{\partial a}{\partial x} + a \frac{\partial u}{\partial t} + 2ua \frac{\partial u}{\partial x} = S_2 + g_c a \frac{\partial z_b}{\partial x}
\]

After this system has been solved, the new bed level is calculated with

\[
a \psi \frac{\partial u}{\partial x} + \frac{\partial z_b}{\partial t} = S_3
\]

(8)

With respect to the calculation of the water movement, the celerities are (again with opposite sign used in this report)

\[c_t = -u \pm \sqrt{-g_c a}\]  

(9)
The compatibility equations are

\[ c_t \frac{da}{d\tau} + c_f(c_t + u) \frac{du}{d\tau} = O \]

\[ O = c_f(g_z a - u(c_t + u))S_1 + c_f(c_t + u) \left( S_2 + g_z a \frac{dz_b}{dx} \right) \]  

(10)

Integration along a characteristic \( c_n \) and substitution of non-uniform initial conditions yields

\[ g_z c_n \Delta a + c_n(c_n + u) \Delta u = \]

\[ \Delta t \tau_1 O + g_z \Delta t \ c_n^2 \frac{\partial a}{\partial x} + \Delta t \ c_n^2(c_n + u) \frac{\partial u}{\partial x} \]

(11)

Now, changes in \( a \), \( u \) and \( z_b \) can be solved implicitly by integration over \( \Delta t \)

\[
\begin{bmatrix}
\frac{\partial a}{\partial t} \\
\frac{\partial u}{\partial t} \\
\frac{\partial z_b}{\partial t}
\end{bmatrix} = \begin{bmatrix}
-u \\
g_z \frac{\partial a}{\partial x} \\
0
\end{bmatrix} + \begin{bmatrix}
-(c_{t1} + u)(c_{t2} + u) \\
-c_{t1} + c_{t2} + u \\
a \psi
\end{bmatrix} \begin{bmatrix}
\frac{\partial a}{\partial x} \\
\frac{\partial u}{\partial x} \\
\frac{dz_b}{dx}
\end{bmatrix} + \begin{bmatrix}
0 \\
S_1 \\
S_2 - u S_1 \\
\frac{S_1}{S_3}
\end{bmatrix} \]  

(12)

Substitution of the characteristics \( c_n \) yields the system of PDE’s again.

Comparison of the implicit solutions (Eqs 6 and 12) of both models enables derivation of conditions for similarity. Both models yield similar solutions if

\[ (n_{t1} + u) (n_{t2} + u) (n_{t3} + u) = g_z a u \]

\[ n_{t1} + n_{t2} + n_{t3} = -2u \]

\[ n_{t1} n_{t2} n_{t3} = -g_z a u \psi \]

(13)

Substitution proves that these conditions are equivalent to the characteristic equation (Eq.6, Chapter 1), and consequently are satisfied. In Appendix C, similarity conditions are used to analyze some simplifications.
Hence, as can be expected, for *linearized* morphological models, no difference exists between the complete and the decoupled solution of velocity and depth. In other words, linearization eliminates any non-linear interaction for all values of $Fr$. This is in correspondence with remarks of Holly and Rahuel (1990).

Consequently, the error that is introduced due to decoupling must be related to non-linear effects. It can be understood that non-linearities can be reduced by reduction of the time-interval of decoupling.

### 3.3. Second-order gradients in mobile-bed models.

Because effects of decoupling cannot be found when combining linearized models with first-order initial conditions, higher-order terms should be considered. Therefore, contributions of second-order terms on the solution are analyzed.

To analyze effects of decoupling, again linearized models are used; but non-linearity is included by accounting for higher-order gradients in the initial conditions.

The non-uniform initial conditions are again defined with the help of spatial gradients. Second-order effects are taken into account by distinction of up- and downstream gradients. Spatial gradients are now approximated with

$$\frac{\partial U}{\partial x_{down}} = \frac{\partial U}{\partial x} |_{x} + \frac{\Delta x}{2} \frac{\partial^2 U}{\partial x^2} |_{x}$$

$$\frac{\partial U}{\partial x_{up}} = \frac{\partial U}{\partial x} |_{x} - \frac{\Delta x}{2} \frac{\partial^2 U}{\partial x^2} |_{x}$$

The derivation is added in Appendix B.
Substitution of the initial conditions into the solution yields

\[
\begin{bmatrix}
\Delta a \\
\Delta u \\
\Delta z_b
\end{bmatrix}
= \Delta t
\begin{bmatrix}
-u \\
g_z \\
0
\end{bmatrix}
\frac{\partial a}{\partial x} + \frac{\Delta t}{2}
\begin{bmatrix}
-g_z \delta_{a_1} \\
-g_z \delta_{a_2} \\
-u \delta_{a_3}
\end{bmatrix}
\frac{\partial^2 a}{\partial x^2}
\]

\[+ \Delta t
\begin{bmatrix}
-a \\
-u \\
-a \psi
\end{bmatrix}
\frac{\partial u}{\partial x} + \frac{\Delta t}{2}
\begin{bmatrix}
-a \delta_{u_1} \\
-u \delta_{u_2} \\
-a \psi \delta_{a_3}
\end{bmatrix}
\frac{\partial^2 u}{\partial x^2}
\]

\[+ \Delta t
\begin{bmatrix}
0 \\
g_z \\
0
\end{bmatrix}
\frac{\partial z_b}{\partial x} + \frac{\Delta t}{2}
\begin{bmatrix}
0 \\
g_z \delta_{z_1} \\
u \delta_{z_2}
\end{bmatrix}
\frac{\partial^2 z_b}{\partial x^2}
\]

\[+ \Delta t
\begin{bmatrix}
S_1 \\
S_2 - u S_1 \\
S_3
\end{bmatrix}
\frac{a}{a}
\]

(15)

The dimensionless coefficients \( \delta_{ij} \) are constructed in Figures 3.2-a and b. To indicate the effect of sediment transport rate, transport rates are multiplied with a factor 10 in Figure 3.2-b.

![Fig.3.2-a](image1.png)  ![Fig.3.2-b](image2.png)

It is noted that all second-order gradients act as stabilizing diffusion terms on the solution. Clearly three regions can be distinguished; subcritical flow (\( Fr < 0.8 \)), transcritical flow (\( 0.8 \leq Fr \leq 1.2 \)) and supercritical flow (\( Fr > 1.2 \)). For sub- and supercritical flows, the second-order effects are discussed.
\( Fr < 0.8. \)

With respect to water depth, \( \delta_{a1} \) and \( \delta_{z1} \) are larger than \( \delta_{u1} \). Hence, "diffusive" contributions by positive second-order gradients in water depth and bed level dominate.

Analogously, for velocity it can be concluded that \( \delta_{a2} \) can be considered larger than \( \delta_{u2} \) and \( \delta_{z2} \), implying that second-order gradients in velocity affect the solution mainly. It is noted that higher-order interaction between velocity and depth is not significant for low values of \( Fr \).

As far as changes in bed level are concerned, all values of \( \delta_{a3} \), \( \delta_{u3} \) and \( \delta_{z3} \) are of comparable magnitude. This indicates that second-order effects in all variables are equally important for the solution of bed level changes. Based on the magnitudes of \( \delta \) values, it can be concluded that the coupling between morphology and hydraulics is weak for \( Fr < 0.8. \)

\( Fr > 1.2. \)

For supercritical flow, the stabilizing interaction by higher-order gradients between velocity and depth becomes dominant. The effect of second-order gradients in bed level can be considered negligible for lower values of \( s_b/q \).

In contrast with depth of flow, the higher-order contributions to velocity are not only are due to second-order gradients in depth and velocity, but second-order gradients in bed level have become equally important.

It can be concluded that for supercritical flow with \( Fr > 1.2. \), there is a direct interaction between bed level and velocity.
3.4. Second-order gradients in fixed bed models.

Application of an analogous procedure to the quasi-fixed bed model yields

\[
\begin{bmatrix}
\Delta a \\
\Delta u \\
\Delta z_b
\end{bmatrix} = \Delta t \begin{bmatrix}
\begin{array}{c}
-u \\
0 \\
\end{array}
\end{bmatrix} \frac{\partial a}{\partial x} + \frac{\Delta t \Delta x}{2} \begin{bmatrix}
\begin{array}{c}
-u \delta_{a1}' \\
-g_c \delta_{a2}' \\
-u \delta_{a3}'
\end{array}
\end{bmatrix} \frac{\partial^2 a}{\partial x^2} +
\]

\[
+ \Delta t \begin{bmatrix}
\begin{array}{c}
-a \\
u \\
-a \psi
\end{array}
\end{bmatrix} \frac{\partial u}{\partial x} + \frac{\Delta t \Delta x}{2} \begin{bmatrix}
\begin{array}{c}
a \delta_{z1}' \\
u \delta_{z2}' \\
-a \psi \delta_{z3}'
\end{array}
\end{bmatrix} \frac{\partial u}{\partial x^2} +
\]

\[
+ \Delta t \begin{bmatrix}
\begin{array}{c}
g_c \\
0 \\
0
\end{array}
\end{bmatrix} \frac{\partial z_b}{\partial x} + \frac{\Delta t \Delta x}{2} \begin{bmatrix}
\begin{array}{c}
\delta_{z1}' \\
\delta_{z2}' \\
\delta_{z3}'
\end{array}
\end{bmatrix} \frac{\partial^2 z_b}{\partial x^2} + \Delta t \begin{bmatrix}
\begin{array}{c}
S_1 \\
S_2 - u S_1 \\
S_3
\end{array}
\end{bmatrix}
\]

Due to a change of sign in one of the fixed bed characteristics, higher-order effects in sub- and supercritical flows are different. For subcritical flows it can be found

\[
\begin{bmatrix}
\delta_{a1}' \\
\delta_{a2}' \\
\delta_{a3}'
\end{bmatrix} = \begin{bmatrix}
\begin{array}{c}
-1 \\
-1 \\
0
\end{array}
\end{bmatrix} ; \begin{bmatrix}
\delta_{u1}' \\
\delta_{u2}' \\
\delta_{u3}'
\end{bmatrix} = \begin{bmatrix}
\begin{array}{c}
-Fr \\
-1 \\
1
\end{array}
\end{bmatrix} ; \begin{bmatrix}
\delta_{z1}' \\
\delta_{z2}' \\
\delta_{z3}'
\end{bmatrix} = \begin{bmatrix}
\begin{array}{c}
1 \\
0 \\
0
\end{array}
\end{bmatrix}
\]

For supercritical flows it can be found

\[
\begin{bmatrix}
\delta_{a1}' \\
\delta_{a2}' \\
\delta_{a3}'
\end{bmatrix} = \begin{bmatrix}
\begin{array}{c}
-1 \\
-1 \\
0
\end{array}
\end{bmatrix} ; \begin{bmatrix}
\delta_{u1}' \\
\delta_{u2}' \\
\delta_{u3}'
\end{bmatrix} = \begin{bmatrix}
\begin{array}{c}
-1 \\
-1 \\
1
\end{array}
\end{bmatrix} ; \begin{bmatrix}
\delta_{z1}' \\
\delta_{z2}' \\
\delta_{z3}'
\end{bmatrix} = \begin{bmatrix}
\begin{array}{c}
0 \\
-1 \\
0
\end{array}
\end{bmatrix}
\]

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Comparison with the complete model illustrates the effect of the quasi-fixed bed assumption. The error in the solution of depth and velocity in the quasi-fixed bed model changes with flow regime. For subcritical flows \((Fr < 1)\), this error can be written as

\[
\begin{bmatrix}
\Delta a \\
\Delta u
\end{bmatrix} = \frac{\Delta t \Delta x}{2} \begin{bmatrix}
-u(\delta_{a1} + \frac{1}{Fr}) & -a(\delta_{u1} + Fr) & u \left( \delta_{z1} - \frac{1}{Fr} \right) \\
g_z(\delta_{a2} + Fr) & -u(\delta_{u2} + \frac{1}{Fr}) & g_z \delta_{z2}
\end{bmatrix} \begin{bmatrix}
\frac{\partial^2 a}{\partial x^2} \\
\frac{\partial^2 u}{\partial x^2} \\
\frac{\partial^2 z_b}{\partial x^2}
\end{bmatrix}
\] (19)

For supercritical flows, the error in the quasi-fixed bed approach is

\[
\begin{bmatrix}
\Delta a \\
\Delta u
\end{bmatrix} = \frac{\Delta t \Delta x}{2} \begin{bmatrix}
-u(\delta_{a1} + 1) & -a(\delta_{u1} + 1) & u \delta_{z1} \\
g_z(\delta_{a2} + 1) & -u(\delta_{u2} + 1) & g_z(\delta_{z2} + 1)
\end{bmatrix} \begin{bmatrix}
\frac{\partial^2 a}{\partial x^2} \\
\frac{\partial^2 u}{\partial x^2} \\
\frac{\partial^2 z_b}{\partial x^2}
\end{bmatrix}
\] (20)

The resulting relative difference in \(\delta\) values of the complete and quasi-fixed bed model is of the order of \(s/\eta\) or \(s\) (Fig.3.3)
With the sign of the error, effects on the solution of depth and velocity can be summarized.

**Depth.**

Omitting the coupling between hydraulics and morphology implies introduction of a diffusion term in case of second-order gradients in water depth and a destabilizing "negative diffusion" term due to second-order gradients in velocity.

**Velocity.**

With respect to velocity, decoupling causes a diffusive effect in case of positive gradients in water depth and a non-diffusive or destabilizing term by second-order gradients in velocity.

In subcritical flows, positive second-order gradients in bed level diffuse or stabilize changes in water depth and velocity, whereas in supercritical flows, second-order gradients destabilize changes in water depth and velocity.
3.5. Effect of linearization.

In the previous sections, the solution of compatibility equations are assumed to be explicit by using constant, linearized coefficients. In this section, the effect of this linearization is analyzed. To solve the implicit solution, a numerical code is applied based on the Newton-Raphson iteration procedure (e.g., Press et al. 1992).

Numerically, differences in complete and quasi-fixed bed models are constructed as a response to second-order gradients in the initial conditions. In Figures 3.4-a, 3.5-a and 3.6-a, the relative error in bed level is large due to small scaling values. To obtain a better scale, differences in bed level for the complete and decoupled model are scaled with the depth of flow in Figures 3.4-b, 3.5-b and 3.6-b.

If all variables have an equal second-order gradient, the relative difference of the two models is
If second-order effects in bed level are absent, these errors are

![Fig.3.5-a](image)

![Fig.3.5-b](image)

If second-order effects in depth and velocity are absent, this error is

![Fig.3.6-a](image)

![Fig.3.6-b](image)

Comparison with Fig.3.3 enables concluding that the effect of linearizing the compatibility equations is rather small.

### 3.6. Scaling of decoupling effect.

In previous sections, time and length scales have been adjusted to characteristics. However, with respect to morphological changes due to flooding, larger scales
should be considered. Therefore, the significance of the hydraulic-morphological coupling is evaluated with respect to other higher-order terms in the mathematical model.

In the following analysis, the system of three first-order PDE's (Section 1.4) is combined into one (linearized) PDE of third-order (Appendix A).

To illustrate the interaction of changing bed and hydraulic variables, the flow-mass and momentum equation are written as

\[
\frac{\partial a}{\partial t} + u \frac{\partial a}{\partial x} + a \frac{\partial u}{\partial x} = 0
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - g_z \frac{\partial a}{\partial x} = S_2'
\]  \tag{21}

\[
S_2' = g_z \frac{\partial z_b}{\partial x} + S_2
\]

Elimination of \(a\) yields

\[
\frac{\partial^2 u}{\partial t^2} + 2u \frac{\partial^2 u}{\partial x \partial t} + g_z a (1 - Fr^2) \frac{\partial^2 u}{\partial x^2} = \frac{\partial S_2'}{\partial t} + u \frac{\partial S_2'}{\partial x}  \tag{22}
\]

Gradients in the second-order term \(S_2\) can be written as

\[
\frac{\partial S_2'}{\partial t} + u \frac{\partial S_2'}{\partial x} = \frac{2g_z u \frac{\partial u}{\partial t}}{C^2a} + \frac{3g_z u \frac{\partial u}{\partial x}}{C^2a} + g_z \frac{\partial^2 z_b}{\partial x \partial t} + g_z \frac{\partial^2 z_b}{\partial x^2} \tag{23}
\]

It is noted that at the right-hand side, time changes of bed level are introduced into the mass balance and momentum equation for the flow.

If changes in bed level with respect to flow are allowed, substitution, and elimination of velocity gradients with the help of the sediment-mass balance yields

\[
\frac{\partial z_b}{\partial t} + 2u \frac{\partial z_b}{\partial x \partial t} + g_z a (1 + \psi - Fr^2) \frac{\partial^2 z_b}{\partial x^2 \partial t} + g_z a u \psi \frac{\partial z_b}{\partial x^3} - 2 \frac{g_z u}{C^2a} \frac{\partial z_b}{\partial x \partial t} - \frac{3g_z u^2}{C^2a} \frac{\partial^2 z_b}{\partial x^2} = 0  \tag{24}
\]
which is identical to the PDE derived by Barneveld (1988). An identical
equation results for velocity and depth. If spatial gradients in bed level in Eq.23
are neglected, the coupling between flow variables and bed level disappears;
flow-mass balance and flow-momentum equation equal that of fixed bed models.

If, however, bed levels are assumed to be fixed in the momentum equation, the
mass and momentum balances for the flow equal that of a fixed bed model. Again, elimination of velocity results in

$$\frac{\partial^2 z_b}{\partial t^3} + 2u \frac{\partial^3 z_b}{\partial x \partial t^2} + g_z a (1 - Fr^2) \frac{\partial^3 z_b}{\partial x^3 \partial t} + g_z a w \psi \frac{\partial^3 z_b}{\partial x^3} - 2 \frac{g_z u}{C^2 a} \frac{\partial^2 z_b}{\partial t^2} - 3 \frac{g_z u^2}{C^2 a} \frac{\partial^2 z_b}{\partial x \partial t} = 0 \tag{25}$$

Therefore, neglecting the interaction between morphology and hydraulics implies neglecting

$$g_z a \psi \frac{\partial^3 z_b}{\partial x^2 \partial t} = 0 \tag{26}$$

or, using the transformation in Section 1.5, the difference in non-dimensional
format is

$$\frac{a}{T} \frac{\psi}{Fr^2} \frac{1}{T^2} \frac{\partial^3 z_b'}{\partial x^{'}^2 \partial t^{'}} \tag{27}$$

When analyzing characteristics, only first-order effects are considered. This
would lead to the conclusion that this non-linear interaction can be neglected if

$$\psi \ll 1 - Fr^2 \tag{28}$$

This explains the deviation of quasi-fixed bed characteristics for $0.8 < Fr < 1.2$
(Figure C.1).

Satisfying this criterion justifies decoupling even for waves with infinitely short
wave lengths such as discontinuous solutions (e.g. Sieben, 1995).
However, this criterion may be too strict when considering waves with longer wave-lengths (larger values of $E$). If analysis is of different harmonic components is performed (Appendix A), non-linear interaction of flow and morphology can be neglected if

$$\frac{a}{T} \frac{\psi}{Fr^2} \frac{1}{T^2} \frac{\partial^3 z_b'}{\partial x'^2 \partial t'} = 0 \quad (29)$$

with respect to the other terms in the third-order PDE. This hypothesis will be analyzed in Chapter four.
Chapter Four.

Comparison of models.

4.1. Introduction.

In Chapter two, different models have been reviewed and derived. In Chapter three, one of the basic assumptions concerning the interaction between hydraulics and morphology has been analysed. In Appendix A, the analysis of harmonic solutions is described. In this chapter, this analysis is applied to compare solutions of the different models.

The different models reviewed here are

- complete model (reference solution)
- decoupled model (with quasi-fixed bed)
- quasi-steady model (hyperbolic model)
- quasi-steady and quasi-uniform model (parabolic model)
- quasi-steady simple-wave model (no friction)
- quasi-uniform model (telegraph model)
- critical flow model (wave model)

It is noted that zero-order components of flow-variables are assumed uniform and steady (Section 1.5). Hence, conclusions in this chapter refer to first-order perturbations of solutions of the linearized model.

The wave-type solutions of different models are characterized by a propagation rate and attenuation length. These parameters are compared with the complete model in graphs, for different values of $Fr$ and $E$. Values of $i$ have been computed with the help of a representative power law as sediment-transport predictor.
The three different waves that develop due to an initial disturbance at \( t = t_0 \) are identified as in Appendix A (Figure 4.1).

Fig.4.1

4.2. Review of models.

The linearized versions of the different models are summarized below. With the help of the transformation described in Section 1.5, variables are made dimensionless.

complete model

\[
\frac{\partial^3 z_b}{\partial t^3} + 2 \frac{\partial^2 z_b}{\partial x \partial t^2} + \frac{(Fr^2-1-\psi)}{Fr^2} \frac{\partial^2 z_b}{\partial x^2 \partial t} - \psi \frac{\partial^3 z_b}{\partial x^3} + \frac{E}{Fr^2} \frac{\partial^2 z_b}{\partial t^2} + \frac{3}{2} \frac{E}{Fr^2} \frac{\partial^2 z_b}{\partial x \partial t} = 0
\]  

(1)

decoupled model

\[
\frac{\partial^3 z_b}{\partial t^3} + 2 \frac{\partial^2 z_b}{\partial x \partial t^2} + \frac{(Fr^2-1)}{Fr^2} \frac{\partial^2 z_b}{\partial x^2 \partial t} - \psi \frac{\partial^3 z_b}{\partial x^3} + \frac{E}{Fr^2} \frac{\partial^2 z_b}{\partial t^2} + \frac{3}{2} \frac{E}{Fr^2} \frac{\partial^2 z_b}{\partial x \partial t} = 0
\]  

(2)

quasi-steady hyperbolic model (non-uniform and with friction)

\[
\frac{3}{2} \frac{E}{Fr^2} \frac{\partial z_b}{\partial t} - \frac{1-Fr^2}{Fr^2} \frac{\partial^2 z_b}{\partial t \partial x} - \frac{\psi}{Fr^2} \frac{\partial^2 z_b}{\partial x^2} = 0
\]  

(3)

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quasi-steady parabolic model (quasi-uniform)

\[ \frac{\partial z_b}{\partial t} - \frac{2}{3} \frac{1}{E} \frac{\psi}{Fr^2} \frac{\partial^2 z_b}{\partial x^2} = 0 \]  

(4)

quasi-steady simple-wave model (no friction)

\[ \frac{\partial z_b}{\partial t} + \frac{\psi}{1-Fr^2} \frac{\partial z_b}{\partial x} = 0 \]  

(5)

quasi-uniform d’Alembert or wave model (constant friction)

\[ \frac{\partial^2 z_b}{\partial t^2} - \frac{\psi}{Fr^2} \frac{\partial^2 z_b}{\partial x^2} = 0 \]  

(6)

quasi-uniform telegraph model

\[ \frac{3}{2} \frac{E}{\partial t} \frac{\partial z_b}{\partial t} + \frac{\partial^3 z_b}{\partial t^3} - \frac{\psi}{Fr^2} \frac{\partial^2 z_b}{\partial x^2} = 0 \]  

(7)

critical flow model (no friction)

\[ \frac{\partial}{\partial t} \left( \frac{\partial^2 z_b}{\partial t^2} - \frac{\psi}{2Fr^2} \frac{\partial^2 z_b}{\partial x^2} \right) + 2 \frac{\partial}{\partial x} \left( \frac{\partial^2 z_b}{\partial t^2} - \frac{\psi}{2Fr^2} \frac{\partial^2 z_b}{\partial x^2} \right) = 0 \]  

(8)

critical flow (with friction)

\[ \frac{\partial^3 z_b}{\partial t^3} - \frac{\psi}{2Fr^2} \frac{\partial^3 z_b}{\partial x^2} - \frac{\psi}{Fr^2} \frac{\partial^3 z_b}{\partial x^3} + 2 \frac{\partial^2 z_b}{\partial t^2 \partial x} + \]

\[ + E \left( 1 - \frac{\psi}{4} \frac{Fr^2}{2} \right) \frac{\partial^2 z_b}{\partial t \partial x} + E \frac{\partial^2 z_b}{\partial t^2} = 0 \]

(9)
4.3. Conditions for non-trivial solutions.

Substitution of a sinusoidal solution enables derivation of wave-like solutions with corresponding conditions for non-triviality.

complete model

\[ k^3 + \frac{\omega (1+\psi-Fr^2)}{\psi} k^2 + \frac{Fr^2 \omega (3iE-4\omega)}{2\psi} k + \frac{\omega^2 Fr^2 (iE-\omega)}{\psi} = 0 \] (10)

decoupled model

\[ k^3 + \frac{\omega (1-Fr^2)}{\psi} k^2 + \frac{Fr^2 \omega (3iE-4\omega)}{2\psi} k + \frac{\omega^2 Fr^2 (iE-\omega)}{\psi} = 0 \] (11)

quasi-steady hyperbolic model (non-uniform and friction)

\[ k^2 + \frac{(1-Fr^2)}{\psi} k + \frac{3}{2} i\omega E \frac{Fr^2}{\psi} = 0 \] (12)

quasi-steady parabolic model (quasi-uniform)

\[ k^2 + \frac{3}{2} i\omega E \frac{Fr^2}{\psi} = 0 \] (13)

quasi-steady simple wave model (no friction)

\[ k + \frac{\omega (1-Fr^2)}{\psi} = 0 \] (14)

quasi-uniform d'Alembert or wave model (constant friction)

\[ k^2 - \omega^2 \frac{Fr^2}{\psi} = 0 \] (15)
quasi-uniform telegraph model

\[ k^2 + \frac{\omega Fr^2}{\psi} \left( \frac{3}{2} iE - \omega \right) = 0 \]  

(16)

Comparison with Eq.13 illustrates similar behaviour of the telegraph and parabolic model for large values of \(E\).

critical flow

\[ k^3 + \frac{\omega k^2}{2} - 2\omega^2 \frac{Fr^2}{\psi} k - \omega^3 \frac{Fr^2}{\psi} = 0 \]  

(17)

critical-flow with friction

\[ k^3 + \frac{\omega k^2}{2} - 2\omega^2 \frac{Fr^2}{\psi} k + i \frac{EFr^2}{\psi} \left( 1 - \frac{\psi}{4} + \frac{Fr^2}{2} \right) k\omega + Ei\omega^2 \frac{Fr^2}{\psi} - \omega^3 \frac{Fr^2}{\psi} = 0 \]  

(18)

Some convenient solutions of lower-order equations can be compared easily.

quasi-steady parabolic model

\[ k_{1,2} = \pm \frac{Fr}{2} \sqrt{\frac{3\omega E}{\psi}} ; \quad c_w = \pm \frac{2}{Fr} \sqrt{\frac{\omega \psi}{3E}} \]  

(19)

This reveals the diffusive character; equal propagation and attenuation in up- and downstream directions.

simple wave model

\[ k_1 = -\frac{\omega (1-Fr^2)}{\psi} ; \quad k_2 = 0 ; \quad c_w = \frac{\psi}{1-Fr^2} \]  

(20)
quasi-uniform wave model

\[ k_1 = \pm \frac{\omega Fr}{\sqrt{\psi}} \quad ; \quad k_2 = 0 \quad ; \quad c_w = \pm \frac{\sqrt{\psi}}{Fr} \]  \hspace{1cm} (21)

critical flow

\[ \left( k_1 = -\frac{\omega}{2} \quad \vee \quad k_1 = \pm \frac{\omega Fr}{\sqrt{\psi/2}} \right) \quad \wedge \quad k_2 = 0 \]
\[ - \quad c_w = 2 \quad \vee \quad c_w = \mp \frac{\sqrt{\psi/2}}{Fr} \]  \hspace{1cm} (22)

quasi-uniform telegraph model

\[ k_1 = \pm \frac{\omega Fr}{\sqrt{2 \psi}} \left( \sqrt{1 + \frac{4}{9} \frac{E^2 \psi^2}{Fr^4 \omega^2} + 1} \right)^{1/2} \]  \quad ; \quad k_2 = - \frac{\omega E}{3k_1} \]  \hspace{1cm} (23)

Corresponding propagation rates are

\[ c_w = \pm \frac{\sqrt{2 \psi}}{Fr} \left( \sqrt{1 + \frac{4}{9} \frac{E^2 \psi^2}{Fr^4 \omega^2} + 1} \right)^{-1/2} \]  \quad ; \quad k_2 = - \frac{\omega E}{3k_1} \]  \hspace{1cm} (24)

4.4. Quasi-fixed bed model.

In decoupled models, neglected time changes in bed-level gradients results in an error (Section 3.7). The relative magnitude of this error in propagation rate and attenuation length is constructed in Figures 4.2 and 4.3. If propagation rates are compared (Figure 4.2) it can be concluded that the upstream-propagating wave I is affected most by the decoupling procedure. The relative error will be of the order of \( \psi \). In decoupled (or quasi-fixed bed) models, propagation rates of wave II are increased and propagation rates of waves I and III are increased.
For very short waves, a maximum error is found for values of $Fr$ near unity. This confirms the results from the analysis in Chapter three (Sections 3.5 and 3.6). For $E = 0$, the relative error is constructed in broken lines. With increasing values of $E$, the largest error can be found at lower values of $Fr$.

It is noted that the error in the fast, downstream propagating wave III is negligible compared to that in the upstream-propagating wave I. This implies that flood waves in rivers with a relatively uniform geometry can be described with decoupled or quasi-fixed bed models.
With respect to attenuation lengths of waves I and II, it is noted that the sign of the relative error changes from positive to negative (Figure 4.3).

Consequently, for low values of \( Fr \) (subcritical flow), decoupling implies the introduction of a diffusion term, whereas for larger values of \( Fr \) (supercritical flow), this changes into a (destabilizing) "negative-diffusion" term (Section 3.5).

At values of \( Fr \to 2 \), the dissipation for the fast, downstream-propagating wave III diminishes (see also Figure A.2). This effect has been found by Ponce et al. (1978) as well and indicates an unstable wave solution (e.g., roll waves).

4.5. Quasi-steady flow model.

Application of the quasi-steady flow assumption enables derivation of the hyperbolic model. As has been reviewed in Chapter Two, this hyperbolic model changes into the simple-wave (convection) model for short waves with negligible friction loss \( (E = 0) \), and into a diffusive parabolic model for long waves where friction is dominant \( (E \to \infty) \).

Therefore, solutions of the hyperbolic model are constructed for small values of \( E \) (short waves) (Figure 4.4) and large values of \( E \) (long waves) (Figure 4.5).

![Graph](image)

**Fig.4.4**

It can be observed in Figure 4.4 that for low values of \( E \), the simple-wave model breaks down for transcritical flows.

It appears that the performance of the quasi-steady flow approach is related to
the type of wave. For low values of $Fr$ ($Fr < 0.8$), the downstream-propagating wave II is represented well, whereas for higher values of $Fr$ ($Fr > 1.2$) the upstream-propagating wave is represented well.

However, for large values of $E$, the solution of the hyperbolic model is a good approximation of the exact solution (Figure 4.5).

![Graph showing comparison between exact and q-s hyperbolic solutions](image)

**Fig.4.5**

The relation of error and wave type in quasi-steady flow models can be illustrated qualitatively by Figure 4.6. Arrows indicate the conditions for applicability of the quasi-steady model with respect to the wave types I and II.

![Diagram showing wave types I and II](image)

**Fig.4.6**
For low values of \( Fr \), the upstream-propagating wave I represents a change in hydraulic variables mainly. Hence, the quasi-steady error is large. This error diminishes for supercritical flows. The opposite can be observed with respect to downstream-propagating waves. The error in wave II is small for low values of \( Fr \), and increases for supercritical flows. Both waves are equally represented for transcritical flow.

As can be concluded, for linearized models, the assumption of quasi-steady flow is valid for wave I and II for larger values of \( E \), irrespective of flow regime.

To indicate the magnitude of the error in the hyperbolic model, the relative differences in propagation rate and attenuation length are presented in Figures 4.7-a and 4.7-b.

![Fig.4.7-a](image1)
![Fig.4.7-b](image2)

The general trend is a decreasing error with larger values of \( E \), conform the study of Barneveld (1988).

Based on the sign of the error in Figure 4.7-b, it can be concluded that quasi-steady flow concepts introduces a diffusive term in the solution.

**4.6. Quasi-uniform flow model.**

The telegraph model is analysed in Figure 4.8. It resembles the parabolic model (Figure 4.11) for large values of \( E \). Hence, for large values of \( E \), the differences in derivation seem to have no significant effect on the solution.
Relative errors in propagation rate and attenuation length are presented in Figures 4.9-a and 4.9-b. It is noted that propagation rates equal the exact values at critical flow for larger values of $E$. 
For small values of $E$, the telegraph model can be reduced to the wave equation Eq.6. As has been concluded in Section 2.3, propagation rates are overpredicted with a factor $\sqrt{2}$.

![Graph showing the relationship between $|c/u|$ and $Fr$.]

**Fig.4.10**

### 4.7. Quasi-steady and quasi-uniform flow model.

Because values of $E$ tend to be large for typical river-flood waves (Section 1.6), parabolic-type models are often applied. The parabolic model has two propagation rates with opposite sign and equal absolute magnitude.
Differences of hyperbolic and parabolic models can be found by comparison of Figures 4.5 and 4.11.

It is noted that propagation rates of the parabolic model correspond with the average of the absolute values of the exact model. Since with increasing values of $E$ these exact values approach, the approximation of the parabolic model becomes better. At critical flow, propagation rates of the parabolic model equal the exact values.

The relative differences in propagation rate and attenuation length are presented in Figures 4.12-a and 4.12-b.

The error increases for low values of $E$ and $Fr < 0.8$ and $Fr > 1.2$. 
Comparison of the relative errors in the telegraph model (Figs 4.9-a and -b) and parabolic model (Figs 4.12-a and -b) enables concluding that the quasi-uniform flow model performs slightly better.


In Figure 4.13, propagation rates of the critical-flow model (Eq.8 and 9) are constructed.

![Graph](image)

**Fig.4.13**

This model equals the exact solution for values of $Fr$ near unity ($Fr = (1+\psi/2)^{1/2}$). The wave equation, that results from critical-flow conditions can therefore be considered supplementary to the common quasi-steady models.
Relative differences in propagation rate are

![Graph showing relative differences in propagation rate.](image)

Fig. 4.14
Chapter five.

Water-sediment mixtures.

5.1. Introduction.

As reported by Holly and Rahuel (1990), Correia et al. (1992), Sloff (1993a) and others, the formulation of mathematical models such as Eqs 2 and 3 in Chapter one can be extended with respect to flow-sediment interactions. One of the restrictions of the models in the previous chapters is a low concentration of sediment in transport. Changes in flow mass due to changes in suspended load are considered to be negligible. However, although small, these effects do affect the coupling between hydraulics and morphology.

In this chapter, second-order interactions of flow and sediment are taken into account. The objective is not the analysis of sediment-laden flow, but the analysis of effects of a more detailed description of mass conservation and momentum equation at higher values of $Fr$. Therefore, again mass and momentum balances are analysed but now for fluid mixtures of water and sediment. In this report, some details will deviate from the extensive derivation that can be found in Sloff (1993a).

Lateral exchanges of flow and sediment are neglected. Rheological changes other than density differences are not taken into account.

5.2. Mathematical model.

The density of a water-sediment mixture is defined as

$$\rho_m = \rho (1 + \Delta \bar{c})$$  \hspace{1cm} (1)

with $\bar{c}$ the total sediment concentration averaged over the depth.

The mass balance of a water-sediment mixture is

$$\frac{\partial \rho_m a}{\partial t} + \frac{\partial \rho_m u a}{\partial x} - \rho_s \Psi_s - \rho \Psi_w = 0$$  \hspace{1cm} (2)

with $\Psi_s$ and $\Psi_w$ volumetric fluxes of sediment and water from river bed to flow.
The sediment-mass balance of the river bed can be written as

$$\frac{\partial z}{\partial t} \rho (1+\Delta c_0) + \rho \frac{\partial z}{\partial t} \rho \Psi_s + \rho \Psi_w = 0$$  \hspace{1cm} (3)

Changes in density of the river bed are neglected. The sediment fraction in the bed is defined as $c_0 = 1-p$ ($p$ is porosity and approximately 0.4).

With the sediment-mass balance Eq.3, fluxes $\Psi_s$ and $\Psi_w$ can be eliminated from the mixture-mass balance Eq.2 yielding

$$\rho \frac{\partial a}{\partial t} + \rho \frac{\partial u a}{\partial x} + \rho \Delta \frac{\partial ac}{\partial t} + \rho \Delta \frac{\partial uac}{\partial x} + \rho (1+\Delta c_0) \frac{\partial z}{\partial t} = 0$$  \hspace{1cm} (4)

The fraction of water in the total exchange with the bed is defined as

$$\Psi_w = (1-\beta_c) (\Psi_s + \Psi_w) \rightarrow \Psi_w = \frac{1-\beta_c}{\beta_c} \Psi_s$$  \hspace{1cm} (5)

With respect to density, differences can be distinguished in upward and downward fluxes. In case of erosion it can be stated $\beta_c = c_0$. In case of deposition different options exist. Deposition without infiltration implies $\beta_c = 1$.

If, however, "frozen" packages of sediment-water mixture are deposited at the bed, $\beta_c = c$ and the concentration of the mixture will not change.

However, because the density of the bed is assumed constant, it is consistent to use $\beta_c = c_0$.

The depth-integrated sediment-mass balance is

$$\frac{\partial (c_b \delta + c a)}{\partial t} + \frac{\partial (s_b + c u a)}{\partial x} + \beta_c \left( \frac{1+\Delta c_0}{1+\Delta \beta_c} \right) \frac{\partial z}{\partial t} = 0$$  \hspace{1cm} (6)

where $s_b$ is the bed-load transport and $c$ is the relative sediment volume in suspension. For the bed-load rate, instantaneous adaption to local conditions is assumed.

Now, gradients in sediment load can be eliminated from the mixture-mass balance (Eq.4). The result is

$$\frac{\partial a}{\partial t} + \frac{\partial u a}{\partial x} + \left( \frac{1+\Delta c_0}{1+\Delta \beta_c} \right) \frac{\partial z}{\partial t} = 0$$  \hspace{1cm} (7)
The momentum balance of the mixture can be written as

\[
\frac{\partial \rho_m u a}{\partial t} + \frac{\partial \rho_m u^2 a}{\partial x} - \frac{g_z \partial \rho_m a^2}{2 \partial x} - \rho_m g_x a - \rho_m g_z a \frac{\partial z_b}{\partial x} +
\]

\[+ \tau_b - (\rho_s \Psi_s + \rho \Psi_w) u_z = 0 \quad (8) \]

Vertical flux-rates of horizontal momentum by sediment to the flow is represented by a sediment-velocity \( u_z \).

In combination with the mixture-mass balance, the momentum equation can be reduced to

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - g_z \frac{\partial a}{\partial x} - \frac{g_z \Delta a}{2(1+\Delta c)} \frac{\partial c}{\partial x} - \frac{g_z \partial z_b}{\partial x} +
\]

\[+ \frac{\Delta u}{1+\Delta c} \left( \frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} \right) + (1+\Delta c_0) \left( \frac{u \partial a}{1+\Delta c_0} - \frac{u |a|}{1+\Delta \beta_c} \right) \frac{\partial z_b}{\partial t} = g_x - \frac{\tau_b}{\rho_m a} \quad (9) \]

If, for simplicity, the contribution of bed load to density differences is neglected, combination of Eq.7 and 6 yields

\[
\frac{\partial s_b}{\partial x} + a \frac{\partial c}{\partial t} + u a \frac{\partial c}{\partial x} + (\beta_c - c) \left( \frac{1}{1+\Delta \beta_c} \right) \frac{\partial z_b}{\partial t} = 0 \quad (10) \]

Then, the momentum equation can be rewritten with the help of the sediment-mass balance Eq.10

\[
\frac{\partial u}{\partial t} + u \left(1 - \frac{\Delta \psi}{1+\Delta c}\right) \frac{\partial u}{\partial x} - g_z \frac{\partial a}{\partial x} - \frac{g_z \Delta a}{2(1+\Delta c)} \frac{\partial c}{\partial x} - \frac{g_z \partial z_b}{\partial x} +
\]

\[+ \frac{(1+\Delta c_0)}{a} \left( \frac{u \partial a}{1+\Delta c} - \frac{u (1+\Delta (\beta_c - c))}{1+\Delta \beta_c} \right) \frac{\partial z_b}{\partial t} = g_x - \frac{\tau_b}{\rho_m a} \quad (11) \]

In the following, momentum contributions due to moving sediments are neglected \((u_z = 0 \text{ m/s})\).
The approximate, first-order solution of the cross-sectionally integrated balance of suspended sediment is

\[
T_A \frac{\partial c}{\partial t} + L_A \frac{\partial c}{\partial x} + (c - c_e) = 0 \tag{12}
\]

where \( c_e \) is the suspended-load transport-capacity, \( T_A \) and \( L_A \) are the adaption period and length respectively (e.g. Galappatti 1983, Galappatti and Vreugdenhil, 1985).

### 5.3. Characteristic equations.

It is noted that the coupling of changes in flow and bed level has changed due to the presence of additional terms in flow mass and momentum balance. This can be illustrated by analysing the characteristic equations. The celerities are the roots of

\[
\left( \frac{n_t}{u} \right)^3 + \left[ 1 + \zeta - \frac{\psi \xi}{u(\beta_c - c)} \right] \left( \frac{n_t}{u} \right)^2 +
\]

\[
+ \left[ \zeta - \frac{\psi \xi}{u(\beta_c - c)} - \frac{1}{Fr^2} \left( 1 + \frac{\psi(1 - \nu)}{u(\beta_c - c)} \right) \right] \left( \frac{n_t}{u} \right) - \frac{\psi}{Fr^2 u(\beta_c - c)} = 0 \tag{13}
\]

\[
\forall \quad n_t = -\frac{L_A}{T_A}
\]

with

\[
\nu = \left( \frac{1 + \Delta c_0}{1 + \Delta \beta_c} \right) \quad \xi = -\left( 1 + \Delta c_0 \right) \left( 1 - \frac{\Delta c}{1 + \Delta \beta_c} \right) \quad \zeta = 1 - \frac{\Delta \psi}{1 + \Delta c} \tag{14}
\]
If it is used $\beta_c = c_0$ for depositing and eroding beds, the characteristic equation is

$$\left( \frac{n_t}{u} \right)^3 + \left[ 1 + \zeta + \frac{\psi(1+\Delta(c_0-c))}{(c_0-c)} \right] \left( \frac{n_t}{u} \right)^2 +$$

$$+ \left[ \zeta + \frac{\psi(1+\Delta(c_0-c)) - 1}{Fr^2(c_0-c)} \right] \frac{n_t}{u} - \frac{\psi}{Fr^2(c_0-c)} = 0 \quad \forall \quad n_t = \frac{L_A}{T_A}$$

(15)

If, for simplicity, the adaption period and length of the suspended load is neglected, both bed load and suspended load can be combined into a total load description. Then, the momentum equation can be written as

$$\frac{\partial u}{\partial t} + u \left( 1 + \frac{\Delta(\psi-c)}{2Fr^2(1+\Delta c)} \right) \frac{\partial u}{\partial x} - g_z \left( \frac{2+\Delta c}{1+\Delta c} \right) \frac{\partial a}{\partial x} - g_z \frac{\partial z_b}{\partial x} +$$

$$+ \frac{1+\Delta c_0}{a} \left( \frac{u_z}{1+\Delta c} - \frac{u(1+\Delta(\beta_c-c))}{1+\Delta \beta_c} \right) \frac{\partial z_b}{\partial t} = g_x - \frac{\tau_b}{\rho_m a}$$

(16)

Now, the corresponding characteristic equation is

$$\left( \frac{n_t}{u} \right)^3 + \left[ 1 + \zeta + \frac{\psi(1+\Delta(c_0-c))}{(c_0-c)} \right] \left( \frac{n_t}{u} \right)^2 +$$

$$+ \left[ \zeta + \frac{\psi(1+\Delta(c_0-c)) - 1}{Fr^2(c_0-c)} \right] \frac{n_t}{u} - \frac{\psi}{Fr^2(c_0-c)} = 0$$

(17)

with

$$\zeta = 1 + \frac{\Delta(\psi-c)}{2Fr^2(1+\Delta c)}$$

(18)
In Figure 1, the effects of sediment load on the characteristics is shown relative to the "clear" water flow model (Chapter 1, Eq.7).

![Graph showing characteristics](image)

*Fig.5.1*

General trends correspond with Sloff (1993a); smaller celerities in upstream direction, larger celerities in downstream direction. It is noted that despite relatively low concentration of sediment load, the effect of sediment load on the flow increases significantly for larger values of Fr.

The corresponding compatibility equation can be formulated

\[ T_{ai} \frac{da}{d\tau_i} + T_{ui} \frac{du}{d\tau_i} + T_{zi} \frac{dz_b}{d\tau_i} = n_{zi} \left( g_z \frac{\tau_b}{\rho_m a} \right) \]  

with

\[ T_{ai} = \frac{g_z n_{ti}}{2} \left( \frac{2+\Delta c}{1+\Delta c} \right) ; \quad T_{ui} = n_{2i} = n_{b}(n_{ti}+u) ; \quad T_{zi} = g_z(n_{ti}+u) \]  

Hence, time-gradients in \( a \), \( u \) and \( z_b \) can be solved by

\[
\begin{bmatrix}
\frac{\partial a}{\partial t} \\
\frac{\partial u}{\partial t} \\
\frac{\partial z_b}{\partial t}
\end{bmatrix}
= \begin{bmatrix}
-u \\
g_z \alpha_a \\
0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial a}{\partial x} \\
\frac{\partial u}{\partial x} \\
\frac{\partial z_b}{\partial x}
\end{bmatrix}
+ \begin{bmatrix}
\frac{\partial a}{\partial x} \\
\frac{\partial u}{\partial x} \\
\frac{\partial z_b}{\partial x}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial a}{\partial t} \\
\frac{\partial u}{\partial t} \\
\frac{\partial z_b}{\partial t}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
S_z/a
\end{bmatrix}
\]  

(21)
The coefficients in Eq.21 can be compared with that of the "clear" water model (Chapter 3, Eq.6). Together with the characteristics of Figure 5.1, absolute, relative differences of coefficients in compatibility equations are constructed in Figure 5.2.

![Fig.5.2](image)

Obviously, the effect of sediment load increases with \(s/q\). It is noted that the error made in the solution of depth and velocity for values of \(Fr\) near unity is of the order 10 \(\psi\). Hence, despite relatively low values of \(s/q\) (Figure 5.1), the effect of sediment load can be considerable.

The corresponding similarity conditions are

\[
(n_{t1} + u)(n_{t2} + u)(n_{t3} + u) = u^3 \left( \frac{(1-\beta)^2}{4} - \frac{\alpha}{Fr^2} \right)
\]

\[
n_{t1} + n_{t2} + n_{t3} = - (1+\beta) \ u
\]

\[
n_{t1}n_{t2}n_{t3} = - g_\epsilon au \frac{\Psi}{c_0}
\]
For critical flow, these conditions can be reduced to

\[
\frac{n_3}{u} = -1 - \beta ; \quad \frac{n_{12}}{u} = \sqrt{1 + \frac{(1-\beta)^2}{4\beta}} - \frac{\alpha}{Fr^2}
\]

\[
Fr = \frac{\beta \psi/c_0 + (1+\beta)\alpha}{\sqrt{(1+\beta)[\beta + (1-\beta)^2/4]}}
\]

5.4. Harmonic solution.

To analyse waves with periods or lengths longer than that of "infinitely" short characteristic waves, an harmonic solution must be studied. Again, the set of three PDE's can be combined into one third-order PDE. Linearization and transformation into non-dimensional variables yields

\[
\frac{\partial^3 z_b}{\partial t^3} + \left[2 + \beta \right] \frac{\partial^3 z_b}{\partial x \partial t^2} + \left[1 + \beta \right] \frac{\alpha c_0 + \psi(1-\alpha)}{c_0 Fr^2} \frac{\partial^3 z_b}{\partial x^3 \partial t} - \frac{\psi}{Fr^2 c_0} \frac{\partial^3 z_b}{\partial x^3} + \frac{E}{2} \frac{\partial^2 z_b}{\partial t^2} + \frac{E}{2} \left[3 + \frac{\Psi}{c_0} \right] \frac{\partial^2 z_b}{\partial x \partial t} = 0
\]

(24)

with

\[
\alpha = \frac{1}{2} \left( \frac{2 + \Delta c}{1 + \Delta c} \right) ; \quad \beta = \frac{\Delta (\psi - c)}{2Fr^2(1+\Delta c)} + \left(1 + \Delta (c_0 - c)\right) \frac{\psi}{c_0}
\]

or, if sediment transport is predicted with a power-law \( s = mu^n \),

\[
\alpha = \frac{1}{2} \left( \frac{2 + \Delta c}{1 + \Delta c} \right) ; \quad \beta = c \left( \frac{\Delta (n-1)}{2Fr^2(1+\Delta c)} + n \left(1 + \Delta (c_0 - c)\right) \right) \frac{c_0}{c_0}
\]
Hence, for low values of \( c; \alpha \approx 1 \) and \( \beta \approx 0 \) and again, the third-order PDE of the "clear-water" model is obtained

\[
\frac{\partial^3 z_b}{\partial t^3} + 2 \frac{\partial^3 z_b}{\partial x \partial t^2} + \left[ 1 - \frac{(c_0 + \Psi)}{c_0 Fr^2} \right] \frac{\partial^2 z_b}{\partial x^2 \partial t} - \frac{\Psi}{c_0 Fr^2} \frac{\partial^3 z_b}{\partial x^3} + \frac{E}{\partial t^2} + \frac{3}{2} E \frac{\partial^2 z_b}{\partial x \partial t} = 0
\]

(27)

The corresponding characteristic equation is obtained after substitution of a harmonic solution. The response amplitudes of velocity and bed level to a perturbation in depth are

\[
\begin{bmatrix}
\Delta a / a \\
\Delta u / u \\
\Delta z / a
\end{bmatrix} = \frac{\Delta a}{a} \begin{bmatrix}
1 \\
\frac{k' + w'}{k'(\psi/c_0 - 1)} \\
\frac{\Psi}{c_0} \frac{k' + w'}{w'(1 - \psi/c_0)}
\end{bmatrix}
\]

(28)

Responses (amplitude and phase-shifts) are multiplied by \((1 - \psi/c_0)^{-1}\) (relative to the responses as derived in Appendix A).

In the following graphs, the propagation rates of waves with different periods are compared (Figures 5.3-a and 5.4-a). Relative differences are constructed in Figures 5.3-b and 5.4-b. In Figure 5.3 waves for low values of \( E \) are constructed, in Figure 5.4, waves for high values of \( E \) are constructed.

The value of \( \psi \) in Figures 5.3 and 5.4 is 0.01, \( s/q = 0.002 \). Hence, despite low values of sediment-transport rates, differences can be observed at higher values of \( Fr \).
Continuous lines represent the conventional "clear-water" model, without second-order effects of sediment load. Broken lines represent propagation rates of waves in the "water-sediment mixture" model.

![Graph](image)

**Fig. 5.3-a**

In correspondence with Figure 5.1, waves in downstream direction are faster, waves propagating in upstream direction are slower.

It is noted that in correspondence with the error due to decoupling (Chapter Three), the error is again large for waves I and II.

For low values of $E$ (short waves), the error is at maximum for values of $Fr$ near unity. For larger values of $E$ ($E \approx 100$), the maximum error in upstream-propagating waves I can be found at subcritical flows, the maximum error in the downstream-propagating wave II occurs at supercritical flows. The error reduces with increasing values of $E$ (or longer waves), as is shown in Figs 5.4.

![Graph](image)

**Fig. 5.4-a**

This does not seem to correspond to the numerical analysis by Correia et al. (1992), where the error in a wave type II increases with time due to additional terms.
5.5 Simplified models.

In this section, the assumptions of quasi-steady and uniform flow are applied to the model for water-sediment mixtures. Now, the complete mathematical model consists of

\[
\frac{\partial a}{\partial t} + \frac{\partial u a}{\partial x} + \frac{\partial z_b}{\partial t} = 0
\]  \hspace{1cm} (29)

\[
\frac{\partial u}{\partial t} + u \beta \frac{\partial u}{\partial x} - \alpha g_z \frac{\partial a}{\partial x} - g_z \frac{\partial z_b}{\partial x} - \frac{u}{\alpha} \frac{\partial z_b}{\partial t} = g_x + \frac{g_z u^2}{C^2 a}
\]  \hspace{1cm} (30)

\[
a \psi \frac{\partial u}{\partial x} + c_0 \frac{\partial z_b}{\partial t} = 0
\]  \hspace{1cm} (31)

with

\[
\alpha = \frac{1}{2} \left( \frac{2 + \Delta c}{1 + \Delta c} \right) \quad ; \quad \beta = 1 + \frac{\Delta (\psi - c)}{2 Fr^2 (1 + \Delta c)} \quad ; \quad \gamma = \frac{1}{2} \left( \frac{2 + \Delta c}{1 + \Delta c} \right)
\]

After transformation (Section 1.5) into non-dimensional form, the quasi-steady flow approach yields

\[
\frac{3E}{2} \left( 1 - \frac{\psi}{3 c_0} \right) \frac{\partial z}{\partial t} - \left( \beta + \gamma \frac{\psi}{c_0} + \frac{\alpha (\psi / c_0 - 1)}{Fr^2} \right) \frac{\partial^2 z_b}{\partial t \partial x} - \frac{\psi}{c_0 Fr^2} \frac{\partial^2 z_b}{\partial x^2} = 0
\]

which is again a hyperbolic model.

Neglecting friction yields

\[
\frac{\partial^2 z_b}{\partial x \partial t} - \left[ \frac{\psi / c_0}{Fr^2 (\beta + \psi / c_0) - \alpha (1 - \psi / c_0)} \right] \frac{\partial^2 z_b}{\partial x^2} = 0
\]

which is equivalent to the simple-wave model.
Applying the quasi-uniform approach, the result is

$$\frac{\partial}{\partial t} \left( E \frac{\partial z}{\partial t} - \frac{E \psi}{2} \frac{\partial z_b}{\partial x} + \frac{\partial^2 z_b}{\partial t^2} + \frac{\gamma \psi}{c_0} \frac{\partial^2 z_b}{\partial t \partial x} - \frac{\psi}{c_0 Fr^2} \frac{\partial^2 z_b}{\partial x^2} \right) = 0$$

If the quasi-steady and quasi-uniform assumptions are combined, the "clear"-water flow would yield the parabolic model. However, for the sediment-water mixture model, again a hyperbolic model is obtained.

$$\frac{3E}{2} \left( \frac{\psi}{3c_0} - 1 \right) \frac{\partial z}{\partial t} - \frac{\gamma \psi}{c_0} \frac{\partial^2 z_b}{\partial t \partial x} - \frac{\psi}{c_0 Fr^2} \frac{\partial^2 z_b}{\partial x^2} = 0$$
Chapter six.

Conclusions.

6.1. Introduction.

In this study, simplified formulations of mathematical models for river morphology have been analyzed for conditions prevailing in mountain rivers. The simplifications are based on distinction of typical time and length scales. These scales can be defined with the help of characteristics, or more generally with the help of harmonic solutions, and are related to the key parameters $E$ (flow steadiness / uniformity), $Fr$ (ratio of convection terms to gravity terms) and $\psi$ (bed mobility).

The use of analytical methods requires linearization of the nonlinear mathematical model. As a result, conclusions based on harmonic solutions refer to flow with zero-order uniform conditions. Application of conclusions to non-uniform conditions of mountain rivers can be erroneous.

To check the effects of linearization, numerical experiments must be carried out.

6.2. Definition of time scales.

Hydraulic and morphological time scales for values of $Fr < 0.8$ can be distinguished with the help of characteristics (De Vries, 1965). At higher values of $Fr$, however, the coupling between hydraulic and morphological changes has become more pronounced (Sieben, 1994), which prevents distinction of entirely hydraulic and entirely morphological phenomena.

Then, a time scale (valid for both hydraulic and morphological changes) can be found (Figure 2.2) with the help of harmonic solutions. The relation between time and length scale is a function of $Fr$, $\psi$ and $E$, and can be determined by analyzing these harmonic solutions.
6.3. Simplifications.

In Chapter two, some modelling concepts have been summarized. The simplifications reviewed are based on

- decoupling hydraulics from morphology (quasi-fixed bed models)

- quasi-steady flow

- quasi-uniform flow

- critical flow

In addition to the classical morphology-models based on the quasi-steady flow assumption (hyperbolic, simple-wave and parabolic model) a quasi-uniform, telegraph model is proposed. For large values of $E$, the parabolic model approaches this telegraph model. In general, the error in the parabolic model exceeds that in the telegraph model.

It is noted that the classical quasi-steady flow concepts are actually based on a combination of the quasi-fixed bed and the quasi-steady flow assumption. Changes in flow are assumed to adapt instantaneously relative to the modelling time step. During the adaption, the bed is assumed to be fixed.

For $E \to 0$ (short waves, or frictionless flow), the quasi-steady flow concept fails for $0.8 < Fr < 1.2$. This can be explained by comparing quasi-fixed bed models and mobile-bed models. The effect of bed mobility is negligible if

$$\psi < |1 - Fr^2|$$

(Section 3.6).

For these conditions, the critical-flow model (Section 2.7) is suggested. This model equals the exact solution for $Fr = (1 + \psi)^{1/2}$ and consequently provides an approximation for values of $Fr$ near unity.

6.4. Hydraulics-morphology interaction.

The non-linear interaction between hydraulic and morphological changes with respect to the fast, downstream-propagating wave III can be neglected for all values of $Fr$, in the case of zero-order uniform flow. The latter restriction refers
to a rather uniform river geometry.

The error in the hyperbolic model changes with the type of wave that is considered. For low values of \( Fr \), the downstream-propagating wave II is approximated well, whereas for supercritical flows, only the upstream-propagating wave I is represented. This effect has been visualized quantitatively in Figure 4.6.

The linear similarity between quasi-fixed bed and mobile-bed models enables derivation of similarity conditions (Eq.13, Chapter three). These conditions are equivalent to the classical characteristic equation (Eq.6, Chapter one), and enable extended analysis of characteristic roots (Appendix C).

Decoupling, or application of the quasi-fixed bed approach implies the introduction of a stabilizing "diffusion" term in subcritical flows, and a destabilizing "negative-diffusion" term in supercritical flows.

The error made when decoupling changes in hydraulics and morphology is due to non-linearities. This error can be reduced for all flow regimes by selecting a smaller decoupling-time interval. It is noted that this limitation of time step reduces the original merit of efficient morphological computation.

The quasi-steady flow concept implies the introduction of a diffusion term in the solution of depth and velocity.

6.5. Water-sediment mixtures.

For higher values of \( Fr \), the mathematical model becomes sensitive to the contribution of sediment to mass and momentum fluxes of the water-sediment mixture. Despite low values of \( s/q \), second-order effects induce errors due to sediment load (Figure 5.3).

In general, the error in the "clear-water" flow formulation will have a diffusive effect on wave-type solutions.

For quasi-steady flow, again a hyperbolic model and a simple-wave model can be found. The parabolic model, however, cannot be derived without neglecting the second-order effects of sediment load; application of quasi-steady and quasi-uniform assumptions will result in a hyperbolic model.
LL
List of main symbols.

\( a \)  depth \hspace{1cm} [m]  
\( B \)  width \hspace{1cm} [m]  
\( c \)  actual concentration of suspended sediment \hspace{1cm} [-]  
\( \bar{c} \)  depth-averaged concentration of bed-material load \hspace{1cm} [-]  
\( c_b \)  concentration of sediment in the bed-load layer \hspace{1cm} [-]  
\( c_e \)  equilibrium concentration of suspended sediment \hspace{1cm} [-]  
\( c_0 \)  sediment fraction in river bed \hspace{1cm} [-]  
\( C \)  Chézy roughness parameter \hspace{1cm} [m^{1/2}/s]  
\( u \)  velocity \hspace{1cm} [m/s]  
\( Fr \)  Froude number, defined as \( Fr = u\sqrt{-g_c a} \) \hspace{1cm} [-]  
\( i \)  slope of bed level \hspace{1cm} [-]  
\( L_A \)  adaption length of equilibrium concentration-profile \hspace{1cm} [m]  
\( T \)  characteristic wave-period \hspace{1cm} [s]  
\( T_A \)  adaption period of equilibrium concentration-profile \hspace{1cm} [s]  
\( T_h \)  characteristic hydraulic time-scale \hspace{1cm} [s]  
\( T_m \)  characteristic morphological time-scale \hspace{1cm} [s]  
\( g \)  gravitation constant \hspace{1cm} [m/s^2]  
\( g_j \)  component of gravitation constant in \( j \)-direction \hspace{1cm} [m/s^2]  
\( n_t \)  celerity \hspace{1cm} [m/s]  
\( p \)  porosity of sediment \hspace{1cm} [-]  
\( q \)  discharge per meter width in \( x \)-direction defined as \( q = ua \) \hspace{1cm} [m^3/s]  
\( s_b \)  bed-load transport \hspace{1cm} [m^3/s]  
\( s \)  total bed material load \hspace{1cm} [m^3/s]  
\( S_1 \)  transversal flow-mass exchange \hspace{1cm} [m/s]  
\( S_2 \)  boundary flow-momentum exchange \hspace{1cm} [m^2/s^2]  
\( S_3 \)  transversal sediment-mass exchange \hspace{1cm} [m/s]  
\( z_b \)  local bed level \hspace{1cm} [m]  
\( Z_b \)  spatial- and time-averaged bed-level \hspace{1cm} [m]  

\( \Delta \)  relative sediment density defined as \( \Delta = \rho / \rho_s - 1 \) \hspace{1cm} [-]  
\( \tau_b \)  bed shear-stress \hspace{1cm} [kg/ms^2]  
\( \rho \)  density of water \hspace{1cm} [kg/m^3]  
\( \rho_s \)  density of sediment \hspace{1cm} [kg/m^3]
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Appendix A.

Analysis of harmonic flood-wave solutions.


The three first-order partial-differential equations can be combined into one third-order partial-differential equation.

\[
\frac{\partial^3 z_b}{\partial t^3} + 2\alpha \frac{\partial^3 z_b}{\partial x \partial t^2} + \alpha^2 \frac{(Fr^2 - 1 - \psi)}{Fr^2} \frac{\partial^3 z_b}{\partial x^2 \partial t} - \frac{\alpha^2 \psi}{Fr^2} \frac{\partial^3 z_b}{\partial x^3} + \frac{E}{2} \frac{\partial^2 z_b}{\partial x^2 \partial t} = 0
\]  (1)

Now, a sinusoidal solution is considered to represent a flood hydrograph. This sinusoidal solution is defined as

\[
\begin{bmatrix}
  a(x,t) \\
  u(x,t) \\
  z_b(x,t)
\end{bmatrix} = \Delta a \begin{bmatrix}
  1 \\
  -1 - \frac{\omega'}{\alpha k'} \\
  \psi \left(1 + \frac{\alpha k'}{\omega'}\right)
\end{bmatrix} e^{i (k' x' + \omega' t')} ; \quad k = \frac{\alpha k'}{u T} ; \quad \omega = \frac{\omega'}{T}
\]  (2)

with \( \omega = \pi / T \). The complex wave number is defined as \( k = k_1 + i k_2 \).

The resulting flood wave can be characterized by a propagation rate and a damping length. The dimensionless propagation rate of the flood wave is

\[
\frac{c_w}{u_0} = \frac{\omega'}{\alpha k'_1}
\]  (3)

The dimensionless damping length of the flood wave is

\[
L'_d = \frac{1}{k'_2}
\]  (4)

For \( k_2 > 0 \) the wave amplitude attenuates (stable solution), for \( k_2 < 0 \), the wave amplitude increases (unstable solution). Although the scaling parameter \( \alpha \) has no resulting effect on the solution (Eq.5), it does appear in the interpretation of the attenuation length.
For the solution to be non-trivial, the determinant of the set of equations with the substituted solution must be singular, or

\[-\psi \left( \frac{\text{\( \frac{\alpha k}{\omega} \)}}{\omega'} \right)^3 + (\text{\( \frac{\alpha k}{\omega} \)} - 1 - \psi) \left( \frac{\text{\( \frac{\alpha k}{\omega} \)}}{\omega'} \right)^2 + \text{\( Fr^2 \)} \left( \frac{\text{\( \frac{3E}{2\omega} \)}}{\omega'} \right) \left( \frac{\text{\( \frac{\alpha k}{\omega} \)}}{\omega'} \right) + \text{\( Fr^2 \)} \left( \frac{-i \text{\( E \)} - \omega'}{\omega'} \right) = 0\]

(5)

Hence, the key parameters for the behaviour of the solution are \( Fr, \psi \) and \( E/\omega' \).

Eq.13 can be solved analytically (e.g., Press et al. 1992). The roots have been analysed numerically. For \( E = 0 \), Eq.5 is the characteristic equation. Because three roots for the wave number can be found, three solutions of stable waves are represented; two propagating downstream and one propagating upstream. In this study, the waves are identified as

wave I; propagating upstream

wave II; propagating downstream (slow)

wave III; propagating downstream (fast)

A.2. Characterization of flood-wave solution.

The corresponding propagation rates are constructed in Fig.1. It is noted that for longer waves (larger values of \( E \)), lower propagation rates can be found. It can be concluded from Figure A.1, that for low values of \( Fr \), this reduction in propagation rate for larger values of \( Fr \) is negligible, whereas for larger values of \( Fr \), the effect of \( E \) on the solution is dominant.
For large values of $E$, the fast, positive, wave III approaches a propagation rate of $\frac{3}{2} u$, similar to flood waves in the diffusion-analogy models.

Fig.A.1
The dimensionless attenuation length for wave III increases for longer waves and higher values of $Fr$ (less deformation), and decreases for waves I and II (strong deformation).

\[ \psi = 0.01 \]

\[ E = 1000 \]

\[ \begin{array}{c}
L_w \\
Fr
\end{array} \]

Fig.A.2

Based on Figures A.1 and A.2, it can be concluded that for larger values of $E$, the behaviour of wave III is different from that of wave I and II.

Apart from propagation and attenuation rates, amplitudes of model responses to hydrographs can be analysed. The dimensionless amplitudes of velocity and bed level for the different waves can be constructed relative to a prescribed amplitude in water depth.

\[
\begin{vmatrix}
\Delta a \\
\Delta u \\
\Delta z_b
\end{vmatrix} = \Delta a
\begin{vmatrix}
1 \\
-1 - \frac{\omega'k'_1}{\alpha(k_{1}^2 + k_{2}^2)} + i \frac{\omega'k'_2}{\alpha(k_{1}^2 + k_{2}^2)} \\
\psi \left( 1 + \frac{\alpha k'_1}{\omega'} \right) + i \psi \frac{\alpha k'_2}{\omega'}
\end{vmatrix}
\]

(6)

Hence, responses consist of both an amplitude and a phase shift.

In Figure A.3, the amplitude of velocity response for the three different waves are considered.

![Graph showing velocity response](image)

Fig. A.3

For \( Fr > 0.4 \) and \( E > 100 \), the effect of \( Fr \) on the amplitude of velocity diminishes.

The maximum response of velocity to a depth disturbance is

\[
\text{wave I, II} \quad \frac{\Delta u'}{\Delta a'} = a \frac{\Delta u}{u \Delta a} = 1
\]

\[
\text{wave III} \quad \frac{\Delta u'}{\Delta a'} = a \frac{\Delta u}{u \Delta a} = \frac{1}{2}
\]

This implies that without considering phase-shifts, flows at wave I and II approaches quasi-steady flow for the conditions described above. The absolute values of velocity-phase shifts are constructed in Figure A.4. Phase-shifts for the downstream-propagating waves II and III are negative, whereas the phase-shift for the upstream-propagating wave I is positive.
With respect to the fast, downstream-propagating wave III, the phase shift in velocity can be neglected.

**A.4. Response of bed level.**

The response of the bed level to the different waves has been constructed in the Figures A.5 and A.6. The relative amplitudes are as shown in Figure A.5.
It can be concluded that for waves I and II, the maximum response of bed level is similar for larger values of $E$. Then, combination of wave I and II into one wave-type solution is obvious. For low values of $Fr$, the response of the bed level is of the order of $\psi$ with respect to wave III. For larger values of $Fr$, the response of the bed is negligible. Hence, flood waves of wave-III type can be routed assuming fixed beds.

The phase-shifts in bed level response are negative for wave I and II, and positive for wave III.

![Fig.A.6](image)

It is noted here that for values of $Fr$ near unity, features of waves I and II approach each other.
Appendix B

Second-order gradients in compatibility equations.

To analyse higher-order effects in mathematical models for river morphology, the linear compatibility equations are combined with non-uniform initial conditions. To enable any analytical comparison of complete and quasi fixed-bed models, non-linearities are therefore present in the initial conditions only.

The non-uniform initial conditions are defined with the help of spatial gradients. Second-order effects are taken into account by distinction of up- and downstream gradients.

Spatial gradients are now approximated with

\[
\frac{\partial U}{\partial x} \bigg|_{\text{down}} = \frac{\partial U}{\partial x} + \frac{\Delta x}{2} \frac{\partial^2 U}{\partial x^2}; \quad \frac{\partial U}{\partial x} \bigg|_{\text{up}} = \frac{\partial U}{\partial x} - \frac{\Delta x}{2} \frac{\partial^2 U}{\partial x^2} \quad (1)
\]

The initial conditions for the characteristic equation with the upstream-propagating characteristic \(n_1\)

\[
U_1^n = U_0^n + \Delta t \ n_1 \frac{\partial U}{\partial x} \bigg|_{\text{down}} = U_0^n + \Delta t \ n_1 \frac{\partial U}{\partial x} + n_1 \Delta t \Delta x \frac{\partial^2 U}{\partial x^2} \quad (2)
\]

Similarly, the initial conditions for the characteristic equations with the downstream-propagating \(n_2\) and \(n_3\)

\[
U_i^n = U_0^n + \Delta t \ n_i \frac{\partial U}{\partial x} \bigg|_{\text{up}} = U_0^n + \Delta t \ n_i \frac{\partial U}{\partial x} - n_i \Delta t \Delta x \frac{\partial^2 U}{\partial x^2} \quad (3)
\]

with \(i=1,2,3\).

As a result, changes in \(U\) along a characteristic \(n_i\) are

\[
\Delta U_i = U_i^{n+1} - \left(U_0^n + \Delta t \ n_i \frac{\partial U}{\partial x} \bigg|_{\text{downup}}\right) = \Delta U - \Delta t \ n_i \left(\frac{\partial U}{\partial x} \pm \frac{\Delta x}{2} \frac{\partial^2 U}{\partial x^2}\right) \quad (4)
\]
Substitution of the initial conditions into the solution yields

\[
\begin{bmatrix}
\Delta a \\
\Delta u \\
\Delta z_b
\end{bmatrix}
= \Delta t
\begin{bmatrix}
-g_c \frac{\partial a}{\partial x} + \frac{\Delta t \Delta x}{2} g_c \delta_{a1} \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
-\frac{\partial^2 a}{\partial x^2} + \\
-\frac{\partial^2 a}{\partial x^2} + \\
-\frac{\partial^2 a}{\partial x^2} +
\end{bmatrix}
\]

\[+ \Delta t
\begin{bmatrix}
-a \\
-u \\
-a \psi
\end{bmatrix}
\begin{bmatrix}
\frac{\partial u}{\partial x} + \frac{\Delta t \Delta x}{2} u \delta_{u1} \\
\frac{\partial u}{\partial x} + \frac{\Delta t \Delta x}{2} u \delta_{u2} \\
\frac{\partial u}{\partial x} + \frac{\Delta t \Delta x}{2} u \delta_{u3}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial^2 u}{\partial x^2} + \\
\frac{\partial^2 u}{\partial x^2} + \\
\frac{\partial^2 u}{\partial x^2} +
\end{bmatrix}
\]

\[+ \Delta t
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial z_b}{\partial x} + \frac{\Delta t \Delta x}{2} g_c \delta_{z1} \\
\frac{\partial z_b}{\partial x} + \frac{\Delta t \Delta x}{2} g_c \delta_{z2} \\
\frac{\partial z_b}{\partial x} + \frac{\Delta t \Delta x}{2} g_c \delta_{z3}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial^2 z_b}{\partial x^2} + \\
\frac{\partial^2 z_b}{\partial x^2} + \\
\frac{\partial^2 z_b}{\partial x^2} +
\end{bmatrix}
\]

\[+ \Delta t
\begin{bmatrix}
\frac{S_1}{S_2 - u S_1} \\
\frac{S_1}{S_2 - u S_1} \\
\frac{S_1}{S_2 - u S_1}
\end{bmatrix}
\]

\[= \begin{bmatrix}
2 n_1 g_c a \psi \\
2 n_1 g_c a \psi \\
2 n_1 g_c a \psi
\end{bmatrix}
\]

\[
\left[ \frac{\delta_{a1}}{\delta_{a2}} \frac{\delta_{a3}}{\delta_{u1}} \frac{\delta_{u2}}{\delta_{u3}} \right] = \left[ \begin{array}{c}
\frac{u^2 (n_2 - n_1) (n_3 - n_1)}{2 n_2 (n_2 - n_1) (n_3 - n_1)} \\
\frac{-(2 n_2 (n_2 - n_1) (n_3 - n_1)}{2 n_1 (n_2 - n_1) (n_3 - n_1)} \\
\frac{2 n_1 g_c a \psi}{2 n_1 g_c a \psi}
\end{array} \right]
\]

The dimensionless coefficients \( \delta_{\eta} \) are defined as

\[
\left[ \delta_{a1} \delta_{a2} \delta_{a3} \right] = \left[ \begin{array}{c}
\frac{(2 n_2 (n_2 - n_1) (n_3 - n_1)}{n_2 (n_2 - n_1) (n_3 - n_1)} \\
\frac{-n_1 (n_2 - n_1) (n_3 - n_1)}{n_2 (n_2 - n_1) (n_3 - n_1)} \\
\frac{n_1 (n_2 - n_1) (n_3 - n_1)}{n_2 (n_2 - n_1) (n_3 - n_1)}
\end{array} \right]
\]

\[
\left[ \delta_{u1} \delta_{u2} \delta_{u3} \right] = \left[ \begin{array}{c}
\frac{2 n_1 g_c a \psi}{u (n_1 - n_2) (n_3 - n_1)} \\
\frac{n_1 (n_2 - n_1) (n_3 - n_1)}{u (n_1 - n_2) (n_3 - n_1)} \\
\frac{-2 g_c a (n_1 + u) + u (n_1 - n_2) (n_3 - n_1)}{u (n_1 - n_2) (n_3 - n_1)}
\end{array} \right]
\]

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Appendix C.

Analysis of similarity conditions.

C.1. Introduction.

The similarity conditions are based on comparison of fluxes of mass and momentum. In Chapter Three it has been found that quasi-fixed bed models and complete models yield similar solutions if

\[
(n_{t1} + u) (n_{t2} + u) (n_{t3} + u) = g_z a u
\]

\[
n_{t1} + n_{t2} + n_{t3} = -2u
\]

\[
n_{t1} n_{t2} n_{t3} = - g_z a u \Psi
\]

These conditions are equivalent to the characteristic equation in Chapter 1.

With these conditions, characteristic roots can be analysed in a way, somewhat different from the classical approach (e.g. De Vries, 1992). In the following parts, assumptions are combined with the similarity conditions to obtain a corresponding set of characteristics.

C.2. Quasi-fixed bed models.

The first option concerns the assumption of a fixed bed with respect to hydraulic conditions. This changes the similarity conditions into

\[
(n_{t1} + u) (n_{t2} + u) u = g_z a u
\]

\[
n_{t1} + n_{t2} = -2u
\]

\[
n_{t1} n_{t2} n_{t3} = - g_z a u \Psi
\]

which yields

\[
\frac{n_{t1,2}}{u} = -1 \pm \frac{1}{Fr} ; \quad \frac{n_{t3}}{u} = - \frac{\Psi}{1-Fr^2}
\]

These expressions equal indeed the approximative characteristic roots introduced by De Vries (1965). This confirms the conclusion that linearized versions of
complete and quasi-fixed bed models are equivalent. However, it is noted that the approximations derived are unstable for values of $Fr$ near unity (Figure C.1).

Apart from the approximative and exact characteristics, the relative difference $\epsilon$ is constructed.

**Fig.C.1**

### C.3. Quasi-steady flow models.

A second assumption is $|n_\text{l1}| \gg u$, and $|n_\text{l2}| \gg u$. Then, the set of similarity conditions can be approximated with

\[
\begin{align*}
   n_\text{l1} & \quad n_\text{l2} \quad (n_\text{l3} + u) = g_z \quad a \quad u \\
   n_\text{l1} + n_\text{l2} + n_\text{l3} & = -2u \\
   n_{l1} \quad n_{l2} \quad n_{l3} & = -g_z a \quad u \quad \Psi
\end{align*}
\]

This yields

\[
\begin{align*}
   \frac{n_\text{l3}}{u} & = -\frac{\Psi}{1 + \Psi} \\
   \frac{n_{l1,2}}{u} & = -\frac{1}{2} \left( \frac{2 + \Psi}{1 + \Psi} - \sqrt{\left( \frac{2 + \Psi}{1 + \Psi} \right)^2 + 4 \frac{1 + \Psi}{Fr^2}} \right)
\end{align*}
\]

For low values of $Fr$, this resembles the "instantaneous" adaption of hydraulic conditions. The corresponding characteristics approximate the exact characteristics for $Fr < 0.4$ (Figure C.2).
In other words, the assumption of instantaneous hydraulic adaption for (short) waves propagating with a characteristic celerity is valid for $Fr < 0.4$.

An alternative simplification for this case is

$$n_{r1} n_{r2} (n_{r3} + u) = g_z a u$$

$$n_{r1} + n_{r2} + n_{r3} = 0$$

$$n_{r1} n_{r2} n_{r3} = -g_z a u \psi$$

The corresponding solution is

$$\frac{n_{r3}}{u} = -\frac{\psi}{1 + \psi}; \quad \frac{n_{r1,2}}{u} = -\frac{1}{2} \frac{\psi}{1 + \psi} \left(1 \pm \sqrt{1 + 4 \frac{(1 + \psi)^3}{Fr^2 \psi^2 (1 + 2 \psi)}}\right)$$
This approximation is compared with the exact characteristics in Figure C.3.

Similarly, an "instantaneous" adaption of the bed level can be assumed. However, combination of such a set of equations does not result into a convenient equation.


For values of $Fr$ near unity, the characteristics of the quasi-steady flow model fail. Because absolute values of two characteristics approach, $n_{t_1} = -n_{t_2}$ can be substituted into the similarity conditions. This yields

$$(u^2 - n_{t_1}^2)(n_{t_3} + u) = g_z au$$

$$n_{t_3} = -2u$$

$$- n_{t_1}^2 n_{t_3} = -g_z au \psi$$

The solution is

$$Fr = \sqrt{1 + \frac{\psi}{2}} ; \quad \frac{n_{t_1,2}}{u} = \pm \frac{\sqrt{\psi/2}}{Fr} = \pm \sqrt{\frac{\psi}{2 + \psi}} ; \quad \frac{n_{t_3}}{u} = -2$$

This will be used in the analysis of critical flow in Chapter five.
C.5. Approximate solution of characteristics.

Now, as an exercise, the following roots are substituted
\[ n_1 = -u - c_1 ; \quad n_2 = -u + c_2 \]

Substitution into the similarity conditions yields
\[ n_3 = c_1 - c_2 ; \quad c_1^3 + u \quad c_1^2 + g_z a (1+\psi) \quad c_1 + g_z au = 0 \]
\[ c_2 = \frac{c_1 + u}{2} \left( 1 \pm \sqrt{1 - \frac{4}{1+u/c_1} \left( 1 + \frac{g_z a}{c_1^2} \frac{1+\psi}{1+u/c_1} \right)} \right) \]

If expressions for \( c_1 \) are proposed, the error can be calculated with the third-order equation.
\[ c_1 = \sqrt{-g_z a} \quad - \quad \epsilon = (-g_z a)^{3/2} \psi \quad ; \quad c_1 = \sqrt{-g_z a (1 + \psi)} \quad - \quad \epsilon = -g_z au \psi \]
If the first celerity is substituted, the approximated roots will be
\[
\frac{n_{1}}{u} = -1 - \frac{1}{Fr}
\]
\[
\frac{n_{2,3}}{u} = \frac{1-Fr}{2Fr} \pm \frac{1}{2Fr} \sqrt{1 - 2Fr + Fr^2 + 4\psi}
\]

These roots are constructed with the exact characteristics in Figure C.5.

If the second option is used, the characteristics are approximated with
\[
\frac{n_{1}}{u} = -1 - \frac{\sqrt{1+\psi}}{Fr}
\]
\[
\frac{n_{2,3}}{u} = \frac{\sqrt{1+\psi} - Fr}{2Fr} \pm \frac{\sqrt{1+\psi}}{2Fr} \sqrt{\frac{1+\psi - 2Fr}{\sqrt{1+\psi}} + \frac{Fr^2}{1+\psi}}
\]
In Figure C.6, these roots are shown.