STELLINGEN

behoorende bij het proefschrift

"Design of systems for active sound control"

van Niek Doelman.

1 Bij het ontwerp van een regelsysteem voor de actieve onderdrukking van geluid is het raadzaam naast de vooruitregeling ook een teruggekoppelde regeling te implementeren (dit proefschrift).

2 Het gebruik van een adapterende regellus in anti-geluid systemen wordt in veel gevallen ten onrechte als noodzakelijk gezien (dit proefschrift).

3 Het is opmerkelijk dat de nauw verwante disciplines van anti-geluid en anti-trilling op een zo verschillende regeltechnische wijze worden aangepakt.

4 Met de techniek van anti-geluid probeert de mens een symptoom van zijn dichtbevolkte, gehaaste en lawaaïge wereld te bestrijden. In dat opzicht is het te hopen dat de toepassing van anti-geluid beperkt blijft (omslag van dit proefschrift).

5 Met het uitvoeren van simulatie experimenten kan slechts worden aangetoond dat een idee in de praktijk niet werkt.

6 Toen filmmaker Sergei Eisenstein in de jaren 20 de kracht van montage vatte in de expressie $1 + 1 = 3$, had hij nog geen weet van de naoorlogse superproducties uit Hollywood, die dit corrigeerden tot $1 + 1 = 2$.

7 De huidige televisie-cultuur – met zijn hoge informatiedichtheid en zapp faciliteiten – leidt tot een ernstige vernauwing van het bewustzijn van de kijker.

8 Bij de zaalsport Squash kan een speler, als antwoord op een korte bal, kiezen uit o.a. een gewone lengte-slag, een cross court lengte-slag, een drop-shot, en een lob. Van deze opties is de vaak gespeelde cross court lengte-slag een hachelijke onderneming. De hoge lob daarentegen is betrouwbaar en efficiënt, en als zodanig ondergewaardeerd.

9 Een evenwichtiger verdeling van mannelijke en vrouwelijke studenten aan een Technische Universiteit is wellicht te bereiken, door in de eerste maanden van de studie een verplichte cursus sociale vaardigheden in het programma op te nemen.

10 Met het oog op de aktieve rol die Nederland in de internationale politiek wil spelen, is het raadzaam een bewindsman – zoals de minister-president en de minister van financiën – ook te selecteren op een naam, die in het Engels serieus overkomt.

11 Ondanks de verkorting van de duur van het promotie-onderzoek blijft het nodig het apparaat van de reclassering ook in te richten voor net gepromoveerden.

Delft, 2 december 1993.
Design of Systems for Active Sound Control
Design of Systems for Active Sound Control

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prof. ir. K.F. Wakker,
in het openbaar te verdedigen ten overstaan van een commissie,
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prof. dr. ir. A.J. Berkhout
prof. ir. G. Honderd
Abstract

In this thesis the complete trajectory of designing a control system for active sound reduction is described. Starting with the fundamental acoustical theory, the design deals with discrete source and sensor distributions, discrete acoustic processes, control strategies and, finally, a large number of practical experiments.

The acoustical essence of active sound reduction is based on a distribution of Huygens’ sources, that has the potential to reconstruct a primary sound field with equal strength but opposite phase. A continuous distribution of monopoles and dipoles on a closed surface is shown to be the ideal anti-sound source distribution. It is capable of completely attenuating a sound field inside the closed surface. In order to arrive at a practical number of anti-sound or so-called secondary sources, suitable cost functions are defined. A suitable cost function represents the nuisance of a sound field and this could - in its simplest form - be the total acoustic energy or the total radiated acoustic power. The optimal strength of the secondary sources then follows from a minimisation of the cost function.

The acoustic part of the design is completed by addressing the distribution of a limited number of sensors. Two types of sensors are required. First, a distribution of detection sensors measures the incident primary sound field. Second, a distribution of error sensors measures the residual sound field. It is shown that the space average squared pressure at well-placed error sensors is a suitable cost function in practice. The acoustic design of source and sensor distributions is worked out for the free field, a multi-modal duct and a reverberant enclosure.

Given a source-sensor distribution, a digital control strategy has to be designed. A first step in this is to model the discrete acoustic processes or transfer functions. It is demonstrated that a discrete acoustic process is likely to have a non-minimum phase behaviour. This has to be accounted for in the design of the controller. The Generalised Minimum Variance control strategy is proposed as a suitable way to design the control system for active sound reduction. It can be used to design a combined feedforward and feedback regulator. The main objective of such a combination is to improve the robustness of the feedforward link. Although a feedforward regulator may achieve a substantial reduction of a measurable disturbance, it unfortunately shows a poor robustness.

In a series of practical experiments both the acoustical and control design guidelines are tested. The experiments are carried out in an echo-free chamber, a duct and reverberant enclosure. The results of the experiments confirm the proposed design methods to a large extent.
Samenvatting

In dit proefschrift wordt het ontwerptrject van een systeem voor actieve geluidonderdrukking beschreven. Het ontwerp begint met de fundamentele akoestische theorie en behandelt vervolgens discrete verdelingen van bronnen en sensoren, discrete akoestische processen, regelstrategiën en uiteindelijk een groot aantal praktische experimenten.

Het akoestisch principe van actieve geluidonderdrukking is gebaseerd op een verdeling van Huygens bronnen. Een dergelijke bronverdeling is in staat om een primair geluidveld te reconstrueren met een gelijke sterkte maar met een tegengestelde fase. Aangetoond wordt dat een continue verdeling van monopolen en dipolen op een gesloten oppervlak de ideale anti-geluid bronverdeling is. Om tot een praktischere verdeling van anti-geluid of secundaire bronnen te komen, worden geschikte kosten functies gedefinieerd. Een geschikte kosten functie geeft de hinder van een geluidveld weer en dit zou in z’n eenvoudigste vorm de totale akoestische energie of het totaal afgetraalde akoestisch vermogen kunnen zijn. De optimale sterkte van de secundaire bronnen wordt dan gevonden door de kosten functie te minimaliseren. Het akoestische gedeelte van het ontwerp wordt besloten met de verdelingen van de sensoren. Er zijn twee typen sensoren nodig. Ten eerste een verdeling van detectie-sensoren, die het invallend primaire geluidsveld meet. En ten tweede een verdeling van fout-sensoren die het residu geluidsveld meet. Het blijkt dat de gemiddelde kwadratische druk op goed geplaatste fout-sensoren een geschikte en praktische kosten functie is. Het akoestisch ontwerp van de verdeling van bronnen en sensoren is uitgewerkt voor het vrije veld, een multi-modal kanaal en een galmende, omsloten ruimte.

Op basis van de verdeling van bronnen en sensoren dient er een digital regelstrategie ontworpen te worden. De eerste stap hierin is het modelleren van de discrete akoestische processen of overdachtspuncties. Het is aangetoond dat een discreet akoestisch proces waarschijnlijk niet-minimum fase gedrag vertoont. Met deze eigenschap dient rekening gehouden te worden in het ontwerp van de regelaar. De Gegenormaliseerde Minimum Variante (GMV) regelstrategie wordt voorgesteld als een geschikte methode voor het ontwerp van het regelsysteem voor actieve geluidonderdrukking. De strategie kan gebruikt worden voor de toepassing van een gecombineerde vooruitregeling en teruggekoppelde regeling. Het voornaamste doel van een dergelijke combinatie is het verbeteren van de robuustheid van de vooruitregeling. Hoewel een vooruitregeling een aanzienlijk onderdrukking van het, van te voren meetbare, lawaai kan bereiken, vertoont het een grote gevoeligheid voor systeem verstoringen.

De akoestische en regeltechnische ontwerpregels zijn getest in een aantal praktische experimenten. Deze zijn uitgevoerd in een galmvrije kamer, een kanaal en een galmende omsloten ruimte. De resultaten van de experimenten bevestigen de ontwerpregels in grote mate.
The doctoral study as described in this thesis started in August 1989, when a research proposal in the program of the Fundamental Research on Matter (FOM) was granted by the Netherlands Technology Foundation (STW). The actual research has been carried out at the TNO Institute of Applied Physics (TPD) in the Department of Active Noise and Vibration Control. The scientific supervision has been taken care of by prof. Berkhout of the Group of Seismics and Acoustics, Faculty of Applied Physics of the Delft University of Technology. Later on he was joined by prof. Honderd of the Control Laboratory of the Faculty of Electrical Engineering.

First of all I want to acknowledge the TPD for the facilities and time I was given to finish the doctoral thesis. Also, I owe tanks to my colleagues who had to cope with the familiar whims of a doctoral student.

I am grateful to my promotor prof. Berkhout for continuously pointing out to me possible improvements on the thesis. Certainly, it takes a broad knowledge and perspective to come up with these numerous recommendations. Moreover, he has the ability to encourage the student to really carry through the ideas.
I also want to thank prof. Honderd. Although his competence was called in at a late stage, the cooperation has been very fruitful. I enjoyed the discussions on the control aspects of anti-noise, looking at it from different perspectives.
<table>
<thead>
<tr>
<th>Chapter 1; Introduction to Active Sound Reduction</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1 Active and Passive Noise Reduction</td>
<td>2</td>
</tr>
<tr>
<td>1.2 Reconstruction of sound fields</td>
<td>3</td>
</tr>
<tr>
<td>1.3 Objective of the thesis</td>
<td>14</td>
</tr>
<tr>
<td>1.4 References</td>
<td>17</td>
</tr>
<tr>
<td>Chapter 2; Acoustical Principles of Active Sound Reduction</td>
<td>19</td>
</tr>
<tr>
<td>2.1 The fundamental equations of linear acoustics</td>
<td>19</td>
</tr>
<tr>
<td>2.2 Sources of acoustic wave fields</td>
<td>21</td>
</tr>
<tr>
<td>2.3 Non-uniqueness of source distributions</td>
<td>23</td>
</tr>
<tr>
<td>2.4 The Helmholtz equation and Green’s function</td>
<td>24</td>
</tr>
<tr>
<td>2.5 Applications of the generalised Kirchhoff Integral</td>
<td>27</td>
</tr>
<tr>
<td>2.6 Kirchhoff solutions for active sound control</td>
<td>33</td>
</tr>
<tr>
<td>2.7 Energetic aspects of Kirchhoff solutions</td>
<td>42</td>
</tr>
<tr>
<td>2.8 References</td>
<td>44</td>
</tr>
<tr>
<td>Chapter 3; Acoustic Cost Functions and Discrete Source Distributions</td>
<td>45</td>
</tr>
<tr>
<td>3.1 Design of acoustic cost functions</td>
<td>45</td>
</tr>
<tr>
<td>3.2 Active reduction of free field radiation</td>
<td>48</td>
</tr>
<tr>
<td>3.3 Case study of a one-dimensional enclosure</td>
<td>57</td>
</tr>
<tr>
<td>3.4 Active reduction of sound fields in a duct</td>
<td>75</td>
</tr>
<tr>
<td>3.5 Active reduction of enclosed sound fields</td>
<td>82</td>
</tr>
<tr>
<td>3.6 The minimisation of the maximum of the squared pressure distribution</td>
<td>91</td>
</tr>
<tr>
<td>3.7 References</td>
<td>96</td>
</tr>
<tr>
<td>A3.1 Minimisation of quadratic Hermitian forms</td>
<td>99</td>
</tr>
<tr>
<td>A3.2 Eigenfunctions expansions for enclosed sound fields</td>
<td>105</td>
</tr>
<tr>
<td>A3.3 An eigenfunction expansion for a general impedance boundary condition</td>
<td>109</td>
</tr>
<tr>
<td>A3.4 An eigenfunction expansion for a lightly damped enclosure</td>
<td>110</td>
</tr>
<tr>
<td>A3.5 Minimisation of the infinity norms</td>
<td>111</td>
</tr>
<tr>
<td>Chapter 4; The Distribution of Sensors and the Constraint of Causality</td>
<td>113</td>
</tr>
<tr>
<td>4.1 The discretisation of acoustic cost functions</td>
<td>113</td>
</tr>
<tr>
<td>4.2 The theory of multivariate optimal linear prediction</td>
<td>127</td>
</tr>
<tr>
<td>4.3 Detection of the primary sound field</td>
<td>135</td>
</tr>
<tr>
<td>4.4 Acoustic design procedures for active sound reduction systems</td>
<td>138</td>
</tr>
<tr>
<td>4.5 References</td>
<td>140</td>
</tr>
<tr>
<td>Chapter 5; Modelling and Control of Acoustic Disturbances</td>
<td>141</td>
</tr>
<tr>
<td>5.1 Concepts of control systems</td>
<td>142</td>
</tr>
<tr>
<td>5.2 The acoustic modal state-space representation</td>
<td>143</td>
</tr>
<tr>
<td>5.3 Discrete-time transfer functions models</td>
<td>146</td>
</tr>
</tbody>
</table>
5.4 Acoustic disturbance models 152
5.5 Feedforward and feedback control of disturbances 157
5.6 The overall model for the acoustic system 160
5.7 The LQG control strategy 163
5.8 References 167

Chapter 6; Self-Tuning Control Strategies 169
6.1 Single-stage criterion functions 169
6.2 Self-tuning implementations of minimum variance control strategies 177
6.3 Recursive parameter estimation 178
6.4 Characterisation of existing active sound control strategies 180
6.5 References 186

Chapter 7; Experiments on Active Sound Reduction 189
7.1 Feedforward control of broadband sound 189
7.2 Feedback control of broadband sound 192
7.3 Robustness of the controllers for broadband sound 196
7.4 Feedforward control of harmonic sound 198
7.5 Feedback control of harmonic sound 200
7.6 Robustness of the controllers for harmonic sound 201
7.7 Adaptive feedforward control of time-varying harmonic sound 202
7.8 Multi-channel reduction of harmonic sound in an enclosure 204
7.9 Multi-channel reduction of random sound in an enclosure - electric detection signal 207
7.10 Multi-channel reduction of random sound in an enclosure - acoustic detection signal 209
7.11 Multi-channel reduction of random sound in duct 211
7.12 Multi-channel reduction of sound in an echo-free chamber 213
7.13 References 215

Chapter 8; Conclusions and Discussion 217
8.1 Summary 217
8.2 Discussion and future research 219
8.3 References 222

Published Papers 223

Curriculum Vitae 225
CHAPTER 1

Introduction to Active Sound Reduction

Sound can be described by small, time-varying deviations of the static pressure in a fluid: compressions and rarefactions of fluid particles. Within the frequency range of about 30 - 16000 Hz sound is audible. Although it is a subjective matter, in some cases 'sound' is considered to be 'noise'. Often noise can be described as sound with a high amplitude or sound with disturbing frequencies. Also, characteristics like duration and number of occurrences turn out to be important.

With the principles of sound in mind, the effect of noise can be attenuated by reducing the compressions and rarefactions of air near the ear. In 1933 the German scientist Paul Lueg formulated a very special way to achieve this:

- an original sound wave can be reduced by generating another sound wave that has equal amplitude but opposite polarity; i.e. a rarefaction where the original wave has a compression and vice versa.

![Diagram of a system for active sound reduction.](Image)

Figure 1.1: Schematic diagram of a system for active sound reduction.

This principle of destructive interference of sound waves is called active sound(noise) reduction or simply anti-noise. The range of action of the counteracting or secondary wave field to reduce the original or primary wave field may vary from a point (e.g. at the ear) to a complete room or even outdoors. Figure 1.1 shows the acoustic structure of such a system. The control sources generate the anti-sound field, which has to cancel the primary sound field in the region...
of silence. The sensors which provide the controller's input measure the characteristics of the primary wave field. It should be noted that the principle mentioned above also applies to the reduction of vibrations. Though the principle is quite simple, the realisation is much more complicated. This is the main reason why Lueg's idea has come to practice only in the late 70's. It was the rapidly evolving digital electronics together with the new insights in acoustic control that enabled the first practical implementation of systems for active sound reduction (ASR). Nowadays, the state of the modern processors still is a key factor in the capabilities of systems for ASR.

1.1 Active and Passive Noise Reduction

The classical way to deal with noise problems is by utilising absorption and/or reflection. The widely used absorptive mufflers convert the acoustic energy of the incident wave into thermal energy. And a sound wall prevents noise to enter a residential area by reflecting the incident wave field. As no secondary sound field is generated here, these methods are denoted as passive sound reduction. Let us compare the specific features of active and passive systems:

- Systems for active sound reduction show a good performance for the low frequency band; say below 1 kHz. The performance of passive systems is poor for the band below 500 Hz. A reasonable reduction by passive systems would require heavy and voluminous attenuators. For frequencies higher than 1 kHz the performance of active systems deteriorate since sound fields are more complex at higher frequencies. Conversely, passive systems show an improved performance and do not require much volume or weight for the higher frequency band. So, active and passive systems for noise reduction are complementary.

- Systems for ASR can be implemented such that only a specific noise component of the sound field is attenuated. A feedforward ASR system e.g will only reduce that part of the sound field that is correlated with its input signal. Also, ASR systems can be tuned to operate on a specific frequency band only. By this feature we can improve e.g. the speech intelligibility or the perception of music and warning signals.

- As passive sound reduction systems have to absorb and/or reflect sound, they may cause a drop in the flow rate when used in ventilation systems, jets or exhaust pipes. This deteriorates the original performance of the relevant (noisy) machine.

- In a polluted (flour or soot) environment the operation of a muffler is gradually deteriorating and should be replaced periodically. A system for ASR can be installed such that it is unaffected by the environment. On the other hand the performance of these systems may be affected by high temperatures or electromagnetic fields. Moreover, as ASR systems are partly electronic they demand regular servicing.
Section 1.2  Reconstruction of sound fields

- Poorly designed ASR systems may amplify the original sound field.
- Although a lot of effort is put into the product development of ASR systems, special problems require a lot of research beforehand. Therefore, the ASR solution usually is more expensive than the passive solution.
- A control system for ASR communicates with the sound field by means of sensors and actuators. In certain cases suitable transducers may not be available to realise an active sound reduction system; e.g. low frequency sound sources that can generate high sound levels as required in large Roots blowers.

With the above in mind we can define the following application areas for active sound reduction systems:

1) ducts; this includes fan noise, engine exhaust noise, compressor inlet noise and noise in industrial or domestic blowers.
2) (small) enclosures; this area concerns headsets, cabins of aircraft (propeller, jet or helicopter), vehicles (cars, trucks and buses) and control rooms.
3) exteriors; ASR systems can be employed to reduce the radiation from e.g. transformers or gas turbines.

The third application area of ASR systems is not as important as the other two. Furthermore, in application areas 2 and 3 solutions based on active reduction of vibration are gaining attention.

1.2 Reconstruction of sound fields

Reduction of sound fields is closely related to reconstruction of sound fields. More precisely, reduction of a sound field means reconstruction of a sound field with opposite polarity. Therefore, the terms reconstruction and reduction are used interchangeably in this Section. A distinction is made between temporal and spatial reconstruction. The degree of temporal or spatial reconstruction is determined by several properties of the primary sound field. Before addressing these relations, first two basic system setups for sound control are shown.

The first principal system to obtain reduction of the primary noise field is depicted in Figure 1.2. The noise field is modelled as being excited by a disturbance generator. This generator may stand for any kind of noise source.

The contribution of the primary noise in the region of silence is related to the disturbance generator through a transfer function. The principle of this ASR system is that both the disturbance signal and the transfer function are assumed to be known. In order to cancel the primary noise in the appointed region the ASR system has to simulate the transfer function. If the simulated transfer function exactly equals the primary transfer function in opposite phase, the anti-sound completely cancels the primary sound in the appointed region.
In a more practical situation the knowledge of the disturbance generator and transfer function may not be exact or complete. Then, attenuation can still be obtained. This non-ideal performance depends on:

1) The cross-correlation properties of the input signal to the ASR system and the primary noise in the region of silence. If the disturbance generator is only partly known or other generators are involved as well, complete cancellation is no longer achievable. The correlation between the measured disturbance signal and the total primary noise therefore determine the reduction.

2) The accuracy of the simulation of the primary transfer function. In general, the primary transfer function is characterised by a set of parameters, that needs to be estimated before or during the reduction process. An accurate simulation of the primary transfer function requires accurate estimation.

These two characteristics are common for control systems based on the feedforward principle.

The second principal setup to achieve attenuation of the primary noise is illustrated in Figure 1.3. Here, there is no knowledge about neither the disturbance generator nor the primary transfer function. The ASR system learns about the primary noise by measuring it in the region of silence. By feeding this information back to the region, the ASR system confronts the noise with its own time history. In this way reduction of the primary noise is obtained for those components that are correlated with the primary noise of earlier instants. In fact, the ASR system predicts the primary noise in the region of silence as well as possible. So the performance of this feedback ASR system depends on the auto-correlation properties or predictability of the primary sound in the region of silence.
Figure 1.3: A feedback configuration of an active sound reduction system.

Note that the behaviour of the feedback ASR system is determined by the signal characteristics (second order statistics) of the primary noise. This is in contrast with the feedforward ASR system, which is determined by the system characteristics (transfer function) of the primary sound.

In the forthcoming analysis on temporal reconstruction the performance of the two principal systems will be discussed separately. It will turn out that the characteristics of the primary noise and the acoustical environment have a distinct influence on the performance of the two systems.

**temporal reconstruction I; correlation characteristics**

The temporal characteristics of the primary sound field are determined by two parameters;

1) The temporal characteristics of the disturbance signal.

2) The reverberation properties of the region of silence contained in the transfer function.

The temporal performance of the two basic ASR systems is analysed for four special cases of these parameters; see Figure 1.4.

The acoustical free field is described by the property that all frequency components of a sound field are treated equally with respect to their amplitude. A reverberant enclosure, however, shows a strong preference for specific frequencies (resonances) whereas other frequencies (anti-resonances) are suppressed. As the region of silence can be interpreted as a filter operating upon the disturbance signal, the free field does not change the temporal characteristics of the disturbance signal. A poorly damped enclosure, however, can substantially alter the characteristics of the disturbance.

The amount in which the feedforward system can reconstruct the primary signal depends on the cross-correlation function of the primary signal and the detected disturbance signal. Ideally, this is not related to the spectral properties of the noise nor the damping properties of the region. The performance of the feedback system does directly relate to these parameters and will be
addressed in more detail below. In order to give this analysis practical relevance, we have to assume a certain time delay between the measurement of primary noise in the region of silence and the feedback controller's response to it.

<table>
<thead>
<tr>
<th>Case</th>
<th>Region of Silence</th>
<th>Disturbance Signal</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>Free field</td>
<td>White noise</td>
</tr>
<tr>
<td>2</td>
<td>Free field</td>
<td>Narrowband noise</td>
</tr>
<tr>
<td>3</td>
<td>Reverberant enclosure</td>
<td>White noise</td>
</tr>
<tr>
<td>4</td>
<td>Reverberant enclosure</td>
<td>Narrowband noise</td>
</tr>
</tbody>
</table>

*Figure 1.4: Characteristics of the primary field in 4 extreme cases.*

CASE 1
Here the feedback ASR system cannot reduce anything because the auto-correlation properties of the primary sound do not allow any prediction; theoretically the auto-correlation function consists of a single peak at lag zero. Therefore, the white noise in the region of silence is completely unpredictable.

CASE 2
Narrowband noise in the free field is well predictable and therefore gives the feedback ASR system the opportunity to achieve a substantial reduction.

CASE 3
Although the disturbance signal is unpredictable a reasonable reduction is still possible. Because the enclosure amplifies certain frequencies and attenuates others, the primary sound becomes partly predictable. The feedback ASR system is capable of reducing the resonances. The 'colouring' of the white noise by the enclosure will be counteracted by the ASR system. The feedback controller 'whitens' the primary sound.

CASE 4
This is the most favourable situation for the feedback ASR system. Both the enclosure and the disturbance signal give rise to a well-predictable primary sound. A very good performance can therefore be expected here. A rare exception to this rule occurs when the enclosure's geometry is such that it suppresses all (harmonic) components of the disturbance signal; anti-resonances at the disturbance frequencies.
temporal reconstruction II; the inversion of the secondary process

In the above analysis the controller response in the region of silence was assumed to be equal to the controller output. In general however, the response is a filtered version of the controller output. The so-called *secondary transfer function* or *secondary process* relates the controller output to its response in the region of silence (the anti-sound field). Usually, the secondary transfer function is built up of D-A converter, low-pass filter, loudspeaker, acoustic path and microphone. The role of the secondary transfer function is depicted in Figures 1.5a and 1.5b for both systems.

![Figure 1.5a](image1)  Feedforward ASR configuration with secondary transfer function.

![Figure 1.5b](image2)  Feedback ASR configuration with secondary transfer function.

In order to create the desired response in the region of silence, the controller output has to be corrected for the secondary transfer function. In principle, the *inverse* of the secondary process has to be part of the controller. Let us examine the consequences of this requirement for two extreme types of primary signals.
NARROWBAND SOUND
In this case knowledge of the secondary transfer function is only required for the frequency band of the disturbance signal. Usually, correcting for this transfer function is straightforward, as no true inversion has to be carried out. Fortunately, the practical cases of narrowband noise are numerous. An operating machine with rotating or reciprocating parts is likely to produce noise of an harmonic nature. The frequency of the fundamental harmonic is directly related to the revolution speed of the relevant rotating part.

BROADBAND SOUND
Here the secondary transfer function has to be known for the broad frequency band. The inversion of this transfer function may be troublesome. If an acoustic process has non-minimum phase behaviour, a stable and causal inverse does not exist. As will be shown in Chapter 5 it is quite common for discrete-time acoustic processes to have unstable zeros. In that case the ASR system has to correct for the secondary transfer function by using an approximate inverse. The effect of this approximation on the performance depends on the acoustical properties of the region of silence. Let us discuss this for the two extreme cases of the region of silence:

1) THE FREE FIELD: It turns out that the probability of unstable zeros in the secondary process is low in well-damped regions. However in the case of one or more unstable zeros, the degradation in performance by implementing an approximate inverse can be quite significant.

2) THE REVERBERANT ENCLOSURE: In Chapter 5 it is shown that the stronger the resonances, the higher the probability (> 0.5) of a non-minimum phase process. The stabilised approximation of the inverse of a non-minimum phase process, however, may only slightly degrade the performance. This is due to the fact that the resonances can still be well attenuated.

*temporal reconstruction III: the delay in the secondary process*
Apart from a few other properties, a feedforward and a feedback controller differ in the way the primary field is measured. The feedforward principle is based on gathering time-advanced information on the primary sound. If there is a time delay between the primary sound measured by the detection sensor(s) and that measured by the sensors in the region of silence, a feedforward controller always is more up to date than a feedback controller.

In fact, a feedback controller should always react to a sensor signal as fast as possible. In doing so its response is slowed down by the delay in the secondary process. Once more, the
Section 1.2  Reconstruction of sound fields

performance of a feedback controller depends on the delayed auto-correlation characteristics of
the primary sound.
As a feedforward controller is supplied with advanced primary information, the secondary
delay probably does not have such a grave influence on the system's performance. However, if
the secondary delay exceeds the delay of the primary transfer function, the early primary
response cannot be cancelled by a feedforward controller either. The ideal controller should be
non-causal in such a configuration. The limits to performance depend on the importance of the
early, primary response. Again, in a poorly damped enclosure the resonances can still be
attenuated. In the free field, however, the early primary response can represent the majority of
primary sound. In that case the causality requirement has far-reaching consequences on the
performance.

spatial reconstruction

The performance of an ASR system to reconstruct a sound field spatially, mainly depends on
the following items:
1) the extent of the region of silence with respect to the disturbance wavelength.
2) the damping properties of the region of silence.
3) the spectral contents of the primary sound
4) the extent of primary source distribution with respect to the wavelength.

Spatial reconstruction of a sound field is closely related to its complexity. Therefore, an
important aspect of the primary noise field - either in free or enclosed space - is its number of
degrees of freedom. A low-frequency source in a wave guide e.g. can only excite plane waves
travelling upstream or downstream. A similar source in free space, however, many excite plane
waves in all possible (3-dimensional) directions. In an enclosed space the eigenmodes can be
interpreted as degrees of freedom. It seems obvious that the number of degrees of freedom of a
noise field determines how well or easily it can be reconstructed, and how many control sources
and sensor are required. In general, the degrees of freedom of a sound field are directly related
to the ratio of wavelength and size of the relevant region. Together with items 2, 3 and 4 it plays
an important role in how many control sources and sensors are needed for a reasonable field
reconstruction.

THE EXTENT OF THE REGION OF SILENCE
This parameter determines the upper limit of the degrees of freedom of a sound field in the
region. In an lightly damped enclosure e.g. every eigenmode has a cut-on frequency or
resonance frequency, below which it hardly contributes to the sound field. As a rule of thumb
we may presume that an ASR system needs one (independent) source per (dominant) degree of
freedom. So, the size of the region of silence with respect to the wavelength determines the
number of sources and sensors of the ASR system. Therefore, a large region like a vehicle’s cabin requires more anti-sound sources than a small volume, like in an ear-defender. As a consequence, in practice the performance of ASR systems usually is better for small region problems than for large region problems.

THE DAMPING IN THE REGION OF SILENCE

The primary wave field in a closed volume consists of spatial pressure maxima and minima. This spatial distribution of acoustic energy depends on the reverberating characteristics of the volume. In the previous paragraph it was stated that every resonance or eigenmode only contributes above its cut-on frequency. Essentially, this holds for lightly damped enclosures only. The more the absorption at the boundaries of the region, the more an eigenmode contributes to the sound field below its cut-on frequency. Besides, it holds that the more absorption the less pronounced the resonances and anti-resonances. In the extreme case - the free space - we can no longer speak of maxima and minima or resonance frequencies. This leads us to the rule of thumb that the less damping in the region of silence, the less eigenmodes dominate the sound field, and - as a consequence - the less sources and sensors an ASR system needs to obtain reasonable attenuation. In other words, given a fixed number of sources and sensors an ASR system will achieve the best results in a lightly damped space (with a small modal overlap).

THE SPECTRUM OF THE NOISE

In the two paragraphs above discussing the influence of acoustic parameters on spatial reconstruction, it was assumed that the primary noise has a broadband nature. In that case an ASR system has to cope with all resonances and anti-resonance occurring in the relevant frequency band. Obviously, if the primary noise is narrowband an ASR system might be simplified just to deal with the few modes excited by the primary source. Especially, if the primary sound resonates in the enclosure only a small amount of control sources and sensors can achieve substantial attenuation. Moreover, one can imagine that for a given ASR system the optimal spatial distribution of sources and sensors is frequency dependent. Therefore, if the primary noise is narrowband one can optimise the configuration of the ASR system.

THE EXTENT OF THE PRIMARY SOURCE DISTRIBUTION

The reconstruction of the primary sound field by means of a distribution of secondary sources can also be interpreted as reconstructing the primary source. If the primary source is compact it may be very effective to locate a set of secondary sources nearby. In this way reduction of the primary field is achieved by designing the distribution of secondary sources such, that the total distribution has poor radiation characteristics. The applications of ASR in exhausts or gas turbines (see e.g. [Ff81]) can be interpreted to be designed by this criterion. In the case of a
distributed primary source this direct approach becomes much less feasible. Usually, the
distribution of secondary sources is placed nearby the region of silence. This simplifies the
design of the ASR system at the cost of a limited region of sound reduction.

**stability and robustness**

In the beginning of this Section two principal ASR systems were described - the feedforward
and the feedback based approach - that have a distinct behaviour with respect to the temporal
reconstruction (reduction) of a primary noise field. This is due to the way in which information
on the primary noise is detected. Based on the analysis of temporal reconstruction a
feedforward controller must be expected to achieve the best performance; i.e. if the primary
sound can be detected in advance.

However, a controller should not be judged on its optimal performance only. Let us again
compare the two control setups; see Figure 1.6. Note that both controllers are non-adaptive.

![Block diagram of an open-loop controller (a) and a closed-loop controller (b).](image)

*Figure 1.6; Block diagram of an open-loop controller (a) and a closed-loop controller (b).*

The feedforward controller in Figure 1.6 can be compared to an open-loop controller. An
open-loop controller generates its control signal on the basis of past and present values of the
input signal only, which is in this case the signal from the detection sensor. No information
about the acoustic process is fed to the open-loop controller except for what is available before
the control action starts. A closed-loop controller - the feedback controller - can take advantage
of the information about the process, as this is contained in the input signal. Although a
feedback controller may not reject disturbances as well as a feedforward controller, it does
"observe" the acoustic process characteristics. This essential distinction between a feedforward
and a feedback controller has far-reaching consequences on their robustness properties.
ROBUSTNESS OF PERFORMANCE
The robustness of control performance is related to the sensitivity of performance to unmodelled or changing influences. For the acoustic case we can mention the following perturbations:

1) Unknown or extraneous disturbances.
2) Variations in the acoustical characteristics.
3) Detection sensor observation noise.
4) Output sensor observation noise.

In the following Table a comparison is drawn between the behaviour of feedforward and feedback controllers with respect to these perturbations. The Table shows the proportion in which each of the four influences may deteriorate the performance of the specific controller. These characteristics can be found in many of the textbooks on control theory; see e.g. [KS72].

<table>
<thead>
<tr>
<th>perturbation</th>
<th>open-loop or feedforward control</th>
<th>closed-loop or feedback control</th>
</tr>
</thead>
<tbody>
<tr>
<td>unmodeled disturbances</td>
<td>full effect on residual</td>
<td>effect on residual can be reduced</td>
</tr>
<tr>
<td>acoustic process variations</td>
<td>full effect on residual</td>
<td>effect on residual can be reduced</td>
</tr>
<tr>
<td>detection sensor observation noise</td>
<td>full effect on residual</td>
<td>no effect on residual</td>
</tr>
<tr>
<td>output sensor observation noise</td>
<td>no effect on residual</td>
<td>effect on residual</td>
</tr>
</tbody>
</table>

Figure 1.7: Overview of the sensitivity of two types of controllers to four kinds of perturbations.

The overview 1.7 shows the possibility to design a feedback controller such that degradation of performance can be diminished when unmodelled influences occur. The structure of a feedforward controller is not suitable for robust design. The reader should realise, however, that a robust design of a feedback controller generally implies only a modest reduction of the modelled disturbance. If a feedback controller is designed to have an optimal reduction of the modelled disturbance it will likely be very sensitive to the influences mentioned above. So we may complete the robustness analysis with the remark that a combined feedforward-feedback controller can achieve a robust reduction of measurable (using a detection sensor) primary noise. This principle will be worked out in the thesis.
ROBUSTNESS OF STABILITY
Taking a close look at the structure of the feedback controller in Figure 1.5b reveals that the closed-loop characteristics should be carefully designed in order to prevent instability. Moreover, if the secondary acoustic process is time-variant an initially stable closed-loop might grow unstable. So for a feedback controller the robustness of stability is an important issue. Note that a stable feedforward controller can never give rise to instability as long as the acoustic process is stable, which is what we presume throughout. It turns out that stability robustness and performance robustness of a feedback controller are conflicting requirements. This unfavourable - though familiar - physical phenomenon can be easily imagined, as the most robust stability is obtained by not to control.

The robustness of stability is also related to the acoustical characteristics of the region of silence. In a hard-walled wave guide, for example, the secondary acoustic process is bound to show strong resonances and anti-resonances. This corresponds to both poles and zeros of the process close to the stability border. In order to create a significant reduction a feedback controller has to cancel the undamped zeros by means of poles in the control function. Then only a slight change in the acoustic process can make the closed-loop unstable.

![Diagram](image)

*Figure 1.8: Block diagram of adaptive control of disturbances; (a) adaptive feedforward control and (b) adaptive feedback control.*

ADAPTIVE CONTROL
In particular cases it may suffice to measure the characteristics of the primary sound and the transfer function between control transducers beforehand. Subsequently, the design of the fixed-time control systems is carried out and the device is switched on. The operation conditions of many practical noise source mechanisms, however, are bound to change with time. This usually leads to varying characteristics of noise and acoustics. As mentioned above the effects of these changes can be reduced by designing robust controllers. This may however
not be sufficient to maintain the required performance and stability. It might be very effective to implement an on-line adaptive control design that keeps up with the occurring changes. This leads to the structure of Figure 1.8.

The adaptive function analyses the controller output signal and the residual signal. Recursively an update of the controller characteristics is calculated and loaded down to the controller. In the case of a time-varying secondary acoustic process two main approaches can be utilised:

1) INDIRECT ADAPTIVE CONTROL. By means of its input signals the adaptive procedure continuously estimates the secondary acoustic process. Based on the most recent estimate the corresponding controller characteristics are calculated.

2) DIRECT ADAPTIVE CONTROL. By means of the input signals the adaptive procedure directly estimates the controller parameters. This approach requires to rewrite the system equation into an expression in terms of the controller parameters, taking into account the specific control objective.

Especially for a feedforward controller an adaptation loop can significantly improve the performance robustness. Actually, an adaptation loop can be viewed upon as a ‘soft’ closed-loop with all its features. Still, the designer of control systems for active noise reduction should be reluctant to use adaptive control from the very start. An adaptive procedure may draw heavily on the system’s hardware and software. One should employ the robustness improving benefits of a feedback loop first.

1.3 Objective of the thesis

A system for active sound reduction comprises three main disciplines, acoustics, control and hardware/software engineering. Although the problem of noise and the solution of anti-noise is purely acoustic, the auxiliary disciplines of control and system engineering should have an equal share of attention in the overall system design. Obviously, one cannot design the overall system without gearing the individual disciplines for one another. If we consider the literature on active sound reduction a full integration of these three aspects is not often found. A main objective of the thesis is to integrate these fields, especially the relation between acoustics and control.

The activities on ASR in 60’s and 70’s were mainly based on an acoustic point of view. In order to realise systems the theory of ‘adaptive filtering’ was applied to the problem. This development was inspired by the work of Widrow, who reported on successful experiments with adaptive noise cancellation in electronic systems; [Wi75]. Nowadays, a quick glance at the literature shows that adaptive filtering still is the main topic in designing the control system. One may even get the impression that the world of ASR is nothing but adaptive filtering with
Widrow's LMS algorithm and variants thereof. Unlike the field of active vibration reduction, a fundamental control approach to active sound reduction is seldom employed. The recently published textbook [NE91] on active sound reduction gives an excellent overview of the state of the art. It is therefore unavoidable that specific parts of the thesis correspond to the material in this textbook. The main contribution of the thesis, however, is the introduction of modern control theory and the demonstration of its applicability for the design of ASR systems. Existing techniques in acoustics and control are linked together in a unique and powerful way.

![Diagram](image)

*Figure 1.9: Design trajectory of a system for active sound reduction.*

The thesis is set-up in a way that runs parallel to the design trajectory. It starts with the acoustical principle of complete attenuation by so-called Huygens' sources. After that, the acoustic design is discussed by introducing a finite number of sources and sensors. The leitmotiv in the acoustical chapters is the cost function, which represents the annoyance of the noise field. In every step towards a practical realisation this cost function has to be adapted while maintaining its original goal. Having specified the acoustical characteristics (sensors and sources) of the system, the controller has to be designed such that the acoustic aim is fulfilled. In most cases, the acoustic cost function has to be approximated again in order to achieve a
good performance together with stability and robustness. The Chapters on control cover the path from optimal control to a realisable algorithm, that can be implemented on hardware. The block diagram in Figure 1.9 resumes the trajectory of ASR system design.

Finally, an overview of the thesis' contents is given:

- Chapter 2 addresses the acoustic principles for active sound reduction. It is shown that the Kirchhoff integral is the key to complete primary field cancellation by means of a space-continuous distribution of control sources.
- In Chapter 3 the possible performance of a finite number of control sources is analysed. The measure of performance is given by an acoustic cost function, which has to represent the nuisance of noise. Special attention is paid to the active sound reduction in the exterior, in a duct and in enclosures.
- Chapter 4 completes the acoustic design by introducing sensors to measure the primary sound and the performance. The requirement of causal control action deserves special attention.
- In Chapter 5 the principles of controlling acoustic disturbances are shown. This requires discrete-time models of both the acoustic plant and the disturbance signals. The chapter concludes by showing the performance of LQG-control. The characteristics of feedforward and feedback control are evaluated in this analysis.
- Chapter 6 is to design practical, real-time control strategies for specific applications of ASR. Most of the control algorithms are based on Generalised Minimum Variance (GMV) control, as this is easy to implement and does not require the enormous amount of computational effort as the LQG-strategy. The popular strategy of adaptive filtering is shown to be a special case of the proposed control strategy. As most of the resulting control strategies are adaptive, the issue of parameter estimation is discussed as well.
- The final Chapter 7 deals with a couple of practical experiments to verify the theory of the preceding chapters. The application of ASR is tested in a multi-modal duct, an echo-free chamber, and a reverberant enclosure. Moreover, the features of feedback control and combined feedforward/feedback control are demonstrated in a single-channel experiment.
1.4 References


CHAPTER 2

Acoustical Principles of Active Sound Reduction

In this Chapter the theoretical acoustic framework for active sound control is formulated. Most of the acoustical relations used can be found in the textbooks on theoretical acoustics like [Pi81] or [MF53]. Therefore, the fundamental theory and equations are given in a concise way. It will be demonstrated that a given wave field in an arbitrary closed volume can be exactly reconstructed by a suitably chosen source distribution. This phenomenon is closely related to Huygens' principle of wave field propagation. The potential to reconstruct a sound field of a distribution of Huygens' sources will turn out to be the acoustic essence of active sound reduction.

2.1 The fundamental equations of linear acoustics

This Section presents the two fundamental equations describing the acoustical wave theory used in this thesis. The relevant acoustic medium is a homogeneous, isotropic and quiescent fluid. Moreover, it is assumed to be inviscid and non-thermally conducting. The equation for conservation of mass describes the physical requirement that the time rate of change of mass inside an arbitrary volume equals the net mass per unit time entering the volume. Expressed in acoustic field quantities the linear approximation of the law of mass conservation for the relevant fluid reads

\[ \frac{1}{c_0^2} \frac{\partial p(\mathbf{r}, t)}{\partial t} + \rho_0 \mathbf{\nabla} \cdot \mathbf{v}(\mathbf{r}, t) = 0 \]  

(2.1)

in which the scalar \( p \) and vector \( \mathbf{v} \) are the acoustic fluid pressure and velocity as a function of position in scalar coordinates, \( \mathbf{r} = (x, y, z)^T \), and time \( t \). The quantities \( c_0 \) and \( \rho_0 \) stand for ambient speed of sound and ambient density respectively. This expression corresponds to Hooke's law for a unit mass per unit volume [Be87].

Second, Euler's equation of motion gives the relation between the spatial variation of pressure and the corresponding particle acceleration. The linearised form of this momentum conservation law is given by

\[ \rho_0 \frac{\partial \mathbf{v}(\mathbf{r}, t)}{\partial t} + \nabla p(\mathbf{r}, t) = 0 \]  

(2.2)

A useful third relation emerges from eq. (2.2) if one realises that the curl of a gradient field equals zero. Therefore it holds for the time derivative of the so-called vorticity
\[ \frac{\partial}{\partial t} [\mathbf{V} \times \mathbf{v}(r, t)] = 0 \quad (2.3) \]

Combining the linearised conservation expressions (2.1) and (2.2) gives the following scalar wave equation:

\[ \nabla^2 p(r, t) - \frac{1}{c_0^2} \frac{\partial^2 p(r, t)}{\partial t^2} = 0 \quad (2.4) \]

An identical relation can be given for each of the cartesian components of particle velocity \( \mathbf{v} \).

An important energetic relation can be derived from (2.1) and (2.2), when taking the dot product of \( \mathbf{v} \) and (2.2), the product of \( p \) and (2.1) and adding the results

\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \rho_0 |\mathbf{v}|^2 + \frac{1}{2} \frac{p^2}{\rho_0 c_0^2} \right) + \mathbf{V} \cdot (p \mathbf{v}) = 0 \quad (2.5) \]

The term \( \frac{1}{2} \rho_0 |\mathbf{v}|^2 \) is usually referred to as the acoustic kinetic energy density, \( \frac{1}{2} p^2 / (\rho_0 c_0^2) \) as the acoustic potential energy density, and \( p \mathbf{v} \) as the instantaneous acoustic intensity \( \mathbf{I} \).

Applying the divergence theorem

\[ \int_V \nabla \cdot \mathbf{a} \, dV = \oint_S \mathbf{a} \cdot \mathbf{n} \, dS \quad (2.6) \]

integration of (2.5) over an arbitrary volume \( V \) gives the energy conservation law:

\[ \frac{\partial}{\partial t} (E_{\text{kin}} + E_{\text{pot}}) + \oint_S \mathbf{I} \cdot \mathbf{n} \, dS = 0 \quad (2.7) \]

in which a capital \( E \) stands for the total energy of the relevant type in \( V \) and \( S \) encloses the volume \( V \), as shown in Figure 2.1. Equation (2.7) expresses the requirement, that the time rate of change of total acoustic energy inside \( V \) equals the net acoustic energy flux out of \( V \) into \( \overline{V} \).

---

Figure 2.1: Diagram of the geometry of volumes \( V \) and complementary volume \( \overline{V} \), surface \( S \) and normal vector \( \mathbf{n} \).
2.2 Sources of acoustic wave fields

The fundamental equations of linear acoustics presented in the previous Section all apply to a sourceless medium. In order to include an arbitrary distribution of acoustical sources into the analysis, the fundamental equations have to be extended. Two basic kinds of sources will be considered. A source that introduces a fluctuating volume flow into the medium, like a radially pulsating sphere, can be governed by the following equation of mass conservation:

$$\frac{1}{c_0^2} \frac{\partial p(r,t)}{\partial t} + \rho_0 \nabla \cdot v(r,t) = \rho_0 q(r,t)$$  \hspace{1cm} (2.8)

in which $q$ stands for the source strength density with dimension [1/s]. The product $\rho_0 q$ represents the rate of mass introduction per unit volume. Practical examples of a volume source are the acoustic siren and a loudspeaker in a closed box.

When a source of fluctuating force acts on the medium, the linearised equation of momentum conservation has to be rewritten as

$$\rho_0 \frac{\partial v(r,t)}{\partial t} + \nabla p(r,t) = f(r,t)$$  \hspace{1cm} (2.9)

The vector $f$ is the external force per unit volume. Practical examples of a force source are a transversely oscillating thin disk in a fluid [Pi81], and an open loudspeaker.

The scalar wave equation corresponding to the inhomogeneous conservation equations (2.8) and (2.9) is given by

$$\nabla^2 p(r,t) - \frac{1}{c_0^2} \frac{\partial^2 p(r,t)}{\partial t^2} = \nabla \cdot f - \rho_0 \frac{\partial q(r,t)}{\partial t}$$  \hspace{1cm} (2.10)

Apart from the scalar wave equation (2.10), an acoustic wave field is determined by the boundary and initial conditions. A general form of boundary conditions will be given in Section 2.4. Here, we only mention the particular boundary condition for an infinite homogeneous medium. Then Sommerfeld’s radiation condition holds, which is in spherical coordinates

$$\lim_{r \to \infty} \left[ r \left( \frac{\partial p(r,t)}{\partial r} + \frac{1}{c_0} \frac{\partial p(r,t)}{\partial t} \right) \right] = 0$$  \hspace{1cm} (2.11)

This condition requires that no waves are reflected at the boundary (at infinity); merely outgoing waves are allowed. The so-called free field solution for wave fields obeys the Sommerfeld radiation condition.

A special kind of source is the monopole. The concept of this point source at $r_0$ is governed by expressing it as a volume source with strength distribution

$$q_m(r,t) = u(t) \delta(r - r_0)$$  \hspace{1cm} (2.12)

where $u(t)$ represents the temporal characteristics of the source strength. The spatial delta function represents a volume density of time rate of volume injection, that equals zero
everywhere except at \( r_0 \). Although the use of a generalised function as the Dirac \( \delta \)-function is non-physical, it is an attractive tool in the mathematical analysis. Moreover, as the bandwidth under consideration is limited, there is a finite resolving power. This means that the delta function can be interpreted as one of its, physically existing, smoothed and bounded equivalents. Here, the point monopole source can be considered as a limit of a radially pulsating sphere. The solution of the inhomogeneous wave equation for the free field boundary condition (2.11) and a monopole source can be found to be [MF53]:

\[
p_m(r | r_0, t) = \rho_0 \frac{S(t - \frac{|r - r_0|}{c_0})}{4\pi |r - r_0|}
\]

(2.13)

in which the source function \( S(t) = \frac{\partial}{\partial t} u(t) \). The monopole pressure field appears to be spherically symmetrical. An observer at \( r \) receives the source information retarded by travel time \( |r - r_0|/c_0 \) and attenuated by a factor proportional to the distance.

![Figure 2.2: Schematic illustration of the dipole source.](image)

Another important type of source is the dipole; see Figure 2.2. Two monopoles with opposite source strengths located at short distance \( |\Delta r_0| \ll \Delta r_0 \) from each other yield the following superimposed field

\[
p_d(r, t) = \rho_0 S(t - \frac{|r - (r_0 + \frac{1}{2}\Delta r_0)|}{c_0}) - \rho_0 S(t - \frac{|r - (r_0 - \frac{1}{2}\Delta r_0)|}{c_0})
\]

(2.14)

In the limit of small \( \Delta r_0 \), expansion of (2.14) in a Taylor series around \( r_0 \) yields the following dipole field (powers of \( \Delta r_0 \) higher than two neglected)

\[
p_d(r, t) = \Delta r_0 \cdot \nabla \rho_0 \frac{S(t - \frac{|r - r_0|}{c_0})}{4\pi |r - r_0|} = -\Delta r_0 \cdot \nabla p_m(r, t)
\]

(2.15)

Note that the gradient \( \nabla \rho_0 \) is taken with respect to the source coordinates. Evaluating the scalar wave equation for the dipole field in (2.15) gives
\[ \nabla^2 p_d(r,t) - \frac{1}{c_0^2} \frac{\partial^2 p_d(r,t)}{\partial t^2} = \rho_0 \frac{\partial}{\partial t} [\Delta r_0 \cdot \nabla q_m(r,t)] \]  

(2.16)

Comparing the source term in eq. (2.16) to that in (2.10) and inserting (2.12) renders for the dipole wave equation

\[ \nabla^2 p_d(r,t) - \frac{1}{c_0^2} \frac{\partial^2 p_d(r,t)}{\partial t^2} = \nabla \cdot f_d(r,t) \]  

(2.17)

in which

\[ f_d(r,t) = \rho_0 s(t) \delta(r - r_0) \Delta r_0 \]  

(2.18)

with \( s(t) \) as the source function. So a dipole source is equivalent to a source of force. Note that the direction of dipole force is parallel to its axis.

The balance equation for acoustic energy, that was shown in the previous Section can be adapted to include the effects of the \( q \) and \( f \) source distribution. We find

\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \rho_0 |\mathbf{v}|^2 + \frac{1}{2} \frac{p^2}{\rho_0 c_0^2} \right) + \nabla \cdot \mathbf{I} = pq + \mathbf{v} \cdot \mathbf{f} \]  

(2.19)

The products \( pq \) and \( \mathbf{v} \cdot \mathbf{f} \) represent the acoustic power per unit volume supplied by the volume and force source distribution. With the use of eq. (2.19) it is straightforward to evaluate the total radiated power of e.g. a monopole. Inserting eq. (2.12) and integrating over a volume including the source gives

\[ \frac{\partial}{\partial t} E_{\text{tot}} + \oint S \cdot \mathbf{n} \, dS = u(t) \rho(r_0,t) \]  

(2.20)

For the monopole the radiated acoustic power is determined by the product of \( u(t) \) and pressure at the very source point. In [Ne87] expression (2.20) is evaluated for a monopole source radiating sound of random character. It follows that the time average power output of the source is finite, although the pressure is singular at the source point.

2.3 Non-uniqueness of source distributions

In this Section the inhomogeneous wave equation (2.10) will be examined. A lucid reasoning will show, that the source distribution of a given sound field in an arbitrary volume \( V \) cannot be determined in a unique way. The following example of the non-uniqueness is well-known and can be found in [MG72]. Introduce a spatial weighting function \( M(r) \) and define the weighted wave field \( \tilde{M}p \). Suppose that the weighting function equals unity inside a certain volume \( V \) and zero in \( V^c \); see Figure 2.3.
Then the weighted wave field $Mp$ is equal to the original wave field inside $V$. Evaluating the homogeneous scalar wave equation for the weighted wave field yields

$$\nabla^2(Mp) - \frac{1}{c_0^2} \frac{\partial^2(Mp)}{\partial t^2} = \nabla \cdot (p
abla M) + \nabla p \cdot \nabla M$$  \hspace{1cm} (2.21)

This inhomogeneous wave equation clearly reveals the sources for the sound field $Mp$; equal to $p$ inside $V$ and a zero field in $\overline{V}$. Comparing (2.21) to (2.10) we may conclude, that the sound field $Mp$ can be generated by distributions of force sources with strength density $f = p \nabla M$ and volume sources with a strength density given by $\rho_0 \frac{\partial}{\partial t} q = -\nabla p \cdot \nabla M$. In this case the sources are located on the surface $S$ only, since $\nabla M$ is vanishing everywhere else. The possibility of sound field reconstruction will be further addressed in Section 2.5.

The mathematical manipulations shown above represent the physical principle, that a sound field in a given volume can be generated by many different acoustic source distributions. The non-uniqueness of a source distribution for a sound field gives the opportunity to reconstruct the original sound field in the given volume. In fact, one can transform the sound field into any other sound field by means of an additional source distribution. The sources for the transformed field can be found by evaluating (2.21). An important application of this principle is active sound reduction, in which the original sound field is reconstructed in $V$ with equal amplitude but opposite polarity. So, the superimposition of the sound fields yields a null field.

### 2.4 The Helmholtz equation and Green’s function

In this Section a general solution will be derived for the acoustic wave field generated by an arbitrary source distribution. This solution again will demonstrate the non-uniqueness of acoustic sources for a given sound field in volume $V$. In order to simplify the mathematical analysis required, first the Fourier transform is defined as; [Be87]
Section 2.4  The Helmholtz equation and Green’s function

\[ P(\omega) = \int_{-\infty}^{\infty} p(t) e^{-j\omega t} dt \]  
(2.22)

and the inverse transform is given by

\[ p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\omega) e^{j\omega t} d\omega \]  
(2.23)

The Fourier transform (2.22) can be interpreted as a decomposition of functions of variable \( t \) into functions of \( e^{j\omega t} \), in which \( \omega \) stands for the angular frequency. Throughout the thesis the Fourier transform of a function \( p(t) \) will be represented by a capital: \( P(\omega) \).

The Fourier transform can be used to derive the Helmholtz equation. Applying (2.22) to (2.4) results in

\[ \nabla^2 P(\mathbf{r}, \omega) + k^2 P(\mathbf{r}, \omega) = 0 \]  
(2.24)

the homogeneous Helmholtz equation. The variable \( k \) is called the wavenumber and equals \( \omega/c_0 \). The acoustic pressure field \( P(\mathbf{r}, \omega) \) has now been decomposed in monochromatic components. The inhomogeneous Helmholtz equation follows from eq. (2.10)

\[ \nabla^2 P(\mathbf{r}, \omega) + k^2 P(\mathbf{r}, \omega) = \nabla \cdot \mathbf{F}(\mathbf{r}, \omega) - j \omega \rho_0 Q(\mathbf{r}, \omega) \]  
(2.25)

Next, Green’s function is introduced as being the solution of eq. (2.25) with an elementary source term

\[ \nabla^2 G(\mathbf{r}|\mathbf{r}_0, \omega) + k^2 G(\mathbf{r}|\mathbf{r}_0, \omega) = -\delta(\mathbf{r} - \mathbf{r}_0) \]  
(2.26)

So Green’s function stands for the pressure field response of the relevant medium for a monopole source at \( \mathbf{r}_0 \) with source strength \( 1/j \omega \rho_0 \). As a consequence, Green’s function is a very useful tool for the analysis of acoustic wave fields generated by arbitrary source distributions. For the free field condition, (2.11), the solution for Green’s function is

\[ G(\mathbf{r}|\mathbf{r}_0, \omega) = \frac{e^{-jk|r-r_0|}}{4\pi|r-r_0|} \]  
(2.27)

As can be seen in (2.27) Green’s function remains unchanged if the positions of source and observer are reversed.

\[ G(\mathbf{r}|\mathbf{r}_0, \omega) = G(\mathbf{r}_0|\mathbf{r}, \omega) \]  
(2.28)

Though not proved here, this spatial reciprocity relation is a universal property of Green’s function; see e.g. [Pi81].

As mentioned above, the monopole field for the Helmholtz equation is closely related to Green’s function

\[ P_m(\mathbf{r}, \omega) = j \omega \rho_0 U(\omega) G(\mathbf{r}|\mathbf{r}_0, \omega) \]  
(2.29)

Consequently, a dipole field and Green’s function are related by; see eq. (2.15).
\[ P_t(r, \omega) = j \omega \rho_0 U(\omega) \Delta r_0 \cdot \nabla_0 G(r \mid r_0, \omega) \]  

Expression (2.27) for Green's function is valid for an infinite homogeneous volume and consequently obeys Sommerfeld's radiation condition (2.11). A more general form of a passive boundary condition is given by

\[ \alpha(r_B, \omega) P(r_B, \omega) + \beta(r_B, \omega) \nabla P(r_B, \omega) \cdot \mathbf{n} = 0 \]  

(2.31)

where \( r_B \) denotes the spatial variable for the boundary and \( \mathbf{n} \) is the outward unit normal vector. The coefficients \( \alpha \) and \( \beta \) are generally complex-valued. Special cases of (2.31) are the so-called Dirichlet and Neumann conditions, and Sommerfeld's radiation condition (2.11):

\begin{enumerate}
  \item[a)] \( P(r_B, \omega) = 0 \) \hspace{2cm} \text{Dirichlet}
  \item[b)] \( \nabla P(r_B, \omega) \cdot \mathbf{n} = 0 \) \hspace{2cm} \text{Neumann}
  \item[c)] \( \lim_{|r_B| \to \infty} \left[ \frac{1}{r_B} \left( jk P(r_B, \omega) + \nabla P(r_B, \omega) \cdot \mathbf{n} \right) \right] = 0 \) \hspace{2cm} \text{Sommerfeld}
\end{enumerate}

(2.32)

Boundary condition (2.31) applies to a locally reacting boundary of the relevant homogeneous medium. The specific normal impedance \( Z \) is determined by the coefficients \( \alpha \) and \( \beta \):

\[ Z(r_B, \omega) = j \omega \rho_0 \frac{\beta(r_B, \omega)}{\alpha(r_B, \omega)} \]  

(2.33)

With the use of Green's function a general solution can be derived for the acoustic wave field inside a given volume \( V \). The geometrical specifications for the following analysis is drawn in Figure 2.4.

![Diagram of volume V and a distribution of sources, partly inside and partly outside V.](image)

**Figure 2.4:** Diagram of volume \( V \) and a distribution of sources, partly inside and partly outside \( V \).

For the sake of clarity, throughout the analysis the volume \( V \) lies inside \( \Omega \) and is enclosed by surface \( S \). The volume \( \bar{V} \) is the spatial complement of \( V \) and \( S \) in \( \Omega \). Volume \( \Omega \) itself may be bounded or unbounded. The unit normal vectors \( \mathbf{n} \) and \( \bar{n} \) of closed surfaces \( V \) and \( \bar{V} \) respectively always point outward. Let the acoustic pressure field in \( \Omega \) be determined by
\[ \nabla^2 P(r, \omega) + k^2 P(r, \omega) = -\Psi_{\text{tot}}(r, \omega) \tag{2.34} \]

in which \( \Psi_{\text{tot}} \) represents all acoustic sources present in the medium. For convenience of legibility the argument \( \omega \) will be omitted in most cases from now on.

Multiplying eq. (2.34) by \( G(r | r_0) \), eq. (2.26) by \( P(r) \) and taking the difference yields

\[ G(r | r_0) \nabla^2 P(r) - P(r) \nabla^2 G(r | r_0) = -G(r | r_0) \Psi_{\text{tot}}(r) + P(r) S(r - r_0) \tag{2.35} \]

and integrating the result over \( V \)

\[ \oint [G(r | r_0) \nabla^2 P(r) - P(r) \nabla^2 G(r | r_0)] dV \]

\[ + \oint \Psi_{\text{tot}}(r) G(r | r_0) dV = \begin{cases} P(r_0) & \text{for } r_0 \in V \\ \frac{1}{2} P(r_0) & \text{for } r_0 \text{ on } S \\ 0 & \text{for } r_0 \in \overline{V} \end{cases} \tag{2.36} \]

The factor 1/2 for the surface point value is merely valid for \( S \) having a unique tangent plane in \( r_0 \); [dHo91]. In the case of a less regularly shaped surface another weighting on the local pressure has to be applied. Application of the divergence theorem of Gauss (2.6) and the reciprocity relation (2.28) results in the **generalised Kirchhoff integral**:

\[ P(r_0) = \oint \Psi_{\text{tot}}(r) G(r_0 | r) dV + \oint [G(r_0 | r) \nabla P(r) - P(r) \nabla G(r_0 | r)] \cdot n dS \tag{2.37} \]

for \( r_0 \in V \). The generalised Kirchhoff integral describes the wave field inside an arbitrary volume. According to eq. (2.37) the wave field inside a volume \( V \) can be split up into two contributions. First, the direct response to the sources within \( V \) and second, an indirect response depending on \( P \) and \( \nabla P \) on the enclosing surface. Green’s function \( G \) and its gradient \( \nabla G \cdot n \) play the role of transfer functions. In the next Section the indirect response will be shown to correspond to a distribution of monopoles and dipoles. In both the volume and surface integral (2.37) Green’s function plays an essential role. It should be noted that the choice for \( G \) is not unique. Since Green’s function has to obey eq. (2.26) inside \( V \) only, the medium for \( G \) beyond \( V \) can be chosen arbitrarily. This means that the boundary conditions for \( G(r_0 | r) \) on \( S \) do not necessarily correspond to those for \( P(r) \).

### 2.5 Applications of the Generalised Kirchhoff Integral

The fundamental expression for the sound field inside an arbitrary volume, as developed in the previous Section, will be used for three main applications:

1. The reconstruction of sound fields inside a sourceless volume.
2. The radiation of multipoles.
3. The reconstruction of scattered sound fields.
Kirchhoff-Helmholtz integral

Suppose the volume $V$ and surface $S$ are chosen to be free of acoustic sources, as depicted in Figure 2.5. Then, eqs. (2.36) and (2.37) can be written as

$$P(r_0) = \int_{\Sigma} \left[ G(r_0 | r) \nabla P(r) - P(r) \nabla G(r_0 | r) \right] \cdot \mathbf{n} \, dS \quad \text{for} \quad r_0 \in V \quad (2.38)$$

The wave field inside a sourceless volume is completely determined by its values of $P$ and $\nabla P$ on the enclosing surface. This integral is called the Kirchhoff-Helmholtz integral and can be considered as a reconstruction theorem of a 3-dimensional sound field based on surface data. Evaluation of the equation gives the insight, that the source for sound field reconstruction consists of a continuous monopole and dipole layer. This is equivalent to the result found in Section 2.3, eq. (2.21). With the aid of the equation of conservation of momentum, (2.38) can be rewritten as

$$P(r_0) = -\int_{\Sigma} \left[ G(r_0 | r) j \omega \rho_0 V(r) \cdot \mathbf{n} + P(r) \nabla G(r_0 | r) \cdot \mathbf{n} \right] dS \quad \text{for} \quad r_0 \in V \quad (2.39)$$

With reference to eqs. (2.29) and (2.30) this integral represents a distribution of monopole sources, with source strength $Q(r_s, \omega) = -V(r_s, \omega) \cdot \mathbf{n}$ per unit area, and dipole sources, with equivalent force $F(r_s, \omega) = -P(r_s, \omega) \mathbf{n}$ per unit area, on $S$. Clearly, eq. (2.39) is a mathematical representation of Huygens' principle of wave field propagation.

Figure 2.5 suggests that the surface of monopoles and dipoles should enclose the volume of reconstruction. However, the surface can also be turned inside out and likewise enclose the primary source region. This solution is illustrated in Figure 2.6. Given a volume $V'$ being enclosed by two separate surfaces $S$ and $S'$, in which $S$ encloses the whole primary source region. Again, the unit normal vector $\mathbf{n}'$ points outward. The Kirchhoff-Helmholtz accordingly reads
Section 2.5 Applications of the Generalised Kirchhoff Integral

\[ P(r_0) = \oint_S \left[ G(r_0 \mid r) \nabla P(r) - P(r) \nabla G(r_0 \mid r) \right] \cdot \mathbf{n} \, dS + \oint_{S'} \left[ G(r_0 \mid r) \nabla P(r) - P(r) \nabla G(r_0 \mid r) \right] \cdot \mathbf{n'} \, dS' \quad \text{for} \quad r_0 \in V' \]  
\hspace{1cm} (2.40)

![Diagram of the volume V' and its two enclosing surfaces S and S'](image)

**Figure 2.6:** Diagram of the volume V' and its two enclosing surfaces S and S'.

If we let surface S' coincide with the boundaries of the homogeneous medium, condition (2.31) applies to both pressure P and Green's function G. Then, it can be verified that the surface integral over S' vanishes. Hence, the sound field outside the source region is fully determined by its values of P and \nabla P on the surface S enclosing the source region:

\[ P(r_0) = \oint_S \left[ G(r_0 \mid r) \nabla P(r) - P(r) \nabla G(r_0 \mid r) \right] \cdot \mathbf{n} \, dS \quad \text{for} \quad r_0 \in V' \]  
\hspace{1cm} (2.41)

Even if the homogeneous medium is unbounded, the above reasoning is valid. Letting S' tend to infinity and applying the Sommerfeld radiation condition, as given in eq. (2.32), results in a zero contribution of the S'-integral to wave field inside V'. In Figures 2.7a and 2.7b the two mentioned cases of the Kirchhoff-Helmholtz integral are depicted. Note that V' coincides with volume \( \overline{V} \).

So the Kirchhoff-Helmholtz integral shows that a sound field in a sourceless volume can be represented as being generated by a continuous distribution of monopoles and dipoles. Therefore, the integral is also called the *representation theorem*. It should be realised, that the Kirchhoff integral (2.38) or (2.41) yields a null value for points inside the domain of primary sources. Apparently, the combination of a monopole and a dipole source distribution gives a unidirectional behaviour and cannot cover space outside the relevant volume in representing the original sound field. Inside the sourceless domain the monopole and dipole distribution interfere in a constructive manner, whereas outside this domain the two source distributions interfere in a destructive manner.
source radiation

The second application of the generalised Kirchhoff integral (2.37) follows when the relevant volume is chosen to occupy the whole homogeneous medium $\Omega$. By similar reasons as given above the surface integral will vanish in that case. For both a bounded and unbounded (Sommerfeld’s radiation condition) homogenous medium Green’s function can be chosen to satisfy the same boundary condition as the pressure field $P$. Evaluating (2.37) reduces to the following radiation expression

$$P(x_0) = \int_{\Omega} \Psi_{\text{tot}}(x) G(x_0 | r) \, d\Omega$$

(2.42)

The resultant pressure field can be interpreted as the (infinite) summation over the individual responses ($G(x_0 | r)$) of each point source in $\Psi_{\text{tot}}$.
Figure 2.8: Geometry for which the Kirchhoff-Helmholtz integral over \( S \) yields a zero contribution to the sound field outside \( \Omega \).

Next, consider the configuration as depicted in Figure 2.8, where \( \Omega \) is the volume of interest. The generalised Kirchhoff integral gives the sound field in \( \Omega \) as both a surface \( S \) response and a direct response. The radiation expression (2.42), however, requires the direct response only. Therefore, it holds that

- The Kirchhoff-Helmholtz integral (2.38) equals the original wave field if the target point is on the opposite side of the surface as the primary sources.
- The Kirchhoff-Helmholtz integral (2.38) vanishes if the target point is on the same side of \( S \) as the primary sources.

So, here we have a physical validation of the unidirectional behaviour of Huygens' sources.

**scattered field reconstruction**

The calculation of radiated field becomes more complicated if a scattering object is present within \( \Omega \); see Figure 2.9. Let us denote the resultant sound field as the superposition of an incident field \( P_{\text{inc}} \) and a field reflected from the object, \( P_{\text{scat}} \).

\[
P = P_{\text{inc}} + P_{\text{scat}}
\]

(2.43)

where the incident field \( P_{\text{inc}} \) is defined as the field that would exist, if the scatterer were not present. From the discussion on source radiation it follows that eq. (2.42) describes this incident field. Next, the generalised Kirchhoff integral (2.37) is evaluated for this situation in which the volume of interest is \( \Omega \).

\[
P(r_0) = P_{\text{inc}}(r_0) + \oint_S [G(r_0 \mid r) \nabla(P_{\text{inc}}(r) + P_{\text{scat}}(r)) - [P_{\text{inc}}(r) + P_{\text{scat}}(r)] \nabla G(r_0 \mid r)] \cdot \mathbf{n} \, dS
\]

(2.44)

where \( r_0 \in \Omega \). So, the resultant sound field consists of a direct source response and a response from the boundary of the scattering object. Note that by virtue of eq. (2.41) and the relevant geometry the surface integral over the \( \Omega \)-boundary vanishes. As the scattered field \( P_{\text{scat}} \) is
unknown and appears on both sides of (2.44) we face an integral equation. It should be realised that eq. (2.44) is not a correct description of the sound field inside the scattering object.

![Diagram](image.png)

**Figure 2.9:** Illustration of the geometry of primary source and scattering object.

The surface integral in (2.44) over the incident field $P_{\text{inc}}$ corresponds to the situation as depicted in Figure 2.8 and characterised by eq. (2.38). This means that this part of the integral has no contribution to the left hand side of (2.44). As a consequence, the integral equation can also be written as

$$P_{\text{scat}}(\mathbf{r}_0) = \oint_S \left[ G(\mathbf{r}_0 \mid \mathbf{r}) \nabla P_{\text{scat}}(\mathbf{r}) - P_{\text{scat}}(\mathbf{r}) \nabla G(\mathbf{r}_0 \mid \mathbf{r}) \right] \cdot \hat{n} \, dS \quad \text{for} \quad \mathbf{r}_0 \in \overline{V} \quad (2.45)$$

which corresponds to the Kirchhoff-Helmholtz integral as given by (2.38). For this representation of $P_{\text{scat}}$ it is useful to specify the field's (notional) source. Given the Helmholtz equation for the scattered field

$$\nabla^2 P_{\text{scat}}(\mathbf{r}, \omega) + k^2 P_{\text{scat}}(\mathbf{r}, \omega) = -\Psi_{\text{scat}}(\mathbf{r}, \omega) \quad (2.46)$$

in which $\Psi_{\text{scat}}$ is the notional source distribution for $P_{\text{scat}}$, the generalised Kirchhoff integral follows as

$$P_{\text{scat}}(\mathbf{r}_0) = \int_{\overline{V}} \Psi_{\text{scat}}(\mathbf{r}) G(\mathbf{r}_0 \mid \mathbf{r}) \, d\overline{V} + \oint_S \left[ G(\mathbf{r}_0 \mid \mathbf{r}) \nabla P_{\text{scat}}(\mathbf{r}) - P_{\text{scat}}(\mathbf{r}) \nabla G(\mathbf{r}_0 \mid \mathbf{r}) \right] \cdot \hat{n} \, dS \quad (2.47)$$

where $\mathbf{r}_0 \in \overline{V}$. Comparing eq. (2.45) to (2.47) provides the insight that the source distribution $\Psi_{\text{scat}}$ of $P_{\text{scat}}$ described by eqs. (2.45) and (2.46), is located entirely within $\overline{V}$ - the domain of the scatterer.

In the absence of the contrasting object, the scattered field in $\overline{V}$ can be reconstructed by installing a layer of monopoles and dipoles on $S$. The required source strength densities are given by
\[
\begin{align*}
Q_m(r, \omega) &= -V_{\text{scat}}(r, \omega) \cdot \vec{n} \\
F_d(r, \omega) &= -P_{\text{scat}}(r, \omega) \vec{n}
\end{align*}
\]

for \( r \in S \) \hfill (2.48)

The layer of reconstruction sources generates a field for which it holds

\[
\oint_S \left[ G(r_0 \mid r) j \omega \rho_0 Q_m(r) + F_d(r) \nabla G(r_0 \mid r) \right] dS = \begin{cases} 0 & r_0 \in V \\ P_{\text{scat}}(r_0) & r_0 \in \overline{V} \end{cases}
\]

(2.49)

In the next Section we will pay further attention to the above description of scattering problems.

---

**Figure 2.10:** Illustration of Kirchhoff's solution to active sound reduction (active receiver shielding); a) the primary field, b) the secondary field of the monopole (+) and dipole (+−) layer and c) the superimposed field.

### 2.6 Kirchhoff solutions for active sound control

The relevance of the principle of sound field reconstruction for active sound control is quite obvious. Assume that an arbitrary primary source inside \( \overline{V} \) generates a wave field \( P_{\text{prim}} \); see Figure 2.10. In addition, a continuous distribution of monopoles and dipoles on \( S \) renders a secondary wave field \( P_{\text{sec}} \). By inspection of eq. (2.39) we find, that if the following holds for the monopole and dipole strengths per unit area
\[ Q_m(r, \omega) = V_{\text{prim}}(r, \omega) \cdot n \]
\[ F_d(r, \omega) = P_{\text{prim}}(r, \omega) n \]
\text{for } r \in S \quad (2.50)

the superimposed wave field is
\[ \int_V \frac{G(r_0 \mid r)}{V} \Psi_{\text{prim}}(r) dV + \oint_S \left[ G(r_0 \mid r) j \omega \rho_0 Q_m(r) + F_d(r) \cdot \nabla G(r_0 \mid r) \right] dS \]
\[ = \begin{cases} 0 & r_0 \in V \\ \frac{1}{2} P_{\text{prim}}(r_0) & r_0 \in S \\ P_{\text{prim}}(r_0) & r_0 \in V \end{cases} \quad (2.51) \]

in which volume \( V \) contains the primary source and volume \( V \) contains no sources. Hence, complete suppression of the primary wave field in \( V \) can be acquired by the continuous monopole and dipole distribution on the surface. This source surface can be chosen to enclose either the region of silence (active receiver shielding) or the region of primary sources (active source shielding). The latter solution follows from the reasoning in the previous Section, eq.(2.41), and is shown in Figure 2.11.

![Figure 2.11](image)

**Figure 2.11:** Active shielding of the primary source by means of a continuous monopole and dipole distribution.

Besides the perfect cancellation inside \( V \), the Kirchhoff sources leave the primary wave field outside \( V \) undisturbed. Though this is an elegant feature, it is of secondary importance. It will be shown below, that a continuous distribution of either monopoles or dipoles is capable of complete reduction of the primary wave field inside \( V \) at the cost of a disturbance of the primary wave field outside \( V \).

As is well-known, for the purpose of numerical sound field reconstruction or extrapolation, the Kirchhoff-Helmholtz integral can be simplified by choosing an appropriate Green's function. If a Green's function can be found that satisfies the Dirichlet or Neumann boundary conditions,
eq. (2.32), the integral expression reduces to a so-called Rayleigh integral. Then, knowledge of only \( P \) or \( \mathbf{V} \cdot \mathbf{n} \) on the surface is required in order to calculate the inner \( V \) field. In the case of active sound control, however, Green’s function and its normal gradient are directly related to true monopole and dipole source responses respectively. Nevertheless, if we drop the requirement that the secondary sources generate a null field outside \( V \), there is a latitude in the selection of the source strengths.

Consider the following two applications of the Kirchhoff-Helmholtz integral and Figure 2.12.

\[
\Psi_{\text{prim}} \rightarrow \mathbf{V} \rightarrow \Omega
\]

**Figure 2.12:** Geometry for the cases of active receiver shielding and scattered field reconstruction by means of a distribution of monopoles and dipoles.

1) The situation of active receiver shielding is formulated by

\[
\oint_S [G(r_0 \mid r) j \omega \rho_0 \mathbf{V}_{\text{prim}}(r) + P_{\text{prim}}(r) \nabla G(r_0 \mid r)] \cdot \mathbf{n} \, dS = \begin{cases} -P_{\text{prim}}(r_0) & r_0 \in V \\ -\frac{1}{2} P_{\text{prim}}(r_0) & r_0 \in \overline{V} \\ 0 & \text{otherwise} \end{cases}
\]

This secondary field follows immediately from eq. (2.51).

2) The second situation concerns the reconstruction of the scattered primary wave field \( P_{\text{scat}} \) as was addressed in Section 2.5, eqs. (2.48) and (2.49) and Figure 2.9. Here, the scattering object coincides exactly with volume \( V \). In the absence of the scatterer, the field \( P_{\text{scat}} \) can be reconstructed by a continuous distribution of monopoles and dipoles on \( S \).
\[ -\oint_S \left[ G(r_0 | r) j \omega \rho_0 V_{\text{scat}}(r) + P_{\text{scat}}(r) V G(r_0 | r) \right] \cdot \overline{n} \ dS = \begin{cases} 0 & r_0 \in V \\ \frac{1}{2} P_{\text{scat}}(r_0) & r_0 \in S \\ P_{\text{scat}}(r_0) & r_0 \in \overline{V} \end{cases} \]  

with \( \overline{n}(r) = -n(r) \).

If the two tasks of the monopole and dipole distribution are carried out simultaneously

\[
\begin{align*}
Q_m(r, \omega) &= V_{\text{prim}}(r, \omega) \cdot n + V_{\text{scat}}(r, \omega) \cdot n \\
E_d(r, \omega) &= P_{\text{prim}}(r, \omega) n + P_{\text{scat}}(r, \omega) n
\end{align*}
\]

for \( r \in S \)  

the resultant secondary field \( P_{\text{sec}} \) can be obtained as (with the help of eqs. (2.52) and (2.53)):

\[
P_{\text{sec}}(r_0) = \oint_S \left[ G(r_0 | r) j \omega \rho_0 Q_m(r) + E_d(r) \cdot V G(r_0 | r) \right] dS
\]

\[
= \begin{cases} -P_{\text{prim}}(r_0) & r_0 \in V \\
\frac{1}{2} [P_{\text{scat}}(r_0) - P_{\text{prim}}(r_0)] & r_0 \in S \\
P_{\text{scat}}(r_0) & r_0 \in \overline{V} \end{cases} \]  

So the superimposed secondary source surface is capable of generating \(-P_{\text{prim}}\) inside \( V \) and \( P_{\text{scat}} \) in \( \overline{V} \). Now suppose that the reconstructed scattered field is such that one of the following two conditions holds

\[
\begin{align*}
1 & \quad P_{\text{prim}}(r, \omega) n + P_{\text{scat}}(r, \omega) n = 0 \\
2 & \quad V_{\text{prim}}(r, \omega) \cdot n + V_{\text{scat}}(r, \omega) \cdot n = 0
\end{align*}
\]

for \( r \in S \)  

Since the primary field \( P_{\text{prim}} \) here equals the incident field \( P_{\text{inc}} \) for the scattering problem, equation (2.56) specifies the desired acoustic behaviour on the boundary of the notional scatterer. The first condition corresponds to the boundary condition of pressure release, whereas the second condition represents a rigid boundary. This means that we want our distribution of monopoles and dipoles to reconstruct the scattered primary field \( P_{\text{scat}} \) from an acoustically impenetrable object. Then it follows from eqs. (2.54) and (2.56), that the primary field inside \( V \) can be annihilated by using either a monopole or the dipole distribution. The superposition of the primary field and secondary field, as given by eq. (2.55), has the following characteristics

\[
P_{\text{tot}}(r_0) = P_{\text{prim}}(r_0) + P_{\text{sec}}(r_0) = \begin{cases} 0 & r_0 \in V \\
\frac{1}{2} [P_{\text{prim}}(r_0) + P_{\text{scat}}(r_0)] & r_0 \in S \\
P_{\text{prim}}(r_0) + P_{\text{scat}}(r_0) & r_0 \in \overline{V} \end{cases}
\]

with \( 1 \quad P_{\text{tot}}(r) = 0 \) 

or \( 2 \quad V_{\text{tot}}(r) \cdot n = 0 \) 

for \( r \in S \)

Note that from the perspective of the primary source the action of the secondary sources cannot be distinguished from that of the notional, completely reflective scatterer. Both the pure
monopole as the pure dipole solution reflect the incident primary field. The monopole solution is illustrated in Figure 2.13. Clearly, the approach described here also holds for the case of active source shielding, as was shown in Figure 2.11.

Figure 2.13: Illustration of the monopole solution to active sound reduction; a) the primary field, b) the notional scatterer and scattered field, c) the secondary field of the monopole layer and d) the superimposed field.

The two Kirchhoff solutions of one source type can be formulated as:

- the Kirchhoff monopole solution:

  Choose the notional scatterer such that the sum of $P_{\text{prim}}$ and $P_{\text{scat}}$ vanishes on $S$. If the monopole source strength $Q_m = (V_{\text{prim}} + V_{\text{scat}}) \cdot n$, we obtain the following result

  \[
  \left\{ \begin{array}{ll}
  0 & r_0 \in V \\
  \frac{1}{2} [P_{\text{prim}}(r_0) + P_{\text{scat}}(r_0)] & r_0 \in S \\
  P_{\text{prim}}(r_0) + P_{\text{scat}}(r_0) & r_0 \in \overline{V}
  \end{array} \right.
  \]

  \begin{equation}
  (2.58)
  \end{equation}
the Kirchhoff dipole solution:

If it holds at the notional scatterer’s boundary that \((V_{\text{prim}} + V_{\text{scat}}) \cdot n = 0\), then choose \(F_d = (P_{\text{prim}} + P_{\text{scat}}) n\) which results in

\[
\int_{\mathcal{V}} G(r_0 \mid r) \nabla \cdot [P_{\text{prim}}(r) + P_{\text{scat}}(r)] n \, dS
\]

\[
= \begin{cases} 
0 & r_0 \in \mathcal{V} \\
\frac{1}{2} [P_{\text{prim}}(r_0) + P_{\text{scat}}(r_0)] & r_0 \in \mathcal{S} \\
[P_{\text{prim}}(r_0) + P_{\text{scat}}(r_0)] & r_0 \in \mathcal{V}
\end{cases}
\]  

\tag{2.59}

Clearly, the price for the simplification of the secondary source distribution is the disturbance of the primary field outside \(V\). In Section 2.7 the acoustical aspects of the solutions of one source type are discussed in further detail. The application of the Kirchhoff integral equations on active sound reduction will be illustrated by the following three examples.

**monopole field attenuation by a spherical source surface**

Imagine an unbounded homogeneous medium with a primary field generated by a monopole source with strength \(Q_{\text{prim}}\) at \(r = 0\). On a sphere with radius \(a\) around \(r = 0\), a thin layer with a continuous distribution of monopoles and dipoles is set up, in order to shield the primary source from the surroundings, as shown in Figure 2.14.

In spherical coordinates the primary wave field is given by

\[
P_{\text{prim}}(r, \omega) = j \omega \rho_0 Q_{\text{prim}} \frac{e^{-jkr}}{4\pi r}
\]

\tag{2.60}

Using the full Kirchhoff-Helmholtz solution (2.50), it is easily found that the required monopole and dipole source strengths per unit area are

\[
Q_m(\omega) = -Q_{\text{prim}} \frac{1 + jka}{4\pi a^2} e^{-jka}
\]

\[
F_d(\omega) = j \omega \rho_0 Q_{\text{prim}} \frac{e^{-jka}}{4\pi a} n
\]

\tag{2.61}

Integrating over this source distribution yields for the secondary wave field

\[
P_{\text{sec}}(r, \omega) = \begin{cases} 
-j \omega \rho_0 Q_{\text{prim}} \frac{e^{-jkr}}{4\pi r} & r > a \\
0 & r < a
\end{cases}
\]

\tag{2.62}

which cancels the primary field outside the sphere.
As discussed above the Kirchhoff solutions of one source type can be found with the use of a special scattered wave field, that obeys one of the two conditions in eq. (2.56). The mere monopole solution follows from the primary field in the sphere with a ‘pressure release’ boundary. This field can be found as

\[ P(r, \omega) = P_{\text{inc}}(r, \omega) + P_{\text{scat}}(r, \omega) = j \omega \rho_0 Q_{\text{prim}}(\omega) \frac{\sin[k(a - r)]}{4 \pi r \sin(ka)}, \quad r \leq a \]  

(2.63)

where \( P \) vanishes for \( r = a \). From (2.54) we have for the optimal monopole source strength

\[ Q_m(\omega) = V(r, \omega) \cdot n |_{r=a} = - \frac{Q_{\text{prim}}(\omega)}{4 \pi a^2} \frac{ka}{\sin(ka)} \]  

(2.64)

which solution considerably differs from the complete Kirchhoff solution. The resultant inner sphere field generated by primary and secondary sources is identical to the notional field of eq. (2.63).

The primary field that would exist inside the sphere if it had a rigid boundary is given by

\[ P(r, \omega) = j \omega \rho_0 Q_{\text{prim}}(\omega) k a \cos[k(a - r)] - \sin[k(a - r)] \frac{ka}{k a \cos(ka) - \sin(ka)}, \quad r \leq a \]  

(2.65)

Consequently, the monopole source strength vanishes and the dipole strength reads

\[ F_d(\omega) = P(r, \omega) n |_{r=a} = j \omega \rho_0 Q_{\text{prim}}(\omega) k a^2 \frac{ka^2}{k a \cos(ka) - \sin(ka)} \]  

(2.66)

Again, the superposition of primary field (2.60) and secondary field equals the notional field inside the rigid sphere (2.65) and a null field outside the sphere.
Generally, every linear combination of the Kirchhoff monopole and the Kirchhoff dipole solution, \(\gamma Q_m, (1-\gamma)F_d\), results in reduction of the primary field in the appointed area. Only for a single value of \(\gamma\) the primary field remains undisturbed outside the region of silence. For the specific geometry of Figure 2.14 this parameter equals

\[
\gamma(ka) = (1 + jka)e^{-\frac{jka \sin(ka)}{ka}}
\]

(2.67)

**Figure 2.15:** Diagram of a primary monopole at \(r = 0\) and a secondary source plane at \(x = x_0\). The dotted area represents the region of silence.

**monopole field attenuation by a planar source surface**

Another example with geometrical simplicity emerges, if we try to prevent radiation of a primary monopole source at \(r = 0\) into a semi-infinite half-space; see Figure 2.15. Again, the twofold Kirchhoff solution for the secondary sources is found by simply evaluating the primary wave field on the source layer for \(x = x_0\). The notional scattered wave field \(P_{\text{scat}}\) required for the source solutions of one type can be found by putting an 'image monopole source' with strength \(Q_{\text{scat}}\) at \(x = 2x_0\). It turns out that the total dipole strength vanishes if \(Q_{\text{scat}} = -Q_{\text{prim}}\), whereas \(Q_{\text{scat}} = Q_{\text{prim}}\) eliminates the total monopole strength. An overview of the three different solutions is given below.

- complete Kirchhoff solution
Kirchhoff solutions for active sound control

\[ Q_m(x_0, y, z, \omega) = -Q_{\text{prim}} \frac{1 + jkr}{4\pi r^3} x_0 e^{-jkr} \]

\[ F_d(x_0, y, z, \omega) = j\omega p_0 Q_{\text{prim}} \frac{e^{-jkr}}{4\pi r} \mathbf{n} \tag{2.68} \]

with \( r = \sqrt{x_0^2 + y^2 + z^2} \)

- Kirchhoff monopole solution

The monopole source strength density is given by

\[ Q_m(x_0, y, z, \omega) = -2Q_{\text{prim}} \frac{1 + jkr}{4\pi r^3} x_0 e^{-jkr} \tag{2.69} \]

The total wave field for \( x < x_0 \) appears to be the primary field completely reflected at \( x = x_0 \)

\[ P_{\text{tot}}(r, \omega) = j\omega p_0 Q_{\text{prim}} \left[ \frac{e^{-jkr}}{4\pi r} - \frac{e^{-jkr'}}{4\pi r'} \right], \quad x < x_0 \]

with \( r' = \sqrt{(2x_0 - x)^2 + y^2 + z^2} \) \tag{2.70}

- Kirchhoff dipole solution

The dipole layer is specified by

\[ F_d(x_0, y, z, \omega) = 2j\omega p_0 Q_{\text{prim}} \frac{e^{-jkr}}{4\pi r} \mathbf{n} \tag{2.71} \]

Again, the primary wave field is virtually reflected at the secondary source plane:

\[ P_{\text{tot}}(r, \omega) = j\omega p_0 Q_{\text{prim}} \left[ \frac{e^{-jkr}}{4\pi r} + \frac{e^{-jkr'}}{4\pi r'} \right], \quad x < x_0 \tag{2.72} \]

For this geometry it follows that the complete Kirchhoff solution (2.68) is 1/2 times the combination of the monopole solution (2.69) and the dipole solution (2.71); so the ideal \( \gamma \) equals 1/2.
Figure 2.16: Diagram of two primary monopoles and two planar surfaces of secondary sources. The region of silence lies in between the surfaces.

*primary field attenuation by two planar source surfaces*

A slightly more complicated geometry is depicted in Figure 2.16. Two infinite layers of monopole and/or dipole sources have to attenuate the sound of two monopoles. Without going into mathematical details, an interesting remark can be made on this geometry. The complete Kirchhoff solution requires the field values at the two planar surfaces from both primary sources. The one source type solutions on the planes, however, are merely influenced by the primary source field on the same side of the clamped region. The condition of complete reflection on both plates prevents the primary sources to reach the medium beyond the region of silence. Drawing conclusions on this Section, it can be stated that the Kirchhoff integral equation gives the recipe for designing an active sound control system, that obtains complete suppression in a given area. Simplification of the solution by eliminating the monopole or dipole sources (solutions of one source type) will disturb the primary field outside the area of silence with a scattered field.

### 2.7 Energetic aspects of Kirchhoff solutions

As revealed in the previous Section, the three integral expressions to design a continuous distribution of monopoles and/or dipoles have a dissimilar impact on the original primary field outside the region of silence. In this Section the radiated power into the region of silence is calculated for the primary source and the three secondary source distributions.
As shown in [Pi81], the radiated power through a surface $S$ for the complex representation of harmonically varying quantities equals

$$\Pi(\omega) = \oint_S I(\mathbf{r}, \omega) \cdot \mathbf{n} \, dS = \frac{1}{2} \text{Re} \oint_S P^* (\mathbf{r}, \omega) V(\mathbf{r}, \omega) \cdot \mathbf{n} \, dS$$  \hspace{1cm} (2.73)$$

in which $I(\omega)$ stands for the time-averaged active intensity. So, the radiated primary power $\Pi_{\text{prim}}$ into the region of silence - in the absence of secondary sources - can be calculated as

$$\Pi_{\text{prim}} = \oint_S \mathbf{I}_{\text{prim}} \cdot \mathbf{n} \, dS = -\frac{1}{2} \text{Re} \oint_S P^*_{\text{prim}} V_{\text{prim}} \cdot \mathbf{n} \, dS$$  \hspace{1cm} (2.74)$$

where the minus sign is due to $\mathbf{n}$, which points outward of the volume of silence. For the relevant geometry the reader is referred to e.g. Figure 2.12. The radiated primary power (2.74) may not be the total power output of the primary source, as $S$ not necessarily encloses the primary source. The total power output of the secondary sources, however, is an interesting quantity. Recall the time-domain energy balance equation (2.19).

$$\frac{\partial}{\partial t} e_\text{tot} + \mathbf{V} \cdot \mathbf{I}(t) = pq + \mathbf{v} \cdot \mathbf{f}$$  \hspace{1cm} (2.75)$$

in which $\mathbf{I}(t)$ represents the instantaneous power flow vector. The frequency domain version of (2.75) can be written as

$$\mathbf{V} \cdot \mathbf{I}(\mathbf{r}, \omega) = \frac{1}{2} \text{Re} \left\{ P^* Q + V^* \cdot F \right\}$$  \hspace{1cm} (2.76)$$

which again holds for the complex representation of harmonically varying quantities. As a consequence the average acoustic power output $\Pi$ of a source distribution is given by

$$\Pi_{\text{source}} = \oint_{V_1} \oint_{V_i} \mathbf{V} \cdot \mathbf{I} \, dV_i = \frac{1}{2} \text{Re} \left\{ \oint_{V_1} \oint_{V_i} \left[ P^* Q + V^* \cdot F \right] \, dV_i \right\}$$  \hspace{1cm} (2.77)$$

where $V_1$ contains the entire source.

So the acoustic power output of the secondary sources can be evaluated with knowledge of the pressure and velocity in the source area. For the full Kirchhoff solution it was found that $Q=V^*_{\text{prim}} \cdot \mathbf{n}$ and $F=P^*_{\text{prim}} \cdot \mathbf{n}$; see eq. (2.50). For the total pressure and particle velocity on the surface of (transparent) point sources, we have in accordance with eq. (2.51)

$$P_{\text{tot}}(\mathbf{r}, \omega) = \frac{1}{2} P_{\text{prim}}(\mathbf{r}, \omega)$$

$$V_{\text{tot}}(\mathbf{r}, \omega) = \frac{1}{2} V_{\text{prim}}(\mathbf{r}, \omega)$$  \hspace{1cm} (2.78)$$

Evaluating the volume integral (2.77) over the secondary source area yields for the total power output

$$\Pi_{\text{KH}} = \frac{1}{2} \text{Re} \left\{ \oint_{V_1} \oint_{V_i} \left[ P_{\text{tot}}^* Q + V_{\text{tot}}^* \cdot F \right] \, dV_i \right\} = \frac{1}{2} \text{Re} \oint_S P^*_{\text{prim}} V_{\text{prim}} \cdot \mathbf{n} \, dS = -\Pi_{\text{prim}}$$  \hspace{1cm} (2.79)$$

The monopole-dipole layer absorbs exactly the primary power input to the region of silence. The radiation impedance as experienced by the primary source has not changed by the introduction of the secondary sources.
For the one type surfaces of monopole or dipole sources a simple conclusion on the power absorption can be drawn:

\[
\Pi_m = \frac{1}{2} \text{Re} \int_S P_{tot}^* Q \, dS = 0
\]

\[
\Pi_d = \frac{1}{2} \text{Re} \int_S V_{tot}^* \cdot F \, dS = 0
\]  (2.80)

since either the effective pressure or particle velocity on \( S \) is cancelled by virtue of the scattered wave field \( P_{\text{scat}} \). The simplified Kirchhoff solutions for active sound control do not absorb the primary power but reflect it. In terms of specific acoustic impedance \( Z \) this means

\[
Z_m(\mathbf{r}_s) = \frac{P(\mathbf{r}_s)}{V(\mathbf{r}_s) \cdot \mathbf{n}} = 0
\]

\[
Z_d(\mathbf{r}_s) = \frac{P(\mathbf{r}_s)}{V(\mathbf{r}_s) \cdot \mathbf{n}} \rightarrow \infty
\]  (2.81)

The continuous distribution of monopoles creates a pressure release on the boundary, whereas the dipole distribution acts as a perfectly rigid wall. Both solutions of one source type prevent the primary source to radiate acoustic power into the region of silence. In the perspective of the primary source the region has turned into an acoustically impenetrable medium. In accordance with this the primary field scatters at the region’s boundary. With the linear combination of the Kirchhoff monopole and dipole solutions, as discussed in the previous Section, it is possible to create an arbitrary acoustic impedance at the boundary of the region of silence. This impedance is exactly the same as the impedance of the notionall scatterer in Section 2.6.

2.8 References


CHAPTER 3

Acoustic Cost Functions and Discrete Source Distributions

In the previous Chapter the fundamental principles of active sound control have been addressed. It has been demonstrated that the Kirchhoff-Helmholtz integral leads to the possibility of active sound control by means of a continuous surface distribution of monopole and dipole sources. This solution achieves absolute suppression of the primary sound field in an appointed area, and leaves the primary sound field outside the area undisturbed. Simplification of the Kirchhoff solution (of two source types) yields solutions of one source type. In that case either a continuous monopole or a continuous dipole distribution completely cancels the primary field. A drawback of the homogeneous source solutions is the disturbance of the primary sound field outside the region of silence. Such a source distribution acts like a completely reflecting boundary.

In this Chapter we will investigate the performance of a discretised distribution of secondary sources. Under which requirements is it still possible to achieve an acceptable attenuation of the primary field? The measure of performance is determined by an a priori defined cost function. Therefore, an important aspect of this Chapter will be the definition of suitable acoustic cost functions. The optimal configuration of discrete source distributions emerges by minimising the specific cost function. The attention will be focussed on the following three types of acoustic media: the free field, the duct and the enclosure.

3.1 Design of acoustic cost functions

The Kirchhoff solutions to active sound reduction as presented in the previous Chapter are far from a practical implementation. This follows from the simple observation that a continuous distribution of control sources cannot be realised. The first step to a practical ASR system is to find a way to approximate the Kirchhoff solution using a finite number of secondary sources. The performance of the continuous-space Kirchhoff and related homogeneous source solutions is perfect. We might describe their performance as:

- A complete reduction of the acoustic energy in the region of silence; potential, kinetic and total acoustic energy.
- A complete absorption of power by the solution of two source types and complete reflection of power by the solutions of one source type.
If we want to approximate the continuous Kirchhoff solution as well as possible by a discrete source distribution, it is most effective to approximate the Kirchhoff performance. This approach seems more pragmatic than attempting to approximate the continuous source distribution itself. In the early 70’s it was the ‘Russian School’ that studied the latter alternative. Results on the discrete approximation of continuous Kirchhoff surfaces can be found in [Ko79] and [KF87]. The approach undertaken by Konyaev is that of finding a discrete approximation based on Gaussian quadrature formulae.

Another approach could be to apply the spatial sampling theory to the continuous-space solution. As is well-known, a wave field can be discretised if it is spatially bandlimited [Be87]. A spatial discretisation of the Kirchhoff reconstruction integral would lead to a discrete source distribution that can almost exactly reconstruct a bandlimited wave field; usually it is exact apart from an evanescent field. The main requirement is given by the secondary source spacing, which is a function of the smallest apparent wave length. The drawback of this approach is that it only gives a requirement on source spacing in order to achieve perfect reduction. It does not indicate the degradation in performance when the source spacing is chosen too coarse. Additionally, the application of the sampling theory does not provide a ‘cheap’ source configuration in cases that only a modest reduction is desired.

So, the more pragmatic approach to the approximation of the Kirchhoff solution is to define a measure of performance or cost function and optimise the secondary source distribution with respect to this measure. If we consider the features of the continuous Kirchhoff solution above, we observe a complete annihilation of the primary field or acoustic energy. This leads to the definition of the fundamental cost function

$$J_{\text{tot}}(\omega) = \int_V \left[ \frac{|P(r,\omega)|^2}{4\rho_0 c_0^2} + \frac{\rho_0 |V(r,\omega)|^2}{4} \right] dV$$

(3.1)

the total acoustic energy in the region of ‘silence’ V. The optimal strengths for any number of secondary sources can be found by minimising this cost function. Theoretically, the total acoustic energy is an appealing cost function. Therefore, it will be the principal measure of performance throughout the Chapter. In a medium free of reflections the total acoustic energy is proportional to the total power output of the sources. In the next Section, we will minimise the source power output in order to minimise the far field acoustic energy. This indirect way is chosen because the source power output is easier to evaluate in this specific case.

On the practical side, however, the total acoustic energy may turn out to be a less favourable choice:

- Realising that the common acoustical sensors measure the pressure field, it may be hard to evaluate the kinetic energy part in the total acoustic energy. It would require the difference of two nearby measured pressure field values to obtain an estimate of the
velocity field value in a certain direction. So of the components of the total acoustic energy the potential energy is far the most easy to be measured.

- One might wonder whether the minimisation of total acoustic energy corresponds to a minimisation of the nuisance of noise. It should be recalled that the human ear is a sensor of pressure. Although the perception of a noise field is a very complicated process - requiring multidisciplinary research - we may state that the acoustic kinetic energy is not involved here.

So we propose as a more practically orientated alternative for the total acoustic energy the space average squared pressure in the region of silence

$$J_p(\omega) = \frac{1}{V} \int \left| P(r, \omega) \right|^2 dV$$

(3.2)

In this Chapter we will study the minimisation of both the total acoustic energy and the space average squared pressure. This provides a good insight into the difference between minimising a theoretically and a practically orientated cost function. Obviously, minimising the space average squared pressure is equivalent to minimising the acoustic potential energy, as these differ by a factor $4 \rho_0 c_0^2 / V$.

Again, we should realise that the space average squared pressure may be a poor representative of the perceived annoyance of noise. If we were to apply the common criterion used in legislation, the cost function would read

$$J_{dB}(\omega) = \frac{1}{V} \int F(\omega) 10 \log_{10} \left[ \left| P(r, \omega) \right|^2 \right] dV$$

(3.3)

in which the function $F(\omega)$ represents a specific weighting over the frequency band; like the A, B, C or D weighting function. Still we cannot assume that this a good measure of the nuisance of noise. Though the aspects of noise perception grow beyond the scope of this thesis, it is certainly an essential subject of future research.

The acoustic energy - total or potential - is a global quantity which gives information on the average spatial behaviour of acoustical quantities. Although the minimisation of an energetic quantity leads to reduction in the mean, somewhere in the region the primary energy may have increased. If we want to prevent a local increase of the primary field, we should minimise the maximum value of the squared pressure in a certain area $V$. This obviously is a local quantity and can be mathematically written as

$$J_\infty(\omega) = \| P(r, \omega) \|_2^2 = \max_{r \in V} \left\{ \left| P(r, \omega) \right|^2 \right\}$$

(3.4)

Minimising this cost function means that the secondary source strengths are such, that they yield the smallest maximum value of the squared pressure over $V$ out of all possible values for the strengths. The minimisation of eq. (3.4) is also known as a minimax problem and can be interpreted as worst-case design. Note that the solution of (3.4) also minimises the $\infty$-norm
\|P(\mathbf{r}, \omega)\|_{\infty}. \) In the last Section of the Chapter we will treat the minimisation of this interesting criterion alternative.

### 3.2 Active reduction of free field radiation

In this Section the performance of a discrete distribution of secondary sources is examined for the free field. The main issue is the design of a suitable cost function. Based on the cost function the performance of a distribution of secondary sources can be identified. Consequently, the strengths and positioning of the secondary sources can be optimised.

In a free field situation the region of silence is generally located far from the primary source. One might think of a residential area disturbed by the noise from plants; like a gas turbine or a transformer. In these cases, we should place the secondary sources close to the primary source and aim at the solution of active source shielding, as was shown in the previous Chapter. Based on the discussion in Section 3.1 the principal criterion function is the total acoustic energy in the region of silence. This leads us to the following alternatives:

1) the total acoustic energy in the Fraunhofer area.
2) the total acoustic energy in the Fraunhofer area corresponding to a limited arc of the field.

In the forthcoming analysis the features of the two proposed cost functions will be discussed.

**the sound field in the Fraunhofer area**

In order to evaluate the cost functions let us characterise the far field of an arbitrary source. As given in [Be87] and [JF72], the sound field in the Fraunhofer area of any finite source distribution satisfies the following criteria:

a) the pressure amplitude decays as \(1/r\); with \(r\) being the distance to the source center.

b) the angular \((\theta, \varphi)\) dependence of the pressure amplitude does not vary with \(r\).

c) the specific acoustic impedance equals \(\rho_0 c_0\).

Mathematically, these criteria are governed by

\[
P_{\text{ff}}(r, \theta, \varphi, \omega) = S(\omega)D(\theta, \varphi, \omega)\frac{e^{-jkr}}{r}
\]  \(\text{(3.5)}\)

in which \(D\) represents the directivity characteristics. By virtue of criterion (c) and eq. (2.5) the kinetic energy density equals the potential energy density. As a consequence, the far field total acoustic energy density reads

\[
e_{\text{tot}}^{\text{ff}}(r, \omega) = \frac{1}{2\rho_0 c_0^2} |P_{\text{ff}}(r, \omega)|^2
\]  \(\text{(3.6)}\)

So in order to evaluate the total acoustic energy knowledge of only the pressure field \(P_{\text{ff}}\) in the Fraunhofer area is required. This means that minimising the total acoustic energy is equivalent to minimising the potential acoustic energy in the far field. So here the two alternatives of Section 3.1 coincide.
Section 3.2  Active reduction of free field radiation

Let the relevant volume of ‘silence’ $V$ be enclosed by two concentric spheres with all sources in the common center; see Figure 3.1.

![Diagram of far field volume of silence V.](image)

**Figure 3.1:** Diagram of far field volume of silence $V$.

Using eqs. (3.5) and (3.6) the total acoustic energy in $V$ for this geometry can be written as

$$
\mathcal{E}_{\text{tot}}^{\text{ff}}(\omega) = \int_{V} e_{\text{tot}}^{\text{ff}}(r, \omega) \, dV = \frac{(r_2 - r_1)}{2 \rho_0 c_0^3} \int_{S} \left| \mathbf{p}^{\text{ff}}(r, \omega) \right|^2 \, dS
$$

(3.7)

where $r_1$ and $r_2$ denote the inner and outer sphere radius respectively and $r_1 \in S$. For the special choice of volume $V$ the total acoustic energy is determined by the far field pressure at the border $S$ of $V$. The total acoustic power output of the distribution of sources is defined by a similar relation:

$$
\Pi(\omega) = \frac{1}{S} \mathbf{I} \cdot \mathbf{n} \, dS = \frac{1}{2 \rho_0 c_0^3} \int_{S} \left| \mathbf{p}^{\text{ff}}(r, \omega) \right|^2 \, dS
$$

(3.8)

This means that the total acoustic energy in $V$ is proportional to the radiated power of all present sources. The latter quantity may be easy to evaluate in the case of a limited number of point sources; see eq. (2.77). The total power output of $N$ monopoles, for example, can be evaluated using

$$
\Pi(\omega) = \frac{1}{S} \mathbf{I} \cdot \mathbf{n} \, dS = \frac{1}{2} \text{Re} \sum_{i=1}^{N} P_{\text{tot}}^{\star}(r_i, \omega) U_i(\omega)
$$

(3.9)

where $r_i$ denotes the position of the $i$-th monopole with source strength $U_i$. The total far field energy is directly related to the source strengths and the pressure field in the source region. For numerical convenience the near field expression (3.9) is preferable to the far field expression (3.8). Dipole sources can be easily included in this approach; see Section 2.7.

**Minimisation of the total radiated power**

The active shielding of a monopole field by means of a spherical surface of secondary sources was already introduced in Section 2.6, Example 1. In Figure 3.2 four different secondary source
geometries are shown. As depicted, active reduction of the primary field is attempted by means of only 1, 2, 4 and 6 secondary monopoles.

Figure 3.2: Four different configurations of secondary source distributions.

Based on the discussion in the previous subsection the total radiated power can be chosen as an representative of the total acoustic energy or potential energy in $V$. So the criterion function reads

$$f_1(\omega) = \frac{1}{2} \text{Re} \left\{ \left[ P_{\text{prim}}^*(r_s, \omega) + P_{\text{sec}}^*(r_s, \omega) \right] \left[ V_{\text{prim}}(r_s, \omega) + V_{\text{sec}}(r_s, \omega) \right] \cdot \mathbf{n} \, dS \right\}$$  \hspace{1cm} (3.10)

By writing the secondary pressure field $P_{\text{sec}}$ and velocity field $V_{\text{sec}}$ as a function of the secondary source strengths, the criterion (3.10) can be minimised with respect to the source strengths. This approach to active sound reduction in the free field was first proposed in [NE86]. A thorough treatment of this optimisation procedure is given in Appendix A3.1. Here we mainly concentrate on the solution for the secondary sources and the minimum value of $f_1$.

For the configuration with one secondary source - Figure 3.2 - we have

$$U_s = -U_p \text{sinc}(ka)$$

$$J_{\text{min}} = J_p \left[ 1 - \text{sinc}^2(ka) \right]$$  \hspace{1cm} (3.11)

where $J_p$ is the radiated power by the primary source. Evaluating eq. (3.9) for this configuration we find

$$J(\omega) = \frac{1}{2} \text{Re} \left\{ P_{\text{tot}}^*(r_p, \omega) U_p(\omega) + P_{\text{tot}}^*(r_s, \omega) U_s(\omega) \right\}$$  \hspace{1cm} (3.12)

From this expression we conclude that minimising the radiated power of two point sources implies minimising the product of total pressure and source strength. Bearing this principle in mind the solution (3.11) can be explained as follows:
1) For any \( ka \) it holds that the pressure response of the secondary source at the primary source's location is in anti-phase with primary pressure and vice versa. This means that for the low frequency band \( (ka < \pi \text{ or } \lambda > 2a) \) the secondary source acts in anti-phase with the primary source. For \( ka > \pi \), the secondary source may also be in phase with the primary source. Both sources are hindering each others power radiation by reducing the pressure field at the very source positions; recall eq. (3.9). The reduction for \( ka > \pi \), however, is little as is shown in Figure 3.3.

2) The secondary source strength diminishes with its distance (in wavelength \( \lambda \)) to the primary source. The higher the \( ka \) value the lower the responses of the sources at each others location. The \( (ka)^{-1} \) factor ensures a (small) reduction of the primary radiated power without having a high radiated power of the secondary source.

For two secondary sources - see Figure 3.2 -we find

\[
U_s = -U_p \frac{\text{sinc}(ka)}{1 + \text{sinc}(2ka)}
\]

\[
J_{\text{min}} = J_p[1 - \frac{2\text{sinc}^2(ka)}{1 + \text{sinc}(2ka)}]
\]  

(3.13)

Here, we notice the similarity of solution (3.13) with that for one secondary source (3.11). The source strengths also have the \( \text{sinc}(ka) \) behaviour but are corrected for each others action by the denominator; note that they are separated \( 2a \) from each other. The solution for the configurations with 4 and 6 sources - given in Figure 3.2 - exhibits a similar behaviour: the basic term \( \text{sinc}(ka) \) corrected for the contributions of the other sources. The optimal source strength for 4 secondary monopoles is

\[
U_s = -U_p \frac{\text{sinc}(ka)}{1 + 3\text{sinc}\left(\frac{2}{3}ka\sqrt{6}\right)}
\]  

(3.14)

Note that for this tetrahedron configuration the sources are located \( (2a\sqrt{6})/3 \) from each other. For 6 secondary monopoles we find

\[
U_s = -U_p \frac{\text{sinc}(ka)}{1 + \text{sinc}(2ka) + 4\text{sinc}(ka\sqrt{2})}
\]  

(3.15)

In the octahedron configuration the opposite source is located at \( 2a \) and the other four at \( a\sqrt{2} \). The results of power minimisation for the four different source distributions are shown in Figure 3.3. The label on the vertical axis 'normalised residual power' stands for the residual power with respect to the primary power.

On this simple experiment the following conclusions can be drawn:

- The reduction of the primary radiation improves with the number of secondary sources.
- Above \( ka = \pi \) the optimal solution hardly achieves reduction of the radiated power. This higher frequency band corresponds to \( a > \lambda/2 \).
Let us visualise the polar characteristics of the radiated intensity; see Figure 3.4. The polar diagrams show the local intensity at the far field surface S. For the configurations with one or two secondary sources the diagrams are radially symmetric with the source axis and only depend on \( \theta \).

Figure 3.4; Geometry for the polar diagrams of radiated intensity. The configuration with one secondary source is depicted. For the configuration with two secondary sources the geometry corresponds to that of Figure 3.2.

Figure 3.5 shows the primary and residual intensity values as a function of \( \theta \) for \( ka = 0.5\pi \) and \( 1.5\pi \) for the configurations with one and two secondary monopoles. For both frequencies the ASR systems obtain a global reduction of the radiated intensity, as can be detected in Figure 3.3.
Figure 3.5; Polar diagram of primary and residual radiated intensity for

a) one secondary source and $ka = \pi/2$

b) one secondary source and $ka = 3\pi/2$

c) two secondary sources and $ka = \pi/2$

d) two secondary sources and $ka = 3\pi/2$

In the polar diagrams we notice a less favourable effect of minimising the total radiated power of a set of monopoles. For certain angles the acoustic intensity has increased after optimisation! This is due to the global character of the cost function. Apparently, the systems cannot reduce the radiated intensity in one direction without amplifying the intensity in another direction. Later in this Chapter we will examine a cost function that prevents the increase of primary radiated intensity after control.

**minimisation of the radiated power over a limited arc of the field**

The results of the minimisation of the total radiated power are quite modest. Only for the very low frequency band ($ka < \pi$) the secondary monopole(s) can obtain a substantial reduction of the primary field. This is partly due to the requirement of reducing the field in every direction. It is obvious that this leads to conflicting requirements for the phase of the secondary source’s
strength. In practice the required region of silence probably comprises only the Fraunhofer area over a limited arc of the field. In Figure 3.6 this is shown for 1/2 arc and 1/4 arc. The corresponding cost function is given by

$$J_2(\omega, \theta_{max}) = \frac{1}{2} \text{Re} \left\{ [P^*_{\text{prim}}(r_s, \omega) + P^*_{\text{sec}}(r_s, \omega)] [V_{\text{prim}}(r_s, \omega) + V_{\text{sec}}(r_s, \omega)] \cdot n \, dS_g \right\}$$

(3.16)

For the two depicted examples the angle $\theta_{max}$ would be $\pi/2$ and $\pi/4$ respectively. Note that for the far field criterion function (3.16) a near field representative like (3.12) is not available.

![Figure 3.6: Diagrams of the limited arc of the field for radiated power minimisation](image)

(a) 1/2 arc minimisation  
(b) 1/4 arc minimisation

As is proven in Appendix A3.1 the result for minimising criterion (3.16) with one secondary source is

$$U_s = -\frac{jU_p}{2ka'} e^{-jka} \left[ 1 - e^{2jka'} \right]$$

$$J_{\text{min}} = J_p \left[ 1 - \text{sinc}^2(ka') \right]$$

with $a' = \frac{1}{2} a[1 - \cos \theta_{max}]$

(3.17)

In Figure 3.7 graphs of the reduction factor $J_{\text{min}}/J_p$ is given for four choices of subarea of silence. The analytical solution for the minimum value of radiated power over a limited arc contains a surprising characteristic.

- The performance of the ASR system minimising the radiated power over a limited arc (see Figure 3.6) equals that of minimising the total radiated power with the secondary monopole located $(1 - \cos \theta_{max})/2$ times closer to the primary source.

Note that the factor $(1 - \cos \theta_{max})/2$ is equal to the fraction of the surface of the limited arc sphere and the surface of the whole sphere. From the simulation experiment of minimising the
total radiated power it became apparent that the separation of the sources expressed in wavelength $\lambda$ determines the performance. In reducing the arc over which the radiated power is to be minimised, the secondary monopole is better capable of reconstructing the primary field; both amplitude and phase demands are less varying over the surface. Compared with the full surface case the secondary source may therefore seem to be located closer to the primary source. Note that the closer the secondary source is to the primary, the better it can reconstruct its field. Moreover, the performance of a secondary monopole to reduce the primary total acoustic energy can be enhanced by restricting the region of silence as much as possible.

It should be realised, however, that for the geometry of Figure 3.6 the reduction results probably are not the best that can be achieved. It can be expected that for a limited arc around $\theta = \pi/2$, the residual power may be less than shown in Figure 3.7.

![Graph showing normalized residual power vs. normalized wavenumber $ka$](image)

**Figure 3.7:** Residual radiated power of a primary monopole and a secondary monopole for 4 choices for the region of silence.

For the configuration with two secondary sources we find a result similar to (3.17); see Appendix A3.1.

\[
J_{\text{min}} = J_p \left[ 1 - \frac{2 \text{sinc}^2(ka')}{{\text{sinc}}(2ka')} \right] ;
\]

with \( a' = \frac{1}{2} a_0 \left[ 1 - \cos \theta_{\text{max}} \right] \)  \hspace{1cm} (3.18)

compare with eq. (3.13).

It is interesting to see at which expense the reduction over the limited arc is obtained. The polar diagrams of Figure 3.8 reveal an increase of intensity for the irrelevant directions. The secondary source distribution is well capable of focussing its control effort. The smaller the arc of the region of silence, the higher the increase of intensity for other directions. This effect is
illustrated in Figure 3.9 as well, where the total the radiated power is evaluated for several choices of limited arc. Obviously, minimising the power over a limited arc is obtained at the cost of an increase of the total radiated power.

Figure 3.8; 
Polar diagram of primary and residual radiated intensity for the configuration of one secondary source.

a) 1/2 arc minimisation and $ka = \pi/2$
b) 1/4 arc minimisation and $ka = \pi/2$
c) 1/2 arc minimisation and $ka = 3\pi/2$
d) 1/4 arc minimisation and $ka = 3\pi/2$
Section 3.3 Case study of a one-dimensional enclosure

Figure 3.9: Residual radiated power over the complete arc for a primary and a secondary monopole. The source strength of the secondary monopole was chosen to minimise the radiated power over a limited arc.

3.3 Case study of a one-dimensional enclosure

Before treating the practical cases of active sound reduction in a duct or a room, first a fundamental problem is studied. This case-study will reveal the principles of active sound reduction in an enclosure in a most comprehensive way. Consider a finite length \((L)\), hard-walled duct of which the cross-section is such, that the cut-on frequency of the first transverse mode lies well above the frequency band of interest. Then, the sound field can be considered to be acoustically one-dimensional, as merely plane wave propagation is possible in the duct. Furthermore, the left and right end terminations are characterised by reflection coefficient \(R\). For the sake of simplicity it is chosen to be constant with frequency. Then, the sound field generated by an arbitrary (one-dimensional) monopole source at \(x = x_0\) with source strength \(U\) is given as the solution of

\[
\frac{\partial^2 P}{\partial x^2} + k^2 P = -j \omega \rho_0 U(\omega) \delta(x - x_0)
\]

\[
P - \frac{1 + R_0}{1 - R_0} \frac{1}{jk} \frac{\partial P}{\partial x} = 0 \quad ; \quad x = 0
\]

\[
P + \frac{1 + R_L}{1 - R_L} \frac{1}{jk} \frac{\partial P}{\partial x} = 0 \quad ; \quad x = L
\]

(3.19)

The closed-form solution of eq. (3.19) can be found to be
for \( x < x_0 \)
\[
P(x, \omega) = \frac{\rho_0 c_0 U(\omega)}{2A} e^{-jk(x_0 - x)} \frac{[1 + R_0 e^{-2jkx_0}] [1 + R_f e^{-2jk(L-x_0)}]}{1 - R_0 R_f e^{-2jkL}}
\]

for \( x \geq x_0 \)
\[
P(x, \omega) = \frac{\rho_0 c_0 U(\omega)}{2A} e^{-jk(x-x_0)} \frac{[1 + R_0 e^{-2jkx_0}] [1 + R_f e^{-2jk(L-x)}]}{1 - R_0 R_f e^{-2jkL}}
\]

in which \( A \) is the cross-section area of the duct. In the case-study to follow a single primary source is located at \( x = 0 \), and a secondary source is located at an arbitrary \( x = x_s \), as shown in Figure 3.10. Both reflection coefficients are set equal to \( R; R_0 = R_L = R \).

![Diagram of the one-dimensional enclosure.](image)

**Figure 3.10:** Diagram of the one-dimensional enclosure.

As mentioned in Section 3.1, the fundamental cost function for active sound reduction is given by the total acoustic energy in the region of interest. Apart from the total energy the space average squared pressure as a criterion will be studied as well. The total acoustic energy density generated by the primary source with source strength \( U_p \) can be found using (3.1) and (3.20) as
\[
E_{\text{prim}}(x, kL) = \frac{\rho_0 |U_p|^2}{8A^2} \frac{1 + |R|^2}{1 - R^2 e^{-2jkL}} (1 + |R|^2)
\]

which is constant with \( x \).

In Figures 3.11a and 3.11b, graphs of the total acoustic energy density due to the primary source are given for low \( kL \) and several values of \( R \) (0.99, 0.6, 0, -0.6); the factor \( \rho_0 |U_p|^2/(8A^2) \) is set equal to \( 1.0 \times 10^{-3} \). We may conclude that for the total acoustic energy of a monopole source:

a) The spectrum exhibits resonances at equidistant frequencies.

b) The higher the absolute value of \( R \), the stronger the resonance peaks.

c) The resonance frequencies depend on the phase of \( R \) and the duct length \( L \); this follows from eq. (3.21).
As a first experiment we will minimise the total acoustic energy in the region left of the secondary source; region I in Figure 3.10. Employing eq. (3.20) to calculate the total acoustic energy in region I, and minimising it with respect to the source strength $U_s$, yields the following result:
\[ e_{\text{min}}(x, kL) = \frac{\rho_0 |U_p|^2 |1 + R|^2}{8A^2 \left| 1 + R \right|^2}; \quad \text{for } x < x_s \quad \text{(region I)} \]

\[ e_{\text{right}}(x, kL) = \frac{\rho_0 |U_p|^2 |1 + R|^2 |1 - Re^{-2jk(L-x_s)}|^2}{8A^2 \left| 1 + Re^{-2jk(L-x_s)} \right|^2}; \quad \text{for } x > x_s \quad \text{(region II)} \]

\[ U_s(kL) = \frac{1 + R}{1 + |R|^2} \frac{Re^{jkx_s} + Re^{-jk(2L-x_s)}}{1 + Re^{-2jk(L-x_s)}} U_p(kL) \]

**Figure 3.12a; Residual total acoustic energy density in regions I and II for R = 0.99. The total acoustic energy is minimised in region I.**

This means for the left area minimisation:

- The secondary source is capable of eliminating the resonance peaks of the original wave field. The residual wave field has a travelling character with an energy density uniform over \( kL \); see Figures 3.12a and 3.12b.
- For a purely travelling primary field, \( R = 0 \), there is no reduction of total acoustic energy.
- The total acoustic energy at the opposite side of the secondary source (region II) has increased. Here, a new standing wave field is created by means of the impedance of the right termination and a virtual nil impedance \( (R_s = -1) \) of the source. This effect is shown in Figures 3.12a and 3.12b.
If we focus on minimisation of the total acoustic energy in region II, another interesting result is obtained:

\[ e_{\text{left}}(x, kL) = \frac{\rho_0 |U_p|^2}{4A^2} \frac{|1 + R|^2}{|1 + Re^{-2j\alpha x_s}|^2}; \quad \text{for} \quad x < x_s \quad (\text{region I}) \]

\[ e_{\text{min}}(x, kL) = 0; \quad \text{for} \quad x > x_s \quad (\text{region II}) \]  

(3.23)

\[ U_s(kL) = -\frac{(1 + R)e^{-j\alpha x_s}}{1 + Re^{-2j\alpha x_s}} U_p(kL) \]
Figure 3.14: Reduction (fraction of residual and primary) of total acoustic energy as a function of $kL$ and $x_s$ for

a) $R = 0$

b) $R = 0.2$

c) $R = 0.5$

d) $R = 1.0$
Now, we conclude that

- The secondary source is capable of completely reducing the primary total acoustic energy downstream of the sources; region II. The downstream travelling plane wave of the primary source can perfectly be cancelled by the secondary plane wave.

- Again, on the other side (region I) a new standing wave field is created. At $x = x_s$ the primary field experiences a pressure release boundary condition ($R_s = -1$); see also Figure 3.13.

The first two experiments indicate that reduction of the primary total acoustic energy is possible on one side of the secondary source at the cost of a high level on the other side. Reduction of the primary field implies that the secondary source acts as a virtual reflecting boundary of the region of silence. For an enclosure, however, we are usually faced with the problem of reducing the total acoustic energy in the entire space. The following experiment is to examine how well a single source can reduce the primary energy in the whole duct. There are three main parameters, that determine the minimum total acoustic energy in the enclosure:

1) the position of the secondary source $x_s$, 
2) the reflection coefficient $R$, 
3) the normalised wavenumber $kL$.

In Figure 3.14 the reduction results are shown for several values of $R$ and the low $kL$-band.

From these graphs we derive the following characteristics:

a) The closer the secondary source to the primary source, the better the results for the whole frequency band. This corresponds to experiment 2 described above.

b) For a high value of $R$ the results strongly depend on position and wavenumber:

- There is very good reduction at resonance frequencies, even far from the primary source; except if the source is located at a node or close to a node. In Figure 3.14d the nodes manifest themselves as dots when $kL$ is an integer multiple of $\pi$.
- There is little reduction in between the resonance frequencies.

c) For a low $R$-value the results vary with $x_s$ and $kL$ in a smooth manner:

- Only if the secondary source is placed close to the primary a substantial reduction can be obtained.
- For $R = 0$ the reduction chart shows great similarity with the results of free field radiated power reduction. Recall from Section 3.2 that the reduction diminished proportionally with distance between the sources.

d) If the secondary source is located at the right end $L_s$, it is never confronted with a node.

e) For a low value of $R$ the reduction mainly depends on $x_s$, whereas for a high value of $R$ it is more dependent on $kL$. 
It turns out that an explanation of the above behaviour can be suitably given by using the approach of modal expansion. In Appendix A3.3 a modal expansion is derived for Green’s function in the 1-dimensional enclosure. This leads to the representation for the pressure and velocity

\[
P(x, \omega) = -j\rho_0 c U(\omega) \sum_{n=-\infty}^{+\infty} \frac{\varphi_n(x_0)}{k - k_n} \varphi_n(x)
\]

\[
V(x, \omega) = -jU(\omega) \sum_{n=-\infty}^{+\infty} \frac{\varphi_n(x_0)}{k - k_n} \psi_n(x)
\]

(3.24)

where the eigenfunctions are given by

\[
\varphi_n(x) = \frac{1}{2\sqrt{L}} \left[ \sqrt{\Re e^{-jk_nx}} + \frac{1}{\sqrt{R}} e^{jk_nx} \right]
\]

\[
\psi_n(x) = \frac{1}{2\sqrt{L}} \left[ \sqrt{\Re e^{-jk_nx}} - \frac{1}{\sqrt{R}} e^{jk_nx} \right]
\]

(3.25)

and the eigenvalues by

\[
k_n = \frac{n\pi}{L} - \frac{j}{L} \log_e(R)
\]

(3.26)

Note the expression (3.24) describes a sound field as the sum of standing waves (modes). The same sound field can be expressed as the sum of downstream and upstream travelling waves, as is shown in (3.20).

The frequency at which a given mode \( \varphi_n \) resonates is

\[
\omega_{res}(n) = c_0 \Re(\text{Re}(k_n))
\]

(3.27)

From eqs. (3.24), (3.25) and (3.26) we derive that the modal excitation coefficient is determined by

- the position of the secondary source in \( \varphi_n(x_s) \).
- the reflection coefficient \( R \) in the imaginary part of \( k_n \).
- the frequency in the denominator \( (k - k_n)^{-1} \).

With reference to eq. (3.24) the primary and secondary fields can be denoted as

\[
P_{prim}(x, \omega) = U_p \sum_{n=-\infty}^{\infty} a_n \varphi_n(0) \varphi_n(x)
\]

\[
P_{sec}(x, \omega) = U_s \sum_{n=-\infty}^{\infty} a_n \varphi_n(x_s) \varphi_n(x)
\]

(3.28)

A similar relation holds for the velocity field:
\[ V_{\text{prim}}(x, \omega) = \frac{1}{\rho_0 \omega} U_p \sum_{n=-\infty}^{\infty} a_n \varphi_n(0) \psi_n(x) \]
\[ V_{\text{sec}}(x, \omega) = \frac{1}{\rho_0 \omega} U_s \sum_{n=-\infty}^{\infty} a_n \varphi_n(x_s) \psi_n(x) \] (3.29)

Note that the mere difference in the modal excitation coefficients for the primary field and the secondary field is due to the location of the sources; \( a_n \varphi_n(0) \) against \( a_n \varphi_n(x_s) \). Minimising the total acoustic energy in the enclosure implies that the secondary source has to generate a wave field that matches (in anti-phase) the primary field as much as possible. Consequently, the modal excitation coefficients have to match as much as possible. Note that the modal excitation coefficients are the same for the pressure and velocity field. Realising this criterion, the characteristics of Figure 3.14 can be explained as follows:

- If the secondary source is close to the primary, the modal terms \( \varphi_n(x_s) \) will approximate those of the primary field \( \varphi_n(0) \). Then a choice of \( U_s \) close to \( -U_p \) will yield a substantial reduction. If the set of \( \varphi_n(x_s) \) differs from the set \( \varphi_n(0) \) a substantial reduction is only feasible in special cases.

- A reflection coefficient \( R \) close to unity means modal wave numbers \( k_n \) with a small imaginary part; see eq. (3.26). At the resonance frequency \( \omega_0 \) the modal term \( a_n \) will be quite large compared to the other modes. An example is shown in Figure 3.17. In this case, in which the wave field is dominated by a single mode, the secondary source can simply generate this mode in anti-phase without having to deal with the other modes. In fact a secondary source at any position can eliminate a dominant mode except when it is located in the vicinity of a node where \( \varphi_n(x_s) \ll 1 \). At this position the secondary source strength has to be large, which leads to a strong excitation of other modes. Because of this spill-over the overall reduction will be little.

- If the reflection coefficient is close to unity, but the excitation frequency is in between resonance frequencies, the distribution of the modal terms \( a_n \) is far less favourable; see Figure 3.18. Now at least two modes dominate the wave field. Reducing two modes becomes impossible if the signs of \( \varphi_i(0)/\varphi_i(x_s) \) and \( \varphi_j(0)/\varphi_j(x_s) \) differ.

- For a small value of \( R \) the wave field is always determined by at least a couple of modes; see Figures 3.19a and 3.19b. In this case the position of the secondary source is more important than for high \( R \) values. It should be placed close to the primary source in this case.
Figure 3.15: Reduction (fraction of residual and primary) of space average squared pressure as a function of $kL$ and $x_s$ for

a) $R = 0$

b) $R = 0.2$

c) $R = 0.5$

d) $R = 1.0$
Figure 3.16: Graphs of the increase of the fraction of residual and primary total acoustic energy, when the space average squared pressure is minimised, for several values of $x_s$ and $kL$.

a) $R = 0$
b) $R = 0.2$
c) $R = 0.5$
d) $R = 1.0$
Figure 3.17: The distribution of mode terms $a_n$ for $kL = 3\pi$ and $R = 0.99$.

Figure 3.18: The distribution of mode coefficients for $kL = 2.5\pi$ and $R = 0.99$.

Figure 3.19a: The distribution of mode coefficients for $kL = 3\pi$ and $R = 0.2$. 
Section 3.3 Case study of a one-dimensional enclosure

Figure 3.19b: The distribution of mode coefficients for $kL = 2.5\pi$ and $R = 0.2$.

It should be realised that the favourable results for secondary source close to the primary source are due to the fact that the primary source is also a monopole! For another kind of primary source the modal coefficients of primary and secondary field would be different if $x_s$ were close to $x_p$.

**minimisation of the space average squared pressure**

In Section 3.1 the space average squared pressure was proposed as a practical alternative for the total acoustic energy. Now let us examine the reduction of space average squared pressure in the duct as a function of reflection coefficient $R$, the position of the secondary source $x_s$ and the normalised wavenumber $kL$; see Figure 3.15.

The principal conclusion is that the characteristics of minimising the space average squared pressure very well correspond to those of minimising the total acoustic energy. There is only one striking difference:

- For the low frequency band $-(kL < \pi)$- the reduction of space average squared pressure is significantly better than that of total energy. When the duct is shorter than half the wavelength $\lambda$, minimisation of both the potential energy and the kinetic energy has a conflicting demand on the secondary source strength. The secondary source is capable of reducing the primary field such, that mainly one type of motion remains in the fluid: mainly translation (kinetic energy) or mainly compression and expansion (potential energy). Therefore, the reduction of the space average squared pressure is better for the low $kL$ band.

For a complete treatment of the characteristics of Figure 3.15 the reader is referred to the analysis given for the total acoustic energy. The interpretation of the results using the modal expansion model is fully transferable to the case of minimising the space average squared pressure. Inspection of eqs. (3.28) and (3.29) reveals that the modal excitation coefficients are the same for both the pressure field and the velocity field.
Although the characteristics of minimising the total acoustic energy or the space average squared pressure in the enclosure are quite similar, both criteria are not necessarily interchangeable. Minimising one cost function does not imply (nearly) minimisation of the other. However, we can conclude that the minimisation of the space average squared pressure inherently leads to an approximate minimisation of the total acoustic energy in two specific cases:

1) For a high value of $R$ the modal summation is dominated by a single mode for most frequencies. Then, minimisation of both the space average squared pressure and the total energy implies minimisation of the total modal excitation coefficient of the dominant mode.

2) For a low value of $R$ the wave field consists of travelling waves. This means that if the secondary source is close to the primary source, the total velocity field is approximately linearly related to the total pressure field (a factor $\rho_0 c_0$). In this case minimisation of the space average squared pressure inherently leads to an (approximate) minimisation of the total acoustic energy.

In order to examine the approximate equivalence further, the reduction of total acoustic energy is calculated for a secondary source strength optimised for the space average squared pressure. In Figure 3.16 the deterioration in performance of the ASR system when the non-optimal source strength is used. Here we observe two main features:

- For the most of the frequency band the solution to minimise the space average squared pressure very well approaches the absolute minimum of the total acoustic energy. In this region the two criteria are as good as equivalent irrespective of the termination impedance.
- For the low $kL (< \pi)$ band there is a clear deviation of the optimal residual. As pointed out above minimisation of the pressure field in the low $kL$ band will give an increase of the velocity field. This phenomenon is clearly exhibited in Figure 3.16. The residual total acoustic energy is far from optimal for $kL < \pi$, except for the secondary source close to the primary and when the field is dominated by a single mode (high $R$).

So, the two suggestions made above prove to be correct.

So at this stage we can report the good news that both the characteristics and the solutions of the theoretical cost function (total acoustic energy) and the practical cost function (space average squared pressure) show great similarities.

**multiple secondary sources**

As follows from the foregoing, the reduction of the total acoustic energy may be improved when using more than one secondary source. In fact it can be demonstrated that under soft
constraints, increasing the number of secondary sources always gives a lower minimum value of the total acoustic energy; see Appendix A3.1. Here we will illustrate the principle of active sound reduction using multiple sources.
Consider a secondary monopole at $x_s = 0.50L$ in a duct with strong reflective end terminations; $R = 0.99$. In Figure 3.20a the primary and residual total acoustic energy for this configuration are shown.

![Figure 3.20a](image1)

*Figure 3.20a: The primary and residual total acoustic energy using a secondary source at 0.5L.*

![Figure 3.20b](image2)

*Figure 3.20b: The primary and residual total acoustic energy using a secondary source at 0.75L.*
The secondary source is capable of reducing only the even modes ($\varphi_0$, $\varphi_2$, $\varphi_2$, ...) of the primary field. For the odd modes ($\varphi_1$, $\varphi_3$, $\varphi_5$, ...) the source is located at a node and therefore cannot cancel them without strongly exciting other modes. If we place the secondary source at $0.75L$, a similar phenomenon emerges (Figure 3.20b).

This secondary source configuration cannot well excite the second mode ($\varphi_2$) and therefore does not reduce it. This also holds for higher modes like $\varphi_6$, $\varphi_{10}$ and so on. However, the configuration does attenuate the odd modes in contrast to the previous configuration. Therefore, if we combine the two sources, the active control system should be able to reduce the first five modes shown. In Figure 3.21 the performance of two secondary sources (at $0.50L$ and $0.75L$) can be found. As expected the two sources are able to remove all resonances.

![Diagram](image)

**Figure 3.21;** The primary and residual total acoustic energy using two secondary sources at $0.5L$ and $0.75L$.

Reduction of these low-frequency resonances could also have been obtained by using just one secondary source at $x_s = L$. At this position, the secondary source is capable of exciting all duct modes. This is exhibited in Figure 3.22.

So, reduction of resonances can easily be obtained by using 1 or 2 secondary sources. The anti-resonances are much harder to attenuate, as these are built up of contribution of many modes. As a consequence, the combination of three sources does not yield a substantially better reduction; see Figure 3.23.
Section 3.3  Case study of a one-dimensional enclosure

Figure 3.22;  The primary and residual total acoustic energy using a secondary sources at L.

Figure 3.23;  The primary and residual total acoustic energy using three secondary sources at 0.5L, 0.75L and L.

the space average squared pressure

As mentioned in Section 3.1 the choice of the acoustic cost function is fundamental. Although the total acoustic energy is a very natural measure of the global strength of the primary field, it does not necessarily represent our acoustic aim. A more practically oriented alternative of the total acoustic energy is the space average squared pressure. A near similarity between these two criteria was shown in the beginning of this Section for a single source configuration. Let us try to minimise the primary squared pressure by using the secondary sources at 0.5L, 0.75L and L. Figures 3.24a, 3.24b and 3.24c show the reduction results.
Figure 3.24a: The primary and residual space average squared pressure using a single secondary source at $0.5L$.

Figure 3.24b: The primary and residual space average squared pressure using two secondary sources at $0.5L$ and $0.75L$.

As expected, the minimisation of the squared pressure or the total energy are quite similar for a multiple source configuration. Also, we can improve the performance by adding more sources to the system. The mere difference is the behaviour for $kL < \pi$. In that frequency band a substantial reduction is achievable by letting the secondary source(s) move in opposite phase with the primary source. This results in a mere translation of the air volume in the duct which corresponds to kinetic energy only.
Figure 3.24c: The primary and residual space average squared pressure using three secondary sources at 0.5L, 0.75L and L.

3.4 Active Reduction of Sound Fields in a Duct

A duct or a waveguide is characterised by one dimension, the duct's axis, being much larger than the other two. The axis is usually denoted as the travelling direction. For the sake of simplicity the cross-section of the duct is assumed to have a constant shape and acoustic behaviour

\[ \alpha(r_B)P(r_B, \omega) + \beta(r_B)\nabla P(r_B, \omega) \cdot n = 0 \]  \hspace{1cm} (3.30)

in which \( r_B \) is an element of the duct wall. This type of boundary condition was introduced in Section 2.4. The complex coefficients \( \alpha \) and \( \beta \) are functions of the transverse coordinates \( y \) and \( z \) only. For the duct geometry it is convenient to express the wave field as the product of two separate functions

\[ P(r, \omega) = X(x, \omega)\Phi(y, z) \]  \hspace{1cm} (3.31)

for which it holds that

\[ \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)\Phi(y, z) + \gamma^2 \Phi(y, z) = 0 \]

\[ \frac{d^2}{dx^2}X(x) + (k^2 - \gamma^2)X(x) = 0 \]  \hspace{1cm} (3.32)

For the cross-section area we can use the eigenfunction representation; see Appendix A3.2. Hence, the variation of the duct field over the cross-section can be described as an infinite weighted sum of eigenfunctions. For the variation of the pressure field along the axis we use right(+) and left(−) travelling waves. This yields
\[ P(r, \omega) = \sum_{n} P_n^+(\omega) \varphi_n(y, z) e^{-jk_{mn}x} + P_n^-(\omega) \varphi_n(y, z) e^{+jk_{mn}x} \]

with \[ k_{mn}^2 = k^2 - k_n^2 \]  

The model (3.33) is valid for any type of waveguide with boundary conditions, eq. (3.55), constant with the axis. It can be derived that the wave field in an infinite duct generated by a monopole with source strength \( U \) at \( r_0 \) equals

\[ P_m(r, \omega) = j\omega \rho U(\omega) \sum_n \frac{\varphi_n(y, z) \varphi_n(y_0, z_0) e^{-jk_{mn}x-x_0}}{2jk_{mn}} \]  

Let us examine a special case of a monopole field in a rectangular duct -with cross sizes \( L_y \) and \( L_z \)- and rigid walls; so in eq. (3.30) \( \alpha(r_B) = 0 \) and \( \beta(r_B) \neq 0 \). Then, the normalised acoustic modes are specified as

\[ \varphi_{mn}(y, z) = \sqrt{\frac{\varepsilon_m l_n}{L_y L_z}} \cos(\frac{m\pi y}{L_y}) \cos(\frac{n\pi z}{L_z}) \]

with \[ \varepsilon_m = 2 - \delta_{0m}; \ m, n = \{0, 1, 2, 3, \ldots\} \]  

The corresponding wavenumber for the propagation direction is

\[ k_{mn} = \begin{cases} \sqrt{k^2 - \gamma_{mn}^2} \quad & \text{for } k^2 \geq \gamma_{mn}^2 \\ -j\sqrt{\gamma_{mn}^2 - k^2} \quad & \text{for } k^2 < \gamma_{mn}^2 \end{cases} \]  

where

\[ \gamma_{mn} = \sqrt{\left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_z}\right)^2} \]  

The relation eq. (3.36) for the axial wavenumber has a twofold impact:

1) The wave field falls apart into propagating (in the x-direction) and evanescent waves. If the eigenvalue \( \gamma_{mn} \) of an acoustic mode exceeds the wavenumber \( k \), the mode exhibits an exponentially decaying amplitude in the x-direction. For a given frequency only a finite number of acoustic modes is capable of travelling along the duct. Therefore the frequency for which \( k \) equals \( \gamma_{mn} \) is called the cut-on frequency of the mode \( mn \).

2) Each travelling mode displays a different group velocity in the x-direction

\[ c_{\text{group}}(m, n) = \left(\frac{\partial k_{mn}}{\partial \omega}\right)^{-1} = c_0 \sqrt{1 - \frac{\gamma_{mn}^2}{k^2}} \]  

So every higher \( (m>0, n>0) \) mode propagates through the duct at a group velocity smaller than the velocity of sound. The plane wave mode travels fastest through the duct. The dispersive property of higher modes can easily be explained by visualising an acoustic mode as a skew travelling plane wave, reflected by the duct walls; see Figure 3.25. The effective group velocity in the axial direction depends on the angle of incidence and therefore on the modal index.
Figure 3.25: Graphical interpretation of higher order mode propagation.

The dispersion relation eq. (3.38) also means that each mode travels at a different speed for different frequencies. For a frequency much higher than its cut-on frequency the group velocity of a mode approximates \( c_0 \), whereas at the very cut-on frequency the group velocity is nil.

**Acoustic cost functions for duct noise**

In this Section we aim at the reduction of the sound radiated from the duct’s mouth. Usually, the sound field inside an exhaust or duct is of minor importance in relation to the exterior sound field. As it is hard to evaluate the total acoustic energy inside a room as a function of the duct interior sound field, we need to define a criterion related to the total acoustic energy:

1. Reduction of the sound field outside the duct generated by the duct source.
   Here we propose the total radiated acoustic power from the duct’s mouth.
   \[
   J_1 = \Pi(\omega) = \frac{1}{2} \text{Re} \int_S P(\mathbf{r}, \omega) V^\ast(\mathbf{r}, \omega) \, dS
   \]  
   (3.39)

   Similar to the fundamental cost function - the total acoustic energy - this criterion involves the measurement of the velocity field. This may lead to complications when it comes to practical realisation. Therefore, in this Section as well the space average squared pressure is examined upon its suitability to serve as a practical alternative. This is represented by the second criterion function.

2. Reduction of the space average squared pressure in a region near the duct’s end.
   \[
   J_2 = \overline{\mathcal{P}(\omega)} = \frac{1}{V} \int_V |P(\mathbf{r}, \omega)|^2 \, dV
   \]  
   (3.40)

Let us consider the sound field in a rigid-walled duct with anechoic end terminations - worst case for radiated power. In this case the eigenfunctions are real-valued. The pressure field downstream of a monopole at \( r_0 \) is given in eq. (3.34). Here we use the notation

\[
P(\mathbf{r}, \omega) = \sum_n a_n(\omega) \varphi_n(y, z) e^{-jk_w(x-x_0)}
\]  
(3.41)

In order to calculate the acoustic power flow through the duct we merely need the \( x \)-component of the particle velocity.
\[ V_z(r, \omega) = \frac{k x_n}{\omega \rho_0} \sum_n a_n(\omega) \phi_n(y, z)e^{-jkx_r(x-x_0)} \]  \hspace{1cm} (3.42)

The mean acoustic power radiating down the duct can be obtained by the expression

\[ \Pi(r, \omega) = \frac{1}{2} \text{Re} \int \int PV_z^* d\gamma d\zeta = \frac{1}{\rho_0 c_0} \sum_{n=0}^{N-1} \frac{k x_n}{k} |a_n|^2 \]  \hspace{1cm} (3.43)

In order to investigate the features of reducing radiated power using a few secondary sources, a simulation experiment is set up. Consider a rectangular, lightly damped duct, as depicted in Figure 3.26. In accordance with the previous Section the control sources are positioned such, that they are not close to node points of the low frequency modes. For this hard-walled waveguide the corners turn out to be 'safe' positions.

![Figure 3.26: Geometry of the duct, sources and region of silence.](image)

\[ L_x = 5; \quad L_y = 0.85; \quad L_z = 0.33 \]

\[ P \text{ at } (0, 0.05, 0.05) \]

\[ S_1 \text{ at } (2, 0.05, 0.28); \quad S_2 \text{ at } (2, 0.8, 0.05); \quad S_3 \text{ at } (2, 0.8, 0.28) \]

\[ \text{region of silence} \]

![Figure 3.27a: Primary and residual radiated power using secondary source 1.](image)
The primary monopole is located such that it generates all of the low frequency modes. For the configuration as in Figure 3.26 the results of radiated power minimisation beyond the secondary sources can be found in Figures 3.27a, b and c; see Appendix A3.1 for details on the minimisation procedure.

![Figure 3.27b](image1)

*Figure 3.27b: Primary and residual radiated power using secondary source 1 and 2.*

![Figure 3.27c](image2)

*Figure 3.27c: Primary and residual radiated power using secondary source 1, 2 and 3.*
From these graphs we extract the following characteristics:

1) The resonance peaks can be well reduced except if the secondary source is located near a nodal line of the specific mode. Although it is not specifically examined it holds that placing a secondary source near a nodal line of a mode, the corresponding resonance cannot be attenuated. This statement follows directly from the modal expansion (3.41) and the analysis in the previous Section.

2) If the number of excited modes is higher than the number of secondary sources, there is only a modest reduction in between resonances.

3) If the number of secondary sources is larger or equal the number of excited modes, an absolute reduction is obtained.

The absolute reduction of the primary power for the low frequencies is in full agreement with experiment 1 in Section 3.3. In that case a single secondary source completely reduced the downstream plane wave field. Here, a few secondary sources are capable of completely reducing the radiated power of as many components (plane wave and a few higher modes).

*minimisation of the space average squared pressure*

Based on the modal expression (3.41) we find for the squared pressure density in a cross-section

$$|P(x,\omega)|^2 = \sum_{n=0}^{N-1} |a_n|^2 + \sum_{n=N}^{\infty} |a_n|^2 e^{-2jk_n(x-x_0)}$$

(3.44)

So the squared pressure density of the wave field falls into two contributions:

1) The contribution of a propagating mode ($n < N$) is independent of the position in the wave guide. The coefficient $a_n$ depends - among other things - on the position of the source.

2) Modes with a resonance frequency higher than the excitation frequency have an imaginary propagation wavenumber - see eq. (3.36) - and are therefore evanescent. At close range of the source, however, the evanescent field significantly contributes to the total acoustic energy.

This squared pressure density (3.44) differs from the radiated power (3.43) in a twofold manner. First, the summation both includes the propagating and evanescent modes. For the radiated power the pressure and axial velocity of the evanescent modes appear to be 90 degrees out of phase. Second, the individual modal weighting factor differs by a factor $k_{xp}/k$.

In order to study the suitability of the space average squared pressure to ‘replace’ the radiated power as a criterion, the following experiment was carried out:

- For three secondary source configurations the strengths are calculated such that the space average squared pressure is minimised.
- For these source strengths the residual radiated power is calculated and compared with the optimum.

In Figure 3.28a the residual radiated power is shown, where the secondary sources are tuned such that the space average squared pressure in region between $x = 2$ and $x = 5$ is minimised.

![Figure 3.28a: Primary, optimal and residual radiated power using source 1.](image1)

![Figure 3.28b: Primary, optimal and residual radiated power using source 1 and 2.](image2)
Comparing the optimal reduction of the radiated power with that achieved by using a secondary source strength that minimises the space average squared pressure, we find

a) The space average squared pressure can well be used as an alternative criterion instead of the radiated power. The reduction obtained is quite close to the optimal reduction—i.e. substantial reduction of radiated power for the first modes.

b) As becomes apparent when comparing expression (3.44) to (3.43), the solution minimising the space average squared pressure suffers from the influence of the evanescent modes. If the region of silence were chosen further away from the sources, this influence would have been smaller.

Principally, the results of this simulation experiment are in full agreement with the analysis of the 1-dimensional enclosure.

### 3.5 Active reduction of enclosed sound fields

In this Section active control of sound in an enclosure is examined. Again, we will analyse the performance of a small distribution of secondary sources. Here an enclosure $\Omega$ is specified as a homogeneous medium limited in all three dimensions. A wave field in the enclosure obeys the following set of differential equation

$$\nabla^2 P(r, \omega) + k^2 P(r, \omega) = -\Psi_{\text{tot}}(r, \omega)$$  \hspace{1cm} (3.45)$$

and boundary condition:
\[ \alpha(r_B) P(r_B, \omega) + \beta(r_B) \nabla P(r_B, \omega) \cdot n = 0 \]  
(3.46)

in which \( r_B \) denotes a point on the enclosure boundary. The function \( \Psi_{\text{tot}} \) represents the distribution of both volume and force sources. As demonstrated in Section 3.3 dropping the \( \omega \)-dependence of \( \alpha \) and \( \beta \) in the boundary condition simplifies the wave field modelling without loosing the essential features. More details on this can be found in Appendix A3.2. It is convenient to express an enclosed wave field in terms of eigenfunctions. Consider the three-dimensional set of eigenfunctions that obey the equation

\[ \nabla^2 \varphi_n(r, \omega) + k_n^2 \varphi_n(r, \omega) = 0 \]  
(3.47)

and the boundary condition (3.46). Again, the infinite set of eigenfunctions is complete for any square-integrable function in \( \Omega \) that satisfies (3.46). The wave field generated by a monopole at \( r_0 \), \( \Psi_{\text{tot}} = j \omega \rho_0 U(\omega) \delta(r-r_0) \), can be expanded as follows

\[ P(r, \omega) = -j \omega \rho_0 U(\omega) \sum_n \frac{\varphi_n(r_0) \varphi_n(r)}{k_n^2 - k_n^2} \]  
(3.48)

The expansion (3.48) follows directly from that for Green’s function derived in Appendix A3.2. The representation is also called a spectral decomposition of the wave field. The field is decomposed in eigenfunctions or acoustic modes, \( \varphi_n(r) \), with amplitudes \( P_n \). Modes with a resonance frequency close to the excitation frequency dominate the field. In principle each acoustic mode contributes to the total wave field, although modes excited far from resonance, \( |k_n^2 - k_n^2| >> 1 \), may be neglected.

Concerning the reconstruction of the primary field in the enclosure, the analysis is similar to that of the 1-dimensional enclosure (Section 3.3) and the duct (Section 3.4). The primary field can be reconstructed properly, if the distribution of acoustic modes that count, can be generated by a set of secondary sources in an identical way. Usually, this demands as many secondary sources as relevant modes located independently with respect to the mode coefficients. Since in practice the number of required sources may be too high, we have to define appropriate cost functions in order to optimise a chosen set of control sources.

**minimisation of global acoustic energy**

Again, the total acoustic energy in the enclosure is proposed as a suitable cost function. The time-averaged potential and kinetic energy are given by

\[ E_{\text{pot}} = \frac{1}{4 \rho_0 c_0^2} \oint P P^* d\Omega \]  
(3.49)

\[ E_{\text{kin}} = \frac{1}{4 \rho_0 c_0^2 k_n^2} \oint \nabla P \cdot \nabla P^* d\Omega \]

In evaluating (3.49) the pressure field is denoted as the weighted sum of eigenfunctions.
\[ P(r, \omega) = \sum_n a_n(\omega) \phi_n(r) \] (3.50)

Inserting eq. (3.50) into eq. (3.49) usually yields a double sum over the eigenfunctions. For real-valued \( \phi \) however, the orthogonality relation (A3.45) holds, which simplifies the expression (3.49). For a rigid-walled enclosure, a reverberation chamber, the total acoustic energy can be expressed as

\[ E_{\text{tot}} = \frac{1}{4\rho_0 c_0^2} \sum_n |a_n|^2 (1 + \frac{k_n^2}{k^2}) \] (3.51)

in which the kinetic energy is responsible for the \( (k_n/k)^2 \). It is pointed out by Pierce [Pi85] that the kinetic energy is not finite as \( n \) goes to infinity. This is due to the unlimited contribution near the volume source(s). The region close to the sources can be excluded from the integral, however, by truncating the summation. The potential energy is hardly influenced by this approximation. In a very extensive treatment [Bu87] on active sound reduction in enclosures the acoustic potential energy is chosen as cost function. In this reference the reader can find more details on the truncation of the modal summation.

![Figure 3.29: Top view diagram of geometry of enclosure and sources.](image)

The relevant enclosure for the simulation experiment is given by a hard-walled rectangular box with dimensions 1.00x0.85x0.08m. To keep resonances limited a small damping factor is included in the model; see Appendix A3.4 for more details. The dimensions are such that for low frequencies the enclosure behaves acoustically two-dimensional; oblique modes are far from being excited below 700 Hz. In Figure 3.29 the distribution of primary and secondary sources is depicted. Again, the secondary sources are placed such that they well couple into the modes of the relevant frequency band. For numerical convenience a source comprises a squared piston of dimensions 0.1x 0.1m. All sources are located at \( z = 0.08 \text{m} \). The primary source strength is chosen to be constant with frequency.
In Figure 3.30 the global acoustic energy is given as a function of the frequency band of interest. In order to arrive at this result the summation (3.51) was carried out over 2,000 modes. The graph clearly reveals the resonance frequencies of the enclosure modes. As the frequency increases the resonances become distributed in a denser way. This phenomenon did not occur in the one-dimensional enclosure of Section 3.3. The increase of modal density with frequency in a three-dimensional enclosure is due to the existence of tangential and eventually oblique modes apart from the one-dimensional axial modes.

![Graph showing acoustic energy](image)

*Figure 3.30: The primary total acoustic energy.*

![Graph showing cross-sectional view](image)

*Figure 3.31a: Cross-sectional view of primary squared pressure at 98 Hz.*

It is interesting to observe the influence of the several modes on the wave field in the enclosure. At 98 Hz e.g. the wave field is dominated by a contribution of the (0, 0, 0) mode - which is plane - and some higher order modes. The resultant distribution of squared pressure is shown in
Figure 3.31a. A small ripple is present near the primary source. This indicates that the higher order modes already influence the field far below their resonance frequency. At 170 Hz the first mode in the x-direction - the (1, 0, 0) mode - is in resonance. Due to the low damping properties of the enclosure, the mode completely dominates the wave field. Figure 3.31b clearly exhibits a distribution of energy with a nodal line at $x = L_y/2$ and maxima at the walls.

![Graphical representation](image)

*Figure 3.31b; Cross-sectional view of primary squared pressure at 170 Hz.*

The next mode in resonance - the (0, 1, 0) mode - has an identical behaviour in the y-direction. In between the two resonance frequencies the wave field is mainly determined by these two modes. This gives rise to a diagonal nodal line and maxima in the corners; see Figure 3.31c.

![Graphical representation](image)

*Figure 3.31c; Cross-sectional view of primary squared pressure at 185 Hz.*
minimisation of the total acoustic energy

If the total acoustic energy of the primary source and the nearby secondary source $s_4$ is minimised the following result is obtained; Figure 3.32. Since the secondary source is close to the primary the total acoustic energy is very well reduced for the lower part of the frequency band. For the lower part of the frequency band the source excites the modes with the same phase as the primary source. The first problem emerges around 400 Hz where two resonance frequencies coincide; $f_{210} = 395$ Hz and $f_{020} = 401$ Hz. The piston $s_4$ lies on a nodal line of the (2, 1, 0) mode.

![Figure 3.32: Primary and residual total acoustic energy using source 4.](image)

The reduction achieved by using corner source $s_1$ is far less favourable; see Figure 3.33. Generally, this source can well reduce isolated resonance peaks. For frequency components
with a higher modal density there is hardly reduction. The performance improves by adding corner sources $s_2$; see Figure 3.34.

![Graph showing primary and residual total acoustic energy using corner sources 1 and 2.](image)

*Figure 3.34; Primary and residual total acoustic energy using corner sources 1 and 2.*

The combination of the sources is still not able to attenuate the two coinciding modes around 400 Hz. Both sources are placed at positions where the relevant modes (the $(2, 1, 0)$-mode and the $(0, 2, 0)$-mode) have opposite phase in relation to the position of the primary source. Therefore, cancellation of the two modes has a conflicting phase demand of the sources; also see Figure 3.35.

![Diagram showing nodal lines and relative phases for two modes with close resonance frequencies.](image)

*Figure 3.35; The nodal lines and relative phases for two modes with close resonance frequencies*

a) the $(2, 1, 0)$ mode with resonance frequency $395$ Hz.

b) the $(0, 2, 0)$ mode with resonance frequency $401$ Hz.
In this graph it is also seen that for the \((2, 1, 0)\) mode source \(s_3\) has another phase relation. When adding source \(s_3\) to the active sound control system the two troublesome modes can be well reduced; see Figure 3.36.

\[
\begin{array}{c}
\text{Figure 3.36: Primary and residual total acoustic energy using corner sources 1, 2 and 3.}
\end{array}
\]

The main conclusions on total acoustic energy reduction in an reverberating enclosure are:

- A favourable reduction is achieved if the coupling between the primary source distribution and the secondary source distribution is strong. This occurs in the following cases:
  a) The sound field is dominated by as many as modes as secondary sources; this is likely to happen at low frequencies and at resonances.
  b) One secondary source is of the same type and at close range \((kr < 1)\) of the primary source.

- A poor reduction occurs in the following cases:
  a) There are more dominating modes than secondary sources; especially when the relative mode phases for the secondary sources differ from those for the primary.
  b) The secondary source distribution is not capable of exciting a mode in resonance; source is located at nodal line.
  c) The secondary source distribution generates too much of the non-dominant modes. This occurs when the distribution is located near a nodal line.

In general, the characteristics of actively reducing the global acoustic energy in an enclosure are similar to those of the 1-dimensional enclosure. In fact, the main difference is the increase
of modal density with frequency in a three-dimensional room. This phenomenon sets a limit to the band of operation for active sound reduction. As one might expect, the principles of minimising the total acoustic energy agree with those of minimising the space average squared pressure; see also [Bu87]. Here we show the results of minimising the space average squared pressure for the configurations with the corner sources.

![Graph](image1)

*Figure 3.37: Primary and residual space average squared pressure using corner source 1.*

![Graph](image2)

*Figure 3.38: Primary and residual space average squared pressure using corner sources 1 and 2.*
Figure 3.39: Primary and residual space average squared pressure using corner sources 1, 2 and 3.

Again, we conclude that the principles of minimising both criteria very well agree. The main difference occurs for the low frequency band in which the (0, 0, 0) - mode is dominant. In that case the space average squared pressure can be substantially attenuated at the cost of an increase of the acoustic kinetic energy.

3.6 The minimisation of the maximum of the squared pressure distribution

The cost functions utilised in this Chapter so far all have a global character. Minimising one of these criteria implies reduction of the primary sound in the spatial mean. They do not provide information on the local performance of the active sound reduction system. In order to prevent a local amplification of the primary field and to guarantee reduction in the whole area, a novel cost function can be defined as

- The maximum value of the squared pressure in the region of silence.

If we choose the secondary source strength to minimise the maximum value of the squared pressure, we face a so-called minimax optimisation problem. This Section is one of the first steps into the use of minimax cost functions for active sound reduction. For two specific cases some principles of minimisation the maximum of the squared pressure are shown.

minimax optimisation of radiated intensity

The polar diagrams of the residual intensity resulting from the minimisation of the radiated power displayed an unfavourable side-effect; see Figure 3.5, Section 3.2. Minimisation of the globally radiated intensity may comprise an increase of the locally radiated intensity. This increase can be circumvented by choosing the following cost function
\[ J_\infty(\omega) = \max_{r_s} \{ I_\infty(r_s, \omega) \} \tag{3.52} \]

which has a more local nature. The optimisation is carried out over \( r_s \) denoting the coordinates of the far field surface \( S \); see Figure 3.1.

**Figure 3.40a:** Normalised maximum values of angular radiated power; one secondary monopole.

**Figure 3.40b:** Normalised maximum values of angular radiated power; two secondary monopoles.

As can be found in Section 3.2, minimising the maximum value of the radiated intensity indirectly means the minimax value of the far field squared pressure. The maximum value for the radial intensity is the maximum of the primary intensity, which is uniform over \( S \). This means that the solution of the minimax optimisation might be to turn off the secondary sources.
in order to prevent a local increase of primary intensity. In Appendix A3.5 more details on the minimax optimisation can be found. The results of minimising the maximum value of the acoustic intensity at any angle are depicted in Figures 3.40a and 3.40b.

In the graphs the solution emerging from minimising the radiated power is denoted as the $L_2$ solution; see Section 3.1. The striking difference between an $L_2$ and an $L_{\infty}$ solution is the behaviour for higher $ka$ bands. At a certain $ka$ value ($0.5\pi$) it is no longer possible to reduce the radiated power without increasing the intensity for certain angles. Therefore, cost function (3.52) implies to refrain from control for the higher $ka$ values.

---

Figure 3.41: Polar diagram of primary and residual radiated intensity for two different solutions and $ka = 0.45\pi$.

- a) $L_2$ - solution for one secondary source
- b) $L_{\infty}$ - solution for one secondary source
- c) $L_2$ - solution for two secondary sources
- d) $L_{\infty}$ - solution for two secondary sources
The mechanism behind the minimisation of the maximum radiated intensity can be studied in the polar diagrams of Figure 3.41. Here, the global solutions have their maximum intensity for $\theta = 0$ and $\theta = \pi$. For the other angles the intensity is much lower. The minimax solutions, however, reduce the value for the unfavourable angles at the cost of an increase for the other angles. The resulting intensity shows a less fluctuating behaviour with $\theta$. The main difference between the $L_{\infty}$ and the $L_2$ solution occurs for the frequency band $ka > \pi/2$; global reduction can only be obtained at the cost of a local increase of intensity. For these frequencies the minimax optimised radiated power equals the primary.

**the minimax optimisation of the squared pressure in the 1-dimensional duct**

In the 1-dimensional enclosure - see Section 3.3 - we studied the minimisation of the space average squared pressure. Mathematically, this can be written as the squared 2-norm of the pressure field

$$J_2 = \|P\|_2^2 = \int_0^L P(x)^2 \, dx$$

(3.53)

The criterion studied here is the maximum value of the squared pressure, which can be denoted as the squared $\infty$-norm:

$$J_{\infty} = \|P\|_{\infty}^2 = \max_{0 \leq x \leq L} \{ |P(x)|^2 \}$$

(3.54)

The optimal secondary source strength is such, that the maximum of the residual energy density is the lowest for all possible values of the source strength. For a primary source at $x = 0$ and a secondary source at $x = L$ the squared pressure is given by:

$$|P(x, kL)|^2 = \frac{(\rho_0 c_0)^2}{4A^2} \left| 1 + R \right|^2 \frac{U_p [1 + Re^{-2jk(L-x)}] e^{-jkr} + U_s [1 + Re^{-2jkx}] e^{-j(kL-x)}^2}{1 - R^2 e^{-2jkL}}$$

(3.55)

Generally, a minimax problem is not easy to solve in an analytical way. In this case a numerical algorithm has been employed to find the optimal value of $U_s$ in (3.55). Figures 3.42a to 3.42c show the minimum maximum values of the squared pressure for the configuration described above. Also, the maximum values due to minimisation of the space average squared pressure are depicted.
Figure 3.42a: Maximum values of squared pressure for two different solutions and $R = 0$.

Figure 3.42b: Maximum values of squared pressure for two different solutions and $R = 0.6$.

Drawing a comparison between the 2-norm and $\infty$-norm of the pressure field we may conclude:

a) The difference in maximum values of the squared pressure is moderate. For a low $R$ the 2-norm minimisation leads to an increase of the primary maximum. Note the similarity of Figure 3.42a and that obtained in the free field case (Figure 3.40a). Obviously, the $\infty$-norm minimisation yields a maximum which is never larger than the primary.

b) At resonance frequencies both cost functions give the most similar results. A mode in resonance dominates both the maximum value and the average value of the squared pressure.
c) At anti-resonances the distinction between the impact of the two cost functions is most striking. Apparently, the minimisation of frequency components with high modal density can only be obtained at the cost of amplifying the primary field somewhere in the duct.

Concluding, the $\infty$-norm might be a useful extension to the set of suitable cost functions. A thorough study on the characteristics of this cost function is still to be carried out, however. In addition, the practical implementation is expected to be more complicated compared to the implementation of quadratic cost functions. This idea is shared in [El87].

3.7 References


Appendix A3.1 Minimisation of quadratic Hermitian forms

In this Section the minimisation of functions of the following type is addressed

\[ J = [d - Gu]^H[d - Gu] \]  
(A3.1)

in which \( J \) is to be minimised with respect to \( u \). In fact, \( J \) is the squared Euclidean norm of an \( N \)-dimensional vector, where \( N \) is the dimension of \( d \). Criterion function (A3.1) can also be written as the square of a vector length

\[ J = \|d - Gu\|^2 \]  
(A3.2)

Assuming that the dimension of \( d \) exceeds that of \( u \), the optimal solution can be found as

\[ u_{\text{opt}} = [G^HG]^{-1}G^Hd \]  
(A3.3)

\[ J_{\text{min}} = d^H[1 - G[G^HG]^{-1}G^H]d \]

It is most clarifying to interpret the result in a geometrical way. The primary vector \( d \) denotes a point in the \( N \)-dimensional space \( \Omega \). Next, we want to approximate \( d \) as close as possible by means of vector \( Gu \). The points that can be reached by \( Gu \) lie in the subspace \( S \) spanned by the columns of \( G \). Hence, the distance between \( Gu \) and \( d \) is as small as possible, if \( Gu \) equals the orthogonal projection of \( d \) on the subspace \( S \). The matrix \( P_0 \) for projection on \( S \) is given by - see e.g. [ND88] -

\[ P_0 = G[G^HG]^{-1}G^H \]  
(A3.4)

yielding the residual vector

\[ d_0 = (I - P_0)d \]  
(A3.5)

which is perpendicular to the subspace \( S \). The projection matrix \( P_0 \) is idempotent with eigenvalues equal to 1 or 0.

Now let us extend the \( M \)-dimensional \( (M < N) \) vector \( u \) with an additional component and investigate the consequences on the minimum length problem. The secondary vector has become

\[ (Gu)_{M+1} = \begin{pmatrix} G & g_{M+1} \end{pmatrix} \begin{pmatrix} u \\ u_{M+1} \end{pmatrix} \]  
(A3.6)

The extended matrix \( G_{M+1} \) spans apart from \( S \) the additional subspace \( S_1 \). This additional subspace is created by the part of \( g_{M+1} \) which is orthogonal to \( S \). So the vector \( s_x \) that spans \( S_1 \) equals

\[ s_x = (I - P_0)g_{M+1} \]  
(A3.7)

This means that we can approximate the primary vector \( d \) closer by its orthogonal projection \( P_x \) on \( S_1 \):
\[ d_x = s_x (s_x^H s_x)^{-1} s_x^H d_0 = P_x d_0 \]  
(A3.8)

As a consequence, the new residual vector \( d_1 \) can be written as

\[ d_1 = (I - P_0 - P_x)d \]  
(A3.9)

The improvement in squared length reduction equals

\[
\left( \|d_0 - d_1\|_2 \right)^2 = (P_x d)^H P_x d = (P_x d_0)^H P_x d_0
\]  
(A3.10)

which is the squared length of the projection of \( d \) on \( S_1 \). The following conclusions on the efficacy of extending the secondary vector can be drawn:

1) The additional \( u \)-component \( u_{M+1} \) is optimally effective if the residual \( d_0 \) lies in \( S_1 \), the subspace spanned by that part of \( G_{M+1} \), which is orthogonal to \( S \).

2) No reduction is obtained if \( G_{M+1} \) is an element of \( S \) or its orthogonal (to \( S \)) component is orthogonal to \( d_0 \) as well.

Now the projection theory is applied on several minimisation problems of active sound reduction. As wave fields add, we have to replace the minus sign in eq. (A3.2)

\[ J = [d + Gu]^H [d + Gu] \]  
(A3.11)

which implies complete validity of the original theory if we change the sign of \( u \).

Next, let us examine the minimisation of quadratic cost functions appearing in Chapter 3. In addition, it may be effective to know under which conditions an extra source improves the performance best. The quadratic cost functions in Chapter 3 are of the same type as eq. (A3.11) except for a possible use of a weighting matrix \( W \)

\[ J = [d + Gu]^H W [d + Gu] \]  
(A3.12)

On physical grounds the weighting functions are Hermitian; \( W^H = W \). This feature enables us to apply the Cholesky decomposition

\[ W = F^H F \]  
(A3.13)

where \( F \) is an upper triangular matrix. Evaluating eq. (A3.12) gives

\[ J = [d + Gu]^H F^H F [d + Gu] = [d' + G'x]^H [d' + G'x] \]

with \( d' = Fd \); \( G' = FG \)  
(A3.14)

So the weighted minimisation can be rewritten in an unweighted form with transformed components \( d' \) and \( G' \). The optimal solution can be found as

\[ u_{\text{opt}} = -[G^H W G]^{-1} G^H W d \]

\[ J_{\text{min}} = d^H [I - W G [G^H W G]^{-1} G^H] W d \]  
(A3.15)

This minimum can be achieved if the matrix \([G^H W G]\) is invertible. This is the case when \( G \) is full rank: the sources are positioned independently with respect to their contribution to \( J \).
Generally, the matrix $[G^H W G]$ is likely to be invertible and therefore positive definite. Consider the case that $d=0$ which yields for the energetic quantity $J$

$$J = u^H (G^H W G) u$$

(A.3.16)

If $G$ is not of full rank the matrix $[G^H W G]$ is positive semidefinite. In respect to (A.3.16) this means that the secondary sources are located such, that there exist a non-zero vector of strengths ($u \neq 0$) that gives rise to no field in the sense of $J$. An dependently designed configuration of secondary sources is capable of cancelling its own contribution to the cost function.

Based on the structure (A.3.12) for a cost function, below the quadratic minimisation problems of this Chapter will be treated. For each case the components $d$, $G$ and $W$ are specified. The vector $u$ represents the strengths of the secondary sources. The optimal solution to each of the immunization problems can be found in (A.3.15).

**free field power output**

With reference to Section 3.2 it holds for the radiated power of a primary and $N$ secondary monopoles

$$\Pi_{\text{rad}} = \frac{1}{2} \text{Re} \left\{ p^* (r_p) U_p + p_{\text{tot}}^H u \right\}$$

(A.3.17)

where $p$ is an $N$-dimensional vector containing the values of effective pressure at the source points and $u$ is the vector of secondary source strengths. The first term represents the radiated power of the primary source. Writing the free field acoustic pressures as a function of primary and secondary source strengths results in the quadratic form

$$J = \Pi_{\text{rad}} = d^H d + d^H G u + u^H G^H d + u^H G^H G u$$

$$\begin{align*}
(G^H G)_{i,j} &= \frac{\omega^2 \rho_0}{8\pi \omega_0} \frac{\sin(k |r_i - r_j|)}{k |r_i - r_j|} \\
(G^H d)_i &= U_p \frac{\omega^2 \rho_0}{8\pi \omega_0} \frac{\sin(k |r_i - r_p|)}{k |r_i - r_p|} \\
d^H d &= \frac{\omega^2 \rho_0}{8\pi \omega_0} |U_p|^2
\end{align*}$$

(A.3.18)

where the matrix $G^H G$ contains the real parts of the acoustic impedances relating source strengths and pressures.

If we place an extra secondary monopole, the reduction of radiated acoustic power improves if the new part of the impedance matrix is independent of the original set of sources and is not orthogonal to that of the primary vector, as indicated by the earlier reasoning.
radiated intensity over a limited arc

For this special cost function the components are only given for the configuration of two secondary sources. Use has been made of the typical Fraunhofer area expressions for the pressure field of the set of monopoles - with reference to Figure 3.6 -

\[ P^f(r, \omega) = j \omega \rho_0 \frac{e^{-jkr}}{4\pi r} \left[ U_p + U_1 e^{j k a \cos \theta} + U_2 e^{-j k a \cos \theta} \right] \]  \hspace{1cm} (A3.19)

The radial component of the far field velocity field reads

\[ V_r^{f^*}(r, \omega) = \frac{1 + j k r e^{-j k r}}{4\pi r} \left[ U_p + U_1 e^{j k a \cos \theta} + U_2 e^{-j k a \cos \theta} \right] \]  \hspace{1cm} (A3.20)

This yields for the far field intensity

\[ I_r^{f^*}(r, \omega) = \frac{1}{2} \text{Re} \left[ P^{f^*} V^{f^*} \right] = \frac{\omega^2 \rho_0}{32 \pi^2 c_0} \frac{1}{r^2} \left[ U_p + U_1 e^{j k a \cos \theta} + U_2 e^{-j k a \cos \theta} \right]^2 \]  \hspace{1cm} (A3.21)

The cost function - radiated power over a limited arc of a far field sphere - can be evaluated by integrating the intensity function (A3.21) over part of a far field sphere

\[ \Pi_{rad}(\theta_{\text{max}}, \omega) = \int_{\theta=0}^{\theta_{\text{max}}} \int_{\phi=0}^{2\pi} I_r^{f^*}(r, \omega) r^2 \sin \theta \, d\phi \, d\theta \]  \hspace{1cm} (A3.22)

Bearing in mind the identity

\[ \int_{\theta=0}^{\theta_{\text{max}}} e^{j k a \cos \theta} \sin \theta \, d\theta = \frac{1}{n k a} \left[ e^{j k a \cos \theta_{\text{max}}} - e^{j k a} \right] \]  \hspace{1cm} (A3.23)

the radiated power can be written in the form of (A3.11) as

\[
\begin{align*}
J = \Pi_{rad} &= d^H d + d^H G u + u^H G^H d + u^H G^H G u \\
\mathbf{G}^H G &= \frac{\omega^2 \rho_0}{16 \pi c_0} \begin{pmatrix} 1 - \cos \theta_{\text{max}} & j \frac{2 k a}{e^{j k a \cos \theta_{\text{max}}} - e^{j k a}} \\
\frac{j}{2 k a} [e^{-2 j k a} - e^{-j k a \cos \theta_{\text{max}}}] & 1 - \cos \theta_{\text{max}} \end{pmatrix} \\
\mathbf{G}^H d &= U_p \frac{\omega^2 \rho_0}{16 \pi c_0} \begin{pmatrix} e^{j k a} - e^{-j k a \cos \theta_{\text{max}}} \\
e^{j k a \cos \theta_{\text{max}}} - e^{j k a} \end{pmatrix} \\
d^H d &= \frac{\omega^2 \rho_0}{16 \pi c_0} [1 - \cos(\theta_{\text{max}})] |U_p|^2
\end{align*}
\]  \hspace{1cm} (A3.24)

Now, the solution for the source strengths and the minimum criterion value readily follow from eq. (A3.15). These are

\[
\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = -\frac{j U_p}{2 k a [1 + \text{sinc}(2 k a)]} \left( e^{-j k a [1 - e^{2 j k a}]} \right) \]  \hspace{1cm} (A3.25)

and
\[
J_{\text{min}} = J_p \left[ 1 - \frac{2 \text{sinc}^2(ka')}{1 + \text{sinc}(2ka')} \right]
\]  \hspace{1cm} (A3.26)

where

\[
J_p = \frac{|U_p|^2 \omega^2 \rho_0}{16 \pi \omega_0} \quad \text{and} \quad a' = \frac{1}{2} a [1 - \cos \theta_{\text{max}}]
\]

The solution for the one secondary source problem can be found by simply leaving out the second rows and columns in (A3.24). This would lead to

\[
U_1 = -\frac{j U_p}{2ka} e^{-jka} \left[ 1 - e^{2jka'} \right]
\]

\[
J_{\text{min}} = J_p \left\{ 1 - \text{sinc}^2(ka') \right\}
\]  \hspace{1cm} (A3.27)

**radiated power in a duct**

The pressure field in hard-walled duct can be written as the infinite sum over travelling waves with different cross-sectional behaviour. For the wave field of a monopole source at \((x_0, y_0, z_0)\) we arrive at

\[
P(r, \omega) = \rho_0 c_0 U(\omega) \sum_n \frac{k}{2k_{xn}} \varphi_n(y_0, z_0) \varphi_n(y, z) e^{-jk_{xn}(x-x_0)}
\]  \hspace{1cm} (A3.28)

The radiated power through a downstream cross-section can be evaluated as

\[
\Pi(\omega) = \frac{1}{2} \text{Re} \int_{yz} P V_x^* \, dy \, dz = \frac{\rho_0 c_0}{8} |U(\omega)|^2 \sum_{n=0}^{N-1} \frac{k}{k_{xn}} \varphi_n^2(y_0, z_0)
\]  \hspace{1cm} (A3.29)

The evanescent waves do not contribute to the radiated power. Therefore, we have a finite summation over the acoustic modes; the \(N\)th mode has the smallest resonance frequency above the frequency band of interest. For a primary source at \(r_0\) and \(M\) secondary monopoles at \(r_1, r_2, \ldots, r_M\) the total radiated power reads

\[
J = J_{\text{rad}} = \frac{\rho_0 c_0}{8} [d + Gu]^H W [d + Gu]
\]

\[
d = U_p \left[ \varphi_0(y_0, z_0) \quad \varphi_1(y_0, z_0) \cdots \varphi_{N-1}(y_0, z_0) \right]^T
\]

\[
G = \left( \begin{array}{cccc}
\varphi_0(y_1, z_1) & \varphi_0(y_2, z_2) & \cdots & \varphi_0(y_M, z_M) \\
\varphi_1(y_1, z_1) & \varphi_1(y_2, z_2) & \cdots & \varphi_1(y_M, z_M) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_{N-1}(y_0, z_0) & \cdots & \cdots & \varphi_{N-1}(y_M, z_M)
\end{array} \right)
\]

\[
W = k \left( \begin{array}{cccc}
k_{x_1}^{-1} & 0 & \cdots & 0 \\
0 & k_{x_2}^{-1} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & k_{x_{N-1}}^{-1}
\end{array} \right)
\]  \hspace{1cm} (A3.30)
**space average squared pressure in a hard-walled duct**

An expression for the downstream pressure field is given in eq. (A3.28). For 'travelling' modes with a resonance frequency above the frequency band of interest the amplitude decays exponentially with distance to the source. In the case study of Section 3.4 the downstream termination of the duct is non-reflective, which means that potential and kinetic acoustic energy are equal. The space average squared pressure in a volume between \(x_a\) and \(x_b\) with unit cross-section can be written as:

\[
\mathbf{d} = U_k \begin{bmatrix} \varphi_0(y_0,z_0) & \varphi_1(y_0,z_0) & \cdots & \varphi_{N-1}(y_0,z_0) & \varphi_N(y_0,z_0) e^{i k N z_0} & \varphi_{N+1}(y_0,z_0) e^{i k (N+1) z_0} & \cdots \end{bmatrix}^T
\]

\[
\mathbf{G} = \begin{bmatrix}
\varphi_0(y_1,z_1) & \varphi_0(y_2,z_2) & \cdots & \varphi_0(y_M,z_M) \\
\varphi_1(y_1,z_1) & \varphi_1(y_2,z_2) & \cdots & \varphi_1(y_M,z_M) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_N(y_1,z_1) e^{i k N z_1} & \varphi_N(y_2,z_2) e^{i k N z_2} & \cdots & \varphi_N(y_M,z_M) e^{i k N z_M} \\
\varphi_{N+1}(y_1,z_1) e^{i k (N+1) z_1} & \varphi_{N+1}(y_2,z_2) e^{i k (N+1) z_2} & \cdots & \varphi_{N+1}(y_M,z_M) e^{i k (N+1) z_M} \\
\vdots & \vdots & \ddots & \vdots 
\end{bmatrix}
\]

\[
W = k^2 \begin{bmatrix}
k_{x_1}^{-2}(x_b - x_a) & 0 & \cdots & 0 & 0 & 0 & \cdots \\
0 & k_{x_2}^{-2}(x_b - x_a) & \cdots & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & \cdots & k_{x_{N+1}}^{-2}(x_b - x_a) & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & \frac{1}{2} k_{x_N}^{-3} [e^{-2 i k N z_N} - e^{-2 i k N z_b}] & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \frac{1}{2} k_{x_{N+1}}^{-3} [e^{-2 i k (N+1) z_{N+1}} - e^{-2 i k (N+1) z_b}] & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots 
\end{bmatrix}
\]

(A3.31)

The primary source is located at \(r_0\) and the secondary sources at \(r_1, r_2, \ldots, r_M\). The \(n^{th}\) row of \(d\), \(G\) and \(W\) gives the contribution of the \(n^{th}\) mode to the total energy. The number of contributing modes is in principle infinite. In the minimisation procedure the modal can be truncated by leaving out those modes for which the weighting matrix \(W\) has a negligible factor; \(W_{i,j} \ll 1\).

**total acoustic energy in a hard-walled enclosure**

For this volume it is most effective to express the pressure field in terms of an eigenfunction expansion. For a monopole source at \(r_0\) this renders
\[ P(\mathbf{r}, \omega) = -j \omega \rho_0 U(\omega) \sum_n \frac{\varphi_n(\mathbf{r}_0) \varphi_n(\mathbf{r})}{k^2 - k_n^2} \]  

(A3.32)

For a rigid walled enclosure the eigenfunctions are real and orthonormal under the conventional inner product. This property enables us to formulate the following vector expression for the total energy in the enclosure generated by a primary monopole and \( M \) secondary monopoles:

\[
J = E_{\text{tot}} = \frac{\rho_0}{4} [\mathbf{d} + \mathbf{G}u]^H \mathbf{W}[\mathbf{d} + \mathbf{G}u] \\
\mathbf{d} = \mathbf{U}_p [\varphi_0(\mathbf{r}_0) \hspace{1cm} \varphi_1(\mathbf{r}_0) \hspace{1cm} \cdots \hspace{1cm} \varphi_n(\mathbf{r}_0) \hspace{1cm} \cdots]^T \\
\mathbf{G} = \\
\begin{pmatrix}
\varphi_0(\mathbf{r}_1) & \varphi_0(\mathbf{r}_2) & \cdots & \varphi_0(\mathbf{r}_M) \\
\varphi_1(\mathbf{r}_1) & \varphi_1(\mathbf{r}_2) & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_n(\mathbf{r}_1) & \cdots & \cdots & \varphi_n(\mathbf{r}_M) \\
\vdots & \vdots & \cdots & \vdots 
\end{pmatrix} \\
\mathbf{W} = \\
\begin{pmatrix}
\frac{k^2 + k_0^2}{ik^2 - k_0^2} & 0 & \cdots & 0 & \cdots \\
0 & \frac{k^2 + k_0^2}{ik^2 - k_1^2} & \cdots & \vdots & \cdots \\
\vdots & \vdots & \ddots & \vdots & \cdots \\
0 & \cdots & \cdots & \frac{k^2 + k_M^2}{ik^2 - k_n^2} & \cdots 
\end{pmatrix} 
\]

(A3.33)

The primary source is located at \( \mathbf{r}_0 \) and the secondary sources are \( \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_M \). The \( n \)th row of \( \mathbf{d} \), \( \mathbf{G} \) and \( \mathbf{W} \) gives the contribution of the \( n \)th mode to the total energy. The number of modes is in principle infinite. In the minimisation procedure the modal can be truncated by leaving out those modes for which the weighting matrix \( \mathbf{W} \) has a negligible factor; \( \mathbf{W}_{NN} \ll 1 \). For minimisation problems described above it holds that addition of a control source may improve the performance. Obviously, the requirement for an improvement is that the extra source is capable of exciting (part of) the residual wave field.

**Appendix A3.2  Eigenfunctions expansions for enclosed sound fields**

In this Appendix the concept of eigenfunction modelling is introduced. For a wide variety of applications a representation of a wave field in terms of eigenfunctions is very convenient. The main principles of eigenfunction modelling will be revealed in following one-dimensional study. For a more thorough mathematical treatment the reader is referred to [Fr56] and [MF53]. Consider a finite homogeneous medium with boundary points \( x_1 \) and \( x_2 \) (\( x_2 > x_1 \)). The homogeneous Helmholtz equation and accompanying boundary conditions for this medium read
\[
\frac{\partial^2}{\partial x^2} P(x, \omega) + k^2 P(x, \omega) = 0
\]  \hspace{1cm} (A3.34)

\[
\alpha_1(k) P(x_1, \omega) - \beta_1(k) \frac{\partial}{\partial x} P(x_1, \omega) = 0
\]

\[
\alpha_2(k) P(x_2, \omega) + \beta_2(k) \frac{\partial}{\partial x} P(x_2, \omega) = 0
\]  \hspace{1cm} (A3.35)

Note that the \(\alpha\)'s and \(\beta\)'s may be complex-valued. The boundary conditions need to depend on the wavenumber \(k\) in order to represent a locally reacting boundary with impedance \(Z\). This is given by

\[
P(\omega) + \frac{Z(\omega)}{\rho_0 c_0} \frac{1}{jk} \frac{\partial}{\partial x} P(\omega) = 0
\]  \hspace{1cm} (A3.36)

In principle, an infinite number of functions \(\varphi_n\) can be found that obey the Helmholtz equation (A3.34) and the boundary conditions (A3.35). These functions are called characteristic, natural or eigenfunctions. An eigenfunction represents a free acoustic wave motion or acoustic mode for the relevant geometry and acoustic properties of the boundary. For the elementary Neumann (hard-walled) and Dirichlet (soft-walled) boundary conditions the eigenfunctions look like

**Neumann:** \[ \frac{\partial}{\partial x} P(x_1, \omega) = \frac{\partial}{\partial x} P(x_2, \omega) = 0 \]

\[ \varphi_n(x) = \cos\left(\frac{n \pi x}{x_2 - x_1}\right) \]  \hspace{1cm} (A3.37)

**Dirichlet:** \[ P(x_1, \omega) = P(x_2, \omega) = 0 \]

\[ \varphi_n(x) = \sin\left(\frac{n \pi x}{x_2 - x_1}\right) \]  \hspace{1cm} (A3.38)

In practice, however, it may be hard to find analytical expressions for the eigenfunctions. Still, the following characteristics can be specified

a) The eigenfunctions satisfy the equation

\[
\frac{d^2}{dx^2} \varphi_n(x) + k_n^2 \varphi_n(x) = 0
\]  \hspace{1cm} (A3.39)

and the boundary conditions

\[
\alpha_1(k_n) \varphi_n(x_1) - \beta_1(k_n) \frac{\partial}{\partial x} \varphi_n(x_1) = 0
\]

\[
\alpha_2(k_n) \varphi_n(x_2) + \beta_2(k_n) \frac{\partial}{\partial x} \varphi_n(x_2) = 0
\]  \hspace{1cm} (A3.40)

The constant \(k_n^2\) denotes the eigenvalue. Generally, the eigenfunctions and eigenvalues are frequency-dependent! For a bounded medium the distribution of eigenvalues is discrete. The radial frequency \(\omega_n = \text{Real}(k_n c_0)\) stands for the resonance frequency of the acoustic mode \(\varphi_n\).
b) For the second feature we have to distinguish between two cases:

- If the boundary coefficients $\alpha$ and $\beta$ do not depend on $k$ the following holds. For each set of eigenfunctions $\{\phi_n\}$ there is another set $\{\psi_n\}$ such that

$$\langle \phi_n, \psi_m \rangle = \int_{x_1}^{x_2} \phi_n \psi_m^* \, dx = 0 \quad \text{for} \quad n \neq m \quad (A3.41)$$

in which $\langle , \rangle$ stands for inner product. The set $\psi_n$ is called the bi-orthogonal conjugate of $\phi_n$. The bi-orthogonal conjugate set is the solution of the adjoint eigenvalue problem. The adjoint problem is intrinsically defined by

$$\frac{d^2}{dx^2} \psi_n + \lambda_n^2 \psi_n = 0 \quad (A3.42)$$

$$\langle \psi, \frac{d^2}{dx^2} \phi \rangle = \langle \frac{d^2}{dx^2} \psi, \phi \rangle \quad (A3.43)$$

in which the latter equality determines the boundary conditions for $\psi$. An eigenvalue problem is called self-adjoint if the adjoint operator and boundary conditions are equal to their originals. The Neumann and Dirichlet problems e.g. are self-adjoint. For the type of boundary conditions (A3.35), it can be shown that [MF53]

$$\psi_n^* = \phi_n \quad \text{for each} \ n \quad (A3.44)$$

Consequently, if the eigenfunctions are real they are orthogonal

$$\langle \phi_n, \phi_m \rangle = \int_{x_1}^{x_2} \phi_n \phi_m^* \, dx = 0 \quad \text{for} \quad n \neq m \quad (A3.45)$$

This property holds in general for self-adjoint problems.

- Generally, the boundary coefficients do depend on $k$, which severely complicates the issue of (bi-)orthogonality. There still exists a set of functions with

$$\langle \phi_n, \psi_m \rangle = 0 \quad \text{for} \quad n \neq m \quad (A3.46)$$

The inner product $\langle , \rangle$, however, likely differs from the conventional Euclidean form. An example of an appropriately defined inner product is given in the following Appendix.

Finally, the orthogonality relations (A3.41), (A3.45) and (A3.46) are extended to

$$\langle \phi_n, \psi_m \rangle = \delta_{nm} \quad (A3.47)$$

This orthonormality relation is preserved throughout as a convention.

c) The set of eigenfunctions $\phi_n$ is assumed to be complete for any square integrable function $\phi(x)$ on the relevant interval. This means that the quantity
\[
\sum_{n=1}^{N} a_n \varphi_n - f(x) dx
\]

(A3.48)

approaches zero as \(N\) approaches infinity. The eigenfunction expansion is a least-squares fit for \(\varphi(x)\). Employing the general relation (A3.46) the expansion can be specified as

\[
f(x) = \sum_{n} a_n \varphi_n(x)
\]

\[
a_n = \langle f, \psi_n \rangle
\]

(A3.49)

The expansion in eigenfunctions will prove to be useful in the description of low frequency sound fields in bounded media. When the dimension of the relevant medium tends to infinity the distribution of eigenvalues becomes continuous. In that case the summation sign should be replaced by an integral sign. In the free field case a representation of a wave field in the wavenumber domain is in fact an expansion of the field in plane waves. In the free field plane waves play the role of eigenfunctions.

The importance of Green’s function in the analysis of wave field is clearly presented in Chapter 2. Now the eigenfunction expansion theory will be used to represent Green’s function, for which it holds

\[
\frac{\partial^2}{\partial x^2} G(x|x_0, \omega) + k^2 G(x|x_0, \omega) = -\delta(x-x_0)
\]

(A3.50)

and the general boundary conditions (A3.35). For the sake of simplicity the boundary coefficients are assumed to be constant with \(k\). In Appendix A3.3 a more complicated problem is dealt with. The expansion is given by

\[
G(x|x_0, \omega) = \sum_{n} a_n(x_0, \omega) \varphi_n(x)
\]

(A3.51)

Multiplying (A3.51) by \(\psi_m\), (A3.42) by \(G\), taking the difference and integrating over the volume yields

\[
G(x|x_0, \omega) = -\sum_{n} \frac{\varphi_n(x_0) \varphi_n(x)}{k^2 - k_n^2}
\]

(A3.52)

Out of this expansion for Green’s function the following properties can be observed:

- the reciprocity relation; \(G(x_0|x) = G(x|x_0)\);
- At the resonance frequency \(\omega = \text{Real}(k_n c_0)\) the acoustic mode \(n\) is dominant. The smaller the imaginary part of \(k_n\) the more pronounced the resonance.
- In principle all acoustic modes contribute to Green’s function. However, far from resonance a mode’s contribution is negligible.
As a final remark it should be noted that the results of the analysis above usually can be carried over to the multidimensional case. This is to be discussed in further detail for the two- and three-dimensional case in Sections 3.4 and 3.5.

Appendix A3.3 An eigenfunction expansion for a general impedance boundary condition

In this Section we will derive an eigenfunction expansion for a monopole field in a one-dimensional enclosure with the following boundary conditions:

\[
P - \frac{Z_0}{\rho_0 c_0} \frac{1}{j k} \frac{\partial}{\partial x} P = 0 \quad ; \quad x = 0
\]

\[
P + \frac{Z_L}{\rho_0 c_0} \frac{1}{j k} \frac{\partial}{\partial x} P = 0 \quad ; \quad x = L
\]

(A3.53)

The impedances \(Z_0\) and \(Z_L\) are assumed to be constant with \(\alpha\). The eigenfunctions obeying these conditions and the one-dimensional Helmholtz equation can be found as

\[
\varphi_n(x) = \frac{1}{2 \sqrt{L}} \left[ \sqrt{R_0} e^{-jk_n x} + \frac{1}{\sqrt{R_0}} e^{jk_n x} \right]
\]

with

\[
k_n = \frac{n \pi}{L} + \frac{1}{2L} \log \left\{ R_0 R_L \right\}
\]

(A3.54)

\[
R_0 = \frac{Z_0 - \rho_0 c_0}{Z_0 + \rho_0 c_0} ; \quad R_L = \frac{Z_L - \rho_0 c_0}{Z_L + \rho_0 c_0}
\]

Note that the quantities \(R_0\) and \(R_L\) denote the pressure reflection coefficients for the enclosure terminations. Under the conventional inner product the eigenfunctions are not orthogonal. In the coming analysis a new inner product will be introduced under which a bi-orthogonality relation holds.

Define a two-component vector of pressure and particle velocity \([P \ V]^T\) and an operator \(\Gamma\) for which it holds

\[
\Gamma \begin{pmatrix} P \\ V \end{pmatrix} = \begin{pmatrix} j \rho_0 c_0 V \\ - \rho_0 c_0 P \end{pmatrix} = k \begin{pmatrix} P \\ V \end{pmatrix}
\]

(A3.55)

together with the boundary conditions (A3.53). This expression (A3.55) contains the Helmholtz equation for both pressure \(P\) and velocity \(V\).

Next, define the inner product as

\[
\langle u, w \rangle = \int_0^L [u_1 w_1 - (\rho_0 c_0)^2 u_2 w_2] \, dx
\]

(A3.56)

Under this inner product the operator \(\Gamma\) is self-adjoint

\[
\langle w, \Gamma u \rangle = \langle \Gamma w, u \rangle
\]

(A3.57)

which means that the eigenfunctions given by
\[ \varphi_n(x) = \begin{pmatrix} p_n \\ v_n \end{pmatrix} = \frac{1}{2iL} \begin{pmatrix} \sqrt{R_0} e^{-jk_n x} + \frac{1}{\sqrt{R_0}} e^{jk_n x} \\ \frac{1}{\rho_0 c_0} \left[ \sqrt{R_0} e^{-jk_n x} - \frac{1}{\sqrt{R_0}} e^{jk_n x} \right] \end{pmatrix} \]  

(A3.58)

are orthonormal under (A3.56):

\[ \langle \varphi_n, \varphi_m \rangle = \int_0^L \left[ p_n p_m - (\rho_0 c_0)^2 v_n v_m \right] dx = \delta_{nm} \]  

(A3.59)

Obviously, the modified inner product relation holds only for the eigenfunctions of this specific boundary value problem. In [Fr56] and [Ma91], which inspired the solution above, other examples of redefining the inner product can be found.

The appropriately chosen inner product is an essential tool in finding the eigenfunction expansion of Green's function for the medium in question. Here, we have to decompose Green's function in a pressure and a velocity component

\[ kP = j\rho_0 c_0 \frac{\partial V}{\partial x} - \frac{1}{k} \delta(x - x_0) \]  

\[ kV = -j \frac{\partial P}{\rho_0 c_0} \frac{\partial}{\partial x} \]  

(A3.60)

The expansion equation for the Green's vector is given by

\[ \begin{pmatrix} P(x) \\ V(x) \end{pmatrix} = \sum_{n=-\infty}^{\infty} a_n \varphi_n(x) \]  

(A3.61)

where the eigenfunction vector is defined in eq. (A3.58). Inserting eq. (A3.61) into (A3.60) results in

\[ \sum_{n=-\infty}^{\infty} a_n (k - k_n) \varphi_n(x) = \begin{pmatrix} -\frac{1}{k} \delta(x - x_0) \\ 0 \end{pmatrix} \]  

(A3.62)

The modal coefficients can now be found easily by using the inner product relation (A3.59)

\[ a_n = -\frac{\varphi_n(x_0)}{k(k - k_n)} \]  

(A3.63)

Note that eqs. (A3.61) and (A3.63) constitute the eigenfunction expansion for both the pressure and the velocity field.

**Appendix A3.4  An eigenfunction expansion for a lightly damped enclosure**

In Appendix A3.2 an eigenfunction expansion for the one-dimensional Green's function is derived. For a three-dimensional rectangular enclosure a similar relation holds
\[ G(r|\mathbf{r}_0, \omega) = -\sum_n \frac{\phi_n(r_0)\phi_n(r)}{k^2 - k_n^2} \]  
(A3.64)

if the boundary conditions are constant with \( k \). In general an analytical expression for the eigenfunctions is not available. For the special case of rigid walls, nevertheless, we have

\[ \phi_n(r) = \sqrt{\frac{\epsilon_n}{L_x L_y L_z}} \cos\left(\frac{n_x \pi x}{L_x}\right) \cos\left(\frac{n_y \pi y}{L_y}\right) \cos\left(\frac{n_z \pi z}{L_z}\right) \]  
(A3.65)

with \( \epsilon_n = 2 - \delta_{0n} \)

Now, the problem is self-adjoint and yields real-valued eigenfunctions and eigenvalues. For a practical simulation, however, the physical unrealistic case of an undamped resonance emerges. In order to avoid this problem the following approximation, originally developed by Morse [Mo48], is used. Suppose there is a small damping at the walls

\[ k_n = \frac{\omega_n}{c_0} + j \frac{\zeta_n}{c_0}; \quad \text{with} \quad \zeta_n << \omega_n \]  
(A3.66)

Consequently, the expansion is written as

\[ G(r|\mathbf{r}_0, \omega) = c_0^2 \sum_n \frac{\phi_n(r_0)\phi_n(r)}{(\omega_n^2 - \omega^2) + 2j \omega_n \zeta_n} \]  
(A3.67)

where the damping is assumed to be such small, that the eigenfunctions can be reasonably well approximated by the cosines in eq. (A3.65). In this way we have limited resonances, but still real-valued and orthonormal eigenfunctions.

**Appendix A3.5 Minimisation of the infinity norms**

For a few acoustic cases the cost function is defined as the infinity norm of an energetic quantity. The objective of such a cost function is to minimise the maximum value of the energy density. An optimisation problem like this is also referred to as a minimax problem. It is rather complicated to solve a minimax problem analytically. The results as presented in the thesis are obtained by using the ‘Optimisation’ Toolbox of MATLAB. The drawback of such a numerical approach is twofold

1) The relevant energy density has to be discretised before feeding it to a computer. This means that the acquired results are close to optimal. The chosen sampling interval, however, always was on the safe side (\( \Delta r < \lambda /10 \)).

2) The numerical routine may find a local minimum instead of a global minimum. This event is hard to detect by the user. The mere indication for a global solution is that the maximum value for another solution is higher than the obtained ‘minimum’ maximum.

In a single occasion the optimisation was suitable to be conducted analytically; for the one-dimensional enclosure and \( R = 1 \). The numerical answer agreed with the analytical in that case.
CHAPTER 4

The Distribution of Sensors and the Constraint of Causality

In the previous Chapter an extensive analysis was conducted on how to reduce a primary sound field by means of a limited number of secondary sources. The transition from the 'ideal situation', as described in Chapter 2, to the more practical approach of Chapter 3 introduced a severe restriction on the possible performance of systems for active sound reduction. In the Sections to come we will fulfill the practical design of active sound reduction systems by addressing the need for acoustic sensors. Additionally, we will return to the time-domain to deal with the constraint of causality. Both these practical aspects will generally take us further away from the ideal system's performance, as derived in Chapter 2.

4.1 The discretisation of acoustic cost functions

The necessity of defining acoustic cost functions is discussed in Section 3.1. Likewise, the optimal strength for the secondary source distribution is found by minimising the relevant cost function. In the case studies of Chapter 3 the following theoretical criteria - quadratic functions of the secondary source strengths - were introduced

- the far field total acoustic energy in the free space.
- the total acoustic energy in a 1-dimensional enclosure.
- the radiated power through a plane in a multi-modal duct.
- the total acoustic energy in a 3-dimensional enclosure.

All these cost functions involve knowledge of the sound field over a continuous two- or three-dimensional space. As this is a highly non-practical demand, it is useful to examine the approximation of the energetic quantities by evaluating the sound field at a limited number of points.

All cost functions listed above include both the pressure and velocity components of sound fields. In Chapter 3 the space average squared pressure was examined as a practical alternative to the theoretical orientated cost functions. It was shown that the characteristics of space average squared pressure minimisation very well agreed to those of total energy minimisation. In fact, there are sensible reasons to use the space average squared pressure as cost function. First, it should be recalled that the human ear is a sensor of pressure. Furthermore, it is technically easy to measure pressure, as the microphone is a sensor of pressure as well. So, preluding the practical implementation of systems for active sound reduction, here we will use the space average squared pressure as the basic cost function.
The main issue in the approximation of continuous-space cost functions is the number and the position of the required sensors. If the continuous cost function is given by

\[ J_{\text{con}} = \frac{1}{V} \int_V |P(r, \omega)|^2 \, dV \] \hspace{1cm} (4.1)

the discrete approximation reads

\[ J_{\text{dis}} = \frac{1}{N} \sum_{i=1}^N |P(r_i, \omega)|^2 \] \hspace{1cm} (4.2)

It seems reasonable to state that the required number of sensors is related to the spatial complexity of the pressure field. A single measurement point e.g. gives sufficient information on the transverse characteristics of a plane wave. Later in this Section the necessary number of sensors and the modal density of enclosed sound fields will be linked. Let us, however, assume that a number of \( M \) sensors has been installed and we want to find the optimal strengths of \( N \) secondary (volume) sources. The estimate of the space average squared pressure can be written as

\[ \hat{\rho}_{av}^2 = J = (p_p + Hu_s)^H (p_p + Hu_s) \] \hspace{1cm} (4.3)

with reference to Appendix 3.1. The vector \( p_p \) denotes the vector primary pressure values at the sensors, \( u_s \) the secondary source strengths and the matrix \( H \) contains the transfer functions from source strength to sensor pressure. Concerning the minimisation of (4.3) three distinct cases are possible:

a) \( M < N \); the minimisation problem is underdetermined. An infinite number of solutions of \( u \) can be found that minimise \( J \) to zero. This situation can be employed to simultaneously minimise the control effort; see [NE91]. This is, however, not discussed in this Chapter.

b) \( M = N \); the fully determined or ‘square’ problem. In principle, it is possible to drive the discretised cost function to zero here. So, one must be sure that the approximation of the global energy is accurate.

c) \( M > N \); the overdetermined case and the most useful configuration. The optimal source strengths for this case can be found as

\[ u_s = [H^H H]^{-1} H^H p_p \] \hspace{1cm} (4.4)

Clearly, one should be aware that \([H^H H]\) may be ill-conditioned. The condition number of the matrix is closely related to the positions of sources and sensors.

Next, the approximation of the original cost function by measuring the sound field at a discrete number of positions is analysed using the 1-dimensional enclosure, as introduced in Section 3.3. First, a few simulation experiments are shown and consequently the results are explained by means of a modal analysis. The procedure for the simulation experiments is as follows:
1) Determine the primary and secondary sound field at the sensors.
2) Minimise the discrete-space criterion function with respect to the secondary source strengths.
3) Calculate the continuous-space criterion function for this solution of source strengths.
4) Compare the residual criterion value with the optimum residual of the continuous-space function.

**minimisation of space average squared pressure in a 1-dimensional enclosure**

Recall from Section 3.3 that a one-dimensional duct is introduced as a case study for active sound reduction in enclosures. Extensive simulations are carried out to investigate the influence of source positioning, the end termination reflection, the modal density, the number of sources and frequency on the performance. Again, this case study is used to delineate and comprehend the practical cases of the multi-modal duct and the reverberation chamber. In the following the sensors measuring the cost function are sometimes referred to as error sensors, which originates from the terminology of recursive optimisation.

![Graph of space average squared pressure over the whole enclosure for a configuration of a secondary source at x = 0.7 and an error sensor at x = 0.9. The optimum curve results from minimisation of the global space average squared pressure, the residual by using the error sensor.](image)

**Figure 4.1a:** Graph of space average squared pressure over the whole enclosure for a configuration of a secondary source at \( x = 0.7 \) and an error sensor at \( x = 0.9 \). The optimum curve results from minimisation of the global space average squared pressure, the residual by using the error sensor.
Figure 4.1b: Graph of space average squared pressure for a configuration of a secondary source at $x = 0.7$ and an error sensor at $x = 0.3$.

Figure 4.2: Graph of space average squared pressure for a configuration of a secondary source at $x = 0.7$ and error sensors at $x = 0.3$ and 0.9.

In the first experiment a single error sensor is placed at the left or right side of the secondary source. The residuals as given in Figures 4.1a and 4.1b, show a reduction of primary resonances, but the occurrence of new resonances. This phenomenon directly follows from Section 3.3, in which we minimised the acoustic energy in the right- or left-side region of the
secondary source. In exchange for a substantial reduction in the appointed area, the level in the other area increased. By positioning a single sensor in one area, the secondary source is bound to arrive at the same effect. Two error sensors equally divided over both regions prevents the introduction of new resonances while maintaining the original suppression of primary resonances; see Figure 4.2. Moreover, the residual level well approximates the optimum, which is obtained by minimising the global space average squared pressure.

![Graph of space average squared pressure](image)

*Figure 4.3a:* Graph of space average squared pressure for a configuration of a secondary source at $x = 1.0$ and an error sensor at $x = 0.5$.

Next, the secondary source is positioned on the other side of the enclosure, $x = L$, and an error sensor is put in the middle, $x = 0.5$. The resultant minimised energy, as shown in Figure 4.3a, clearly reveals the inability of this sensor the detect the odd modes. Consequently, the secondary source cannot attack them and even amplifies these resonances. Addition of an extra sensor that does observe these modes, dramatically improves the performance; see Figure 4.3b. A set of four evenly spaced sensors leads to a solution that as good as equals the optimal reduction for this secondary source; Figure 4.3c.
Figure 4.3b: Graph of space average squared pressure for a configuration of a secondary source at $x = 1.0$ and error sensors at $x = 0.5$ and 0.7.

Figure 4.3c: Graph of space average squared pressure for a configuration of a secondary source at $x = 1.0$ and error sensors at $x = 0.3, 0.5, 0.7$ and 0.9.
Figure 4.4a: Graph of space average squared pressure for a configuration of secondary sources at $x = 0.6$ and $1.0$ and error sensors at $x = 0.3$ and $0.9$.

Figure 4.4b: Graph of space average squared pressure for a configuration of secondary sources at $x = 0.6$ and $1.0$ and error sensors at $x = 0.3$, $0.7$ and $0.9$. 
The last simulation experiment comprises the use of two control sources and sets of 2, 3 and 4 error sensors. In Figures 4.4a, 4.4b and 4.4c the reduction results can be found. In contrast to the above simulation, the use of two sensors at 0.3 and 0.9 does not lead to a favourable result. Although the pressure level at both sensors completely vanishes, the reduction system generates new resonances. The two samples of the sound field may be a good representative of the global field behaviour, as long as they are not both located near a node. Apparently, they do not observe the new resonance. The set of three sensors solves this problem and use of four sensors again yields a performance close to optimal.

![Graph of space average squared pressure for a configuration of secondary sources at x =0.6 and 1.0 and error sensors at x = 0.3, 0.4, 0.7 and 0.9.](image)

**Figure 4.4c:** Graph of space average squared pressure for a configuration of secondary sources at \( x = 0.6 \) and 1.0 and error sensors at \( x = 0.3, 0.4, 0.7 \) and 0.9.

**The observation of dominant modes**

In Section 3.3 a modal analysis was introduced to comprehend the acoustic mechanisms of reducing an enclosed sound field using a few secondary monopoles. Concisely, it turned out that a substantial reduction of the primary field is possible, if one uses as many well-placed sources as there are dominant acoustic modes. Here, we will come up with a similar conclusion for the error sensors.

Consider a sound field that can well approximated as the sum of a limited number of modes:

\[
P(x, \omega) = \sum_{n=1}^{N} a_n(\omega) \varphi_n(x) \tag{4.5}
\]
The modes can be seen as to constitute a basis for an \(N\)-dimensional space of sound fields. As a consequence, the pressure field equals zero over the volume if all \(N\) coefficients \(a_n\) equal zero. Complete information on the status of the \(a_n\)'s can only be extracted from a set of at least \(N\) sensors. The sound field is capable of being zero at \(M (<N)\) points and still yield high levels in between. In order to have control over an \(N\)-dimensional field, one should observe it at minimally \(N\) points.

For the 1-dimensional enclosure a reasonable estimation of the acoustic potential is obtained if as many sensors as dominant modes are well-positioned. At resonance frequencies a single sensor suffices, whereas at anti-resonances much more sensors are required. In this analysis the graphs of the mode coefficients distribution, Figures 3.17 to 3.19, prove to be useful. Combining the above analysis and simulation experiments in the 1-dimensional duct, we arrive at the following features of space average squared pressure estimation using error sensors.

- It is preferable to use more sensors than secondary sources in order to obtain a stable solution of the least-squares problem. If there are as many sensors as sources one can drive the pressure to zero at the sensors. This, however, is no guarantee for global reduction.
- Using a single secondary source and sensor it is possible to reduce the resonance peaks of the space average squared pressure. In between resonances serious amplifications of the original field can be expected.
- For a substantial reduction of the primary space average squared pressure over the whole frequency range, one should employ as many sources as there are dominant modes. The sensors should be located such that all relevant modes are observed at all frequencies. For the 1-dimensional enclosure a sensor at \(x = 0\) lies on the maximum of all acoustic modes.

The modal model again provides a transparent insight into the mechanisms of active sound reduction in an enclosed space. It is straightforward to apply the above reasoning to the other case-studies in a duct, an enclosure or the free field. In the simulation experiments to follow the performance of active sound reduction systems using a few sensors is compared with the performance of the systems in Chapter 3 - having full global knowledge of the cost function.

radiated power minimisation in a multi-modal duct

The number of transverse modes contributing to the radiated power in a duct is small for the low frequency band; see Section 3.4. As a consequence, the number of required sensors is expected to be small as well. For the hard-walled duct - the same as in Figure 3.26 - it is most appropriate to place observation sensors near the wall, as there all modes have an extreme value. This is depicted in Figure 4.5.
In Figure 4.6a an unfavourable result is depicted using only 1 sensor. From the third higher mode on, however, an amplification of the primary power occurs. Employing four transverse plane corner microphones improves the reduction for low frequencies and diminishes the amplification effects for higher frequencies; see Figure 4.6b.
Figure 4.6b: Graph of radiated power for a configuration of a secondary source at $(2, 0.05, 0.05)$ and 4 error sensors in the corners at $x = 5$.

**space average squared pressure minimisation in an enclosure**

In the hard-walled box of Section 3.5 (Figure 3.29) the number of modes contributing to the cost function is higher than in the waveguide. Here, we have to deal with the infinite summation over almost negligible residual modes. Again, the corners are suitable positions for the cost function sensors. The configuration of the primary source ($P$), a single secondary source ($S$) and 4 error sensors ($M$) is show in Figure 4.7.

Figure 4.7: Geometry of sources and sensor in the hard-walled enclosure.

Using only one sensor does not yield very favourable results again; see Figure 4.8a. The residual level significantly amplifies the primary level at anti-resonances. Adding a microphone in another corner substantially improves the performance. For this secondary source 4 error sensors in all corners do not prove to do better than 2 sensors; see Figure 4.8b and 4.8c.
Figure 4.8a: Primary, residual and optimal space average squared pressure for a configuration of a single secondary source S and error sensor M₁.

Figure 4.8b: Primary, residual and optimal space average squared pressure for a configuration of a single secondary source S and error sensors M₁ and M₂.

**Radiated power minimisation of a free field monopole**

As mentioned in Section 3.3 the ‘modes’ of free space are plane waves travelling in any direction. Therefore, we may expect a free sound field to have more degrees of freedom than a field enclosed in a duct or a box. In order to reduce the primary field a larger number of far field minimisation sensors will be required to ensure a global reduction. Here, the problem of reducing the far field total acoustic energy of a primary source by means of a single secondary monopole is studied.
Section 4.1  The discretisation of acoustic cost functions

Figure 4.8c; Primary, residual and optimal space average squared pressure for a configuration of a single secondary source S and error sensors $M_1$, $M_2$, $M_3$ and $M_4$.

Figure 4.9a; Graph of normalised radiated power for a configuration of a secondary source and 1 or 2 error sensors.

In Figures 4.9a and 4.9b the results are shown for 1, 2, 4 and 12 error microphones. The sensors were equally divided over the far field arc. So this means that 1 error sensor was located at $\theta = \pi/2$, two error sensors at angles $\pi/4$ and $3\pi/4$, four error sensors at angles $\pi/8$, $3\pi/8$, $5\pi/8$ and $7\pi/8$, and so on. These locations are not necessarily the best for the error sensors. It follows that increasing the number of sensors extends the area of favourable operation for the secondary source. Using four sensors only however, already yields a satisfactorily performance as for higher $ka$-values the maximum attainable reduction is negligible.
Figure 4.9b; Graph of normalised radiated power for a configuration of a secondary source and 4 or 12 error sensors.

Figure 4.10a; Graph of normalised radiated power into half-space for a configuration of a secondary source and 1 or 2 error sensors.

In Section 3.2 the active reduction of radiated power over a limited arc was studied as well. Restricting the region of silence has a clear positive effect on the performance of the active sound reduction system. In Figures 4.10a and 4.10b it is shown that limiting the region also improves the performance when using a few sensors to measure the cost function. Again, the sensors were equally divided over the limited arc. For the half-space case - as depicted below - the reduction curve using 4 sensors is already close to the optimal curve. In the full space case this holds for 12 sensors.
The theory of multivariate optimal linear prediction

In the preceding Chapters and in Section 4.1 the problem of finding the optimal strength for a given secondary source distribution has been carried out in the frequency domain. In fact, the problem is decomposed into a set of independent monochromatic events. Although this is a mathematically convenient approach, it does not cover the whole range of constraints. In several analytically solved cases the optimal secondary source strength is expressed in terms of a filtered version of the completely known primary source strength. Knowledge of the primary source is in most cases a pure academic assumption. In practice it may be hard - what's more impossible - to model the primary noise source as a linear transducer with a measurable strength signal. Besides this, it can be verified that most of these secondary source solutions - as given in Chapter 3 - are non-causal functions of the primary strength. So here two new practical problems emerge:

1) the detection of the primary source strength.
2) the constraint of causal action of the control sources.

The impact of both issues will be analysed using the theory of optimal filtering as developed by Wiener in the 1940's. A concise treatment of this theory can be found in e.g. the monograph of Priestley [Pr81].

Let us consider the single-channel case of reducing the primary signal $p(t)$ at the error sensor; see Figure 4.11. The signal $x(t)$ stands for the detection of the primary sound using an acoustic sensor, the impulse responses $c(t)$ represents the controller and $h(t)$ is the impulse response relating the secondary source strength and its sensor response. Note that the term controller - to
be used frequently - embodies the electronic processing of the detection signal \( x(t) \) until it is fed to the secondary source.

![Figure 4.11: Schematic diagram of signals and systems in the optimal prediction problem.](image)

The signal from the error sensor can be written as

\[
e(t) = p(t) + h(t) \ast [c(t) \ast x(t)]
\]  \hspace{1cm} (4.6)

in which \( \ast \) is the notation for a continuous-time convolution. If we assume the cost function for this single-point cancellation is quadratic

\[
J = E[|e(t)|^2]
\]  \hspace{1cm} (4.7)

the problem is almost similar to that of optimal filtering. The main discrepancies are the inclusion of a impulse response \( h \) and the summation instead of difference in the expression for the error equation (4.6). So strictly speaking we face the problem of optimal negative prediction or rather optimal cancellation.

The appearance of the acoustic transfer function can be accounted for in a straightforward manner. Since both the controller and the acoustic impulse response are assumed to be time-invariant systems, the order of the impulse response appearances in (4.6) can be reversed

\[
e(t) = p(t) + c(t) \ast [h(t) \ast x(t)] = p(t) + c(t) \ast r(t)
\]  \hspace{1cm} (4.8)

in which \( r(t) \) is the filtered detection signal. Evaluating cost function (4.7) yields

\[
J = \phi_{pp}(0) + \int_{\tau}^{\infty} \phi_{rp}(\tau) c^*(\tau) d\tau + \int_{\tau}^{\infty} \phi_{rr}(\tau) c(\tau) d\tau
\]  \hspace{1cm} (4.9)

in which a cross-correlation function \( \phi_{xy}(t) \) is defined as

\[
\phi_{xy}(\tau) = E[x^*(t)y(t + \tau)]
\]  \hspace{1cm} (4.10)

The error criterion is minimised if the following holds; see [Pr81]

\[
\phi_{pr}(\tau_1) + \int_{0}^{\infty} c(\tau_2)\phi_{rr}(\tau_1 - \tau_2)d\tau_2 = 0, \hspace{1cm} \tau_1 \geq 0
\]  \hspace{1cm} (4.11)
a Wiener-Hopf equation. At first sight it seems that this equation is easily solved by transforming it to the frequency or Laplace domain. However, since eq. (4.11) merely holds for non-negative $\tau_1$, the problem is more complicated. Taking a Laplace Transform of (4.11)

$$S_{rr}(s) + C(s)S_{rr}(s) = A(s)$$  \hspace{1cm} (4.12)

in which $A(s)$ is the transform of an arbitrary function $a(t)$ that equals zero for $t \geq 0$. Under broad conditions, [Pr81], the auto power spectrum $S_{rr}$ can be spectrally factorised into

$$S_{rr}(s) = S^+_{rr}(s)S^-_{rr}(s)$$  \hspace{1cm} (4.13)

in which the superscript $+$ indicates the factored part of $S_{rr}$ with all its poles and zeros in the left-half plane. The inverse transform of $S^+_{rr}$ is a positive-time function. Likewise, $S^-_{rr}$ has all poles and zeros in the right-half plane and is the transform of a negative-time function. Let us rewrite eq. (4.12) as

$$C(s)S^+_{rr}(s) = \frac{A(s)}{S^-_{rr}(s)} - \frac{S_{pp}(s)}{S^-_{rr}(s)}$$  \hspace{1cm} (4.14)

In this expression the left hand side is a positive-time function, the first term on the right hand side is a negative-time function and the second on the right hand side is of mixed type. Therefore, when taking the positive-time part of eq. (4.14), the causal controller follows as

$$C(s) = -\frac{1}{S^+_{rr}(s)} \left[ \frac{S_{pp}(s)}{S^-_{rr}(s)} \right]^+$$  \hspace{1cm} (4.15)

The corresponding minimum value of the cost function equals

$$J_{\text{min}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} [S_{pp}(\omega) - S^*_p(\omega)C(\omega)] d\omega$$  \hspace{1cm} (4.16)

From the solution of the optimal prediction problem - eqs. (4.15) and (4.16) - we conclude that the performance depends on two aspects:

1) The correlation between the primary signal and the filtered detection signal - as denoted by the cross-power function $S_{rp}$ - directly influences the minimum value of the criterion function.

2) The causality of the cross-correlation function of $r$ and $p$ determines the control function. If this is a positive-time function ($S^-_{rp} = 1$) the truncation in eq. (4.15) does not need to be carried out in order to create a causal controller.

If the filtered detection signal $r(t)$ is linearly related to $p(t)$ and their correlation function is causal, it is possible to have a perfect prediction (cancellation) of the primary signal.
example 1; causal cancellation of the primary sound in the 1-dimensional enclosure

Recall from Section 4.1 that substantial reductions of the global space average squared pressure in the 1-dimensional duct can be obtained using only a few secondary sources and error sensors. These results, however, are based on full knowledge of the primary source strength. In practice this is generally not available and one or more detection sensors have to be placed in the duct. For a first test we put a (white noise $\xi(t)$) primary source at $x = 0$, a detection sensor at $x_d$, a secondary source at $x_s$ and an error sensor at $x_e$; see Figure 4.12. In order to find the effect of the causality constraint the secondary source is closer to the primary source than the detection sensor is. The primary response at the error sensor $P(x_e, s)$, the detection signal $X(x_d, s)$ and the transfer function $H$ can be found with the help of Section 3.3 as

$$P(x_e, s) = e^{-s\tau_3} \frac{[1 + R][1 + Re^{-2s(\tau_4 - \tau_3)}]}{1 - R^2e^{-2s\tau_4}} \Xi(s)$$

$$X(x_d, s) = e^{-s\tau_2} \frac{[1 + Re^{-2s\tau_2}][1 + Re^{-2s(\tau_4 - \tau_2)}]}{1 - R^2e^{-2s\tau_4}} \Xi(s)$$

$$H(x_e | x_s, s) = e^{-s(\tau_3 - \tau_1)} \frac{[1 + Re^{-2s\tau_1}][1 + Re^{-2s(\tau_4 - \tau_3)}]}{1 - R^2e^{-2s\tau_4}}$$

(4.17)

For the auto- and crosspower functions we find:

$$S_{pp}(s) = H(-s)X(-s)P(s)$$

$$S_{pr}(s) = H(s)X(s)H(-s)X(-s)$$

(4.18)

Note that both $P(s)$ and $H(s)X(s)$ are positive-time functions, whereas $H(-s)X(-s)$ is a negative-time function. This leads to the solution as given by (4.15)

$$C(s) = -\frac{1}{e^{s(\tau_2 + \tau_3 - \tau_1)} H(s)X(s)} \left[ e^{s(\tau_2 + \tau_3 - \tau_1)} P(s) \right]^+$$

(4.19)

which is causal and stable if $R < 1$. Here, we notice that the difference in time-delays for the primary path ($\tau_3$) and secondary path ($\tau_2 + \tau_3 - \tau_1$) plays a very important role in the prediction
problem. The early response of the primary sound cannot be predicted and therefore is left as residual signal

$$E_0(s) = e^{-s(\tau_2+\tau_3-\tau_1)} \left[ e^{s(\tau_2+\tau_3-\tau_1)} P(s) \right]^-$$  \hspace{1cm} (4.20)

Apart from the fact that the reduction of the primary energy at $x_e$ depends on the difference in time delays $\tau_3$ and $(\tau_2+\tau_3-\tau_1)$, it also depends on the length of the primary impulse response.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.13a.png}
\caption{The advanced primary impulse response at the error sensor for $R = -0.9$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.13b.png}
\caption{The advanced primary impulse response at the error sensor for $R = -0.3$.}
\end{figure}

How important is the contribution of the early primary response, that cannot be cancelled? In Figures 4.13a and 4.13b an impression of the advanced (by $\tau_2-\tau_1$) primary impulse response is
depicted for reflection coefficients $R = -0.9$ and $R = -0.3$ (and $\tau_2 - \tau_1 = 0.2 / c_0$). Clearly, it holds that the smaller the reflection at the ducts's ends the more important the anti-causal, first response. In the extreme case of an anechoic duct nothing of primary response can be cancelled with a causal controller.

Note that in the example above the detection sensor provided a signal perfectly correlated with the primary sound. In practice, however, the detection signal may be contaminated with measurement noise. This obviously deteriorates the control performance.

**multivariable prediction**

In the multivariable case the sound at multiple error sensors ($M$) has to be predicted using the information from multiple detection sensors ($K$). For numerical convenience we have a decoupled controller such, that each of the $K$ controllers act on one of the detection signals only. This implies that the vector of error signals $\mathbf{e}(t)$ can be written as

$$
\mathbf{e}(t) = \mathbf{p}(t) + \mathbf{R}(t) \ast \mathbf{c}(t)
$$

with

$$
(R)_{ij} = h_{ij}(t) \ast d_j(t)
$$

(4.21)

Note that we have a matrix of filtered detection signals here. Based on (4.21) the cost function becomes

$$
J = E[\mathbf{e}(t)\mathbf{e}(t)^H] = \phi_{pp}(0) + \int_0^\infty \phi_{pp}(\tau) \mathbf{c}(\tau) d\tau + \int_0^\infty \int_0^\infty \mathbf{c}(\tau_1) \mathbf{c}(\tau_2) \phi_{rr}(\tau_1 - \tau_2) d\tau_2 d\tau_1
$$

(4.22)

in which the correlation quantities are given by

$$
\phi_{pp}(0) = \sum_{m=1}^M \phi_{pm} p_m
$$

$c$ $K \times 1$ vector

$$
\phi_{pp} \text{ $K \times 1$ vector with } (\phi_{pp})_i = \sum_{m=1}^M \phi_{pi} p_m
$$

(4.23)

$$
\phi_{rr} \text{ $K \times K$ matrix with } (\phi_{rr})_{ij} = \sum_{m=1}^M \phi_{ri} r_m p_j
$$

The corresponding Wiener-Hopf equation follows as

$$
\phi_{pp}(t) = \int_0^\infty \phi_{rr}(t - \tau) \mathbf{c}(\tau) d\tau \quad t \geq 0
$$

(4.24)

The implications of the causality condition on the geometry of detection sensors and the effective performance will be demonstrated in the following example.
example 2: reduction of free field random sound fields

Consider the configuration of sources and sensors in Figure 4.14. The primary wave field, i.e. its space average squared pressure, generated by two random noise sources is to be cancelled at the error microphone. Four different cases are examined

a) 1 primary source, 1 detection sensor
b) 2 primary sources, 1 detection sensor
c) 2 primary sources, 2 detection sensors
d) 1 primary source, 2 detection sensors

The primary sources are driven by two separate and independent Gaussian noise signals $\xi_1$ and $\xi_2$. For the primary and filtered detection signals at the error sensor relations we have

\[
p(t) = \frac{1}{\Delta_1} \xi_1(t - \tau_1) + \frac{1}{\Delta_2} \xi_2(t - \tau_2)
\]
\[
r_1(t) = \frac{1}{\Delta_3} \xi_1(t - \tau_3) + \frac{1}{\Delta_4} \xi_2(t - \tau_4) \\
r_2(t) = \frac{1}{\Delta_5} \xi_1(t - \tau_5) + \frac{1}{\Delta_6} \xi_2(t - \tau_6) \tag{4.25}
\]

Note that only the main features of a free field monopole - amplitude attenuation with distance like $1/\Delta$ and linear phase - are included here.

a) one source, one sensor

For a single primary source and detection sensor the correlation functions are
\[ \phi_p(\tau) = \frac{1}{\Delta_1 \Delta_3} \delta(\tau - \tau_1 + \tau_3) \]
\[ \phi_{rr}(\tau) = \frac{1}{\Delta_3^2} \delta(\tau) \]  \hspace{1cm} (4.26)

Evaluating the Wiener filter solution (4.15), the optimal controller reads
\[ C_1(s) = \begin{cases} \frac{-\Delta_1}{\Delta_1} e^{-s(\tau_1 - \tau_3)} & \tau_3 \leq \tau_1 \\ 0 & \tau_3 > \tau_1 \end{cases} \]  \hspace{1cm} (4.27)

If the information on the primary sound is on time complete cancellation follows. Otherwise, the primary white noise cannot be reduced at all.

b) two sources, a single sensor

If the primary is generated by two independent sources, we find for the Wiener filter
\[ \phi_p(\tau) = \frac{1}{\Delta_1 \Delta_3} \delta(\tau - \tau_1 + \tau_3) + \frac{1}{\Delta_2 \Delta_4} \delta(\tau - \tau_2 + \tau_4) \]
\[ \phi_{rr}(\tau) = \frac{1}{\Delta_3^2} \delta(\tau) + \frac{1}{\Delta_4^2} \delta(\tau) \]  \hspace{1cm} (4.28)
\[ C_1(s) = -\frac{\Delta_1^2 + \Delta_3^4}{\Delta_1 \Delta_3} e^{-s(\tau_1 - \tau_3)} H(\tau_1 - \tau_3) - \frac{\Delta_2^2 + \Delta_4^4}{\Delta_2 \Delta_4} e^{-s(\tau_2 - \tau_4)} H(\tau_2 - \tau_4) \]  \hspace{1cm} (4.29)

Even if the information on both processes arrives in due time, complete field cancellation at the error sensor generally is impossible.

c) two sources, two sensors

In order to reduce the field of two independent sources, two well-placed sensors are required. If \( \tau_1 \) and \( \tau_2 \) are bigger than all other time delays the solution can be found after a Laplace transform of (4.24) and a matrix inversion
\[ \Phi_p(\tau) = \begin{bmatrix} \frac{1}{\Delta_1 \Delta_3} \delta(\tau - \tau_1 + \tau_3) + \frac{1}{\Delta_2 \Delta_4} \delta(\tau - \tau_2 + \tau_4) \\ \frac{1}{\Delta_1 \Delta_3} \delta(\tau - \tau_1 + \tau_5) + \frac{1}{\Delta_2 \Delta_6} \delta(\tau - \tau_2 + \tau_6) \end{bmatrix} \]  \hspace{1cm} (4.30)
\[ \Phi_{rr}(\tau) = \begin{bmatrix} \frac{1}{\Delta_3^2} \delta(\tau) + \frac{1}{\Delta_4^2} \delta(\tau) & \frac{1}{\Delta_3 \Delta_5} \delta(\tau - \tau_3 + \tau_5) + \frac{1}{\Delta_4 \Delta_6} \delta(\tau - \tau_4 + \tau_6) \\ \frac{1}{\Delta_3 \Delta_5} \delta(\tau - \tau_3 + \tau_5) + \frac{1}{\Delta_4 \Delta_6} \delta(\tau - \tau_4 + \tau_6) & \frac{1}{\Delta_3^2} \delta(\tau) + \frac{1}{\Delta_4^2} \delta(\tau) \end{bmatrix} \]
d) one source, two sensors

For this geometry the auto- and cross-correlation quantities are

\[
\Phi_{pp}(\tau) = \begin{bmatrix}
\frac{1}{\Delta t_3}\delta(\tau - \tau_1 + \tau_3) \\
\frac{1}{\Delta t_3}\delta(\tau - \tau_1 + \tau_5)
\end{bmatrix}
\]

\[
\Phi_{rr}(\tau) = \begin{bmatrix}
\frac{1}{\Delta s^2}\delta(\tau) & \frac{1}{\Delta s}\delta(\tau - \tau_5 + \tau_3) \\
\frac{1}{\Delta s}\delta(\tau - \tau_3 + \tau_5) & \frac{1}{\Delta s^2}\delta(\tau)
\end{bmatrix}
\]  \hspace{1cm} (4.31)

from which it follows

\[
C_1(s) = C_2(s) = 0 \quad \text{if} \quad \tau_5 > \tau_1 \wedge \tau_3 > \tau_1
\]

\[
C_1(s) = -\frac{\Delta t}{\Delta t_4} e^{-s(\tau_1 - \tau_3)}; C_2(s) = 0 \quad \text{if} \quad \tau_5 > \tau_1 \wedge \tau_3 \leq \tau_1
\]

\[
C_1(s) = 0; C_2(s) = -\frac{\Delta t}{\Delta t_4} e^{-s(\tau_1 - \tau_3)} \quad \text{if} \quad \tau_5 \leq \tau_1 \wedge \tau_3 > \tau_1
\]

\[
C_1(s) + \frac{\Delta t}{\Delta s} C_2(s)e^{-s(\tau_5 - \tau_3)} + \frac{\Delta t}{\Delta t_4} e^{-s(\tau_1 - \tau_3)} = 0 \quad \text{if} \quad \tau_3 \leq \tau_5 \wedge \tau_3 \leq \tau_1
\]  \hspace{1cm} (4.32)

Clearly, one should remove one sensor in order to avoid an ill-conditioned problem. Two sensors appear to conflict in the case of one independent primary source.

In contrast to example 1, in the free field the ‘timing’ is crucial to the control performance. If the secondary path is longer than the primary nothing of the primary sound can be reduced. In the one-dimensional enclosure the controller is still capable to cancel the reflected - and therefore ‘older’ - primary sound.

\subsection*{4.3 Detection of the primary sound field}

The theory on Wiener filtering or optimal prediction in the previous Section contiguously leads to the main demands for the detection sensor: acquire all and premature information on the primary field. Strictly, the prediction analysis merely holds for so-called ‘general linear processes’ [Pr81]. A general linear process can be written as filtered Gaussian noise:

\[
x(t) = \int_{-\infty}^{\infty} h(\tau)\varepsilon(t - \tau) \, d\tau
\]  \hspace{1cm} (4.33)

in which \(\varepsilon\) again is the noise signal. The acoustically important harmonic process does not belong to this class. However, the constraint of causality can be dropped for harmonic processes. This leaves as a mere condition that all harmonic components of the primary field should be detected. In many practical cases a harmonic sound field is generated by rotating equipment. Useful information on the relevant harmonics may be acquired by just monitoring the axis’ rotation. This technique is to be further addressed in the coming Chapters.
The example of free field sound prediction in Section 4.2 shows a very strict condition for reduction: if the path detection sensor-secondary source-error sensor is longer than the primary path there is no reduction at all. This extreme behaviour is due to the white noise characteristic of the primary signal. In the next analysis it will be demonstrated that the presence of impedance boundaries has a favourable influence on reducing random primary sound.

Finally, we should note that in operation a set of detection sensors both measures the primary and the secondary sound field. This contamination of the detection signal is not included in the analysis here. It is assumed that the secondary contribution to the detection signal is completely known, such that it eliminated from the controller input signal. As for the other relevant transfer functions, this implies exact knowledge of the transfer function relating secondary source signal and detection signal. The actual impact of this input contamination is to be dealt with in the thesis later.

![Diagram](image)

**Figure 4.15a;** Graph of space average squared pressure for a configuration of a secondary source at \( x = 1.0 \), a detection sensor at \( x = 0.6 \) and two error sensors at \( x = 0.5 \) and \( 0.7 \). The reflection coefficient \( R \) equals 0.99.

If we drive the primary source with pure Gaussian noise it is interesting to see the degradation in performance due to the causality constraint. Recall from the simulation example in Section 4.1 the configuration of a single secondary source, \( x_s = 1 \), and two error sensors, at 0.5 and 0.7. For a high value of \( R \) the reduction of global energy turns out to be substantial. If the controller, however, is restricted to causal action, the reduction may diminish. If we put one detection sensor at \( x_d = 0.6 \), part of the primary information will be unpredictable. In Figures 4.15a, 4.15b
and 4.15c the performances of the causal and non-causal controllers are compared for several values of the reflection coefficient $R$. Note that in Figure 4.15c the primary and causally reduced magnitude coincide.

Figure 4.15b; Graph of space average squared pressure for a configuration of a secondary source at $x = 1.0$, a detection sensor at $x = 0.6$ and two error sensors at $x = 0.5$ and $0.7$. The reflection coefficient $R$ equals 0.60.

Figure 4.15c; Graph of space average squared pressure for a configuration of a secondary source at $x = 1.0$, a detection sensor at $x = 0.6$ and two error sensors at $x = 0.5$ and $0.7$. The reflection coefficient $R$ equals 0.
So, we can draw the following conclusions on primary energy reduction in the 1-dimensional enclosure using a detection sensor.

a) Resonances can be well reduced, since for those frequencies the early, non-predictable primary response at the error sensors is only a minor part of the total response. The impedance boundary has a 'colouring' effect on the white noise, which improves the signal's predictability.

b) At anti-resonances the predictable part of the primary signal is far less significant. Still, a modest reduction of space average squared pressure at the error sensors is achievable. For the global space average squared pressure the causality constraint with the use of a local cost function leads to a little amplification of global space average squared pressure.

c) Similar to the previous cases of global space average squared pressure and local space average squared pressure minimisation, the overall reduction improves with the strength of the resonances. On the other hand, if the end termination is non-reflective the controller achieves no reduction at all.

**the use of detection sensors in a multi-modal duct and a three-dimensional enclosure**

The mathematical analysis and simulation experiment in the one-dimensional enclosure again serves as the exemplary case study representing the practice of the duct and the room. The problem of detecting the primary signal for these cases is addressed in short here.
The minimisation of radiated power in a multi-modal waveguide was introduced in Section 3.4. As it has a non-reflective termination, the position of the detection sensors has to be in between the primary source and the secondary sources. This is the most sensible approach, even if the termination does reflect, because then the constraint of causal action does not limit the performance. Note that this favourable behaviour is due to a region of silence which is beyond the set of secondary sources.

In the three-dimensional enclosure the causality constraint does limit the reduction properties, as the region of silence usually comprises the complete volume. Generally, part of the error sensors is closer to the primary source(s) than to the secondary sources. Anyway, the detection sensors should be located as close to the primary sources as possible. Concerning the effective reduction of space average squared pressure we may expect a behaviour similar to that of the one-dimensional enclosure.

**4.4 Acoustic design procedures for active sound reduction systems**

In this and the previous Chapter we discussed how to approximate the active reduction of sound fields, as was introduced in Chapter 2. The practical limitations on the number of control sources and error sensors caused a degradation of the optimal performance of the Kirchhoff
continuous distributions of control sources. The need for well-placed detection sensors followed from the actual unfamiliarity with the primary source and the constraint of causal action.

The case studies as used in Chapters 3 and 4 apply to situations that suit the application of active sound reduction. Below an overview of their relevance is given.

free field: transformer
gas turbine

duct: ventilations systems
diesel exhausts

enclosure: transportation vehicles (engine and road noise)
propeller aircraft

All cases concern low-frequency noise either harmonic or random. Next, the main aspects of the acoustic design of a system for active sound reduction are briefly summarised.

error sensors
First, define a cost function that ideally represents the ‘annoyance’ of the primary noise. This may be an energetic quantity in an area or perhaps a maximum value, as discussed in Chapter 3. Next, find a set of error sensors and positions and define a discretised cost function that well approximates the original, global cost function. The error sensors have to independently observe all ‘modes’ or degrees of freedom of the primary field, such that minimisation of the discrete cost function stands for a global effect.

secondary sources
The control sources have to be capable to generate all dominant ‘modes’ of the primary field with arbitrary amplitude; i.e. a full controllability. This may imply to locate them close to the primary sources, like in the free field. In a reverberating environment positions well away from the primary source are suitable as well. For numerical reasons it is convenient that the secondary sources are outnumbered by the error sensors.

detection sensors
The number and locations of the detection sensors has to be such that all uncorrelated primary sources can be observed individually. Further, it is advisable to locate the sensors as close to the primary sources as possible to eliminate the possibly detrimental effects of the causality
constraint. In certain cases, however, this may be unavoidable. The consequent degradation in performance depends on the reverberating characteristics of the environment.

4.5 References

CHAPTER 5

Modelling and Control of Acoustic Disturbances

In Chapters 2, 3 and 4 a complete analysis was carried out on how to design an acoustic system for active sound reduction. After having defined a configuration of secondary sources, detection and error sensors, the problem emerges on how to control the input (sensor) and output (source) signals. How do we design a controller that realises the planned acoustical performance?

This Chapter will address the general aspects of designing a control system for acoustic noise. Initially, this implies modelling of the acoustic processes and disturbance signals. In the modelling analysis a few essential features of acoustic systems will emerge. It will be demonstrated for example that discrete-time acoustic processes are likely to have non-minimum phase zeros. This phenomenon is hardly recognised in the acoustic control community.

Also, the performance and robustness characteristics of feedforward controllers is discussed. A feedforward controller should take care of the measurable disturbances - as most noise signals are. However, the feedforward principle is not robust. This Chapter will show that the addition of a feedback link can well improve the poor robustness characteristics of a feedforward controller. Furthermore, a feedback controller is capable of (partly) reducing unmeasurable disturbances. Finally, the LQG control strategy is introduced that describes the characteristics of a combined feedforward - feedback controller for acoustic disturbances.

Figure 5.1: Schematic diagram of the acoustic control system.
5.1 Concepts of control systems

In Chapter 4 an acoustical system for sound control emerged as depicted in Figure 5.1. It consists of a distribution of a secondary source, an error sensor and a detection sensor. Obviously, if the complexity of the sound field requires so, the system can be extended to a multiple source - multiple sensor configuration. In terms of control concepts, the system falls into three subsystems: the acoustic plant or process, the disturbance models and the regulator.

- **ACOUSTIC PLANT;** this subsystem refers to the electro-acoustic relations between the available signals. The primary acoustic process describes the relation between the primary signal measured by the error sensor and that measured by the detection sensor. The secondary acoustic process represents the transfer function from control signal to error signal. Finally, the tertiary acoustic process connects the control signal to its contribution to the detection signal - the acoustic feedback of the control sound field.

- **DISTURBANCE MODEL;** this model relates the primary signal at the sensors to some elementary disturbance generator by means of a linear process. This process does not exactly simulate but stands in for the underlying physics of the generation of the primary sound. Generally, the overall disturbance is partly measurable (by the detection sensor). The other part, which is measured by the error sensor only, is called unmeasurable or extraneous.

It should be mentioned that it is assumed, that the measurable disturbance arrives at the detection sensor before it arrives at the error sensor. Then, the primary process - see Figure 5.2 - can be modelled as a causal process. If the measurable disturbance would arrive at the error sensor first, the detection sensor would be of no use. In that case the disturbance is part of the extraneous disturbance.

- **REGULATOR;** usually, the regulator has the task to make the output signal follow a command signal as good as possible. This may be achieved by processing the available sensor signals together with the command signal. For our purpose of sound reduction the command signal is a null signal. The part of the controller that acts on the detection signal is denoted by feedforward controller and the part acting on the output signal is called feedback controller. Note that output signal and error signal are equivalent terms, as the desired or command signal is zero. It is assumed here, that the signal from the detection sensor is free of measurement noise.

If the concepts of a general control system are projected on the sound control system, the following set-up is obtained as depicted in Figure 5.2.
If we take the regulator, the acoustic plant and the disturbance model together, the emerging global process relates the disturbance generators to the output signal. Clearly, the intrinsic regulator has the task of modifying this transfer function towards the desired behaviour: attenuation of the primary sound at the error sensor.

In the forthcoming Sections the characteristics of the acoustic processes and the disturbance models are addressed. Contiguously, the principles of feedforward and feedback control are treated.

### 5.2 The acoustic modal state-space representation

Following the derivation in Appendices 3.2 and 3.3 we find for Green’s function in the one-dimensional enclosure

\[
G(x | x_s, s) = c_0^2 \sum_{n=-\infty}^{\infty} \frac{\varphi_n(x_s) \varphi_n(x)}{s(s - j\omega_n)} = \sum_{n=-\infty}^{\infty} a_n(s) \varphi_n(x)
\]

(5.1)

where \( \varphi_n \) are the eigenfunctions of the enclosure and \( \omega_n \) their complex-valued eigenfrequencies (\( \omega_n = c_0 k_n \)); \( c_0 \) is the speed of sound. Green’s function can be interpreted as a transfer function of the enclosure. The sound field of a monopole source at \( x_s \) with strength
\( U(s) \) follows directly from \( P_m(s) = \rho s U(s) G(s) \). In this respect Green's function represents the sound field generated by a unit point source; \( P_g(s) = G(s) U(s) \). For the modal coefficients \( a_n(s) \) of this sound field the following relation holds

\[
s[s - j\omega_n]a_n(s) = c_0^2 \varphi_n(x_0) U(s)
\]

(5.2)

Transforming this relation to the time-domain renders

\[
\dot{a}_n(t) - j\omega_n a_n(t) = c_0^2 \varphi_n(x_0) u(t)
\]

(5.3)

in which a time-derivative is denoted by an upper dot; \( \dot{a}_n = \frac{d}{dt} a_n \). If we describe the acoustic transfer function (5.1) in a state-space representation

\[
\begin{align*}
\dot{x}(t) &= A x(t) + B u(t) \\
y(t) &= C x(t) + D u(t)
\end{align*}
\]

(5.4)

we find for the coefficient of a single mode \( \varphi_n \)

\[
\begin{bmatrix}
\dot{\tilde{a}}_n(t) \\
\tilde{a}_n(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\omega_n^2 & 0
\end{bmatrix}
\begin{bmatrix}
a_n(t) \\
\dot{\tilde{a}}_n(t)
\end{bmatrix} + c_0^2
\begin{bmatrix}
0 \\
\varphi_n(x_0)
\end{bmatrix}
\begin{bmatrix}
0 \\
a_n(t)
\end{bmatrix}
\]

(5.5)

Here the output signal \( y_n(t) \) is the pressure field measured by a sensor at \( x_m \). If all modal states are included in \( x(t) \) the following model is obtained

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix}
0 & I \\
0 & j\Omega
\end{bmatrix} x(t) + c_0^2
\begin{bmatrix}
0 \\
b
\end{bmatrix}
\begin{bmatrix}
u(t)
\end{bmatrix} \\
y(t) &= c_0^2
\begin{bmatrix}
0 \\
b
\end{bmatrix}
\begin{bmatrix}
x(t)
\end{bmatrix}
\end{align*}
\]

(5.6)

in which \( I \) is an identity matrix and \( \Omega \) a diagonal matrix of all eigenfrequencies. Column vector \( b \) contains the mode functions evaluated at the source location, whereas row vector \( c_0^2 \) contains the mode values for the sensor location. With reference to Figure 5.2 the model (5.6) may function as a description of the primary or secondary acoustic process. Since \( u(t) \) is usually identified as the control signal, we will focus on the secondary process.

In control theory state-space models have proved to be a convenient way to describe the underlying characteristics of the relevant phenomenon. For our model for example it is straightforward to extend it to more sources and sensors

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix}
0 & I \\
0 & j\Omega
\end{bmatrix} x(t) + c_0^2
\begin{bmatrix}
0 \\
b
\end{bmatrix}
\begin{bmatrix}
u(t)
\end{bmatrix} \\
y(t) &= [C, 0] x(t)
\end{align*}
\]

(5.7)

where \( u(t) \) is the vector of source strengths and \( y(t) \) that of sensor signals. Note that the state vector has not changed. Also, state-space models have favourable numerical properties concerning finite-precision arithmetic, compared with polynomial or zero-pole-gain models. A serious disadvantage, however, are the possibly high dimensions of the model's matrices and
vectors. As in the case of eq. (5.6) the model is infinitely-dimensional in principle. In practice the modal superposition (5.1) can be truncated, although this still yields a high-dimensional model.

**controllability and observability**

There are two important issues related to dynamical systems. These are:

- Can we steer the system from an initial state to any arbitrary state?
- Can we determine the system’s state by observing the inputs and outputs?

The answers to these questions can be found in many textbooks on control theory; see e.g. [GS84].

A particular state $x_0$ of the system (5.6) is said to be controllable, if there exists a $T$ and a control function $u(t)$ for $t = 0$ to $T-1$, which drives the system from $x(0)=x_0$ to $x(T) = 0$. A system is called **controllable** if all states are controllable. This is the case if the controllability matrix $W_c$

$$W_c = [B\ AB\ A^2B\ \cdots\ A^{n-1}B]$$  \hspace{1cm} (5.8)

is of full rank, where $n$ is the state dimension. The related and more restrictive concept of **reachability** is concerned with steering the system from the zero state to any other.

A particular state $x_0$ of system (5.6) is said to be unobservable if for any $T > 0$ and $u(t) = 0$, $0 \leq t \leq T$, the initial state $x_0$ produces a zero output: $y(t) = 0$ for $0 \leq t \leq T$. A system is completely **observable** if no state, except the zero state, is unobservable. Then the observability matrix $W_o$

$$W_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$  \hspace{1cm} (5.9)

has full rank.

If the requirements for controllability and observability are applied to the modal state-space representation, a simple conclusion can be drawn, as is demonstrated by Balas [Ba78].

- If the eigenvalues $\omega_k$ have unit multiplicity, the system (5.7) is completely controllable and observable if each row of $B$ and each column of $C$ has at least one non-zero entry. This can be obtained by using one source and one sensor, if they are located away from the zeros of the included mode functions. If there are eigenvalues with multiplicity greater than one, the ranks of the corresponding blocks of $B$ and $C$ should equal this multiplicity.

Physically, these requirements are straightforward. As is shown in the preceding Chapters, reduction of a certain mode is only possible, if we can excite it and measure its contribution to the cost function. If a particular set of modes has the same resonance frequency (strictly
speaking the same eigenvalue) we need as many sensors and sources as these modes to
distinguish between them. Consequently,

- the controllability and observability characteristics of the modal state-space model give
direct information on the quality of the acoustic design (the chosen geometry of sensors
and sources).

The model and corresponding conclusions as given above, are concerned with a
one-dimensional enclosure with arbitrary impedance terminations. In Chapter 3 we also
derived the modal expansion for Green’s function in a three-dimensional enclosure. For the
pressure field of a unit point source we can write

\[ P_\phi(x, t) = \sum_{n=0}^{\infty} a_n(s) \phi_n(x) \]

with \[ a_n(s) = c^2 \frac{\phi_n(x) U(s)}{s^2 + \omega_n^2} \] (5.10)

which holds for the self-adjoint boundary value problem only; see Appendix 3.2. This leads to
a simplification of the expansion structure of eq. (5.1). The corresponding state-space model is
given by

\[ \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\Omega^2 & 0 \end{bmatrix} x(t) + c^2 \begin{bmatrix} 0 \\ b \end{bmatrix} u(t) \]

\[ y(t) = \begin{bmatrix} c & 0 \end{bmatrix} x(t) \] (5.11)

with identical definitions of \( x, \Omega, b \) and \( c \). The requirements for controllability and observability
remain unchanged, which is obvious on physical grounds.

5.3 Discrete-time transfer functions models

In practice the models for the acoustic process have to apply to the discrete-time case. This is
mainly because of the extensive capacity, accuracy and flexibility of digital controllers
nowadays. Moreover, one should realise that the main application area of active sound
reduction systems is the low frequency band (say below 1 kHz). There is no need to include the
ineffective high frequency information.

Let us focus on the sampling process using the single-channel modal-state for the
one-dimensional wave guide as given by eq. (5.6). Apart from this model and the modal
expansion (5.1), also a closed-form solution is known as

\[ P_\phi(s) = \frac{1}{2} \frac{e^{-s\tau_1}}{s} \frac{[1 + R_0 e^{-2s\tau_2}][1 + R_L e^{-2s\tau_3}]}{1 - R_0 R_L e^{-2s\tau_4}} U(s) \] (5.12)
where \( R_0 \) and \( R_L \) are the termination reflection coefficients and the several time delays are clarified in Figure 5.3.

\[
\begin{align*}
\text{delay:} & \quad \tau = \tau_1 \\
\text{poles:} & \quad s_k = \frac{1}{2\tau_4} \left[ \log_e(R_0R_L) + 2j\pi k \right] \quad \text{and} \quad s = 0 \\
\text{zeros:} & \quad s_n = \frac{1}{2\tau_3} \left[ \log_e(R_0R_L) + 2j(n + \frac{1}{2})\pi \right] \\
& \quad s_m = j\left(\frac{2m+1}{2}\right)\pi
\end{align*}
\]

(5.13)

in which \( k, m \) and \( n \) are integers. None of these poles or zeros lie in the right half plane.

Now let us investigate the influence of sampling on the characteristics of the model. As depicted in Figure 5.4, the sampled or discrete-time transfer function gives the relation between the input sequence \( u(kT_s) \) and output sequence \( y(kT_s) \). The acoustic process comprises the acoustic path and the electro-dynamical conversion in the source and the sensor.

\[
\begin{array}{cccc}
\text{controller} & \text{zero-order hold} & \text{acoustic process} & \text{sampler} \\
\hline
u(kT_s) & u(t) & y(t) & y(kT_s) \\
\end{array}
\]

\[\text{Figure 5.4: Block diagram of the relation between discrete-time input and output signal.}\]
The discrete-time zero-order hold equivalent of a continuous state-space model (5.4) is given by [ÅW84]

\[ \begin{align*}
\mathbf{x}(kT_s + T_s) &= \Phi \mathbf{x}(kT_s) + \Gamma u(kT_s) \\
y(kT_s) &= C \mathbf{x}(kT_s) + D u(kT_s)
\end{align*} \]  
(5.14)

with \( \Phi = e^{AT_s}, \Gamma = \int_0^{T_s} e^{At} \, dt \, B \). For reasons of legibility the sampling period \( T_s \) will be omitted hereafter. The discrete-time pulse transfer operator \( H(q^{-1}) \) gives the input-output relation of the system without being concerned with all its internal states

\[ y(k) = H(q^{-1}) u(k) = (C[I - q^{-1}\Phi]^{-1} q^{-1} \Gamma + D) u(k) \]  
(5.15)

in which \( q^{-1} \) is the backward shift operator with \( q^{-1} x(k) = x(k-1) \). From (5.15) it follows that in general \( H(q^{-1}) \) can be written as the fraction of a numerator and denominator polynomial together with a time delay

\[ H(q^{-1}) = \frac{B(q^{-1})}{A(q^{-1})} = q^{-b} \frac{b_0 + b_1 q^{-1} + \cdots + b_{n_B} q^{-n_B}}{1 + a_1 q^{-1} + \cdots + a_{n_A} q^{-n_A}} \]  
(5.16)

with \( b_{n_B} \) and \( a_{n_A} \) unequal to zero. The powers \( n_B \) and \( n_A \) represent the degree of the polynomials. The difference \( (n_A - n_B - b) \) is called the pole excess and should be non-negative to have causal behaviour. The discrete-time delay of the process is \( b \). In acoustic systems the time delay is related to the acoustic travel time from source to sensor. As the time delay is modelled separately, the leading \( B \)-coefficient \( b_0 \) is non-zero.

For the analysis of a system’s characteristics, like time-delay, poles and zeros and frequency response, the pulse transfer function in terms of \( q^{-1} \) or \( z^{-1} \) is an attractive tool. Performing the operations given by eqs. (5.14) and (5.15) on the modal-space (5.6) eventually yields the transfer function

\[ H(q^{-1}) = \sum_n c_n(x_s, x_m) q^{-1} \frac{B_n(q^{-1})}{(1 - q^{-1})(1 - e^{j\omega_T s} q^{-1})} \]  
(5.17)

So, in principle the discrete-time model contains infinitely many poles and zeros. A low order model is obtained if the length \( L \) is an integer multiple of \( c_0 T_s \). In that case the poles of the repeating modes overlap, which leads to a simple form of the pulse transfer function. Let us examine this for a special configuration; \( L = 9c_0 T_s, x_s = 0, x_m \) is variable and \( R_0 = R_L = 0.9 \). This leads to the following poles and zeros; see Figure 5.5.
Section 5.3  Discrete-time transfer functions models

Figure 5.5:  Pole (*) - zero (o) diagram of the pulse transfer function for two different sensor positions.

a)  \( x_m = 0.2 \).

b)  \( x_m = 0.5 \).

Based on eq. (5.17) and Figure 5.5, the following conclusions can be drawn on the characteristics of a discrete-time pulse transfer function in the one-dimensional wave guide:

a)  The acoustic travel time leads to a time-delay which is equal to quotient of travel time and sampling period rounded off upwards to the next integer. Though not proven, this is an obvious conclusion.

b)  Clearly, the poles of the transfer function are determined by the resonances (frequency and amplitude) in the wave guide. So, the locations are related to the eigenvalues of the acoustic modes and the sampling period. More specifically we have  \( p_n = e^{j\omega_k T_s} \). These poles are repeated for special values of sampling period \( T_s \).

c)  The locations of certain zeros of \( H(z^{-1}) \) have no direct physical ground. Although, the continuous-time zeros of \( G(s) \) - (note that of eq. (5.12) \( \tau_2 = 0 \) here)- are all minimum phase, its discrete-time equivalent may contain non-minimum phase zeros as well. It should be realised that in the analysis of the zeros the time delay is not taken into account. The term non-minimum phase applies to the numerator polynomial only. If we strip \( G(s) \) of its multiple reflections

\[
G_{\text{direct}}(s) = \frac{1}{2} \frac{e^{-st_1}}{s} [1 + R_0 e^{-2st_2}][1 + R_L e^{-2st_2}] 
\]

(5.18)
a set of delayed step functions remains. The sampled version of a delayed step function is a well-known example in control textbooks to illustrate the ‘creation’ of non-minimum phase zeros. Now it is apparent how the sampler and zero-order hold circuit determine the transfer function’s numerator of which an example is given in Figure 5.8. So, the sampling process may introduce non-minimum phase zeros; see also [Ås84].
In order to demonstrate the influence of the reflection coefficients and the sampling period on this phenomenon, an extensive statistical simulation has been carried out. In Figure 5.6 the probability of a non-minimum phase acoustic process is depicted as a function of sampling frequency and for 4 values of the reflection coefficients $R_0=R_L=R$. Here we observe that on average the probability increases with:

- the reverberation of the enclosure.
- the sampling frequency; this is in accordance with Åström’s paper [Ås84].

**Figure 5.6:** The probability of one or more non-minimum phase zeros in the discrete-time model of Green’s function in a one-dimensional enclosure for several values of the reflection coefficients.

**Remarks**

In order to give the pole-zero analysis more practical value the response of an ordinary electrodynamic loudspeaker should be included in the overall transfer function. The pressure field of an enclosed loudspeaker in a duct can be written as Green’s function filtered by a second order system

$$P_1(x_1 x_s, s) = \frac{\rho v^2 c_0^2 s}{s^2 + \frac{\omega_0}{q_T} s + \omega_0^2} \sum_n \frac{\Phi_n(x_s)\Phi_n(x)}{(s - j\omega_n)}$$  \hfill (5.19)

in which $\omega_0$ is the loudspeaker’s resonance frequency and $Q_T$ the quality factor. In order to arrive at Figure 5.7 the values $\omega_0 = 0.05 f_s$ and $Q_T = 2$ were used. The influence of the speaker characteristics on the pole-zero pattern is not very significant, as can be observed in Figure 5.7. The major change is given by the creation of a zero at $z = 1$, which is a familiar feature of acoustic sources.
Figure 5.7: Pole (*) - zero (o) diagram of the pulse transfer function of a duct with an electrodynamic loudspeaker for two different sensor positions.

a) $x_m = 0.2$.
b) $x_m = 0.5$.

Figure 5.8: Pole (*) - zero (o) diagram of the low-pass filtered pulse transfer function of an electrodynamic loudspeaker in a duct for two different sensor positions.

a) $x_m = 0.2$.
b) $x_m = 0.5$. 
Furthermore, in practice low-pass filtering should be applied to avoid aliasing effects. An \( n \)-th low-pass Butterworth filter - with cut-off frequency \( s_c \) - has the following transfer function

\[
LP(s) = \prod_{j=1}^{n} \frac{s_c}{s + s_c e^{-\frac{j}{\pi} n (n+2j-1)/n}}
\]  

(5.20)

Adding a second-order Butterworth filter at \( 0.14 f_s \) to the loudspeaker response leads to the pole-zero pattern as given in Figure 5.8. Here, we note the creation of a non-minimum phase zero in the response at \( x_m = 0.5 \). The other discrete-time response already was of non-minimum phase.

**Model order truncation**

For practical reasons finite order models of the acoustic processes have to be used. Usually, the controller degree is linearly related to the degree of process model. For a general sampling period this implies a truncation of the exact, infinite-dimensional model. Two ways of truncation are proposed:

1. If the acoustic behaviour can well be approximated by a limited number of modes, the state-space model can be stripped of its minor modes. This leads to finite degree input-output model with as many poles as acoustic modes accounted for. This truncation is probably most effective in a strong reverberating environment; also see Chapter 3.

2. If the reverberation time is not extraordinary high, a truncated pulse response model can be used. This leads to an all-zero \( (A = 1) \) model of degree: \( n_B = T_{60}/T_s \), in which \( T_{60} \) is a measure of the reverberation time. For acoustic environments in which the direct field dominates, this type of modelling is more effective than the pole-zero model.

For both approximations it is hard determine the characteristics of the zeros. It is easily verified that the zeros of a truncated pole-zero model are calculated to be minimum phase, whereas the true pulse transfer function has non-minimum phase zeros. Current research is focussing on this problem; see [To91].

### 5.4 Acoustic disturbance models

In this Section attention is paid to representation of the disturbance signals. The basic idea behind a disturbance model is the use of an elementary function as the *disturbance generator*. This generator most probably does not correspond to the physical generation of a disturbance
signal. It may, however, result in a suitable description of the dynamical behaviour of the relevant disturbance. The general model for disturbances looks like

$$y(k) = \frac{C(q^{-1})}{A(q^{-1})} \gamma(k)$$  \hspace{1cm} (5.21)

where $y(k)$ is the disturbance as measured by a sensor and $\gamma(k)$ is the disturbance generator. It may be a pulse, a sequence of pulses or white noise; [ÅW84]. Below, a few typical acoustic disturbance signals are listed that are covered by (5.21).

- **deterministic signals**

General deterministic signals can be modelled by (5.21) with $\gamma(k) = \delta(k)$, a Kronecker delta function. This class of stationary signals may contain either harmonic or transient signals. Generally, harmonic signals are generated by rotating or reciprocating machinery, like e.g. a Roots blower, a compressor, a transformer, a vehicle’s engine or an aircraft’s propellers. The latter two, however, create sound fields that are seriously non-stationary and therefore will be treated among the class of piece-wise deterministic signals. A stationary harmonic signal of radial frequency $\omega_0$ fits in the model like

$$y(k) = \frac{c_0 + c_1 q^{-1}}{1 - 2 \cos(\omega_0 T) q^{-1} + q^{-2}} \gamma(k)$$  \hspace{1cm} (5.22)

with $\gamma(k) = \delta(k)$

The source function - being a unit pulse at $k = 0$ - guarantees the stationarity of the disturbance. The numerator polynomial $C(q^{-1})$ determines the amplitude and phase of the harmonic, whereas the denominator $A(q^{-1})$ determines the frequency. The non-decaying behaviour of an harmonic is represented by poles at the unit circle. Note that a signal of the type (5.22) is not necessarily periodic; when sampling period and signal period have no common multiple.

Due to the mechanics of rotating or reciprocating apparatus the signals usually contain a fundamental component and higher harmonics. Obviously, this can be described by just superimposing the individual harmonics of the type (5.22). Transient signals can also be written as filtered Kronecker delta functions. The poles of the transfer function are all inside the unit circle in that case.

- **piecewise deterministic signals**

This type of signal is deterministic during a certain period, after which amplitude and phase characteristics change. This is modelled by eq. (5.21) where $\gamma(k)$ contains a stationary random pulse sequence like a Bernoulli-Gaussian process $r(k)$.

$$r(k) = \tilde{\xi}(k) s(k)$$  \hspace{1cm} (5.23)
in which $\xi(k)$ is a Gaussian sequence and $s(k)$ is a Bernoulli sequence given by $s(k) = 1$ with probability $\lambda$ and $s(k) = 0$ with probability $1 - \lambda$. An harmonic signal that occasionally changes amplitude and phase can thus be represented by

$$y(k) = \frac{c_0 + c_1 q^{-1}}{1 - 2 \cos(\omega_0 T) q^{-1} + q^{-2}} [\delta(k) + \xi_1(k)s(k) + \xi_2(k)s(k-1)]$$  \hspace{1cm} (5.24)

Here the Gaussian sequences $\xi_1(k)$ and $\xi_2(k)$ have zero mean value and a variance that indicates the average (squared) rate of change. Note that two coupled Bernoulli-Gaussian processes are needed to enable a change of phase. Again, the delta function acts as a start-up mechanism. It is felt that one harmonic of the interior cabin sound of a propeller aircraft is well modelled by (5.24).

Unlike the harmonic sound of a propeller aircraft, in which the fundamental frequency is kept as stationary as possible, the sound of a car engine is characterised by a non-stationary fundamental frequency as well. The fundamental frequency drifts away during acceleration or deceleration and changes suddenly during change of gear. In the sense of the model of eq. (5.24) this means that the denominator polynomial $A$ is time-variant.

- stochastic signals
  This includes disturbance signal like e.g. flow noise, ventilation duct noise, noise from combustion instabilities, road noise in a vehicle or extraneous noise in general. Here, we simply set $\gamma(k)$ equal to a white noise sequence. The continuous-time equivalent of this signal was introduced in Section 4.3, eq. (4.33). By virtue of the spectral factorisation theorem [ÅW84] the polynomial $C$ has no roots outside the unit circle. A drifting stochastic disturbance can be modelled by putting one or more roots of $A$ on the unit circle; e.g. $A(q^{-1}) = (1 - q^{-1})A_1(q^{-1})$.

- combinations of stochastic and deterministic signals
  Obviously, this is obtained by adding the individual disturbance types. The sound in a car e.g. consist of the piece-wise deterministic engine sound and the coloured noise of the road and wind. In a ventilation duct usually a few harmonics are present and a substantial level of broadband noise. In fact, in most practical cases of rotating equipment a certain level of background stochastic noise accompanies the harmonics.

The disturbances previously described are generated by a source of random nature $\gamma(k)$. In order to reduce a disturbance the controller requires information on this signal. This could be provided by the error sensors, although the controller response to it is retarded by the secondary process. As one can imagine, in many cases it is feasible to measure the disturbance beforehand by locating one or more sensors near the primary source, as shown in Figure 5.1. This yields more advanced information on the disturbance and probably a better performance of the control
system. The principal requirement for the use of detection sensors is that it leads to more timely information on the disturbance. This also implies that the use of detection sensors may be awkward in certain cases:

- The primary source has a distributed nature and possibly surrounds the error sensors. As a consequence, a large number of detection sensor would be required to outperform the information delivered by the error sensors.
- Apart from sound from the primary source an *acoustical* detection sensor is likely to measure sound from the control sources as well. This acoustic feedback effect complicates the controller design.
- Although, the actual situation is not crystal clear, the use of a sensor to measure the disturbance signal is subject to several worldwide patents. This especially holds for harmonic disturbances.
- The relation between disturbance signal at the error sensor and at a detection sensor close to the primary source is not necessarily described by a linear primary acoustic process. This happens e.g. when placing an accelerometer on the chassis of a car to measure road noise; see [Su91].

*Figure 5.9:* Diagram illustrating the optical monitoring of an engine's axis revolution for the purpose of harmonic disturbance rejection in the exhaust.

Still, in the majority of applications detection sensors are used to provide the regulator with timely information on the imminent disturbance behaviour. A very elegant way of detecting the disturbance is feasible in the case of a rotating machine; see Figure 5.9. By observing the revolution of the machine's axis with the use of a magnetic or optical sensor, a useful detection
signal is acquired. Out of this it is straightforward to derive the fundamental component and higher harmonics of the primary sound. Note that the sensor is inaccessible to the control sound field - there is no acoustic feedback. As mentioned above, however, this method of disturbance detection is subject to patents.

**Prediction of disturbances**

An essential aspect of reducing the acoustic disturbances is the ability to predict them. Let us examine the prediction of random signals in more detail. Assume an output like (5.21) with \( \gamma \) as a random sequence. Next, consider the identity

\[
C(q^{-1}) = M(q^{-1})A(q^{-1}) + q^{-i}N(q^{-1})
\]

\[
\text{with} \quad M(q^{-1}) = m_0 + m_1q^{-1} + \cdots + m_{i-1}q^{-i+1}
\]

Note that \( M \) in fact consists of the first \( i \) points of the pulse response \( C/A \). Multiplying (5.21) by \( Mq^i \) and insertion of (5.25) yields

\[
Cy(k+i) = Ny(k) + MC\gamma(k+i)
\]

(5.26)

Here we have a relation for the future value of \( y \) at \( k+i \). The first term of the right hand side is completely known at \( k \), whereas the second term \( M\gamma \) is completely unknown and of random character. So the optimal \( i \)-step-ahead prediction of \( y(k) \) is given by

\[
y^o(k+i) = \frac{N}{C} y(k)
\]

(5.27)

with the prediction error

\[
e(k+i) = y(k+i) - y^o(k+i) = M\gamma(k+i)
\]

(5.28)

which is minimum with respect to a quadratic criterion. Here, the polynomial \( C \) is assumed to be stable for convenience. When \( \gamma(k) \) is a white noise sequence the minimum prediction error is an \( i \)-points moving average of the original sequence. Alternatively, if the disturbance generator is a sequence of random spikes, the prediction is perfect except for the events of a new spike. Finally, the prediction error vanishes for deterministic signals.

In the case of a measurable disturbance the system equations are assumed to read

\[
y(k) = D E x(k-d)
\]

\[
x(k) = G H y(k)
\]

(5.29)

with \( D \) and \( G \) assumed stable. The optimal prediction of \( y(k+i) \) requires the following two Diophantine equations

\[
D = M_1E + q^{-i}N_1
\]

\[
M_1G = M_2H + q^{-i+d}N_2
\]

(5.30)
in which again the $M_1, M_2, N_1$ and $N_2$ polynomials are the minimal degree solutions. It can be shown that insertion of eq. (5.30) in eq. (5.29) eventually leads to

$$y^e(k+i) = \frac{N_1}{D} y(k) + \frac{N_2}{G} x(k)$$

(5.31)

with prediction error

$$\varepsilon(k+i) = M_2 \gamma(k + i - d)$$

(5.32)

an $(i-d)$ points moving average of the random noise sequence. In general, this prediction error is smaller than that of eq. (5.28); especially for $G = H = 1$, $D = C$, $E = A$ and $d > 0$. The mathematical manipulations above illustrate the conspicuous notion that advance information on $\gamma$ improves its predictability. Note that if $d \geq i$ the polynomial $M_2$ in eq. (5.30) equals zero and so the prediction error (5.32) vanishes.

### 5.5 Feedforward and feedback control of disturbances

Throughout this Chapter a clear distinction has been made between measurable and unmeasurable disturbances. The mere object of the control system is to cancel the total disturbance signal - realise that active sound control systems usually have no servo task and therefore face a pure regulator problem. In the presence of the two types of disturbances the structure of the regulator is as depicted in Figure 5.10.

![Diagram](image)

**Figure 5.10:** Structure of acoustic controller to cancel disturbance signals at the output.
The feedforward link is to provide a direct reduction of the measurable disturbance at the output. Note that in acoustic systems the majority of disturbances can be measured in advance. In the structure of Figure 5.10 the sensor giving a priori information is positioned inside the acoustic process. In certain cases it is possible to place the sensor outside the acoustic plant and it is therefore no longer receptive to the anti-sound. In the configuration of Figure 5.10 the detection sensor measures both primary and secondary sound. This problem, however, will be dealt with later. For the moment we will assume that the advance disturbance information is not contaminated.

If we look at the reduction mechanism of the control system, the feedback link takes care of the extraneous part of the disturbance signal. The performance of the feedback link depends on the auto-correlation characteristics of the output signal. For the feedforward link the cross-correlation between the measured disturbance and the output signal determines the achievable attenuation. Apart from their individual control performance a distinct feedforward and a distinct feedback controller show quite a different behaviour in the case of perturbations. Consider a pure feedback and a pure feedforward controller as shown in Figure 5.11.

![Block diagram of (a) a feedback controller and (b) a feedforward controller. D stands for the disturbance process, H for the controlled process, K for the feedback controller, F for the feedforward controller, x for the measured disturbance and y for the output signal.](image)

The output signal \( y_{fb} \) for the feedback control system can be written as

\[
y_{fb} = \frac{D}{1 + KH} x
\]

(5.33)

From this expression it follows that the sensitivity of the closed-loop system to disturbances is given by

\[
\sigma(z^{-1}) = \frac{1}{1 + K(z^{-1})H(z^{-1})}
\]

(5.34)
where $\sigma$ denotes the \textit{sensitivity function}. For a substantial reduction of unknown disturbances the loop gain $KH$ should be high. In that respect the feedback controller is less sensitive to the unmodelled disturbances than the feedforward controller. We find for $y_{ff}$:

$$y_{ff} = (D + FH)x$$

(5.35)

This implies that every extraneous disturbance - a change in $Dx$ - arrives at the output unaffected. For a good performance a feedforward controller requires accurate knowledge of the acoustic process. The sensitivity function of this controller equals unity.

The sensitivity function is closely related to the issue of robustness. Suppose that the transfer function $H$ is perturbed with $\Delta H$. Then, the relative change in performance for the feedback controller is

$$\lim_{\Delta H \to 0} \left| \frac{\Delta Y_{fb}}{Y_{fb}} \frac{H}{\Delta H} \right| = \left| \frac{KH}{1 + KH} \right|$$

(5.36)

The resultant fraction is called the \textit{complementary sensitivity function} $\tau$ for the obvious reason that

$$\tau(z^{-1}) = \frac{K(z^{-1})H(z^{-1})}{1 + K(z^{-1})H(z^{-1})} = 1 - \sigma(z^{-1})$$

(5.37)

So in order to keep the sensitivity of the performance to process changes small, we should have a low value of $\tau$. Contrarily, to keep the sensitivity of the output small, we should have a low value of $\sigma$. Note that for a high loop gain $\tau$ equals unity - a good reduction implies a high sensitivity to process perturbations.

The performance sensitivity of the feedforward controller to changes in $H$ looks much more unfavourable

$$\lim_{\Delta H \to 0} \left| \frac{\Delta Y_{ff}}{Y_{ff}} \frac{H}{\Delta H} \right| = \left| \frac{FH}{D + FH} \right|$$

(5.38)

This fraction is inversely proportional to the residual level. This means that for feedforward controller the performance sensitivity is bound to be larger than unity.

From the analysis on performance robustness we see that performance and performance robustness are rather conflicting design requirements. The performance robustness of a feedback controller usually is better than that of a feedforward controller. Contrarily, the feedforward controller achieves a better performance for measurable disturbances. A second important issue related to perturbations is the stability robustness. A feedforward controller cannot become instable as long as the control transfer function is stable; an open-loop control property. In a closed loop, however, this no longer holds. Even if a feedback controller is designed properly with respect to stability, perturbation of the process may cause instability. It can be demonstrated - see e.g. [Kw93] - that stability under perturbation of $H$ is maintained if
$$\left| \frac{\Delta H(z^{-1})}{H(z^{-1})} \right| |\tau(z^{-1})| < 1$$

(5.39)

Here again, we note the important role of the complementary sensitivity function $\tau$. A high loop gain or a low sensitivity function gives the control system a poor stability robustness. So, designing a feedback controller on performance and robustness leads to a mixed-sensitivity problem.

If a feedforward and a feedback controller are combined both performance and robustness demands may be served. Without proving this allegation for the moment, it is expected that:

a) The addition of a feedback link to a feedforward controller can reduce the sensitivity to perturbations, while maintaining a good performance. So this improves the performance robustness.

b) The addition of a feedforward controller to a feedback controller can give a high reduction without demanding a high feedback gain. This improves the stability robustness.

Note that even for mere measurable disturbances an additional feedback loop is advisable in the case of changing plant dynamics.

5.6 The overall model for the acoustic system

Based on the models for the acoustic processes and disturbances in Sections 5.3 and 5.4, the overall system model can be set up. The output signal $y(k)$ consist of three parts: the contribution to the control signal $y_u(k)$, to the measurable disturbance $y_x(k)$ and to the unmeasurable or extraneous disturbance $y_\xi(k)$

$$y(k) = y_u(k) + y_x(k) + y_e(k)$$

(5.40)

The individual transfer function relations are given by

$$y_u(k) = q^{-b} \frac{B_o(q^{-1})}{A_o(q^{-1})} u(k)$$

(5.41)

$$y_x(k) = q^{-d} \frac{D_o(q^{-1})}{E(q^{-1})} \xi(k)$$

(5.42)

$$y_\xi(k) = \frac{C_o(q^{-1})}{A_o(q^{-1})} \xi(k)$$

(5.43)

where $u(k)$ is the control signal, $\xi(k)$ the measurable disturbance and $\xi(k)$ the generator for the unmeasurable disturbance. All polynomials are expressed in the backward shift operator $q^{-1}$ and have degree $n_A, n_B$ and so on. Furthermore, the polynomials $A_o, C_o$ and $E$ are monic.
(leading coefficient equals unity) and stable. Like in Section 5.3 the denominator polynomial \( A_o \) is to represent the reverberant characteristics of the acoustic volume. Therefore, \( A_o \) has to be part of any transfer function relating a source to a sensor. The measurable disturbance \( x(k) \) is modelled as

\[
x(k) = \frac{G(q^{-1})}{H(q^{-1})} \gamma(k)
\]

(5.44)

where \( \gamma(k) \) is a disturbance generator. It is assumed that \( \xi(k) \) and \( \gamma(k) \) are mutually uncorrelated. The polynomial \( G \) is monic and stable, whereas \( F \) and \( H \) are monic but may have zeros on the unit circle. Furthermore, the polynomial \( A_o \) is part of \( EH \). In practice, the measurement of disturbance \( x(k) \) may be contaminated by the output of the tertiary acoustic process, resulting in \( z(k) \)

\[
z(k) = x(k) + q^{-\ell} \frac{L(q^{-1})}{A_o(q^{-1})} u(k)
\]

(5.45)

The tertiary process is modelled by a delay \( \ell \) and the enclosure’s acoustics given by \( A_o \). In Figure 5.12 a block diagram of the acoustic plant is depicted.

**Figure 5.12:** Diagram of the general model for the acoustic system.
The polynomials \( E, H \) and \( L \) are completely determined by the type of detection sensor used. In Section 5.4 it was demonstrated that an optical or magnetic sensor could be used in the case of rotating equipment. If we make a rough discrimination between acoustical and non-acoustical detection sensors, the following holds for the relevant polynomials

- If the detection sensor is acoustic and positioned within the acoustic plant, \( H \) should contain polynomial \( A_o \). The primary acoustic process, \( q^{-d}Dq/E \), gives the relation between the disturbance signal at the detection sensor and error sensor. So, we can only specify \( H \) as

\[
H = A_o H_0 \quad (5.46)
\]

- If the detection sensor is non-acoustic, the primary acoustic process has to contain the reverberant characteristics. Moreover, we assume that the detection sensor is insensitive to any secondary sound.

\[
H = H_0; \quad E = A_o; \quad L = 0 \quad (5.47)
\]

Just like \( F \) the polynomial \( H_0 \) may have zeros on the unit circle in order to model drifting or deterministic disturbances. Therefore, we divide them into the product of a stable (having all zeros in the stability region) and an unstable part (having all zeros on or beyond the stability boundary): \( F = F_s F_u \) and \( H_0 = H_s H_u \).

The linear, discrete-time model for the acoustic plant can be written as

\[
A(q^{-1})y(k) = q^{-b}B(q^{-1})u(k) + q^{-d}D(q^{-1})x(k) + C(q^{-1})n(k) \quad (5.48)
\]

where \( x(k) \) is given by eqs. (5.44) and (5.45) and the signal \( n(k) \) by

\[
n(k) = \frac{1}{F_u(q^{-1})} \xi(k) \quad (5.49)
\]

The basic polynomials \( A, B, C \) and \( D \) are related to the original transfer functions as

\[
A = F_s E A_0; \quad B = F_s E B_0; \quad C = E C_0; \quad D = F_s A_0 D_0 \quad (5.50)
\]

Here we note that polynomials \( A \) and \( B \) have the stable common factor \( F_s E \). The model (5.48) will serve as the model of a uniform plant for which control systems for active sound reduction are to be developed.

As a matter of convention polynomials are expressed in terms of the backward shift operator \( q^{-1} \). In specific cases other forms of notation may be useful. The following types of polynomial representations are defined

<table>
<thead>
<tr>
<th>( M )</th>
<th>( M(q^{-1}) = m_0 + m_1 q^{-1} + \cdots + m_m q^{-m} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>conjugate polynomial:</td>
<td>( M_*(q) = m_0 + m_1 q + \cdots + m_m q^m )</td>
</tr>
<tr>
<td>reciprocal polynomial:</td>
<td>( \overline{M}(q^{-1}) = q^{-m} M_*(q) = m_0 q^{-m} + m_1 q^{-m+1} + \cdots + m_m )</td>
</tr>
</tbody>
</table>
Section 5.7 The LQG control strategy

For a transition to the frequency domain the operator \( q \) is replaced by the argument \( z \). In many cases the arguments \( q^{-1} \), \( q \), \( z^{-1} \) and \( z \) are omitted to improve the legibility of expressions.

5.7 The LQG control strategy

In this Section a control strategy is discussed, which can be considered to yield the optimal controller under the assumptions made earlier. Let us define the cost function

\[
J = \lim_{N \to \infty} \frac{1}{2N} \sum_{k=0}^{N} \left( E \left[ |P(q^{-1})y(k)|^2 \right] + \rho E \left[ |Q(q^{-1})F_u(q^{-1})u(k)|^2 \right] \right)
\]  

where both the time-average and the expectation operator are included to deal with deterministic and random signals simultaneously. This cost function represents the time-averaged energy of the filtered output signal, \( Py \), and the filtered control signal, \( QF_uu \). The need for having \( F_u \) in the criterion is elucidated later on. The weighting of control effort is additionally scaled by the parameter \( \rho \).

Since the complete range of sample instants is included in (5.52), the cost function is an infinite-horizon criterion. The control strategy is known as Linear Quadratic Gaussian (LQG) control. This name refers to a linear controller minimising a quadratic criterion, where disturbances of Gaussian nature may occur in the process model. The infinite-stage nature of the criterion will be shown to complicate the control design.

As preserved throughout the thesis, polynomial models are used. The polynomial LQG theory has only been described recently in e.g. the work of Sternad [St87]. Since the theory described in there is well applicable to the acoustic situation, we will cite from Sternad's work in this Section and look at the consequences for the acoustical case. In his thesis Sternad [St87] proposes a regulator with the structure

\[
R(q^{-1})F_u(q^{-1})u(k) = -\frac{W(q^{-1})}{V(q^{-1})} x(k) - S(q^{-1})y(k)
\]

(5.53)

This specific choice - the feedback poles \((RF_u)\) are also part of the feedforward regulator - makes it possible to solve the feedforward and feedback design problems separately.

An important role in finding the solution to the optimal regulator of the type (5.53) is played by the stable, monic polynomial \( \beta \), given by the spectral factorisation

\[
\mu \beta \beta^* = BB^* + \rho AF_u Q Q^* F_u A^*
\]

(5.54)

in which \( \mu \) is a constant and \( \rho \) is the scaling factor of the control effort weighting. Next, if the polynomials \( W, V, R \) and \( S \) are the unique solution of the set of equations
\[
\begin{align*}
\mu \beta_r R - q^{-b+1}BN_s &= \rho \Omega Q_s F_u A_u C \\
\mu \beta_s S + qAF_u N_s &= q^b B_s C \\
V &= G \\
q^{-d+1}D F_u G N_s &= \mu \beta_s W + qCH M_s
\end{align*}
\] (5.55)

the regulator (5.53) attains the global minimum of (5.52), with \( P = 1 \), under the following constraints:

a) The polynomials \( \beta, C \) and \( G \) are stable. This condition is inherently fulfilled, as these polynomials are stable by definition. It will turn out that the closed-loop poles are placed at \( \beta C \) or \( \beta CG \), which explains this constraint.

b) The polynomials \( AF_u \) and \( B \) have no unstable common factors. In other words, system (5.48) with \( \xi(k) \) inserted is completely stabilisable and detectable.

c) The polynomial \( H_u \) is a factor of \( Q F_u D \); this is elucidated later on.

Note that the orders of controller polynomials \( W, V, R \) and \( S \) follow implicitly from (5.55).

**Characteristics of the LQG regulator**

The polynomial solution of the LQG criterion has a number of striking features, which eventually apply to control in general as well.

1) The optimal regulator yields the closed-loop behaviour

\[
y(k) = \frac{q^{-d}D F_u V - q^{-b}B W}{V[AF_u + q^{-b}BS]} x(k) + \frac{CR}{ARF_u + q^{-b}BS} \xi(k)
\] (5.56)

where the control action obeys

\[
u(k) = -\frac{q^{-d}D S V + AW}{V[AF_u + q^{-b}BS]} x(k) - \frac{CS}{ARF_u + q^{-b}BS} n(k)
\] (5.57)

The role of spectral factor \( \beta \) becomes apparent when evaluating the closed-loop poles, as given by eqs. (5.56) and (5.57). With the use of eq. (5.55) the denominator polynomial equals

\[
AF_u R + q^{-b}BS = \beta C
\] (5.58)

So the closed-loop system is stable by definition (requirement a) above). Note that the feedforward controller introduces additional stable poles in the roots of \( G \). On behalf of equality (5.58) the LQG strategy may be interpreted as pole placement control in \( \beta C \). Moreover, if \( AF_u \) and \( B \) have no common factors, the equation can be used for the (fast) calculation of the controller polynomials \( R \) and \( S \).

2) One may have noticed the care with which the unstable factors \( F_u \) and \( H_u \) are to be dealt.
• Polynomial $F_u$ has to be part of the regulator’s denominator polynomial (5.53) in order to be capable of eliminating the unstable part of the extraneous disturbance. The fact that disturbance dynamics have to be part of the regulator is often referred to as the *internal model principle*.

• As a consequence, $F_u$ has to be part of the weighting polynomial of $u(k)$ in the cost function (5.52) in order to keep the criterion value finite.

• The factor $H_u$ of the measurable disturbance has to divide $Q F_u D$, again to keep the criterion within finite bounds; [St91].

3) For the special structure of the controller (5.53) the feedback polynomials $R$ and $S$ have no influence on the optimal reduction of the measurable disturbance, as long as the closed-loop is stable. For any $R$ and $S$ a set of $V$ and $W$ can be found that reach the minimum criterion value (5.52) for $\xi(k) = 0$.

• With respect to the performance (degree of disturbance rejection) feedforward control can be done without a feedback link, if the system is stable - which acoustic systems usually are - and the system model is accurate.

So for the stable system

$$Ay(k) = Bu(k - b) + Dx(k - d)$$

(5.59)

it is never advantageous to use pure feedback control, based on

$$AHy(k) = BHu(k - b) + DG\gamma(k)$$

(5.60)

compared with pure feedforward control. Obviously, for non-zero delay $d$ the feedforward filter has more recent information on the disturbance. The higher $d$ the better the disturbance can be predicted and eliminated. For $d = 0$, however, the feedforward action cannot improve the performance of feedback and vice versa.

• In the absence of feedback control - with $R = 1$, $S = 0$ and $\xi(k) = 0$ - the optimal feedforward regulator $W$, $V$ can be found as

$$V = \beta G$$

$$q^{-d+b}B_\ast DG = \mu_\ast W + qAHM_\ast$$

(5.61)

The characteristics of the optimal feedforward regulator demand solving a spectral factor (5.54) and a Diophantine equation (5.61).

An LQG controller mainly focuses on performance. Although a control designer may improve the robustness by tuning the $\rho$ and $W$, it is not the main control objective. So in the LQG control solution we observe a feedforward link for the measurable disturbance and a feedback link for the extraneous disturbance. As follows from Section 5.5, however, for the relation between feedforward and feedback control it can also be asserted that
• Feedforward control can improve the stability robustness of a feedback controller, as the high gain of the feedback loop for disturbance rejection is no longer required.

• Feedback control should accompany a feedforward controller in practice because significant unmeasurable disturbances may be present. Moreover, a feedback link may be used to improve the performance robustness. In that case the feedback loop acts like a safety net for the effect of changing system dynamics.

This relation is discussed in Chapters 6 and 7 in further detail.

**Minimum variance solutions**

It is of special interest to know the lower bound on the output signal's energy. This limit corresponds to the minimum of the cost function

\[ J_{MV} = \lim_{N \to \infty} \frac{1}{2N} \sum_{k=0}^{N} E\{ y^2(k) \} \]  

(5.62)

the minimum variance (MV) criterion; note that \( P = 1 \) and \( \rho Q = 0 \) in (5.52). As demonstrated in [St87], minimum variance control gives rise to the closed-loop system response

\[ y(k) = \frac{M_1}{B^-} \gamma(k - \min\{b, d\}) + \frac{R_1}{B^-} \xi(k) \]

(5.63)

where \( M_1 \) and \( R_1 \) satisfy

\[
\begin{align*}
B^- C &= AFR_1 + q^{-b} \mu^2 B^- S \\
\mu^{-d+b-1} R_1 F_{u} D G &= q^{-1} \mu^2 B^- W + CHM_i \\
\text{with} & \quad B = \mu^2 B^+ B^-; \quad t = \max\{0, b - d\} \\
& \quad \beta = B^+ B^-
\end{align*}
\]

(5.64)

From these relations we conclude:

• Perfect cancellation of the measurable disturbance implies \( M_1 = 0 \). By inspection of eq. (5.64) this requires \( d \geq b \) and \( B^- \) is part of \( D \). Acoustically, these are rather obvious conditions. First, the information of the detection sensor should lead that of the error sensor. Second, the secondary process has to be minimum phase to enable a stable inversion. The unstable factor \( B^- \) to divide \( D \) is very unlikely to be the case in acoustics.

• For a minimum phase secondary process, with a large delay, \( b > d \), the minimum variance output signal is a moving average of the disturbance generators - \( \{ M_1 \gamma(k-d) + R_1 \xi(k) \} \). Note that \( M_1 \) contains the first \( b-d \) components of disturbance model \( DG/\lambda H \), whereas \( R_1 \) contains the first \( b \) components of model \( C/\lambda F \).

• All stable zeros of \( B \) are cancelled. For the unstable part \( B^- \) a denominator \( B^- \) is introduced in order to create a spectrally neutral factor; \( |B^- (z)|/|B^- (z)| = 1 \) for all \( z \).

• It should be realised that minimum variance LQG control focuses on **performance** - disturbance attenuation - only. The sensitivity function \( \sigma(z^{-1}) \) - as introduced in Section
5.5 - of an LQG controlled system is bound to be small. This implies that small changes in the acoustic process will have a serious influence on the performance and may possibly lead to instability. This characteristic will be experimentally demonstrated in Chapter 7.

5.8 References

[Su91] Sutton, T.J. et al. (1991), Use of Non-Linear Controllers in the Active Attenuation of Road Noise inside Cars, pp. 682-690 in Recent Advances in Active Control of Sound and Vibration, Technomic Publishing.
CHAPTER 6

Self-Tuning Control Strategies

This Chapter presents control strategies that are suitable to be implemented in a real-time environment. In general, the strategies for active sound reduction are based on a couple of simplifications applied to the LQG control strategy in Chapter 5. The main step will be the substitution of the infinite-stage criterion by a single-stage criterion. It will be shown that this may lead to a (minor) deterioration of the LQG performance and problems with the system stability. The performance robustness of the control strategies is enhanced by casting them into a self-tuning form. Also, the commonly used (self-tuning) control strategies for active sound reduction are compared with the proposed schemes.

6.1 Single-stage criterion functions

In Chapter 5 a system equation was derived relating the signals from the error sensor and the detection sensor to the control signal and the disturbance generators. Additionally, a feedforward and a feedback link were proposed in order to obtain robust rejection of the total disturbance at the system output. In Figure 6.1 this basic set-up is depicted.

![Diagram of signals, feedforward and feedback regulator and acoustic process.](image-url)
The design of the regulators was based on minimising the criterion

\[
J_w = \lim_{N \to \infty} E \left\{ \frac{1}{2N} \sum_{i=0}^{N} [P(q^{-1})y(i)]^2 + \lambda[Q(q^{-1})u(i)]^2 \right\}
\]  \hspace{1cm} (6.1)

where for the sake of simplicity we assume that \(F_u = 1\) - no unstable disturbances are present. The strategy of minimising the above function with respect to the regulator parameters is generally referred to as LQG control. The criterion is an infinite-stage function since the variance of \(Py\) and \(Qu\) is included for the complete (up to infinity) time-horizon. As indicated in the previous Chapter, finding the solution for the LQG-feedforward/feedback regulator is a tedious operation. In Section 5.7 a spectral factorisation and three coupled polynomial equations were shown to represent the (implicit) LQG solution. It is therefore most suitable for an off-line regulator design. A periodical update of the regulator characteristics may, however, be required in the case of slowly\(^1\) time-varying acoustic processes. In this Section we will investigate whether it is possible to define a much simpler control strategy without losing too much of the attractive LQG-performance. Instead of the infinite-stage criterion (6.1) consider

\[
J_b = \frac{1}{2} E \left\{ \sum_{i=k}^{k+b} [P(q^{-1})y(i)]^2 + \lambda[Q(q^{-1})u(i)]^2 \right\}
\]  \hspace{1cm} (6.2)

which uses a time-horizon equal to the secondary process delay \(b\). As mentioned in the previous Chapter this delay represents the acoustic travel time from secondary source to error sensor together with delays in AD/DA converters and filters. Also, the well-known one-sample computational delay in the controller is included in \(b\).

In (6.2) the expectation operator is unconditional. This type of criterion was proposed by Clarke and Gawthrop in the mid 70's [CG75]. It should be mentioned that the criterion is used to find \(u(i)\) for \(i = k\) only; determining future values of \(u(i)\) with \(i > k\) using (6.2) would be meaningless. On the other hand, the filtered output \(Py(i)\) is not influenced by \(u(k)\) for \(i < (k + b)\). So the criterion can also be written as

\[
J_{GMV} = \frac{1}{2} E \left\{ [P(q^{-1})y(k+b)]^2 + \lambda[Q(q^{-1})u(k)]^2 | y(i), y(t-1), \ldots; u(t-1), u(t-2), \ldots \right\}
\]  \hspace{1cm} (6.3)

where in contrast with eqs. (6.1) and (6.2) the expectation operator is conditional. The suffix GMV stands for Generalized Minimum Variance like the control strategy referred to in [CG75]. From (6.3) we conclude that the GMV strategy is concerned with a single-stage criterion. It will be demonstrated that for this criterion it is much easier to find the optimal regulator. The weighting polynomials \(P\) and \(Q\) in general consist of a numerator and a denominator polynomial

\[\text{1. By slowly time-varying we mean that the changes in process parameters occur on a time-scale much larger than the process response time.}\]
\[ P(q^{-1}) = \frac{P_n(q^{-1})}{P_d(q^{-1})}; \quad Q(q^{-1}) = \frac{Q_n(q^{-1})}{Q_d(q^{-1})} \]

Without loss of generality \( P_n, P_d \) and \( Q_d \) are assumed monic. The main objective of the \( Q \)-polynomial in GMV control is to retain system stability. As will be demonstrated in the forthcoming analysis, it is obligatory to have \( \lambda_1 Q \neq 0 \) if the secondary process is of non-minimum phase. The weighting polynomials \( P \) and \( Q \) can also be used to provide a desirable closed-loop behaviour. Usually, a desirable closed-loop behaviour is accomplished by having poles close to the origin with a positive real part. In that case the overshoot, the rise time and the settling time are within reasonable limits.

**minimisation of the GMV cost function**

Recall from Section 5.6 that the system equation reads

\[ A(q^{-1})y(k) = B(q^{-1})u(k-b) + D(q^{-1})x(k-d) + C(q^{-1})\xi(k) \quad (6.4) \]

in which for the moment we leave out the unstable disturbance model polynomial \( F_u \) for the sake of simplicity. If we examine the criterion (6.3) it follows from the system equation that \( Py(k+b) \) is unknown at sample instant \( k \). Therefore, it has to be predicted using the available signal samples up to instant \( k \). Consider the polynomial identity

\[ \frac{P_n}{P_d} \frac{C}{A} = M + q^{-b} \frac{N}{P_d A} \quad (6.5) \]

where the polynomial \( M \) is monic and of the order \( (b - 1) \). The \( M \) and \( N \) polynomials are equivalent to those that emerged in Section 5.4 in the context of prediction of disturbances. Having \( M \) operate on the system equation (6.4) and applying (6.5) yields

\[ C \Phi_y(k+b) = \frac{N}{P_d} y(k) + MBu(k) + MDx(k-d+b) + MC\xi(k+b) \quad (6.6) \]

with \( \Phi_y(k+b) = \frac{P_n}{P_d} y(k+b) \)

The auxiliary function \( \Phi_y(k+b) \) is introduced just for convenience. Since \( M\xi(k+b) \) is fully unpredictable at time \( k \), we have for the optimal\(^{(*)} \) prediction of \( \Phi_y(k+b) \)

\[ C \Phi_y^*(k+b) = \frac{N}{P_d} y(k) + MBu(k) + MDx(k-d+b) \quad (6.7) \]

where the prediction error is

\[ \epsilon(k+b) = \Phi_y(k+b) - \Phi_y^*(k+b) = M\xi(k+b) \quad (6.8) \]

If we divide \( Py(k+b) \) into its optimal prediction and related prediction error and insert this in to the criterion function, we obtain
\[ I_{\text{GMV}} = \frac{1}{2} \left[ \Phi_{y}(k+b) \right]^2 + \lambda_1 [Q(q^{-1})u(k)]^2 + \frac{1}{2} \mathbb{E} \left[ M_k^2(k+b) \right] \]  

(6.9)

Note that the conditional expectation operator has been dropped for the first two terms, as these are completely deterministic at instant \( k \). Differentiating (6.9) with respect to \( u(k) \) gives

\[
\frac{\partial I_{\text{GMV}}}{\partial u(k)} = [\Phi_{y}(k+b)] \frac{\partial}{\partial u(k)} [\Phi_{y}(k+b)] + \lambda_1 [Q(q^{-1})u(k)] \frac{\partial}{\partial u(k)} [Q(q^{-1})u(k)]
\]

(6.10)

Evaluating the derivatives with the help of eq. (6.7), the requirement for a stationary point of \( I_{\text{GMV}} \) reads

\[
0 = \frac{N}{P_d} y(k) + [MB + \lambda QC] u(k) + MD x(k-d+b)
\]

with \( \lambda = \frac{\lambda_1 g_{n,0}}{b_0} \)

(6.11)

In fact, the GMV criterion is minimised under condition (6.11); see [CG75]. In this publication on GMV control it is also demonstrated that minimising (6.3) is equivalent to minimising

\[
J = \frac{1}{2} \mathbb{E} \left[ (P(q^{-1})y(k+b) + \lambda Q(q^{-1})u(k))^2 \right] y(t), y(t-1), \ldots, u(t-1), u(t-2), \ldots
\]

(6.12)

Given the solution (6.11) means that, when the general feedforward-feedback regulator

\[
R(q^{-1})u(k) = -\frac{W(q^{-1})}{V(q^{-1})} x(k) - S(q^{-1}) y(k)
\]

(6.13)

is designed using the GMV control strategy, the regulator polynomials can be found to be

<table>
<thead>
<tr>
<th>GMV control</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R = MB + \lambda QC ); ( S = N / P_d )</td>
</tr>
<tr>
<td>( W = q^{-d+b} MD ); ( V = 1 )</td>
</tr>
</tbody>
</table>

(6.14)

**Remarks**

- The fractional solution for \( S \) can be dealt with by defining a filtered output signal \( y^f(k) \)

\[
y^f(k) = \frac{1}{P_d(q^{-1})} y(k)
\]

(6.15)

and having \( S \) operate on \( y^f(k) \).

- Applying the GMV solution (6.14) gives the following closed-loop behaviour

\[
y(k) = \frac{\lambda DQ}{BP + \lambda AQ} x(k-d) + \frac{BM + \lambda CQ}{BP + \lambda AQ} \xi(k)
\]

(6.16)

From this expression the role of the weighting polynomials \( P \) and \( Q \) becomes apparent. Together with the secondary process \( (B \text{ and } A) \) they determine the closed-loop pole locations. So \( P \) and \( Q \) can be employed to create a suitable disturbance response. The main role of \( Q \), however, is concerned with the system stability. The Minimum Variance control strategy \( (P = 1, \lambda Q = 0) \) requires that \( B \) is minimum phase. When controlling
non-minimum phase processes one has to tune $Q$ such that the closed-loop response becomes stable. This can be carried out beforehand after estimation of the process models or - if necessary - $Q$ can be tuned on-line to maintain stable control action. The weighting on $u(k)$ prevents the control effort to become exceedingly large.

- The essence of the GMV control strategy is the $b$-step-ahead prediction of the filtered output signal $Py(k)$. The control signal is chosen such that the optimal prediction of the term $[Py(k + b) + \lambda Qu(k)]$ equals zero. Stability, however, is not guaranteed by just setting $\lambda Q$ equal to small constant. This can be ascribed to the definition of the criterion, in which only the current control effort is weighted. This is a clear disadvantage of single-stage criterion functions.

- When minimum variance control is feasible, the solution for the controller reduces to

$$
\begin{align*}
\text{MV control} \\
R &= MB; \\
S &= N \\
W &= q^{-d+b}MD; \\
V &= 1
\end{align*}
$$

(6.17)

This solution coincides with LQG-solution for minimum variance control and a minimum phase secondary process; see eq. (5.64). Apparently, it does not make any difference to minimise the squared $b$-step-ahead-prediction of $y(k)$ or the infinite-horizon energy of $y(k)$. Clearly, the MV controller is unstable if $B$ contains non-minimum phase zeros. In the following subsection a variant of the GMV control strategy will be described that retains the LQG characteristics when $B$ is not stably invertible.

- In the derivation of the GMV controller it was tacitly assumed that the primary delay $d$ exceeded the secondary delay $b$; delay $d$ is mainly concerned with the acoustic travel time from detection to error sensor. As a result of this assumption the measurable disturbance response at the output sensor $[D/A x(k - d + b)]$ was completely predictable using the detection signal $x(k)$. This assumption may be too optimistic. In general we have to predict this part of the output $y(k + b)$ as well. In particular, the optimal prediction of $\Phi(k+b)$ in eq. (6.7) is not realisable as it may contain future values of $x(k)$. Consider the Diophantine equation

$$
\frac{MD}{C} \frac{G}{H} = M_1 + q^{-b+d} \frac{N_1}{CH}
$$

(6.18)

in which $M_1$ has degree $(b - d - 1)$. If $(b - d) > 0$ insertion of this polynomial equation into the optimal prediction (6.6) yields

$$
C \Phi_y^*(k + b) = Ny^*(k) + MBu(k) + \frac{N_1}{H} y(k) + CM_1 y(k + b - d)
$$

(6.19)

Note that $y(k)$ is the generator of the measurable disturbance; $x = G/H y$. In the $b$-step-ahead prediction the last term consists of future values of $y(k)$ only. This implies that the prediction should be written as
\[ C \Phi_y'(k+b) = Ny^p(k) + MBu(k) + \frac{N_i}{G} x(k) \]  
\[ (6.20) \]

This in turn leads to the GMV controller for a large secondary delay:

<table>
<thead>
<tr>
<th>GMV control</th>
</tr>
</thead>
</table>
| \[ R = MB + \lambda QC; \]  
| \[ S = N / P_d \]  
| \[ W = N_i; \]  
| \[ V = G \]  
\[ (6.21) \]

- In order to design a GMV regulator knowledge of the system dynamics is required. An essential role in the design is played by the secondary delay \( b \). If the delay is underestimated the controller is likely to get unstable. This follows directly from the structure of the criterion (6.2). If \( b \) is smaller than the true secondary delay, one tries to minimise the output signal without experiencing the response of the chosen control action. A minor modification of the GMV strategy, however, can handle uncertainty in the secondary delay; see [Ta85].

**weighted minimum variance control**

The characteristics of the GMV strategy as discussed above, clearly show that the stability of the regulator can only be guaranteed under special conditions. In the case of a non-minimum phase secondary process, the weighting polynomial \( Q \) has to be designed such that the roots of \([BP+ \lambda AQ] \) are placed inside the unit circle. This can be a cumbersome approach, since it requires accurate knowledge of \( B \) and \( A \). Moreover, these system polynomials may (slowly) vary with time and this would imply an on-line design of the weighting polynomials to retain system stability. In [Gr81] a very attractive way is proposed to deal with non-minimum phase processes without having to tune \( Q \). Consider a factorisation of the \( B \) polynomial

\[ B(q^{-1}) = B^+(q^{-1})B^-(q^{-1}) \]  
\[ (6.22) \]

where \( B^+ \) contains all stable and the monic \( B^- \) contains all unstable zeros of \( B \). The idea behind the **Weighted Minimum Variance** control strategy is to include \( B^- \) in the denominator of \( P \), such that detrimental \( B^- \) is no longer part of the closed-loop poles; see eq. (6.16). Consider a revised form of Diophantine equation (6.4)

\[ \frac{B^-}{B^+} = \frac{P_n C}{P_d A} = \frac{M}{B^-} + q^{-b} \frac{N}{P_d A} \]  
\[ (6.23) \]

in which again \( M \) is monic and of order \((b - 1)\). Note that the reciprocal \( B^- \) is included in the weighting on the output \( y \) to neutralise the spectral properties of \( B^- \). Carrying on with the derivation, the auxiliary function \( \Phi_y(k+b) \) is modified into

\[ B^- C \Phi_y(k+b) = B^- Ny^p(k) + B^- MB^+ u(k) + MDx(k-d+b) + MC \xi(k+b) \]

with \[ \Phi_y(k+b) = \frac{B^- P_n}{B^- P_d} y(k+b) \]  
\[ (6.24) \]
Again, the optimal prediction of the auxiliary function would be to leave out the contribution of $\xi$ in (6.24), because this would imply for the prediction error

$$
e(k + b) = \Phi_y(k + b) - \Phi_y^*(k + b) = \frac{M}{B^-} \xi(k + b)$$

(6.25)

If we interpret $B^-$ as a stable, anti-causal polynomial, the prediction error contains only future components of $\xi(k)$. Pursuing the control design as in the GMV case the regulator polynomials eventually turn out to be

$$
\begin{align*}
WMV \text{ control} \\
R &= MB^+ + \lambda QC; \quad S = N / P_d \\
W &= N_1; \quad V = G
\end{align*}
$$

(6.26)

in which the $N_1$ polynomial is the solution of the equation

$$q^{-d + b - n} \frac{MD}{B^- C H} = \frac{M_1}{B^-} + q^{-n} \frac{N_1}{CH}$$

with $n = \max\{0, b - d\}$; $\deg(M_1) = \deg(B^-) - 1 + n$

(6.27)

It can be verified that the WMV control strategy, as given by its solution (6.26), is identical to LQG control for the minimum variance criterion; $P = 1$, $\lambda Q = 0$. We conclude that WMV control solves the problem of designing $Q$ when $B$ is of non-minimum phase, at the cost of a factorisation of $B$.

**feedforward minimum variance control**

In the case of absence of an extraneous disturbance signal - $C = 0$ in (6.4) - a feedback link might be eliminated with respect to the rejection of the disturbance signal. In both the GMV solution (6.21) and the WMV solution (6.26) the $S$-polynomial vanishes and the future output can be written as

$$y(k + b) = \frac{B}{A} u(k) + \frac{D}{A} x(k - b + d)$$

(6.28)

If we assume for the moment that the primary delay is larger than the secondary delay, $d > b$, the future output signal is completely predictable. In the terminology of the GMV strategy this means that if we have a minimum variance criterion, the optimal prediction of the auxiliary function is

$$\Phi^*(k + b) = \frac{B}{A} u(k) + \frac{D}{A} x(k - b + d)$$

(6.29)

such that the control law would read

$$W = q^{-d + b}D; \quad V = B$$

(6.30)

However, this approach is not feasible when the secondary process is not stably invertible. Then, the feedforward regulator is not allowed to cancel all zeros of $B$, as eq. (6.30) proposes.
It will be shown below that for harmonic sound the controller (6.30) is realisable in a slightly modified version. For broadband sound we have to utilise the idea of WMV control. Let the new auxiliary function be given by

$$
\Phi(k + b) = \frac{B^{-}y(k + b)}{B^{-}} = \frac{B^{-}B^{+}}{A} u(k) + \frac{B^{-}D}{B^{-}A} x(k - b + d) \tag{6.31}
$$

Again, if $B^{-}$ is interpreted as a stable, anti-causal operator, the second term on the right hand side is no longer fully predictable. Define a suitable Diophantine equation

$$
\frac{B^{-}DG}{B^{-}AH} = \frac{M_{2}}{B^{-}} + q^{-d+b} \frac{N_{2}}{AH} \tag{6.32}
$$

Insertion of this identity into (6.31) and using $x = (G/H)\gamma$ yields

$$
\Phi(k + b) = \frac{B^{-}}{B^{-}} y(k + b) = \frac{B^{-}B^{+}}{A} u(k) + \frac{N_{2}}{AG} x(k) + \frac{M_{2}}{B^{-}} \gamma(k + b - d) \tag{6.33}
$$

Based on the idea behind WMV the third term on the right of (6.33) can be considered as consisting of future random samples only. Therefore, the optimal prediction of the auxiliary function is

$$
\Phi^{\circ}(k + b) = \frac{B^{-}B^{+}}{A} u(k) + \frac{N_{2}}{AG} x(k) \tag{6.34}
$$

And this in turn leads to the WMV feedforward controller for broadband sound:

$$
\begin{array}{|c|}
\hline
\text{WMV feedforward control} \\
W = N_{2}; \quad V = B^{+}B^{-}G \\
\hline
\end{array} \tag{6.35}
$$

Note that the WMV control law suggest to cancel only the stable zeros of $B$.

**REMARKS**

- So far the tertiary process has been ignored in the control design. Nevertheless, all the control strategies also apply to the case that the detection signal is contaminated with secondary sound. The fact is that when it comes to optimal prediction, it is rather awkward to use a system equation in which the detection signal is correlated with the control signal. In order to compensate for the tertiary process the control law should be modified into

$$
u(k) = -\frac{A_{0}W}{A_{0}V - \gamma^{2}LW} z(k) \tag{6.36}
$$

with reference to eq. (5.45). It should be realised that the modified denominator does not determine the controller's stability. Neither does the tertiary process influence the residual criterion value.
• As mentioned above the WMV controller (6.35) is based on the assumption that \( b < d \). In the case of a large secondary delay the derivation is similar to the one that led to (6.21).
• Again the WMV solution is identical to the LQG solution as described by eq. (5.61).

**HARMONIC SOUND**

Returning to the optimal prediction of the future output signal eq. (6.29) we propose to set regulator polynomial \( V = 1 \) and next a solution for \( W \) is sought for the relation:

\[
[-q^{-b}BW + q^{-d}D]\chi(k) = 0
\]

(6.37)

As \( x \) is harmonic the composite polynomial between square brackets needs to have a zero on the unit circle at frequency component of \( x \). The corresponding optimal prediction of the auxiliary function is

\[
\Phi^*(k + b) = \frac{B}{A} [u(k) + Wx(k)]
\]

(6.38)

So for harmonic and measurable disturbances the problems with a non-minimum phase secondary process can be avoided by choosing an FIR controller.

### 6.2 Self-tuning implementations of minimum variance control strategies

As mentioned in the previous Section, designing a regulator based on a single-stage criterion (GMV or WMV) requires knowledge of all system dynamics involved. From that perspective it does not offer any advantage over the use of a strategy based on an infinite-stage criterion (LQG). This Section will demonstrate that minimising a single stage criterion is ideally suited to be implemented in a recursive manner. This will require only minor knowledge of the relevant system dynamics.

Define the total auxiliary function \( \Phi(k) \) as

\[
\Phi(k) = \frac{B^{-1}P_d}{B^{-1}P_d} y(k) + \frac{\lambda}{Q_d} u(k - b)
\]

(6.39)

According to the WMV theory, eqs. (6.24) and (6.27) in particular, the optimal prediction \( \Phi^* \) of this auxiliary function \( \Phi \) is

\[
CB^{-1}\Phi^*(k) = B^{-1}Ny^P(k - b) + B^{-1}[MB^P + \lambda CQ]u(k - b) + B^{-1}\frac{N_d}{G}x(k - b)
\]

(6.40)

The corresponding prediction error

\[
\epsilon(k) = \Phi(k) - \Phi^*(k) = \frac{M}{B} \gamma(k - b + n) + \frac{M}{B} \xi(k)
\]

(6.41)

contains merely future values (at time \( k - b \)) of both disturbance generators and is therefore uncorrelated with the prediction of the auxiliary function (6.40). It was also shown in the previous Section, that minimising the single-stage criterion (6.3) is equivalent to setting the
optimal prediction of the auxiliary function equal to zero. This observation leads to the
following recursive implementation of the WMV control strategy

1) Read new samples of \( y(k) \) and \( x(k) \).
2) Calculate the auxiliary function \( B^{-}\Phi(k) \), where \( \Phi(k) \) is given by (6.39).
3) Estimate the regulator polynomials in the expression

\[
B^{-}\Phi^s(k) = S y^{pb}(k - b) + Ru^b(k - b) + \frac{W}{V} x^b(k - b) + [1 - C] B^{-}\Phi^s(k)
\]

where \( y^{pb}(k) = B^{-} y^p(k) \), \( u^b(k) = B^{-} u(k) \), \( x^b(k) = B^{-} x(k) \)
with the estimation error equals \([B^{-}\Phi(k) - B^{-}\Phi^s(k)]\). Suitable estimators might be the
stochastic gradient algorithm or the Extended Least Squares (ELS) algorithm.
4) Apply control according to

\[
R(k, q^{-1}) u(k) = -\frac{W(k, q^{-1})}{V(k, q^{-1})} x(k) - S(k, q^{-1}) y(k)
\]

5) Return to Step 1.

As observed by Åström [ÅW73] the inclusion of the C-polynomial and the auxiliary function
estimate \( \Phi^s(k) \) in the parameter estimation is not obligatory. This is justified by the fact that the
control law implies equating this function to zero. The simplification is only allowed in the
absence of a set-point.
The self-tuning implementation of the WMV strategy seems promising, although its
convergence features heavily depend on the knowledge of two process characteristics:

1) THE SECONDARY DELAY \( b \). A GMV control strategy in the case of an unknown or
variable secondary delay can be found in [Ta85].
2) THE UNSTABLE PART OF THE SECONDARY PROCESS \( B^{-} \). A very efficient algorithm for
spectral factorisation of polynomials is described in [Ku79].

6.3 Recursive parameter estimation

There are several ways to find the optimal regulator parameters in a sequential way. The method
of (linear) regression is in most cases a suitable approach. The general parameter update
equation is [LS83]

\[
\theta(k) = \theta(k - 1) + \mu(k) R^{-1}(k) \psi(k) \epsilon(k)
\]

in which
\( \theta(k) \): Vector of the estimated parameters at time \( k \).
\( \epsilon(k) \): Error at time \( k \). This could be the difference between the actual output at time \( k \) and
its predicted value, or filtered versions of these signals. In active sound control
applications it is sometimes the plain error sensor signal \( y(k) \). The error may be a function of the latest parameter set \( \theta(k - 1) \) only. In general, however, it is a function of the history of vector \( \theta \).

\( \psi(k) \): The regression vector at time \( k \). It usually consists of observed input and output signal samples, or filtered versions thereof. Ideally, it is related to the error \( e \) in the sense

\[
\psi(k) = -\frac{\partial e(k)}{\partial \theta(k - 1)}
\]

\( R(k) \): Positive-definite matrix that modifies the parameter update direction (of \( \psi \)) in order to improve convergence.

\( \mu(k) \): Step-size parameter for update at time \( k \).

Next, consider the estimation of the parameters of the optimal prediction of \( \Phi(k) \). For reasons of simplicity only a pure feedback regulator is considered. Let us denote the parameter vector by

\[
\theta(k) = [s_0(k) \ s_1(k) \ldots s_{n_s}(k) \ r_0(k) \ r_1(k) \ldots r_{n_r}(k)]^T
\]

and define a vector \( \phi \) of signal samples as

\[
\phi(k) = [y^P(k) \ y^P(k - 1) \ldots y^P(k - n_s) \ u(k) \ u(k - 1) \ldots u(k - n_r)]^T
\]

Using \( \phi(k) \) the regression vector can be written as

\[
\psi(k) = B^{-1}(q^{-1})\phi(k - b)
\]

This leads to the parameter update equation

\[
\theta(k) = \theta(k - 1) + \mu(k)R^{-1}(k)\psi(k)[B^{-1}\Phi(k) - \psi(k)^T\theta(k - 1)]
\]

and a short notation for the control law

\[
\phi(k)^T\theta(k) = 0
\]

Two specific update algorithms will be mentioned here

<table>
<thead>
<tr>
<th>Projection or Gradient Descent Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu(k) = \mu )</td>
</tr>
<tr>
<td>( R(k) = [c + \psi(k)^T\psi(k)]I )</td>
</tr>
</tbody>
</table>

This algorithm can characterised by its update direction, which is in the direction of the negative gradient of the error criterion with respect to \( \theta \). In special cases this leads to very slow parameter convergence. The main advantage of the projection algorithm is its simplicity, which may be a decisive factor when it comes to real-time implementation of an estimator. The constant \( c \ (0 < c << 1) \) in \( R \) is only meant to prevent dividing by zero. In the context of adaptive filtering the algorithm is also known as the Normalised Least-Mean-Squares (NLMS).
The Recursive Least Squares (RLS) algorithm has the characteristics

\[
\begin{align*}
\text{Least-Squares Algorithm} \\
\mu(k) &= 1 \\
R(k) &= R(k-1) + \frac{1}{\lambda} [y(k)\psi(k)^T - R(k-1)]
\end{align*}
\]  

(6.51)

In general the RLS algorithm exhibits a fast convergence of the parameters. This is accomplished by the matrix \( R \), that rotates the gradient direction towards the minimum of the criterion. A major drawback is the computational complexity of this estimator. In the presence of high level extraneous noise or in the case of time-varying parameters, the RLS algorithm loses its convergence advantage over the projection algorithm; see e.g. [EF86].

Mainly, because of its complexity the RLS algorithm has not been employed in the experiments of Chapter 7. Moreover, it turned out that a variant of the projection algorithm proved to converge reasonably fast in most cases.

### 6.4 Characterisation of existing active sound control strategies

In the preceding Sections we tried to tune existing control strategies (LQG, GMV) to the specific problem of active sound reduction. As most principal characteristics of acoustic control problems were mentioned

- the absence of a command signal.
- the possibility to measure the disturbace signals beforehand.

This usually leads to a regulator structure in which there is a feedforward part. Concerning the features of acoustic processes we have the following three specifications:

1. a delay time (due to the acoustic travel time),
2. possibly non-minimum phase zeros (due to sampling),
3. usually poorly damped poles (a reverberating enclosure).

The first two acoustic properties seriously complicate the regulator design problem.

If we examine the modern approaches to active sound reduction in literature, it turns out that the majority of the utilised strategies to control sound are based on the technique of *adaptive filtering*. Let us try to characterise the properties in more detail:

- First of all it is assumed that the signal from the detection sensor - whether a microphone or a tachometer - is completely and linearly correlated with the primary signal at the error sensor. The possibility of extraneous disturbances at the error sensor is usually ignored. This implies that pure feedforward control is utilised. An important advantage of this approach is that the stability problems as experienced with feedback controllers do not occur.
• The presence of non-minimum phase zeros in the secondary process is not taken into account in the control design. For the filtered $x$ LMS algorithm reducing harmonic sound the non-minimum phase behaviour does not count. Usually, the secondary process is modelled as an FIR filter; i.e. $A = 1$.

• In many cases the criterion applies to the minimum variance strategy.

• The coefficients of the adaptive filter (adaptive feedforward regulator) are found by explicitly minimising the criterion function \cite{Tr83} by means of a recursive estimator. In this way the sound control problem is made comparable with the problem of recursive system identification.

• In order to retain the convergence of the filter parameters, sometimes an on-line estimation procedure for the secondary process is included, when the acoustics are time-varying. This may require injection of a low level random signal to improve identifiability of the secondary process; see e.g. \cite{Er91}. If the acoustic properties are assumed to be as good as stationary, one usually estimates the secondary process beforehand. During control the regression vector is thereupon filtered by this fixed-time estimate.

• For the purpose of an efficient real-time implementation the projection algorithm is frequently utilised. In combination with the filtering of the regression vector this algorithm is often referred to as filtered $x$ LMS algorithm and was first introduced by Morgan \cite{Mo80}.

The "frequently used" adaptive filtering strategies fall into two separate cases: one for harmonic sound and one for random sound. Both will be discussed in this Section.

**explicit criterion minimisation**

First, let us give a unified design strategy for the existing controllers for active sound reduction. Let us presume that all disturbances are measurable such that the output can be written as

$$A(q^{-1})y(k) = B(q^{-1})u(k-b) + D(q^{-1})z(k-d)$$  \quad (6.52)

Based on the discussion of the LQG regulator, in this case feedforward control can be done without a feedback link when only considering disturbance rejection. Consider the minimum variance cost function

$$J_{MV} (\theta) = \frac{1}{2} E \{ y(k, \theta)^2 \}$$  \quad (6.53)

which is equivalent to the minimum variance LQ-cost function - eq. (5.62) - in the case of ergodic signals. In (6.53) the vector $\theta$ contains the regulator parameters of an a priori defined regulator

$$u(k) = - \frac{W(q^{-1})}{V(q^{-1})} z(k)$$  \quad (6.54)
which is a pure feedforward control law. Recall that detection signal $z(k)$ contains primary and secondary sound

$$z(k) = x(k) + q^{-\ell} \frac{L(q^{-1})}{A_0(q^{-1})} u(k)$$

(6.55)

If we are dealing with purely harmonic sound, the feedforward control law (6.54) can be reduced to

$$u(k) = -W(q^{-1})x(k)$$

(6.56)

The design method of explicit criterion minimisation attempts to minimise the cost function for the given regulator in recursive manner. This control strategy has been extensively described in the thesis of Trulsson [Tr83] for feedback regulators and afterwards extended for feedforward regulators by Sternad [St87]. An important property of the explicit criterion minimisation (ECM) strategy is that under certain conditions it yields the LQG-solution; see Section 5.7. Below, the frequently employed control strategies for active sound reduction will be shown to be special cases of the ECM strategy.

The main principle of the ECM strategy is to find a stationary point of the criterion function. Whether this stationary point represents a global minimum will depend on the selected orders of the regulator parameters. So the implicit solution is given by

$$\frac{\partial J(\theta_0)}{\partial \theta_0} = 0$$

(6.57)

where $\theta_0$ indicates the solution parameter vector. A recursive way to arrive at his solution can be accomplished by iteratively updating the parameter estimates in the (approximate) direction of the negative gradient of the cost function. The stochastic gradient algorithm, as given by eq. (6.50), is one of the alternatives for the parameter update algorithm

$$\theta(k) = \theta(k-1) + \mu \psi(k)y(k, \theta(k-1))$$

(6.58)

Here the regression vector $\psi(k)$ is the approximate gradient - with respect to $\theta(k-1)$ - of the criterion function. The true gradient reads

$$\frac{\partial J_{MV}}{\partial \theta} = E\left\{y(k, \theta) \frac{\partial y(k, \theta)}{\partial \theta}\right\}$$

(6.59)

As this is an expected value it cannot be observed directly. Now, the evaluation of the gradient is simplified using the approximations:

1) The gradient (6.59) is calculated using current signal samples.

2) The controller parameters are assumed to vary slowly with time with respect to the response time of the process.

Given the control law (6.53), the expression for the detection signal (6.54) and the system equation (6.52), the controlled output signal can be written as
\[ y(k) = \left[ q^{-d} \frac{D}{A} - q^{-b} \frac{B}{A(VA_o + q^{-\ell}WL)} \right] x(k) \]  
(6.60)

Differentiating with respect to the regulator parameters and obeying the above approximations eventually leads to the expression for the approximate gradient

\[ \Psi(k) = \frac{B}{A} \frac{A_o}{[VxA_o + q^{-\ell}WxL]} \phi(k - b) \]  
(6.61)

where vectors \( \phi(k) \) of signal samples and \( \theta \) of regulator parameters in (6.58) are

\[ \phi(k) = [z(k) z(k-1) \cdots z(k-n_m) \mid u(k-1) \cdots u(k-n_y)]^T \]
\[ \theta(k) = [w_o(k) \ w_1(k) \cdots w_{n_w}(k) \mid v_1(k) \cdots v_{n_v}(k)]^T \]

and the update algorithm

\[ \theta(k) = \theta(k-1) + \mu \phi(k) y(k) \]  
(6.63)

In [Fl91] a gradient similar to (6.61) has been derived for the case without a secondary path. In [Tr83] and [St87] it has been demonstrated that the ECM strategy converges to the LQG-solution under the conditions:

- The system polynomials \( B, A, L, A_o \) used in the calculation of the regression vector are accurately estimated or “converge to their true values”.
- Strictly, the step-size parameter \( \mu \) should decay monotonically with time to assure parameter convergence. This makes the algorithm insensitive to residual sound.
- The orders of the regulator polynomials are chosen in accordance with the LQG orders. These follow from solution eq. (5.61). Note that this condition also implies the absence of any extraneous noise.

If the orders are chosen properly the criterion function \( J \) has a single unique minimum.

Restricting the regulator’s complexity gives rise to local minima of \( J \).

Below it will be show how two frequently used control strategies for ASR fit into the framework of the ECM strategy.

**Recursive minimisation of broadband sound**

The approach to reduce sound of random nature was proposed by Eriksson. The most extensive description can be found in [Er91]. A feedforward regulator of the structure (6.54) is employed where the detection sensor signal is contaminated with secondary sound; see eq. (6.55). The update algorithm (6.63) together with similar definitions for parameter and signal vector (6.62) are given in [Er91]. However, Eriksson utilises a simplified version of the regression vector

\[ \phi'(k) = q^{-b} \frac{B(q^{-1})}{A(q^{-1})} \phi(k) \]  
(6.64)

In the context of parameter estimation it is well-known that simplification of the gradient vector is allowed under certain conditions. These recursive estimators are generally referred to as
Pseudo Linear Regression (PLR) algorithms; see e.g. [LS83]. Note that the original regression vector (6.61) reveals that the gradient is a non-linear function of the regulator parameters. The simplified gradient (6.64) feigns this relation is linear. Use of the PLR algorithm can only be justified, when the average direction of the simplified regression vector does not deviate more than 90° from the true instantaneous direction. Mathematically, this condition can be written as

\[
\text{Re} \left\{ \frac{A_0(e^{j\omega})}{A_0(e^{j\omega})V(e^{j\omega}) - e^{j\omega}L(e^{j\omega})W(e^{j\omega})} \right\} > 0 \quad \text{for all } \omega < \omega_{\text{nyq}}
\]  

(6.65)

for “sufficiently small” \(\mu\). The requirement (6.65) is also known as the Strictly Positive Real (SPR) condition and is common in the context of PLR algorithms [LS83]. As long as the real part of the term between brackets is positive for all frequencies, the approximate gradient will not deviate more than 90° from the original gradient for all frequencies.

A serious void in all studies on the above algorithm is knowledge of the stationary point of \(J_{MV}\) to which the regulator polynomials \(W\) and \(V\) eventually converge. It is by no means conceivable that the solution as mentioned in [Er91]:

\[
W(q^{-1}) = -q^{-d+b} \frac{D(q^{-1})}{B(q^{-1})}
\]

\[
V(q^{-1}) = 1 - q^{-d-c} \frac{D(q^{-1})L(q^{-1})}{B(q^{-1})A_0(q^{-1})}
\]

(6.66)

can be implemented due to the probability of non-minimum phase behaviour in \(B\). The true optimal solution has been derived in Section 6.1.

**Recursive minimisation of harmonic sound**

This approach has been extensively described by Elliott in [El87]. Since then, the control strategy has been utilised successfully in the interior of a car (engine noise) and the interior of propeller driven aircraft. Here, we will shortly discuss a single-channel version. The feedforward regulator is

\[
u(k) = -W(q^{-1})x(k)
\]

(6.67)

If the harmonic signal \(x(k)\) consists of \(N\) independent harmonics, the regulator order needs to be \((2N-1)\) to create any possible transfer function for the harmonics. Hence, the vector \(\theta\) contains \(2N\) \(\omega\)-parameters. Ensuing the derivation of the ECM algorithm leads to the filtered \(x\) LMS algorithm

\[
\theta(k) = \theta(k-1) + \mu \psi(k)y(k)
\]

(6.68)

where now

\[
\theta(k) = [w_0(k) \quad w_1(k) \cdots w_{2N-1}(k)]^T
\]

\[
\psi(k) = q^{-b} \frac{B(q^{-1})}{A(q^{-1})} [x(k) \quad x(k-1) \cdots x(k-2N+1)]^T
\]

(6.69)
For this algorithm no specific SPR condition applies as the gradient is a true linear function of the regulator parameters. In the case of time-varying process characteristics however, it is obvious that the estimate of the $B/A$ may not differ more than 90° from the true process; i.e. for the relevant frequencies and a "sufficiently small" $\mu \downarrow 0$.

**prediction error LMS algorithm**

In the update equations (6.68) and (6.63) for harmonic and random sound the calculation of the gradient is based on the assumption that the controller is only slowly time-varying. In practice this assumption implies that the step-size parameter may not be chosen too large to prevent the algorithm from divergence. As a consequence, the speed of convergence of the filtered-$x$ LMS algorithm is limited. If however, we would use the error $y(k; \theta)$ instead of $y(k)$ in the update equation an accelerated convergence might be accomplished. The error $y(k; \theta)$ can be interpreted as the prediction error of the actual controller parameters $\theta(k-1)$. The fact is that a fixed-time controller with parameters $\theta(k-1)$ would give rise to the output

$$y(k; \theta) = \frac{D(q^{-1})}{A(q^{-1})} x(k-d) - \frac{W(q^{-1})}{V(q^{-1})} \frac{B(q^{-1})}{A(q^{-1})} z(k-b) \quad (6.70)$$

in which the order of controller polynomials and process polynomials is reversed, because both are time-invariant. In (6.70) the primary signal is unknown. It can, however, be estimated by taking the difference of the measured output $y(k)$ and the modelled controller response $(\hat{B} / \hat{A}) u(k-b)$. The estimate of the prediction error can now be written as

$$\hat{y}(k, \theta) = y(k) - \left[ \frac{\hat{B}}{\hat{A}} u(k-\hat{b}) + \frac{W_k}{V_k} \frac{\hat{B}}{\hat{A}} z(k-\hat{b}) \right] \quad (6.71)$$

The difference between using the predicted output and the true output is their relation to the controller coefficients. The measured output $y(k)$ is a function of the history of the controller coefficients. The predicted output $y(k; \theta)$ is a function of the actual parameters only. This is a clear advantage as the update of the parameters should be based on their error only. The update equation for the prediction error LMS algorithm is

$$\theta(k) = \theta(k-1) + \mu \psi(k) \hat{y}(k, \theta) \quad (6.72)$$

with suitable coefficient and regression vectors; see eq. (6.69) for harmonic sound and eq. (6.62) for random sound. So the mere difference with filtered $x$ LMS algorithm is the error used in the update. The regression vectors are equivalent regardless of the accuracy of the model of the secondary process $(\hat{B} / \hat{A})$. In the case of the inaccurate model, however, the step-size parameter should be chosen small in order to avoid divergence. The smaller the step-size parameter the closer the behaviour of the prediction error LMS algorithm to that of the filtered $x$ LMS algorithm. So the use of an error prediction in the update equation (6.72) flourishes with an accurate model of the secondary process. The improved rate of convergence is achieved at the cost of an increase of computational effort.
adaptive control

Since both the strategies based on GMV control and those based on explicit criterion minimisation employ a recursive estimator to find the regulator, they are partly adaptive. Here we define adaptive feature with respect to two kinds of perturbations

1) CHANGE IN THE PRIMARY SIGNAL; By this change the residual will grow and consequently the regulator parameters are updated towards the new solution through the update equation. Since the filters of the signal vector \( \phi \) do not depend on the primary process, the gradient keeps pointing in the right direction.

2) CHANGE IN THE SECONDARY PROCESS; This perturbation is more complicated, as in this case the gradient of the cost function requires an estimate of the secondary process. The most robust way to obtain an on-line adaptation of a time-varying process is described in [Er91]. Because control signal and sensor signals are (partly) correlated, it is not possible to estimate both regulator and process simultaneously using the same test signal. Adding a low level noise signal to the control signal makes it possible to discriminate between controller response and process identification test signal.

6.5 References


CHAPTER 7

Experiments on Active Sound Reduction

Concluding the design of systems for active sound reduction, a number of principal experiments has been conducted. These experiments serve to confirm the design guidelines built up in the previous Chapters. The first 7 basic experiments were carried out for a single-channel system, as depicted in Figure 7.1. Apart from this one-dimensional enclosure, also a three-dimensional enclosure, a waveguide and an echo-free chamber have served as "region of silence". In each experiment the sampling frequency was 1 kHz.

![Diagram of the single-channel experiment enclosure]

*Figure 7.1: Geometry of the single-channel experiment enclosure.*

For all experiments it will be shortly indicated what the expected results are, based on the theory of the previous Chapters.

7.1 Feedforward control of broadband sound

The problem of reducing broadband sound in a duct has fascinated the researchers in the field from the early beginning. In the 80's Roure [Ro85] and afterwards Eriksson [Er91] developed effective control strategies and achieved impressive results. However, as turned out in Chapter 6, the control mechanism behind active sound reduction using a detection sensor, seems still not fully understood. Here, we will compare the use of an electrical and an acoustical detection signal and their relation to the controller structure.

The first experiment gives the results for an ASR system making use of an electrical detection signal. Although this is unlikely to be the case in practice, it is quite an interesting experiment from a control point of view. The algorithm used is similar to that of explicit criterion minimisation using a prediction error; see Section 6.4 for more details. In Figure 7.2 a striking difference is shown between using a (60, 60) points IIR regulator and a 240 points FIR
regulator. Whereas the IIR regulator reduces the primary sound down to the extraneous noise level, the FIR regulator has trouble in compensating for the secondary process. The fact is that the FIR residual peaks in Figure 7.2 correspond to anti-resonances of the secondary transfer function. One can also say that the FIR length is too short to model all acoustic multiples contained in the recursive part of the IIR solution. The IIR regulator appears to handle the non-minimum secondary process very well. As mentioned in Chapter 5, a non-minimum phase process in a reverberant enclosure does not lead to a serious deterioration of the performance.

Figure 7.2; Primary, IIR residual and FIR residual levels for an ASR system using an electrical detection signal.

Figure 7.3; Primary, IIR residual and FIR residual levels for an ASR system using an acoustical detection signal.
In practical applications of an ASR system a microphone is probably used to measure the primary sound. In this experiment a "unidirectional" Sennheiser ME40 has been placed near the primary source as depicted in Figure 7.1. Obviously, it was directed towards the primary source. This type of microphone has a poor response below 50 Hz. Therefore, the performance is not expected to achieve that of Figure 7.2 for the low-frequency band.

As the IIR solution does not contain poles as poorly damped as in the previous solution, the FIR regulator is expected to do better now. The results in Figure 7.2 confirm this. In this configuration the IIR regulator’s only merit is the number of points used - 120 IIR points to 240 FIR points. As a comparison between using an electrical and an acoustical detection signal, the characteristics of both FIR regulators are depicted in Figure 7.4.

![Figure 7.4](image)

**Figure 7.4:** Impulse response of the FIR regulator using a electrical detection signal (left) and an acoustical detection signal (right).

![Figure 7.5](image)

**Figure 7.5:** Primary and residual level at the error sensor after extending the duct length.
For the configuration with detection microphone and IIR regulator the following robustness test was carried out:

- Let the recursive control strategy converge to the solution of the original duct.
- Fix the regulator's characteristics.
- Extend the duct length between detection sensor and secondary source with 0.1 m.
- Measure the perturbed reduction.

As Figure 7.5 indicates, the slight change in geometry has a far-reaching influence on the performance of the feedforward regulator. This is in full agreement with the results in Chapter 5, where it was stated that the feedforward principle is not robust.

7.2 Feedback control of broadband sound

Before attempting to reject disturbances at the error sensor using a feedback link, first the main characteristics of two specific feedback regulators are shown. As can be found in Chapters 5 and 6, a feedback regulator \((R, S)\) gives rise to closed-loop poles described by

\[
A(q^{-1})R(q^{-1}) + q^{-B}(q^{-1})S(q^{-1}) = T(q^{-1})
\]

where \(A\) and \(B\) constitute the secondary process model. In Chapter 5 is was shown that the closed-loop poles of the LQG regulator are given by

\[
T(q^{-1}) = \beta(q^{-1})C(q^{-1})
\]

where \(\beta\) is defined as

\[
\mu \beta = BB + \rho AQQ_A^*.
\]

So in the minimum variance case - \(Q = 0\) - the LQG controller places the closed-loop poles at \(B^*B^{-1}C\), just like the WMV controller in Section 6.1. This specific control action is denoted as suboptimal minimum variance control, as in spite of a non-minimum phase process, as many as possible zeros are cancelled.

Another special pole assignment controller is the dead-beat controller. A dead-beat controller attempts to place all closed-loop poles at the origin. So, now we have

\[
A(q^{-1})R(q^{-1}) + q^{-b}B(q^{-1})S(q^{-1}) = 1
\]

The dead-beat controller does not specifically cancel zeros of \(B\). Both the LQG and the dead-beat regulator can be constructed using the GMV/WMV strategy by a suitable choice of the weighting polynomials \(P\) and \(Q\); see eq. (6.16).

In Figure 7.6 the effect of an LQG pole placement (eq. (7.2)) on the secondary transfer function (from control source to error sensor in the duct) is demonstrated. The total delay involved here equals about 5 ms. Nonetheless, the influence of both poles and zeros in the transfer function is very well reduced. Only some higher frequency zeros had to be released, because of a small
inaccuracy in the estimation of the secondary process. Therefore, the factor $\beta$ was calculated with little weighting on the control effort; $\rho = 0.1$ and $Q = 1$ in eq. (7.3). The LQG controller was tempered by this and restrained from unstable behaviour.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure7.6.png}
\caption{Original and LQG controlled secondary transfer function.}
\end{figure}

The dead-beat regulator creates a less significant change in the transfer function. It removes all poles but leaves the zeros. Therefore, the controlled transfer function looks like a typical notch-filter.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure7.7.png}
\caption{Original and dead-beat controlled secondary transfer function.}
\end{figure}

If we drive the primary source with white noise, it can be investigated how well a feedback controller can reduce coloured noise at the error sensor. As was shown in the previous Chapters the performance of a feedback controller depends on the autocorrelation characteristics of the
primary noise. Due to the delay in the secondary path the controller needs a reverberating signal, such that its dated information can still reduce some of the noise. Based on the experiment on pole-zero cancellation we expect a better reduction of the LQG-regulator compared with that of the dead-beat regulator; see Figures 7.8 and 7.9.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{primary_residual.png}
\caption{Primary and residual level of the error signal when applying LQG feedback control.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{primary_residual.png}
\caption{Primary and residual level of the error signal when applying dead-beat feedback control.}
\end{figure}

Again, the LQG regulator achieves far more equilisation of the spectrum than the dead-beat regulator. The fact is that the two control strategies lead to a different loop gain. This gain is given by
Section 7.2  Feedback control of broadband sound

\[
K(e^{j\omega}) = \frac{Y_{\text{prim}}(e^{j\omega}) - Y_{\text{res}}(e^{j\omega})}{Y_{\text{res}}(e^{j\omega})}
\]  \hspace{1cm} (7.5)

where \( Y \) is the Discrete Fourier Transform of the error signal. So the higher the reduction of \( y \), the higher the loop gain. This is depicted in Figure 7.10.

![Figure 7.10](image)

*Figure 7.10:  Absolute values of the loop gain for LQG and dead-beat control.*

A high loop gain, however, is not the ultimate design goal. As was discussed in Section 5.5 a high loop gain gives rise to:

- a high sensitivity of the performance to system changes.
- a poor robustness of stability.

![Figure 7.11](image)

*Figure 7.11:  Primary and residual level at the error sensor with an LQG regulator after changing the duct length.*
Both detrimental features show up as soon as a small change in the secondary process takes place. In Figure 7.11 it is illustrated that the non-adaptive LQG regulator looses much of its performance, and even gets unstable when the duct length is extended by 0.1 m. The performance of the dead-beat regulator is not that much affected and the closed-loop remains stable. Obviously, the dead-beat regulator did not risk as much by having a lower loop gain.

So, this experiment has demonstrated the pros and cons of using a “high-gain” controller like the LQG controller. It may achieve a substantial reduction of extraneous noise, but it is hard to maintain this reduction. Although the “medium-gain” dead-beat controller does not achieve a very high reduction, it is more robust. The dead-beat regulator e.g. has proven to operate satisfactorily on low frequency narrowband sound in a passenger car [OD91].

7.3 Robustness of the controllers for broadband sound

In Section 7.1 it turned out that the performance of the feedforward controller for broadband sound is very sensitive to changes in the acoustic processes. In this Section the robustness of the feedforward regulator is improved by adding a dead-beat feedback link to the control system. The experimental procedure is as follows:

1) Find a dead-beat regulator with suitable weighting of control effort, that can handle the change in duct length; i.e. without going unstable.
2) Add the feedforward link and recursively optimise the reduction of primary sound in the original duct.
3) Fix the regulator parameters and change the duct length.
4) Measure the deterioration in performance and compare with Figure 7.5.
For the original duct the dead-beat and the feedforward controller achieve a substantial reduction of the primary noise; see Figure 7.13.

**Figure 7.13:** Primary and residual level of a combined feedback-feedforward regulator.

And due to the addition of the feedback loop the degradation in performance of the controller is far less than found in Section 7.1, Figure 7.5. Here, we arrive at two important aspects of a combined feedforward-feedback regulator.

1) Addition of a feedback link to a feedforward link can lead to robust rejection of a measurable disturbance.
2) Addition of a feedforward link to a feedback link may improve the stability robustness, as now the feedback regulator can have a lower loop gain, because the feedforward link provides most of the disturbance rejection.

In the coming three Sections a similar conclusion will follow for the active reduction of harmonic sound.

7.4 Feedforward control of harmonic sound

The active reduction of harmonic sound has gained considerable attention during the ten past years. Efficient and satisfactory solutions have been developed to attenuate the harmonic noise in e.g. vehicles, propeller aircraft and in exhausts. In this Section, however, we will make a step backwards and consider a non-adaptive implementation of a feedforward controller for harmonic sound. The procedure is as follows:

- Let the recursive control strategy converge to the solution of a given harmonic disturbance.
- Fix the regulator’s characteristics; a 2-points FIR filter.
- Change the disturbance characteristics into broadband noise.
- Measure the controller’s sensitivity to broadband noise.

The harmonic disturbances for this experiment were chosen to be two sinusoids, one with its frequency at an anti-resonance (144 Hz) of the secondary transfer function and one at a resonance (217 Hz). Observing the power spectra of the error signal both disturbance frequencies appear to be completely cancelled; see Figure 7.15.

![Figure 7.15](image)

*Figure 7.15; Primary and residual level of an harmonic disturbance at the error sensor; 144 Hz (left) and 217 Hz (right).*
Figure 7.16: Primary and residual level at the error sensor. The feedforward controller is designed to cancel harmonic sound of 144 Hz.

The broadband sensitivity spectra for the 144 Hz controller and the 217 Hz controller are nevertheless quite different. This is clarified in Figures 7.16 and 7.17. In order to cancel the 144 Hz harmonic completely, the feedforward regulator needs to have a high gain since at this specific frequency the acoustic process has an anti-resonance. As the other disturbance harmonic lies at a process's resonance, the controller gain can be much lower. The sensitivity of the fixed-time 144 Hz controller to other disturbances is far too high. The first remedy will be to add a feedback link to the control system.

Figure 7.17: Primary and residual level at the error sensor. The feedforward controller is designed to cancel harmonic sound of 217 Hz.
7.5 Feedback control of harmonic sound

With the GMV control strategy in mind it is straightforward to design a feedback regulator for harmonic sound. As an harmonic disturbance is completely predictable, a feedback controller should achieve a reduction comparable to that of a feedforward controller. An interesting aspect here is that the controller has to generate an harmonic disturbance itself. This implies that in order to regulate an unstable disturbance - as an harmonic officially is - the disturbance dynamics have to be part of the controller dynamics. This rule is better known as the internal model principle. In that case the loop gain is made infinite at the disturbance frequencies.

Figure 7.18; Primary and residual level at the error sensor. The feedback controller is designed to cancel harmonic sound of 144 Hz.

Figure 7.19; Primary and residual level at the error sensor. The feedback controller is designed to cancel harmonic sound of 217 Hz.
Together with disturbance rejection a feedback controller can also modify the closed-loop poles. Therefore, a dead-beat controller is proposed here to limit the sensitivity to other disturbances. This means that a GMV controller is designed with polynomials $P$ and $Q$ such that the closed-loop poles are placed at the origin. Figures 7.18 and 7.19 show the result. Contrary to the feedforward case, quite a neutral attitude can be observed of the dead-beat regulator with respect to other disturbances. For the specific harmonic disturbance, 144 and 217 Hz, both the feedforward and the feedback controller achieve complete cancellation.

7.6 **Robustness of the controllers for harmonic sound**

This Section is to demonstrate experimentally that a combined feedforward-feedback controller for harmonic sound can be designed with a low sensitivity to other disturbances. The first step is to design a dead-beat regulator with an additional weighting of control effort, which yields a low loop gain and a modest reduction of the harmonic disturbance. The resulting primary and residual response can be found in Figure 7.20, and the corresponding loop gain is depicted in Figure 7.21.

When observing the modest action of the dead-beat regulator, one might wonder whether the feedforward link still does not need a high gain to cancel the disturbance. The fact is, however, that on a linear scale the feedback link has reduced more than half of the disturbance. So, the feedforward link can do with a much lower gain. The benefit of this is illustrated in Figure 7.22, where a favourable sensitivity is revealed together with the absolute rejection of the original harmonic.

![Graph showing primary and residual level on the error sensor. The weighted dead-beat controller is designed to cancel 144 Hz.](image)
7.7 Adaptive feedforward control of time-varying harmonic sound

The objective of the feedback link in the previous Section has been to prevent an amplification of the primary sound. Clearly, it is not feasible to design a fixed-time feedforward-feedback controller that also rejects the unmodelled disturbances. In this experiment the feedforward regulator is extended with an adaptive loop instead of a robust feedback link. A time-varying harmonic - a sweep wave - served as primary sound. The frequency ran from 70 Hz to 250 Hz in about 10 seconds. The feedforward regulator consisted of a 2-points FIR filter. The characteristics of the feedforward filter were updated on-line using two kinds of update
equations. In Figure 7.23 the performance of the filtered $x$ LMS algorithm is shown. The corresponding update equation is given by (6.68). The same sweep wave was controlled using the prediction error LMS algorithm; see eq. (6.72). The result of this test is depicted in Figure 7.24.

Figure 7.23: Envelopes of primary and residual signal for a sweep wave controlled by an adaptive feedforward regulator based on filtered $x$ error LMS.

Figure 7.24: Envelopes of primary and residual signal for a sweep wave controlled by an adaptive feedforward regulator based on prediction error LMS.
Comparing the two we may conclude that:

- Both algorithms give a substantial reduction for the resonance frequencies. At anti-resonances the performance temporarily deteriorates. This is probably due to the fact that at those instants fast changes in phase occur. Also, at the sudden change-over from 250 Hz to 70 Hz - at about 10.3 s - both algorithms exhibit an initial problem to switch back.

- The prediction error LMS algorithm appears to have a better tracking capability in this case. Due to an accurate estimate of the secondary process the step-size parameter $\mu$ for prediction error LMS could be chosen 10 times larger than the $\mu$ for filtered $x$ LMS. This accelerated the feedforward filters reaction to the drifting harmonic sound.

### 7.8 Multi-channel reduction of harmonic sound in an enclosure

In Sections 3.5 and 4.1 the reduction of sound in a reverberant room was discussed extensively. It was shown that substantial reduction of the space average squared pressure could be obtained in the following cases:

- The sound field is dominated by as many modes as secondary sources; this is likely to happen at low frequencies and at resonances.

- One secondary source is of the same type and at close range ($kr < 1$) of the primary source.

![Figure 7.25: Top view diagram of sources (P and S) and sensors (M) in the experiment enclosure.](image)

As the latter case is rather unlikely in practice it has not been examined in this experiment. The experimental enclosure has dimensions 1.0x0.85x0.35m, which means that the $x$ and $y$ sizes are the same as used in the simulations. Since the walls of the enclosure are reasonably rigid - 0.02m MDF - the low frequency resonances can be expected to correspond to those in Figure 3.30. Also, the locations of the control sources are chosen to be similar to that of the
simulations; see Figure 3.29 and Figure 7.25. As turned out in Chapter 3, the control sources should be placed as far as possible from nodal lines of the relevant modes. The same holds for the locations of the error sensors. In agreement with the simulation experiment in Section 4.1 four error sensors are positioned in the upper enclosure corners whereas two others on the nodal lines of the 100 and 010 modes; see Figure 7.25.

As follows from Section 4.1 a suitable cost function is given by the space average squared pressure on the error sensors

$$J(\omega) = \frac{1}{6} \sum_{m=1}^{6} |P(r_m, \omega)|^2$$  \hspace{1cm} (7.6)

Before discussing the results of minimising this criterion in the wooden enclosure, we will first recall the results obtained in the simulation experiments; see Figure 3.33 to 3.36.

1) Using a single corner source yielded reduction of all separate resonances up to 500 Hz. For the overlapping resonances around 400 Hz hardly any attenuation was achieved just as at anti-resonances.

2) Using two corner sources resulted in a slightly better performance at anti-resonances. However, the two coinciding resonances around 400 Hz still could not be reduced. This is due to the unfavourable phase relation of the control sources with respect to the primary source. This is illustrated in Figure 3.35.

3) Using all three corner sources gave a substantial reduction of all resonances up to 500 Hz.

![Figure 7.26: Primary and residual space average squared pressure using source $S_1$.](image)

In the experiments the same source combinations are used. The frequency band of operation is from 150 to 450 Hz. As this experiment concentrates on harmonic sound, it has been carried on
a single frequency components between 150-450 Hz at a time; starting at 150 Hz with steps of 5 Hz up to 450 Hz. The algorithm used is a multivariable version of the prediction error LMS algorithm as described in Section 6.4. In Figure 7.26 the primary and residual space average squared pressure using source $S_1$ are shown.

**Figure 7.27; Primary and residual space average squared pressure using sources $S_1$ and $S_2$.**

For the Figures 7.26 to 7.28 it should be realised that the results have only been measured at multiples of 5 Hz in the band of 150-450 Hz. For all other frequencies shown interpolation has taken place. The obtained reduction using a single secondary source is in full agreement with the expectation; also see Figure 4.8a. There is a substantial reduction of the resonances - apart
from those around 400 Hz - and very little reduction at anti-resonances. In Figures 7.27 and 7.28 the results are shown using 2 and 3 control sources.

Comparing the theoretical and experimental results, the following can be observed:

- Using two control sources improves the reduction at anti-resonances. The coinciding resonances, however are still alive, as predicted above.
- Using all three secondary sources yields a low residual level for all frequencies up to 450 Hz.

So, the main characteristics of multi-channel reduction of harmonic sound in an enclosure very well agree with the theoretical results.

### 7.9 Multi-channel reduction of random sound in an enclosure - electric detection signal

In this experiment the previous one is repeated for broadband primary sound. So, the geometry of sources and sensors is the same as in Figure 7.25. The only difference with Section 7.8 is that now all frequencies are attenuated *simultaneously*. One of the main problems with broadband cancellation is the causality constraint for the controller. In order to make a fair comparison with the previous experiment, the signal to the primary source was therefore delayed, such that the causality constraint did not limit the performance. In other words, the controller was provided with the primary signal beforehand, such that the secondary response at the sensors could lead the primary response. Although this is quite academic, it enables us to compare broadband and harmonic control. In the next Section the practically feasible configuration with a detection sensor will be discussed.

![Graph showing primary and residual magnitude vs frequency](image)

*Figure 7.29: Primary and residual space average squared pressure using source $S_1$.*
Figure 7.30; Primary and residual space average squared pressure using sources $S_1$ and $S_2$

Figure 7.31; Primary and residual space average squared pressure using sources $S_1$, $S_2$ and $S_3$

Due to the detection signal in advance, the reduction results can be expected to be similar to those for harmonic sound. In Figures 7.29, 7.30 and 7.31 the primary and residual space average squared pressure is depicted where the primary sound is driven by white noise. Comparing these with Figures 7.26, 7.27 and 7.28 we find

- The main characteristics of multi-channel reduction of random sound correspond to those of harmonic sound.
- The broadband reduction at a specific frequency, however, is a slightly less on the average than reduction of an harmonic. For this phenomenon two causes are proposed
1) The order of the IIR controller is large enough to achieve a substantial reduction. It is, however, not sufficient to match the harmonic performance for every frequency component simultaneously.

2) Due to the presence of unstable zeros in the secondary processes, the performance of harmonic control is not achievable by a broadband regulator.

7.10 Multi-channel reduction of random sound in an enclosure - acoustic detection signal

It is obvious that the configuration with an detection signal in advance is by no means feasible in practice. Generally, the characteristics of the primary sound should be detected by an acoustical sensor close to the primary source. This configuration is depicted in Figure 7.32.

![Diagram showing source, error sensors, and detection sensor in an enclosure](image)

*Figure 7.32: Top view diagram of sources (P and S), error sensors (M) and the detection sensor (D) in the experiment enclosure.*

The detection microphone $D_1$ is positioned near the primary source in order to provide the controller with the earliest possible primary signal. As was demonstrated in Section 4.3 a single detection sensor suffices as there is only one primary source. For the configuration of Figure 7.32 the primary sound at all error sensors will always lead the secondary sound. The fact is that the secondary response is delayed by:

1) The travel time from primary source to detection sensor.
2) The delay in the low-pass filters together with the computational delay in the controller.
3) The travel time from secondary source to error sensor.

It turned out that the total delay for every source-sensor combination exceeds the primary delay. This implies that the best achievable performance of the ASR system heavily depends on the predictability of the primary sound. In Section 4.3 it is shown that this improves with the reverberation of the enclosure. So for this experiment we can expect reduction of the resonance peaks only. For these frequencies, the early, non-predictable primary response is only a minor
part of the total response. The primary and residual levels of space average squared pressure are given in Figures 7.33, 7.34 and 7.35.

Here, it is observed that:

- The reduction at resonances is reasonable and improves slightly with the number of control sources. When analysing the residual level at individual sensors the sensor closest to the primary source - $M_1$ in Figure 7.25 - appears to be dominating the criterion eq. (7.6). As the primary response at this sensor is the least predictable, the controller will focus on this sensor. Since secondary source $S_1$ is nearest to sensor $M_1$ of all control sources, the other two can hardly improve the single source performance on $M_1$. A small benefit of multiple sources is represented by the improved reduction at higher frequency resonances.

- There is no reduction and may even be amplification at the anti-resonance frequencies. The same phenomenon was observed in Section 4.3, where optimal prediction of primary sound implied an evening out of the spectrum. Note that this is also the case with feedback control of broadband sound; see Section 7.2.

![Graph showing primary and residual space average squared pressure using source $S_1$ and detection sensor $D_1$.](image)

*Figure 7.33; Primary and residual space average squared pressure using source $S_1$ and detection sensor $D_1$.***
Figure 7.34; Primary and residual space average squared pressure using sources $S_1$ and $S_2$ and detection sensor $D_1$.

Figure 7.35; Primary and residual space average squared pressure using sources $S_1$, $S_2$ and $S_3$ and detection sensor $D_1$.

7.11 Multi-channel reduction of random sound in duct

The reduction of sound in a waveguide has been treated in Sections 3.4 and Section 4.1. It was demonstrated that:

- Using 4 error sensors in the corners of a hard-walled duct, the original criterion function
  - the radiated power - can be well approximated by the space average squared pressure.
• A substantial reduction can be obtained below the cut-on frequency of the \( n \)-th mode when \( n \) well-placed control sources are used. Near the cut-on frequency a degradation of performance occurs due to evanescent modes.

The selected configuration of sources and sensors can be found in Figure 7.36. Like the error sensors the secondary sources are positioned near the walls in order to couple well into the relevant modes. The omnidirectional detection sensor is located close to the primary source, such that the controller receives timely information on the primary signal. Again, the primary source was driven by white noise. In accordance with the simulation experiments both duct terminations were attempted to make fully absorptive by using absorption material of 0.4m thickness. The duct walls are made 0.02m MDF and are not completely rigid. In accordance with the simulations the cross-section is 0.85m x 0.35m.

Figure 7.36; Geometry of primary source (P), secondary sources (S), detection sensor (D) and error sensors (M) in the experiment wave guide.

Figure 7.37; Primary and residual space average squared pressure at the error sensors using secondary source \( S_1 \).
The results of minimising the space average squared pressure at all 4 error sensors employing 1 or 2 control sources are shown in Figures 7.37 and 7.38.

The following characteristics are noticed:

- The primary energy does not look at all like that in the simulation experiments; see for example Figures 4.9a and 4.9b. This is probably due to limited absorption of the duct terminations. The cut-on frequencies of modes (1, 0) and (2, 0), however, can be expected to be 200 Hz and 400 Hz.
- Assuming these modal resonance frequencies, the experimental results are in reasonable agreement with the theoretical results. Using a single control source there is reduction up to 200 Hz, whereas two control sources extend this region to 400 Hz. Near the end of the reduction regions the performance degrades due to the generation of higher order modes by the control the source(s).

7.12 Multi-channel reduction of sound in an echo-free chamber

The active attenuation of sound in the free field has been treated in Sections 3.2 and 4.1. The main characteristics can be summarised as

- Only below \( ka = \pi \) a substantial reduction of the total radiated power can be achieved, where \( a \) is the distance of the secondary source(s) to the primary source.
- If the radiated power is minimised over a limited arc of the field, the virtual distance between the sources is reduced as given by eqs. (3.17) and (3.18).
- Using only 4 sensors the radiated power can be well approximated for \( ka < \pi \).
In an echo-free chamber of 10 m³ a configuration of a primary source, a secondary source and four error sensors was installed; see Figures 7.39 and 7.40. The distance between the two sources equals 1.2 m and the distance between an error sensors and the primary source about 6 m. Again, the experiment consisted of a number of monochromatic tests, ranging from 80 to 250 Hz with a 10 Hz interval. Note that for this frequency band the error sensors are located quite far from the primary source: $kr > 8.7$. In the simulation experiments the sensors were placed in the far field.

*Figure 7.39:* Geometry of primary source (P), secondary source (S) and error sensors (M) in the echo-free chamber.

*Figure 7.40:* Picture of the geometry of the primary source (back) and secondary source (front) in the echo-free chamber.
If we compare the measured residual with the predicted residual a fair similarity is found in Figure 7.41. The slight difference might be due to the circumstance that the error sensors are still too close to the sources.

![Figure 7.41](image)

*Figure 7.41; Measured, predicted and optimal residual space average squared pressure using 1 secondary source and 4 error sensors in an echo-free chamber.*

7.13 References


CHAPTER 8

Conclusions and Discussion

8.1 Summary

The thesis aims at a broad discussion on both the acoustical and control aspects of designing a system for active sound reduction. Several approaches have been proposed depending on important parameters like for example: the size of the region of silence (with respect to the wavelength), the damping properties of the region of silence, the possibility to measure the primary sound beforehand. The application of active sound reduction demands an integrated approach to acoustics and control. The input to the control discussion is the acoustical configuration of sources and sensors (both error and detection). The result of the control design is a suitable algorithm that follows a minimisation of the original acoustic criterion function as well possible. Since the thesis can be viewed as a step-by-step design guide, the conclusions are given in a similar chronological way. Each design step implies an adaptation of the criterion function in order to cope with practical constraints.

The acoustical principle of active sound reduction lies in wave field reconstruction as given by the generalised Kirchhoff integral. It is shown in Chapter 2 that the use of a continuous distribution of monopoles and dipoles on a closed surface can yield a complete annihilation of the original sound field inside the surface. Moreover, it is demonstrated that a solution of one source type - only monopoles or only dipoles - also may lead to a complete annihilation of the original sound field. These solutions, however, create complete reflection of the incident primary field at the closed surface. For the remainder of the acoustical analysis - Chapters 2 and 3 - the all monopole solution is discussed in more detail.

The first design step is to approximate the continuous monopole distribution by a discrete distribution. An essential problem here is the choice of the criterion function which represents the ‘annoyance’ of the sound field. It is shown in Chapter 3 that the space average squared pressure is a very appealing choice. This holds for both the free field and the reverberant enclosure. The configuration of control sources, however, heavily depends on the damping properties of the region of silence. In general, the secondary sources should be placed as close as possible to the primary source. In that case one attempts to modify the source of the primary field. In a reverberating environment it may also be possible to attenuate the primary wave field with secondary sources well away from the primary source. This is due to the fact that at resonance frequencies wave fields are dominated by only a few or even a single acoustic mode.
In these situations a small number of control sources can yield a substantial reduction of a reverberant sound field. In Chapter 3 the analysis has been focussed on a one-dimensional enclosure with arbitrary termination conditions. This reveals many of the principles of active sound reduction in the free field, a multi-modal wave guide and a reverberant enclosure. The acoustical design can be completed by adding a configuration of sensors. These have a twofold goal. The so-called error sensors measure the sound field in the region of silence. In this way a measure of the performance is obtained. The error sensors can also be employed to provide the controller with information on the primary sound. In many cases, however, it is more effective to measure the primary sound outside the region of silence or close to the primary source. These sensors are denoted as detection sensors. It follows from Chapter 4 that one detection sensor is needed per independent primary source.

By means of the error sensors we have to estimate the original continuous criterion function. Therefore, their configuration has to be such that minimisation of the sound field at the sensors represents a similar global effect. The mean squared pressure again is a suitable choice. The sensors should be placed far from nodal lines and outnumber the secondary sources.

Given its input signals, measured by detection and error sensors, the controller has to generate the control signals to drive the secondary sources. In doing so it has to take into account the transfer functions between sources and sensors. In Chapter 5 it is shown that a discrete transfer function has the following three characteristics: a time delay caused by the acoustic travel time, poles which may lie close to the unit circle and zeros which may even lie outside the unit circle. The last characteristic seriously complicates the control design. For many criterion functions it means that weighting of the control energy is required in order to end up with a stable controller. Here the phenomenon of a non-minimum phase process leads to deterioration of the performance. This is most clearly seen when applying the Generalised Minimum Variance (GMV) control strategy, where weighting polynomial $Q$ has to be tuned to create stable closed-loop poles.

The GMV control strategy and a variant thereof Weighted Minimum Variance (WMV) are proposed in the thesis as suitable design methods to arrive at a controller for active sound reduction. This controller may consist of a feedforward link and a feedback link. With respect to performance the feedforward link takes care of the measurable disturbance, whereas the feedback link can reduce part of the extraneous disturbance. A pure feedforward GMV regulator shows great similarity with the existing regulators for harmonic and broadband sound. Although the latter two originate from the strategy of explicit criterion minimisation, the resultant self-tuning algorithms are largely equivalent. Still, an important flaw for Eriksson's regulator is revealed. Due to a non-minimum phase secondary process the optimal solution cannot fully invert for this process.
With respect to robustness, the roles of a feedforward and a feedback link are quite different. The principle of feedforward control is very sensitive to system perturbations. Contrarily, a feedback controller can be designed such that it is far less sensitive to perturbations. Robustness and performance, however, are conflicting design demands. Therefore, a feedback link can improve the performance robustness of a feedforward regulator without rejecting the disturbance itself. On the other hand, a feedforward link added to a high-gain feedback regulator can improve the stability robustness. This follows from the fact that a feedforward link may take over (part of) the rejection and because of that lower the closed-loop gain. In Chapter 7 the favourable robustness properties of a combined controller are demonstrated by means of practical experiments. It is also shown that the sensitivity to perturbations of a feedforward controller can be reduced by implementing an adaptive procedure. In this way changes in the secondary process can be tracked.

The final Chapter, discussing the experiments, has verified many of the theoretical characteristics mentioned earlier in the thesis. It is demonstrated that in the free field, the multi-modal wave guide as well as in the reverberant enclosure the practical results well agree with the theoretical. Also a comparison is drawn between the control of harmonic sound and broadband sound. It follows that a broadband controller does not perform as well as a series of harmonic controllers reducing the same noise.

Furthermore, two kinds of feedback controllers are tested in practice. Their ability of rejecting harmonic sound and the coloured components of random sound is demonstrated. Also, improvement of the performance robustness of a feedforward link by adding a feedback link is experimentally shown. This was tested for both a change in secondary process and a change in disturbance signal. It has turned out that for small perturbations the robust feedforward-feedback controller performed satisfactorily. For more significant perturbations the use of an adaptive loop is recommended.

### 8.2 Discussion and future research

As mentioned in Chapter 1 the main contribution of the thesis is the introduction of modern control theory and the experimental demonstration of its possibilities for active sound reduction. This has led to a number of interesting results ranging from principal guideline to minor improvement. Below these results and others are discussed in chronological order.

Starting with the theoretical idea of a continuous distribution of point sources, one immediately is faced with the problem of choosing a suitable cost function. This choice is discussed throughout Chapters 3 and 4. Unlike many of the publications on active sound reduction the criterion of space average squared pressure is not taken for granted. It is compared with other cost functions like total acoustic energy, radiated power and even maximum squared pressure.
Although, in the end the space average squared pressure is selected as a practical alternative, the discussion is not yet completed. As stated earlier the aspects of perception of noise deserve a thorough research effort. This includes both the frequency-dependent and the space-dependent perception of a sound field. For the latter a small impulse has been given in Section 3.6 with the minimax criterion function. The merit of such a cost function is, for example, that after minimisation nowhere in the region of silence the primary field has increased. The feasibility of the minimax criterion, however, is far from demonstrated in the thesis.

In the minimisation of far field acoustic energy an interesting result has been found for the minimisation over a limited arc. With respect to performance, reducing the region of silence can be interpreted as putting the secondary sources closer to the primary source. This could be of importance for the application of anti-sound to transformer noise.

The one-dimensional enclosure with arbitrary end impedances of Section 3.3 has been used as an “experimental garden” to derive ASR characteristics for the free field, the multi-modal wave guide and the reverberant enclosure. The possibilities of the one-dimensional enclosure have not been completely exploited, however. Too much attention has been focussed on hard-walled volumes, like the hard-walled wave guide and the reverberant enclosure. In a reverberant environment the application of active sound reduction is most effective and fairly simple. The application in an environment with significant wall damping is more troublesome. The one-dimensional enclosure can also be employed to gain fundamental insight into this, more practical, application area.

Before designing a control system knowledge has to be gained on the processes and disturbances. The possibility of non-minimum phase behaviour has been demonstrated for an acoustic process. This is an important result, since the control design has to account for this behaviour. The control designer cannot rely on the (small) probability of dealing with a minimum phase process. Apart from feedforward control of harmonic sound, minimum variance control is therefore not feasible. For this reason the (sub)optimal solution has been derived for a feedforward regulator of broadband sound. This result is believed to be new, although - under a number of constraints - the common recursive algorithm may converge to this solution.

For the well-known filtered x LMS algorithm a modification has been found based on a prediction error. In an experiment of reducing a time-varying harmonic the so-called prediction error LMS algorithm has shown to have favourable tracking properties. This speeding-up of the LMS algorithm is obtained at the cost of a increased computational complexity. In the case of harmonic sound the tracking properties can improved even more. This is achieved by choosing a special filter based on a monochromatic decomposition; see [Ov93].
Section 8.2 Discussion and future research

Joining a feedback link to a feedforward regulator may serve two purposes; (1) rejection of extraneous disturbances and (2) improvement of the performance robustness. The application area of the first purpose is rather small. One might think of low frequency sound reduction in a car. Still, in most cases the primary sound can be detected beforehand.

The second purpose of adding a feedback link seems to be far more important. The aim of a combined controller is to achieve a robust sound reduction. Robustness of a control system is always related to perturbation bounds. When a specific perturbation exceeds its a priori modelled bound, the robust controller loses its favourable characteristics. In the case of large perturbations an adaptive control loop may be recommendable. It should be realised, however, that an adaptive feedforward controller should be judged upon its performance robustness and its stability robustness. It is therefore not imaginary that under certain time-varying conditions a robust, adaptive controller is required.

Finally, the four most essential fields of future research are listed:

- **ACOUSTIC COST FUNCTIONS**: As the human being plays the role of receiver in many noise problems, it is important to fit the perception of noise into the acoustic cost function. It is unlikely that an A-weighting will be sufficient here. This is partly prompted by the fact that ASR systems are usually designed for low frequency noise.
  
  Besides the perceptual aspects, it is also of interest to see whether a minimum of spatial average of the squared pressure really meets our needs of global minimisation.

- **STATE-SPACE DESCRIPTION**: In the field of active control of vibration a state-space approach is quite common. The acoustic case lends itself perfectly to be described in this way; see Section 5.2. When applying a state-space description a control designer retains a good insight in the physical behaviour of system, as all internal states are modelled. With the concepts of reachability and observability, for example, it is straightforward the evaluate the configuration of sources and sensors. Moreover, a more fundamental approach can be used to design this configuration.

- **ROBUST AND ADAPTIVE CONTROL**: In the field of active sound reduction the adaptive procedure is widely used. This is probably due to the fact that the majority of control algorithms originates from Widrow's theory of adaptive filtering [WS85]. As clearly indicated in the thesis a fixed robust controller may perform satisfactorily. It should be realised that an adaptive loop may heavily call on the design of a controller. Therefore, it is essential to discriminate between acoustical situations requiring fixed-time robust control, adaptive control and robust, adaptive control and develop suitable control strategies for each case.

- **IMPLEMENTATION**: The issue of implementing a control algorithm is hardly addressed in the thesis. It is, however, of significant importance, since the polynomial description -
which is used in the thesis - is known to have unfavourable numerical properties. Since ASR systems are designed to operate night and day and the model orders are relatively high, it may be inevitable to use other, numerically more stable implementation forms. These could be the lattice form or the $\delta$-operator form; see [MG90].

8.3 References


Published Papers

During the research presented in the thesis the following papers have been published:


## Curriculum Vitae

<table>
<thead>
<tr>
<th>Naam</th>
<th>Doelman, Nicolaas Jan</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geboortedatum</td>
<td>25 februari 1964</td>
</tr>
<tr>
<td>Geboorteplaats</td>
<td>Leiderdorp</td>
</tr>
</tbody>
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1976 - 1982
Atheneum B aan het Christelijk Lyceum te Alphen aan den Rijn.

1982 - 1987
Technische Natuurkunde aan de Technische Universiteit Delft.
Afstudeeronderzoek bij de Vakgroep Akoestiek en Seismiek in samenwerking met de Technische Physische Dienst (TPD-TNO-TU);
"Adaptive, Active Sound Attenuation of Stochastic Sound in a Cylindrical Waveguide".

1987 - 1989
Vervangende dienstplicht bij de Technische Physische Dienst;
onderzoek op het gebied van extrapolatie van geluidvelden in onmsloten ruimten.

1989 - 1993
Wetenschappelijk medewerker bij de Technisch Physische Dienst;
onderzoek mede gesteund door STW (DTN 99.1698).