Replication and risks of the ATM Forward Percentage Call Spread

(Dutch title: Replicatie en risico's van de ATM Forward Percentage Call Spread)

BACHELOR THESIS
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"Replication and risks of the ATM Forward Percentage Call Spread"
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Contents

1 Introduction ................................................. 7

2 Description of the Derivative ................................ 9
  2.1 Standard Call Option ................................... 9
  2.2 ATM Forward Percentage Call Spread ................... 10
  2.3 Relevance ............................................. 13

3 Binomial Pricing Model ...................................... 15
  3.1 Two-Period Binomial Model ............................. 15
  3.2 Multi-period Binomial Model ........................... 18
  3.3 Derivation of parameters .............................. 20
  3.4 Explanatory remarks on the code ...................... 21

4 Calibration of the Binomial Model .......................... 23
  4.1 Analysis ............................................. 23
  4.2 Determination of Market Parameters .................. 24
  4.3 Pricing ATM forward percentage call spreads ......... 27

5 Black-Scholes Pricing Model ................................. 29
  5.1 The Black-Scholes Equation ............................. 29
  5.2 Black-Scholes for the ATM forward percentage call spread 31
  5.3 Pricing ATM forward percentage call spreads ......... 32

6 Replication strategies ....................................... 35
  6.1 Hedging with index futures ............................ 35
  6.2 Hedging with index options ........................... 37
  6.3 Replication strategy evaluation ........................ 38

7 Conclusion and further research ............................ 41

A Matlab code ................................................. 45
  A.1 Binomial Model ....................................... 45
  A.2 Analytical price calculation ........................... 46
  A.3 Iterative Implied Volatility calculation ............... 47
  A.4 Matlab code Black-Scholes pricing model ............. 47

B Data ....................................................... 49
  B.1 Quoted AEX CALL data ................................ 49
  B.2 Quoted AEX data 1998-2008 ........................... 50
Chapter 1

Introduction

Options are financial instruments that convey the right, but not the obligation, to make a future transaction on some underlying asset. The underlying asset typically is a stock, a share of a company, or any commodity. This explains why options are often referred to as derivative contracts or derivative securities. There are many types of options, each with different contract features. Options are actively traded on financial markets around the globe and they are often embedded in other financial products.

One might now ask oneself how this can possibly be related to Applied Mathematics. Well, the answer is simple. An option will always have a nonnegative value, which depends on its features and is determined on the financial market. The theoretical value of an option can be determined by a variety of Mathematical models. These models are developed by quantitative analysts, which is where the need for Mathematicians steps in. Financial models can predict how the value of an option will change as related conditions change, so that the risks associated with trading can be understood and managed.

The main goal of this project is to study the 'At-the-money forward percentage call spread', which is a special type of option. The properties of the ATM forward percentage call spread will be described in the next chapter. In chapter 3, a simple model will be introduced to price the option, after which the model will be calibrated by deriving the necessary parameters from market data (chapter 4). In chapter 5 we will proceed by introducing the Black-Scholes pricing model for the ATM forward percentage call spread. Finally we will discuss a strategy that hedges the risk involved with issuing an ATM forward percentage call spread.

Because this topic requires specific knowledge that is not covered by the Bachelor program Applied Mathematics, this report attempts to give the reader an overview of the used financial theory and terminology, without going into unnecessary details. The interested reader is invited to consult referenced literature for more detailed theory.
Chapter 2

Description of the Derivative

To get a clear understanding of what an option is and how it works, an important example will be introduced first: the European call option.

2.1 Standard Call Option

Suppose we decide to start up a little business in selling laptops to students. Regarding the 'laptop project' which is being spread out over Delft UT, this might turn out to be a very lucrative business. We have found a supplier in the United States that can deliver us the estimated demand of 3000 laptops in 6 months, and who we have signed a contract with today $(t = 0)$. The price per laptop is $700,-$ which means that we will have to pay $2,100,000 at the date of delivery $(t = T)$.

One problem with this contract is that it involves a certain currency risk. We don’t know what the currency rate will be six months from now, so we do not know how much EUR we will have to pay for our order. At the current rate of $0.64\text{€}$/\$, the price of the laptops would be €1,337,580 at $t = T$. However, as can be seen in table 2.1, this price can widely differ.

<table>
<thead>
<tr>
<th>exchange rate at $t = T$</th>
<th>price at $t = T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.85 \text{€}$/$</td>
<td>€1,794,872</td>
</tr>
<tr>
<td>0.73 \text{€}$/$</td>
<td>€1,532,847</td>
</tr>
<tr>
<td>0.64 \text{€}$/$</td>
<td>€1,337,580</td>
</tr>
<tr>
<td>0.56 \text{€}$/$</td>
<td>€1,186,441</td>
</tr>
</tbody>
</table>

Table 2.1: The changing currency rate involves a risk

We could think of different strategies that eliminate this risk.

1. The first one is to buy $2,100,000 right now and keep it for six months. Some drawbacks are that we would need this huge amount of money right now and we would have to tie it up for six months. Besides, buying the dollars at the prevailing rate would imply not taking advantage of a lower exchange rate at time $T$.

2. Another possibility would be to buy a forward contract for $2,100,000. Such a contract can be negotiated with a bank and holds the following two things:

...
• In six months from now \((t = T)\), the bank will deliver us $2,100,000.

• At \(t = T\) we will pay for these dollars at the rate of \(K \in \$/\), where the forward price \(K\) is determined at \(t = 0\). The costs of entering a forward contract are zero by definition.

However, also this contract does not allow taking advantage of a lower exchange rate at time \(t = T\).

3. What we would like to have is a contract which guards us against the risk of a higher exchange rate at \(t - T\), while still allowing us to take advantage of a low rate. Such contracts do exist and are called European call options. A formal definition is the following:

**Definition 2.1.1.** A European call option on the amount of \(X\) US dollars, with strike price \(K \in \$/\) and maturity \(T\) is a contract written at \(t = 0\) with the following properties:

- The holder of the contract has the right to buy \(X\) US dollars at the price \(K \in \$/\), at \(t = T\).
- The holder of the option has no obligation to buy the dollars.

So this contract gives its holder the right, but not the obligation, to buy some underlying asset, US dollars in this case. This example illustrates that a call option has a positive value when the underlying asset has a price \(S_T\) above the strike price \(K\) at time \(T\). A call option is said to be in-the-money when the strike price is below the market price of the underlying. It is clear that the option will not be exercised unless it ends in-the-money. Only then the option has a positive value at the expiration date \(t = T\). So the value of the option is a function of \(S_T\), and this payoff function is given by

\[
max\{S_T - K, 0\}.
\]

Accordingly, the value \(V_T\) of a call option at maturity date \(T\) is

\[
V_T = (S_T - K)^+,
\]

where the notation \((\ldots)^+\) indicates that we take the maximum of the expression between the parentheses and zero. The payoff diagram of a call option for our laptop example is shown in figure 2.1.

As mentioned before, there are many types of options. A call option gives the holder the right to buy, whereas a put option gives the holder the right to sell an underlying asset at the strike price. Furthermore, the prefix European means that the option can only be exercised at a prespecified date, the date of expiration. In the next section we will explain the properties of the ATM forward percentage call spread.

### 2.2 ATM Forward Percentage Call Spread

As can be deduced from its name, the option that we will be studying in this project is a special type of call option, and can be seen as a forward start option. A forward start option is an advance purchase of a put or call option that will become active at some specified future time. The price of the option is paid at time zero, and also the underlying security, the starting time and time to expiration are specified at that time.
2.2. ATM FORWARD PERCENTAGE CALL SPREAD

![Graph showing the payoff of an ATM forward percentage call spread.]

Figure 2.1: Payoff of a European call option

For the ATM forward percentage call spread, the following properties hold. The option has a start date $T_{\text{start}} > 0$ and an expiration date $T_{\text{final}} > T_{\text{start}}$. Both $T_{\text{start}}$ and $T_{\text{final}}$ are specified at the time of purchase $t = 0$, and the option can only be exercised on time $t = T_{\text{final}}$. From now on $T_s$ and $T_f$ will be used when referring to $T_{\text{start}}$ and $T_{\text{final}}$.

\[ t = 0 \quad t = T_s \quad t = T_f \]

At time $t = T_f$, this option pays $\mathcal{E}1$, for every percent that the underlying asset increases with respect to $T_s$, limited by a cap $C$. Also $C$ is a prespecified constant. A formal definition is:

**Definition 2.2.1.** An ATM forward percentage call spread on an underlying security, with cap $C$, start date $T_s$ and exercise date $T_f$, is a contract written at $t = 0$ with the following properties:

- At $t = 0$ the contract with a prespecified $C$ can be purchased on payment of its price.
- At $t = T_f > T_s$ the payoff is $\mathcal{E}1$ per percentage the price of the underlying increased since $t = T_s$, with a maximum of $C$.

Also this option will clearly be worthless if the market price of the underlying security decreases between $T_s$ and $T_f$. Its payoff is independent of the price development until $T_s$, and is described by:

\[ V = (\min\{100\frac{S_f - S_s}{S_s}, C\})^+ = (\min\{100\frac{S_f}{S_s} - 100, C\})^+ \quad (2.3) \]

where $S_s - S_{T_f}$ and $S_f - S_{T_f}$.

Looking at our laptop example, the working of the ATM forward percentage call spread could be interpreted as follows. If at $T_f$ the price of the dollars has increased, then we will receive a payoff proportional to the increase of the dollar price since $T_s$, with a maximum of $C$. This
could be regarded as buying the dollars at time \( T_f \) at the prevailing market price, with a certain discount given by the payoff. Since the payoff depends on both the market price of the underlying at \( T_s \) and \( T_f \), the payoff can best be visualized in a three-dimensional diagram. The payoff of an ATM forward percentage call spread on $2,100,000 can be seen in figure 2.2. In this figure we choose \( C = 100 \), which means that the payoff could rise up until 100 if the dollar price increased 100 percent or more. This example may seem unrealistic, but it gives a good impression of the shape of the payoff as a function of \( T_s \) and \( T_f \).

![Figure 2.2: Payoff of an ATM forward percentage call spread on $2,100,100](image)

The payoff of the ATM forward percentage call spread can also be seen as a combination of the payoffs of two European call options. Consider a European Call option written at time \( T_s \), with strike price \( K_1 \) and expiration date \( T_f \). Its payoff at \( T_f \) is then given by \((S_f - K_1)^+\). To let the payoff of the ATM forward percentage call spread be determined by this call option, its payoff function has to equal (2.3). This can be done by multiplying the final stock price with \( \frac{100}{S_s} \) and taking \( K_1 \rightarrow 100 \), obtaining the payoff function \((\frac{100}{S_s}S_f - 100)^+\).

A second European Call option (on the same underlying security) is used to simulate the cap. This option is also written at \( T_s \) and expires at \( T_f \), but has a different strike price \( K_2 > K_1 \) and payoff \((S_f - K_2)^+\). By also using the multiplied stock price here and subtracting this payoff from the payoff of the first option, the following expression is obtained.

\[
V = (\frac{100}{S_s}S_f - 100)^+ - (\frac{100}{S_s}S_f - K_2)^+ \tag{2.4}
\]

Both adapted payoff's and their combination are shown in figure 2.3. In this figure we choose \( K_2 = 250 \). This corresponds to an increase of 150% with respect to the strike price of our first option, and results in a cap for the ATM forward percentage call spread of 250. Choosing \( K_2 - 110 \) the resulting cap \( C \) would be 10.

From this we can conclude that the payoff of an ATM forward percentage call spread with maximum payoff \( C \) and time zero equal to \( T_s \) can be composed using two European Call options with strikes \( K_1 = 100 \) and \( K_2 = 100 + C \), and maturity \( T_f \). Equivalently, we can say that the value of an ATM forward percentage call spread with \( T_s \geq 0 \), can be calculated by discounting the time \( T_s \) value of two European Call options combined as described above. This result will prove very useful in chapter 6.
2.3 Relevance

The ATM forward percentage call spread is an existing option, and typically has the AEX index as its underlying. It is not traded on the market itself, but it does play a role in so-called structured products. These are financial products, often offered by banks, with which individuals buy a complete construction of options. Examples of products in which one comes upon this option go under the name ratchet option, cliquet option or reset option. These type of products lock in gains based on a time cycle, a year for example.

The product we will consider, resets the initial price periodically over the options lifetime, to bring out-of-the-money options back to being at-the-money. This is based on the believe that even if the market closes lower at maturity, it should close higher at the end of one or more sub periods. So it is a periodically reset option with multiple payouts, which are paid at the end of each reset period. The examples in figures 2.4 and 2.5 show how an option with sub periods can generate a higher cash flow than a standard option.

Composing structured products mostly consists of pricing exotic options, such as the ATM forward percentage call spread. In this project we will study how to price and replicate a periodically reset option consisting of ATM forward percentage call spreads with different starting times. For now we choose $T_f - T_s = 1$ year. The goal of this project is to answer the following questions.

1. How to price a periodically reset option consisting of ATM forward percentage call spreads?

2. What replication strategy hedges the risk involved with issuing a periodically reset option consisting of ATM forward percentage call spreads?
Figure 2.4: Example of reset option payoffs with ATM forward percentage call spreads on historic AEX data, with maturity 2 years and reset time 1 year

Figure 2.5: Example of reset option payoffs with ATM forward percentage call spreads on historic AEX data, with maturity 10 years and reset time 2 year
Chapter 3

Binomial Pricing Model

Prior to exercise, the option value, and therefore the price, varies with the price of the underlying asset in time. The price of the option must reflect the chance of the option to finish in-the-money. The price should thus be higher with more time to expiry and with a more volatile underlying instrument. Of course, the buyer and seller must agree on the initial value, otherwise the exchange of the option will not take place. The science of determining this value is the central tenet of financial mathematics. The most common method, which will be introduced in chapter 5, is to use the Black-Scholes formula. However, to understand how to compose a pricing model for our derivative, we will first introduce a more simple model: the Binomial pricing model. Mathematically, the Binomial model can be considered as a finite difference approximation of the Black-Scholes model.

3.1 Two-Period Binomial Model

The binomial asset-pricing model (also referred to as the Cox-Ross-Rubinstein model) provides a powerful tool to understand arbitrage pricing theory. In this section the simple binomial model as described in Shreve [6] will be adapted to our option, starting with only two periods.

The first period starts at time \( t = 0 \) and ends at \( t = T_s \), whereas the second period runs from \( t = T_s \) until \( t = T_f \). We consider one share of stock which price at time zero is denoted by \( S_0 \). The market price of the stock changes in time, so after one period its price will have another value. In the binomial model we make the following assumptions:

**Assumption 3.1.1 (binomial model).**

1. The price \( S \) over one period can only have two possible outcomes. An initial value \( S \) evolves either up or down.

2. The factor for an upward movement over the first period is \( u_1 \), for a downward movement \( d_1 \), where \( 0 < d_1 < u_1 \).

3. The factor for an upward movement over the second period is \( u_2 \), for a downward movement \( d_2 \), where \( 0 < d_2 < u_2 \).

4. The probability of a movement up (down) is \( p \) \((q - 1 - p)\), with \( p, q > 0 \).

For the two-period model this means that the price can take two different values after the first period, which we denote by \( S_s(H) \) and \( S_s(T) \). After the second period the price can take four different values, \( S_f( HH), S_f(HT), S_f(TH) \) and \( S_f(TT) \), as shown in figure 3.1. \( H \) and \( T \) stand for Heads and Tails, so imagine that the random mechanism of the price process is
determined by coin tossing. The probabilities of head and tails are \( p \), respectively \( q = 1 - p \).

\[
\begin{align*}
S_f(\text{HH}) &= u_1 u_2 S_0 \\
S_f(\text{HT}) &= u_1 d_2 S_0 \\
S_f(\text{TH}) &= d_1 u_2 S_0 \\
S_f(\text{TT}) &= d_1 d_2 S_0
\end{align*}
\]

Figure 3.1: Two period binomial model for the ATM forward percentage call spread

The up and down factors mentioned in the assumptions are

\[
u_1 - \frac{S_f(\text{HH})}{S_0}, \quad u_2 - \frac{S_f(\text{TH})}{S_0} = \frac{S_f(\text{TT})}{S_0}, \tag{3.1}\]

and

\[
d_1 - \frac{S_f(\text{HT})}{S_0}, \quad d_2 - \frac{S_f(\text{TT})}{S_0} = \frac{S_f(\text{HT})}{S_0}, \tag{3.2}\]

where \( d_1 < u_1 \) and \( d_2 < u_2 \).

We also introduce the risk-free interest rate \( r \) per year. One euro invested in the money market at time zero (by for instance putting it on a bank account) will yield \( 1 + r \) euro after a period of one year. For simplicity, we assume that both periods in the two-period binomial model are of equal length of one year. An essential feature of an efficient market is that if a trading strategy begins with no money and has zero probability of losing money, then it should also have zero probability of making money. Otherwise it would be possible to generate money out of nothing without any risk, which we call arbitrage. In reality, arbitrage opportunities do appear. Their existence however is of short duration because the market rebalances itself. Therefore, mathematical models admitting arbitrage cannot be used for analysis. If an arbitrage opportunity appears in reality aTo rule out arbitrage in the two-period model we must assume

\[
0 < d_3 < 1 + r < u_3, \tag{3.3}
\]

\[
0 < d_2 < 1 + r < u_1. \tag{3.4}
\]

It can easily be verified that there exists an arbitrage possibility if conditions (3.3) and (3.4) are not satisfied. Suppose that \( 1 + r < d \). Then we could borrow money from the bank in order to buy stock. Even if the stock would go down at time \( T_f \) it would be worth enough to pay off our debt. On the other hand, if \( 1 + r \geq u \), we could sell the stock short and invest the earned money in the money market. Even if the stock would rise, the value of the invested money would be enough to replace the stock, which again provides an arbitrage possibility.

As mentioned, the option value varies with the price of the underlying asset in time. Our aim is to determine the price of the option at the time it is issued, and therefore calculate \( V_0 \).
3.1. **TWO-PERIOD BINOMIAL MODEL**

The value $V_f$ at the final time can easily be calculated for each possible outcome of the stock price $S_f$ by using (2.3). Looking at figure 3.1, we can see that from the four possible values $V_f$, the expected value of our option at time $T$ can be calculated, discounted by an interest rate factor. For $V_s$ this would be

$$V_s(u) = \frac{1}{1 + r} E[V_f(u)] = \frac{1}{1 + r} [pV_f(TH) + qV_f(TT)], \tag{3.5}$$

$$V_s(T) = \frac{1}{1 + r} E[V_f(T)] = \frac{1}{1 + r} [pV_f(TH)] + qV_f(TT)]. \tag{3.6}$$

Using (3.5) and (3.6) we can now calculate the value of our option at $t = T$ in the same way:

$$V_0 - \frac{1}{1 + r} E[V_s] - \frac{1}{1 + r} [pV_s(H) + qV_s(T)]. \tag{3.7}$$

The numbers $\hat{p}$ and $\hat{q}$ here are chosen in such a way that

$$S_0 = \frac{1}{1 + r} [pS_s(H) + qS_s(T)]. \tag{3.8}$$

They are not the real probabilities, which we call $p$ and $q$, but rather the so-called *risk-neutral probabilities*, that make the rate of growth of the stock appear to be equal to the rate of growth of money on a bank account. Under the actual probabilities, the average rate of growth of the stock is typically greater than the rate of growth of an investment in the money market. Otherwise no one would be willing to take the risk of investing in the stock.

**Example 3.1.1.** For the two-period model of figure 3.1, let $S_0 = 40$, $u = 1.5$, $d = \frac{1}{2}$ and $r = \frac{1}{1}$. Depending on the result of the coin toss the stock price can take two values at $T$: $S_s(H) = 60$ and $S_s(T) = 20$. If we now let $u_s = 2$ and $d_s = \frac{1}{2}$, then $S_f(HH) = 120$, $S_f(HT) = 20$, $S_f(TH) = 40$ and $S_f(TT) = 6\frac{2}{3}$. Using these variables, we can solve equation (3.8) for the risk-neutral probabilities obtaining $\hat{p} = \frac{3}{4}$ and $\hat{q} = \frac{1}{4}$ between $t = T$, and $\hat{p} = \frac{11}{20}$ and $\hat{q} = \frac{9}{20}$ for the period between $T_s$ and $T_f$ (see figure 3.2).

![Figure 3.2: Two period binomial model for example 3.1.1.](image)

*At expiration, the payoff of is $V_f = (\min(100S_f, C))^+$. If we choose $C = 20$ we can compute the payoff $V_f$ of the option for each possible outcome of the coin toss (see figure).*
Using (3.5) and (3.6) it is now easy to calculate the possible values of our option at \( T \). For both values we find
\[
V_s = \frac{11}{5} \cdot \frac{11}{20} = 8.8.
\]  \hfill (3.9)

Applying (3.7) to \( V_s \), the value of this option is found to be \( V_0 = 7.01 \).

\[\square\]

### 3.2 Multi-period Binomial Model

In this section, the two-period model will be extended to a more realistic multi-period binomial model. Again we will use assumption 3.1.1, and toss a coin repeatedly. Whenever we throw Heads, the stock price moves up by the factor \( u \), whereas the price moves down by the factor \( d \) whenever we get a Tails. In this model, the same \( u \) and \( d \) will be used for every period. Because of the special role of \( t = T \), we will treat the multi-period binomial model in two separate parts. First we will discuss the model for the period between \( t = 0 \) and \( T \), second the model between \( T \) and \( T_f \) will be considered.

Also the risk-free interest rate per year \( r \) will be used, which is a constant per period. Because the option value will be discounted over multiple (let's say \( m \)) time steps of length \( \delta t \) per period, the interest rate per time step is \( \frac{r}{m} \). The growth factor over the whole period then becomes \( \left( \frac{1 + \frac{r}{m}}{2} \right)^{mt} \), where \( t \) is the time variable. Increasing the number of time steps, leads to the idea of a continuous model. If we let \( m \to \infty \), we get the following expression for the growth rate:

\[
\lim_{m \to \infty} \left( \frac{1 + \frac{r}{m}}{2} \right)^{mt} = e^{rt}.
\]  \hfill (3.10)

Equation (3.8) can now be written in the more general form

\[
S_n = e^{-r \delta t} E[S_{n+1}].
\]  \hfill (3.11)

In the next section we will show how to use this equation to derive parameters for the multi-period binomial model. Furthermore we assume that the multi-period binomial model satisfies the no-arbitrage condition \( d \leq e^{r \delta t} \leq u \).

For the period between \( t = 0 \) and \( T \), the initial stock price is denoted by \( S_0 \), which has a positive value. At time \( \delta t \), the price is denoted by \( S_1(H) = uS_0 \) or by \( S_1(T) = dS_0 \), depending on the outcome of the toss. After the second toss, the price will be one of:

\[
S_2(HH) = uS_1(H) = u^2 S_0, S_2(HT) = dS_1(H) = udS_0, S_2(TH) = uS_1(T) = u^2 S_0, S_2(HT) = dS_1(T) = d^2 S_0.
\]  \hfill (3.12)

After three tosses there will be eight possible outcomes, although not all of them result in different stock prices (see figure 3.1). At \( T \), there will be \( m + 1 \) different outcomes, where \( m \) denotes the number of time steps or coin tosses between time zero and \( T \). Using the described binomial tree we can simulate all the possible outcomes of \( S \), depending on the number of time steps.
3.2. MULTI-PERIOD BINOMIAL MODEL

\[ S_1(H) = uS_0, \quad S_1(T) = dS_0 \]

\[ S_2(HT) - udS_0 = S_2(TH) \]

\[ S_3(HH) = u^2S_0, \quad S_3(HTH) = S_3(THH) \]

\[ S_4(HHT) = u^3dS_0 \]

\[ S_5(HHHH) - u^3S_0 \]

\[ S_6(HTT) = u^2dS_0 - S_6(THT) \]

\[ S_7(TTT) = d^3S_0 \]

Figure 3.3: Multi-period binomial model between \( S_0 \) and \( S_s \)

Suppose now that the value of the option \( V_s \) is known for every possible price \( S_s \). Then we can calculate the value \( V_0 \) of our option at \( t = 0 \), using the same algorithm introduced for the two-period model. A general expression for this recursive algorithm is

\[ V_n = e^{-r\delta t} E[V_{n+1}] = e^{-r\delta t}[pV_{n+1}(H) + \hat{q}V_{n+1}(T)]. \]  \hspace{1cm} (3.14)

For the option of our example (3.14) can be written as

\[ V_0 = e^{-rT_f} E[V_T]. \]  \hspace{1cm} (3.15)

As in previous section, \( \hat{p} \) and \( \hat{q} \) are risk neutral probabilities such that (3.11) is satisfied.

So to find \( V_0 \), all possible values \( V_s \) of the option have to be known. To determine these, we will now proceed with the model for the period between \( T_s \) and \( T_f \). Because the payoff of the ATM forward percentage call spread depends on the difference between the prices on \( T_s \) and \( T_f \), we must obtain a different binomial tree for each possible stock price \( S_s \). For each of these (identical) trees, the initial stock price is denoted by one outcome of \( S_s \). As in the first tree, there are two possible outcomes for each coin toss, so the price at time \( T_{s+1} \) is denoted by \( S_{s+1}(H) = uS_s \) or by \( S_{s+1}(T) = dS_s \) (see figure 3.4). If we let the number of time steps between \( T_s \) and \( T_f \) be denoted by \( n \), then there will be \( n + 1 \) possible outcomes per outcome of \( S_s \). The time steps are of length \( \delta t \), equal to the first tree. The resulting tree is similar to the one for the first period.

\[ S_s \]

\[ S_{s+1}(H) = uS_s, \quad S_{s+1}(T) = dS_s \]

\[ S_{s+2}(HT) = udS_s, \quad S_{s+2}(TH) = S_{s+2}(THT) \]

\[ S_{s+3}(HH) = u^2S_s, \quad S_{s+3}(HTH) = S_{s+3}(THH) \]

\[ S_{s+4}(HHT) = u^3dS_s \]

\[ S_{s+5}(HHH) = u^3S_s \]

Figure 3.4: Multi-period binomial model between \( S_s \) and \( S_f \)

Attaching this tree to each possible outcome of \( S_s \), a total of \((m+1) \cdot (n+1)\) possible outcomes for \( S_f \) is obtained. For each of these outcomes the payoff can be calculated using (2.3).
Note that this payoff depends on both $S_n$ and $S_f$, which is why a separate tree is used for every possible $S_n$. As concluded before, all possible values of the option at $T^*$ are needed in order to calculate its price $V_0$. Applying formula (3.14) on the second binomial tree we can easily find all the values $V_n$ that are necessary to find $V_0$.

### 3.3 Derivation of parameters

So far, the values of the parameters $u$, $d$ and $\tilde{p}$ have remained undetermined. We will now fix them so that the model is consistent with the continuous case. From section 3.1 we know that the probabilities $\tilde{p}$ and $\tilde{q}$ are chosen in such a way that the rate of growth of the stock (equation (3.8)) appears to be equal to the (continuous) rate of growth of the money market. For the discrete model from previous section we can write

$$E[S_{n+1}] = \tilde{p}S_n + \tilde{q}S_{n+1} = \tilde{p}uS_n + \tilde{q}dS_n.$$  \hfill (3.16)

Equating to equation (3.11) we obtain an expression for $\tilde{p}$

$$e^{\sigma \delta t} - \tilde{p}u + \tilde{q}d = (1 - \tilde{p})d,$$  \hfill (3.17)

so

$$\tilde{p} = \frac{e^{\sigma \delta t} - d}{u - d}. \hfill (3.18)$$

To find expressions for $u$ and $d$, we equate the variance of the discrete model to the variance of the continuous model that will be introduced in chapter 5. In the variance of the continuous model we encounter the parameter $\alpha$, which denotes the volatility and can be seen as a measure for the standard deviation of the fluctuation in stock price. In the discrete model we have that

$$\text{Var}(S_{n+1}) = E[S_{n+1}^2] - (E[S_{n+1}])^2 = \tilde{p}(S_n)^2 + (1 - \tilde{p})(S_n)^2\tilde{d}^2 = S_0^2(\tilde{p}u + (1 - \tilde{p})d)^2.$$  \hfill (3.19)

In the continuous case the variance is

$$\text{Var}(S_{t+1}) = S_0^2e^{2\sigma^2\delta t}(e^{2\sigma^2\delta t} - 1).$$  \hfill (3.20)

The derivations of those variances can be looked up in Searle [5]. Equating and using (3.17) gives:

$$\tilde{p}u^2 + (1 - \tilde{p})d^2 = e^{2\sigma^2\delta t}e^{\sigma^2\delta t}.$$  \hfill (3.21)

If we now substitute $\alpha = e^{\sigma \delta t}$ and use $u = 1/d$, this equation can be written as:

$$u\alpha - 1 + d\alpha - \alpha^2e^{\sigma^2\delta t}.$$  \hfill (3.22)

Dividing by $\alpha$ and multiplying by $u$ gives the quadratic form

$$u^2 - u(\alpha^{-1} + \alpha e^{\sigma^2\delta t}) + 1 - 0, \hfill (3.23)$$

with solutions $u = \frac{1}{2}(\alpha^{-1} + \alpha e^{\sigma^2\delta t})$. Summarizing, the parameters $u$, $d$ and $p$ are now given by:

| $\beta$ | $\frac{1}{2}(e^{-r\delta t} + e^{(r+\sigma^2)\delta t})$ |
| $u - \beta$ | $\sqrt{\beta^2 - 1}$ |
| $d$ | $1/u = \beta - \sqrt{\beta^2 - 1}$ |
| $p$ | $\frac{e^{r\delta t} - d}{u - d}$ |
3.4 Explanatory remarks on the code

In order to calculate a price for the ATM forward percentage call spread, the multi-period binomial model introduced in section 3.2 is modeled in Matlab. In this section we will briefly go into the code (see Appendix A.1).

The first remark we will make is about the time mesh. As can be seen in the first two lines, the starting time $T_s$ and final time $T_f$ can be entered. Furthermore the number of time steps, which can be interpreted as the number of coin tosses, is a multiple of the least common divisor of $T_s$ and $T_f$, to make sure that the number of time steps in both intervals are integers. The variable $dt$ indicates the length of each time step and is used in the parameters derived in last section. The variables $L_s$ and $L_f$ indicate the number of possible outcomes for the stock price at $T_s$ respectively $T_f$.

$$Ts = \text{input('Tstart')}$$
$$Tf = \text{input('Tfinal')}$$
$$L = 1cm(Ts,Tf)*10 \text{ number of time steps}$$
$$dt = Tf/L; \text{ length time step}$$
$$Ls = 1+(Ts/dt); \text{ number of possible outcomes at } T_s$$
$$Lf = 1+((Tf-Ts)/dt); \text{ number of possible outcomes for } S_f$$

Using the time variables and the derived parameters, all the possible combinations of $S_s$ and $S_f$ are calculated, as well as their corresponding payoffs. Next, each of the $L_s$ possible values of the option at time $T_s$ are calculated:

$$Vs=[zeros(Ls,1); \text{ vector with the option values at } T_s$$
while $i<L_s$
$$V=SV((i-1)*Ls+1:i*Ls,2);$$
$$j=Lf;$$
while $j>=2$
$$k=[1:j-1];$$
$$V(k)=\exp(-r*dt)*(p2*V(k)+(1-p2)*V(k+1));$$
$$j=j-1;$$
cend;
$$Vs(i)=V(1);$$
$$i=i+1;$$
end;

The first while-loop indicates which out of the $L_s$ possible values for $S_s$ is calculated. The second while-loop takes the vector of let’s say $n$ possible values at time $t$ | $1$ and calculates a new vector with the $n - 1$ possible values at time $t$. This is repeated for all the time steps between $T_s$ and $T_f$, until the value of the option at $T_f$ is obtained. After the vector with possible values at $T_s$ is filled, a similar loop calculates the value of the option at $t = 0$, which is the price of the option we are looking for.
Chapter 4
Calibration of the Binomial Model

To see whether the binomial pricing model gives correct outcomes, first an analytical approach to the calculation of the option price will be derived. After making sure that the implemented model gives correct prices, the needed parameters will be derived from real market data in order to make our model fit reality.

4.1 Analysis

In this section we will derive an analytical expression for the option price, based on the binomial pricing model introduced in the preceding chapter. We will see that the binomial pricing model enables us to analytically find \( V_0 \). The reason why to use the recursive model introduced, is that it gives an intuitive approach of the course of a stock price. Besides, the number of time steps we can use in the analytical calculations is much lower because of the factorial operation necessary to calculate the binomial coefficient.

Suppose we have \( m \) time steps between \( t = 0 \) and \( T_x \), and \( n \) time steps between \( T_x \) and \( T_f \). Then the \( n + 1 \) possible stock prices for each \( S_x \) are given by \( S_f = u^j d^{n-j} S_x \) for \( j = 0, 1, \ldots, n \). If we leave the payoff limit \( K \) out of the calculations, the payoff at time \( T_f \) is given by

\[
V_f = (100\left(\frac{S_f}{S_x}\right))^+ - (100(u^j d^{n-j} - 1))^+, \quad j = 0, 1, \ldots, n \tag{4.1}
\]

The reason to leave \( K \) out of the calculations is to simplify the analysis. Using a very high \( K \) in our Matlab program must lead to the same results, enabling us to verify the model's accuracy.

From (4.1) we can see that an identical vector with \( n + 1 \) payoffs is obtained for every possible \( S_x \), independent of the stock price \( S_x \) itself. Recalling that \( u = 1/d \) in the model, it is easy to see that (4.1) will only give a positive payoff for \( j > n - j \), so only the payoffs with \( j > n/2 \) have to be considered. To calculate the value of the option for each possible \( S_x \), we have to multiply each value \( V_f \) with its probability and the binomial coefficient, sum over all \( j > n/2 \) and discount over the \( n \) time steps:

\[
V_x = 100e^{-rn \delta t} \sum_{j=[n/2]}^{n} \binom{n}{j} (u^j d^{n-j} - 1) p^j (1 - p)^{n-j}. \tag{4.2}
\]

The payoff, so also the value of the option, is independent of the course of the stock price between \( t = 0 \) and \( T_x \). So to find an expression for the price, (4.2) only needs to be discounted.
over \( m \) time steps:

\[
V_0 = 100e^{-r(m+1)n\Delta t} \sum_{j=-[n/2]+1}^{n} \binom{n}{j} (p^n n^j - 1)p^j(1-p)^n \frac{V_j}{n}.
\] (4.3)

To verify whether the recursive program gives correct outcomes, equation (4.3) was implemented (see appendix A.2) and its output was compared to the result of the binomial model. For certain \( r \) and \( \sigma \), several numbers of time steps and different \( T_k \) and \( T_f \), the calculated value \( V_0 \) was equal for both programs, so we assume our implementation is correct.

4.2 Determination of Market Parameters

In order to get realistic prices from our model, we need to have numerical estimates for all the input parameters. In our model, the input consists of \( T_k \), \( T_f \), \( S_0 \), \( C \), \( \sigma \) and \( r \). For \( T_k \), \( T_f \) and \( C \), we can choose what values to use, depending on the option. \( S_0 \) is the 'price' of the underlying at time zero. Since the underlying in this case is the AEX-index, \( S_0 \) is chosen equal to the height of the AEX, so \( S_0 = 395 \). The only parameters still unknown are the market parameters \( r \) and \( \sigma \).

For \( r \) we will take the spot rates, which are the basic interest rates. The spot rate \( r_t \) is the rate of interest, expressed in yearly terms, charged for the money held from time zero until time \( t \). Under a continuous model, the spot rate \( s_t \) is defined so that the growth rate corresponds with

\[
S_t = e^{s_t} S_0.
\] (4.4)

Spot rates can be measured using bonds. Bonds can be regarded as being derivative to interest rates. In particular, the price of a risk-free zero-coupon bond with maturity in \( n \) years, is a direct measure of the \( n \)-year interest rate. So the obvious way to determine spot rates is to find the prices of a series of zero-coupon bonds. Unfortunately, the set of available zero-coupon bonds is rather sparse. However, spot rates can be determined by the prices of coupon-bearing bonds by starting with short maturities and working forward toward longer maturities. We will do this using Staats Obligaties-data (Dutch treasury bills) from June 25th (see table 4.1).

The first things to determine for every bond are the time \( t \) left to maturity (using the number of left over trading days) and the number of remaining coupon payments. We assume that there are 250 trading days per year, and that coupons are paid once per year, on exactly the same date the bond was issued. Furthermore, note that the face value of every bond is \( F = 100 \), the price \( P \) is the last value marked to the market, and its coupon payment \( C \) is a certain percentage of the face value, depending per bond.

As mentioned above, we will now start by determining the spot rate for the bond with shortest time to maturity and zero coupon payments left. For this bond, the price should equal \( P = (100 + C)e^{-s_1 F} \) so the spot rate \( s_t \) is determined by

\[
s_t = \ln(\frac{100 + C}{P} F).
\] (4.5)

From table 4.1 we can see that \( t = 0.05 \), \( P = 100.02 \), and the coupon value is 5.25%. Suppose that someone would purchase this specific bond today. Within one month, this person would already receive the final payment including the last coupon payment of 0.0525\(F \). However, considering that the bond pays out coupons only once per year, the new owner would then
receive interest that was theoretically earned by the previous owner. For this reason, purchasing the bond today means that the coupon payment will only partly go to the new owner, while the first 237.5 days' interest accrues to the previous owner. In order to account for this accrued interest, the coupon payment is multiplied by the time left to maturity, giving $0.05C - 0.2625$.

This brings us to the following spot price value.

\[
s_{0.05} = \frac{1}{0.05} \ln\left(\frac{100 + 0.2625}{100.02}\right) = 0.0484.
\]

(4.6)

Using the same approach we find $s_{0.56}$ and $s_{0.82}$ as in the last column of table 4.1.

Next consider a bond with maturity in 1.05 years and one coupon payment left. This bond will make two more payments, for which we can write

\[
P = 0.05Cc^{s_{0.05}0.05} + (100 + C)c^{s_{1.05}1.05}.
\]

(4.7)

Since $s_{0.05}$ is already known, we can solve this equation for $s_{1.05}$. This same method can be used to determine $s_{1.56}$ and $s_{1.82}$. Next considering the bonds with two more coupon payments left, we can determine $s_{2.05}$ and $s_{2.56}$, and so forth. The results shown in the last column of table 4.1, result in the spot rate curve shown in figure 4.1.

The spot rates that will be used in our further analysis can be read from the graph, and are following.

\[
\begin{array}{cccccccccccc}
<table>
<thead>
<tr>
<th>s0</th>
<th>s1</th>
<th>s2</th>
<th>s3</th>
<th>s4</th>
<th>s5</th>
<th>s6</th>
<th>s7</th>
<th>s8</th>
<th>s9</th>
<th>s10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.048</td>
<td>0.045</td>
<td>0.046</td>
<td>0.046</td>
<td>0.047</td>
<td>0.046</td>
<td>0.046</td>
<td>0.046</td>
<td>0.047</td>
<td>0.047</td>
<td>0.047</td>
</tr>
</tbody>
</table>
\end{array}
\]

Table 4.2: Spotrates

More details about spot rates can be found in Luenberger [4].
The last parameter to be determined is the volatility $\sigma$. The volatility cannot be observed directly, but can be extracted from observed market data. One approach to do this, is to use historic data of the option involved, over a period of the same length as the time to maturity. The obtained volatility is called a historic volatility. Next to the fact that this approach does not take the expectation of the market into account, we also do not have historic data of the ATM forward percentage call spread to our disposal.

Another approach is to, given a quoted option value, find the $\sigma$ that leads to this value. A $\sigma$ computed this way is known as an implied volatility. $\sigma$ is implied by option value data in the market. Having found $\sigma$ using for example European Call options, we may use our model to value other options on the same asset, such as the ATM forward percentage call spread. Using this idea, we will show how to find the implied volatility for quoted AEX call options of different maturities. The resulting volatilities will be used to price ATM forward percentage call spreads of corresponding maturity.

For European call options, we take the Black-Scholes valuation formula (see Ch.5) of the type $V = v(T, S_0, K, r, \sigma)$. In this calibration approach, the unknown parameter $\sigma$ is calculated iteratively as a solution of the implicit equation

$$f(\sigma) = V - v(T, S_0, K, r, \sigma) = 0.$$  \hspace{1cm} (4.8)

Given a quoted option value $V$, our task is to find the implied volatility $\sigma^*$ that solves (4.8). Using the Newton-Raphson method, the form

$$\sigma_{n+1} = \sigma_n - \frac{f(\sigma_n)}{f'(\sigma_n)}$$  \hspace{1cm} (4.9)

is obtained, where $f'(\sigma_n)$ is minus the derivative of $v$, and for which the following relation holds.

$$\lim_{n \to \infty} |\sigma_n - \sigma^*| = 0$$  \hspace{1cm} (4.10)

To execute the iteration, a numerical value for $\sigma_0$ is needed. Without further explanation we will use

$$\sigma_0 = \sqrt{\frac{2 \log(S_0/K) + r(T - t)}{T - t}}.$$  \hspace{1cm} (4.31)

The interested reader is encouraged to read the related derivations in Higham [3]. Note that to use Newton-Raphson we need the derivative of the option price. For the Black-Scholes formula this is known, and we can use this. But for pricing formulas like the binomial, where the partial
derivatives are not that easy to calculate, simple bisection is the preferred algorithm.

![Figure 4.2: Implied volatility and volatility surface for AEX CALL options of different maturity](image)

With the use of iteration (4.9) and quoted options' data, it is easy to find $\sigma^*$. The market data used for this can be seen in appendix B.1, and the implementation of the iterative algorithm is included in appendix A.3. A plot of volatility curves for options of different maturities is shown in figure 4.2. We see characteristic differences in implied volatility. Options that mature earlier exhibit a larger swing in implied volatility than options with longer maturities. This swing shows a pattern in which at-the-money options tend to have lower implied volatilities than other options. This tells that there is a bigger demand for in-the-money and out-of-the-money options than for at-the-money-options. The resulting graph is often referred to by the term volatility smile. If we plot implied volatility as a function of both strike price and time to maturity, an implied volatility surface as shown on the right hand side of figure 4.2 is obtained. Considering the displayed results, we will fix $\sigma = 0.2$ for our further calculations, regardless of the time to maturity.

### 4.3 Pricing ATM forward percentage call spreads

In this section prices will be determined of options with different $C$, $T_s$ and $T_f$, using the binomial model and the parameters determined in previous section. Only ATM forward percentage call spread options with $T_f = T_s + 1$ will be considered, since these will be used to compose periodically reset options.

The price as a function of $T_s$ and $C$ is displayed in figure 4.3.
<table>
<thead>
<tr>
<th>$T_s$</th>
<th>$T_f$</th>
<th>$r$</th>
<th>Price</th>
<th>$C = 5$</th>
<th>$C = 10$</th>
<th>$C = 15$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.045</td>
<td>2.4133</td>
<td>4.3802</td>
<td>5.9398</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.046</td>
<td>2.2913</td>
<td>4.1540</td>
<td>5.6298</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0.046</td>
<td>2.1883</td>
<td>3.9672</td>
<td>5.3799</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0.047</td>
<td>2.0877</td>
<td>3.7898</td>
<td>5.1416</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>0.046</td>
<td>1.9897</td>
<td>3.6163</td>
<td>4.9078</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>0.046</td>
<td>1.9008</td>
<td>3.4506</td>
<td>4.6792</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>0.046</td>
<td>1.8187</td>
<td>3.3002</td>
<td>4.4700</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>0.046</td>
<td>1.7400</td>
<td>3.1565</td>
<td>4.2822</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>9</td>
<td>0.047</td>
<td>1.6482</td>
<td>2.9889</td>
<td>4.0565</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>0.047</td>
<td>1.5805</td>
<td>2.8707</td>
<td>3.8797</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.3: ATM forward percentage call spread prices

![Graph showing the binomial determined price as a function of $T_s$ and cap $C$.](image)
Chapter 5

Black-Scholes Pricing Model

As mentioned, the most common used method for option pricing is the Black-Scholes formula. In this chapter we will show how this formula can be used to determine the price of an ATM forward percentage call spread. In the first section we will give an overview of the Black-Scholes theory, applied on the Europopn Call option. For a complete analysis and derivation of the used results we refer to Björk [2]. In the second section of this chapter, the Black-Scholes formula will be adapted to the ATM forward percentage call spread.

5.1 The Black-Scholes Equation

We start by assuming that the market consists of two assets:

- a risk free asset with price process $B$ and dynamics

$$dB(t) = rB(t)dt,$$

(5.1)

corresponding to a bank with interest rate $r$, and

- a stock with stock price dynamics given by

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t),$$

(5.2)

where $\alpha$ and $\sigma$ are deterministic constants. The diffusion term $\sigma$ is the volatility we introduced earlier, while the drift term $\alpha$ of the process is the local rate of return.

Since $S$ is a Markov process we can make the following assumptions

**Assumptions 5.1.1 (Black-Scholes model).**

1. The derivative instrument in question can be bought and sold on a market.
2. The market is free of arbitrage.
3. The price process for the derivative asset is of the form

$$V(t) = F(t, S(t)),$$

(5.3)

where $F$ is some smooth function.

Our task is now to determine what $F$ might look like. Applying the Itô formula to (5.2) and (5.3) and denoting the relative portfolio as described in Björk [2], the following important proposition can be derived.
**Theorem 5.1.2** (Black-Scholes Equation). Assume that the market is specified by equations (5.1) and (5.2). Then the only pricing function of the form (5.3) which is consistent with the absence of arbitrage is when $F$ is the solution of the following boundary value problem in the domain $[0, T] \times \mathbb{R}_+$:

\[
\begin{align*}
F_t(t, s) &+ r s F_s(t, s) + \frac{1}{2} s^2 \sigma^2(t, s) F_{ss}(t, s) = r F(t, s) = 0, \quad (5.4) \\
F(T, s) &= V(T). \quad (5.5)
\end{align*}
\]

The question now is how to solve the pricing equation (5.4). Using the Feynman-Kac stochastic representation formula, we can see that the solution is given by

\[
F(t, s) = e^{-r(T-t)} E^\mathbb{P}_t [V(X(T))],
\]

where $X$ is a process defined by

\[
\begin{align*}
dX(u) &= rX(u)du + X(u)\sigma(u, X(u))d\tilde{W}(u), \quad (5.7) \\
X(t) &= s,
\end{align*}
\]

and where $\tilde{W}$ is a Wiener process. Note that the SDE (5.7) looks exactly like the price process (5.2), except for one important difference. Whereas $S$ has local rate of return $\alpha$, the $X$-process has the interest rate $r$ as its local rate of return. In section 3.1 we choose the probability $\tilde{\mathbb{P}}$ such that under this $\tilde{\mathbb{P}}$-measure the rate of growth of the stock price appeared to be equal to the rate of growth of the money market, the interest rate $r$. In order to do the same for the price process in (5.2), we write $S$ instead of $X$ in (5.7) and define a $\mathbb{P}$-Wiener process $\tilde{W}$. Then the $\mathbb{P}$-dynamics of $S$ is defined as

\[
dS(t) = rS(t)dt + S(t)\sigma(t, S(t))d\tilde{W}(t). \quad (5.9)
\]

We may now state the following result for derivative pricing:

**Theorem 5.1.3** (Risk Neutral Valuation). The arbitrage free price of an option with payoff $V(S(T))$ is given by $V(t) = F(t, S(t))$, where $F$ is given by the formula

\[
F(t, s) = e^{-r(T-t)} E^\mathbb{P}_t [V(S(T))],
\]

and the $\mathbb{P}$-dynamics of $S$ are as in (5.9).

The SDE (5.9) can be recognized as geometric Brownian motion, thus the solution can be written as

\[
S(T) = s \exp\left\{ (r - \frac{1}{2} \sigma^2) (T - t) + \sigma \sqrt{T - t} \right\} W(T) - W(t) \right\}.
\]

Substituting this result into (5.10) we obtain the pricing formula

\[
F(t, s) = e^{-r(T-t)} \int_{-\infty}^{\infty} V(se^z)f(z)dz,
\]

where $Z$ is a stochastic variable with the distribution $N((r - \frac{1}{2} \sigma^2)(T - t), \sigma \sqrt{T - t})$, and $f$ is the corresponding density function. The function $V$ in the integral is the payoff function which depends on the contract. For the European call option, where $V = (S - K)^+$, we can evaluate (5.12) analytically. After some calculations for which we refer to Björk [2], we are left with the following famous result:
5.2 Black-Scholes for the ATM forward percentage call spread

In section 2.2 we have shown how the value of an ATM forward percentage call spread with cap C can be calculated by discounting the time $T_s$ value of two combined European Call options: one with payoff $(\frac{100}{S_0} S_T - 100)^+$ and one with payoff $(\frac{100}{S_0} S_T - 100 + C)^+$. Thanks to this result, we can use the Black-Scholes formula for European call options to find a price for the ATM forward percentage call spread.

Suppose we want to price an ATM forward percentage call spread with certain start and final times $T_s$ and $T_f$, and cap C. Using the Black-Scholes formula, a time $T_s$ value can be calculated for the two European call options introduced earlier. For our 'adapted' European call options, (5.13) becomes:

\[
V_s = 100 N(d_1(T_s, S_0)) c \frac{r(T_f, T_s) cN(d_2(T_s, S_0))}{\sigma \sqrt{T_f - T_s}} \{ \ln \left( \frac{100}{C} \right) + (r + \frac{1}{2} \sigma^2) (T_f - T_s) \},
\]

\[
d_1(T_s, S_0) = \frac{1}{\sigma \sqrt{T_f - T_s}} \{ \ln \left( \frac{100}{C} \right) + (r + \frac{1}{2} \sigma^2) (T_f - T_s) \},
\]

\[
d_2(T_s, S_0) = d_1(T_s, S_0) - \sigma \sqrt{T_f - T_s}.
\]

After obtaining the prices $V_1$ and $V_2$ of both European call options at time $T_s$, we can easily find the value of the ATM forward percentage call spread $V$ by applying equation (2.4). In order to find the value of our option at time zero, the time $T_s$ price has to be discounted by the growth rate. After all, our model is based on that assumption, so

\[
V_0 = e^{-rT_s} (V_1 - V_2).
\]

This leads to our next proposition.

**Proposition 5.2.1** (Black-Scholes price). The price of an ATM forward percentage call spread with cap C, start time $T_s$ and final time $T_f$ is given by the formula $V_0 = e^{-rT_s} (V_1 - V_2)$, where

- $V_1$ and $V_2$ are both Black-Scholes prices of European call options on the AEX-index, written at $T_s$ and with maturity $T_f$,
- Option 1 has strike price $K_1 = 100$ and payoff $(\frac{100}{S_0} S_T - 100)^+$, and
- Option 2 has strike price $K_2 = 100 + C$ and payoff $(\frac{100}{S_0} S_T - 100 - C)^+.$
5.3 Pricing ATM forward percentage call spreads

As in section 4.3 we can now determine prices of ATM forward percentage call spread options. The resulting prices (table 5.1) appear to be similar to the prices produced by the Binomial model. As already mentioned in chapter 3, the Binomial model can in fact be interpreted as a numerical method to solve the Black-Scholes equation.

<table>
<thead>
<tr>
<th>$T_s$</th>
<th>$T_f$</th>
<th>$r$</th>
<th>Price</th>
<th>$C = 5$</th>
<th>$C = 10$</th>
<th>$C = 15$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.045</td>
<td>2.3939</td>
<td>4.3387</td>
<td>5.8771</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
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<td>2.2885</td>
<td>4.1477</td>
<td>5.6185</td>
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<tr>
<td>2</td>
<td>3</td>
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<td>2.1878</td>
<td>3.9652</td>
<td>5.3712</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0.047</td>
<td>2.0916</td>
<td>3.7908</td>
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<td></td>
</tr>
<tr>
<td>4</td>
<td>5</td>
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<td>1.9905</td>
<td>3.6240</td>
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</tr>
<tr>
<td>5</td>
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<tr>
<td>6</td>
<td>7</td>
<td>0.046</td>
<td>1.8274</td>
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<tr>
<td>7</td>
<td>8</td>
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<td>8</td>
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<td>1.6701</td>
<td>3.0270</td>
<td>4.1003</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>0.047</td>
<td>1.5966</td>
<td>2.8938</td>
<td>3.9199</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.1: ATM forward percentage call spread prices

Figure 5.1: Black-Scholes price as a function of $T_s$ and cap $C$
5.3. PRICING ATM FORWARD PERCENTAGE CALL SPREADS

We have now found a method that easily prices ATM forward percentage call spreads. Using this result we can formulate an answer to the question 'how to price a periodically reset option consisting of ATM forward percentage call spreads'. The periodically reset options considered here have maturities varying from $T = 1$ till $T = 10$, where an option with maturity $n$ years is a product consisting of $n$ ATM forward percentage call spreads, of which each covers one year. For example, an option with maturity time 3 consists of three ATM forward percentage call spreads, one with $T_s = 0$ and $T_f = 1$, a second with $T_s = 1$ and $T_f = 2$, and a third one with $T_s = 2$ and $T_f = 3$. Prices of the periodically reset options are determined by simply taking the cumulative sums of the ATM forward percentage call spreads. The results for different caps $C$ are shown in table 5.2

<table>
<thead>
<tr>
<th>maturity</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$C = 5$</td>
</tr>
<tr>
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<td>2.3939</td>
</tr>
<tr>
<td>2</td>
<td>4.6824</td>
</tr>
<tr>
<td>3</td>
<td>6.8702</td>
</tr>
<tr>
<td>4</td>
<td>8.9618</td>
</tr>
<tr>
<td>5</td>
<td>10.9613</td>
</tr>
<tr>
<td>6</td>
<td>12.8728</td>
</tr>
<tr>
<td>7</td>
<td>14.7002</td>
</tr>
<tr>
<td>8</td>
<td>16.4472</td>
</tr>
<tr>
<td>9</td>
<td>18.1173</td>
</tr>
<tr>
<td>10</td>
<td>19.7139</td>
</tr>
</tbody>
</table>

Table 5.2: Prices of periodically reset options consisting of ATM forward percentage call spreads
Chapter 6

Replication strategies

In finance, a hedge is an investment that is taken out to reduce the risk in another investment. If properly hedged, the impact of a negative event is reduced. Financial institutions offer a variety of option products to their clients. Exposure to the risk involved with selling such options is often hedged by replicating the option using other derivatives, and offset a long position in this replicating portfolio. In the case of the ATM forward percentage call spread, the option does not correspond to standardized products traded by exchanges, but it does closely follow the AEX index. We will therefore show how to use AEX index futures to hedge the ATM forward percentage call spread. Also a hedging method using options will be shortly introduced.

In the next few sections we use as an example a financial institution that has sold 1000 periodically reset options with cap C = 15 and maturity T = 2 years, for a total of €15000. As calculated in previous chapter, the Black-Scholes price of this product is about €1150, so the financial institution has sold the options for €3500 more than its theoretical value. However, it is now faced with the problem of hedging its risks.

One strategy, referred to as a naked position, is to do nothing. If in our example the sum of the relative growth of the AEX over both years does not exceed 3.5%, this strategy works pretty well. It does not work as well if the AEX increases over 3.5% until maturity. For example, if over the first year the AEX increases 17%, and another 8% over the second year. Then the option costs the financial institution $1000 \times (€15 - €8) = €23000$, which is considerably larger than the €15000 charged for it.

6.1 Hedging with index futures

In section 2.1 we mentioned a forward contract on U.S. dollars. Like forward contracts, futures contracts are agreements to buy or sell an asset at a future time for a certain price. The main difference is that futures are traded on organized exchanges, and the contract terms are organized by that exchange. Every investor in futures markets keeps a so-called margin account. This account is adjusted daily to reflect gains or losses. The fundamentals and applications of futures markets are extensively treated in Bailey [1].

A futures contract on a stock index can be seen as the agreement to buy or sell the hypothetical portfolio of stocks traced by the index. Financial futures on stock indices are settled in cash because it is inconvenient or impossible to deliver the underlying asset. The contract size of an AEX index future is 200, so having a long position, a one point decrease of the AEX will result in a €200 loss. Available maturities range from one month up to one year. As futures contracts on stock indices in general, AEX futures are settled in cash. It is important to realize that a hedge using futures can result in a decrease as well as in a increase of profits, compared to the
situation without the hedge.
A reasonable criterion for the success of a hedge is that the value of the hedged portfolio should increase by an amount equal to the risk-free rate of return over the life of the hedge. In order to use AEX-index futures contracts as a hedge instrument, it is necessary to establish a link between the return on the portfolio to be hedged and the return on the index. Because our portfolio consists of 1-year ATM forward percentage call spreads for subsequent years, we will set up a hedge that it is rebalanced every year at the time our periodically reset option pays off.

Consider a period of one year, from \(t\) to \(t+1\). According to the market model (see Lucenberger [4]), the rate of return of the portfolio \(r_P\) is given by

\[
r_P = r + (d + \frac{S_{t+1}}{S_t} - r) \beta + \epsilon,
\]

where \(r\) is the risk-free interest rate, \(d\) is the dividend return of the index (4.02 for the AEX\(^1\)), \(\epsilon\) is a random error with \(E[\epsilon] = 0\), and \(\beta\) is the beta-coefficient of the ATM forward percentage call spread. If the portfolio exactly mirrors the index, the beta-coefficient equals one. In our case however, the option mirrors the index’ relative growth, so an equal increase of the index has a bigger impact for low values of the portfolio. This leads to the idea of a beta-coefficient depending on \(V_0\). For our calculations we will use \(\beta = \frac{\beta}{V_0}\).

Equation (6.1) can now be simplified as follows:

\[
r_P = r + (\frac{dS_t + S_{t+1} - S_t - rS_t}{S_t} - r) \beta + \epsilon
\]

\[
= r + (\frac{S_{t+1} - (d + r)S_t}{S_t}) \beta + \epsilon
\]

\[
= r + (\frac{f_{t+1} - f_t}{S_t}) \beta + \epsilon
\]

with \(f_t\) the futures price at time \(t\). Multiplying the obtained expression with the value of the portfolio \(V_0\) gives the estimated return of the portfolio between \(t\) and \(t+1\). The payoff on the AEX-futures contracts is given by \(200 \cdot n(f_{t+1} - f_t)\), where \(n\) is the number of contracts to be purchased. The payoff from the hedge strategy over the first year equals the sum of the payoff from the futures position and the return on the portfolio itself:

\[
200 \cdot n(f_{t+1} - f_t) + rV_0 + (\frac{f_{t+1}}{S_t} - f_t) \beta V_0 + \epsilon V_0
\]

(6.2)

If we choose the number of contracts for hedging to satisfy \(n = \frac{\beta V_0}{200S_t}\), the return on the hedged portfolio equals \((r + \epsilon)V_0\). Since the expectation of \(\epsilon\) is zero, the return is \(rV_0\) and equals the risk-free rate.

Recall our example, and suppose that after one year the AEX-index reaches 420 points. The costs of the option will be 1000 \(\cdot\) 100 \(\cdot\) 420 \(\cdot\) 0.5 = €6329.11. The chosen number of futures contracts for hedging is \(\beta = \frac{\frac{420-395}{395}}{200} = 1.27\), which is rounded to 1. This leads to a payoff of 200(420-397) = €4600 on the futures. After the second year, an AEX-level of 425 points is observed. The option will cost €1190.48, and with the purchase of again one futures contracts the gain on this futures position is €600. The total costs of the option are €7519.59, while the chosen hedging strategy reduces the total costs of the portfolio to €2319.59.

\(^1\)Taken from dividendpagina.nl
6.2 HEDGING WITH INDEX OPTIONS

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>305</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_0$</td>
<td>11500.00</td>
</tr>
<tr>
<td>$S_1$</td>
<td>420</td>
</tr>
<tr>
<td>$S_2$</td>
<td>425</td>
</tr>
</tbody>
</table>

| Number futures contracts $T_1$ | 1 | 1 | 1 | 1 | 1 | 1 |
| Payoff option $T_1$ | -6329.11 | -13924.05 | -15000.00 | 0.00 | 0.00 | 0.00 |
| Payoff futures $T_1$ | 4600.00 | 10600.00 | 20600.00 | 19400.00 | 14400.00 | 9400.00 |
| Increase AEX 1st yr | 6.33% | 13.92% | 26.58% | -24.05% | -17.72% | -11.39% |
| Hedging strategy payoff $T_1$ | -1729.11 | -3324.05 | 5600.00 | -19400.00 | -14400.00 | -9400.00 |

| Number futures contracts $T_2$ | 1 | 1 | 2 | 2 | 1 |
| Payoff option $T_2$ | -1190.48 | -4444.44 | -2000.00 | -15000.00 | -15000.00 |
| Payoff futures $T_2$ | 600.00 | 3600.00 | 1600.00 | 79200.00 | 89200.00 | 49600.00 |
| Increase AEX 2nd yr | 1.19% | 4.14% | 2.00% | 66.67% | 69.28% | 71.43% |
| Hedging strategy payoff $T_2$ | -590.48 | -844.44 | -400.00 | 64200.00 | 74200.00 | 34600.00 |

| Costs without hedging | 7519.59 | 18368.50 | 17000.00 | 15000.00 | 15000.00 |
| Costs hedged portfolio | 2319.59 | 4168.50 | -5200.00 | -44800.00 | -59800.00 |
| Gain hedge | 5200.00 | 14200.00 | 22200.00 | 59800.00 | 71800.00 |

Table 6.1: Performance of AEX-futures hedge

Table 6.1 summarizes these calculations together with similar calculations for other values of the AEX-index at $T_1$ and $T_2$. From the table we see that for our example, this strategy generally reduces the impact of growth of the AEX-index. Looking at the performance of this strategy per year we notice that the strategy works pretty good for swings of the AEX between 0% and 15%. This is not a very surprising observation, since the payoff of the ATM forward percentage call spread is limited by these two percentages. Summing the hedging strategy payoff over both periods can lead to good results in other cases as well. It is however important to note that the opposite might happen as well for different examples. In the last section of this chapter we will test the performance of this strategy on historical AEX-data.

6.2 Hedging with index options

According to the Black-Scholes theory (section 5.1), an option behaves like a weighted portfolio of risky stock and risk-free bonds. Theoretically you can therefore own a portfolio of stock and bonds, achieving exactly the same return. To do so, the weights in the portfolio need to be adjusted continuously. This is referred to as dynamic option replication. Traders often hedge the risk involved with options, by offsetting a reverse position in the dynamic replication portfolio. Because it is theoretically impossible to continuously adjust the portfolio, and there are transaction costs associated with adjusting the weights of the portfolio which grow as the frequency of adjustment grows, this strategy involves a certain inaccuracy. Most traders rebalance their portfolio only once a week. Dynamic hedging strategies involve the use of the Greeks, which are measures to an options’ risk in different dimensions.

In this section we will consider a dynamic hedge involving the use of delta. The delta ($\Delta$) of an option is defined as the rate of change of the option price with respect to the price of the underlying asset. In general,

$$\Delta = \frac{\delta V}{\delta S}.$$  (6.3)
where \( V \) is the earlier introduced option price and \( S \) the stock price. Consider a short position in an option with price \( V \). Suppose now that the stock price changes \( \Delta S \). Then the impact of this change in stock price on the option is \( \Delta \Delta S \). If we initially purchase \( \Delta \) shares of stock, the impact of the change in stock price on our stock position equals \( \Delta \Delta S \). The delta of the investor’s overall position is therefore zero, and the position is referred to as being delta neutral. This procedure of keeping a portfolio delta neutral is called delta hedging.

For a short position in one European call option, it can be shown that

\[
\Delta(\text{call}) = N(d_1),
\]

where \( d_1 \) is defined as in equation (5.14). Using delta hedging for a short position in a European call option, involves maintaining a long position of \( N(d_1) \) for each option sold. If we now consider an ATM forward percentage call spread a combination of two European call options as described in section 2.2, we can use delta hedging. The delta of an ATM forward percentage call spread would then be a combination of the delta’s of the two European call options. For \( C = 15 \) we have performed the calculations. In the last column of table 6.2 one can see how many AEX calls to purchase in order to hedge a short position in one AEX forward percentage call spread of corresponding start en final times.

<table>
<thead>
<tr>
<th>( T_0 )</th>
<th>( T_f )</th>
<th>delta</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>9.8256</td>
</tr>
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<td>9.8206</td>
</tr>
</tbody>
</table>

### 6.3 Replication strategy evaluation

To evaluate the performance of the hedging strategy presented in section 6.1, we will use the AEX data from the last 10 years as shown in figure 6.1 and the table. To evaluate the delta hedging method from previous section, it should be further worked out.

Now recall the periodically reset options of different maturities, presented in section 5.3. For each of these options, the performance of the hedging strategy introduced in this chapter will be calculated. We assume that all options matured on July 25, 2008. So the option with maturity 10 years was purchased in 1998. As in the example at the beginning of this chapter, the overall payoff of both the option and the hedged portfolio are calculated for different values of the cap \( C \). As a measure of the performance of the hedge we use the following relation:

\[
\text{performance} = (\text{payoff portfolio} - \text{payoff hedged portfolio})/\text{payoff portfolio}
\]

So the performance measures the relative gain of the hedge. A negative performance is an indication that the hedged portfolio costs more than the same portfolio without the hedge. A
performance of for example 0.5 means that the hedge reduces the loss with 50% compared to the situation without a hedge. The results are summarized in figure 6.3.

Looking at the figure we can note that the hedge performs better as the maturity of the periodically reset option increases. For maturities lower than 4, the performance is negative, from which we have to conclude that the presented strategy makes no sense in those cases. For periodically reset options with maturities greater than 3, the hedge seems to reduce losses, for high maturities even more than 50%. Also, the performance for all maturities increases as the cap increases. This has a logical explanation. The performance of the hedge is inversely proportional to the payoff of the hedged portfolio divided by the payoff of the portfolio without the hedge. Since we have not accounted for the presence of a cap in the determination of the number of futures contracts for hedging, both payoffs are proportional. Therefore a low cap results in a higher ratio between both payoffs, so in a lower performance.
Figure 6.2: Hedging strategy performance
Chapter 7

Conclusion and further research

In chapter two, the goal of this project was formulated as to answer the following questions.

1. How to price a periodically reset option consisting of ATM forward percentage call spreads?

2. What replication strategy hedges the risk involved with issuing a periodically reset option consisting of ATM forward percentage call spreads?

In the first part of this report, two pricing-models have been developed for ATM forward percentage call spreads of different caps and maturities. Prices resulting from both the binomial and the Black-Scholes pricing model did correspond, from which we conclude that the models give realistic prices. In chapter four, real market data have been used to derive the parameters necessary for our models. Composing periodically reset options as a combination of several ATM forward percentage call spreads, a price of these options was found by summing prices of the corresponding ATM forward percentage call spreads. We have thus answered the first question.

An answer to the second question turned out to be a bit more complicated, since there is not always an unambiguous recipe for setting up successful hedges in finance. Therefore it is hard to judge whether the method presented in section 6.1 performs good enough. Considering its low performance for low caps and maturities, it is recommendable to investigate other hedging possibilities a bit further. The second method introduced will probably perform much better, since delta hedging proved itself a successful strategy. A further elaboration on the method and an evaluation of its performance should prove this. For now we have to conclude that have not yet found a reliable hedging strategy, however there are many opportunities left open.

A continuation of this project would first of all require more tests of the presented hedging strategy on several simulated data sets. Furthermore, a comparison to alternative approaches to the problem of hedging exposure, should prove the reliability of our strategy. Alternative approaches include the dynamic hedging strategy introduced in section 6.2. It would be interesting to further develop the idea of delta hedging for the ATM forward percentage call spread, and evaluate its performance for periodically reset options.

In short, there is plenty of room left for elaboration on the replication and risks of the ATM forward percentage call spread. This report offers a substantial basis for any future research on this topic.
Bibliography


Appendix A

Matlab code

A.1 Binomial Model

% time mesh
Ts = input('Tstart');
Tf = input('Tfinal');
L = lcm(Ts,Tf)*10; % number of time steps
dt = Tf/L; % length time step
Ls = 1+(Ts/dt); % number of possible outcomes Ss
Lf = 1+(Tf-Ts)/dt; % number of possible outcomes for Ss

% parameters
S0 = input('Stock price at time 0');
sigma = 0.2;
K = input('strike price K');
r = 1/b; % input('interest rate')
beta = (exp(-r*dt) + exp((r + sigma^2)*dt))/2;
u1 = beta + sqrt(betav2-1); % up factor 1
d1 = 1/u1; % down factor 2
u2 = beta - sqrt(betav2-1); % up factor 2
d2 = 1/u2; % down factor 2
p1 = (exp(r*dt) - d1)/(u1-d1);
p2 = (exp(r*dt) - d2)/(u2-d2);

% Calculating binomial tree
k = [1:Ls];
l = [1:Lf];
Ss(1:Ls) = S0*u1.^(Ls-k).*d1.^k; % prices at Sstart
Ssf(1:Lf) = u2.^(Lf-l).*d2.^(l-1); % price factor between Sstart and Sfinal

% Calculating Sfinal and Vfinal
SV=zeros(Ls*Lf,2);
for i=1:Ls
    for j=1:Lf
        c=(i-1)*Lf+j;
        SV(c,1)=Ss(i)*Ssf(j); % calculates final stock price Sf
        SV(c,2)= max(min(100*(SV(c,1)-Ss(i))/Ss(i),K),0); % calculates payoff Vf
end;
end;

% calculate the values at Tstart, backward recursive
i=1;
Vs=zeros(Ls,1); % vector with the option values at Ts
while i<=Ls
    V=SV((i-1)*Lf+1:i*Lf,2);
    j=Lf;
    while j>=2
        k=[1:j-1];
        V(k)=exp(-r*dt)*(p2*V(k)+(1-p2)*V(k+1));
        j=j-1;
    end;
    Vs(i)=V(1);
    i=i+1;
end;

% calculating the value at t0, backward recursive
V=Vs;
l=Ls;
while 1>=2
    k=[1:l-1];
    V(k)=exp(-r*dt)*(p1*V(k)+(1-p1)*V(k+1));
    l=l-1;
end;
V0=V(1) % price of the option

A.2 Analytical price calculation

% time mesh
Ts = 7; % input('Tstart')
Tf = 11; % input('Tfinal')
L = lcm(Ts,Tf)*2; % aantal tijdstappen
dt = Tf/L; % lengte tijdstap
m=Ta/dt;
N=(Tf-Ts)/dt;

% parameters
r = 1/5; % input('Interest rate')
sigma = 0.2;
beta = (exp(-r*dt) + exp((r + sigma^2)*dt))/2;
u = beta + sqrt(beta^2-1); % up factor
d = 1/u; % down factor
p = (exp(r*dt) - d)/(u-d);

fn=floor(n/2)+1; % eerste relevant up
j=[fn:n];
\[ V_*(1:n-fn+1) = 100 * (\text{factorial}(n) / (\text{factorial}(j) * \text{factorial}(n-j))) \* ((u^j) * (d^(n-j)) - 1) * ((p^j) * (1-p)^{(n-j)}); \]

\[ V_0 = \exp(-r*(m*n)*dt)*\text{sum}(V_*); \]

\[ V_0 \]

### A.3 Iterative Implied Volatility calculation

```plaintext
% parameters
T = input('Tstart');
Tf = input('Tfinal');
r = input('Interest rate');
K = input('strike price K');
V = 440; % current price

n=20; % number of days
sigma=zeros(1,n);
sigma(1)=sqrt(2*abs((log(S0/K)+r*(T-t))/(T-t)));

for i=2:n
    d1 = (1/(sigma(i-1)*sqrt(T-t)))*((log(S0/K)+(r+0.5*sigma(i-1)^2)*(T-t)));
    d2 = d1-(sigma(i-1)*sqrt(T-t));
    Vbs = S0*(cdf('norm',d1,0,1))-exp(-r*(T-t))*K*(cdf('norm',d2,0,1))
    f = V-Vbs;
    df=(-S0*sqrt(i-t)/sqrt(2*pi))*exp(-0.5*d1^2);
    sigma(i)=sigma(i-1)-(f/df);
end

sigma
```

### A.4 Matlab code Black-Scholes pricing model

```plaintext
% parameters
Ts = 0; % input('Tstart')
Tf = 1; % input('Tfinal')
sigma = 0.3;
r = 0.05; % input('Interest rate')
K = 9; % input('max payoff K')

% Calculate SS
S0 = 5; % input('Stock price at time 0')
SS=exp(r*Ts)*S0;

% Calculate first option price
K1 = 4;
d1 = (1/(sigma*sqrt(Tf-Ts)))*((log(SS/K1)+(r+0.5*sigma^2)*(Tf-Ts)));

V1 = SS*(cdf('norm',d1,0,1))-exp(-r*(Tf-Ts))*K1*(cdf('norm',d2,0,1));
```
Calculate second option price

K2 = 100*K;

c1 = (1/(sigma*sqrt(Tf-Ts)))*(log(Ss/K2) + (r+0.5*sigma^2)*(Tf-Ts));

c2 = d1 - sigma*sqrt(Tf-Ts);

V2 = Ss*(cdf('norm',c1,0,1))-exp(-r*(Tf-Ts))*K2*(cdf('norm',c2,0,1));

V0=exp(-r*Tt)*(V1-V2); % price of the option
Appendix B

Data

B.1 Quoted AEX CALL data

AEX CALL JUL 2008
S0 T r K V volatility
391.98 0.05 0.048 365 32.2 0.4187
391.98 0.05 0.048 370 25.9 0.3195
391.98 0.05 0.048 380 17.15 0.2711
391.98 0.05 0.048 385 13.95 0.2739
391.98 0.05 0.048 390 8.35 0.1950
391.98 0.05 0.048 400 3.5 0.1821
391.98 0.05 0.048 410 1.1 0.1749
391.98 0.05 0.048 420 0.3 0.1764
391.98 0.05 0.048 430 0.10 0.1884
391.98 0.05 0.048 440 0.05 0.2093
391.98 0.05 0.048 450 0.05 0.2452
391.98 0.05 0.048 460 0.05 0.2798
391.98 0.05 0.048 470 0.05 0.3130
391.98 0.05 0.048 475 0.05 0.3293
391.98 0.05 0.048 490 0.05 0.3916
391.98 0.05 0.048 520 0.05 0.4646

AEX CALL DEC 2008
S0 T r K V volatility
391.98 0.42 0.045 300 99.5 0.2949
391.98 0.42 0.045 320 81.9 0.2919
391.98 0.42 0.045 340 67.0 0.3097
391.98 0.42 0.045 380 36.8 0.2649
391.98 0.42 0.045 390 32.56 0.2767
391.98 0.42 0.045 400 24.85 0.2479
391.98 0.42 0.045 410 19.75 0.2396
391.98 0.42 0.045 420 15.35 0.2321
391.98 0.42 0.045 430 11.50 0.2237
391.98 0.42 0.045 440 8.5 0.2178
391.98 0.42 0.045 450 6 0.2108
391.98 0.42 0.045 460 4.3 0.2078
391.98 0.42 0.045 480 1.9 0.1984
391.98 0.42 0.045 500 0.9 0.1976
391.98 0.42 0.045 520 0.9 0.1966
391.98 0.42 0.045 540 0.2 0.2000
391.98 0.42 0.045 560 0.15 0.2120
391.98 0.42 0.045 580 0.15 0.2318
391.98 0.42 0.045 600 0.06 0.2260

AEX CALL JUN 2009
80 T r K V volatility
391.98 0.94 0.045 360 60 0.2188
391.98 0.94 0.045 380 46.35 0.2090
391.98 0.94 0.045 400 36.2 0.2121
391.98 0.94 0.045 420 26.5 0.2095
391.98 0.94 0.045 440 18.25 0.1967
391.98 0.94 0.045 460 12 0.1893
391.98 0.94 0.045 480 9.15 0.1977
391.98 0.94 0.045 500 6.0 0.1793
391.98 0.94 0.045 520 3 0.1805
391.98 0.94 0.045 540 1.5 0.1724
391.98 0.94 0.045 560 0.9 0.1728
391.98 0.94 0.045 600 0.25 0.1688

B.2 Quoted AEX data 1998-2008

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\(^1\text{from Euroinvestor.com}\)