THE POST-BUCKLING STIFFNESS OF RECTANGULAR SIMPLY SUPPORTED PLATES

by

Prof. dr. ir. A. van der Neut

Delft - Nederland
October 1962
THE POST-BUCKLING STIFFNESS OF RECTANGULAR SIMPLY SUPPORTED PLATES

by

Prof. dr. ir. A. van der Neut

DELFt - NEDERLAND
October 1962
Summary.

In stability investigations of composite structures, containing plates in the post-buckling stage, the behaviour of the plates with respect to deformation increments should be known. This report gives the numerical values of the stiffness matrix for plates, the initial deformation of which is simultaneous longitudinal and transversal compression or extension. The data have been established on the basis of W.T.Koiter's shear field theory.
Symbols:

\[ a_{ij} = \frac{1}{\kappa} \frac{\partial \sigma_i}{\partial \varepsilon_j} \], elements of the stiffness matrix.

\[ \varepsilon = \frac{\varepsilon_e}{\varepsilon} \], normalized strain.

\[ f \], amplitude of wave pattern (fig. 1)

\[ i, j \], suffixes denoting (1) longitudinal, (2) transversal and (3) shear strain or stress.

\[ l \], panel length.

\[ m \], parameter indicating the slope of the nodal line of the wave pattern (fig. 1)

\[ s = \frac{\sigma}{E \varepsilon} \], normalized dimensionless stress.

\[ t \], plate thickness.

\[ w \], plate width (fig. 1)

\[ x, y \], coordinates in the plane of the plate (fig. 1)

\[ z \], deflection of the plate

\[ A, B, C \], coefficients of the eq. (22), defined by (23).

\[ D = \frac{w^2}{L^2} \], wave length parameter.

\[ E \], Young's modul.

\[ L \], half wave length in the direction of x (fig. 1).

\[ \alpha \], wave shape parameter (fig. 1).

\[ \varepsilon \], strain.

\[ \varepsilon^* = \frac{\pi^2}{3(1-\mu)} \frac{t^2}{w} \], critical compressive longitudinal strain, unit of strain.

\[ \varphi = \frac{3}{4}(1-\mu) \frac{t^2}{w} \], wave depth parameter.

\[ \mu \], Poisson's ratio.

\[ \sigma \], stress.
1. Introduction.

When investigating the stability of composite structures, containing plate panels which are themselves in the post-buckling stage, knowledge of the behaviour of the panels with respect to deformation increments is needed. This is the problem of the "tangential modulus" for buckled plates.

The need for data of this kind has existed ever since the application of stressed-skin construction confronted the stress analyst with the problem of the buckling stress of the stiffeners and his question was: what is the effective width of skin that should be added to the stiffener when establishing the bending stiffness of the stringer-skin-combination. The answer to this question was published, as far as the author knows, for the first time by H.Ebner (ref.1). It is simply that the effective skin section which works together with the stringer section is \( \frac{1}{E} \frac{d \delta}{d \epsilon} \), where \( \epsilon \) is the overall compressive strain of the panel and \( P \) is the load carried by the panel.

The same question emerges, though in a more complex situation, when the deformation increment is not mainly a longitudinal compression but in addition a variation of lateral compression or extension and of shear. This situation is present when the general instability of stiffened shell structures is under consideration. Then apart from the tangent-modulus with respect to longitudinal compression, the tangent moduli for lateral compression and shear affect the stability of the structure and in addition "cross stiffnesses" like f.i. the longitudinal load increment resulting from lateral deformation increment. This means that the complete stiffness matrix of the buckled panel should be known.

The present report establishes the equations from which the elements of this stiffness matrix can be computed for flat plates of constant width, the initial deformation of which is simultaneous longitudinal and lateral compression (or extension), therefore without shear. Graphs are given of the matrix elements as a function of the 2 initial extensions and the restrictions on their validity is discussed.

The derivation of the equations is based on the shear field theory developed by W.T.Koiter (ref.2). This theory was evaluated numerically by W.K.G.Floor and T.J.Burgerhout (ref.3); their work resulted in diagrams giving the relations between the 3 strain and the 3 average
stress components. Since this evaluation was meant to cover more in particular the range of large values of the strain to buckling strain ratio it has been repeated and extended for the purpose of this report so as to find more accurate results for the smaller strain to buckling strain ratios. These data are needed in stability calculations for defining the initial, pre-buckling state of stress and strain.

2. Definition of the stiffness matrix:

The deformations of the rectangular plate, length \( l \), width \( w \), are described by the relative displacements of the edges of the plate: the longitudinal elongation \( \varepsilon_1 \) where \( \varepsilon_1 \) defines the overall longitudinal strain, the lateral elongation \( \varepsilon_2 \) where \( \varepsilon_2 \) defines the overall lateral strain, and the shear angle \( \varepsilon_3 \) of perpendicular edges of the panel. The edges are assumed to remain straight and to be simply supported. The load carried by the plate is described by the average stresses: the longitudinal tensile stress \( \sigma_1 \), the lateral tensile stress \( \sigma_2 \) and the shear stress \( \sigma_3 \).

In stead of these strains and stresses we introduce the normalised strains

\[
\varepsilon_i = \varepsilon_i / \varepsilon^*, \quad i = 1, 2, 3, \tag{1}
\]

and the dimensionless normalized stresses

\[
s_i = \sigma_i / (B \varepsilon^*), \quad i = 1, 2, 3, \tag{2}
\]

where

\[
\varepsilon^* = \frac{\pi^2}{3(1-\mu^2)} \left( \frac{t}{w} \right)^2, \tag{3}
\]

the critical compressive strain of longitudinally compressed simply supported plates is taken to be the unit of strain.

When the 3 strain components \( \varepsilon_i \) are given it is assumed that the 3 stress components \( s_i \) are unambiguously defined, though it must be admitted that under certain conditions more than one wave pattern may be possible for the same set of values \( \varepsilon_i \) (see par. 6).
The relations between \( e_i \) and \( s_i \) are taken from ref. 2 (They are also given in ref. 3. Appendices A and B eqs A5, A6, B5/3).

The problem of this report is to find the increments of \( s_i \) induced by increments of the strains \( e_i \).

The stress increments are

\[
d\sigma_i = \sum_{j=1}^{3} \frac{\partial \sigma_i}{\partial e_j} d_j, \quad i = 1, 2, 3,
\]

or using the normalised strains and stresses (1) and (2)

\[
ds_i = \sum_{j=1}^{3} \frac{1}{E} \frac{\partial \sigma_i}{\partial e_j} de_j
\]

With

\[
a_{ij} = \frac{1}{E} \frac{\partial \sigma_i}{\partial e_j} = \frac{\partial s_i}{\partial e_j}
\]

the stress-strain relations are

\[
ds_i = \sum_{j=1}^{3} a_{ij} de_j
\]

Then the stiffness matrix with respect to incremental deformation is

\[
a = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
\]

The elements of this matrix have to be established.

3. The shear field theory.

Koiter's shear field theory (ref. 2) accounts for large strain to critical strain ratio's by assuming deflection patterns in which a part of the panel width around the center of the panel is developable and the remaining edge strips are double curved (fig. 1):

\[
0 < y < \frac{1}{2} \omega \quad : \quad z = F(y) \sin \frac{\pi y}{L} \quad x - \eta(y)
\]

\[
\frac{1}{2} \omega < y < (1-\frac{1}{2}) \omega \quad : \quad z = f \sin \frac{\pi y}{L} \quad x - \eta(y)
\]
Three sets of functions $F(y)$ and $\eta(y)$ have been taken for the plate with simply supported edges. In this report, "wave shape 1" is being used, where

$$F(y) = f \sin \frac{\pi y}{L}, \quad \eta(y) = m$$ (7a, b)

Thus the nodal lines are assumed to be straight and the wave pattern is governed by the 4 parameters $f$ wave depth, $L$ longitudinal half wave length, $m$ cotangent of the angle between the nodal lines and the longitudinal axis and $\alpha$ the relative width of the double curved edge zones.

With increasing compressive strain to critical strain ratio $f$ will usually increase, $L$ decreases - continuously since the plate length is assumed to be infinite -, $m$ depends on the ratio of the three strains, and $\alpha$ decreases but may remain constant, equal to unity, for the lower strain to critical strain ratios.

Some simplifying assumptions on the displacement components in the plane of the plate have been made, which together with the assumed deflection $z$ and the edge displacements following from $e_1$, $e_2$, $e_3$ yield plausible expressions for the membrane strains. Then the strain energy can be determined and minimized with respect to the four parameters $f$, $L$, $m$, $\alpha$. This yields 4 equations from which for given $e_1$, $e_2$, $e_3$ these parameters can be solved. Next the load parameters $s_1$, $s_2$, $s_3$ pertaining to $e_1$, $e_2$, $e_3$ can be established.

After introduction of the dimensionless parameters

$$D = \frac{w}{L^2}, \quad \varphi = \frac{3}{4} (1-\mu^2)^2 f^2 / t^2$$

the four equations are

$$\varphi = \frac{4(1-\alpha)}{\alpha^2} \left[ \frac{\alpha(1+m^2)D^2}{2(1-\alpha)} + \frac{2}{2(1+3m^2)D} + \frac{1}{\alpha^2(1-\alpha)} \right] \left(3 - \frac{2\alpha}{2-\alpha} \right) D - \frac{1}{(1-\mu)^2}$$ (9a)

$$s_2 = -\frac{1}{2\alpha} \left[ \varphi \left( \frac{1}{1-\mu} \right) - \frac{1}{2(2-\alpha)} \right]$$ (9b)

$$s_3 = -m \left[ s_2 + \frac{1}{2}(1+m^2)D + \frac{3}{2\alpha(2-\alpha)} \right]$$ (9c)

$$s_1 = ms_3 - \frac{\alpha(3-2\alpha)}{4(2-\alpha)} D \varphi - \frac{1}{2}(1+m^2)D - \frac{1}{2\alpha(2-\alpha)}$$ (9d)
The first one of these equations follows from minimizing the strain energy with respect to \( \alpha \); therefore it is not valid for the range where \( \alpha = 1 \).

The relations between the strains \( e_1, e_2, e_3 \) and the stresses \( s_1, s_2, s_3 \) are

\[
\begin{align*}
    s_1 - \mu s_2 &= e_1 + (1 - \frac{1}{2}\alpha) D\varphi \quad (10a) \\
    s_2 - \mu s_1 &= e_2 + (1 - \frac{1}{2}\alpha)m^2 D\varphi + \frac{1}{2\alpha} \varphi \quad (10b) \\
    2(1+\mu)s_3 &= e_3 - 2(1-\frac{1}{2}\alpha)m D\varphi \quad (10c)
\end{align*}
\]

The 7 equations (9) and (10) will yield for a given system of edge displacements \( e_1, e_2, e_3 \) the corresponding stress components \( s_1, s_2, s_3 \) and the parameters \( \varphi, D, m, \alpha \). Numerically much more simple is the process where for a given set of values \( D, m, \alpha \) from the equations (9) \( \varphi, s_2, s_3 \) and \( s_1 \) are solved and next \( e_1, e_2, e_3 \) are computed from (10).

4. The state of stress of the plate not loaded in shear.

When in the initial state \( s_3 = 0 \) it follows from (9c) that \( m = 0 \). It is obvious, that in that case the nodal lines must be normal to the edges. Then the equations (9) and (10) valid for \( \alpha \ll 1 \) become

\[
\begin{align*}
    \varphi &= \frac{4(1-\alpha)}{\alpha^2} \left[ \varphi D^2 + \frac{2}{2-\alpha} D + \frac{1}{\alpha^2(1-\alpha)} \right] \left[ (3 - \frac{2\alpha}{2-\alpha}) D^2 - \frac{1}{(1-\mu)} \alpha^2 \right] \quad (11a) \\
    s_1 &= -\frac{1}{2} \left[ \alpha (3-2\alpha) \right] \varphi D D\varphi + D + \frac{1}{\alpha (2-\alpha)} \quad (11b) \\
    s_2 &= -\frac{1}{4\alpha^2} \left[ \frac{\varphi}{2} - \alpha^3 (2-\alpha) D^2 + 1 \right] \quad (11c) \\
    e_1 &= s_1 - \mu s_2 - (1-\frac{1}{2}\alpha) D\varphi \quad (12a) \\
    e_2 &= s_2 - \mu s_1 - \frac{\varphi}{2\alpha} \quad (12b)
\end{align*}
\]

For given \( D \) and \( \alpha \) - such that \( \varphi > 0 \) - these equations yield \( \varphi, s_1, s_2, e_1, e_2 \) respectively.
For the smaller strain to critical strain ratios, where \( \alpha \equiv 1 \) the equation (11a) is no longer valid and the other equations become

\[
\begin{align*}
    s_1 &= -\frac{1}{2} \left[ \frac{1}{2} D \varphi + D + 1 \right] \quad (13a) \\
    s_2 &= -\frac{1}{4} \left[ \frac{1}{1-\mu^2} \varphi - D^2 + 1 \right] \quad (13b) \\
    e_1 &= s_1 - \mu s_2 - \frac{1}{2} D \varphi \quad (13c) \\
    e_2 &= s_2 - \mu s_1 - \frac{1}{2} \varphi \quad (13d)
\end{align*}
\]

For given \( D \) and \( s_2 \) - such that \( \varphi > 0 \) - these equations yield \( \varphi, s_1, e_1, e_2 \) respectively.

Figs. 2 and 3 give \( s_1, s_2 \) versus \( e_1, e_2 \) both for the range \( \alpha \equiv 1 \) and \( \alpha < 1 \); \( \mu \) has been taken to be 0,3.

From the equations (13) the critical combination of stresses \( s_1, s_2 \) can be found by putting \( \varphi = 0 \). Then

\[
\begin{align*}
    s_1 &= -\frac{1}{2} (D+1) \\
    s_2 &= -\frac{1}{4} (1-D^2) \quad (14)
\end{align*}
\]

Since \( D \) should be positive its smallest value is zero, which yields \( s_1 = -\frac{1}{2}, s_2 = -\frac{1}{4} \).

For \( D > 0 \) elimination of \( D \) from (14) yields

\[
\begin{align*}
    s_2 &= s_1^2 + s_1, \quad \text{valid for } s_2 > -\frac{1}{4} \quad (15)
\end{align*}
\]

This defines one branch of the buckling curve.

For \( D \equiv 0 \) \( s_1 \) does not affect the stability, therefore

\[
\begin{align*}
    s_1 &> -\frac{1}{2}, \quad s_2 = -\frac{1}{4} \quad (16)
\end{align*}
\]

is the other branch of the buckling curve. By means of (13c, d) the critical combination of stresses can be translated into critical combination of strains. The branch corresponding to (15) has been drawn in fig. 7a; the line for \( s_{33} = 0,335 \) to the left of \( e_1 = -0,425 \). The 2 branches have been given in fig. 8, together with the parameters \( \varphi, D \) and \( \alpha \) as function of \( e_1 \) and \( e_2 \).
5. The tangential stiffness of the plate not loaded in shear.

For reasons of symmetry the derivatives of \( m \) to \( e_1 \) and \( e_2 \) must vanish. However the derivative of \( m \) to \( e_3 \) does not vanish because \( de_3 \) will change the inclination of the nodal lines. Therefore, when determining \( a_{ij} \) from (9) and (10), \( m \) has to be maintained in these formula when differentiating to \( e_3 \).

In what follows derivatives of \( \varphi, D, m \) and \( \lambda \) to \( e_1 \) will be denoted by adding the suffix 1 to these 4 functions.

Differentiation of (10), thereby using (4) and solving for \( a_{ij} \) yields

\[
\begin{align*}
a_{11}(1-\mu^2) &= 1 + (1 - \frac{1}{2} \alpha)(D \varphi)_1 + \frac{1}{2\alpha} \varphi_1 - \frac{1}{2} \varphi(D + \frac{1}{\alpha^2}) \psi_1 & (17a) \\
a_{12}(1-\mu^2) &= \mu + (1 - \frac{1}{2} \alpha)(D \varphi)_2 + \frac{1}{2 \alpha} \varphi_2 - \frac{1}{2} \varphi(D + \frac{1}{\alpha^2}) \psi_2 & (17b) \\
a_{21}(1-\mu^2) &= \mu + (1 - \frac{1}{2} \alpha)(D \varphi)_1 + \frac{1}{2 \alpha} \varphi_1 - \frac{1}{2} \varphi(D + \frac{1}{\alpha^2}) \psi_1 & (17c) \\
a_{22}(1-\mu^2) &= 1 + (1 - \frac{1}{2} \alpha)(D \varphi)_2 + \frac{1}{2 \alpha} \varphi_2 - \frac{1}{2} \varphi(D + \frac{1}{\alpha^2}) \psi_2 & (17d) \\
2(1+\mu)a_{33} &= 1 - (2 - \alpha) D \varphi m_3 & (17e) \\
\end{align*}
\]

and

\[
\begin{align*}
a_{13} - \mu a_{23} &= (1 - \frac{\alpha}{2})(D \varphi)_3 - \frac{1}{2} D \varphi \alpha \psi_3 & (17f) \\
a_{23} - \mu a_{13} &= \frac{1}{2\alpha} (\varphi_3 - \frac{\alpha}{\alpha} \varphi \psi_3) & (17g) \\
2(1 + \mu)a_{31} &= -(2 - \alpha) D \varphi m_1 & (17h) \\
2(1 + \mu)a_{32} &= -(2 - \alpha) D \varphi m_2 & (17i) \\
\end{align*}
\]

Since \( m_1 = m_2 = 0 \) it follows

\[
a_{31} = a_{32} = 0
\]

This result again is obvious, since the increments of \( e_1 \) and \( e_2 \) leave the plate in a symmetrical condition so that the anti-symmetrical stress...
s₃ cannot occur.

From Maxwell's reciprocity theorem it follows that \( a_{ij} = a_{ji} \)
therefore

\[
a_{13} = a_{23} = 0
\]  

(19)

This can also be derived directly from considerations on symmetry:
application to the symmetrical state \( s_3 = 0 \) of the anti-symmetrical
increment \( -de_3 \) brings the plate into the same state as application of
the increment \( -de_3 \), so that in these 2 cases \( s_1 \) and \( s_2 \) must be equal
which requires that \( a_{13} \) and \( a_{23} \) vanish.

Hence (17f,g) yield

\[
\varphi_3 = \frac{\varphi}{\alpha} \alpha_3
\]

(20a)

\[
D_3 = -D \frac{2(1-\alpha)}{\alpha(2-\alpha)} \alpha_3
\]

(20b)

At this stage it can already be concluded that \( \varphi_3 = D_3 = \alpha_3 = 0 \),
since \( \alpha \) is a symmetrical geometrical quantity which must be equal
at the strain increments \( de_3 \) and \( -de_3 \).

From Maxwell's reciprocity theorem follows \( a_{12} = a_{21} \). This cannot
be concluded from (17b) and (17c), nor does it follow from the equations
in a later stage. Therefore the 2 expressions for \( a_{12} \) and \( a_{21} \) have been
maintained and have been used as a check on the numerical computations.

So far only the equations (10) have been used to find for \( a_{ij} \) the
formula (17a/e). Since \( \varphi, D \) and \( \alpha \) are governed by the minimum energy
conditions (9), these latter equations have to be used in order to
establish \( \varphi, D \) and \( \alpha \) and their derivatives.

Differentiation of (9b,c,d) yields, taking into account \( m = m_1 = m_2 = 0 \),

\[
a_{11} = -\frac{\alpha(3-2\alpha)}{4(2-\alpha)} (D\varphi)_1 - \frac{1}{2} D_1 + \left\{ -\frac{1}{(2-\alpha)^2} - 1 \right\} \frac{1}{2} D\varphi + \frac{1-\alpha}{\alpha^2(2-\alpha)^2} \alpha_1
\]

(21a)

\[
a_{12} = -\frac{\alpha(3-2\alpha)}{4(2-\alpha)} (D\varphi)_2 - \frac{1}{2} D_2 + \left\{ -\frac{1}{(2-\alpha)^2} - 1 \right\} \frac{1}{2} D\varphi + \frac{1-\alpha}{\alpha^2(2-\alpha)^2} \alpha_2
\]

(21b)
\[ a_{21} = - \frac{1}{4(1-\mu)} \frac{1}{\alpha} \varphi_1 + \alpha(1-\alpha) DD_1 + \left\{ \frac{1}{2(1-\mu)} \frac{\varphi}{\alpha^3} + \frac{1}{2}(1-\alpha)D^2 + \frac{1}{2\alpha^3} \right\} \alpha_1 \]

\[ a_{22} = - \frac{1}{4(1-\mu)} \frac{1}{\alpha} \varphi_2 + \alpha(1-\alpha) DD_2 + \left\{ \frac{1}{2(1-\mu)} \frac{\varphi}{\alpha^3} + \frac{1}{2}(1-\alpha)D^2 + \frac{1}{2\alpha^3} \right\} \alpha_2 \]

\[ a_{33} = \left[ s_2 + \frac{3}{2\alpha(2-\alpha)} \right] m_3 \]

\[ a_{13} = - \frac{\alpha(3-2\alpha)}{4(2-\alpha)} D\varphi_3 - \frac{1}{2} D_3 + \left\{ \left[ \frac{1}{(2-\alpha)^2} - 1 \right] \frac{1}{2} D\varphi + \frac{1-\alpha}{\alpha^2(2-\alpha)^2} \right\} \alpha_3 \]

\[ a_{23} = - \frac{1}{4(1-\mu)} \frac{1}{\alpha} \varphi_3 + \alpha(1-\alpha) DD_3 + \left\{ \frac{1}{2(1-\mu)} \frac{\varphi}{\alpha^3} + \frac{1}{2}(1-\alpha)D^2 + \frac{1}{2\alpha^3} \right\} \alpha_3 \]

\[ a_{31} = 0 \]

\[ a_{32} = 0 \]

Substituting from (20a) (20b) and putting \( a_{13} = a_{23} = 0 \) it follows from (21f,g), as has already been concluded from symmetry considerations, that \( \alpha_3 = 0 \).

The remaining equations are (17a/e, 21a/e). From these equations can be solved \( \varphi_1, D_1, \alpha_1, m_3 \). By elimination of \( a_{1j} \) between (17a, 21a) etc. linear equations are obtained in \( \varphi_1, D_1, \alpha_1, m_3 \) \( i = 1,2 \). There are 7 differential quotients to be determined, whereas (17) and (21) yield 5 equations. The 2 missing equations for the solution of \( \varphi_1, D_1, \alpha_1, m_3 \) \( i = 1,2 \) can be found by using (9a) and differentiating it to \( e_1 \) and \( e_2 \). This yields

\[ \varphi_1 = A_D D_1 + A_\alpha \alpha_1, \]

\[ \varphi_2 = A_D D_2 + A_\alpha \alpha_2. \]
where

\[
A_D = - \left( \frac{5}{4} - \frac{1}{2-\alpha} \right) \left( \frac{1-\alpha}{2-\alpha} \alpha^2 D^2 + D \right) + \frac{1-\alpha}{4(1-\mu)^2 \alpha^2} \left( \nu D + \frac{1}{2-\alpha} \right) \left( \frac{5}{4} - \frac{1}{2-\alpha} \right) \alpha^2 D^2 + \frac{1}{4(1-\mu)^2 \alpha^2} \right)^2 \right)
\]

\[
A_\alpha = - \left( \frac{5}{4} - \frac{2-\alpha^2}{(2-\alpha)^2} \alpha^2 D^3 + \left( \frac{5}{4} - \frac{19}{2(2-\alpha)} + \frac{5}{2(2-\alpha)} \frac{D^2}{D} + \left( \frac{5}{4} - \frac{3}{2-\alpha} - \frac{2}{(2-\alpha)^2} \frac{4(1-\mu)^2 \alpha}{(2-\alpha)} \right) \frac{D}{D^2} + \frac{1}{2-\alpha} \right)^2 \right)
\]

\[
E_{\text{lem}} \frac{a_{ij}}{a_{ij}} \text{ from (17a) and (21a) (17b) and (21b) etc., yields}
\]

\[
B_\varphi \varphi_1 + B_D D_1 + B_\alpha \alpha_1 = -\mu, \quad (22c)
\]

\[
C_\varphi \varphi_2 + C_D D_2 + C_\alpha \alpha_2 = -\mu, \quad (22d)
\]

\[
C_\varphi \varphi_1 + C_D D_1 + C_\alpha \alpha_1 = -1, \quad (22e)
\]

\[
B_\varphi \varphi_2 + B_D D_2 + B_\alpha \alpha_2 = -1, \quad (22f)
\]

where

\[
B_\varphi = \mu(1-\frac{1}{2-\alpha})D + \frac{1}{2-\alpha}(1+\frac{1}{2-\alpha}) \quad (23c)
\]

\[
B_D = \mu(1-\frac{1}{2-\alpha})\varphi - (1-\mu^2)(1-\frac{1}{2-\alpha})D \quad (23d)
\]

\[
B_\alpha = \frac{1}{2} \left[ \mu D \phi + \left( 1-\mu^2 \right) \left( (1-\alpha)D^2 + \frac{1}{\alpha^2} \right) \alpha + \mu \phi \right] \quad (23e)
\]

\[
C_\varphi = \left( 1 - \frac{1}{2} \alpha + \frac{1}{4} \mu^2 \alpha (2-\frac{1}{2-\alpha}) \right) D + \frac{\mu}{2\alpha} \quad (23f)
\]

\[
C_D = \left( 1 - \frac{1}{2} \alpha + \frac{1}{4} \mu^2 \alpha (2-\frac{1}{2-\alpha}) \right) \varphi + \frac{\mu}{2} \quad (23g)
\]

\[
C_\alpha = \frac{1}{2} \left[ \left( \mu^2 + \frac{1}{(2-\alpha)^2} \right) D \varphi + \left( \mu \varphi + \frac{1}{2} \mu \alpha \frac{1}{2} \right) \left( 1-\frac{1}{2} \alpha \right) \right] \quad (23h)
\]

From the equations (22) \( \varphi_i, D_i, \alpha_i \) (i=1,2) can be solved. Next a \( a_{ij} \) (i=1,2; j=1,2) can be established by means of (17a/d).
$m_3$ follows after elimination of $a_{33}$ from (17e) and (21e)

$$m_3 = 1 : \left[ (2-\alpha)D\varphi + 2(1+\mu)(s_2 + \frac{1}{2}D + \frac{3}{2\alpha(2-\alpha)}) \right]$$  \hspace{1cm} (24a)

Then from (17e)

$$a_{33} = \frac{1}{2(1+\mu)} \left[ 1 - \frac{(2-\alpha)D\varphi}{(2-\alpha)D\varphi + (1+\mu)(2s_2 + D + \frac{3}{\alpha(2-\alpha)})} \right]$$  \hspace{1cm} (24b)

Graphs for $a_{ij}$ have been given in figs 4, 5, 6 and 7. They have been established by computing for given $D$ and $\alpha$ the corresponding $e_1$, $e_2$, $\varphi$ and $a_{ij}$, and interpolating between these $a_{ij}$ values. $\mu$ has been assumed to be 0.2.

The foregoing equations apply to $\alpha < 1$. For the range $\alpha = 1$ the deduction is analogous. Then the equations (17) and (21) have to be changed by putting $\alpha = 1$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Again $a_{13} = a_{23} = 0$ yields $\varphi_3 = D_3 = 0$.

The equations (22) do not apply since equation (9) from which they are derived is not valid for $\alpha = 1$.

Elimination of $a_{ij}$ from (17) and (21) yields the equations (22c/f) where $\alpha_1 = \alpha_2 = 0$ and $B$ and $C$ can be taken from (23c,d,f,g) by putting $\alpha = 1$.

Graphs for $a_{ij}$ in the range $\alpha = 1$ have been given in figs 4, 5, 6 and 7.

6. Discussion of results.

In practical conditions $s_2 < -C_{25}$ will never occur, so that $e_2$ (see fig.3) will always be above -$6.5$ to -$1.5$, depending on the magnitude of $e_1$. For this practical range of $e_2$ the validity of the curves for $a_{ij}$ is well established.

At the transition between the range $\alpha = 1$ and the range $\alpha < 1$ $a_{22}$ and $a_{33}$ appear to be continuous; however small discontinuities appear with $a_{11}$ and $a_{12}$. These discontinuities, which cannot exist in reality, are a consequence of the approximative character of the assumed wave-form. A similar discontinuity is known to exist in $d\varphi/de_1$ with the classical case $s_2 = 0$. A discontinuity of $\alpha_1$ will cause discontinuities of $\varphi_1$ and $D_1$ (see eq. 22) and next discontinuities of $a_{11}$ and $a_{21}$ (see eq. 17).
The cross-stiffness $a_{12}$ (fig. 5) appears to be a small negative number; in the unbuckled state $a_{12} = \mu/(1-\mu^2) = 0.330$.

$a_{22}$ (fig.6) appears to be a large number, of equal order of magnitude as its value in the unbuckled state, which is $a_{22} = 1/(1-\mu^2) = 1.10$.

$a_{33}$ (fig.7) decreases gradually with increasing longitudinal compression, starting at the onset of buckling with its value in the unbuckled state $a_{33} = 1/(2(1+\mu)) = 0.3846$.

The behaviour of $a_{11}$ (fig.4) is known for the case $s_2 = 0$ from earlier work. In this case (5) yields

$$ds_1 = a_{11}de_1 + a_{12}de_2$$
$$ds_2 = a_{21}de_1 + a_{22}de_2 = 0,$$ therefore $de_2 = -\frac{a_{21}}{a_{22}}de_1$, hence
$$\frac{ds_1}{de_1} = a_{11} - \frac{a_{12}}{a_{22}}.$$

Since $a_{12}$ is very small, $a_{11}$ is about equal to the tangential stiffness in the case $s_2 = 0$. Therefore the results shown for $a_{11}$ correspond to what could be conjectured.

The validity of the curves in the range of large negative values of $e_2$ is suspect. A large negative value of $e_2$ means a large negative value of $s_2$ (see fig.3). However the buckling load for lateral compression is $s_2 = -0.25$ and the corresponding buckling mode shows $L = \infty$ ($D = 0$).

The equations of ref.2 yield for $D = 0$ a linear relation between $e_1$ and $e_2$, which has been drawn in fig. 7a: the line for $a_{33} = 0.335$ to the right of $e_1 = -0.425$, and fig. 6. This line is not the continuation of the buckling curve, discussed at the end of para.4; it represents states of deformation with $\phi \neq 0$ and it refers therefore to postbuckling conditions. It follows then from (13b) that in the post-buckling state with $D = 0$ $-s_2$ would be greater than $\frac{1}{4}$.

This conclusion cannot be correct. The plate at the mode $D = 0$ behaves in just the same way as a column. Under its buckling load $-s_2 = \frac{1}{4}$ the plate is in equilibrium also when the deflection is finite ($\phi \neq 0$). The addition of $s_2$ does not affect the equilibrium when $D = 0$, it affects only $e_1$ and $e_2$. 
This inconsistency of the theory of ref.2 is due to the fact that \( s_2 \) is the average stress over the half wave length \( L \). Applying the theory to \( D = 0 \), where \( L \) is infinite, it is being assumed that the deflection, which is \( f \) at the region of the plate under consideration, decreases to zero in infinity. Then the stress \( s_2 \), following from the theory is the average over this infinitely long half wave and it is different from the stress at the region under consideration. Therefore the theory (equations 13) loses its validity for \( D = 0 \) and is of doubtful validity for values of \( D \) close to zero.

The theory is based on the assumption that the edges of the panel remain straight. This assumption is completely satisfied if the structure consists of an infinite number of panels in the direction of \( y \). With a stiffened plate of finite width the buckling pattern will cause lateral displacements of the free edges with the half wave length \( \frac{1}{2} L \) (fig.9). This means that the edge panels are being bent in their plane; the panels next to the edge panels are also bent but to a lesser degree. This bending is opposed by the stiffness of the plate and more when the number of panels is greater and \( \frac{1}{2} L/w \) is smaller. With \( D = 0.3 \) follows \( \frac{1}{2} L/w = 0.9 \), which is already fairly small. This suggests that for \( D > 0.3 \) the theory is no longer subject to the objection which was justified at \( D = 0 \) and at very small values of \( D \).

However there is another reason for suspicion with regard to that part of the figs 2 to 7, where \( D > 0.3 \) and \( s_2 < -0.25 \).

The curves refer to wave patterns where \( L \) is finite (\( D \geq 0 \)). If in fact \( L \) is finite the plate, corrugated by its bulges, appears to be able to carry compressive stresses \( -s_2 \) in excess of \( \frac{1}{4} \) even with the addition of compressive stresses \( -s_1 \). It is however open to doubt whether the plate will buckle according to this mode with finite \( L \) or that it will assume \( L = 0 \). There seems to be a duality of possible buckling patterns.

It can be imagined that the decision on the choice of wave pattern depends on the load history of the panel. If the compression \( -e_2 \) is applied first the plate buckles with \( D = 0 \) and this wave pattern will possibly hold on when subsequently \( -e_1 \) is applied. If however a compression \( -e_1 \) is applied first a short wave pattern will occur, which possibly is stable when thereupon \( -e_2 \) is applied, though with some
adjustment of the wave length. It could also be imagined that the buckling mode is unstable when $-e_2$ is applied which would involve that the wave pattern explosively changes into $D = 0$. The problem of the stability of the post-buckling stage has not been solved so far.

These arguments mean that the $s_1$, $s_2$ values shown in figs 2 and 3 and the $a_{ij}$ values shown in figs 3 to 7 cannot be considered to be conclusive when $s_2 < -0.25$. Beside these data others should be considered which refer to the wave pattern $D = 0$.

With $D = 0$ the strain increment $d_{e_1}$ changes - due to Poisson's ratio - the wave depth $f$ without change of $s_2$; therefore $a_{11} = 1$ and $a_{21} = 0$. The strain increment $d_{e_2}$ again changes the wave depth without change of $s_2$; therefore $a_{22} = 0$ and since there is no Poisson contraction $a_{12} = 0$. With respect to additional shear $d_{e_3}$ the curved plate offers the same stiffness as a flat plate, which means $a_{33} = \frac{1}{2(1+\mu)}$.

This brings to the conclusion that for $e_1$, $e_2$ combinations where according to fig. 3 $s_2 < -0.25$, beside the stiffness data given in fig. 3 to 7, also the following stiffnesses, relating to $D = 0$, should be considered:

$$a_{11} = 1, \quad a_{12} = a_{21} = a_{22} = 0, \quad a_{33} = \frac{1}{2(1+\mu)}$$

(25)

Acknowledgment.

The author wishes to thank Mrs. J.C. van Poeteren, who programmed the computations for the Zebra-computer of the Technological University, Delft, and Mr. A.C.W. Michon for his careful work in translating the computed data into graphs.

October 1962.
References.


Fig. 1: Buckling pattern $0 < y < \frac{1}{2} \alpha w$

$$z = f \sin \frac{\pi y}{\alpha w} \sin \frac{\pi}{L} (x - my)$$

$$\frac{1}{2} \alpha w < y < (1 - \frac{1}{2} \alpha w)$$

$$z = f \sin \frac{\pi}{L} (x - my)$$
FIG. 2. $s_1$, VERSUS $e_1$, $e_2$. 

$\alpha < 1 \quad \Rightarrow \quad \alpha = 1$
FIG. 3: $s_2$, VERSUS $e_1, e_2$. 

$\alpha < 1 \quad \leftrightarrow \quad \alpha = 1$
FIG. 4. $a_{11}$, VERSUS $e_1, e_2$
FIG. 5a. $\alpha = \alpha_{12,21}$ VERSUS $e_1', e_2'$ FOR SMALL REDUCED STRAINS.
FIG. 5b. $\alpha = \alpha_{12/21}$ VERSUS $e_{1}, e_{2}$ FOR ADVANCED POST-BUCKLING STATE.
FIG. 6. $\alpha_{22}^*$ VERSUS $e_1^*, e_2^*$. 

\[ \alpha < 1 \quad \text{and} \quad \alpha \equiv 1 \]
FIG. 7a. $\alpha_{33}$ VERSUS $e_1$, $e_2$ FOR SMALL REDUCED STRAINS.
FIG. 7b. $a_{33}$ VERSUS $e_1, e_2$ FOR ADVANCED POST-BUCKLING STATE.
Fig. 8a: Parameters \( D, \varphi \) versus \( e_1, e_2 \).
Fig. 8b: Parameter $\alpha$ versus $e_1, e_2$. 
Lateral edge displacement due to buckling of a single panel.

Bending up the outmost panels due to buckling for a plate consisting of 3 panels.

Fig. 9: Illustration of the bending of the panel which would result when the lateral membrane stress would be constant.