Martingale decompositions and weak differential subordination in UMD Banach spaces

Yaroslavtsev, Ivan S.

DOI

[10.3150/18-BEJ1031](https://doi.org/10.3150/18-BEJ1031)

Publication date

2019

Document Version

Final published version

Published in

Bernoulli

Citation (APA)

Yaroslavtsev, I. S. (2019). Martingale decompositions and weak differential subordination in UMD Banach spaces. *Bernoulli*, 25(3), 1659-1689. <https://doi.org/10.3150/18-BEJ1031>

Important note

To cite this publication, please use the final published version (if applicable).
Please check the document version above.

Copyright

Other than for strictly personal use, it is not permitted to download, forward or distribute the text or part of it, without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license such as Creative Commons.

Takedown policy

Please contact us and provide details if you believe this document breaches copyrights.
We will remove access to the work immediately and investigate your claim.

Martingale decompositions and weak differential subordination in UMD Banach spaces

IVAN S. YAROSLAVTSEV

Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands. E-mail: I.S.Yaroslavtsev@tudelft.nl

In this paper, we consider Meyer–Yoeurp decompositions for UMD Banach space-valued martingales. Namely, we prove that X is a UMD Banach space if and only if for any fixed $p \in (1, \infty)$, any X -valued L^p -martingale M has a unique decomposition $M = M^d + M^c$ such that M^d is a purely discontinuous martingale, M^c is a continuous martingale, $M_0^c = 0$ and

$$\mathbb{E}\|M_\infty^d\|^p + \mathbb{E}\|M_\infty^c\|^p \leq c_{p,X} \mathbb{E}\|M_\infty\|^p.$$

An analogous assertion is shown for the Yoeurp decomposition of a purely discontinuous martingales into a sum of a quasi-left continuous martingale and a martingale with accessible jumps.

As an application, we show that X is a UMD Banach space if and only if for any fixed $p \in (1, \infty)$ and for all X -valued martingales M and N such that N is weakly differentially subordinated to M , one has the estimate $\mathbb{E}\|N_\infty\|^p \leq C_{p,X} \mathbb{E}\|M_\infty\|^p$.

Keywords: accessible jumps; Brownian representation; Burkholder function; canonical decomposition of martingales; continuous martingales; differential subordination; Meyer–Yoeurp decomposition; purely discontinuous martingales; quasi-left continuous; stochastic integration; UMD Banach spaces; weak differential subordination; Yoeurp decomposition

1. Introduction

It is well known from the fundamental paper of Itô [20] on the real-valued case, and several works [1,2,5,13,32] on the vector-valued case, that for any Banach space X , any centered X -valued Lévy process has a unique decomposition $L = W + \tilde{N}$, where W is an X -valued Wiener process, and $\tilde{N}$ is an X -valued weak integral with respect to a certain compensated Poisson random measure. Moreover, W and $\tilde{N}$ are independent, and therefore since W is symmetric, for each $1 < p < \infty$ and $t \geq 0$,

$$\mathbb{E}\|\tilde{N}_t\|^p \leq \mathbb{E}\|L_t\|^p. \quad (1.1)$$

The natural generalization of this result to general martingales in the real-valued setting was provided by Meyer in [29] and Yoeurp in [44]. Namely, it was shown that any real-valued martingale M can be uniquely decomposed into a sum of two martingales M^d and M^c such that M^d is purely discontinuous (i.e., the quadratic variation $[M^d]$ has a pure jump version), and M^c is

continuous with $M_0^c = 0$. The reason why they needed such a decomposition is a further decomposition of a semimartingale, and finding an exponent of a semimartingale (we refer the reader to [23] and [44] for the details on this approach). In the present article, we extend Meyer–Yoeurp theorem to the vector-valued setting, and provide extension of (1.1) for a general martingale (see Section 3.1). Namely, we prove that for any UMD Banach space X and any $1 < p < \infty$, an X -valued L^p -martingale M can be uniquely decomposed into a sum of two martingales M^d and M^c such that M^d is purely discontinuous (i.e., $\langle M^d, x^* \rangle$ is purely discontinuous for each $x^* \in X^*$), and M^c is continuous with $M_0^c = 0$. Moreover, then for each $t \geq 0$,

$$(\mathbb{E} \|M_t^d\|^p)^{\frac{1}{p}} \leq \beta_{p,X} (\mathbb{E} \|M_t\|^p)^{\frac{1}{p}}, \quad (\mathbb{E} \|M_t^c\|^p)^{\frac{1}{p}} \leq \beta_{p,X} (\mathbb{E} \|M_t\|^p)^{\frac{1}{p}}, \quad (1.2)$$

where $\beta_{p,X}$ is the UMD $_p$ constant of X (see Section 2.1). Theorem 3.33 shows that such a decomposition together with L^p -estimates of type (1.2) is possible if and only if X has the UMD property.

The purely discontinuous part can be further decomposed: in [44] Yoeurp proved that any real-valued purely discontinuous M^d can be uniquely decomposed into a sum of a purely discontinuous quasi-left continuous martingale M^q (analogous to the “compensated Poisson part”, which does not jump at predictable stopping times), and a purely discontinuous martingale with accessible jumps M^a (analogous to the “discrete part”, which jumps only at certain predictable stopping times). In Section 3.2, we extend this result to a UMD space-valued setting with appropriate estimates. Namely, we prove that for each $1 < p < \infty$ the same type of decomposition is possible and unique for an X -valued purely discontinuous L^p -martingale M^d , and then for each $t \geq 0$,

$$(\mathbb{E} \|M_t^q\|^p)^{\frac{1}{p}} \leq \beta_{p,X} (\mathbb{E} \|M_t^d\|^p)^{\frac{1}{p}}, \quad (\mathbb{E} \|M_t^a\|^p)^{\frac{1}{p}} \leq \beta_{p,X} (\mathbb{E} \|M_t^d\|^p)^{\frac{1}{p}}. \quad (1.3)$$

Again as Theorem 3.33 shows, the (1.3)-type estimates are a possible only in UMD Banach spaces.

Even though the Meyer–Yoeurp and Yoeurp decompositions can be easily extended from the real-valued case to a Hilbert space case, the author could not find the corresponding estimates of type (1.2)–(1.3) in the literature, so we wish to present this special issue here. If H is a Hilbert space, $M : \mathbb{R}_+ \times \Omega \rightarrow H$ is a martingale, then there exists a unique decomposition of M into a continuous part M^c , a purely discontinuous quasi-left continuous part M^q , and a purely discontinuous part M^a with accessible jumps. Moreover, then for each $1 < p < \infty$, and for $i = c, q, a$,

$$(\mathbb{E} \|M_t^i\|^p)^{\frac{1}{p}} \leq (p^* - 1) (\mathbb{E} \|M_t\|^p)^{\frac{1}{p}}, \quad (1.4)$$

where $p^* = \max\{p, \frac{p}{p-1}\}$. Notice that though (1.4) follows from (1.2)–(1.3) since $\beta_{p,H} = p^* - 1$, it can be easily derived from the differential subordination estimates for Hilbert space-valued martingales obtained by Wang in [38].

Both the Meyer–Yoeurp and Yoeurp decompositions play a significant rôle in stochastic integration: if $M = M^c + M^q + M^a$ is a decomposition of an H -valued martingale M into continuous, purely discontinuous quasi-left continuous and purely discontinuous with accessible jumps parts, and if $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ is elementary predictable for some UMD Banach space X ,

then the decomposition $\Phi \cdot M = \Phi \cdot M^c + \Phi \cdot M^q + \Phi \cdot M^a$ of a stochastic integral $\Phi \cdot M$ is a decomposition of the martingale $\Phi \cdot M$ into continuous, purely discontinuous quasi-left continuous and purely discontinuous with accessible jumps parts, and for any $1 < p < \infty$ we have that

$$\mathbb{E} \|(\Phi \cdot M)_\infty\|^p \approx_{p,X} \mathbb{E} \|(\Phi \cdot M^c)_\infty\|^p + \mathbb{E} \|(\Phi \cdot M^q)_\infty\|^p + \mathbb{E} \|(\Phi \cdot M^a)_\infty\|^p.$$

The corresponding Itô isomorphism for $\Phi \cdot M^c$ for a general UMD Banach space X was derived by Veraar and the author in [37], while Itô isomorphisms for $\Phi \cdot M^q$ and $\Phi \cdot M^a$ have been shown by Dirksen and the author in [14] for the case $X = L^r(S)$, $1 < r < \infty$.

The major underlying techniques involved in the proofs of (1.2) and (1.3) are rather different from the original methods of Meyer in [29] and Yoeurp in [44]. They include the results on the differentiability of the Burkholder function of any finite dimensional Banach space, which have been proven recently in [41] and which allow us to use Itô formula in order to show the desired inequalities in the same way as it was demonstrated by Wang in [38].

The main application of the Meyer–Yoeurp decomposition are L^p -estimates for weakly differentially subordinated martingales. The weak differential subordination property was introduced by the author in [41], and can be described in the following way: an X -valued martingale N is weakly differentially subordinated to an X -valued martingale M if for each $x^* \in X^*$ a.s. $|\langle N_0, x^* \rangle| \leq |\langle M_0, x^* \rangle|$ and for each $t \geq s \geq 0$

$$[\langle N, x^* \rangle]_t - [\langle N, x^* \rangle]_s \leq [\langle M, x^* \rangle]_t - [\langle M, x^* \rangle]_s.$$

If both M and N are purely discontinuous, and if X is a UMD Banach space, then by [41], for each $1 < p < \infty$ we have that $\mathbb{E} \|N_\infty\|^p \leq \beta_{p,X}^p \mathbb{E} \|M_\infty\|^p$. Section 4 is devoted to the generalization of this result to continuous and general martingales. There we show that if both M and N are continuous, then $\mathbb{E} \|N_\infty\|^p \leq c_{p,X}^p \mathbb{E} \|M_\infty\|^p$, where the least admissible $c_{p,X}$ is within the interval $[\beta_{p,X}, \beta_{p,X}^2]$. Furthermore, using the Meyer–Yoeurp decomposition and estimates (1.2) we show that for general X -valued martingales M and N such that N is weakly differentially subordinated to M the following holds

$$(\mathbb{E} \|N_\infty\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2 (\beta_{p,X} + 1) (\mathbb{E} \|M_\infty\|^p)^{\frac{1}{p}}.$$

The weak differential subordination as a stronger version of the differential subordination is of interest in Harmonic Analysis. For instance, it was shown in [41] that sharp L^p -estimates for weakly differentially subordinated purely discontinuous martingales imply sharp estimates for the norms of a broad class of Fourier multipliers on $L^p(\mathbb{R}^d; X)$. Also there is a strong connection between the weak differential subordination of continuous martingales and the norm of the Hilbert transform on $L^p(\mathbb{R}; X)$ (see [41] and Remark 4.6).

Alternative approaches to Fourier multipliers for functions with values in UMD spaces have been constructed from the differential subordination for purely discontinuous martingales (see Bañuelos and Bogdan [4], Bañuelos, Bogdan and Bielaszewski [3], and recent work [41]), and for continuous martingales (see McConnell [26] and Geiss, Montgomery-Smith and Saksman [18]). It remains open whether one can combine these two approaches using the general weak differential subordination theory.

2. Preliminaries

In the sequel, we will omit proofs of some statements marked with a star (e.g., Lemma*, Theorem*, etc.). Please find the corresponding proofs in the Supplement [43].

We set the scalar field to be $\mathbb{R}$. We will use the *Kronecker symbol* δ_{ij} , which is defined in the following way: $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. For each $p \in (1, \infty)$ we set $p' \in (1, \infty)$ and $p^* \in [2, \infty)$ to be such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $p^* = \max\{p, p'\}$. We set $\mathbb{R}_+ := [0, \infty)$.

2.1. UMD Banach spaces

A Banach space X is called a *UMD space* if for some (equivalently, for all) $p \in (1, \infty)$ there exists a constant $\beta > 0$ such that for every $n \geq 1$, every martingale difference sequence $(d_j)_{j=1}^n$ in $L^p(\Omega; X)$, and every $\{-1, 1\}$ -valued sequence $(\varepsilon_j)_{j=1}^n$ we have

$$\left(\mathbb{E} \left\| \sum_{j=1}^n \varepsilon_j d_j \right\|^p \right)^{\frac{1}{p}} \leq \beta \left(\mathbb{E} \left\| \sum_{j=1}^n d_j \right\|^p \right)^{\frac{1}{p}}.$$

The least admissible constant β is denoted by $\beta_{p,X}$ and is called the *UMD constant*. It is well known (see [19], Chapter 4) that $\beta_{p,X} \geq p^* - 1$ and that $\beta_{p,H} = p^* - 1$ for a Hilbert space H . We refer the reader to [10,19,30,33] for details.

The following proposition is a vector-valued version of [11], Theorem 4.1.

Proposition 2.1. *Let X be a Banach space, $p \in (1, \infty)$. Then X has the UMD property if and only if there exists $C > 0$ such that for each $n \geq 1$, for every martingale difference sequence $(d_j)_{j=1}^n$ in $L^p(\Omega; X)$, and every sequence $(\varepsilon_j)_{j=1}^n$ such that $\varepsilon_j \in \{0, 1\}$ for each $j = 1, \dots, n$ we have*

$$\left(\mathbb{E} \left\| \sum_{j=1}^n \varepsilon_j d_j \right\|^p \right)^{\frac{1}{p}} \leq C \left(\mathbb{E} \left\| \sum_{j=1}^n d_j \right\|^p \right)^{\frac{1}{p}}.$$

If this is the case, then the least admissible C is in the interval $[\frac{\beta_{p,X}-1}{2}, \beta_{p,X}]$.

2.2. Martingales and stopping times in continuous time

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ which satisfies the usual conditions. Then $\mathbb{F}$ is right-continuous, and the following proposition holds (see [41]).

Proposition 2.2. *Let X be a Banach space. Then any martingale $M : \mathbb{R}_+ \times \Omega \rightarrow X$ has a càdlàg version*

Let $1 \leq p \leq \infty$. A martingale $M : \mathbb{R}_+ \times \Omega \rightarrow X$ is called an L^p -martingale if $M_t \in L^p(\Omega; X)$ for each $t \geq 0$, there exists an a.s. limit $M_\infty := \lim_{t \rightarrow \infty} M_t$, $M_\infty \in L^p(\Omega; X)$ and $M_t \rightarrow M_\infty$ in $L^p(\Omega; X)$ as $t \rightarrow \infty$. We will denote the space of all X -valued L^p -martingales on Ω by $\mathcal{M}_X^p(\Omega)$. For brevity, we will use $\mathcal{M}_X^p$ instead. Notice that $\mathcal{M}_X^p$ is a Banach space with the given norm: $\|M\|_{\mathcal{M}_X^p} := \|M_\infty\|_{L^p(\Omega; X)}$ (see [21,23] and [19], Chapter 1).

Proposition* 2.3. *Let X be a Banach space with the Radon–Nikodým property (e.g., reflexive), $1 < p < \infty$. Then $(\mathcal{M}_X^p)^* = \mathcal{M}_{X^*}^{p'}$, and $\|M\|_{(\mathcal{M}_X^p)^*} = \|M\|_{\mathcal{M}_{X^*}^{p'}}$ for each $M \in \mathcal{M}_{X^*}^{p'}$.*

A random variable $\tau : \Omega \rightarrow \mathbb{R}_+$ is called an *optional stopping time* (or just a *stopping time*) if $\{\tau \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$. With an optional stopping time τ , we associate a σ -field $\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t, t \in \mathbb{R}_+\}$. Note that M_τ is strongly $\mathcal{F}_\tau$ -measurable for any local martingale M . We refer to [23], Chapter 7, for details.

Due to the existence of a càdlàg version of a martingale $M : \mathbb{R}_+ \times \Omega \rightarrow X$, we can define an X -valued random variables $M_{\tau-}$ and ΔM_τ for any stopping time τ in the following way: $M_{\tau-} = \lim_{\varepsilon \rightarrow 0} M_{(\tau-\varepsilon) \vee 0}$, $\Delta M_\tau = M_\tau - M_{\tau-}$.

2.3. Quadratic variation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ that satisfies the usual conditions, H be a Hilbert space. Let $M : \mathbb{R}_+ \times \Omega \rightarrow H$ be a local martingale. We define a *quadratic variation* of M in the following way:

$$[M]_t := \mathbb{P} - \lim_{\text{mesh} \rightarrow 0} \sum_{n=1}^N \|M(t_n) - M(t_{n-1})\|^2, \quad (2.1)$$

where the limit in probability is taken over partitions $0 = t_0 < \dots < t_N = t$. Note that $[M]$ exists and is nondecreasing a.s. The reader can find more on quadratic variations in [27,28] for the vector-valued setting, and in [23,28,31] for the real-valued setting.

For any martingales $M, N : \mathbb{R}_+ \times \Omega \rightarrow H$ we can define a *covariation* $[M, N] : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ as $[M, N] := \frac{1}{4}([M+N] - [M-N])$. Since M and N have càdlàg versions, $[M, N]$ has a càdlàg version as well (see [22] and [27], Theorem I.4.47).

Remark 2.4 ([27]). The process $\langle M, N \rangle - [M, N]$ is a local martingale.

2.4. Continuous martingales

Let X be a Banach space. A martingale $M : \mathbb{R}_+ \times \Omega \rightarrow X$ is called *continuous* if M has continuous paths.

Remark 2.5 ([23,28]). If X is a Hilbert space, $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$ are continuous martingales, then $[M, N]$ has a continuous version.

Let $1 \leq p \leq \infty$. We will denote the linear space of all continuous X -valued L^p -martingales on Ω which start at zero by $\mathcal{M}_X^{p,c}(\Omega)$. For brevity we will write $\mathcal{M}_X^{p,c}$ instead of $\mathcal{M}_X^{p,c}(\Omega)$ since Ω is fixed. Analogously to [23], Lemma 17.4, by applying Doob's maximal inequality [19], Theorem 3.2.2, one can show the following proposition.

Proposition 2.6. *Let X be a Banach space, $p \in (1, \infty)$. Then $\mathcal{M}_X^{p,c}$ is a Banach space with the following norm: $\|M\|_{\mathcal{M}_X^{p,c}} := \|M_\infty\|_{L^p(\Omega; X)}$.*

2.5. Purely discontinuous martingales

An increasing càdlàg process $A : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ is called *pure jump* if a.s. for each $t \geq 0$, $A_t = A_0 + \sum_{s=0}^t \Delta A_s$. A local martingale $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ is called *purely discontinuous* if $[M]$ is a pure jump process. The reader can find more on purely discontinuous martingales in [22,23]. We leave the following evident lemma without proof.

Lemma 2.7. *Let $A : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$ be an increasing adapted càdlàg process such that $A_0 = 0$. Then there exist unique up to indistinguishability increasing adapted càdlàg processes $A^c, A^d : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$ such that A^c is continuous a.s., A^d is pure jump a.s., $A_0^c = A_0^d = 0$ and $A = A^c + A^d$.*

Remark 2.8. According to the works [29] by Meyer and [44] by Yoeurp (see also [23], Theorem 26.14), any martingale $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ can be uniquely decomposed into a sum of a purely discontinuous local martingale M^d and a continuous local martingale M^c such that $M_0^c = 0$. Moreover, $[M]^c = [M^c]$ and $[M]^d = [M^d]$, where $[M]^c$ and $[M]^d$ are defined as in Lemma 2.7.

Corollary 2.9. *Let $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ be a martingale which is both continuous and purely discontinuous. Then $M = M_0$ a.s.*

Proposition* 2.10. *A martingale $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ is purely discontinuous if and only if MN is a martingale for any continuous bounded martingale $N : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ with $N_0 = 0$.*

Note that some authors take this equivalent condition as the definition of a purely discontinuous martingale, see, for example, [22], Definition I.4.11, and [21], Chapter I.

Definition 2.11. Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \rightarrow X$ be a local martingale. Then M is called *purely discontinuous* if for each $x^* \in X^*$ the local martingale $\langle M, x^* \rangle$ is purely discontinuous.

Remark 2.12. Let X be finite dimensional. Then similarly to Remark 2.8 any martingale $M : \mathbb{R}_+ \times \Omega \rightarrow X$ can be uniquely decomposed into a sum of a purely discontinuous local martingale M^d and a continuous local martingale M^c such that $M_0^c = 0$.

Remark 2.13. Analogously to Proposition 2.10, a martingale $M : \mathbb{R}_+ \times \Omega \rightarrow X$ is purely discontinuous if and only if $\langle M, x^* \rangle N$ is a martingale for any $x^* \in X^*$ and any continuous bounded martingale $N : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ such that $N_0 = 0$.

Let $p \in [1, \infty]$. We will denote the linear space of all purely discontinuous X -valued L^p -martingales on Ω by $\mathcal{M}_X^{p,d}(\Omega)$. Since Ω is fixed, we will use $\mathcal{M}_X^{p,d}$ instead. The scalar case of the next result have been presented in [21], Lemme I.2.12.

Proposition 2.14. *Let X be a Banach space, $p \in (1, \infty)$. Then $\mathcal{M}_X^{p,d}$ is a Banach space with a norm defined as follows: $\|M\|_{\mathcal{M}_X^{p,d}} := \|M_\infty\|_{L^p(\Omega; X)}$.*

Proof. Let $(M^n)_{n \geq 1}$ be a sequence of purely discontinuous X -valued L^p -martingales such that $(M_\infty^n)_{n \geq 1}$ is a Cauchy sequence in $L^p(\Omega; X)$. Let $\xi \in L^p(\Omega; X)$ be such that $\lim_{n \rightarrow \infty} M_\infty^n = \xi$. Define a martingale $M : \mathbb{R}_+ \times \Omega \rightarrow X$ as follows: $M = (M_s)_{s \geq 0} = (\mathbb{E}(\xi | \mathcal{F}_s))_{s \geq 0}$. Let us show that $M \in \mathcal{M}_X^{p,d}$. First, notice that $\|M_\infty\|_{L^p(\Omega; X)} = \|\xi\|_{L^p(\Omega; X)} < \infty$. Further for each $x^* \in X^*$ by [21], Lemme I.2.12, we have that $\langle M, x^* \rangle$ as a limit of real-valued purely discontinuous martingales $(\langle M^n, x^* \rangle)_{n \geq 1}$ in $\mathcal{M}_{\mathbb{R}}^p$ is purely discontinuous. Therefore, M is purely discontinuous by the definition. $\square$

Lemma 2.15. *Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \rightarrow X$ be a martingale such that M is both continuous and purely discontinuous. Then $M = M_0$ a.s.*

Proof. Follows analogously Corollary 2.9. $\square$

2.6. Time-change

A nondecreasing, right-continuous family of stopping times $\tau = (\tau_s)_{s \geq 0}$ is called a *random time-change*. If $\mathbb{F}$ is right-continuous, then according to [23], Lemma 7.3, the same holds true for the induced filtration $\mathbb{G} = (\mathcal{G}_s)_{s \geq 0} = (\mathcal{F}_{\tau_s})_{s \geq 0}$ (see more in [23], Chapter 7). Let X be a Banach space. A martingale $M : \mathbb{R}_+ \times \Omega \rightarrow X$ is said to be τ -continuous if M is an a.s. constant on every interval $[\tau_{s-}, \tau_s]$, $s \geq 0$, where we let $\tau_{0-} = 0$.

Theorem* 2.16. *Let $A : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$ be a strictly increasing continuous predictable process such that $A_0 = 0$ and $A_t \rightarrow \infty$ as $t \rightarrow \infty$ a.s. Let $\tau = (\tau_s)_{s \geq 0}$ be a random time-change defined as $\tau_s := \{t : A_t = s\}$, $s \geq 0$. Then $(A \circ \tau)(t) = (\tau \circ A)(t) = t$ a.s. for each $t \geq 0$. Let $\mathbb{G} = (\mathcal{G}_s)_{s \geq 0} = (\mathcal{F}_{\tau_s})_{s \geq 0}$ be the induced filtration. Then $(A_t)_{t \geq 0}$ is a random time-change with respect to $\mathbb{G}$ and for any $\mathbb{F}$ -martingale $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ the following holds*

- (i) $M \circ \tau$ is a continuous $\mathbb{G}$ -martingale if and only if M is continuous, and
- (ii) $M \circ \tau$ is a purely discontinuous $\mathbb{G}$ -martingale if and only if M is purely discontinuous.

2.7. Stochastic integration

Let X be a Banach space, H be a Hilbert space. For each $h \in H$, $x \in X$ we denote the linear operator $g \mapsto \langle g, h \rangle x$, $g \in H$, by $h \otimes x$. The process $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ is called *elementary progressive* with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ if it is of the form

$$\Phi(t, \omega) = \sum_{k=1}^K \sum_{m=1}^M \mathbf{1}_{(t_{k-1}, t_k] \times B_{mk}}(t, \omega) \sum_{n=1}^N h_n \otimes x_{kmn}, \quad t \geq 0, \omega \in \Omega, \quad (2.2)$$

where $0 \leq t_0 < \dots < t_K < \infty$, for each $k = 1, \dots, K$ the sets $B_{1k}, \dots, B_{Mk}$ are in $\mathcal{F}_{t_{k-1}}$ and the vectors $h_1, \dots, h_N$ are orthogonal. Let $M : \mathbb{R}_+ \times \Omega \rightarrow H$ be a martingale. Then we define the *stochastic integral* $\Phi \cdot M : \mathbb{R}_+ \times \Omega \rightarrow X$ of Φ with respect to M as follows:

$$(\Phi \cdot M)_t = \sum_{k=1}^K \sum_{m=1}^M \mathbf{1}_{B_{mk}} \sum_{n=1}^N \left((M(t_k \wedge t) - M(t_{k-1} \wedge t)), h_n \right) x_{kmn}, \quad t \geq 0. \quad (2.3)$$

We will need the following lemma on stochastic integration (see [41]).

Lemma 2.17. *Let d be a natural number, H be a d -dimensional Hilbert space, $p \in (1, \infty)$, $M, N : \mathbb{R}_+ \times \Omega \rightarrow H$ be L^p -martingales, $F : H \rightarrow H$ be a measurable function such that $\|F(h)\| \leq C\|h\|^{p-1}$ for each $h \in H$ and some $C > 0$. Define $N_- : \mathbb{R}_+ \times \Omega \rightarrow H$ by $(N_-)_t = N_{t-}$, $t \geq 0$. Then $F(N_-) \cdot M$ is a martingale and for each $t \geq 0$*

$$\mathbb{E} \left| (F(N_-) \cdot M)_t \right| \lesssim_{p,d} C (\mathbb{E} \|N_t\|^p)^{\frac{p-1}{p}} (\mathbb{E} \|M_t\|^p)^{\frac{1}{p}}. \quad (2.4)$$

2.8. Multidimensional Wiener process

Let d be a natural number. $W : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^d$ is called a *standard d -dimensional Wiener process* if $\langle W, h \rangle$ is a standard Wiener process for each $h \in \mathbb{R}^d$ such that $\|h\| = 1$. The following lemma is a multidimensional variation of [24], (3.2.19).

Lemma 2.18. *Let $X = \mathbb{R}$, $d \geq 1$, W be a standard d -dimensional Wiener process, $\Phi, \Psi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R})$ be elementary progressive. Then for all $t \geq 0$ a.s.*

$$[\Phi \cdot W, \Psi \cdot W]_t = \int_0^t \langle \Phi^*(s), \Psi^*(s) \rangle ds.$$

The reader can find more on stochastic integration with respect to a Wiener process in the Hilbert space case in [12], in the case of Banach spaces with a martingale type 2 in [7], and in the UMD case in [35]. Notice that the last mentioned work provides sharp L^p -estimates for stochastic integrals for the broadest till now known class of spaces.

2.9. Brownian representation

The following theorem can be found in [24], Theorem 3.4.2 (see also [34,39]).

Theorem 2.19. *Let $d \geq 1$, $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^d$ be a continuous martingale such that $[M]$ is a.s. absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. Then there exist an enlarged probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with an enlarged filtration $\tilde{\mathbb{F}} = (\tilde{F}_t)_{t \geq 0}$, a d -dimensional standard Wiener process $W : \mathbb{R}_+ \times \tilde{\Omega} \rightarrow \mathbb{R}^d$ which is defined on the filtration $\tilde{\mathbb{F}}$, and an $\tilde{\mathbb{F}}$ -progressively measurable $\Phi : \mathbb{R}_+ \times \tilde{\Omega} \rightarrow \mathcal{L}(\mathbb{R}^d)$ such that $M = \Phi \cdot W$.*

2.10. Lebesgue measure

Let X be a finite dimensional Banach space. Then according to Theorem 2.20 and Proposition 2.21 in [16] there exists a unique translation-invariant measure λ_X on X such that $\lambda_X(\mathbb{B}_X) = 1$ for the unit ball $\mathbb{B}_X$ of X . We will call λ_X the *Lebesgue measure*.

3. UMD Banach spaces and martingale decompositions

Let X be a Banach space, $1 < p < \infty$. In this section, we will show that the Meyer–Yoeurp and Yoeurp decompositions for X -valued L^p -martingales take place if and only if X has the UMD property.

3.1. Meyer–Yoeurp decomposition in UMD case

This subsection is devoted to the generalization of Meyer–Yoeurp decomposition (see Remark 2.8) to the UMD Banach space case:

Theorem 3.1 (Meyer–Yoeurp decomposition). *Let X be a UMD Banach space, $p \in (1, \infty)$, $M : \mathbb{R}_+ \times \Omega \rightarrow X$ be an L^p -martingale. Then there exist unique martingales $M^d, M^c : \mathbb{R}_+ \times \Omega \rightarrow X$ such that M^d is purely discontinuous, M^c is continuous, $M_0^c = 0$ and $M = M^d + M^c$. Moreover, then for all $t \geq 0$*

$$(\mathbb{E} \|M_t^d\|^p)^{\frac{1}{p}} \leq \beta_{p,X} (\mathbb{E} \|M_t\|^p)^{\frac{1}{p}}, \quad (\mathbb{E} \|M_t^c\|^p)^{\frac{1}{p}} \leq \beta_{p,X} (\mathbb{E} \|M_t\|^p)^{\frac{1}{p}}. \quad (3.1)$$

The proof of the theorem consists of several steps. First we introduce the main tool of our proof – the Burkholder function.

Definition 3.2. Let E be a linear space with a scalar field $\mathbb{R}$.

- (i) A function $f : E \times E \rightarrow \mathbb{R}$ is called *biconcave* if for each $x, y \in E$ one has that the mappings $e \mapsto f(x, e)$ and $e \mapsto f(e, y)$ are concave.
- (ii) A function $f : E \times E \rightarrow \mathbb{R}$ is called *zigzag-concave* if for each $x, y \in E$ and $\varepsilon \in \mathbb{R}$ such that $|\varepsilon| \leq 1$, the function $z \mapsto f(x + z, y + \varepsilon z)$ is concave.

The following theorem is a small variation of [9] and [19], Theorem 4.5.6, and has been proven in [41].

Theorem 3.3 (Burkholder). *For a Banach space X the following are equivalent*

1. X is a UMD Banach space;
2. for each $p \in (1, \infty)$ there exists a constant β and a zigzag-concave function $U : X \times X \rightarrow \mathbb{R}$ such that

$$U(x, y) \geq \|y\|^p - \beta^p \|x\|^p, \quad x, y \in X. \quad (3.2)$$

The smallest admissible β for which such U exists is $\beta_{p,X}$.

Remark 3.4. Fix a UMD space X and $p \in (1, \infty)$. A special zigzag-concave function U from Theorem 3.3 have been obtained in [19], Theorem 4.5.6. We will call this function *the Burkholder function*. For the convenience of the reader we leave out the construction of the Burkholder function. The following properties of the Burkholder function U were demonstrated in [41], Section 3:

- (A) $U(\alpha x, \alpha y) = |\alpha|^p U(x, y)$ for all $x, y \in X, \alpha \in \mathbb{R}$.
- (B) $U(x, \alpha x) \leq 0$ for all $x \in X, \alpha \in [-1, 1]$.
- (C) U is continuous.

Remark 3.5. Fix a UMD space X and $p \in (1, \infty)$. Let the Burkholder function U be as in Remark 3.4. Then there exists a biconcave function $V : X \times X \rightarrow \mathbb{R}$ such that

$$V(x, y) = U\left(\frac{x-y}{2}, \frac{x+y}{2}\right), \quad x, y \in X. \quad (3.3)$$

In [41], Section 3, the following properties of V have been explored:

- (A) For each $x, y \in X$ and $a, b \in \mathbb{R}$ such that $|a+b| \leq |a-b|$ one has that the function

$$z \mapsto V(x+az, y+bz) = U\left(\frac{x-y}{2} + \frac{(a-b)z}{2}, \frac{x+y}{2} + \frac{(a+b)z}{2}\right)$$

is concave.

- (B) V is continuous.
- (C) Let X be finite dimensional. Then $x \mapsto V(x, y)$ and $y \mapsto V(x, y)$ are a.s. Fréchet-differentiable with respect to the Lebesgue measure λ_X , and for a.a. $(x, y) \in X \times X$ for each $u, v \in X$ there exists the directional derivative $\frac{\partial V(x+tu, y+tv)}{\partial t}$. Moreover,

$$\frac{\partial V(x+tu, y+tv)}{\partial t} = \langle \partial_x V(x, y), u \rangle + \langle \partial_y V(x, y), v \rangle, \quad (3.4)$$

where $\partial_x V$ and $\partial_y V$ are the corresponding Fréchet derivatives with respect to the first and the second variable.

- (D) Let X be finite dimensional. Then for a.e. $(x, y) \in X \times X$, for all $z \in X$ and real-valued a and b such that $|a + b| \leq |a - b|$

$$\begin{aligned} V(x + az, y + bz) &\leq V(x, y) + \frac{\partial V(x + atz, y + btz)}{\partial t} \\ &= V(x, y) + a \langle \partial_x V(x, y), z \rangle + b \langle \partial_y V(x, y), z \rangle. \end{aligned} \quad (3.5)$$

- (E) Let X be finite dimensional. Then there exists $C > 0$ which depends only on V such that for a.e. pair $x, y \in X$, $\|\partial_x V(x, y)\|, \|\partial_y V(x, y)\| \leq C(\|x\|^{p-1} + \|y\|^{p-1})$.

Definition 3.6. Let d be a natural number, E be a d -dimensional linear space, $(e_n)_{n=1}^d$ be a basis of E . Then $(e_n^*)_{n=1}^d \subset E^*$ is called the *corresponding dual basis* of $(e_n)_{n=1}^d$ if $\langle e_n, e_m^* \rangle = \delta_{nm}$ for each $m, n = 1, \dots, d$.

Note that the corresponding dual basis is uniquely determined. Moreover, if $(e_n^*)_{n=1}^d$ is the corresponding dual basis of $(e_n)_{n=1}^d$, then, the other way around, $(e_n)_{n=1}^d$ is the corresponding dual basis of $(e_n^*)_{n=1}^d$ (here we identify E^{**} with E in the natural way).

Lemma* 3.7. Let d be a natural number, E be a d -dimensional linear space. Let $V : E \times E \rightarrow \mathbb{R}$ and $W : E^* \times E^* \rightarrow \mathbb{R}$ be two bilinear functions. Then the expression

$$\sum_{n,m=1}^d V(e_n, e_m) W(e_n^*, e_m^*) \quad (3.6)$$

does not depend on the choice of basis $(e_n)_{n=1}^d$ of E (here $(e_n^*)_{n=1}^d$ is the corresponding dual basis of $(e_n)_{n=1}^d$).

The following Itô formula is a version of [23], Theorem 26.7, that does not use the Euclidean structure of a finite dimensional Banach space. The proof can be found in [41].

Theorem 3.8 (Itô formula). Let d be a natural number, X be a d -dimensional Banach space, $f \in C^2(X)$, $M : \mathbb{R}_+ \times \Omega \rightarrow X$ be a martingale. Let $(x_n)_{n=1}^d$ be a basis of X , $(x_n^*)_{n=1}^d$ be the corresponding dual basis. Then for each $t \geq 0$

$$\begin{aligned} f(M_t) &= f(M_0) + \int_0^t \langle \partial_x f(M_{s-}), dM_s \rangle \\ &\quad + \frac{1}{2} \int_0^t \sum_{n,m=1}^d f_{x_n, x_m}(M_{s-}) d[[M, x_n^*], [M, x_m^*]]_s^c \\ &\quad + \sum_{s \leq t} (\Delta f(M_s) - \langle \partial_x f(M_{s-}), \Delta M_s \rangle). \end{aligned} \quad (3.7)$$

Proposition 3.9. *Let X be a finite dimensional Banach space, $p \in (1, \infty)$. Let $Y = X \oplus \mathbb{R}$ be a Banach space such that $\|(x, r)\|_Y = (\|x\|_X^p + |r|^p)^{\frac{1}{p}}$. Then $\beta_{p,Y} = \beta_{p,X}$. Moreover, if $M : \mathbb{R}_+ \times \Omega \rightarrow X$ is a martingale on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, then there exists a sequence $(M^m)_{m \geq 1}$ of Y -valued martingales on an enlarged probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ with an enlarged filtration $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t \geq 0}$ such that*

1. M_t^m has absolutely continuous distributions with respect to the Lebesgue measure on Y for each $m \geq 1$ and $t \geq 0$;
2. $M_t^m \rightarrow (M_t, 0)$ pointwise as $m \rightarrow \infty$ for each $t \geq 0$;
3. if for some $t \geq 0$ $\mathbb{E}\|M_t\|^p < \infty$, then for each $m \geq 1$ one has that $\mathbb{E}\|M_t^m\|^p < \infty$ and $\mathbb{E}\|M_t^m - (M_t, 0)\|^p \rightarrow 0$ as $m \rightarrow \infty$;
4. if M is continuous, then $(M^m)_{m \geq 1}$ are continuous as well,
5. if M is purely discontinuous, then $(M^m)_{m \geq 1}$ are purely discontinuous as well.

Proof. The proof of (1)–(3) follows from [41], while (4) and (5) follow from the construction of M^m and N^m given in [41]. $\square$

Remark 3.10. Notice that the construction in [41] also allows us to sum these approximations for different martingales. Namely, if M and N are two X -valued martingales, then we can construct the corresponding Y -valued martingales $(M^m)_{m \geq 1}$ and $(N^m)_{m \geq 1}$ as in Proposition 3.9 in such a way that $M_t^m + N_t^m$ has an absolutely continuous distribution for each $t \geq 0$ and $m \geq 1$.

Proof of Theorem 3.1. *Step 1: finite dimensional case.* Let X be finite dimensional. Then M^d and M^c exist due to Remark 2.12. Without loss of generality $\mathcal{F}_t = \mathcal{F}_\infty$, $M_t^d = M_\infty^d$ and $M_t^c = M_\infty^c$. Let d be the dimension of X .

Let $\|\cdot\|$ be a Euclidean norm on X . Then $(X, \|\cdot\|)$ is a Hilbert space, and by Remark 2.5 the quadratic variation $[M^c]$ exists and has a continuous version. Let us show that without loss of generality we can suppose that $[M^c]$ is a.s. absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. Let $A : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$ be as follows: $A_t = [M^c]_t + t$. Then A is strictly increasing continuous, $A_0 = 0$ and $A_\infty = \infty$ a.s. Let the time-change $\tau = (\tau_s)_{s \geq 1}$ be defined as in Theorem 2.16. Then by Theorem 2.16, $M^c \circ \tau$ is a continuous martingale, $M^d \circ \tau$ is a purely discontinuous martingale, $(M^c \circ \tau)_0 = 0$, $(M^d \circ \tau)_0 = M_0^d$ and due to the Kazamaki theorem [23], Theorem 17.24, $[M^c \circ \tau] = [M^c] \circ \tau$. Therefore for all $t > s \geq 0$ by Theorem 2.16 and the fact that $\tau_t \geq \tau_s$ a.s.

$$\begin{aligned} [M^c \circ \tau]_t - [M^c \circ \tau]_s &= [M^c]_{\tau_t} - [M^c]_{\tau_s} \leq [M^c]_{\tau_t} - [M^c]_{\tau_s} + (\tau_t - \tau_s) \\ &= ([M^c]_{\tau_t} + \tau_t) - ([M^c]_{\tau_s} + \tau_s) \\ &= A_{\tau_t} - A_{\tau_s} = t - s. \end{aligned}$$

Hence $[M^c \circ \tau]$ is a.s. absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. Moreover, $(M^i \circ \tau)_\infty = M_\infty^i$, $i \in \{c, d\}$, so this time-change argument does not affect (3.1). Hence we can redefine $M^c := M^c \circ \tau$, $M^d := M^d \circ \tau$, $\mathbb{F} = (\mathcal{F}_s)_{s \geq 0} := \mathbb{G} = (\mathcal{F}_{\tau_s})_{s \geq 0}$.

Since $[M^c]$ is a.s. absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$ and thanks to Theorem 2.19, we can extend Ω and find a d -dimensional Wiener process W :

$\mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^d$ and a stochastically integrable progressively measurable function $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(\mathbb{R}^d, X)$ such that $M^c = \Phi \cdot W$.

Let $U : X \times X \rightarrow \mathbb{R}$ be the Burkholder function that was discussed in Remark 3.4 and Remark 3.5. Let us show that $\mathbb{E}U(M_t, M_t^d) \leq 0$.

Due to Proposition 3.9 and Remark 3.10 we can assume that M_s^c , M_s^d and $M_s = M_s^d + M_s^c$ have absolutely continuous distributions with respect to the Lebesgue measure λ_X on X for each $s \geq 0$. Let $(x_n)_{n=1}^d$ be a basis of X , $(x_n^*)_{n=1}^d$ be the corresponding dual basis of X^* (see Definition 3.6). By the Itô formula (3.7),

$$\begin{aligned} \mathbb{E}U(M_t, M_t^d) &= \mathbb{E}U(M_0, M_0^d) + \mathbb{E} \int_0^t \langle \partial_x U(M_{s-}, M_{s-}^d), dM_s \rangle \\ &\quad + \mathbb{E} \int_0^t \langle \partial_y U(M_{s-}, M_{s-}^d), dM_s^d \rangle + \mathbb{E}I_1 + \mathbb{E}I_2, \end{aligned} \quad (3.8)$$

where

$$\begin{aligned} I_1 &= \sum_{0 < s \leq t} [\Delta U(M_s, M_s^d) - \langle \partial_x U(M_{s-}, M_{s-}^d), \Delta M_s \rangle - \langle \partial_y U(M_{s-}, M_{s-}^d), \Delta M_s^d \rangle], \\ I_2 &= \frac{1}{2} \int_0^t \sum_{i,j=1}^d U_{x_i, x_j}(M_{s-}, M_{s-}^d) d[\langle M, x_i^* \rangle, \langle M, x_j^* \rangle]_s^c \\ &= \frac{1}{2} \int_0^t \sum_{i,j=1}^d U_{x_i, x_j}(M_{s-}, M_{s-}^d) \langle \Phi^*(s)x_i^*, \Phi^*(s)x_j^* \rangle ds. \end{aligned}$$

(Recall that by (3.3) and Remark 3.5(C), U is Fréchet-differentiable a.s. on $X \times X$, hence $\partial_x U$ and $\partial_y U$ are well-defined. Moreover, U is zigzag-concave, so U is concave in the first variable, and therefore the second-order derivatives U_{x_i, x_j} in the first variable are well-defined and exist a.s. on $X \times X$ by the Alexandrov theorem [15], Theorem 6.4.1.) The last equality holds due to Theorem 3.8 and the fact that by Lemma 2.18 for all $s \geq 0$ a.s.

$$\begin{aligned} [\langle M, x_i^* \rangle, \langle M, x_j^* \rangle]_s^c &= [\langle \Phi \cdot W, x_i^* \rangle, \langle \Phi \cdot W, x_j^* \rangle]_s \\ &= [(\Phi^* x_i^*) \cdot W, (\Phi^* x_j^*) \cdot W]_s \\ &= \int_0^s \langle \Phi^*(r)x_i^*, \Phi^*(r)x_j^* \rangle dr. \end{aligned}$$

Let us first show that $I_1 \leq 0$ a.s. Indeed, since M^d is a purely discontinuous part of M , then by Definition 2.11 $\langle M^d, x^* \rangle$ is a purely discontinuous part of $\langle M, x^* \rangle$, and due to Remark 2.8 a.s. for each $t \geq 0$

$$\Delta |\langle M^d, x^* \rangle|_t^2 = \Delta [\langle M^d, x^* \rangle]_t = \Delta [\langle M, x^* \rangle]_t = \Delta |\langle M, x^* \rangle|_t^2$$

for each $x^* \in X^*$. Thus for each $s \geq 0$ by (3.4) and (3.5) $\mathbb{P}$ -a.s.

$$\begin{aligned} & \Delta U(M_s, M_s^d) - \langle \partial_x U(M_{s-}, M_{s-}^d), \Delta M_s \rangle - \langle \partial_y U(M_{s-}, M_{s-}^d), \Delta M_s^d \rangle \\ &= V(M_{s-} + M_{s-}^d + 2\Delta M_s, M_{s-}^d - M_{s-}) - V(M_{s-} + M_{s-}^d, M_{s-}^d - M_{s-}) \\ &\quad - \langle \partial_x V(M_{s-} + M_{s-}^d, M_{s-}^d - M_{s-}), 2\Delta M_s \rangle \\ &\leq 0, \end{aligned}$$

so $I_1 \leq 0$ a.s., and $\mathbb{E}I_1 \leq 0$. Now we show that

$$\mathbb{E} \left(\int_0^t \langle \partial_x U(M_{s-}, M_{s-}^d), dM_s \rangle + \int_0^t \langle \partial_y U(M_{s-}, M_{s-}^d), dM_s^d \rangle \right) = 0.$$

Indeed,

$$\begin{aligned} & \int_0^t \langle \partial_x U(M_{s-}, M_{s-}^d), dM_s \rangle + \int_0^t \langle \partial_y U(M_{s-}, M_{s-}^d), dM_s^d \rangle \\ &= \int_0^t \langle \partial_x V(M_{s-} + M_{s-}^d, M_{s-}^d - M_{s-}), d(M_s + M_s^d) \rangle \\ &\quad + \int_0^t \langle \partial_y V(M_{s-} + M_{s-}^d, M_{s-}^d - M_{s-}), d(M_s^d - M_s) \rangle \end{aligned}$$

so by Lemma 2.17 and Remark 3.5(E) it is a martingale which starts at zero, hence its expectation is zero.

Finally, let us show that $I_2 \leq 0$ a.s. Fix $s \in [0, t]$ and $\omega \in \Omega$. Then $x^* \mapsto \|\Phi^*(s, \omega)x^*\|^2$ defines a nonnegative definite quadratic form on X^* , and since any nonnegative quadratic form defines a Euclidean seminorm, there exists a basis $(\tilde{x}_n^*)_{n=1}^d$ of X^* and a $\{0, 1\}$ -valued sequence $(a_n)_{n=1}^d$ such that

$$\langle \Phi^*(s, \omega)\tilde{x}_n^*, \Phi^*(s, \omega)\tilde{x}_m^* \rangle = a_n \delta_{mn}, \quad m, n = 1, \dots, d.$$

Let $(\tilde{x}_n)_{n=1}^d$ be the corresponding dual basis of X as it is defined in Definition 3.6. Then due to Lemma 3.7 and the linearity of Φ and directional derivatives of U (we skip s and ω for the simplicity of the expressions)

$$\begin{aligned} \sum_{i,j=1}^d U_{x_i, x_j}(M_{s-}, M_{s-}^d) \langle \Phi^* x_i^*, \Phi^* x_j^* \rangle &= \sum_{i,j=1}^d U_{\tilde{x}_i, \tilde{x}_j}(M_{s-}, M_{s-}^d) \langle \Phi^* \tilde{x}_i^*, \Phi^* \tilde{x}_j^* \rangle \\ &= \sum_{i=1}^d U_{\tilde{x}_i, \tilde{x}_i}(M_{s-}, M_{s-}^d) \|\Phi^* \tilde{x}_i^*\|^2. \end{aligned}$$

Recall that U is zigzag-concave, so $t \mapsto U(x + t\tilde{x}_i, y)$ is concave for each $x, y \in X, i = 1, \dots, d$. Therefore $U_{\tilde{x}_i, \tilde{x}_i}(M_{s-}, M_{s-}^d) \leq 0$ a.s., and a.s.

$$\sum_{i=1}^d U_{\tilde{x}_i, \tilde{x}_i}(M_{s-}(\omega), M_{s-}^d(\omega)) \|\Phi^*(s, \omega) \tilde{x}_i^*\|^2 \leq 0.$$

Consequently, $I_2 \leq 0$ a.s., and by (3.8), Remark 3.4(B) and the fact that $M_0^d = M_0$

$$\mathbb{E}U(M_t, M_t^d) \leq \mathbb{E}U(M_0, M_0) \leq 0.$$

By (3.2), $\mathbb{E}\|M_t^d\|^p - \beta_{p,X}^p \mathbb{E}\|M_t\|^p \leq \mathbb{E}U(M_t, M_t^d) \leq 0$, so the first part of (3.1) holds.

The second part of (3.1) follows from the same machinery applied for V . Namely, one can analogously show that

$$\mathbb{E}\|M_t^c\|^p - \beta_{p,X}^p \mathbb{E}\|M_t\|^p \leq \mathbb{E}U(M_t, M_t^c) = \mathbb{E}V(M^d + 2M^c, -M^d) \leq 0$$

by using a V -version of (3.8), inequality (3.5), and the fact that V is concave in the first variable a.s. on $X \times X$.

Step 2: general case. Without loss of generality, we set $\mathcal{F}_\infty = \mathcal{F}_t$. Let $M_t = \xi$. If ξ is a simple function, then it takes its values in a finite dimensional subspace X_0 of X , and therefore $(M_s)_{s \geq 0} = (\mathbb{E}(\xi | \mathcal{F}_s))_{s \geq 0}$ takes its values in X_0 as well, so the theorem and (3.1) follow from Step 1.

Now let ξ be general. Let $(\xi_n)_{n \geq 1}$ be a sequence of simple $\mathcal{F}_t$ -measurable functions in $L^p(\Omega; X)$ such that $\xi_n \rightarrow \xi$ as $n \rightarrow \infty$ in $L^p(\Omega; X)$. For each $n \geq 1$ define $\mathcal{F}_t$ -measurable ξ_n^d and ξ_n^c such that

$$\begin{aligned} M^{d,n} &= (M_s^{d,n})_{s \geq 0} = (\mathbb{E}(\xi_n^d | \mathcal{F}_s))_{s \geq 0}, \\ M^{c,n} &= (M_s^{c,n})_{s \geq 0} = (\mathbb{E}(\xi_n^c | \mathcal{F}_s))_{s \geq 0} \end{aligned} \tag{3.9}$$

are the respectively purely discontinuous and continuous parts of martingale $M^n = (\mathbb{E}(\xi_n | \mathcal{F}_s))_{s \geq 0}$ as in Remark 2.12. Then due to Step 1 and (3.1), $(\xi_n^d)_{n \geq 1}$ and $(\xi_n^c)_{n \geq 1}$ are Cauchy sequences in $L^p(\Omega; X)$. Let $\xi^c := L^p - \lim_{n \rightarrow \infty} \xi_n^c$ and $\xi^d := L^p - \lim_{n \rightarrow \infty} \xi_n^d$. Define the X -valued L^p -martingales M^d and M^c by

$$M^d = (M_s^d)_{s \geq 0} := (\mathbb{E}(\xi^d | \mathcal{F}_s))_{s \geq 0}, \quad M^c = (M_s^c)_{s \geq 0} := (\mathbb{E}(\xi^c | \mathcal{F}_s))_{s \geq 0}.$$

Thanks to Proposition 2.14, M^d is purely discontinuous, and due to Proposition 2.6 M^c is continuous and $M_0^c = 0$, so $M = M^d + M^c$ is the desired decomposition.

The uniqueness of the decomposition follows from Lemma 2.15. For estimates (3.1), we note that by Step 1, (3.1) applied for Step 1, and [19], Proposition 4.2.17, for each $n \geq 1$

$$(\mathbb{E}\|\xi_n^d\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|\xi_n\|^p)^{\frac{1}{p}}, \quad (\mathbb{E}\|\xi_n^c\|^p)^{\frac{1}{p}} \leq \beta_{p,X}(\mathbb{E}\|\xi_n\|^p)^{\frac{1}{p}},$$

and it remains to let $n \rightarrow \infty$. □

Remark 3.11. Let X be a UMD Banach space, $1 < p < \infty$, $M : \mathbb{R}_+ \times \Omega \rightarrow X$ be continuous (resp. purely discontinuous) L^p -martingale. Then there exists a sequence $(M^n)_{n \geq 1}$ of continuous (resp. purely discontinuous) X -valued L^p -martingales such that M^n takes its values in a finite dimensional subspace of X for each $n \geq 1$ and $M_\infty^n \rightarrow M_\infty$ in $L^p(\Omega; X)$ as $n \rightarrow \infty$. Such a sequence can be provided e.g. by (3.9).

We have proven the Meyer–Yoeurp decomposition in the UMD setting. Next, we prove a converse result which shows the necessity of the UMD property.

Theorem 3.12. Let X be a finite dimensional Banach space, $p \in (1, \infty)$, $\delta \in (0, (\beta_{p,X} - 1) \wedge 1)$. Then there exist a purely discontinuous martingale $M^d : \mathbb{R}_+ \times \Omega \rightarrow X$, a continuous martingale $M^c : \mathbb{R}_+ \times \Omega \rightarrow X$ such that $\mathbb{E}\|M_\infty^d\|^p, \mathbb{E}\|M_\infty^c\|^p < \infty$, $M_0^d = M_0^c = 0$, and for $M = M^d + M^c$ and $i \in \{c, d\}$ the following hold

$$(\mathbb{E}\|M_\infty^i\|^p)^{\frac{1}{p}} \geq \left(\frac{\beta_{p,X} - 1}{2} - \delta \right) (\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}. \quad (3.10)$$

Recall that by [19], Proposition 4.2.17, $\beta_{p,X} \geq \beta_{p,\mathbb{R}} = p^* - 1 \geq 1$ for any UMD Banach space X and $1 < p < \infty$.

Definition 3.13. A random variable $r : \Omega \rightarrow \{-1, 1\}$ is called a *Rademacher variable* if $\mathbb{P}(r = 1) = \mathbb{P}(r = -1) = \frac{1}{2}$.

Lemma* 3.14. Let $\varepsilon > 0$, $p \in (1, \infty)$. Then there exists a continuous martingale $M : [0, 1] \times \Omega \rightarrow [-1, 1]$ with a symmetric distribution such that $\text{sign } M_1$ is a Rademacher random variable and

$$\|M_1 - \text{sign } M_1\|_{L^p(\Omega)} < \varepsilon. \quad (3.11)$$

We will need a definition of a Paley–Walsh martingale.

Definition 3.15 (Paley–Walsh martingales). Let X be a Banach space. A discrete X -valued martingale $(f_n)_{n \geq 0}$ is called a *Paley–Walsh martingale* if there exist a sequence of independent Rademacher variables $(r_n)_{n \geq 1}$, a function $\phi_n : \{-1, 1\}^{n-1} \rightarrow X$ for each $n \geq 2$ and $\phi_1 \in X$ such that $df_n = r_n \phi_n(r_1, \dots, r_{n-1})$ for each $n \geq 2$ and $df_1 = r_1 \phi_1$.

Remark 3.16. Let X be a UMD space, $1 < p < \infty$, $\delta > 0$. Then using Proposition 2.1 one can construct a martingale difference sequence $(d_j)_{j=1}^n \in L^p(\Omega; X)$ and a $\{-1, 1\}$ -valued sequence $(\varepsilon_j)_{j=1}^n$ such that

$$\left(\mathbb{E} \left\| \sum_{j=1}^n \frac{\varepsilon_j \pm 1}{2} d_j \right\|^p \right)^{\frac{1}{p}} \geq \frac{\beta_{p,X} - \delta - 1}{2} \left(\mathbb{E} \left\| \sum_{j=1}^n d_j \right\|^p \right)^{\frac{1}{p}}.$$

Proof of Theorem 3.12. Denote $\frac{\beta_{p,X}-\delta-1}{2}$ by $\gamma_{p,X}^\delta$. By Proposition 2.1, there exists a natural number $N \geq 1$, a discrete X -valued martingale $(f_n)_{n=0}^N$ such that $f_0 = 0$, and a sequence of scalars $(\varepsilon_n)_{n=1}^N$ such that $\varepsilon_n \in \{0, 1\}$ for each $n = 1, \dots, N$, such that

$$\left(\mathbb{E} \left\| \sum_{n=1}^N \varepsilon_n df_n \right\|^p \right)^{\frac{1}{p}} \geq \gamma_{p,X}^\delta (\mathbb{E} \|f_N\|^p)^{\frac{1}{p}}. \quad (3.12)$$

According to [19], Theorem 3.6.1, we can assume that $(f_n)_{n=0}^N$ is a Paley–Walsh martingale. Let $(r_n)_{n=1}^N$ be a sequence of Rademacher variables and $(\phi_n)_{n=1}^N$ be a sequence of functions as in Definition 3.15, that is, be such that $f_n = \sum_{k=2}^n r_k \phi_k(r_1, \dots, r_{k-1}) + r_1 \phi_1$ for each $n = 1, \dots, N$. Without loss of generality, we assume that

$$(\mathbb{E} \|f_N\|^p)^{\frac{1}{p}} \geq 2. \quad (3.13)$$

For each $n = 1, \dots, N$ define a continuous martingale $M^n : [0, 1] \times \Omega \rightarrow [-1, 1]$ as in Lemma 3.14, that is, a martingale M^n with a symmetric distribution such that $\text{sign } M_1^n$ is a Rademacher variable and

$$\|M_1^n - \text{sign } M_1^n\|_{L^p(\Omega)} < \frac{\delta}{KL}, \quad (3.14)$$

where $K = \beta_{p,X} N \max\{\|\phi_1\|, \|\phi_2\|_\infty, \dots, \|\phi_N\|_\infty\}$, and $L = 2\beta_{p,X}$. Without loss of generality, suppose that $(M^n)_{n=1}^N$ are independent. For each $n = 1, \dots, N$ set $\sigma_n = \text{sign } M_1^n$. Define a martingale $M : [0, N+1] \times \Omega \rightarrow X$ in the following way:

$$M_t = \begin{cases} 0, & \text{if } 0 \leq t < 1; \\ M_{n-} + M_{t-n}^n \phi_n(\sigma_1, \dots, \sigma_{n-1}), & \text{if } t \in [n, n+1) \text{ and } \varepsilon_n = 0; \\ M_{n-} + \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}), & \text{if } t \in [n, n+1) \text{ and } \varepsilon_n = 1. \end{cases}$$

Let $M = M^d + M^c$ be the decomposition of Theorem 3.1. Then

$$\begin{aligned} M_{N+1}^c &= \sum_{n=1}^N M_1^n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \mathbf{1}_{\varepsilon_n=0}, \\ M_{N+1}^d &= \sum_{n=1}^N \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \mathbf{1}_{\varepsilon_n=1} = \sum_{n=1}^N \varepsilon_n \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}). \end{aligned}$$

Notice that $(\sigma_n)_{n=1}^N$ is a sequence of independent Rademacher variables, so by (3.12) and the discussion thereafter

$$\left(\mathbb{E} \left\| \sum_{n=1}^N \varepsilon_n \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \right\|^p \right)^{\frac{1}{p}} \geq \gamma_{p,X}^\delta \left(\mathbb{E} \left\| \sum_{n=1}^N \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \right\|^p \right)^{\frac{1}{p}}. \quad (3.15)$$

Let us first show (3.10) with $i = d$. Note that by the triangle inequality, (3.13) and (3.14)

$$\begin{aligned} (\mathbb{E}\|M_{N+1}\|^p)^{\frac{1}{p}} &\geq (\mathbb{E}\|f_N\|^p)^{\frac{1}{p}} - \sum_{n=1}^N (\mathbb{E}\|(M_1^n - \sigma_n)\phi_n(\sigma_1, \dots, \sigma_{n-1})\|^p)^{\frac{1}{p}} \\ &\geq 2 - \frac{\delta}{KL} \cdot N \cdot \max\{\|\phi_1\|, \|\phi_2\|_\infty, \dots, \|\phi_N\|_\infty\} > 1. \end{aligned} \quad (3.16)$$

Therefore,

$$\begin{aligned} (\mathbb{E}\|M_{N+1}^d\|^p)^{\frac{1}{p}} &= \left(\mathbb{E} \left\| \sum_{n=1}^N \varepsilon_n \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \right\|^p \right)^{\frac{1}{p}} \\ &\stackrel{(i)}{\geq} \gamma_{p,X}^\delta \left(\mathbb{E} \left\| \sum_{n=1}^N \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \right\|^p \right)^{\frac{1}{p}} \\ &\stackrel{(ii)}{\geq} \gamma_{p,X}^\delta \left(\mathbb{E} \left\| \sum_{n=1}^N \mathbf{1}_{\varepsilon_n=1} \sigma_n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \right. \right. \\ &\quad \left. \left. + \sum_{n=1}^N \mathbf{1}_{\varepsilon_n=0} M_1^n \phi_n(\sigma_1, \dots, \sigma_{n-1}) \right\|^p \right)^{\frac{1}{p}} \\ &\quad - \gamma_{p,X}^\delta \sum_{n=1}^N (\mathbb{E}\|(M_1^n - \sigma_n)\phi_n(\sigma_1, \dots, \sigma_{n-1})\|^p)^{\frac{1}{p}} \\ &\stackrel{(iii)}{\geq} \gamma_{p,X}^\delta (\mathbb{E}\|M_{N+1}\|^p)^{\frac{1}{p}} - \frac{\delta}{L} \stackrel{(iv)}{\geq} \left(\frac{\beta_{p,X} - 1}{2} - \delta \right) (\mathbb{E}\|M_{N+1}\|^p)^{\frac{1}{p}}, \end{aligned}$$

where (i) follows from (3.15), (ii) holds by the triangle inequality, (iii) holds by (3.14), and (iv) follows from (3.16). By the same reason and Remark 3.16, (3.10) holds for $i = c$. $\square$

Let $p \in (1, \infty)$. Recall that $\mathcal{M}_X^p$ is a space of all X -valued L^p -martingales, $\mathcal{M}_X^{p,d}, \mathcal{M}_X^{p,c} \subset \mathcal{M}_X^p$ are its subspaces of purely discontinuous martingales and continuous martingales that start at zero respectively (see Sections 2.2, 2.4, and 2.5).

Theorem* 3.17. *Let X be a Banach space. Then X is UMD if and only if for some (or, equivalently, for all) $p \in (1, \infty)$, for any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with any filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ that satisfies the usual conditions, $\mathcal{M}_X^p = \mathcal{M}_X^{p,d} \oplus \mathcal{M}_X^{p,c}$, and there exist projections $A^d, A^c \in \mathcal{L}(\mathcal{M}_X^p)$ such that $\text{ran } A^d = \mathcal{M}_X^{p,d}$, $\text{ran } A^c = \mathcal{M}_X^{p,c}$, and for any $M \in \mathcal{M}_X^p$ the decomposition $M = A^d M + A^c M$ is the Meyer–Yoeurp decomposition from Theorem 3.1. If this is the case, then*

$$\|A^d\| \leq \beta_{p,X} \quad \text{and} \quad \|A^c\| \leq \beta_{p,X}. \quad (3.17)$$

Moreover, there exist $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ such that

$$|A^d|, |A^c| \geq \frac{\beta_{p,X} - 1}{2} \vee 1. \quad (3.18)$$

Corollary 3.18. *Let X be a UMD Banach space, $p \in (1, \infty)$. Let $i \in \{c, d\}$. Then $(\mathcal{M}_X^{p,i})^* \simeq \mathcal{M}_{X^*}^{p',i}$, and for each $M \in \mathcal{M}_{X^*}^{p',i}$ and $N \in \mathcal{M}_X^{p,i}$*

$$\langle M, N \rangle := \mathbb{E}\langle M_\infty, N_\infty \rangle, \quad \|M\|_{(\mathcal{M}_X^{p,i})^*} \widetilde{\sim}_{p,X} \|M\|_{\mathcal{M}_{X^*}^{p',i}}.$$

To prove the corollary above we will need the following lemma.

Lemma 3.19. *Let X be a UMD Banach space, $p \in (1, \infty)$, $M \in \mathcal{M}_X^{p,d}$, $N \in \mathcal{M}_{X^*}^{p',c}$. Then $\mathbb{E}\langle M_\infty, N_\infty \rangle = 0$.*

Proof. First, suppose that N_∞ takes its values in a finite dimensional subspace Y of X^* . Let $d \geq 1$ be the dimension of Y , $(y_k)_{k=1}^d$ be the basis of Y . Then there exist $N^1, \dots, N^d \in \mathcal{M}_{\mathbb{R}}^{p',c}$ such that $N = \sum_{k=1}^d N^k y_k$. Hence,

$$E\langle M_\infty, N_\infty \rangle = E\left\langle M_\infty, \sum_{k=1}^d N_\infty^k y_k \right\rangle = \sum_{k=1}^d \mathbb{E}\langle M_\infty, y_k \rangle N_\infty^k \stackrel{(*)}{=} 0, \quad (3.19)$$

where $(*)$ holds due to Proposition 2.10.

Now turn to the general case. By Remark 3.11 for each $N \in \mathcal{M}_{X^*}^{p',c}$ there exists a sequence $(N^n)_{n \geq 1}$ of continuous martingales such that each of N^n is in $\mathcal{M}_{X^*}^{p',c}$ and takes its values in a finite dimensional subspace of X^* , and $N_\infty^n \rightarrow N_\infty$ in $L^{p'}(\Omega; X^*)$ as $n \rightarrow \infty$. Then due to (3.19), $E\langle M_\infty, N_\infty \rangle = \lim_{n \rightarrow \infty} E\langle M_\infty, N_\infty^n \rangle = 0$, so the lemma holds. $\square$

Proof of Corollary 3.18. We will show only the case $i = d$, the case $i = c$ can be shown analogously.

$\mathcal{M}_{X^*}^{p',d} \subset (\mathcal{M}_X^{p,d})^*$ and $\|M\|_{(\mathcal{M}_X^{p,d})^*} \leq \|M\|_{\mathcal{M}_{X^*}^{p',d}}$ for each $M \in \mathcal{M}_{X^*}^{p',d}$ thanks to the Hölder inequality. Now let us show the inverse. Let $f \in (\mathcal{M}_X^{p,d})^*$. Since due to Proposition 2.14 $\mathcal{M}_X^{p,d}$ is a closed subspace of $\mathcal{M}_X^p$, by the Hahn-Banach theorem and Proposition 2.3 there exists $L \in \mathcal{M}_{X^*}^{p'}$ such that $\mathbb{E}\langle L_\infty, N_\infty \rangle = f(N)$ for any $N \in \mathcal{M}_X^{p,d}$, and $\|L\|_{\mathcal{M}_{X^*}^{p'}} = \|f\|_{(\mathcal{M}_X^{p,d})^*}$. Let $L = L^d + L^c$ be the Meyer–Yoeurp decomposition of L as in Theorem 3.1. Then by (3.1)

$$\|L^d\|_{\mathcal{M}_{X^*}^{p',d}} \lesssim_{p,X} \|L\|_{\mathcal{M}_{X^*}^{p'}} = \|f\|_{(\mathcal{M}_X^{p,d})^*}$$

and $\mathbb{E}\langle L_\infty^d, N_\infty \rangle = \mathbb{E}\langle L_\infty, N_\infty \rangle$, so the theorem holds. $\square$

3.2. Yoeurp decomposition of purely discontinuous martingales

As Yoeurp shown in [44], one can provide further decomposition of a purely discontinuous martingale into two parts: a martingale with accessible jumps and a quasi-left continuous martingale. This subsection is devoted to the generalization of this result to a UMD case.

Definition 3.20. Let τ be a stopping time. Then τ is called a *predictable stopping time* if there exists a sequence of stopping times $(\tau_n)_{n \geq 1}$ such that $\tau_n < \tau$ a.s. on $\{\tau > 0\}$ for each $n \geq 1$ and $\tau_n \nearrow \tau$ a.s.

Definition 3.21. Let τ be a stopping time. Then τ is called a *totally inaccessible stopping time* if $\mathbb{P}\{\tau = \sigma < \infty\} = 0$ for each predictable stopping time σ .

Definition 3.22. Let $A : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ be an adapted càdlàg process. A has *accessible jumps* if $\Delta A_\tau = 0$ a.s. for any totally inaccessible stopping time τ . A is called *quasi-left continuous* if $\Delta A_\tau = 0$ a.s. for any predictable stopping time τ .

For the further information on the definitions given, we refer the reader to [23].

Remark 3.23. According to [23], Proposition 25.17, one can show that for any pure jump increasing adapted càdlàg process $A : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ there exist unique increasing adapted càdlàg processes $A^a, A^q : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ such that A^a has accessible jumps, A^q is quasi-left continuous, $A_0^q = 0$ and $A = A^a + A^q$.

The following decomposition theorem was shown by Yoeurp in [44] (see also [23], Corollary 26.16):

Theorem 3.24. Let $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ be a purely discontinuous martingale. Then there exist unique purely discontinuous martingales $M^a, M^q : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ such that M^a has accessible jumps, M^q is quasi-left continuous, $M_0^q = 0$ and $M = M^a + M^q$. Moreover, then $[M^a] = [M]^a$ and $[M^q] = [M]^q$.

Corollary 3.25. Let $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ be a purely discontinuous martingale which is both with accessible jumps and quasi-left continuous. Then $M = M_0$ a.s.

Proof. Without loss of generality, we can set $M_0 = 0$. Then $M = M + 0 = 0 + M$ are decompositions of M into a sum of a martingale with accessible jumps and a quasi-left continuous martingale. Since by Theorem 3.24 this decomposition is unique, $M = 0$ a.s. $\square$

Proposition* 3.26. Let $1 < p < \infty$, $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ be a purely discontinuous L^p -martingale. Let $(M^n)_{n \geq 1}$ be a sequence of purely discontinuous martingales such that $M_\infty^n \rightarrow M_\infty$ in $L^p(\Omega)$. Then the following assertions hold

- (a) if $(M^n)_{n \geq 1}$ have accessible jumps, then M has accessible jumps as well;
- (b) if $(M^n)_{n \geq 1}$ are quasi-left continuous martingales, then M is quasi-left continuous as well.

Definition 3.27. Let X be a Banach space. A martingale $M : \mathbb{R}_+ \times \Omega \rightarrow X$ has *accessible jumps* if $\Delta M_\tau = 0$ a.s. for any totally inaccessible stopping time τ . A martingale $M : \mathbb{R}_+ \times \Omega \rightarrow X$ is called *quasi-left continuous* if $\Delta M_\tau = 0$ a.s. for any predictable stopping time τ .

Lemma* 3.28. Let X be a reflexive Banach space, $M : \mathbb{R}_+ \times \Omega \rightarrow X$ be a purely discontinuous martingale.

- (i) M has accessible jumps if and only if for each $x^* \in X^*$ the martingale $\langle M, x^* \rangle$ has accessible jumps;
- (ii) M is quasi-left continuous if and only if for each $x^* \in X^*$ the martingale $\langle M, x^* \rangle$ is quasi-left continuous.

Definition 3.29. Let X be a Banach space, $p \in (1, \infty)$. Then we define $\mathcal{M}_X^{p,q} \subset \mathcal{M}_X^{p,d}$ as a linear space of all X -valued purely discontinuous quasi-left continuous L^p -martingales which start at 0. We define $\mathcal{M}_X^{p,a} \subset \mathcal{M}_X^{p,d}$ as a linear space of all X -valued purely discontinuous L^p -martingales with accessible jumps.

Proposition* 3.30. Let X be a Banach space, $1 < p < \infty$. Then $\mathcal{M}_X^{p,q}$ and $\mathcal{M}_X^{p,a}$ are closed subspaces of $\mathcal{M}_X^{p,d}$.

The following lemma follows from Corollary 3.25.

Lemma* 3.31. Let X be a Banach space, $M : \mathbb{R}_+ \times \Omega \rightarrow X$ be a purely discontinuous martingale. Let M be both with accessible jumps and quasi-left continuous. Then $M = M_0$ a.s. In other words, $\mathcal{M}_X^{p,q} \cap \mathcal{M}_X^{p,a} = 0$.

The main theorem of this subsection is the following UMD variant of Theorem 3.24.

Theorem 3.32. Let X be a UMD Banach space, $M : \mathbb{R}_+ \times \Omega \rightarrow X$ be a purely discontinuous L^p -martingale. Then there exist unique purely discontinuous martingales $M^a, M^q : \mathbb{R}_+ \times \Omega \rightarrow X$ such that M^a has accessible jumps, M^q is quasi-left continuous, $M_0^q = 0$ and $M = M^a + M^q$. Moreover, if this is the case, then for $i \in \{a, q\}$

$$(\mathbb{E} \|M_\infty^i\|^p)^{\frac{1}{p}} \leq \beta_{p,X} (\mathbb{E} \|M_\infty\|^p)^{\frac{1}{p}}. \quad (3.20)$$

Proof. *Step 1: finite dimensional case.* First, assume that X is finite dimensional. Then M^a and M^q exist and unique due to coordinate-wise applying of Theorem 3.24. Let $M = M^a + M^q$, $N = M^a$. Then for any $x^* \in X^*$, $t \geq 0$ by Theorem 3.24 and Lemma 3.28 a.s.

$$[\langle M, x^* \rangle]_t = [\langle M, x^* \rangle]_t^a + [\langle M, x^* \rangle]_t^q = [\langle M^a, x^* \rangle]_t + [\langle M^q, x^* \rangle]_t,$$

and

$$[\langle N, x^* \rangle]_t = [\langle N, x^* \rangle]_t^a + [\langle N, x^* \rangle]_t^q = [\langle M^a, x^* \rangle]_t.$$

Therefore, a.s.

$$[\langle N, x^* \rangle]_t - [\langle N, x^* \rangle]_s \leq [\langle M, x^* \rangle]_t - [\langle M, x^* \rangle]_s, \quad 0 \leq s < t.$$

Moreover $M_0 = N_0$. Hence, N is weakly differentially subordinated to M (see Section 4), and (3.20) for $i = a$ follows from [41]. By the same reason and since $M_0^q = 0$, (3.20) holds true for $i = q$.

Step 2: general case. Now let X be general. Let $\xi = M_\infty$. Without loss of generality, we set $\mathcal{F}_\infty = \mathcal{F}_t$. Let $(\xi_n)_{n \geq 1}$ be a sequence of simple $\mathcal{F}_t$ -measurable functions in $L^p(\Omega; X)$ such that $\xi_n \rightarrow \xi$ as $n \rightarrow \infty$ in $L^p(\Omega; X)$. For each $n \geq 1$ define $\mathcal{F}_t$ -measurable ξ_n^d and ξ_n^c such that $M^{d,n} = (\mathbb{E}(\xi_n^d | \mathcal{F}_s))_{s \geq 0}$ and $M^{c,n} = (\mathbb{E}(\xi_n^c | \mathcal{F}_s))_{s \geq 0}$ are respectively, purely discontinuous and continuous parts of a martingale $(\mathbb{E}(\xi_n | \mathcal{F}_s))_{s \geq 0}$ as in Remark 2.12. Then thanks to Theorem 3.1, $\xi_n^d \rightarrow \xi$ and $\xi_n^c \rightarrow 0$ in $L^p(\Omega; X)$ as $n \rightarrow \infty$ since M is purely discontinuous.

Since for each $n \geq 1$ the random variable ξ_n^d takes its values in a finite dimensional space, by Theorem 3.24 there exist $\mathcal{F}_t$ -measurable $\xi_n^a, \xi_n^q \in L^p(\Omega; X)$ such that purely discontinuous martingales $M^{a,n} = (\mathbb{E}(\xi_n^a | \mathcal{F}_s))_{s \geq 0}$ and $M^{q,n} = (\mathbb{E}(\xi_n^q | \mathcal{F}_s))_{s \geq 0}$ are respectively with accessible jumps and quasi-left continuous, $\mathbb{E}(\xi_n^q | \mathcal{F}_0) = 0$, and the decomposition $M^{d,n} = M^{a,n} + M^{q,n}$ is as in Theorem 3.24. Since $(\xi_n^d)_{n \geq 1}$ is a Cauchy sequence in $L^p(\Omega; X)$, by Step 1 both $(\xi_n^a)_{n \geq 1}$ and $(\xi_n^q)_{n \geq 1}$ are Cauchy in $L^p(\Omega; X)$ as well. Let ξ^a and ξ^q be their limits. Define martingales $M^a, M^q : \mathbb{R}_+ \times \Omega \rightarrow X$ in the following way:

$$M_s^a := \mathbb{E}(\xi^a | \mathcal{F}_s), \quad M_s^q := \mathbb{E}(\xi^q | \mathcal{F}_s), \quad s \geq 0.$$

By Proposition 3.30 M^a is a martingale with accessible jumps, M^q is quasi-left continuous, $M_0^q = 0$ a.s., and therefore $M = M^a + M^q$ is the desired decomposition. Moreover, by Step 1 for each $n \geq 1$ and $i \in \{a, q\}$, $(\mathbb{E} \|\xi_n^i\|^p)^{\frac{1}{p}} \leq \beta_{p,X} (\mathbb{E} \|\xi_n^d\|^p)^{\frac{1}{p}}$, and hence the estimate (3.20) follows by letting n to infinity.

The uniqueness of the decomposition follows from Lemma 3.31. $\square$

The following theorem, as Theorem 3.12, illustrates that the decomposition in Theorem 3.32 takes place only in the UMD space case.

Theorem 3.33. *Let X be a finite dimensional Banach space, $p \in (1, \infty)$, $\delta \in (0, \frac{\beta_{p,X}-1}{2})$. Then there exist purely discontinuous martingales $M^a, M^q : \mathbb{R}_+ \times \Omega \rightarrow X$ such that M^a has accessible jumps, M^q is quasi-left continuous, $\mathbb{E} \|M_\infty^a\|^p, \mathbb{E} \|M_\infty^q\|^p < \infty$, $M_0^a = M_0^q = 0$, and for $M = M^a + M^q$ and $i \in \{a, q\}$ the following holds*

$$(\mathbb{E} \|M_\infty^i\|^p)^{\frac{1}{p}} \geq \left(\frac{\beta_{p,X}-1}{2} - \delta \right) (\mathbb{E} \|M_\infty\|^p)^{\frac{1}{p}}. \quad (3.21)$$

For the proof, we will need the following lemma.

Lemma 3.34. *Let $\varepsilon \in (0, \frac{1}{2})$, $p \in (1, \infty)$. Then there exist martingales $M, M^a, M^q : [0, 1] \times \Omega \rightarrow [-1 - \varepsilon, 1 + \varepsilon]$ with symmetric distributions such that M^a is a martingale with accessible*

jumps, $\|M_1^a\|_{L^p(\Omega)} < \varepsilon$, M^q is a quasi-left continuous martingale, $M_0^q = 0$ a.s., $M = M^a + M^q$, $\text{sign } M_1$ is a Rademacher random variable and

$$\|M_1 - \text{sign } M_1\|_{L^p(\Omega)} < \varepsilon. \quad (3.22)$$

Proof. Let $N^+, N^- : [0, 1] \times \Omega \rightarrow \mathbb{R}$ be independent Poisson processes with the same intensity λ_ε such that $\mathbb{P}(N_1^+ = 0) = \mathbb{P}(N_1^- = 0) < \frac{\varepsilon^p}{2^p}$ (such λ_ε exists since N_1^+ and N_1^- have Poisson distributions, see [25]). Define a stopping time τ in the following way:

$$\tau = \inf\{t : N_t^+ \geq 1\} \wedge \inf\{t : N_t^- \geq 1\} \wedge 1.$$

Let $M_t^q := N_{t \wedge \tau}^+ - N_{t \wedge \tau}^-$, $t \in [0, 1]$. Then M^q is quasi-left continuous with a symmetric distribution. Let r be an independent Rademacher variable, $M_t^a = \frac{\varepsilon}{2}r$ for each $t \in [0, 1]$. Then M^a is a martingale with accessible jumps and symmetric distribution, and $\|M_1^a\|_{L^p(\Omega)} = \frac{\varepsilon}{2} < \varepsilon$. Let $M = M^a + M^q$. Then a.s.

$$M_1 \in \left\{ -1 - \frac{\varepsilon}{2}, -1 + \frac{\varepsilon}{2}, -\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}, 1 + \frac{\varepsilon}{2} \right\}, \quad (3.23)$$

so $\mathbb{P}(M_1 = 0) = 0$, and therefore $\text{sign } M_1$ is a Rademacher random variable. Let us prove (3.22). Notice that due to (3.23) if $|M_1^q| = 1$, then $|M_1 - \text{sign } M_1| < \frac{\varepsilon}{2}$, and if $|M_1^q| = 0$, then $|M_1 - \text{sign } M_1| < 1$. Therefore,

$$\begin{aligned} \mathbb{E}|M_1 - \text{sign } M_1|^p &= \mathbb{E}|M_1 - \text{sign } M_1|^p \mathbf{1}_{|M_1^q|=1} + \mathbb{E}|M_1 - \text{sign } M_1|^p \mathbf{1}_{|M_1^q|=0} \\ &< \frac{\varepsilon^p}{2^p} + \frac{\varepsilon^p}{2^p} < \varepsilon^p, \end{aligned}$$

so (3.22) holds. $\square$

Proof of Theorem 3.33. The proof is analogous to the proof of Theorem 3.12, while one has to use Lemma 3.34 instead of Lemma 3.14. $\square$

Theorem 3.33 yields the following characterization of the UMD property.

Theorem 3.35. Let X be a Banach space. Then X is a UMD Banach space if and only if for some (equivalently, for all) $p \in (1, \infty)$ there exists $c_{p,X} > 0$ such that for any L^p -martingale $M : \mathbb{R}_+ \times \Omega \rightarrow X$ there exist unique martingales $M^c, M^q, M^a : \mathbb{R}_+ \times \Omega \rightarrow X$ such that $M_0^c = M_0^q = 0$, M^c is continuous, M^q is purely discontinuous quasi-left continuous, M^a is purely discontinuous with accessible jumps, $M = M^c + M^q + M^a$, and

$$(\mathbb{E}\|M_\infty^c\|^p)^{\frac{1}{p}} + (\mathbb{E}\|M_\infty^q\|^p)^{\frac{1}{p}} + (\mathbb{E}\|M_\infty^a\|^p)^{\frac{1}{p}} \leq c_{p,X}(\mathbb{E}\|M_\infty\|^p)^{\frac{1}{p}}. \quad (3.24)$$

If this is the case, then the least admissible $c_{p,X}$ is in the interval $[\frac{3\beta_{p,X}-3}{2} \vee 1, 3\beta_{p,X}]$.

The decomposition $M = M^c + M^q + M^a$ is called the *canonical decomposition* of the martingale M (see [14,23,44]).

Proof. The “if and only if” part follows from Theorem 3.17, Theorem 3.32 and Theorem 3.33. The estimate $c_{p,X} \leq 3\beta_{p,X}$ follows from (3.1) and (3.20). The estimate $c_{p,X} \geq \frac{3\beta_{p,X}-3}{2} \vee 1$ follows from (3.10) and (3.21). $\square$

Corollary 3.36. *Let X be a Banach space. Then X is a UMD Banach space if and only if $\mathcal{M}_X^{p,d} = \mathcal{M}_X^{p,a} \oplus \mathcal{M}_X^{p,q}$ and $\mathcal{M}_X^p = \mathcal{M}_X^{p,c} \oplus \mathcal{M}_X^{p,q} \oplus \mathcal{M}_X^{p,a}$ for any filtration that satisfies the usual conditions.*

Proof. The corollary follows from Theorem 3.32, Theorem 3.33 and Theorem 3.35. $\square$

3.3. Stochastic integration

The current subsection is devoted to application of Theorem 3.35 to stochastic integration with respect to a general martingale.

Proposition* 3.37. *Let H be a Hilbert space, X be a Banach space, $M : \mathbb{R}_+ \times \Omega \rightarrow H$ be a martingale, $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ be elementary progressive. Then*

- (i) *if M is continuous, then $\Phi \cdot M$ is continuous;*
- (ii) *if M is purely discontinuous, then $\Phi \cdot M$ is purely discontinuous;*
- (iii) *if M has accessible jumps, then $\Phi \cdot M$ has accessible jumps;*
- (iv) *if M is quasi-left continuous, then $\Phi \cdot M$ is quasi-left continuous.*

Proposition 3.38. *Let H be a Hilbert space, $M : \mathbb{R}_+ \times \Omega \rightarrow H$ be a local martingale. Then there exist unique martingales $M^c, M^q, M^a : \mathbb{R}_+ \times \Omega \rightarrow H$ such that M^c is continuous, M^q and M^a are purely discontinuous, M^q is quasi-left continuous, M^a has accessible jumps, $M_0^c = M_0^q = 0$ a.s., and $M = M^c + M^q + M^a$.*

Proof. Analogously to Theorem 26.14 and Corollary 26.16 in [23]. $\square$

Theorem 3.39. *Let H be a Hilbert space, X be a UMD Banach space, $p \in (1, \infty)$, $M : \mathbb{R}_+ \times \Omega \rightarrow H$ be a local martingale, $\Phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ be elementary progressive. Let $M = M^c + M^q + M^a$ be the canonical decomposition from Proposition 3.38. Then*

$$\mathbb{E} \|(\Phi \cdot M)_\infty\|^p \approx_{p,X} \mathbb{E} \|(\Phi \cdot M^c)_\infty\|^p + \mathbb{E} \|(\Phi \cdot M^q)_\infty\|^p + \mathbb{E} \|(\Phi \cdot M^a)_\infty\|^p \quad (3.25)$$

and if $(\Phi \cdot M)_\infty \in L^p(\Omega; X)$, then $\Phi \cdot M = \Phi \cdot M^c + \Phi \cdot M^q + \Phi \cdot M^a$ is the canonical decomposition from Theorem 3.35.

Proof. The statement that $\Phi \cdot M = \Phi \cdot M^c + \Phi \cdot M^q + \Phi \cdot M^a$ is the canonical decomposition follows from Proposition 3.37, Theorem 3.35 and the fact that a.s. $(\Phi \cdot M)_0 = (\Phi \cdot M^c)_0 = (\Phi \cdot M^q)_0 = 0$. (3.25) follows then from (3.24) and the triangle inequality. $\square$

Remark 3.40. Notice that the Itô isomorphism for the term $\Phi \cdot M^c$ from (3.25) was explored in [37]. It remains open what to do with the other two terms, but positive results in this direction were obtained in the case of $X = L^q(S)$ in [14].

4. Weak differential subordination and general martingales

This subsection is devoted to the generalization of the main theorem in work [41]. Namely, here we show the L^p -estimates for general X -valued weakly differentially subordinated martingales.

Definition 4.1. Let X be a Banach space, $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$ be local martingales. Then N is *weakly differentially subordinated* to M if $[\langle M, x^* \rangle] - [\langle N, x^* \rangle]$ is an increasing process a.s. for each $x^* \in X^*$.

The following theorem have been proven in [41].

Theorem 4.2. *Let X be a Banach space. Then X has the UMD property if and only if for some (equivalently, for all) $p \in (1, \infty)$ there exists $\beta > 0$ such that for each pair of purely discontinuous martingales $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$ such that N is weakly differentially subordinated to M one has that*

$$(\mathbb{E} \|N_\infty\|^p)^{\frac{1}{p}} \leq \beta (\mathbb{E} \|M_\infty\|^p)^{\frac{1}{p}}.$$

If this is the case, then the least admissible β is the UMD constant $\beta_{p,X}$.

The main goal of the current section is to prove the following generalization of Theorem 4.2 to the case of arbitrary martingales.

Theorem 4.3. *Let X be a UMD Banach space, $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$ be two martingales such that N is weakly differentially subordinated to M . Then for each $p \in (1, \infty)$, $t \geq 0$,*

$$(\mathbb{E} \|N_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2 (\beta_{p,X} + 1) (\mathbb{E} \|M_t\|^p)^{\frac{1}{p}}. \quad (4.1)$$

The proof will be done in several steps. First, we show an analogue of Theorem 4.2 for continuous martingales.

Theorem* 4.4. *Let X be a Banach space. Then X is a UMD Banach space if and only if for some (equivalently, for all) $p \in (1, \infty)$ there exists $c > 0$ such that for any continuous martingales $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$ such that N is weakly differentially subordinated to M , $M_0 = N_0 = 0$, one has that*

$$(\mathbb{E} \|N_\infty\|^p)^{\frac{1}{p}} \leq c_{p,X} (\mathbb{E} \|M_\infty\|^p)^{\frac{1}{p}}. \quad (4.2)$$

If this is the case, then the least admissible $c_{p,X}$ is in the segment $[\beta_{p,X}, \beta_{p,X}^2]$.

For the proof, we will need the following proposition, which demonstrates that one needs a slightly weaker assumption rather than in Theorem 4.4 so that the estimate (4.2) holds in a UMD Banach space.

Proposition 4.5. *Let X be a UMD Banach space, $1 < p < \infty$, $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$ be continuous L^p -martingales s.t. $M_0 = N_0 = 0$ and for each $x^* \in X^*$ a.s. for each $t \geq 0$*

$$[[N, x^*]]_t \leq [[M, x^*]]_t. \quad (4.3)$$

Then for each $t \geq 0$

$$(\mathbb{E} \|N_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2 (\mathbb{E} \|M_t\|^p)^{\frac{1}{p}}. \quad (4.4)$$

Proof. Without loss of generality by a stopping time argument, we assume that M and N are bounded and that $M_\infty = M_t$ and $N_\infty = N_t$.

One can also restrict to a finite dimensional case. Indeed, since X is a separable reflexive space, X^* is separable as well. Let $(Y_m)_{m \geq 1}$ be an increasing sequence of finite-dimensional subspaces of X^* such that $\overline{\bigcup_m Y_m} = X^*$ and $\|\cdot\|_{Y_m} = \|\cdot\|_{X^*|_{Y_m}}$ for each $m \geq 1$. Then for each fixed $m \geq 1$ there exists a linear operator $P_m : X \rightarrow Y_m^*$ of norm 1 defined as follows: $\langle P_m x, y \rangle = \langle x, y \rangle$ for each $x \in X, y \in Y_m$. Therefore $P_m M$ and $P_m N$ are Y_m^* -valued martingales. Moreover, (4.3) holds for $P_m M$ and $P_m N$ since there exists $P_m^* : Y_m \rightarrow X^*$, and for each $y \in Y_m$ we have that $\langle P_m M, y \rangle = \langle M, P_m y \rangle$ and $\langle P_m N, y \rangle = \langle N, P_m y \rangle$. Since Y_m is a closed subspace of X^* , [19], Proposition 4.2.17, yields $\beta_{p,Y_m^*} \leq \beta_{p',X^*}$, consequently again by [19], Proposition 4.2.17, $\beta_{p,Y_m^*} \leq \beta_{p,X^{**}} = \beta_{p,X}$. So if we prove the finite dimensional version, then

$$(\mathbb{E} \|P_m N_t\|^p)^{\frac{1}{p}} \leq \beta_{p,Y_m^*}^2 (\mathbb{E} \|P_m M_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2 (\mathbb{E} \|P_m M_t\|^p)^{\frac{1}{p}},$$

and (4.4) with $c_{p,X} = \beta_{p,X}^2$ will follow by letting $m \rightarrow \infty$.

Let d be the dimension of X , $\|\cdot\|$ be a Euclidean norm on $X \times X$. Let $L = (M, N) : \mathbb{R}_+ \times \Omega \rightarrow X \times X$ be a continuous martingale. Since $(X \times X, \|\cdot\|)$ is a Hilbert space, L has a continuous quadratic variation $[L] : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$ (see Remark 2.5). Let $A : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$ be such that $A_s = [L]_s + s$ for each $s \geq 0$. Then A is continuous strictly increasing predictable. Define a random time-change $(\tau_s)_{s \geq 0}$ as in Theorem 2.16. Let $\mathbb{G} = (\mathcal{G}_s)_{s \geq 0} = (\mathcal{F}_{\tau_s})_{s \geq 0}$ be the induced filtration. Then thanks to the Kazamaki theorem [23], Theorem 17.24, $\tilde{L} = L \circ \tau$ is a G -martingale, and $[\tilde{L}] = [L] \circ \tau$. Notice that $\tilde{L} = (\tilde{M}, \tilde{N})$ with $\tilde{M} = M \circ \tau$, $\tilde{N} = N \circ \tau$, and since by Kazamaki theorem [23], Theorem 17.24, $[M \circ \tau] = [M] \circ \tau$, $[N \circ \tau] = [N] \circ \tau$, and $(M \circ \tau)_0 = (N \circ \tau)_0 = 0$, we have that by (4.3) for each $x^* \in X^*$ a.s. for each $s \geq 0$

$$[[\tilde{N}, x^*]]_s = [[N, x^*]]_{\tau_s} \leq [[M, x^*]]_{\tau_s} = [[\tilde{M}, x^*]]_s. \quad (4.5)$$

Moreover, for all $0 \leq u < s$ we have that a.s.

$$\begin{aligned} [\tilde{L}]_s - [\tilde{L}]_u &= ([L] \circ \tau)_s - ([L] \circ \tau)_u \leq ([L] \circ \tau)_s + \tau_s - ([L] \circ \tau)_u - \tau_u \\ &= ([L]_{\tau_s} + \tau_s) - ([L]_{\tau_u} + \tau_u) = s - u. \end{aligned}$$

Therefore $[\tilde{L}]$ is a.s. absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. Consequently, due to Theorem 2.19, there exists an enlarged probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with an enlarged filtration $\tilde{\mathbb{G}} = (\tilde{\mathcal{G}}_s)_{s \geq 0}$, a $2d$ -dimensional standard Wiener process W , which is defined on $\tilde{\mathbb{G}}$, and a stochastically integrable progressively measurable function $f : \mathbb{R}_+ \times \tilde{\Omega} \rightarrow \mathcal{L}(\mathbb{R}^{2d}, X \times X)$ such that $\tilde{L} = f \cdot W$. Let $f^M, f^N : \mathbb{R}_+ \times \tilde{\Omega} \rightarrow \mathcal{L}(\mathbb{R}^{2d}, X)$ be such that $f = (f^M, f^N)$. Then $\tilde{M} = f^M \cdot W$ and $\tilde{N} = f^N \cdot W$. Let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ be an independent probability space with a filtration $\bar{\mathbb{G}}$ and a $2d$ -dimensional Wiener process $\bar{W}$ on it. Denote by $\bar{\mathbb{E}}$ the expectation on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$. Then because of the decoupling theorem [19], Theorem 4.4.1, for each $s \geq 0$

$$\begin{aligned} (\mathbb{E} \|\tilde{N}_s\|^p)^{\frac{1}{p}} &= (\mathbb{E} \|(f^N \cdot W)_s\|^p)^{\frac{1}{p}} \leq \beta_{p,X} (\mathbb{E} \bar{\mathbb{E}} \|(f^N \cdot \bar{W})_s\|^p)^{\frac{1}{p}}, \\ \frac{1}{\beta_{p,X}} (\mathbb{E} \bar{\mathbb{E}} \|(f^M \cdot \bar{W})_s\|^p)^{\frac{1}{p}} &\leq (\mathbb{E} \|(f^M \cdot W)_s\|^p)^{\frac{1}{p}} = (\mathbb{E} \|\tilde{M}_s\|^p)^{\frac{1}{p}}. \end{aligned} \quad (4.6)$$

Due to the multidimensional version of [23], Theorem 17.11, and (4.5) for each $x^* \in X^*$ we have that

$$s \mapsto [(\tilde{M}, x^*)]_s - [(\tilde{N}, x^*)]_s = \int_0^s (|\langle x^*, f^M(r) \rangle|^2 - |\langle x^*, f^N(r) \rangle|^2) dr \quad (4.7)$$

is nonnegative and absolutely continuous a.s. Since X is separable, we can fix a set $\tilde{\Omega}_0 \subset \tilde{\Omega}$ of full measure on which the function (4.7) is nonnegative for each $s \geq 0$.

Now fix $\omega \in \tilde{\Omega}_0$ and $s \geq 0$. Let us prove that

$$\bar{\mathbb{E}} \|(f^N(\omega) \cdot \bar{W})_s\|^p \leq \bar{\mathbb{E}} \|(f^M(\omega) \cdot \bar{W})_s\|^p.$$

Since $f^M(\omega)$ and $f^N(\omega)$ are deterministic on $\bar{\Omega}$, and since due to (4.7) for each $x^* \in X^*$

$$\begin{aligned} \bar{\mathbb{E}} |(\langle f^N(\omega) \cdot \bar{W} \rangle_s, x^*)|^2 &= \int_0^s |\langle x^*, f^N(r, \omega) \rangle|^2 dr \\ &\leq \int_0^s |\langle x^*, f^M(r, \omega) \rangle|^2 dr = \bar{\mathbb{E}} |(\langle f^M(\omega) \cdot \bar{W} \rangle_s, x^*)|^2, \end{aligned}$$

by [36], Corollary 4.4, we have that $\bar{\mathbb{E}} \|(f^N(\omega) \cdot \bar{W})_s\|^p \leq \bar{\mathbb{E}} \|(f^M(\omega) \cdot \bar{W})_s\|^p$. Consequently, due to (4.6) and the fact that $\bar{\mathbb{P}}(\Omega_0) = 1$

$$(\mathbb{E} \|\tilde{N}_s\|^p)^{\frac{1}{p}} \leq \beta_{p,X} (\mathbb{E} \bar{\mathbb{E}} \|(f^N \cdot \bar{W})_s\|^p)^{\frac{1}{p}} \leq \beta_{p,X} (\mathbb{E} \bar{\mathbb{E}} \|(f^M \cdot \bar{W})_s\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2 (\mathbb{E} \|\tilde{M}_s\|^p)^{\frac{1}{p}}.$$

Recall that $\tilde{M}$ and $\tilde{N}$ are bounded, so thanks to the dominated convergence theorem one gets (4.4) with $c_{p,X} = \beta_{p,X}^2$ by letting s to infinity. $\square$

Proof of Theorem 4.4. The “only if” part & the upper bound of $c_{p,X}$: The “only if” part and the estimate $c_{p,X} \leq \beta_{p,X}^2$ follows from Proposition 4.5 since (4.3) holds for M and N because N is weakly differentially subordinated to M .

The “if” part & the lower bound of $c_{p,X}$: See the supplement [43]. $\square$

Remark 4.6. Let X be a Banach space. Then according to [6,8,17] the Hilbert transform $\mathcal{H}_X$ can be extended to $L^p(\mathbb{R}; X)$ for each $1 < p < \infty$ if and only if X is a UMD Banach space. Moreover, if this is the case, then

$$\sqrt{\beta_{p,X}} \leq \|\mathcal{H}_X\|_{\mathcal{L}(L^p(\mathbb{R}; X))} \leq \beta_{p,X}^2.$$

As it was shown in [41], the upper bound $\beta_{p,X}^2$ can be also directly derived from the upper bound for $c_{p,X}$ in Theorem 4.4. The sharp upper bound for $\|\mathcal{H}_X\|_{\mathcal{L}(L^p(\mathbb{R}; X))}$ remains an open question (see [19], pp. 496–497), so the sharp upper bound for $c_{p,X}$ is of interest.

Lemma* 4.7. Let X be a Banach space, $M^c, N^c : \mathbb{R}_+ \times \Omega \rightarrow X$ be continuous martingales, $M^d, N^d : \mathbb{R}_+ \times \Omega \rightarrow X$ be purely discontinuous martingales, $M_0^c = N_0^c = 0$. Let $M := M^c + M^d$, $N := N^c + N^d$. Suppose that N is weakly differentially subordinated to M . Then N^c is weakly differentially subordinated to M^c , and N^d is weakly differentially subordinated to M^d .

Proof of Theorem 4.3. By Theorem 3.1 there exist martingales $M^d, M^c, N^d, N^c : \mathbb{R}_+ \times \Omega \rightarrow X$ such that M^d and N^d are purely discontinuous, M^c and N^c are continuous, $M_0^c = N_0^c = 0$, and $M = M^d + M^c$ and $N = N^d + N^c$. By Lemma 4.7, N^d is weakly differentially subordinated to M^d and N^c is weakly differentially subordinated to M^c . Therefore, for each $t \geq 0$

$$\begin{aligned} (\mathbb{E}\|N_t\|^p)^{\frac{1}{p}} &\stackrel{(i)}{\leq} (\mathbb{E}\|N_t^d\|^p)^{\frac{1}{p}} + (\mathbb{E}\|N_t^c\|^p)^{\frac{1}{p}} \stackrel{(ii)}{\leq} \beta_{p,X}^2 (\mathbb{E}\|M_t^d\|^p)^{\frac{1}{p}} + \beta_{p,X} (\mathbb{E}\|M_t^c\|^p)^{\frac{1}{p}} \\ &\stackrel{(iii)}{\leq} \beta_{p,X}^2 (\beta_{p,X} + 1) (\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}, \end{aligned}$$

where (i) holds thanks to the triangle inequality, (ii) follows from Theorem 4.2 and Theorem 4.4, and (iii) follows from (3.1). $\square$

Remark 4.8. It is worth noticing that in a view of recent results the sharp constant in (3.1) and (3.20) can be derived and equals the $UMD_p^{(0,1)}$ constant $\beta_{p,X}^{(0,1)}$. In order to show that this is the right upper bound one needs to use a $\{0, 1\}$ -Burkholder function instead of the Burkholder function, while the sharpness follows analogously Theorem 3.12 and 3.33. See [40] for details.

Remark 4.9. In the recent paper, [42] the existence of the canonical decomposition of a general local martingale together with the corresponding weak L^1 -estimates were shown. Again existence of the canonical decomposition of any X -valued martingale is equivalent to X having the UMD property.

Acknowledgements

The author would like to thank Mark Veraar for helpful comments; in particular for showing him [36], Corollary 4.4. The author thanks Jan van Neerven for careful reading of parts of this article and useful suggestions. The author thanks the anonymous referee for his/her valuable comments.

Supplementary Material

Some proofs (DOI: [10.3150/18-BEJ1031SUPP](https://doi.org/10.3150/18-BEJ1031SUPP); .pdf). Recall that throughout the paper many technical proofs have been omitted. The reader can find those proofs in the supplementary file.

References

- [1] Albeverio, S. and Rüdiger, B. (2005). Stochastic integrals and the Lévy–Itô decomposition theorem on separable Banach spaces. *Stoch. Anal. Appl.* **23** 217–253. [MR2130348](#)
- [2] Applebaum, D. (2007). Lévy processes and stochastic integrals in Banach spaces. *Probab. Math. Statist.* **27** 75–88. [MR2353272](#)
- [3] Bañuelos, R., Bielaszewski, A. and Bogdan, K. (2011). Fourier multipliers for non-symmetric Lévy processes. In *Marcinkiewicz Centenary Volume. Banach Center Publ.* **95** 9–25. Warsaw: Polish Acad. Sci. Inst. Math. [MR2918086](#)
- [4] Bañuelos, R. and Bogdan, K. (2007). Lévy processes and Fourier multipliers. *J. Funct. Anal.* **250** 197–213. [MR2345912](#)
- [5] Baumgartner, F. (2015). Lévy processes with values in locally convex Suslin spaces. Available at [arXiv:1510.00538](https://arxiv.org/abs/1510.00538).
- [6] Bourgain, J. (1983). Some remarks on Banach spaces in which martingale difference sequences are unconditional. *Ark. Mat.* **21** 163–168. [MR0727340](#)
- [7] Brzeźniak, Z. (1995). Stochastic partial differential equations in M-type 2 Banach spaces. *Potential Anal.* **4** 1–45. [MR1313905](#)
- [8] Burkholder, D.L. (1983). A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions. In *Conference on Harmonic Analysis in Honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981)*. Wadsworth Math. Ser. 270–286. Belmont, CA: Wadsworth. [MR0730072](#)
- [9] Burkholder, D.L. (1986). Martingales and Fourier analysis in Banach spaces. In *Probability and Analysis (Varenna, 1985)*. *Lecture Notes in Math.* **1206** 61–108. Berlin: Springer. [MR0864712](#)
- [10] Burkholder, D.L. (2001). Martingales and singular integrals in Banach spaces. In *Handbook of the Geometry of Banach Spaces, Vol. I* 233–269. Amsterdam: North-Holland. [MR1863694](#)
- [11] Choi, K.P. (1992). A sharp inequality for martingale transforms and the unconditional basis constant of a monotone basis in $L^p(0, 1)$. *Trans. Amer. Math. Soc.* **330** 509–529. [MR1034661](#)
- [12] Da Prato, G. and Zabczyk, J. (2014). *Stochastic Equations in Infinite Dimensions*, 2nd ed. *Encyclopedia of Mathematics and Its Applications* **152**. Cambridge: Cambridge Univ. Press. [MR3236753](#)
- [13] Dettweiler, E. (1983). Banach space valued processes with independent increments and stochastic integration. In *Probability in Banach Spaces, IV (Oberwolfach, 1982)*. *Lecture Notes in Math.* **990** 54–83. Berlin: Springer. [MR0707509](#)
- [14] Dirksen, S. and Yaroslavtsev, I.S. (2017). L^q -valued Burkholder–Rosenthal inequalities and sharp estimates for stochastic integrals. Available at [arXiv:1707.00109](https://arxiv.org/abs/1707.00109).
- [15] Evans, L.C. and Gariepy, R.F. (1992). *Measure Theory and Fine Properties of Functions*. *Studies in Advanced Mathematics*. Boca Raton, FL: CRC Press. [MR1158660](#)
- [16] Folland, G.B. (1995). *A Course in Abstract Harmonic Analysis*. *Studies in Advanced Mathematics*. Boca Raton, FL: CRC Press. [MR1397028](#)
- [17] Garling, D.J.H. (1986). Brownian motion and UMD-spaces. In *Probability and Banach Spaces (Zaragoza, 1985)*. *Lecture Notes in Math.* **1221** 36–49. Berlin: Springer. [MR0875006](#)
- [18] Geiss, S., Montgomery-Smith, S. and Saksman, E. (2010). On singular integral and martingale transforms. *Trans. Amer. Math. Soc.* **362** 553–575. [MR2551497](#)

- [19] Hytönen, T., van Neerven, J., Veraar, M. and Weis, L. (2016). *Analysis in Banach Spaces. Vol. I. Martingales and Littlewood–Paley Theory. Ergebnisse der Mathematik und Ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]* **63**. Cham: Springer. [MR3617205](#)
- [20] Itô, K. (1942). On stochastic processes. I. (Infinitely divisible laws of probability). *Jpn. J. Math.* **18** 261–301. [MR0014629](#)
- [21] Jacod, J. (1979). *Calcul Stochastique et Problèmes de Martingales. Lecture Notes in Math.* **714**. Berlin: Springer. [MR0542115](#)
- [22] Jacod, J. and Shiryaev, A.N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **288**. Berlin: Springer. [MR1943877](#)
- [23] Kallenberg, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications (New York)*. New York: Springer. [MR1876169](#)
- [24] Karatzas, I. and Shreve, S.E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. New York: Springer. [MR1121940](#)
- [25] Kingman, J.F.C. (1993). *Poisson Processes. Oxford Studies in Probability* **3**. New York: The Clarendon Press, Oxford Univ. Press. [MR1207584](#)
- [26] McConnell, T.R. (1984). On Fourier multiplier transformations of Banach-valued functions. *Trans. Amer. Math. Soc.* **285** 739–757. [MR0752501](#)
- [27] Métivier, M. (1982). *Semimartingales: A Course on Stochastic Processes. de Gruyter Studies in Mathematics* **2**. Berlin: Walter de Gruyter & Co. [MR0688144](#)
- [28] Métivier, M. and Pellaumail, J. (1980). *Stochastic Integration: Probability and Mathematical Statistics*. New York: Academic Press. [MR0578177](#)
- [29] Meyer, P.A. (1976). Un cours sur les intégrales stochastiques. In *Séminaire de Probabilités, X. Lecture Notes in Math.* **511** 245–400. Berlin: Springer. [MR0501332](#)
- [30] Pisier, G. (2016). *Martingales in Banach Spaces. Cambridge Studies in Advanced Mathematics* **155**. Cambridge: Cambridge Univ. Press. [MR3617459](#)
- [31] Protter, P.E. (2005). *Stochastic Integration and Differential Equations. Stochastic Modelling and Applied Probability* **21**. Berlin: Springer. Second edition. Version 2.1, corrected third printing. [MR2273672](#)
- [32] Riedle, M. and van Gaans, O. (2009). Stochastic integration for Lévy processes with values in Banach spaces. *Stochastic Process. Appl.* **119** 1952–1974. [MR2519352](#)
- [33] Rubio de Francia, J.L. (1986). Martingale and integral transforms of Banach space valued functions. In *Probability and Banach Spaces (Zaragoza, 1985). Lecture Notes in Math.* **1221** 195–222. Berlin: Springer. [MR0875011](#)
- [34] Stroock, D.W. and Varadhan, S.R.S. (2006). *Multidimensional Diffusion Processes. Classics in Mathematics*. Berlin: Springer. Reprint of the 1997 edition. [MR2190038](#)
- [35] van Neerven, J.M.A.M., Veraar, M.C. and Weis, L. (2007). Stochastic integration in UMD Banach spaces. *Ann. Probab.* **35** 1438–1478. [MR2330977](#)
- [36] van Neerven, J.M.A.M. and Weis, L. (2005). Stochastic integration of functions with values in a Banach space. *Studia Math.* **166** 131–170. [MR2109586](#)
- [37] Veraar, M. and Yaroslavl'tsev, I. (2016). Cylindrical continuous martingales and stochastic integration in infinite dimensions. *Electron. J. Probab.* **21** Paper No. 59. [MR3563887](#)
- [38] Wang, G. (1995). Differential subordination and strong differential subordination for continuous-time martingales and related sharp inequalities. *Ann. Probab.* **23** 522–551. [MR1334160](#)
- [39] Yaroslavl'tsev, I.S. (2016). Brownian representations of cylindrical continuous local martingales. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* To appear. Available at [arXiv:1605.06946](#).

- [40] Yaroslavtsev, I.S. (2017). Even Fourier multipliers and martingale transforms in infinite dimensions. Available at [arXiv:1710.04958](#).
- [41] Yaroslavtsev, I.S. (2017). Fourier multipliers and weak differential subordination of martingales in UMD Banach spaces. *Studia Math.* Online first version. Available at [arXiv:1703.07817](#).
- [42] Yaroslavtsev, I.S. (2017). On the martingale decompositions of Gundy, Meyer, and Yoeurp in infinite dimensions. Available at [arXiv:1712.00401](#).
- [43] Yaroslavtsev, I.S. (2019). Supplement to “Martingale decompositions and weak differential subordination in UMD Banach spaces”. DOI:[10.3150/18-BEJ1031SUPP](#).
- [44] Yoeurp, Ch. (1976). Décompositions des martingales locales et formules exponentielles. In *Séminaire de Probabilités, X. Lecture Notes in Math.* **511** 432–480. Berlin: Springer. [MR0451395](#)

Received June 2017 and revised February 2018