Stellingen
behorende bij het proefschrift:
"Acoustic monitoring of hydraulic fracture growth"

I

De schijnbare dispersie van de golfpuls, die geobserveerd wordt bij transmissies door een hydraulische scheur, kan verklaard worden door een eenvoudig model bestaande uit een dunne vloeistooflaag zonder intrinsieke verliezen. Deze dispersie is het gevolg van constructieve interferentie van de directe golf met de transmissies die meervoudig gereflecteerd zijn in de laag.

Dit proefschrift, Hoofdstuk 7.

II

De wijde van een hydraulische scheur kan gedurende het openen en sluiten met 'time-lapse' akoestische metingen bepaald worden met een invallende golflengte die een orde groter is dan die scheurwijdte.

Dit proefschrift, Hoofdstuk 11.

III

Voor 'time-lapse' akoestische metingen door scheuren verdienen transmissiemetingen de voorkeur boven reflectiemetingen, omdat de transmissie door het medium voordat de scheur gecreëerd is gebruikt kan worden voor het kalibreren van de amplitude en aankomsttijd van de golfpuls. Bij reflecties is die mogelijkheid afwezig.

Dit proefschrift, Hoofdstuk 6 en Hoofdstuk 11.
IV

Voor dunne hydraulische scheuren die mechanisch open zijn bestaan twee types geleide golven (guided waves); ten eerste een gegeneraliseerde Rayleigh golf en ten tweede een dispersieve golf die zich voornamelijk in de vloestof voortplant, langzamer dan de gegeneraliseerde Rayleigh golf. Deze langzame vloestofgolf is afwezig in het linear-slip model.


Dit proefschrift, Hoofdstuk 7, Hoofdstuk 8 en Appendix D.

V

Het feit dat de sterkte van de diffracties afkomstig van de scheurtip afhankelijk is van de injectiesnelheid van de vloestof die de hydraulische scheurgrond induceert, laat zien dat voor een correcte beschrijving van het amplitudgedrag van diffracties het wijdteprofiel rond de tip meegenomen moet worden.

Dit proefschrift, Hoofdstuk 10.

VI

Bij de numerieke modelering met gridmethodes van de golfpropagatie door een medium waarin de schaal van de inhomogeniteiten sterk varieert, moet de gridverdeling locaal aangepast worden aan die schaalvariatie om de rekentijd binnen praktische grenzen te houden en tevens aan de eisen van de numerieke nauwkeurigheid te voldoen.


Dit proefschrift, Hoofdstuk 8.
VII

De jump/average relatie van het golfveld voor dunne lagen, geïntroduceerd in Hoofdstuk 5 van dit proefschrift, is een nuttige generalisatie van het linear-slip model, omdat feitelijk het linear-slip model alleen geldig is voor lagen die oneindig dun zijn.

Dit proefschrift, Hoofdstuk 5.

VIII

De voortdurende meningsverschillen in de hydraulische breukmechanica over de theoretische beschrijving van scheurgroei is vooral het gevolg van een gebrek aan gegevens over de scheurgeometrie. Bepaling van de scheurgeometrie door middel van akoestische metingen kan de discussie verhelderen, omdat die discussie vervolgens gebaseerd kan worden op objectieve gegevens in plaats van theoretische speculaties.

IX

Nauwkeurige theoretische beschrijving van een fysisch proces is essentieel om meetresultaten te begrijpen. Door de beperking van de meting zijn benaderingen noodzakelijk om tot praktische meetresultaten te komen.
Door de vooruitgang van de communicatie-technologie zijn er steeds meer mogelijkheden om het werkelijke doel van communicatie te vermijden.


Een wetenschappelijke discussie dient gevoerd te worden op basis van de geaccepteerde wetenschappelijke norm. Als deze norm onderdeel wordt van de discussie dan is de wetenschapper nog niet ontslagen van altijd geldende sociale normen.


Een wetenschappelijke verklaring reduceert onbegrepen verschijnselen tot wetmatigheden. Het erkennen van die wetmatigheid kan onze ervaring van de werkelijkheid veranderen, wat het gevaar in zich heeft dat de oorspronkelijke onbevangenheid en verwondering van de ervaring verloren gaat.


De overtuiging dat de vrije wil niet bestaat kan nooit je eigen mening zijn.
ACOUSTIC MONITORING
OF
HYDRAULIC FRACTURE GROWTH
Dit proefschrift is goedgekeurd door de promotor:
Prof. dr. ir. J.T. Fokkema

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"The universe is made of stories, not atoms"

Muriel Rukeyser

aan mijn ouders
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Abstract

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Introduction

1.1 Principles of hydraulic fracturing

Hydraulic fracturing refers to the process of creating planar discontinuities in more or less competent solid material by injecting fluid under high pressure, thus inducing failure. We will focus mainly on its application in the oil and gas industry, where it is used to improve the inflow of hydrocarbons at production wells, as well as the injectivity of water at injection wells. But applications of this technique are also found in the recovery of geothermal energy, where hydraulically induced fractures at local hotspots permit the circulation of water, enabling the extraction of heat. Furthermore, hydraulic fracturing is used for bulk waste disposal and the exploitation of groundwater and coalbed methane.

Hydraulic fracturing as an oil and gas reservoir stimulation technique aims at creating large fractures around the borehole, which will facilitate the inflow of the hydrocarbons. Under high pressure a suitable fluid is pumped into the reservoir and as a result the formation is ruptured. When the fractures are expected to be of a certain desired size, a propping agent, usually a well-sorted coarse sand, is mixed with the injection fluid. The resulting slurry will then fill the fracture volume. After the pumps are shut in, the remaining fluid in the fracture leaks away into the formation and the fracture closes on
the proppant. In the end, the proppant-filled fracture provides the desired highly permeable path. In addition, a hydraulic fracture enables bypass of near-wellbore damage that was caused by the drilling and completion fluids. Hydraulic fracturing has been shown in many cases to increase the yield of a well three- to tenfold, which makes the technique economically attractive. The main fracture treatment is frequently preceded by mini-fracture treatments which are mainly used to estimate fluid leak-off into the reservoir and the in-situ stress state of the reservoir, which determines the closure pressure of the fracture. During fracturing the fluid pressure in the well is measured. All this information is used by the fracture engineer in the design of the fracture treatment. The treatment design also takes into account theoretical experience as well as knowledge of the physics of fracturing. The four important coupled physical processes that control fracture growth are: brittle failure, fluid flow in the fracture, elastic deformation of the reservoir and fluid leak-off into the formation. Because of the complexity of the interaction of the different processes, a variety of mathematical models has been developed that simulate the fracturing process. During the past 20 years, these mathematical models evolved greatly, slowly replacing guess work by sound engineering practice [Cleary, 1988]. The prediction of the fracture dimensions, i.e. height, length and width, by computer models assists the engineer in the design of the fracture treatment.

1.2 Motivation of the research

Despite fifty years of industry experience with hydraulic fracturing, during which more than one million fracture treatments were carried out, there are still situations in which the fracture treatment fails. This is mainly due to the limited amount of control and understanding by reservoir engineers into the actual processes at depth. If the fracture hits a water-bearing formation, the well might start to produce water as a result. Misjudgement whether the fracture is fully developed may lead to failure in placing the proppant into the fracture. This undesired outcome may also be caused by a complicated curvature or multiplicity of the fracture near the wellbore, known as near-wellbore friction.

The experience with failed fracture treatments has led to the view that the success of a treatment depends critically on the ability to predict or estimate the shape and size of the induced hydraulic fracture. Since current theoretical fracture-growth models differ, mainly in the prediction of the pressure level
1.2 Motivation of the research

at the well and the pressure-gradient along the fracture [WARPINSKI, 1985],
research is needed to establish the credibility of competing mathematical
models that are used. Using incorrect predictions of the pressure drop along
the fracture, when extrapolating wellbore pressures, may lead to under-
or overestimates of fracture length and width. Note that any additional mea-
surement (i.e. by acoustic measurements) of the dimension of the fracture,
whether it is its width, height or length, severely limits the uncertainty in
the prediction of its final shape.

The possibility to improve the effectiveness of hydraulic fracture treatments,
by increasing the reliability of theoretical models, led to the initiation of the
research project Geometry of Hydraulic Fractures in 1989. The objective
of this research was to carry out small-scale fracturing experiments in the
laboratory which could form the basis for further analysis of the physics of
fracture mechanics. The benefits of small-scale fracturing laboratory exper-
iments is evaluated by comparing their costs and easiness of repeating or
changing the experiment relative to carrying out actual field experiments.
The controlled laboratory environment provides the opportunity to enhance
our understanding of the fracturing process and to validate the predictions
of the theoretical models by comparing them with the physical experiments.
During the first years of the project an experimental set-up was built and
optimized to carry out small-scale fracturing experiments. A true tri-axial
pressure machine in the laboratory of Rock Physics, at the Faculty of Applied
Earth Science is used to pressurise cubed-shaped rock samples with sides of
30 cm to realistic confining pressures of 10–30 MPa [KAMP AND COCKRAM,
1990]. The blocks are fractured by a highly viscous fracturing fluid, which
slows down the actual fracture growth rate to result in stable and controlled
fracture growth.

Because of fast changes in production technology, for example the dramatic
increase in the number of horizontal and deviated wells, the conditions for
hydraulic fracture treatments changed accordingly, necessitating continuation
of hydraulic fracturing research.
1.3 Objectives of acoustic monitoring

It is desirable to observe the fracturing process during the experiment, but this only possible using non-destructive monitoring techniques. In the previous phase of the project, acoustic acoustic\textsuperscript{1} measurements with active sources were selected for this purpose. In the field, more experience has been gained with acoustic measurements with passive sources [Majer and Doe, 1986; Adair and Sorrels, 1991; Mahrer, 1991; Warpinski et al., 1997], supplied by micro-earthquakes, which are induced by fracturing. The method is especially suited to estimate fracture azimuth and dip. For an accurate determination of the position of the fracture tip the method generally suffers from a lack of redundancy and robustness. This is mainly due to the fact that the micro-earthquakes seem to occur in a rather diffuse zone along the fracture.

Less experience exists in performing acoustic experiments with active sources, as a result of the difficult acquisition geometry related to the considerable depth of most fracture treatments. The high costs of such experiments has limited the number of experiments. Nevertheless, successful results have been published for vertical seismic profiling (VSP) experiments in Turpening [1984] and Meadows and Winterstein [1994] and for cross-well seismic experiments [Aki and Fehler, 1982; Wills et al., 1992].

The advantage of using active acoustic sources while monitoring the fracture growth is the repeatability of the source and the amount of control we have over the source position. The repeatability of the source enables us to relate small changes seen in the measurements over time, to structural changes in the probing medium related to fracture growth. The control of the source position enables us to focus on a specific target of the fracture, whether it is the fracture tip, fracture interface or its borehole intersection.

Early work in the research laboratory of Mobil [Medlin and Massé, 1984] showed the potential of acoustic measurements with active sources. In the first phase of the Geometry of Hydraulic Fractures project, a data acquisition system was built by M. Savić and co-workers, completely integrated with the tri-axial pressure machine [Savić et al., 1990; Savić, 1995]. As a result of that, events were detected that had been scattered by the hydraulic fracture

\textsuperscript{1}To avoid confusion we stress that we use the term "acoustic" measurements or wave to refer to any type of measurement or wave in solid media, including all elastic phenomena like shear waves or interface waves. Only when discussing theoretical concepts, the term "acoustic wave equation" refers to the wave equation for the restricted class of elastodynamic media which cannot sustain shear stresses.
[Savić et al., 1991, 1993; Savić, 1995]. The first measurement configuration consisted of 32 compressional broad-band piezo-electric contact transducers, which can act as a source as well as a receiver.

The radius of the fractures that are created in the rock samples is up to 10 centimetres or more. Because of the viscosity of the fluid, the fracture grows slowly at a rate on the order of 0.1 mm/sec. After a certain time the pump is shut-in after which the fracture slowly closed. The total duration of the experiment is on the order of hours. During all stages of fracture initiation, growth and closure, acoustic waves repeatedly scan the block with an time interval around 30 seconds. Since the duration of a complete scan is well below 5 seconds, we can consider each scan as being obtained from a still fracture.

When discussing the experimental data, we will divide the data into three types\(^2\): transmission measurements detect the waves from a source and receiver transducer pair placed on facing sides of the block. Reflection measurements detect the waves from a source and receiver transducer pair placed on the same side of the block. Diffraction measurement focus on detecting diffractions scattered from the tip of the fracture by selecting a source and receiver transducer pair placed at adjoining block sides at right angles with respect to each other, which configuration optimally detects wave scattering under a relatively large angle. A graphical illustration of this distinction is shown in Figure 1.1.

The main achievement of the first acoustic measurements was the unexpected strength of diffractions, scattered from the perimeter of the fracture [see Savić, 1995]. Initial scepticism whether the thin fractures would be observable in any way was proven to be unjustified. By migrating the diffracted events, the arrival time of these events could be converted to fracture radii estimates as a function of experiment time. This was the first successful application of acoustic measurements. Theoretical fracture-growth curves could be compared with measured fracture-growth curves.

As a step further, we would also like to estimate the width profile of the hydraulic fractures. The width profile is directly linked to the pressure distribution in the fracture which is of primary importance in extrapolating the wellbore pressure to estimate the size of the fracture. To obtain a complete geometrical description of the hydraulic fracture, a width measurement would

---

\(^2\)Note that this distinction is first of all based on the acquisition geometry and not on the type of event. Although a diffraction measurement is optimized for detecting tip diffractions, in addition energy may be measured that is reflected from the fracture.
Figure 1.1: The distinction between different types of measurements; 1: the fracture, 2: the borehole, 3: a piezo-electric transducer, showing the three types of measurement configurations, i.e. 4: transmission, 5: reflection, and 6: diffraction.

present a diagnostic tool to distinguish reliable from less useful theoretical models [WARPINSKI, 1985]. Hence, the main objective of current acoustic research is to assess the feasibility of monitoring the width profile of hydraulic fractures.

Preliminary analysis on the transmission measurements of compressional waves that have propagated across the hydraulic fracture [see also DE PATER ET AL., 1996] shows that when the fracture opens the transmissions are slightly delayed in time and attenuated, compared to the transmission measurement before fracturing. This is illustrated in Figure 1.2(a) and (b) on page 7, which shows the width at the borehole, as measured with a displacement transducer and the compressional transmission. At the moment of fracture closure, the original transmission measurement before fracturing is almost completely restored. Since these transmissions apparently contain so much information on opening and closure of a hydraulic fracture, we postulate that these measurements may indirectly contain information on its width.

The standard method to determine the width of a certain structure with acoustic waves is to estimate the difference of the arrival time of the reflec-
Figure 1.2: Width at the borehole (a), measured by a displacement transducer (LVDT), the compressional transmission (b) and a shear transmission (c) during opening and closure of the fracture.
tion at its upper and lower boundary. Since the centre wavelength in the solid embedding is on the order of 8 mm and the fracture width is below 300 micrometer we do not expect to distinguish these events, which renders this method useless in practice. Hence we have to find a theoretical framework that enables us to predict the time delay and attenuation seen in the transmissions as the fracture opens and closes. More generally, we try to determine how to characterize the hydraulic fracture in terms of its acoustic response and which parameters mainly influence this response. Because of the strong impedance contrast and sharp tip of the fracture, which acts as a diffractor, the scattering response of a hydraulic fracture will be inherently elastic, i.e. no acoustic approximation can be made. The laboratory experiments were previously limited to measurements with compressional transducers [SAVIĆ, 1995]. To obtain full information on the acoustic behaviour of the fracture we selected as our objective to extend the experiment with shear-wave measurements. This forced us to extensively redesign the experimental set-up for the acoustic measurements.

To highlight the main result of the first successful shear-wave measurement, in Figure 1.2(c) on page 7 the shear-wave transmission through a hydraulic fracture is shown, together with the width of the hydraulic fracture, as is measured by a displacement transducer or Linear Variable Differential Transformer (LVDT) clamped inside the borehole (see Figure 1.2(a)). Obviously, in the entire range of 50–250 micrometer, we observe complete shear-wave shadowing. The shadowing of the shear wave was also observed in the field for a differential vertical seismic profiling experiment [TURPENING, 1984] and during cross-well seismic field experiments [AKI AND FEHLER, 1982; WILLS ET AL., 1992]. Recently, also partial shadowing of compressional waves for a cross-well experiment has been measured, during air injection [MAJER ET AL., 1997].

1.4 Solution method

A fracture can be defined as a narrow zone with weakened resistance to deformation as a result of reduced or complete loss of cohesion in the solid material containing the fracture. More specifically we may list the following fracture properties:

- geometrical properties, such as fracture length, height, and curvature, which exist on a scale above the wavelength, i.e. \( L \gg \lambda \).
• sub-wavelength geometrical properties, i.e. $L \ll \lambda$, such as fracture width, interface roughness, or the occurrence of asperities which cause the fracture faces to make mechanical contact.

• constitutive properties such as the fact whether the fracture is dry or filled with fluid. In the latter case the compressibility and viscosity of the fluid should also be included.

Any useful model which describes the interaction of acoustic waves with fractures should in one way or another include the influence of its geometrical and constitutive parameters.

In literature, the standard approach to describe fractures is by considering them as displacement discontinuities [cf. JONES AND WHITTIER, 1967; SCHOENBERG, 1980; FELLER, 1982; PYRAK-NOLTE ET AL., 1990; ROKHLIN AND WANG, 1991]. The fractures are considered as interfaces of vanishing width across which the displacement field jumps by a finite amount. The linear-slip model predicts that this jump in the displacement field at a certain time and position at the fracture interface is determined by the product of the interface traction at the same location and instant, and a fracture parameter called the fracture compliance. For this reason, we designate the linear-slip theory as a linear, instantaneous and local filter model. The fracture compliance is an effective parameter which lumps together the influence of all microscopic geometrical and constitutive properties. Unlike the displacement, the traction is assumed to be continuous across the fracture, in accordance with the assumption that the width of a fracture is negligible.

The original motivation behind the linear-slip theory stemmed from a low-frequency approximation of a low-impedance layer [JONES AND WHITTIER, 1967; SCHOENBERG, 1980], while in addition some empirical evidence was found studying transmission across natural fractures [PYRAK-NOLTE ET AL., 1990].

In a sense, the linear-slip approach with fractures of vanishing width, clashes with our main objective to determine the fracture width. To resolve this issue we will need to review the applicability of linear-slip theory, specifically for hydraulic fractures and the interpretation of its fracture compliance. Since the opening and closing of fractures is expressed in compressional transmission measurements, in the linear-slip model a relationship should exist between (normal) fracture compliance and fracture width.

The fact that we observe shear-wave shadowing means that the hydraulic fracture, at least in the laboratory, cannot transfer any shear deformation. In terms of the linear-slip model this means that the fractures do not have any
transversal fracture stiffness\(^3\). If the fractures were in mechanical contact we would expect at least some shear stiffness of the fracture related to friction of the fracture faces. We conclude that for hydraulic fractures in the laboratory an important characteristic is that the fractures are mechanically open. Here, we define open fractures as fractures where no friction occurs as a result of mechanical contact of the fracture faces. Unlike hydraulically induced fractures, most natural fractures, dry or fluid-filled, are not open. For these kind of fractures the effect of friction or any other contact mechanism should hence be incorporated in the fracture compliance.

As a result of the observations discussed in Figure 1.2 on page 7, the strategy to achieve the goals as set, will be as follows. First, in Chapter 2 we will define some notation conventions and the basic mathematical tools. Next, in Chapter 3 we will describe the physical framework. Since we extend the experiment with shear waves, the material response to shear stresses is described. Since fracturing fluids are often rheologically complex, we have chosen to include viscoelasticity in the description of the constitution of the fluids. After reviewing the theory of viscoelasticity in short, we present the wave equations and reciprocity theorems for elastic or viscoelastic media. The ideal fluid case is treated as a special case.

Because the occurrence of friction of fracture faces seems to appear negligible for the hydraulic fractures in the laboratory, we will start with a simplified fracture model, consisting of a thin fluid-filled layer of infinite lateral extent embedded in an elastic medium. We initially choose to neglect fractur surface roughness, because we want to start with the simplest model and will improve our fracture model when the measured data force us to. Visual inspection of the fracture interfaces in the laboratory shows that the actual fracture faces are razor sharp and have a fracture roughness on the order of the grain size of the cement (≤ 1\(\mu\)m). A discussion on the theoretical effects of surface roughness can be found in Liu et al. [1995].

For the layer of infinite lateral extent in Chapter 4 we will develop the scattering theory of wave propagation through layered viscoelastic media. We include a discussion on the description of the action of the piezo-electric source contact transducer for two polarizations in the generation of the wavefield and the piezo-electric receiver transducer to measure the resulting wavefield. Reflection and transmission at an interface, separating a solid from an ideal fluid is treated separately in Appendix C.

\(^3\)The fracture stiffness is defined as the inverse of the fracture compliance.
Having described scattering by a viscoelastic layer, in Chapter 5 we focus on the specific property that in most cases the fracture can be regarded as thin with respect to the wavelength. Since the original motivation for the linear-slip model stemmed from a thin-layer approximation, we will review the assumptions that have been made to reduce the thin-layer to a linear-slip model. For this purpose we present a generalization of the linear-slip model, which we will refer to as a jump-average relationship. The jump-average relationship is the relationship of the jump of the field quantities across the layer in terms of the average field quantities. This relationship is shown to be valid for general thin layers, whether they are of low or high impedance, in a band-limited domain for the frequency as well as the horizontal slowness. This jump-average relationship proves to be useful to clarify the additional assumptions of the linear-slip model, related to the low impedance of the layer. The analysis shows that only under special conditions we come up with a local and instantaneous filter model for the fracture. Furthermore, expressions for the transmission and reflection in case of a linear-slip interface will be derived.

Once the influence of the layer width is clear from a theoretical point of view, in Chapter 6 we present a fast and practical approach to relate the transmission measurement with a fracture to the transmission measurement without a fracture. In this approach we claim that the amount of dispersion of the transmission through the fracture relative to the transmission measurement before fracturing, can be predicted by convolving the latter transmission measurement with the transmission coefficient of the fracture, for an unknown fracture width. This so-called convolutional model will be of great value for the inversion of transmission measurements to estimates of the fracture width, because in the convolutional model the width estimate is independent on the location of the fracture and the source and receiver parameters.

Related with our time-lapse measurements, the transmission record before fracturing provides us with a very accurate timing as well as amplitude calibration. The small change in arrival time and waveform when the fracture is created can then be related to the width of the fracture. This also explains why we will focus on transmission measurements in this thesis. For reflection measurements the accuracy we strive after will not be attained, because before fracturing no reflection occurs and hence the calibration that we need is absent.
Transmission and reflection at perpendicular incidence is discussed as a special case of scattering of a general incident plane wave, because it elegantly shows the dispersion that is observed in the transmission through the fracture, compared to the measurement before fracturing.

To obtain understanding of all the scattering phenomena that take place at a thin layer or linear-slip interface, we will show the results of a numerical code which models the full waveforms that are measured for this idealized configuration. This enables us to illustrate a few characteristics that we observe in the actual measurements. In particular we investigate the transmission of compressional and shear waves and the excitation and propagation of guided waves along the fracture. The guided waves that occur for a thin fluid-filled layer embedded in a homogeneous solid are treated separately in Appendix D. The existence of guided waves for linear-slip interfaces has been shown in Pyrak-Nolte et al. [1992] and Nihei et al. [1995]. We will show that with respect to interface waves, the thin-layer and linear-slip model diverge as a result of the existence of a slow and highly dispersive guided wave in the thin-layer model, which is not contained in the linear-slip model. The bulk of the energy of the slow wave is contained in the deformation of the fluid.

This slow wave has been detected for tremors in volcanic magma lenses [see e.g. Chouet, 1986; Ferrazzini and Aki, 1987] and is used to estimate the length and width of the magma lens. Moreover, in the field technique called Hydraulic Impedance Testing (HIT) [Paige et al., 1995], guided modes are sent down the borehole, to try to excite slow channel waves in the fracture. When the reflections of the fracture tip can be measured at the surface, the time-delay can be used to estimate the dimensions of the fracture.

We proceed modelling the scattering by a hydraulic fracture in Chapter 8 by discussing the difficulties that arise when the fracture is of finite lateral extent. We will show that in principle the scattering problem of a fracture with a finite size and width can be formulated in terms of a coupled set of two boundary integrals, one for the field inside the fluid inclusion and one for the field in the solid around it. Although the full numerical solution of this scattering problem is beyond the scope of this thesis, we argue that this formulation shows that the fracture compliance of the linear-slip model, is not only a structural parameter reflecting the sub-wavelength geometrical and constitutive parameters of the fracture, but also a macroscopic geometrical property and that the actual apparent fracture compliance distribution is part of the solution to the full scattering problem. In Chapter 8 we argue that
direct application of the linear-slip model will break down in the vicinity of the tip of the fracture. Additional arguments are presented why direct application of the linear-slip model will not result in a correct description to the diffractions from the tip of the fracture. As a practical alternative we suggest that the well-known finite-difference modelling technique can be used to model the scattering by a thin fracture, when we are able to locally adjust the grid spacing to the small scale of the inhomogeneity, in this case the thin fracture. Recent developments in finite-difference codes with varying grid spacing [Falk et al., 1996] shows the potential of this method.

An additional advantage of the finite-difference method is that we can easily include interaction of the wavefield with the borehole in the model. We will show some successful examples of modelling scattering by a fracture with a finite width and lateral extent, including borehole interaction, along with a physical interpretation of the different scattered wave types.

In the last part of this thesis we give an overview of the acoustic data that we acquired in the laboratory along with their practical application. We start in Chapter 9 with an overview of the improvements in the experimental set-up, compared to the set-up described in [Savić, 1995].

In Chapter 10 we show that the diffractions scattered from the tip of the fracture can be used to estimate the size of the fracture. Until now, the interpretation of the various scattered events that we measure has not been well understood. Hence, we elaborate on the physical interpretation of the scattered events that have been measured in the laboratory and show that they are in line with the modelling results that have been obtained using the finite-difference code in Chapter 8. In Chapter 11 we will use compressional transmissions to estimate the fracture width profile of the fracture during its growth.

Chapter 12 summarises the most important conclusions and observations contained in this thesis.
Part I

Theory

“The papalagi really seems to be "The Breakers of the Heavens", the messengers of the Gods, because of their mastery over earth and sky. The papalagi is like a fish, a bird, a worm and a horse, at the same time. He drills into the ground, through the soil and he digs tunnels under the widest freshwater streams. He crawls through mountains and rocks, he ties iron wheels to his feet and speeds off, faster than the fastest horse. He takes off into the air, he can fly! I've seen him glide through the air like a seagull. He has a big canoe for on top of the water and also one for under the water. He sails his canoe from cloud to cloud.”

from 'The Papalagi - Speeches by Tuiavii of Tiavea, a Samoan chief'. These speeches were strictly meant for the Samoan people and were made after visiting the western society in the beginning of this century. Papalagi refers to the white man, literally "Breaker of the Heavens". Originally published in German language by Erich Scheurmann (1920). English version published by Real Free Press (1976) translated by Martin Beumer.
Mathematical Framework

First, we will define the space-time configuration in which we will state our problem. The spatial position vector and the time coordinate will be introduced. In this thesis the physical quantities will be described by Cartesian tensors. We will assume that the reader is familiar with elementary tensor algebra. Nevertheless, to avoid confusion in its use and notation we will explicitly clarify the use of the summation convention. Next, some elementary integral transformations of mathematical physics are defined along with their inverse transformations. The one-sided Laplace transformation is used for causal functions of time. For a bounded or unbounded domain in space we use the two-dimensional spatial Fourier transformation. The reason for this choice stems from the fact that a large part of our analysis is based on simplified fracture models in three-dimensional space, which are invariant in the horizontal direction. We can take advantage of this property by transforming the wavefield into its plane-wave components, which separately are more easily analysed than its complete spatial domain counterpart.

2.1 General description

Vectors in three-dimensional Euclidean space are denoted with bold face symbols. For example, to locate a position in space we use the vector $\mathbf{x}$. By
choosing a *Euclidean* space it is implied that as far as its metrical properties are concerned the physical space is isotropic. In other words, the norm of a vector is *invariant* for a rotation of the chosen reference frame. To specify a given vector, in particular the position vector \( \mathbf{x} \), the three coordinates in relation to a reference frame consisting of three mutually perpendicular unit vectors \( \mathbf{i}_1, \mathbf{i}_2 \) and \( \mathbf{i}_3 \) must be given. Hereby an *orthogonal Cartesian* reference frame is used, opposed to more general skew or curvilinear coordinate systems. A given set of coordinates \( x_i \), is labelled with a subscript index such as \( i, j, k, l, m, n, p, q, r, s \), which ranges from one to three. The symbol \( t \) is exclusively reserved for the time coordinate. The relationship between the vector \( \mathbf{x} \) and its coordinates \( x_i \) is given by

\[
\mathbf{x} = x_1 \mathbf{i}_1 + x_2 \mathbf{i}_2 + x_3 \mathbf{i}_3. \tag{2.1}
\]

In the following chapters we often distinguish the horizontal and vertical direction, where the vertical direction is chosen in direction of the unit vector \( \mathbf{i}_3 \). We introduce the horizontal projection of the position vectors \( \mathbf{x} \) as \( \mathbf{x}_T \), after which we can write the position vector \( \mathbf{x} \) as

\[
\mathbf{x} = \mathbf{x}_T + x_3 \mathbf{i}_3, \tag{2.2a}
\]

with

\[
\mathbf{x}_T = x_1 \mathbf{i}_1 + x_2 \mathbf{i}_2. \tag{2.2b}
\]

### 2.2 The summation convention

The summation convention (also known as Einstein’s summation convention) is a shorthand notation to simplify the notation of tensor equations. The convention prescribes that to every repeated subscript in a product of tensors, the values one to three (the number of dimensions) are successively to be assigned, while after each assignment the result is added. Repeated subscripts are denoted as dummy subscripts. Non-repeated subscripts are denoted as free subscripts. Examples:

\[
a_m b_m \overset{\text{def}}{=} \sum_{m=1}^{3} a_m b_m,
\]

\[
a_m \sigma_{mn} c_n \overset{\text{def}}{=} \sum_{m=1}^{3} \sum_{n=1}^{3} a_m \sigma_{mn} c_n.
\]

Whenever necessary, lowercase *Greek* subscripts are used to indicate the *horizontal* components of a Cartesian tensors. For the restricted two-dimensional
horizontal space, the subscripts can then be assigned the values one and two. The vertical coordinate is explicitly denoted with a subscript with the value three. The summation convention for repeated lowercase Greek indices is accommodated by limiting the summation to the values one and two. The other Roman indices are still connected to the three-dimensional Euclidean space. Examples:

\[
\begin{align*}
a_\alpha b_\alpha \overset{\text{def}}{=} & \sum_{\alpha=1}^{2} a_\alpha b_\alpha, \\
\partial_\alpha \tau_{\alpha i} \overset{\text{def}}{=} & \sum_{\alpha=1}^{2} \partial_\alpha \tau_{\alpha i}, \quad \text{for } i \text{ is } 1, 2 \text{ or } 3.
\end{align*}
\]

From here on the summation convention will be employed, unless specified otherwise.

2.3 The Laplace transformation

Let us assume that the source that generates the wave motion is switched on at the instant \( t = 0 \). In view of the causality condition the source affects the wavefield in the interval \( T \) defined as

\[
T = \{ t \in \mathbb{R} ; \ t > 0 \}. \tag{2.3a}
\]

The complement \( T' \) of the interval \( T \) and the boundary or instant \( \partial T \) between the two intervals are defined accordingly,

\[
\begin{align*}
T' & = \{ t \in \mathbb{R} ; \ t < 0 \}, \tag{2.3b} \\
\partial T & = \{ t \in \mathbb{R} ; \ t = 0 \}. \tag{2.3c}
\end{align*}
\]

We introduce the characteristic function \( \chi_T(t) \) of the set \( T \), illustrated in Figure 2.1, as

\[
\chi_T(t) = \{ 1, \frac{1}{2}, 0 \} \quad \text{when} \quad t \in \{ T, \partial T, T' \}. \tag{2.4}
\]

Now, we define the one-sided Laplace transform of some causal space-time quantity \( u(x, t) \) as

\[
\begin{align*}
\hat{u}(x, s) & = \mathcal{L}_t \{ u \}(x, s) \\
& = \int_{t \in \mathbb{R}} \exp(-st) \chi_T(t) u(x, t) \, dt. \tag{2.5}
\end{align*}
\]
A function in the time domain can be reconstructed from its Laplace transform by explicitly evaluating the Bromwich integral, which acts as the inverse Laplace transformation [Bracewell, 1978]. The Bromwich integral is expressed as

\[
\frac{1}{2\pi j} \int_{s_0-j\infty}^{s_0+j\infty} \exp(st)\hat{u}(x, s) \, ds = \chi_T(t)u(x, t),
\]

where the path of integration of the Bromwich integral is parallel to the imaginary axis of the complex s-domain \(s = s_0\), \(s_0 \in \mathbb{R}\) situated in the right half of the complex s-domain where the Laplace transform itself is analytic \((s_0 > 0)\).

The one-sided Laplace transform of the time-derivative of a function is found by the chain rule of differentiation. The simplest form is found when we interpret the operation of differentiation of \(u(x, t)\) in the distributional sense. We can extend the domain of integration to the whole domain \(\mathbb{R}\) and write

\[
\int_{t \in \mathbb{R}} \exp(-st)\partial_t[\chi_T(t)u(x, t)] \, dt
\]

\[
= \int_{t \in \mathbb{R}} \partial_t[\exp(-st)\chi_T(t)u(x, t)] \, dt - \int_{t \in \mathbb{R}} \partial_t[\exp(-st)]\chi_T(t)u(x, t) \, dt.
\]

The first term on the right-hand side of Eq. (2.7) vanishes, while the second term can be differentiated explicitly leading to

\[
\int_{t \in \mathbb{R}} \exp(-st)\partial_t[\chi_T(t)u(x, t)] \, dt = s\hat{u}(x, s).
\]

Figure 2.1: Domain \(\chi_T(t)\) and its characteristic function.
Hence, the rule applies that the Laplace domain equivalent of the operation of time differentiation is the multiplication by a factor of $s$. For the Laplace transform of the time derivative of $u(x,t)$ over the restricted domain $T$, a correction must be applied for the jump of this function when passing the instant $t = 0$ as follows

$$
\int_{t \in \mathbb{R}} \exp(-st) \chi_T(t) \partial_t [u(x,t)] \, dt = \int_{t \in \mathbb{R}} \exp(-st) \partial_t [\chi_T(t) u(x,t)] \, dt - \int_{t \in \mathbb{R}} \exp(-st) \partial_t [\chi_T(t)] u(x,t) \, dt
$$

$$
= s \hat{u}(x,s) - \lim_{t \downarrow 0} u(x,t). \tag{2.9}
$$

### 2.4 The temporal Fourier transformation

We define the temporal Fourier transformation acting on some causal space-time quantity $u(x,t)$ as the limiting behaviour of the Laplace transformation for imaginary transform parameter, $s \rightarrow j\omega$, leading to

$$
\hat{u}(x,j\omega) = \mathcal{F}_t\{u\}(x,j\omega)
$$

$$
= \int_{t \in \mathbb{R}} \exp(-j\omega t) \chi_T(t) u(x,t) \, dt. \tag{2.10}
$$

A sufficient condition for the convergence of the integral in Eq. (2.10) is the absolute integrability of $u(x,t)$ over the domain $T$. The inverse Fourier transformation is found by taking the inverse Laplace transformation in the limit of imaginary transform parameter resulting in

$$
\mathcal{F}_t^{-1}\{\hat{u}\}(x,t) = \frac{1}{2\pi} \int_{\omega \in \mathbb{R}} \exp(j\omega t) \hat{u}(x,j\omega) \, d\omega
$$

$$
= \chi_T(t) u(x,t). \tag{2.11}
$$

### 2.5 The spatial Fourier transformation

Let a scalar function $\hat{u}(x,s)$ be defined in some bounded or unbounded domain $\mathbb{D}$ in two-dimensional (horizontal) space. The characteristic function $\chi_{\mathbb{D}}$ of its domain, see Figure 2.2, can be defined in $\mathbb{D}$, the boundary surface $\partial \mathbb{D}$ and domain $\mathbb{D}'$, the complement $\mathbb{D} \cup \partial \mathbb{D}$ in $\mathbb{R}^2$, as
\[ \chi_{\mathbb{D}}(x) = \{1, \frac{1}{2}, 0\} \quad \text{when} \quad x_T \in \{\mathbb{D}, \partial\mathbb{D}, \mathbb{D}'\}. \quad (2.12) \]

The real angular horizontal wave vector \( k \) which will act as a transform vector is related to the complex angular horizontal slowness vector \( \alpha \) according to

\[ k = s\alpha = s\alpha_1 i_1 + s\alpha_2 i_2. \quad (2.13) \]

Now we can define the horizontal Fourier transformation acting on a quantity in the Laplace transformed domain \( \hat{u}(x, s) \), defined in an unbounded or bounded domain \( \mathbb{D} \) in terms of the complex angular horizontal slowness, as

\[ \hat{u}(js\alpha, x_3, s) = \mathcal{F}_x\{\hat{u}\}(js\alpha, x_3, s) = \int_{x_T \in \mathbb{R}^2} \exp(js\alpha_{x_3}x_3)\chi_{\mathbb{D}}(x)\hat{u}(x, s) \, dA, \quad s\alpha \in \mathbb{R}^2. \quad (2.14) \]

A sufficient condition for the convergence of the integral in Eq. (2.14) is the absolute integrability of \( u(x, s) \) over the domain \( \mathbb{D} \). The transform vector \( \alpha \) shall be denoted as the complex angular horizontal slowness vector, for reasons to become clear later on. The wave vector \( k \) and hence the product
$s\alpha$ must be real in order to guarantee the convergence of the spatial Fourier transformation for a sufficiently broad class of functions.

The transformation from the angular-slowness domain back to the spatial domain is carried out by employing the Fourier inversion integral

$$\mathcal{F}_x^{-1}\{\hat{u}\}(x, s) = \left(\frac{1}{2\pi}\right)^2 \int_{(s\alpha) \in \mathbb{R}^2} \exp(-j s\alpha x_3) \hat{u}(j s\alpha, x_3, s) \, \mathrm{d}A$$

$$= \chi_{\mathbb{D}}(x)\hat{u}(x, s). \quad (2.15)$$

The horizontal Fourier transform of the horizontal spatial derivative $\partial_\nu u(x, s)$ is found by applying the chain rule of differentiation, similar as in Eq. (2.8). The simplest form again applies when we interpret the operation of differentiation in the distributional sense. We can extend the domain of integration to the domain $\mathbb{R}^2$ and write

$$\int_{x_T \in \mathbb{R}^2} \exp(j s\alpha x_3) \partial_\nu [\chi_{\mathbb{D}}(x)\hat{u}(x, s)] \, \mathrm{d}A$$

$$= \int_{x_T \in \mathbb{R}^2} \partial_\nu [\exp(j s\alpha x_3) \chi_{\mathbb{D}}(x)\hat{u}(x, s)] \, \mathrm{d}A$$

$$- \int_{x_T \in \mathbb{R}^2} \partial_\nu [\exp(j s\alpha x_3)] \chi_{\mathbb{D}}(x)\hat{u}(x, s) \, \mathrm{d}A. \quad (2.16)$$

The first term on the right-hand side of Eq. (2.16) vanishes on account of Gauss' integral theorem [see e.g. Fokkema and van den Berg, 1993], for the evaluation on infinity, expressed as

$$\int_{x_T \in \mathbb{R}^2} \partial_\nu [\exp(j s\alpha x_3) \chi_{\mathbb{D}}(x)\hat{u}(x, s)] \, \mathrm{d}A = 0. \quad (2.17)$$

The second term can be differentiated explicitly. By combining Eqs. (2.16) and (2.17) we conclude

$$\int_{x_T \in \mathbb{R}^2} \exp(j s\alpha x_3) \partial_\nu [\chi_{\mathbb{D}}(x)\hat{u}(x, s)] \, \mathrm{d}A = -js\alpha_\nu \hat{u}(j s\alpha, x_3, s), \quad (2.18)$$

which shows that the operation of horizontal spatial differentiation, $\partial_\nu$, corresponds to multiplication with a factor $-j s\alpha_\nu$ in the transformed domain. The unit vector along the normal to $\partial\mathbb{D}$ pointing away from $\mathbb{D}$ is denoted as $n_\beta$. Using Eq. (2.18) we can evaluate the Fourier transform of the horizontal
spatial derivative of the function over the restricted domain $\mathbb{D}$ as
\begin{equation}
\int_{x_T \in \mathbb{R}^2} \exp(js \alpha x_3) \chi_{\mathbb{D}}(x) \partial_{x_3} \hat{u}(x, s) \, dA \nonumber
\end{equation}
\begin{equation}
= -js \alpha \hat{u}(js \alpha, x_3, s) + \int_{x_T \in \partial \mathbb{D}} \exp(js \alpha x_3) \hat{u}(x, s) n_\nu \, dS, \quad (2.19)
\end{equation}
from which we infer that the jump in $\hat{u}(x, s)$ when passing the boundary $\partial \mathbb{D}$ must be incorporated in the Fourier transform of the horizontal spatial derivative of the function $\hat{u}(x, s)$ over the Domain $\mathbb{D}$ as an addition integral over the boundary of the domain $\mathbb{D}$.

In the following chapters it appears to be convenient to define an alternative slowness vector $\zeta$ to the complex angular horizontal slowness vector $\alpha$ according to
\begin{equation}
\zeta = j\alpha \quad \text{with} \quad -sj\zeta \in \mathbb{R}^2, 
\end{equation}
which we will call the Fourier-slowness vector. Thus, any function in the transformed domain written as $\hat{u}(js \alpha, x_3, s)$ can alternatively be designated as $\hat{u}(s\zeta, x_3, s)$.

The reason for the introduction of the Fourier-slowness vector is twofold. First, we can avoid the extensive use of the imaginary unit $j$, which will repeatedly enter our equations when we apply Eq. (2.18) or Eq. (2.19) to perform the horizontal differentiation, i.e. $\partial_{x_3}$, which now corresponds to multiplication with a factor $-s\zeta_\nu$ in the transformed domain. Second, in the limit $s \to j\omega$, the complex angular slowness vector $\alpha$ becomes imaginary and hence the Fourier-slowness vector $\zeta$ becomes real. The bulk of literature on wave propagation uses a temporal Fourier transformation instead of a Laplace transformation. In that case the Fourier-slowness vector is the more appropriate choice as the spatial transform parameter, hence its name. In other words, in the limit $s \to j\omega$ the limit of the real wavenumber $k$ is found as
\begin{equation}
\lim_{s \to 0} k = \lim_{s \to 0} s\alpha = \omega \zeta, \quad \text{with} \quad \omega \in \mathbb{R}, \quad \zeta \in \mathbb{R}^2. \quad (2.21)
\end{equation}
Our choice is to be persistent in using the complex angular slowness vector $\alpha$ as the transformation vector as in Eq. (2.14), since it is more general. Even so, the Fourier slowness vector $\zeta$ is used merely as an alias for $j\alpha$. In this way, when we take the limit $s \to j\omega$ to limit ourselves to study stationary phenomena in the temporal Fourier domain, we have anticipated on the fact
that the Fourier-slowness vector will become real and a large part of the equations and notation still conform to the bulk of literature.

We will use the Laplace transformation, introduced in this chapter, to transform the wavefield quantities in Chapter 3 to the transformed domain. In Chapter 4 the wave equations in the Laplace transformed domain form the starting point to describe wave propagation in layered viscoelastic media. Further, in Chapter 4 we will exploit the translation invariance of layered media in the horizontal direction by transforming the wave equations to the spatial Fourier domain.
Physical Framework for Elastodynamic Waves

The physical laws for elastodynamic waves are based on conservation laws. The physical quantities that will be conserved are mass, linear momentum, angular momentum and energy. The conservation of linear momentum leads to the equation of motion for elastodynamic waves. The conservation of angular momentum shows that the stress tensor must be symmetrical. When we assume that the spatial changes of the displacement perturbations are small, we can apply the low-amplitude gradient linearizations to the non-linear equation of motion.

In this chapter we will focus on the implementation of viscoelastic material behaviour into the elastodynamic wave equations. The material behaviour is expressed in the constitutive equations relating the internal stress state to the internal strain state. Generally the relationship between the latter is non-linear considering the fact that the material can be confined under large stresses, e.g. in the earth or in the laboratory. However, since the elastodynamic waves that we study consist of small dynamic perturbations on the static stress and strain, a linear relationship is assumed between these stress perturbations and the strain perturbations. All constitutive equations used in this thesis are linear, time invariant and locally reacting, leading to
a linear deformation equation. The special case of isotropic viscoelasticity is described in more detail. We will also review some standard viscoelastic models.

Combination of the linear equation of motion with the linear deformation equation results in a pair of coupled first-order wave equations for the particle velocity and the stress tensor. These equations will be the starting point for the analysis of our wave propagation problem. In case that an ideal fluid is present which cannot sustain any shear stresses, the standard coupled first-order viscoelastic wave equations are not appropriate, since the compliance becomes singular in this formulation. We treat the ideal fluid as a special case and give the reduced coupled first-order acoustic wave equations for the particle velocity and pressure in such an ideal fluid.

Finally, we derive the viscoelastic Betti-Rayleigh reciprocity theorem for linear viscoelastic materials.

### 3.1 Viscoelasticity

#### 3.1.1 The constitutive equations

The equation of motion has to be complemented with relations, describing the constitution of the material. The material behaviour is described by the stress and strain tensor, denoted by the symbols $\tau_{ij}$ and $c_{ij}$, respectively. The stress tensor relates the traction $t_i$ on a certain interface to the normal to that interface $n_j$ according to

$$t_i(x,t) = \tau_{ij}(x,t)n_j,$$  \hfill (3.1)

whereas the linear strain tensor is related to the particle displacement $u_i$ according to

$$c_{ij}(x,t) = \Delta^{+}_{ijkl} \delta_{k} u_l(x,t).$$  \hfill (3.2)

in which the isotropic fourth order tensor $\Delta^{+}_{ijkl}$ is defined [see DE HOOP, 1995] as

$$\Delta^{+}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$  \hfill (3.3)

We can choose to describe the stress response due to a certain internal strain, or conversely describe the strain response due to a certain internal stress. The constitutive relations should be established by means of physical experiments. We assume the following general properties for the material behaviour:
[1] Causality. Only the strain history may influence the present stress on account of the requirement that the material response must be causal. Conversely, only the stress history may influence the present strain.

[2] Locally reacting. The internal stress at a certain fixed position can be related entirely to the strain history at that same position.

[3] Linearity. Generally, the relationship between stress and strain is non-linear [Jeffrey and Engelbrecht, 1994]. However, since the strain perturbations describing the elastodynamic waves are small, a linear relationship is assumed between these strain perturbations and the stress perturbations. The effect of two superposed strain histories is found by adding the resulting stresses for the individual strain histories.

[4] Time-invariance. It is assumed that the material properties and their responses do not change over time resulting in time-invariance of the constitutive equations. Hence the effect of a time delayed strain perturbation is found by retarding the stress response by the same delay in time.

We will first describe the stress response due to a strain source. Consider a strain perturbation $e_{kl}(x,t)$ corresponding to a sudden unit jump at the instant $t = 0$, with strain amplitude $E_{kl}(x)$ written as

$$e_{kl}(x,t) = \chi_T(t)E_{kl}(x).$$

(3.4)

If a certain part of material is placed in this state of strain, an instantaneous stress rise can be followed by a decrease of the stress level. This phenomena is known as stress relaxation and can, in absence of external forces, be described by the stress relaxation tensor $\Gamma_{ijkl}(x,t)$ as

$$\tau_{ij}(x,t) = \chi_T(t)\Gamma_{ijkl}(x,t)E_{kl}(x).$$

(3.5)

The instantaneous stress response is described by the quantity $\Gamma_{ijkl}(x,t \to 0)$. For a general strain history $e_{kl}$ starting at the instant $t = 0$ the stress response can be calculated by virtue of the assumed linearity and time-invariance of the process [Achenbach, 1973] leading to

$$\tau_{ij}(x,t) = \chi_T(t)\Gamma_{ijkl}(x,t)\lim_{t' \to 0}[e_{kl}(x,t')] + \int_0^t \Gamma_{ijkl}(x,t - t')\partial_{t'}[e_{kl}(x,t')]dt'.
$$

(3.6)
The differential operator under the integrand shows that each infinitesimal increment of the internal strain history causes its own retarded stress response. To avoid the differential operator on the strain history, we rewrite Eq. (3.6) by partial integration as

\[
\tau_{ij}(x, t) = \chi_T(t) \lim_{t' \to 0} [\Gamma_{ijkl}(x, t')] e_{kl}(x, t) - \int_0^t \partial_{t'} [\Gamma_{ijkl}(x, t - t')] e_{kl}(x, t') dt'.
\]

(3.7)

We can combine both terms on the right-hand side of Eq. (3.7) by using the generalized definition of time differentiation similar as in Section 2.3, Eq. (2.8), leading to

\[
\tau_{ij}(x, t) = -\int_0^\infty \partial_{t'} [\chi_T(x, t - t') \Gamma_{ijkl}(x, t - t')] e_{kl}(x, t') dt' \\
= +\int_0^\infty \partial_t [\chi_T(x, t - t') \Gamma_{ijkl}(x, t - t')] e_{kl}(x, t') dt',
\]

(3.8)

which shows that the instantaneous response can be brought under the integral sign. It is now convenient to introduce the stiffness tensor \(c_{ijkl}(x, t)\) as

\[
c_{ijkl}(x, t) = \partial_t [\chi_T(t) \Gamma_{ijkl}(x, t)].
\]

(3.9)

Combining Eq. (3.9) with Eq. (3.8) we see that the stress perturbation response is found by convolving the strain history with the stiffness tensor expressed as

\[
\tau_{ij}(x, t) = \int_0^\infty c_{ijkl}(x, t - t') e_{kl}(x, t') dt'.
\]

(3.10)

In the Laplace transformed domain convolution corresponds to a multiplication operation. Hence Eq. (3.10) can alternatively be expressed as

\[
\hat{\tau}_{ij}(x, s) = \hat{c}_{ijkl}(x, s) \hat{e}_{kl}(x, s).
\]

(3.11)

Conversely, the viscoelastic behaviour can be described by the strain creep and compliance tensor. Consider a stress perturbation \(\tau_{kl}(x, t)\), corresponding to a sudden unit jump at the instant \(t = 0\), with constant stress amplitude \(T_{kl}(x)\) expressed as

\[
\tau_{kl}(x, t) = \chi_T(t)T_{kl}(x).
\]

(3.12)
If a certain part of material is placed in this state of stress, the *instantaneous* strain rise can be followed by an increase in strain. This phenomenon is known as (strain) *creep* and can, in absence of external forces, be described by the creep tensor \( J_{ijkl}(x, t) \) as

\[
e_{ij}(x, t) = \chi_T(t) J_{ijkl}(x, t) \mathcal{T}_{kl}(x).
\] (3.13)

The instantaneous strain response is described by the quantity \( J_{ijkl}(x, t = 0) \). For a general stress history \( \tau_{kl}(x, t) \) starting at the instant \( t = 0 \) the strain response can be calculated by virtue of the assumed *linearity* and *time-invariance* of the process leading to

\[
e_{ij}(x, t) = \chi_T(t) J_{ijkl}(x, t) \lim_{t' \to 0} [\tau_{kl}(x, t')] + \int_0^t J_{ijkl}(x, t - t') \partial_t' [\tau_{kl}(x, t')] dt'.
\] (3.14)

Similar arguments as in the discussion on the relaxation function, as expressed in Eq. (3.6) enables us to rewrite Eq. (3.14), c.f. Eq. (3.8), as

\[
e_{ij}(x, t) = \int_0^\infty \partial_t [\chi_T(x, t - t') J_{ijkl}(x, t - t')] \tau_{kl}(x, t') dt'.
\] (3.15)

It is now convenient to introduce the compliance tensor\(^1\) \( S_{ijkl} \) as

\[
S_{ijkl}(x, t) = \partial_t [\chi_T(t) J_{ijkl}(x, t)].
\] (3.16)

Combining Eq. (3.16) with Eq. (3.15) we see that the strain perturbation response is found by convolving the stress history with the compliance tensor,

\[
e_{ij}(x, t) = \int_0^\infty S_{ijkl}(x, t - t') \tau_{kl}(x, t') dt'.
\] (3.17)

In the Laplace transformed domain convolution corresponds to a multiplication operation. Hence Eq. (3.17) can alternatively be expressed as

\[
\hat{e}_{ij}(x, s) = \hat{S}_{ijkl}(x, s) \hat{\tau}_{kl}(x, s).
\] (3.18)

Since by definition, see Eq. (3.2) the strain tensor is symmetric, whereas the stress tensor is symmetric because of the conservation of angular momentum

\(^1\) Note the possible confusion in notation: \( c_{ijkl} \) for Stiffness and \( S_{ijkl} \) for Compliance, which seems to be originated because of historical reasons.
[see de Hoop, 1995], we obtain a set of symmetry properties for the stiffness tensor, which are expressed as

$$\hat{c}_{ijkl}(x, s) = \hat{c}_{jikl}(x, s) = \hat{c}_{ijlk}(x, s) = \hat{c}_{jilk}(x, s)$$  \hspace{1cm} (3.19)

and similarly for the compliance tensor

$$\hat{S}_{ijkl}(x, s) = \hat{S}_{jikl}(x, s) = \hat{S}_{ijlk}(x, s) = \hat{S}_{jilk}(x, s),$$ \hspace{1cm} (3.20)

which reduces the original number of stiffness and compliance functions from 81 to 36.

Combining the relations in Eq. (3.11) and Eq. (3.18) we obtain the intermediate result

$$\hat{c}_{ijpq}(x, s)\hat{S}_{pqkl}(x, s)\hat{\tau}_{kl}(x, s) = \hat{\tau}_{ij}(x, s).$$ \hspace{1cm} (3.21)

Acknowledging the fact that the stress is a symmetric tensor, we find the following identity for the stiffness and compliance tensor

$$\hat{c}_{ijpq}(x, s)\hat{S}_{pqkl}(x, s) = \Delta^+_{ijkl}.$$ \hspace{1cm} (3.22)

Equation (3.22) can be interpreted as the intuitive concept that the stiffness is the inverse of the compliance.

\section*{3.1.2 Isotropic viscoelastic materials}

In case that the material is isotropic, as far as its elastodynamic properties are concerned, the relaxation tensor of rank four is isotropic as well. The most general fourth-rank isotropic tensor can be expressed as a combination of the tensors $\delta_{ij}\delta_{kl}$, $\delta_{ik}\delta_{jl}$ and $\delta_{il}\delta_{jk}$, which comprises all the isotropic tensors of rank four, with three relaxation functions, $\Gamma^\lambda(x, t)$, $\Gamma^\mu(x, t)$ and $\Gamma^\nu(x, t)$ as

$$\Gamma_{ijkl}(x, t) = \Gamma^\lambda(x, t)\delta_{ij}\delta_{kl} + \Gamma^\mu(x, t)\delta_{ik}\delta_{jl} + \Gamma^\nu(x, t)\delta_{il}\delta_{jk}. \hspace{1cm} (3.23)$$

The isotropic stiffness tensor is found with Eq. (3.9) to be

$$c_{ijkl}(x, t) = \lambda(x, t)\delta_{ij}\delta_{kl} - \mu(x, t)\delta_{ik}\delta_{jl} + \nu(x, t)\delta_{il}\delta_{jk},$$ \hspace{1cm} (3.24)

where we have defined $\lambda(x, t)$, $\mu(x, t)$ and $\nu(x, t)$ conform Eq. (3.9) as

$$\lambda(x, t) = \partial_t[\chi_T(t)\Gamma^\lambda(x, t)],$$ \hspace{1cm} (3.25a)

$$\mu(x, t) = \partial_t[\chi_T(t)\Gamma^\mu(x, t)],$$ \hspace{1cm} (3.25b)

$$\nu(x, t) = \partial_t[\chi_T(t)\Gamma^\nu(x, t)].$$ \hspace{1cm} (3.25c)
After Laplace transformations we obtain the stress-strain relation in the complex-frequency domain as

\[
\tilde{\tau}_{ij}(x, s) = [\tilde{\lambda}(x, s)\delta_{ij}\delta_{kl} + \tilde{\mu}(x, s)\delta_{ik}\delta_{jl} + \tilde{\nu}(x, s)\delta_{il}\delta_{jk}]\tilde{\epsilon}_{kl}
\]

\[= \tilde{\lambda}(x, s)\epsilon_{kk}\delta_{ij} + \tilde{\mu}(x, s)\epsilon_{ij} + \tilde{\nu}(x, s)\epsilon_{ji}. \tag{3.26}\]

Because of the fact that the stress and strain tensor are symmetric tensors, we can not distinguish between \(\tilde{\mu}(x, s)\) and \(\tilde{\nu}(x, s)\), i.e.,

\[
\tilde{\nu}(x, s) = \tilde{\mu}(x, s), \tag{3.27}\]

which reduces the number of stiffness functions to two, the elongation modulus \(\lambda(x, t)\) and the shear modulus \(\mu(x, t)\), related to the relaxation functions \(\Gamma^\lambda(x, t)\) and \(\Gamma^\mu(x, t)\), respectively. We will refer to \(\lambda(x, t)\) and \(\mu(x, t)\) as Lamé’s functions. Simplifying Eq. (3.24) we find the isotropic stiffness tensor as

\[
c_{ijk}\epsilon_{kl} = \lambda(x, t)\delta_{ij}\delta_{kl} + \mu(x, t)[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}], \tag{3.28}\]

and the stress-strain relation in the transformed domain with Eq. (3.26) as

\[
\tilde{\tau}_{ij}(x, s) = \tilde{\lambda}(x, s)\epsilon_{kk}\delta_{ij} + 2\tilde{\mu}(x, s)\epsilon_{ij}. \tag{3.29}\]

Equations (3.28) and (3.29) can also be used to model a viscous liquid material with a loss function. In this sense, liquids can temporarily sustain shear stresses and can thus be seen as a generalization of (isotropic) elastic solids. Namely, in case of solids which do not show relaxation, the relaxation functions are constant, i.e.

\[
\Gamma^\lambda(x, t) = \lambda^0(x), \tag{3.30a}\]
\[
\Gamma^\mu(x, t) = \mu^0(x), \tag{3.30b}\]

with stiffness coefficients \(\lambda^0(x)\) and \(\mu^0(x)\). The Lamé’s functions are then obtained with Eqs. (3.25a) and (3.25b) as

\[
\lambda(x, t) = \partial_t[\chi_T(t)\lambda^0(x)]
\]
\[= \lambda^0(x)\delta(t), \tag{3.31a}\]

and similarly for the shear stiffness

\[
\mu(x, t) = \partial_t[\chi_T(t)\mu^0(x)]
\]
\[= \mu^0(x)\delta(t). \tag{3.31b}\]
These relations are most easily expressed in the transformed domain as

\begin{align}
\hat{\lambda}(x, s) &= \lambda^0(x), \\
\hat{\mu}(x, s) &= \mu^0(x),
\end{align}

which then reduces to the stiffness coefficients of a standard perfect linear elastic solid material.

### 3.1.3 Relaxation models

In this section, we shortly discuss different relaxation models for viscoelastic materials. Traditionally, different canonical models have been proposed for the description of relaxation, or conversely creep for visco-elastic materials. Among them are the so called Maxwell, Kelvin-Voigt and Jeffrey’s models, which consist of equivalent representations of springs and dashpots. The combinations of springs and dashpots in series, in parallel or in more complex networks, relate the stress to the strain of the material. A spring represents an element for which the stress is proportional to strain and a stiffness factor, essentially describing a perfect elastic response. A dash pot represents an element in which the stress is proportional to the strain rate and a viscosity factor. For a discussion on the canonical models we refer to GieseKus [1995]. In this thesis we take a slightly different approach. We consider the shear relaxation function \( \Gamma^\mu(x, t) \), since relaxation is most prominently expressed in the shear stress behaviour for viscoelastic fluid, which forms the primary application of viscoelasticity in this thesis. The analysis is completely equivalent for the elongational relaxation. In that case the shear modulus must be replaced by the elongational modulus. A characteristic relaxation behaviour with a single relaxation time \( \tau_r \) and stiffness coefficient \( \mu_\infty \) is given by

\[ \Gamma^\mu(x, t) = \mu_\infty(x) \exp\left(-t/\tau_r(x)\right), \quad t > 0, \]

which we will refer to as a Maxwell model. The stiffness function \( \mu(x, t) \) is then found with Eq. (3.25b) to be

\[ \mu(x, t) = \mu_\infty(x) \tilde{\rho}_\gamma(x) \chi_T(t) \exp\left(-t/\tau_r(x)\right). \]

The Laplace transform of the stiffness function in Eq. (3.34) is found as

\[ \hat{\mu}(x, s) = \mu_\infty(x) \left[ \frac{s\tau_r(x)}{1 + s\tau_r(x)} \right]. \]
To grasp the physical interpretation of the Maxwell model we examine the high- and low-frequency behaviour of this model, and hence take the limit \( s \to j\omega \). We observe that the high-frequency approximation of Eq. (3.35) is given by

\[
\hat{\mu}(x, j\omega) \approx \mu_{\infty}(x), \quad \omega \gg 1/\tau_r(x).
\] (3.36)

Equation (3.36) shows that when the frequency is high compared with the inverse of the relaxation time of the material, the response of such a Maxwell-like material will become effectively elastic, with stiffness \( \mu_{\infty}(x) \). On the other hand, for low frequencies we obtain the approximation

\[
\hat{\mu}(x, j\omega) \approx j\omega \mu_{\infty}(x)\tau_r(x), \quad \omega \ll 1/\tau_r(x). \tag{3.37}
\]

The low-frequency behaviour expressed in Eq. (3.37) approaches that of a Newtonian viscous fluid. For a Newtonian viscous fluid the shear stresses are proportional to the rate of change of the shear deformation [cf. Joseph, 1990], expressed as

\[
\tau_{ij}(x, t) = 2\eta_0(x)\partial_t e_{ij}(x, t), \quad \text{for } i \neq j. \tag{3.38}
\]

with proportionality factor \( \eta_0(x) \), indicating the zero shear-rate viscosity. In the frequency domain temporal differentiation corresponds to multiplication with \( j\omega \), see Eq. (2.8), and hence in the low-frequency limit of Eq. (3.37) the Maxwell model becomes approximately viscous, where the specific zero shear-rate viscosity \( \eta_0 \) of Eq. (3.38) for the Maxwell model, according to Eq. (3.29), corresponds to

\[
\eta_0(x) = \mu_{\infty}(x)\tau_r(x). \tag{3.39}
\]

Generally, the relaxation response of the material can contain more than one relaxation time, corresponding with different molecular or crystal interactions. Therefore we can generalize the Maxwell response to a discrete summation over relaxation times as follows

\[
\Gamma^\mu(x, t) = \sum_i \mu_{\infty}^i(x) \exp\left(-t/\tau_r^i(x)\right), \quad t > 0. \tag{3.40}
\]

Similarly as in Eq. (3.35), the stiffness in the Laplace transformed domain is found to be

\[
\hat{\mu}(x, s) = \sum_i \frac{s\tau_r^i(x)}{1 + s\tau_r^i(x)} \mu_{\infty}^i(x). \tag{3.41}
\]
Again focusing on the frequency response in the limit $s \to j\omega$, we see that when the frequency is high compared with the inverse of all relaxation times, the material will become effectively elastic with stiffness

$$
\mu_\infty(x) = \sum_i \mu_i^\infty(x).
$$

(3.42)

Similarly, when the frequency is low compared to all relaxation times, the material will become effectively viscous with zero shear rate viscosity

$$
\eta_0(x) = \sum_i \mu_i^\infty(x)\tau_i^r(x) = \sum_i \eta_i^0(x).
$$

(3.43)

Sometimes, relaxation times of materials can be extremely small. This means that for most frequencies a set of fast\(^2\) modes mainly contributes to an instantaneous viscosity. This can also be seen by taking a Maxwell response as in Eq. (3.33) with infinitesimal small relaxation time, while keeping the effective zero shear-rate viscosity finite according to Eq. (3.39), formally written [see Gel'fand and Shilov, 1964] as

$$
\Gamma^\mu(x, t) = \lim_{\tau_i \to 0} \frac{\eta_i^0(x)}{\tau_i^r(x)} \exp\left(-\frac{t}{\tau_i^r(x)}\right), \quad t > 0
$$

$$
\to \eta_0(x)\delta(t).
$$

(3.44)

Note that in Eq. (3.44) we achieved to maintain the zero shear-rate viscosity constant, for arbitrarily small relaxation times, by increasing the instantaneous shear modulus to infinitely high values, illustrated with Eq. (3.39) as

$$
\lim_{\tau_i \to 0} \frac{\mu_\infty(x)}{\tau_i^0 \tau_i^r(x)} \to \infty.
$$

(3.45)

This property shows that the Newtonian viscosity model is an idealization and might only be appropriate for certain applications. As the fast modes mainly contribute to an effective viscosity, in a similar manner a set of slow modes mainly contributes to an instantaneous elasticity. In literature [Gieseckus, 1995; Joseph, 1990] the relaxation behaviour is often generalized to include this viscous and elastic behaviour as

$$
\Gamma^\mu(x, t) = \mu_\infty^0(x) + \sum_i \mu_i^\infty(x) \exp\left(-\frac{t}{\tau_i^r(x)}\right) + \eta_0(x)\delta(t), \quad t > 0.
$$

(3.46)

\(^2\)extremely fast shear modes can exist for fluids, with relaxation times of $O(10^{-9}\text{ s})$, also known as glassy modes.
However, we must keep in mind that in Eq. (3.46) the perfectly elastic and viscous part of the causal response consists of high- and low-frequency approximations, which idealise the actual response. The Newtonian viscosity model is an idealization because the disturbance after a sudden deformation of the molecular order, associated with short range forces, will always take some time to relax [Joseph, 1990]. Although the instantaneous approximations might be applicable in many cases, in reality relaxation times are always finite. Essentially we can always design higher frequency experiments for which the viscous term \( \eta_0 \) will start to become viscoelastic or even elastic. We prefer initially to exclude the viscous contribution in Eq. (3.46). Viscosity will not be viewed as a separate physical constant. At a later instant, we can always evaluate the limiting expressions for a Maxwell model with arbitrary small relaxation time to include the Newtonian viscosity model. Another possible generalization is to describe the relaxation in terms of a continuous relaxation time spectrum. Instead of the discrete summation, Eq. (3.40) we obtain a continuous integral as follows

\[
\Gamma^\mu(x, t) = \int_0^\infty \mu_\infty(x, \tau_r) \exp\left(-t/\tau_r\right) d\tau_r, \quad t > 0. \tag{3.47}
\]

The fact that liquids can only temporarily sustain shear stresses must be expressed in the shear relaxation function. This leads to the following constraint for the relaxation function of fluids

\[
\lim_{t \to \infty} \Gamma^\mu(x, t) = 0, \tag{3.48}
\]

and might also be used as a criterion to distinguish viscoelastic fluids from solids. Since resistance against compression is permanent, we require for the elongational relaxation function

\[
\lim_{t \to \infty} \Gamma^\lambda(x, t) \neq 0. \tag{3.49}
\]

### 3.2 The coupled linear wave equations

#### 3.2.1 The viscoelastic wave equations

The equation of motion, combined with the deformation rate equation, based on the constitutive equations in Section 3.1.1 result after the applying low-amplitude gradient linearizations, in two coupled linear first-order differential
equations for the stress \( \tau_{ij} \) and particle velocity \( v_i \). The formulation in space-time domain [see De Hoop, 1995], is expressed as

\[
- \Delta_{ijkl}^{+} \frac{\partial_{j} \tau_{kl}(x, t)}{\partial t} + \rho(x) \frac{\partial_{i} v_{i}(x, t)}{\partial t} = f_{i}(x, t),
\]
\[
\Delta_{ijkl}^{+} \frac{\partial_{k} v_{l}(x, t)}{\partial t} - \partial_{l} \int_{0}^{\infty} S_{ijkl}(x, t - t') \tau_{kl}(x, t') \, dt' = h_{ij}(x, t).
\]

In these equations the wavefield is exited by a force source vector \( f_{i}(x, t) \) and a deformation injection tensor \( h_{ij}(x, t) \). The coupled linear first-order wave equation relates the wavefield consisting of the particle velocity field and the stress field to two tensor quantities, namely the mass density and the compliance of the material. The equivalent formulation in the complex-frequency domain is found by using Eq. (2.8) resulting in

\[
- \Delta_{ijkl}^{+} \frac{\partial_{j} \dot{\tau}_{kl}(x, s)}{\partial t} + s \rho(x) \dot{v}_{i}(x, s) = \dot{f}_{i}(x, s),
\]
\[
\Delta_{ijkl}^{+} \frac{\partial_{k} \dot{v}_{l}(x, s)}{\partial t} - s S_{ijkl}(x, s) \dot{\tau}_{kl}(x, s) = \dot{h}_{ij}(x, s).
\]

The formulation as in Eqs. (3.51a) and (3.51b) is most convenient for deriving the reciprocity theorem. We obtain the formulation with the more familiar stiffness tensor in the frequency transformed domain after contraction of Eq. (3.51b) with the stiffness tensor, and using Eq. (3.22) leading to

\[
- \Delta_{ijkl}^{+} \frac{\partial_{j} \dot{\tau}_{kl}(x, s)}{\partial t} + s \rho(x) \dot{v}_{i}(x, s) = \dot{f}_{i}(x, s),
\]
\[
\hat{c}_{ijkl}(x, s) \frac{\partial_{k} \dot{v}_{l}(x, s)}{\partial t} - s \Delta_{ijkl}^{+} \tau_{kl}(x, s) = \hat{c}_{ijkl}(x, s) \dot{h}_{kl}(x, s),
\]
in which the symmetry property \( \hat{c}_{ijkl} = \hat{c}_{ijlk} \), see Eq. (3.19), has been used. In the following section we will discuss the special case of isotropic viscoelastic materials.

### 3.2.2 The acoustic wave equations

The compliance formulation of the coupled viscoelastic wave equations will appear to be the most convenient for deriving the reciprocity theorem. Because an ideal fluid cannot sustain any shear stresses the shear compliance becomes singular in case of an ideal fluid. Therefore we treat the ideal fluid case as a special case in which we take a priori into account the fact that only isotropic stresses are present in such a fluid. The isotropic stress or pressure in the fluid is defined as

\[
p = -\frac{1}{3} \tau_{kk}.
\]
The coupled linear first-order acoustic wave equations in the space-time domain are then given [Fokkema and van den Berg, 1993] by

\begin{align}
\partial_t p(x, t) + \rho(x) \partial_t v_i(x, t) &= f_i(x, s), \quad (3.54a) \\
\partial_t v_k(x, t) + \kappa(x) \partial_t p(x, t) &= q(x, t), \quad (3.54b)
\end{align}

where \( \kappa \) is the compressibility of the fluid and \( q(x, t) \) is the volume source density of injection rate. The equivalent formulation in the complex-frequency domain is then found with Eq. (2.8) as

\begin{align}
\partial_t \hat{p}(x, s) + s\rho(x) \hat{v}_i(x, s) &= \hat{f}_i(x, s), \quad (3.55a) \\
\partial_t \hat{v}_k(x, s) + s\kappa(x) \hat{p}(x, s) &= \hat{q}(x, s). \quad (3.55b)
\end{align}

### 3.3 The Betti-Rayleigh reciprocity theorem

In this section we discuss the Betti-Rayleigh reciprocity theorem for viscoelastic media. The reciprocity theorem operates on a time-invariant and bounded domain \( \mathbb{V} \) in space, in which two non-identical states can occur. The two states will be distinguished by the superscripts \( A \) and \( B \), respectively. Neither the source distribution of the wavefields nor the viscoelastic properties need to be the same. The construction of the complex-frequency domain reciprocity theorem is based on the complex-frequency domain anisotropic viscoelastic coupled first order wave equations, Eqs. (3.51a) and (3.51b) with the compliance formulation. For state \( A \) we have

\begin{align}
-\Delta^+_{ijkl} \partial_j \hat{\tau}^A_{kl}(x, s) + s \rho^A(x) \hat{v}^A_i(x, s) &= \hat{f}^A_i(x, s), \quad (3.56a) \\
\Delta^+_{ijkl} \partial_k \hat{v}_i^A(x, s) - s \hat{S}^A_{ijkl}(x, s) \hat{\tau}^A_{kl}(x, s) &= \hat{h}^A_{ij}(x, s), \quad (3.56b)
\end{align}

whereas for state \( B \)

\begin{align}
-\Delta^+_{ijkl} \partial_j \hat{\tau}^B_{kl}(x, s) + s \rho^B(x) \hat{v}^B_i(x, s) &= \hat{f}^B_i(x, s), \quad (3.57a) \\
\Delta^+_{ijkl} \partial_k \hat{v}_i^B(x, s) - s \hat{S}^B_{ijkl}(x, s) \hat{\tau}^B_{kl}(x, s) &= \hat{h}^B_{ij}(x, s). \quad (3.57b)
\end{align}

If, in the domain \( \mathbb{V} \), surfaces are present across which the viscoelastic properties are discontinuous, Eqs. (3.56a)–(3.57b) are supplemented with boundary conditions for the viscoelastic field quantities. Let us for the moment suppress the \( (x, s) \)-dependence in the notation. Define the local interaction quantity between the state \( A \) and \( B \) as

\begin{align}
\Delta^+_{ijkl} \partial_i [\hat{v}^A_j \hat{\tau}^B_{kl} - \hat{v}^B_j \hat{\tau}^A_{kl}] &= \\
\Delta^+_{ijkl} \partial_i \hat{\tau}^A_{kl} \hat{v}^B_j + \Delta^+_{ijkl} \partial_i \hat{v}^A_j \hat{\tau}^B_{kl} - \Delta^+_{ijkl} \partial_i \hat{\tau}^B_{kl} \hat{v}^A_j - \Delta^+_{ijkl} \partial_i \hat{v}^A_j \hat{\tau}^B_{kl}. \quad (3.58)
\end{align}
Following from the definition of the tensor $\Delta^{+}_{ijkl}$ in Eq. (3.3) have the equivalence

$$\Delta^{+}_{ijkl} = \Delta^{+}_{jikl} = \Delta^{+}_{ijkl}.$$  \hspace{1cm} (3.59)

Now, by using Eqs. (3.51a) and (3.59), we rewrite the term containing the partial spatial derivative of the stress tensor in Eq. (3.58) as

$$\Delta^{+}_{ijkl} \partial_i \hat{\tau}_{kl} = \Delta^{+}_{jikl} \partial_i \hat{\tau}_{kl}$$

$$= s \rho \hat{v}_j - \hat{f}_j, $$ \hspace{1cm} (3.60)

and similarly by using Eq. (3.51b) for the term containing the partial spatial derivative of the velocity vector is rewritten as

$$\Delta^{+}_{ijkl} \partial_i \hat{v}_j = \Delta^{+}_{klj} \partial_i \hat{v}_j$$

$$= s \hat{S}_{klj} \hat{\tau}_{ij} + \hat{h}_{kl}. $$ \hspace{1cm} (3.61)

Combining Eqs. (3.58), (3.60) and (3.61) for the local interaction quantity of Eq. (3.58) we obtain

$$\Delta^{+}_{ijkl} \partial_i [\hat{\tau}_{kl} \hat{v}_j^B - \hat{\tau}_{kl} \hat{v}_j^A] = -s (\rho^B - \rho^A) \hat{v}_j^B \hat{v}_j^B$$

$$+ s (\hat{S}_{klj} \hat{\tau}_{ij}^B - \hat{S}_{klj} \hat{\tau}_{ij}^A)$$

$$= -(f_j^B \hat{v}_j^B - f_j^B \hat{v}_j^A) - (h_{klj} \hat{\tau}_{kl}^B - h_{klj} \hat{\tau}_{kl}^A). $$ \hspace{1cm} (3.62)

By changing the subscripts in Eq. (3.62) according to

$$\hat{S}_{klj} \hat{\tau}_{ij}^B \hat{\tau}_{kl}^B = \hat{S}_{ijkl} \hat{\tau}_{ij}^A \hat{\tau}_{kl}^A,$$

we can simplify the second term on the right-hand side of Eq. (3.62) as

$$s (\hat{S}_{klj} \hat{\tau}_{ij}^B \hat{\tau}_{kl}^B - \hat{S}_{klj} \hat{\tau}_{ij}^A \hat{\tau}_{kl}^B) = s (\hat{S}_{ijkl} - \hat{S}_{klj}) \hat{\tau}_{ij}^A \hat{\tau}_{kl}^B.$$ \hspace{1cm} (3.64)

Equation (3.62) combined with Eq. (3.64) results in the local form of the complex-frequency domain viscoelastic reciprocity theorem expressed as

$$\Delta^{+}_{ijkl} \partial_i [\hat{\tau}_{kl} \hat{v}_j^B - \hat{\tau}_{kl} \hat{v}_j^A] = -s (\rho^B - \rho^A) \hat{v}_j^B \hat{v}_j^B$$

$$+ s (\hat{S}_{ijkl} - \hat{S}_{klj}) \hat{\tau}_{ij}^A \hat{\tau}_{kl}^B$$

$$= -(f_j^B \hat{v}_j^B - f_j^B \hat{v}_j^A) - (h_{klj} \hat{\tau}_{kl}^B - h_{klj} \hat{\tau}_{kl}^A). $$ \hspace{1cm} (3.65)
The *global* reciprocity theorem can be obtained by integrating the local interaction quantity over the domain \( V \) and applying Gauss’ integral theorem on the local interaction quantity, leading to

\[
\Delta^+_{ij kl} \int_{x \in \partial \Omega} (\hat{\tau}^A_{ji} \hat{v}^B_j - \hat{\tau}^B_{kl} \hat{v}^A_i) n_i \, dA = \\
\int_{x \in V} [-s(\rho^B - \rho^A) \hat{v}^A_j \hat{v}^B_j + s(\hat{S}^B_{ijkl} - \hat{S}^A_{ijkl}) \hat{\tau}^A_{ij} \hat{\tau}^B_{kl}] \, dV \\
- \int_{x \in V} [(\hat{f}^A_{ij} \hat{v}^B_j - \hat{f}^B_{ij} \hat{v}^A_j) + (\hat{h}^A_{kl} \hat{\tau}^B_{ij} - \hat{h}^B_{kl} \hat{\tau}^A_{ij})] \, dV. \quad (3.66)
\]

The first integral on the right-hand side of Eq. (3.66) represents the interaction due to the contrast of the viscoelastic properties in both states. The second integral is related to the excitation of the wavefield by sources in both states. If, due to the medium properties in two states, the contrast interaction vanishes, the two states are denoted as *adjoint* states, which for the compliance and density, respectively, comprises

\[
\hat{S}^A_{ijkl} = \hat{S}^B_{klij}, \quad (3.67a)
\]

and

\[
\rho^A = \rho^B. \quad (3.67b)
\]

If the the medium states in state \( A \) and \( V \) are one and the same, and condition Eq. (3.67a) still holds, the medium is called *self-adjoint* or *reciprocal* [see De Hoop, 1995], which is expressed as the additional symmetry property of the compliance tensor

\[
\hat{S}^{klij} = \hat{S}^{ijkl}. \quad (3.68)
\]

For ideally elastic media, many authors view this property as a consequence of the conservation of energy [cf. Aki and Richards, 1980]. We claim that an analogous derivation in the general viscoelastic case cannot be accomplished. Furthermore, it can be disputed which property is the more fundamental in this case, the self-adjointness or the conservation of energy. Hence, the property of self-adjointness in the general viscoelastic case is disputable on that ground. Nevertheless, the self-adjointness might be a fundamental property, based on other physical principles. In this thesis we will restrict ourselves to isotropic media which are inherently self-adjoint and consequently this fundamental issue is of less relevance for our further discussion.

Many applications in seismic wavefield theory, such as scattering and inverse-scattering problems and wavefield decomposition, are based on the reciprocity
theorem. For a more detailed discussion the reader is referred to DE HOOP [1995] and FOKKEMA AND VAN DEN BERG [1993]. With the Betti-Rayleigh reciprocity theorem we conclude our discussion on the fundamentals of wave propagation in viscoelastic, elastodynamic and acoustic media.

The advantage of introducing viscoelastic fluids as generalized solids, with frequency dependent stiffnesses as has been done in Section 3.1, is that in the frequency domain the treatment of scattering problems is equivalent to the treatment for idealized elastic media. In Chapter 4 we can hence analyse the propagation of the wavefield in layered viscoelastic media, with a formulation that is only slightly different from layered elastic media, where the wave equations of Section 3.2 are the starting point of the analysis. In Chapter 7 we will model the transmission through a single layer and be able to investigate the effect of viscoelasticity on its transmission properties, including the ideal elastic and viscous limit by using the Maxwell viscoelastic model. In this thesis the reciprocity theorems are used to motivate our choice of the specific decomposition of the wavefield into a set of independent eigenfunctions, which describe the standard P-, SV- and SH-polarization modes. Moreover, the reciprocity theorem forms the basis of integral representations and equations [DE HOOP, 1995; FOKKEMA AND VAN DEN BERG, 1993] and the discussion in Chapter 8.
The Layer Model

We propose a thin horizontal viscoelastic layer in a homogeneous elastic environment as a simplified fracture model. In this manner the effect of the contrasting layer, occurring on a small scale, can be investigated. In this chapter we neglect the effect of the roughness of the surface as well as the finite lateral extent of the fracture. We will assume that the theory of elasticity or viscoelasticity can be used to model the wavefield in the embedding and in the fracture itself. We will treat the ideal fluid layer embedded in an elastic medium as a special case.

To exploit the horizontal shift invariance of the infinite flat configuration, we subject the wavefield quantities in the complex-frequency transformed domain to a horizontal spatial Fourier transformation. We will show that six independent eigenfunctions of the wave equations in the time and horizontal space transformed domain can be constructed, which can be interpreted as up- or downgoing waves. We will use these eigenfunctions to compose the wavefield solution out of the set of up- and downgoing waves. Inversely, the decomposition matrix is obtained which decomposes a given wavefield into up- and downgoing waves.

This chapter continues with an exposition of scattering theory. We will demonstrate that by using the decomposition into up- and downgoing waves, along with application of the boundary conditions of the continuity of the
normal traction and the particle velocity, we can establish a downward or upward continuation of the wavefield. For this purpose the field propagator \( P \), the wave propagator \( Q \), and the scattering operator \( S \) are defined. We will deduce the interrelations between the different operators and the composition rules. A composition rule describes the relationship between the propagator for a section in terms of the propagators of the subsections. The composition rule for the scattering operator is known as Redheffer’s star product.

Next, we derive an expression for the scattering operator of an interface as well as the scattering operator of a homogeneous layer. Using the composition rules we obtain the scattering operator of the viscoelastic layer embedded in an elastic space. Special care is given to describe the action of the source and receiver transducers in our measurement configuration. Finally, we transform the inverse spatial Fourier transformation from a Cartesian to a cylindrical coordinate system, which results in an expression for the response, which contains a single integral over the cylindrical wavenumber. This last integral will be evaluated numerically, as described in Chapter 7.

In previous sections we will freely drawn upon the classical work KENNETT [1983] and the clear paper FRYER AND FRAZER [1984]. Further the recursive scattering formulation is used and described in DU CLOUX [1986], although we use a slightly different definition of the elastodynamic field vector, similar as in VAN DER HUIDEN [1987].

### 4.1 The horizontally transformed wave equation

In this chapter we will focus on wave propagation in layered media. For this configuration the horizontal plane is an invariant plane for the medium properties, see Figure 4.1. We will start our analysis by using the anisotropic viscoelastic wave equation in the complex-frequency domain, with the stiffness formulation stated in Eq. (3.52a) and (3.52b) as

\[
-\Delta_{ijkl}^{+} \partial_j \hat{\tau}_{kl}(\mathbf{x}, s) + s\rho(\mathbf{x}) \hat{v}_i(\mathbf{x}, s) = \hat{f}_i(\mathbf{x}, s), \tag{4.1a}
\]

\[
\hat{c}_{ijkl}(\mathbf{x}, s) \partial_k \hat{v}_l(\mathbf{x}, s) - s\Delta_{ijkl}^{+} \hat{\tau}_{kl}(\mathbf{x}, s) = \hat{c}_{ijkl}(\mathbf{x}, s) \hat{h}_{kl}(\mathbf{x}, s). \tag{4.1b}
\]

If we transform Eqs. (4.1a) and (4.1b) to the slowness domain by replacing each horizontal spatial differentiation \( \partial_\alpha \) by \(-s\zeta_\alpha\) and eliminating the horizontal components \( \tau_{i\gamma} \) of the stress tensor, we can cast the viscoelastic field equations for layered media into the following system equation [KENNETT, 1983; VAN DER HUIDEN, 1987],

\[
\partial_\beta \mathbf{b}_I = -s \mathbf{A}_{I,J} \mathbf{b}_J + \mathbf{F}_I. \tag{4.2}
\]
The vector $\mathbf{F}_I$ is the source vector, which we will define in detail later on. The six-dimensional wavefield vector in the slowness-transformed domain, $\mathbf{b}_I$, has been defined as

$$
\mathbf{b}_I(s\mathbf{\zeta}, x_3, s) = \begin{pmatrix} +\tilde{v}_k \\ -\tilde{r}_{i3} \end{pmatrix} = (+\tilde{v}_1, +\tilde{v}_2, +\tilde{v}_3, -\tilde{r}_{13}, -\tilde{r}_{23}, -\tilde{r}_{33})^T.
$$

Note that with the superscript $T$ we indicate the transpose of a vector or matrix. Furthermore we will consistently use capital Roman subscripts when the vector or tensor is six-dimensional and apply the summation convention completely analogous to the definition in Chapter 2 but now for all six dimensions.

At interfaces where the constitutive parameters jump by a finite amount, the wave equation has to be supplemented by boundary conditions. The traction has been defined in Chapter 3, Eq. (3.1). We apply the convention that for layered media the normal vector $\mathbf{n}$ points downward, which results in the following definition of the vertical traction

$$
t_i(x, t) = \tau_{ij}(x, t)n_j = \tau_{i3}(x, t).
$$

Now, the classical continuity boundary conditions for the particle velocity
and traction in the transformed domain are expressed as

\[
\lim_{\epsilon \to 0} \tilde{v}_i(s\zeta, x_3 - \epsilon, s) = \lim_{\epsilon \to 0} \tilde{v}_i(s\zeta, x_3 + \epsilon, s),
\]

\[
\lim_{\epsilon \to 0} \tilde{t}_i(s\zeta, x_3 - \epsilon, s) = \lim_{\epsilon \to 0} \tilde{t}_i(s\zeta, x_3 + \epsilon, s).
\]

(4.5)

(4.6)

Hence, the field vector consists of those field quantities, that are required to be continuous across interfaces.

The six by six matrix \( \mathbf{A} \) is known as the system matrix and can be divided into four three by three submatrices, expressed as

\[
\mathbf{A}_{IJ} = \begin{pmatrix}
\mathbf{A}^{uv} & \mathbf{A}^{vt} \\
\mathbf{A}^{tv} & \mathbf{A}^{tt}
\end{pmatrix},
\]

(4.7)

in which the components of the system submatrices are obtained [see van der Heijden, 1987], in terms of the Fourier-sloveness vector \( \zeta_\alpha \), see Section 2.5, Eq. (2.20) as

\[
\mathbf{A}^{uv} = \begin{pmatrix}
0 & 0 & -\zeta_1 \\
0 & 0 & -\zeta_2 \\
-\frac{\lambda}{\lambda + 2\mu} \zeta_1 & -\frac{\lambda}{\lambda + 2\mu} \zeta_2 & 0
\end{pmatrix},
\]

(4.8a)

\[
\mathbf{A}^{vt} = \begin{pmatrix}
\frac{1}{\mu} & 0 & 0 \\
0 & \frac{1}{\mu} & 0 \\
0 & 0 & \frac{1}{\lambda + 2\mu}
\end{pmatrix},
\]

(4.8b)

\[
\mathbf{A}^{tv} = \begin{pmatrix}
\rho - \rho'_{11} & -\rho'_{12} & 0 \\
-\rho'_{21} & \rho - \rho'_{22} & 0 \\
0 & 0 & \rho
\end{pmatrix},
\]

(4.8c)

where the two-dimensional tensor \( \rho'_{\alpha\beta} \) is defined as

\[
\rho'_{\alpha\beta} = \frac{\hat{\mu}(3\lambda + 2\mu)}{\lambda + 2\hat{\mu}} \zeta_\alpha \zeta_\beta + \hat{\mu} \delta_{\alpha\beta} \zeta_\gamma \zeta_\gamma
\]

(4.8d)

and the system submatrix \( \mathbf{A}^{tt} \) is the transpose of the system submatrix \( \mathbf{A}^{uv} \), i.e.,

\[
\mathbf{A}^{tt} = \begin{pmatrix}
0 & 0 & -\frac{\lambda}{\lambda + 2\mu} \zeta_1 \\
0 & 0 & -\frac{\lambda}{\lambda + 2\mu} \zeta_2 \\
-\zeta_1 & -\zeta_2 & 0
\end{pmatrix} = (\mathbf{A}^{uv})^T.
\]

(4.8e)
Note that due to the fact that we have included the viscoelasticity of the material in the elastic parameters, the system matrix $A$ generally is frequency dependent, whereas in the ideally elastic case this is not the case. The six-dimensional source vector $F$ is split into two three-dimensional vectors, expressed as

$$F_I = \begin{pmatrix} F^v \\ F^\tau \end{pmatrix},$$

(4.9)

where the two three-dimensional subvectors $F^v$ and $F^\tau$ are obtained [see van der Huiden, 1987] as

$$F^v = \begin{pmatrix} \tilde{h}_{31} + \tilde{h}_{13} \\ \tilde{h}_{32} + \tilde{h}_{23} \\ \frac{\lambda}{\lambda + 2\mu} (\tilde{h}_{11} + \tilde{h}_{22}) + \tilde{h}_{33} \end{pmatrix},$$

(4.10a)

and

$$F^\tau = \begin{pmatrix} \tilde{f}_1 + \frac{2\lambda\mu}{\lambda + 2\mu} (\tilde{h}_{11} + \tilde{h}_{22}) \zeta_1 + \tilde{\mu} (\tilde{h}_{1a1} + \tilde{h}_{1a}) \zeta_1 \\ \tilde{f}_2 + \frac{2\lambda\mu}{\lambda + 2\mu} (\tilde{h}_{11} + \tilde{h}_{22}) \zeta_2 + \tilde{\mu} (\tilde{h}_{2a2} + \tilde{h}_{2a}) \zeta_2 \\ \tilde{f}_3 \end{pmatrix}.$$  

(4.10b)

4.2 Wavefield decomposition

We will construct the solution of the wavefield in layered media by using the concept of eigenfunctions of the system equation, formulated in Eq. (4.2). An eigenfunction of the system equation is defined to satisfy the following two conditions:

- The eigenfunction $b^\alpha_I(s\zeta, x_3, s)$ must be a solution to the system equation.

- The result of the differential operation of the system equation, namely vertical differentiation $\partial_3$, on the eigenfunction corresponds to the same vector multiplied with a scalar function, the so-called eigenvalue.

Above conditions can be expressed as

$$\partial_3 b^\alpha_I(s\zeta, x_3, s) = -sA_{IJ}(\zeta, s)b^\beta_J(s\zeta, x_3, s)$$

$$= -s_\zeta^\alpha(s\zeta, s)b^\alpha_I(s\zeta, x_3, s).$$

(4.11)
Hence, an eigenfunction is an independent solution to the system equation. For reasons to become clear later on, we choose the proportionality factor as $s\zeta_3^n(\zeta, s)$, in which we denote $\zeta_3^n(\zeta, s)$ as the \textit{generalized vertical slowness}. The superscript $n$ has been added to the eigenfunction, denoting the $n$-th eigenfunction, since we will show that more than one eigenfunction exists for the system equation in Eq. (4.11). Equation (4.11) suggests that we seek for eigenfunctions of the form

$$b^n_7(s\zeta, x_3, s) = B^n_7(\zeta, s) \exp\left(-s\zeta_3^n(\zeta, s)x_3\right),$$

(4.12)

in which $B^n_7$ represents the polarization vector of the field vector at the reference level $x_3 = 0$. From Eq. (4.11) we extract the condition under which the vector $b^n$ forms an eigenfunction, with eigenvalue $\zeta_3^n$ as

$$(A_{IJ} - \zeta_3^n I_{IJ})B^n_J = 0, \quad n = 1, 2, 3, 4, 5 \text{ or } 6. \quad (4.13)$$

from which can be concluded that

$$A_{IJ}B^n_J = B^n_J \zeta_3^n, \quad n = 1, 2, 3, 4, 5 \text{ or } 6. \quad (4.14)$$

Note that the vertical slownesses $\zeta_3^n(\zeta, s)$ are the eigenvalues of the system matrix $A$, while the polarization vectors $B^n_7(\zeta, s)$ are the eigenvectors of the system matrix $A$. Since the system matrix $A$ in the viscoelastic case is complex and frequency dependent, the vertical slownesses or eigenvalues and polarization matrices are complex and frequency dependent. In the ideally elastic case this would not be the case.

The six eigenvectors that we obtain in elastic isotropic media are related with the up- and downgoing P-, SV- and SH-waves. In viscoelastic media the P- and SV-waves are no longer linearly polarized but elliptically polarized [see Hudson, 1980]. We will order the six eigenvectors of Eq. (4.14), which in elastic isotropic media will correspond to the P-, SV- and SH-waves, into a matrix $D_{IJ}$, which we will denote as the composition matrix, as follows

$$D_{IJ} = (B^{PV}_{I\uparrow}, B^{SV}_{I\uparrow}, B^{SH\uparrow}, B^{PV\downarrow}, B^{SV\downarrow}, B^{SH\downarrow}), \quad (4.15)$$

where the corresponding generalized vertical slownesses of these eigenvectors are given by

$$\begin{pmatrix}
\zeta_3^{PV\uparrow}, & \zeta_3^{SV\uparrow}, & \zeta_3^{SH\uparrow}, & \zeta_3^{PV\downarrow}, & \zeta_3^{SV\downarrow}, & \zeta_3^{SH\downarrow}
\end{pmatrix} = \begin{pmatrix}
-\gamma_P, & -\gamma_S, & -\gamma_S, & +\gamma_P, & +\gamma_S, & +\gamma_S
\end{pmatrix}, \quad (4.16)$$
respectively, in which $\gamma^P$ and $\gamma^S$ are defined as

$$\gamma^P = \left( \frac{\rho}{\lambda + 2\mu} - \zeta_\beta \zeta_\beta \right)^{\frac{1}{2}}, \quad \text{Re}(s\gamma^P) \geq 0 \quad (4.17a)$$

and

$$\gamma^S = \left( \frac{\rho}{\mu} - \zeta_\beta \zeta_\beta \right)^{\frac{1}{2}}, \quad \text{Re}(s\gamma^S) \geq 0. \quad (4.17b)$$

Note that the square-root function is not defined in a unique manner for complex arguments. When the material is ideally elastic and for a horizontal slowness below the critical slownesses, the vertical slowness is real. In that case we can easily interpret an eigenfunction for a single slowness by using the inverse Laplace transformation, expressed in terms of the Bromwich integral in Eq. (2.6) as

$$\frac{1}{2\pi j} \int_{s_{0} - j\infty}^{s_{0} + j\infty} \exp(st)B^n_j(\zeta) \exp(-s\zeta^n_3 x_3) \, ds$$

$$= \frac{1}{2\pi j} \int_{s_{0} - j\infty}^{s_{0} + j\infty} B^n_j(\zeta) \exp(s(t - \zeta^n_3 x_3)) \, ds$$

$$= \chi^T(t)B^n_j(\zeta)\delta(t - \zeta^n_3 x_3), \quad (4.18)$$

Equation (4.18) shows that in the ideally elastic case an eigenfunction for a single horizontal slowness corresponds to a downgoing plane wave, in case $\zeta^n_3$ is positive, and a upgoing plane wave in case $\zeta^n_3$ is negative. In order to extend this interpretation to a viscoelastic medium for which the vertical slowness generally is complex, we have chosen the branchcut of the square-root in Eqs. (4.17a) and (4.17b) corresponding to

$$\text{Re}(s\gamma^P,S) \geq 0, \quad (4.19)$$

which ensures that the damping of the eigenfunctions of Eq. (4.12) is always linked with the direction of propagation in the correct manner.

In the general viscoelastic case the polarization vector and the vertical slowness are complex and frequency dependent. This expresses the fact that waves in viscoelastic media are dispersed while propagating. The interpretation of the eigensolutions as a propagating plane wave with a certain real vertical slowness no longer holds. Nevertheless, we will still use the concept of vertical slowness, but we explicitly denote it as the generalized vertical slowness of the eigenfunction $b^n_j$. 
Note that we have implicitly defined the generalized Fourier-slowness vector $\zeta^n$ of the eigenfunction $b^n_I$ as the vector consisting of the horizontal Fourier-slowness vector $\zeta$ and the generalized vertical slowness $\zeta_3^n$, expressed as

$$\zeta^n = (\zeta_1, \zeta_2, \zeta_3^n)^T. \quad (4.20)$$

From Eq. (4.17a) and Eq. (4.17b) it follows that

$$\zeta_k \zeta_k = \frac{\rho}{\lambda + 2\mu}, \quad \text{for a compressional wave,} \quad (4.21a)$$

and

$$\zeta_k \zeta_k = \frac{\rho}{\mu}, \quad \text{for a shear wave,} \quad (4.21b)$$

Hence we can define the generalized compressional slowness function $\hat{\zeta}_P$ and shear slowness function $\hat{\zeta}_S$, respectively as

$$\hat{\zeta}_P = \left( \frac{\rho}{\lambda + 2\mu} \right)^{\frac{1}{2}}, \quad \text{Re}(s \hat{\zeta}_P) \geq 0 \quad (4.22a)$$

and

$$\hat{\zeta}_S = \left( \frac{\rho}{\mu} \right)^{\frac{1}{2}}, \quad \text{Re}(s \hat{\zeta}_S) \geq 0. \quad (4.22b)$$

The branch-cut has been taken, corresponding to Eqs. (4.17a) and (4.17b). The generalized compressional and shear velocity functions $\hat{c}_P$ and $\hat{c}_S$ are subsequently defined according to

$$\hat{c}_{P,S} = \hat{\zeta}_{P,S}^{-1}. \quad (4.23)$$

Having defined the composition matrix $\mathbf{D}$ in Eq. (4.15), we can summarise the eigenvector equation of Eq. (4.14) for all eigenvectors as

$$\mathbf{A}_{IJ} \mathbf{D}_{JN} = \mathbf{D}_{IJ} \mathbf{A}_{JN}, \quad (4.24)$$

in which the matrix $\mathbf{A}$ is the diagonal matrix containing all eigenvalues.

$$\mathbf{A}_{IJ} = \begin{pmatrix} \mathbf{A}^\dagger & 0 \\ 0 & \mathbf{A}^\dagger \end{pmatrix}. \quad (4.25)$$

The submatrix $\mathbf{A}^\dagger$ related to downgoing waves is defined as

$$\mathbf{A}^\dagger = \begin{pmatrix} +\gamma_P & 0 & 0 \\ 0 & +\gamma_S & 0 \\ 0 & 0 & +\gamma_S \end{pmatrix}. \quad (4.26)$$
and the submatrix $\Lambda^\dagger$ for upgoing waves as

$$
\Lambda^\dagger = \begin{pmatrix}
-\gamma_P & 0 & 0 \\
0 & -\gamma_S & 0 \\
0 & 0 & -\gamma_S \\
\end{pmatrix} = -\Lambda^\perp.
$$

(4.27)

Left-multiplying Eq. (4.24) with the inverse of the composition matrix, we conclude that the matrix of column vectors acts as the transformation matrix which diagonalises the system matrix $A$, expressed as

$$
D_{IK}^{-1}A_{KL}D_{LJ} = \Lambda_{IJ}.
$$

(4.28)

If we now define the wave vector $y$, as the six-dimensional vector consisting of the amplitudes of the generalized up- and downgoing waves at a certain reference level, we can compose the field vector out of the up- and downgoing wave amplitudes by using the composition matrix $D$ as

$$
b_I = D_{IJ}y_J,
$$

(4.29)

while we can decompose the field vector into up- and downgoing waves with the decomposition matrix $D^{-1}$ as

$$
y_I = D_{IJ}^{-1}b_J.
$$

(4.30)

Note that the six-dimensional wave vector is composed of two three-dimensional subvectors, containing the amplitudes of the up- or downgoing waves, i.e.

$$
y_I = \begin{pmatrix}
y_I^\dagger \\
y_I^\perp \\
\end{pmatrix}.
$$

(4.31)

Next, by substituting Eq. (4.29) into Eq. (4.2) we obtain the wave equation for the wave vector $y$ as

$$
\partial_3y_I = -sD_{IK}^{-1}A_{KL}D_{LJ}y_J + D_{IJ}^{-1}F_J
$$

$$
= -s\Lambda_{IJ}y_J + X,
$$

(4.32)

where in the last step we have used the result of Eq. (4.28) that the composition matrix $D$ acts as the transformation matrix which diagonalises the system matrix $A$ and we have defined the source vector $X$ for the wave vector as

$$
X_I = \begin{pmatrix}
X_I^\dagger \\
X_I^\perp \\
\end{pmatrix} = D_{IJ}^{-1}F_J,
$$

(4.33)
Note that because the matrix $\mathbf{A}$, containing the vertical slownesses is diagonal the wave equation for the wave vector is decoupled for the different polarization modes in a homogeneous subdomain. We split the composition matrix into three by three submatrices according to

$$
\mathbf{D} = \begin{pmatrix}
\mathbf{D}^{v,\uparrow} & \mathbf{D}^{v,\downarrow} \\
\mathbf{D}^{\tau,\uparrow} & \mathbf{D}^{\tau,\downarrow}
\end{pmatrix}.
\tag{4.34}
$$

If we define the matrix $\mathbf{N}_v$ as the diagonal matrix containing the normalization constants, which yet have to be determined, as

$$
\mathbf{N}_v = \begin{pmatrix}
n_P & 0 & 0 \\
0 & n_{SV} & 0 \\
0 & 0 & n_{SH}
\end{pmatrix},
\tag{4.35}
$$

then the expressions for the composition submatrices are obtained [see also Van der Hulden, 1987] as

$$
\mathbf{D}^{v,\downarrow} = \begin{pmatrix}
\zeta_1 & \zeta_1\gamma S & -\zeta_2 \\
\zeta_2 & \zeta_2\gamma S & \zeta_1 \\
\gamma P & -\zeta_\alpha\zeta_\alpha & 0
\end{pmatrix} \mathbf{N}_v^{-1}.
\tag{4.36}
$$

while for the particle velocity polarizations of the upgoing waves we have

$$
\mathbf{D}^{v,\uparrow} = \begin{pmatrix}
\zeta_1 & \zeta_1\gamma S & -\zeta_2 \\
\zeta_2 & \zeta_2\gamma S & \zeta_1 \\
-\gamma P & \zeta_\alpha\zeta_\alpha & 0
\end{pmatrix} \mathbf{N}_v^{-1}.
\tag{4.37}
$$

Note Eq. (4.37) shows that the particle velocity polarizations for the upgoing waves are related to the downgoing waves Eq. (4.36) according to

$$
\mathbf{D}^{v,\uparrow} = \mathbf{J}^{\uparrow\downarrow} \mathbf{D}^{v,\downarrow},
\tag{4.38}
$$

where we have defined the unitary matrix $\mathbf{J}^{\uparrow\downarrow}$ as

$$
\mathbf{J}^{\uparrow\downarrow} = \begin{pmatrix}
+1 & 0 & 0 \\
0 & +1 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\tag{4.39}
$$
The composition submatrix for the downgoing traction components [see also Van der Hijden, 1987] is given by

\[
D_{\tau,\downarrow} = \begin{pmatrix}
2\hat{\mu}\zeta_1\gamma P & 2\hat{\mu}\zeta_1\chi & -\hat{\mu}\zeta_2\gamma S \\
2\hat{\mu}\zeta_2\gamma P & 2\hat{\mu}\zeta_2\chi & \hat{\mu}\zeta_1\gamma S \\
2\mu\chi & -2\mu\zeta_\alpha\zeta_\alpha\gamma S & 0
\end{pmatrix} N_v^{-1}, \quad (4.40)
\]

where \(\chi\) is defined as

\[
\chi = \frac{\rho}{2\hat{\mu}} - \zeta_\alpha\zeta_\alpha. \quad (4.41)
\]

For the upgoing components we can write the traction composition submatrix in terms of the unitary matrix \(J_{\tau,\downarrow}^\dagger\) and the downgoing traction composition submatrix \(D_{\tau,\downarrow}\), similar to Eq. (4.38) as

\[
D_{\tau,\uparrow} = -J_{\tau,\downarrow}^\dagger D_{\tau,\downarrow}. \quad (4.42)
\]

Inspection of the polarization submatrices of Eqs. (4.36),(4.38),(4.40) and (4.42) shows that we have incorporated the choice that the horizontal components of the particle velocity of two waves propagating upward and downward are one and the same, while the vertical components of the particle velocities are each other's opposite. As a consequence, the horizontal components of the tractions of two waves propagating upward and downward are each other's opposite, while the vertical components of tractions of two waves propagating upward and downward are one and the same.

Now, we are left to determine the inverse of the composition matrix. In Appendix A we show that based on the Betti-Rayleigh's reciprocity theorem of Eq. (3.66), the polarization matrices related with eigenfunctions in the same medium satisfy certain symmetry relations, the so-called bi-orthogonal relations.

These relations are expressed, see Appendix A, as

\[
(D_{\tau,\downarrow})^T D_{\tau,\uparrow} + (D_{\tau,\uparrow})^T D_{\tau,\downarrow} = -N_D, \quad (4.43a)
\]

\[
(D_{\tau,\downarrow})^T D_{\tau,\downarrow} + (D_{\tau,\downarrow})^T D_{\tau,\downarrow} = 0, \quad (4.43b)
\]

\[
(D_{\tau,\downarrow})^T D_{\tau,\uparrow} + (D_{\tau,\downarrow})^T D_{\tau,\uparrow} = 0, \quad (4.43c)
\]

\[
(D_{\tau,\downarrow})^T D_{\tau,\downarrow} + (D_{\tau,\downarrow})^T D_{\tau,\downarrow} = +N_D, \quad (4.43d)
\]
where the matrix $N_D$ is a diagonal matrix, for which its components are obtained with Eqs. (4.36), (4.37), (4.40) and (4.42) as

$$N_D = \begin{pmatrix} 2\rho \gamma_P & 0 & 0 \\ 0 & 2\rho \zeta_\alpha \zeta_\alpha \gamma_S & 0 \\ 0 & 0 & 2\hat{\mu} \zeta_\alpha \zeta_\alpha \gamma_S \end{pmatrix} (N_v)^{-2}.$$  \hspace{1cm} (4.44)

Now it is convenient to choose the normalization constants contained in $N_v$ such, that the matrix $N_D$ becomes the unity matrix, i.e.

$$N_D = I.$$  \hspace{1cm} (4.45)

Inspecting Eq. (4.44) we see that the normalization constants of Eq. (4.35) have to be taken as

$$n_P = (2\rho \gamma_P)^{1/2},$$  \hspace{1cm} (4.46a)

$$n_{SV} = (2\rho \zeta_\alpha \zeta_\alpha \gamma_S)^{1/2},$$  \hspace{1cm} (4.46b)

$$n_{SH} = (2\hat{\mu} \zeta_\alpha \zeta_\alpha \gamma_S)^{1/2} = \hat{c} s n_{SV}.$$  \hspace{1cm} (4.46c)

To avoid that the inverse of the normalization matrix $N_v$ is ill-defined, we exclude the case that $\zeta_1 = \zeta_2 = 0$ for the moment. This case, which corresponds to plane waves which propagate perpendicular to the layering of the medium, will be treated separately in Chapter 6.

We can now easily construct the decomposition matrix. First, we rewrite Eqs. (4.43a)–(4.43d) with the normalization of Eq. (4.45), in matrix notation as

$$\begin{pmatrix} (D^\gamma_{\uparrow})^T \\ (D^\gamma_{\downarrow})^T \end{pmatrix} \begin{pmatrix} (D^\nu_{\uparrow})^T \\ (D^\nu_{\downarrow})^T \end{pmatrix} \begin{pmatrix} D^\gamma_{\uparrow} & D^\nu_{\uparrow} \\ D^\gamma_{\downarrow} & D^\nu_{\downarrow} \end{pmatrix} = \begin{pmatrix} -I & 0 \\ 0 & +I \end{pmatrix}. \hspace{1cm} (4.47)$$

Next, we identify the decomposition matrix as

$$D^{-1} = \begin{pmatrix} -I & 0 \\ 0 & +I \end{pmatrix} \begin{pmatrix} (D^\gamma_{\uparrow})^T \\ (D^\gamma_{\downarrow})^T \end{pmatrix} \begin{pmatrix} (D^\nu_{\uparrow})^T \\ (D^\nu_{\downarrow})^T \end{pmatrix} = \begin{pmatrix} -(D^\gamma_{\uparrow})^T & -(D^\nu_{\uparrow})^T \\ -(D^\gamma_{\downarrow})^T & -(D^\nu_{\downarrow})^T \end{pmatrix}, \hspace{1cm} (4.48)$$

which completes our analysis on how to decompose the wavefield into up- and downgoing wavefield constituents, and vice versa.
Now, we will elaborate on the interpretation of the symmetry relations expressed in Eqs. (4.43a)–(4.43d). The submatrices $\mathbf{D}$ define the polarizations of the particle velocity and vertical traction vector. For later convenience we define the concept of polarization states. A polarization state is characterised by

- A main propagation direction, where the two admissible states are up- or downgoing.
- A medium state.

Equations (4.43a)–(4.43d) can now be interpreted as a measure of similarity between different polarization states in the same medium. Polarization states with different propagation directions have no similarity. Proper choice of the normalization constants in Eqs. (4.46a)–(4.46c) enables us to normalize the similarity of polarization states with the same propagation direction to unity, according to Eqs. (4.43a)–(4.43d).

We can generalize the concept of similarity of two different polarization states in different media, by constructing the similarity operator which acts on the polarization matrices of two different polarization states. This similarity operator, acting on two polarization states distinguished by the superscripts $A$ and $B$ respectively, is defined as

$$
(D^A \otimes D^B) \equiv (D^{\tau,A})^T D^{v,B} + (D^{v,A})^T D^{\tau,B}.
$$

(4.49)

Note that in the special case of identical medium states, Eqs. (4.43a)–(4.43d) can be written in terms of the similarity operator as

$$
(D^\dagger \otimes D^\dagger) = -\mathbf{I},
$$

(4.50a)

$$
(D^\dagger \otimes D^\dagger) = 0,
$$

(4.50b)

$$
(D^\dagger \otimes D^\dagger) = 0,
$$

(4.50c)

$$
(D^\dagger \otimes D^\dagger) = +\mathbf{I}.
$$

(4.50d)

Note that the similarity operator incorporates the propagation direction in terms of a plus or minus sign. This property proves to be advantageous in our later discussion on transmission and reflection at interfaces between different media. On the other hand this property obstructs us to designate the similarity operator as a proper inner product, since it violates the requirement that an inner product must be positive definite.
Intermezzo: The impedance matrix

We define the downgoing impedance matrix $Z^\downarrow$ as the matrix that relates the downgoing traction components to the downgoing velocity components according to

$$D^{\tau,\downarrow} = Z^\downarrow D^{v,\downarrow}. \quad (4.51)$$

An explicit expression for the downgoing impedance matrix can hence be found as

$$Z^\downarrow = D^{\tau,\downarrow}(D^{v,\downarrow})^{-1}. \quad (4.52)$$

The inverse of $D^{v,\downarrow}$ can be found after algebraic manipulation of Eq. (4.36), resulting in

$$(D^{v,\downarrow})^{-1} = N_v \begin{pmatrix} \Delta_v^{-1} \zeta_1 & \Delta_v^{-1} \zeta_2 & \Delta_v^{-1} \zeta_S \\ \Delta_v^{-1} (\zeta_\alpha \zeta_\alpha)^{-1} \gamma_p \zeta_1 & \Delta_v^{-1} (\zeta_\alpha \zeta_\alpha)^{-1} \gamma_p \zeta_2 & -\Delta_v^{-1} \\ - (\zeta_\alpha \zeta_\alpha)^{-1} \zeta_2 & (\zeta_\alpha \zeta_\alpha)^{-1} \zeta_1 & 0 \end{pmatrix}, \quad (4.53)$$

where the denominator $\Delta_v$ has been defined as

$$\Delta_v = \zeta_\alpha \zeta_\alpha + \gamma_p \gamma_S. \quad (4.54)$$

Substitution of Eqs. (4.40) and (4.53) into Eq. (4.51) yield the following result for the impedance matrix for downgoing waves

$$Z^\downarrow = \rho \begin{pmatrix} \tilde{c}_S^2 \zeta_S + \zeta_1 \zeta_1 \Delta_v^{-1} (\gamma_p - \gamma_S) & \tilde{c}_S^2 \zeta_1 \zeta_2 \Delta_v^{-1} (\gamma_p - \gamma_S) & \zeta_1 \beta \\ \tilde{c}_S^2 \zeta_1 \zeta_2 \Delta_v^{-1} (\gamma_p - \gamma_S) & \tilde{c}_S^2 \gamma_S + \zeta_2 \zeta_2 \Delta_v^{-1} (\gamma_p - \gamma_S) & \zeta_2 \beta \\ -\zeta_1 \beta & -\zeta_2 \beta & \Delta_v^{-1} \gamma_S \end{pmatrix}, \quad (4.55)$$

where $\beta$ has been defined as

$$\beta = 2 \tilde{c}_S^2 - \Delta_v^{-1}. \quad (4.56)$$

For upgoing waves we define the impedance matrix similar to Eq. (4.51) as

$$D^{\tau,\uparrow} = Z^\uparrow D^{v,\uparrow}. \quad (4.57)$$
By using Eqs. (4.38) and (4.42) we rewrite this equation to polarizations for downgoing waves, resulting in

\[-J^\uparrow\downarrow D^\uparrow\downarrow = Z^\uparrow J^\uparrow\downarrow D^\uparrow\downarrow\]  \hspace{1cm} (4.58)

and hence

\[D^\uparrow\downarrow = -J^\uparrow\downarrow Z^\uparrow J^\uparrow\downarrow D^\uparrow\downarrow\]  \hspace{1cm} (4.59)

Comparing this last expression with Eq. (4.51), we conclude that the impedance matrix for upgoing waves is related to the impedance matrix for downgoing waves according to

\[Z^\uparrow = -J^\uparrow\downarrow Z^\downarrow J^\uparrow\downarrow.\]  \hspace{1cm} (4.60)

Due to the special structure of the explicit expression of the impedance matrix for downgoing waves of Eq. (4.55), the latter expression is equivalent to

\[Z^\uparrow = -(Z^\downarrow)^T,\]  \hspace{1cm} (4.61)

which shows that the impedance matrix for upgoing waves is the negative transpose of its downgoing equivalent. For normal incidence Eq. (4.55) reduces to

\[Z^\uparrow(\zeta_1 = 0, \zeta_2 = 0) = \begin{pmatrix} \rho \hat{c}_S & 0 & 0 \\ 0 & \rho \hat{c}_S & 0 \\ 0 & 0 & \rho \hat{c}_P \end{pmatrix}.\]  \hspace{1cm} (4.62)

We conclude that for a plane wave propagating downward, perpendicular to the layering of the medium, the ratio between the traction and the corresponding velocity is given by the normal impedances, $Z_P$ and $Z_S$, defined as

\[Z_{P,S} = \rho \hat{c}_{P,S}.\]  \hspace{1cm} (4.63)

For a plane wave propagating upward, perpendicular to the layering, the normal impedance changes its sign, according to Eq. (4.57). For a more fundamental treatment of the impedance matrix and its applications we refer to De Hon [1996]. For us, the limited discussion offered here suffices.
4.3 Scattering theory

4.3.1 Propagators

Suppose that the field vector, dubbed $b_I$ and transformed into the complex-frequency and angular slowness domain, satisfies the wave equation as expressed in Eq. (4.2). Furthermore, this field vector satisfies the boundary condition at the depth level $x_3 = x_3^a$, specified by $b_J(x_3^a)$. Then we define the field propagator as the operator that transforms the field vector at level $x_3$ to any other depth level $x_3$, which can be expressed as

$$b_I(x_3) = P_{IJK}(x_3, x_3^a)b_J(x_3^a).$$  \hfill (4.64)

Substituting Eq. (4.64) in the source-free wave equation, see Eq. (4.2) leads to

$$\partial_3 P_{IK}(x_3, x_3^a)b_K(x_3^a) = -sA_{IJ} P_{JK}(x_3, x_3^a)b_K(x_3^a),$$  \hfill (4.65)

which shows that the field propagator $P(x_3, x_3^a)$ of the field vector, related to a boundary condition at a level $x_3^a$ is found as the unique and continuous solution of the source-free system equation

$$\partial_3 P_{IK}(x_3, x_3^a) = -sA_{IJ} P_{JK}(x_3, x_3^a),$$  \hfill (4.66)

with the boundary condition

$$P_{IK}(x_3^a, x_3^a) = I_{IK},$$  \hfill (4.67)

where $I$ stands for the $6 \times 6$ identity matrix. By employing the structure of the field vector $b_I$ we can conveniently partition the field propagator into four three by three submatrices expressed as

$$\begin{pmatrix} +\bar{v}_i(x_3) \\ -\bar{\tau}_{i3}(x_3) \end{pmatrix} = \begin{pmatrix} P^{u,u}(x_3, x_3^a) & P^{u,\tau}(x_3, x_3^a) \\ P^{\tau,u}(x_3, x_3^a) & P^{\tau,\tau}(x_3, x_3^a) \end{pmatrix} \begin{pmatrix} +\bar{v}_i(x_3^a) \\ -\bar{\tau}_{i3}(x_3^a) \end{pmatrix}. \hfill (4.68)$$

For $x_3$ at a deeper level, i.e. $x_3 > x_3^a$, the field propagator establishes a downward continuation of the field vector, while conversely for $x_3 < x_3^a$ the field propagator establishes an upward continuation of the wavefield. In the following we use Roman superscripts to distinguish different levels, such as in $x_3^a$ and $x_3^b$. We will implicitly assume that the alphabetic order is representative for their relative level. For example, the use of $x_3^a$ and $x_3^b$ implicitly
states that the level $x_3^b$ is below $x_3^a$, hence $x_3^b > x_3^a$. This choice enables us to call the field propagator $P(x_3^b, x_3^a)$ a downward propagator.

In a similar manner the propagator for the wave vector $y$ can be defined. However, at interfaces where the medium properties jump, the decomposition matrix, which is a function of the medium properties, is discontinuous. Therefore the wave vector and its propagator are discontinuous. When we define the wave propagator across an interface we need to distinguish the level on both sides of the interface. For this purpose we use the following notation; when approached from below we have

$$y_I(x_3^+) \overset{\text{def}}{=} \lim_{\varepsilon \downarrow 0} y_I(x_3 + \varepsilon), \quad (4.69)$$

and when approaching from above

$$y_I(x_3^-) \overset{\text{def}}{=} \lim_{\varepsilon \uparrow 0} y_I(x_3 - \varepsilon). \quad (4.70)$$

Note that the classical boundary condition of the continuity of the field vector of Eqs. (4.5) and (4.6) can be expressed as

$$b_I(x_3) = b_I(x_3^+). \quad (4.71)$$

Now, we define the wave propagator $Q(x_3, x_3^a)$, similar to Eq. (4.64), as

$$y_I(x_3) = Q_{IJ}(x_3, x_3^a) y_J(x_3^a), \quad (4.72)$$

where we keep in mind that if the wave propagator is defined at a discontinuous interface, we have to be explicit from which side of the interface, the specific level is approached. For example, when the level $x_3^a$ is an interface between two different media we have from Eq. (4.71)

$$P(x_3^+, x_3^-) = I, \quad (4.73)$$

whereas for the wave propagator a similar boundary condition does not hold since at the interface reflection and transmission occurs and hence modification of the wave amplitudes, i.e.

$$Q(x_3^+, x_3^-) \neq I. \quad (4.74)$$

With the partitioning of the wave vector as in Eq. (4.31) we can rewrite Eq. (4.72) in terms of submatrices as

$$
\begin{pmatrix}
  y^+(x_3) \\
  y^-(x_3)
\end{pmatrix} =
\begin{pmatrix}
  Q_{11}(x_3, x_3^a) & Q_{12}(x_3, x_3^a) \\
  Q_{21}(x_3, x_3^a) & Q_{22}(x_3, x_3^a)
\end{pmatrix}
\begin{pmatrix}
  y^+(x_3^a) \\
  y^-(x_3^a)
\end{pmatrix}.
\quad (4.75)
$$
Since the field and wave vector are related through the composition operator according to Eq. (4.29), we can rewrite the definition of the wave propagator in Eq. (4.72) as

\[ D^{-1}_{IK}(x_3) b_K(x_3) = Q_{IJ}(x_3, x_3^a) D^{-1}_{JK}(x_3^a) b_K(x_3^a), \]  

(4.76)

which, after left-multiplying with the composition matrix \( D \) at level \( x_3 \), exhibits the relationship between the field propagator and the wave propagator as

\[ P_{IJ}(x_3, x_3^a) = D_{IK}(x_3) Q_{KL}(x_3, x_3^a) D^{-1}_{KJ}(x_3^a). \]  

(4.77)

In this thesis we use the propagator only as a convenient mathematical description to define intermediate result that we find between the wave vectors at different level. The actual implementation and physical interpretation is more elegantly formulated with the use of the scattering formalism.

### 4.3.2 The scattering operator

The scattering operator \( S(x_3^a; x_3^b) \) relates the \textit{outgoing} waves to the \textit{incoming} waves at the levels \( x_3^a \) and \( x_3^b \). Since we have \( x_3^b > x_3^a \), following the contemplation in the previous section, we uniquely define the scattering operator as

\[
\begin{pmatrix}
    y^\uparrow(x_3^a) \\
    y^\downarrow(x_3^b)
\end{pmatrix}
= S(x_3^a; x_3^b)
\begin{pmatrix}
    y^\uparrow(x_3^b) \\
    y^\downarrow(x_3^b)
\end{pmatrix}.
\]  

(4.78)

Owing to the structure of the wave vector on both sides of this equation we partition the scattering operator \( S(x_3^a; x_3^b) \) into the submatrices, which define the reflection and transmission matrices of the interval \( \{x_3^a \leq x_3 \leq x_3^b\} \), i.e.

\[
S(x_3^a; x_3^b) = \begin{pmatrix}
    R^\uparrow(x_3^a; x_3^b) & T^\uparrow(x_3^a; x_3^b) \\
    T^\downarrow(x_3^a; x_3^b) & R^\downarrow(x_3^a; x_3^b)
\end{pmatrix}.
\]  

(4.79)

Figure (4.2) gives an illustration with an explanation of the relevant symbols.

Related with the assertion, that \( x_3^b \) is at a deeper level than \( x_3^a \), as stipulated before, we note that the definition of the incoming and outgoing waves does not change when the order of the arguments is interchanged. To stress this property of the scattering operator we choose to use the notation \( (x_3^a; x_3^b) \) in its argument, instead of \( (x_3^a, x_3^b) \).
4.3 Scattering theory

Figure 4.2: A graphical illustration of the components of the scattering operator, the reflection and transmission matrices.

To shorten our expressions, the following notation is used for the arguments of the wave propagators and the scattering operator; for the downward wave propagator we write

\[ Q_{ba} \overset{\text{def}}{=} Q(x_3^b, x_3^a), \]  
(4.80)

for the upward wave propagator

\[ Q_{ab} \overset{\text{def}}{=} Q(x_3^a, x_3^b), \]  
(4.81)

while for the scattering operator

\[ S_{ab} \overset{\text{def}}{=} S(x_3^a; x_3^b). \]  
(4.82)

Since the order of the level is used in the definition of the scattering operator we do not need to distinguish between \( S(x_3^a; x_3^b) \) and \( S(x_3^b; x_3^a) \) and simply use \( S_{ab} \).

Comparing Eqs. (4.75) and (4.78) we see that the upward wave propagator, downward wave propagator and scattering operator all define a unique and complete relationship between the wave vectors at both levels. Hence, we can always find the up- and downward propagator from the scattering operator and vice-versa. In Appendix B, Section B.1 the relationships between
$Q_{ba}$, $Q_{ab}$ and $S_{ab}$ are derived. The resulting expression for the downward wave propagator in terms of the scattering submatrices, the reflection and transmission matrices, is found as

$$Q_{ba} = \begin{pmatrix}
(T_{ab}^\dagger)^{-1} & -(T_{ab}^\dagger)^{-1}R_{ab} \\
R_{ab}(T_{ab}^\dagger)^{-1} & T_{ab}^\dagger - R_{ab}(T_{ab}^\dagger)^{-1}R_{ab}
\end{pmatrix},$$

(4.83)

whereas the expression for the upward wave propagator in terms of the scattering submatrices is found as

$$Q_{ab} = \begin{pmatrix}
T_{ab}^\dagger - R_{ab}^\dagger(T_{ab}^\dagger)^{-1}R_{ab} & R_{ab}^\dagger(T_{ab}^\dagger)^{-1} \\
-(T_{ab}^\dagger)^{-1}R_{ab} & (T_{ab}^\dagger)^{-1}
\end{pmatrix}. 

(4.84)

Inversely, the scattering operator can be expressed both in terms of the downward propagation submatrices as

$$S_{ab} = \begin{pmatrix}
-(Q_{ba}^{\uparrow\dagger})^{-1}Q_{ba}^{\uparrow\dagger} & (Q_{ba}^{\uparrow\downarrow})^{-1} \\
Q_{ba}^{\downarrow\dagger} - Q_{ba}^{\downarrow\dagger}(Q_{ba}^{\uparrow\dagger})^{-1}Q_{ba}^{\uparrow\dagger} & Q_{ab}^{\dagger}(Q_{ba}^{\uparrow\downarrow})^{-1}
\end{pmatrix},$$

(4.85)

and upward propagation submatrices as

$$S_{ab} = \begin{pmatrix}
Q_{ab}^{\uparrow\dagger}(Q_{ab}^{\downarrow\dagger})^{-1} & Q_{ab}^{\uparrow\downarrow} - Q_{ab}^{\downarrow\dagger}(Q_{ab}^{\uparrow\dagger})^{-1}Q_{ab}^{\uparrow\dagger} \\
(Q_{ab}^{\downarrow\dagger})^{-1} & -(Q_{ab}^{\uparrow\dagger})^{-1}Q_{ab}^{\uparrow\dagger}
\end{pmatrix}. 

(4.86)

It appears that the simplest and most convenient relationships are found when we use a hybrid form, that uses both the up- and downward propagator to determine the scattering operator. Since the right-hand sides of Eqs. (4.85) and (4.86) have to be identical at submatrix level, we can combine both relations in Eq. (4.86) and (4.85) in the hybrid form

$$S_{ab} = \begin{pmatrix}
Q_{ab}^{\uparrow\dagger}(Q_{ab}^{\downarrow\dagger})^{-1} & (Q_{ba}^{\uparrow\dagger})^{-1} \\
(Q_{ab}^{\downarrow\dagger})^{-1} & Q_{ba}^{\dagger}(Q_{ba}^{\uparrow\dagger})^{-1}
\end{pmatrix}. 

(4.87)

\subsection{4.3.3 Composition rules and Redheffer's star product}

In this section we will discuss how the field propagator, wave propagators and scattering operator of a composite interval are composed out of the similar matrices of the subintervals, see Figure (4.3) for an explanation of the various symbols.
Consider the downward field propagators $\mathbf{P}(x_3^b, x_3^b)$ and $\mathbf{P}(x_3^c, x_3^c)$, of the subintervals $\{x_3^b \leq x_3 \leq x_3^c\}$ and $\{x_3^a \leq x_3 \leq x_3^b\}$, respectively. By repeatedly applying the downward propagator as follows

$$
\mathbf{b}(x_3^a) = \mathbf{P}(x_3^c, x_3^b)\mathbf{b}(x_3^b) = \mathbf{P}(x_3^c, x_3^b)\mathbf{P}(x_3^b, x_3^a)\mathbf{b}(x_3^a),
$$

we find the downward propagator of the interval $(x_3^a, x_3^c)$ as

$$
\mathbf{P}(x_3^c, x_3^a) = \mathbf{P}(x_3^c, x_3^b)\mathbf{P}(x_3^b, x_3^a).
$$

A similar procedure can be applied for the wave propagator leading to

$$
\mathbf{Q}(x_3^c, x_3^a) = \mathbf{Q}(x_3^c, x_3^b)\mathbf{Q}(x_3^b, x_3^a).
$$

Note that at any depth level where the wave vector is discontinuous, a distinction must be made between the wave vector and depth level approached from above and the wave vector and depth level approached from below such an interface.
In terms of the submatrices with the abbreviated notation Eq. (4.90) reads

\[
\begin{align*}
Q_{ca}^{\uparrow\uparrow} &= Q_{cb}^{\uparrow\uparrow} Q_{ba}^{\uparrow\uparrow} + Q_{cb}^{\uparrow\downarrow} Q_{ba}^{\downarrow\uparrow}, \\
Q_{ca}^{\uparrow\downarrow} &= Q_{cb}^{\uparrow\uparrow} Q_{ba}^{\uparrow\downarrow} + Q_{cb}^{\uparrow\downarrow} Q_{ba}^{\downarrow\downarrow}, \\
Q_{ca}^{\downarrow\uparrow} &= Q_{cb}^{\uparrow\downarrow} Q_{ba}^{\uparrow\uparrow} + Q_{cb}^{\downarrow\downarrow} Q_{ba}^{\downarrow\uparrow}, \\
Q_{ca}^{\downarrow\downarrow} &= Q_{cb}^{\downarrow\uparrow} Q_{ba}^{\uparrow\downarrow} + Q_{cb}^{\downarrow\downarrow} Q_{ba}^{\downarrow\downarrow}.
\end{align*}
\]  

(4.91a)  
(4.91b)  
(4.91c)  
(4.91d)

Our objective is to derive the scattering operator of the joint interval \(x_3^a \leq x_3 \leq x_3^b\) out of the scattering operators of the subintervals \(x_3^b \leq x_3 \leq x_3^c\) and \(x_3^c \leq x_3 \leq x_3^d\). From Eq. (4.87) we see that for constructing the scatterer product of the interval \(x_3^a \leq x_3 \leq x_3^b\) we need the wave propagators \(Q_{ac}^{\uparrow\downarrow}, Q_{ac}^{\downarrow\uparrow}, Q_{ca}^{\uparrow\uparrow}\) and \(Q_{ca}^{\downarrow\downarrow}\). All these propagators can be composed out of the propagators of the subintervals, as described by Eqs. (4.91a)–(4.91d).

Since the wave propagators of the two subintervals can be rewritten, by using Eqs. (4.83) and (4.84), in terms of the transmission and reflection matrices, we can find the relationship between the wave propagator of the composite interval \(x_3^a \leq x_3 \leq x_3^b\) in terms of the the transmission and reflection matrices of the subintervals. Since we already have found the relationship for the scattering operator in terms of its wave propagators, see Eq. (4.87) we can construct the relationship of the scattering operator of the composite interval \(x_3^a \leq x_3 \leq x_3^b\) in terms of the reflection and transmission matrices of their subintervals. In Appendix B, Section B.2 the procedure described above is applied and this results in the scattering operator composition rule known as Redheffer’s star product [Kennett, 1983]. Redheffer’s star product is expressed as

\[
S(x_3^a; x_3^b) = S_{ab} \ast S_{bc} =
\left(
\begin{array}{cc}
R_{ab}^{\ominus} + T_{ab}^{\uparrow} R_{bc}^{\ominus} (I - R_{ab}^{\ominus} R_{bc}^{\ominus})^{-1} T_{ab}^{\downarrow} & T_{ab}^{\uparrow} (I - R_{ab}^{\ominus} R_{bc}^{\ominus})^{-1} T_{bc}^{\uparrow} \\
T_{bc}^{\downarrow} (I - R_{ab}^{\ominus} R_{bc}^{\ominus})^{-1} T_{ab}^{\uparrow} & R_{bc}^{\ominus} + T_{bc}^{\uparrow} R_{ab}^{\ominus} (I - R_{bc}^{\ominus} R_{ab}^{\ominus})^{-1} T_{bc}^{\downarrow}
\end{array}
\right)
\]  

(4.92)

The physical interpretation of the Redheffer’s star product becomes clear if we construct the inverse operator, appearing in the downgoing transmission matrix \(T_{ac}^{\downarrow}\) on the right-hand side of Eq. (4.92) as

\[
(I - R_{ab}^{\ominus} R_{bc}^{\ominus})^{-1} = I + \sum_{n=1}^{\infty} (R_{ab}^{\ominus} R_{bc}^{\ominus})^n.
\]  

(4.93)
Explicitly writing out the downgoing transmission matrix with Eq. (4.93) we obtain

$$T_{ac} = T_{bc} T_{ab}^{-1} + T_{bc} R_{ab}^{-1} R_{bc} T_{ab}^{-1} + T_{bc} R_{ab}^{-1} R_{bc} R_{ab}^{-1} R_{bc} T_{ab}^{-1} + \ldots$$  \hspace{1cm} (4.94)

Equation (4.94) is interpreted as the multiple expansion of the downward transmission response, where we perceive that the causal hierarchy of these suboperators must be understood acting from the right to the left. In Eq. (4.94) we have cut the multiple train after the second-order multiple. Although the fact that by physical reasoning in terms of multiple summation we can construct Redheffer’s start product, in our derivation in Appendix B we show that in principle this relationship follows directly from the definition of the scattering operator, by using the composition rules for wave propagators.

### 4.3.4 The scattering operator of an interface

In order to obtain the scattering operator of an interface, across which the constitutive parameters jump by a finite amount, we use the continuity of the field vector, which is expressed in Eq. (4.71) as

$$b(x_3^-) = b(x_3^+)$$  \hspace{1cm} (4.95)

Note that the wave vector will be discontinuous. Expressed in terms of the wave vector Eq. (4.95) can be rewritten, using Eq. (4.29) as

$$D(x_3^-) y(x_3^-) = D(x_3^+) y(x_3^+)$$  \hspace{1cm} (4.96)
For the configuration and relevant symbols see Figure (4.4). In abbreviated notation we will use for the composition matrix of the above the interface

$$D_- = D(x_3^-),$$

and similarly for the composition operator below the interface

$$D_+ = D(x_3^+).$$

Using the definition of the wave propagator in Eq. (4.72), the down- and upward wave propagators of the interface, $Q(x_3^+, x_3^-)$ and $Q(x_3^-, x_3^+)$ respectively, are found by applying the decomposition operator to Eq. (4.96), leading to

$$Q(x_3^+, x_3^-) = D^{-1}(x_3^+)D(x_3^-),$$

and for the upward wave vector propagator

$$Q(x_3^-, x_3^+) = D^{-1}(x_3^-)D(x_3^+).$$

By writing out Eqs. (4.100) and (4.100) in terms of submatrices, as indicated by Eqs. (4.34) and (4.48) we obtain

$$Q(x_3^+, x_3^-) = \begin{pmatrix} \begin{pmatrix} -D_{\uparrow, \downarrow}^{\uparrow} \end{pmatrix}^{\top} & -\begin{pmatrix} D_{\uparrow, \downarrow}^{\uparrow} \end{pmatrix}^{\top} \\ \begin{pmatrix} D_{\uparrow, \downarrow}^{\downarrow} \end{pmatrix}^{\top} & \begin{pmatrix} D_{\uparrow, \downarrow}^{\downarrow} \end{pmatrix}^{\top} \end{pmatrix} \begin{pmatrix} D_{\uparrow, \downarrow}^{\uparrow} & D_{\uparrow, \downarrow}^{\downarrow} \\ D_{\uparrow, \downarrow}^{\downarrow} & D_{\uparrow, \downarrow}^{\uparrow} \end{pmatrix}$$

and similarly for the upward propagator

$$Q(x_3^-, x_3^+) = \begin{pmatrix} \begin{pmatrix} -D_{\downarrow, \uparrow}^{\downarrow} \end{pmatrix}^{\top} & -\begin{pmatrix} D_{\downarrow, \uparrow}^{\downarrow} \end{pmatrix}^{\top} \\ \begin{pmatrix} D_{\downarrow, \uparrow}^{\uparrow} \end{pmatrix}^{\top} & \begin{pmatrix} D_{\downarrow, \uparrow}^{\uparrow} \end{pmatrix}^{\top} \end{pmatrix} \begin{pmatrix} D_{\downarrow, \uparrow}^{\downarrow} & D_{\downarrow, \uparrow}^{\uparrow} \\ D_{\downarrow, \uparrow}^{\uparrow} & D_{\downarrow, \uparrow}^{\downarrow} \end{pmatrix}.$$
for the downward propagator and similarly for the upward propagator of the interface

\[
Q(x_3^-, x_3^+) = \begin{pmatrix}
-(D_-^+ \otimes D_+^+)
& -(D_-^+ \otimes D_+^+)
&D_-^+ \otimes D_+^+
&D_-^+ \otimes D_+^+
\end{pmatrix}.
\]

(4.104)

Next, we define the local reflection and transmission coefficients of the interface in terms of the local scattering operator of the interface \(S(x_3^+, x_3^-)\), expressed as

\[
S(x_3^-, x_3^+) = \begin{pmatrix}
\mathbf{r}^\omega(x_3^-; x_3^+)
& \mathbf{t}^\omega(x_3^-; x_3^+)
& \mathbf{t}^\omega(x_3^-; x_3^+)
& \mathbf{r}^\omega(x_3^-; x_3^+)
\end{pmatrix}.
\]

(4.105)

Then with Eqs. (4.103) and (4.104), and using the relationship of the scattering operator in terms of the wave propagators in the hybrid form of Eq. (4.87) we find for the set of local scattering submatrices as

\[
\mathbf{t}^\omega(x_3^-; x_3^+) = -(D_+^+ \otimes D_-^+)^{-1},
\]

(4.106a)

\[
\mathbf{t}^\omega(x_3^-; x_3^+) = + (D_-^- \otimes D_+^-)^{-1},
\]

(4.106b)

\[
\mathbf{r}^\omega(x_3^-; x_3^+) = -(D_-^+ \otimes D_+^+) (D_-^- \otimes D_+^-)^{-1},
\]

(4.106c)

\[
\mathbf{r}^\omega(x_3^-; x_3^+) = -(D_+^+ \otimes D_-^-) (D_+^- \otimes D_-^+)^{-1}.
\]

(4.106d)

Alternative expressions may be find by using Eqs. (4.85) or (4.86) instead of Eq. (4.87) leading to

\[
\mathbf{r}^\omega(x_3^-; x_3^+) = -(D_+^+ \otimes D_-^-)^{-1} (D_+^+ \otimes D_-^-),
\]

(4.107a)

\[
\mathbf{r}^\omega(x_3^-; x_3^+) = -(D_-^- \otimes D_+^+)^{-1} (D_-^- \otimes D_+^+).
\]

(4.107b)

Following from Eqs. (4.50a)–(4.50a described in Section 4.2 we see that when both media opposite of the interface are the same, the similarity of the polarization states with opposite propagation direction vanishes, while for similar propagation direction becomes unity. Hence Eqs. (4.106a)–(4.106d) then show that the local reflection matrices of the interface are zero and that the local transmission matrices are found to be unity. Since there is no contrast we obtain transmission and no reflection. If there is a contrast between the two media then the scattering response can be quantified in terms of the similarity of polarization states in the two media. This is the reason why we denote the cross product in Eq. (4.49) as the similarity operator. Note that
usually the scattering response is given in terms of a contrast or dissimilarity [KENNETT, 1983], which is the opposite of the similarity. To quantify the scattering response in terms of a contrast we rewrite, for the particle case of the down- to upgoing local reflection response in Eq. (4.106c) with the aid of Eqs. (4.38) and (4.42) and the definition of the similarity operator of Eq. (4.49) as

$$r^{\gamma}(x^-; x^+) = \left[ (D^{\tau - \downarrow})^T J^{\uparrow \downarrow} D^{v - \downarrow} + (D^{v - \downarrow})^T J^{\uparrow \downarrow} D^{\tau - \downarrow} \right]^{-1} \left[ (D^{\tau - \downarrow})^T D^{v - \downarrow} + (D^{v - \downarrow})^T D^{\tau - \downarrow} \right]^{-1} \left(4.108\right)$$

and for the up- to downgoing local reflection coefficient with Eq. (4.106d)

$$r^{\gamma}(x^-; x^+) = \left[ (D^{\tau - \downarrow})^T J^{\uparrow \downarrow} D^{v - \downarrow} + (D^{v - \downarrow})^T J^{\uparrow \downarrow} D^{\tau - \downarrow} \right]^{-1} \left[ (D^{\tau - \downarrow})^T D^{v - \downarrow} + (D^{v - \downarrow})^T D^{\tau - \downarrow} \right]^{-1} \left(4.109\right)$$

where on the right-hand side of Eq. (4.109) we additionally used

$$\left[ (D^{\tau - \downarrow})^T D^{v - \downarrow} + (D^{v - \downarrow})^T D^{\tau - \downarrow} \right]^{-1} \left[ (D^{\tau - \downarrow})^T J^{\uparrow \downarrow} D^{v - \downarrow} + (D^{v - \downarrow})^T J^{\uparrow \downarrow} D^{\tau - \downarrow} \right] = - \left[ (D^{\tau - \downarrow})^T J^{\uparrow \downarrow} D^{v - \downarrow} + (D^{v - \downarrow})^T J^{\uparrow \downarrow} D^{\tau - \downarrow} \right] \left(4.110\right)$$

We conclude from Eqs. (4.108) and (4.109) that by interchanging the medium on both sides of the interface we find the up- to downgoing reflection coefficient instead of the down- to upgoing reflection coefficient, and vice versa. Hence, the reflection (and transmission) operators have no sense of direction; they are solely determined by the contrast in medium properties.

In the derivation of the reflection and transmission matrices for an interface, we have applied the continuity boundary condition of the particle velocity and the vertical traction as expressed in Eqs. (4.5) and (4.6). When an ideal fluid medium is in contact with an elastic medium, the horizontal particle displacements, or velocities can slip across the interface, whereas the shear stresses vanish completely, because a ideal fluid cannot sustain any shear stresses. Hence, for this special case we can no longer use Eqs. (4.106a)– (4.106d). We treat the fluid-solid interface separately in Appendix C.

4.3.5 The scattering operator of a homogeneous interval

In this section we will derive the scattering operator of homogeneous interval. For the configuration and the relevant symbols we refer to Figure (4.5).
Figure 4.5: The scattering matrix of a homogeneous subregion, \( \{x_3^a \leq x_3 \leq x_3^b\} \).

We start with the source-free wave equation for the wave vector formulated in Eq. (4.32) as

\[
\partial_3 y_I = -s \Lambda_{IJ} y_J. \tag{4.111}
\]

In a homogeneous subinterval \( \{x_3^a \leq x_3 \leq x_3^b\} \) we can directly solve for the downward wave propagator \( \mathbf{Q}(x_3^b, x_3^a) \), leading to

\[
\mathbf{Q}(x_3, x_3^a) = \exp(-s \mathbf{\Lambda}(x_3 - x_3^a)). \tag{4.112}
\]

By using Eq. (4.112), we can write down the wave propagator of the homogeneous interval \( \{x_3^a \leq x_3 \leq x_3^b\} \) in terms of the submatrices of the vertical slownesses in Eq. (4.25) as

\[
\mathbf{Q}(x_3^b, x_3^a) = \begin{pmatrix}
\exp(-s \mathbf{\Lambda}^\dagger(x_3^b - x_3^a)) & 0 \\
0 & \exp(-s \mathbf{\Lambda}^\dagger(x_3^b - x_3^a))
\end{pmatrix}, \tag{4.113}
\]

which combined with Eq. (4.27) leads to

\[
\mathbf{Q}(x_3^b, x_3^a) = \begin{pmatrix}
\exp(s \mathbf{\Lambda}^\dagger(x_3^b - x_3^a)) & 0 \\
0 & \exp(-s \mathbf{\Lambda}^\dagger(x_3^b - x_3^a))
\end{pmatrix}. \tag{4.114}
\]
Next, the scattering operator of a homogeneous interval can be obtained with Eq. (4.114) and using Eq. (4.85) resulting in

$$S(x_3^a; x_3^b) = \begin{pmatrix} 0 & W(x_3^b - x_3^a) \\ W(x_3^b - x_3^a) & 0 \end{pmatrix}, \quad (4.115)$$

where we have introduced the one-way phase delay operator $W(x_3^b - x_3^a)$, defined as

$$W(x_3^b - x_3^a) = \exp(-s\Lambda^+(x_3^b - x_3^a)). \quad (4.116)$$

Since we have taken the branch-cut of the square-root in Eq. (4.19) such that the one-way phase delay operator cannot blow up in the propagation direction, the scattering operator does not contains any exponentially growing coefficients, while on the other hand the wave propagator contains both exponentially decaying and growing exponentials. Related with this property, although both the propagator matrix and scattering operator formalism are mathematically exact, we prefer the use of the scattering operator formalism, since its numerical implementation is known to lead to more stable results. This is especially true in the evanescent regime, for horizontal slownesses beyond the critical angles, for perfectly elastic media, where the eigenfunctions become exponentially growing and damping solutions. Furthermore, in viscoelastic media the eigensolutions contain damping behaviour for all horizontal slownesses. Hence in that case the use of the scattering formalism becomes even more favourable.

**Intermezzo: The skin depth**

For an elastic or viscous medium in the evanescent regime, the vertical slowness becomes imaginary and hence the vertical propagation is decaying in the up- or downgoing direction. For viscous media, the up- or downgoing waves in the propagating regime contain a damping as well. For a certain vertical distance the influence of such a wave will be negligible. To define a measure of the influence of an up- or downgoing wave we will introduce its skin depth. Since the propagation and damping behaviour of a downgoing wave is given by Eq. (4.12) the wave will be decayed to a factor $\exp(-1)$ when

$$\text{Re}(s\gamma_{P,S}x_3) = 1. \quad (4.117)$$
We define the skin depth $h_{\text{skin}}$ as the depth at which the condition expressed in Eq. (4.117) holds, i.e.

$$h_{\text{skin}} = \Re(s \gamma_{P,S})^{-1}. \quad (4.118)$$

From Eq. (4.118) we see that the skin depth is frequency dependent. In many cases it is more appropriate to investigate the skin depth for a stationary frequency $\omega$, since the main energy is focussed around a centre frequency. Taking the limit $s \to j\omega$ we obtain

$$h_{\text{skin}} = \left|\frac{1}{\omega \Im(\gamma_{P,S})}\right|, \quad (4.119)$$

which shows that for higher frequencies the skin depth decreases. When a fluid becomes dominantly viscous, the damping of the waves can become so strong that their influence will be limited to a small zone, called a boundary layer. The width of this boundary layer is then given in terms of the *viscous* skin depth. For example, in the limit of a purely Newtonian-viscous fluid for a normal horizontal plane wave we have

$$\dot{v} = j\omega \eta_0, \quad (4.120)$$

$$\zeta_1 = \zeta_2 = 0, \quad (4.121)$$

and hence the vertical slowness, with the appropriate branch-cut, is obtained as

$$\gamma_{S} = -(1 + j) \sqrt{\frac{\rho}{2\omega \eta_0}}. \quad (4.122)$$

Combining Eq. (4.118) and (4.122) the *viscous* skin depth is found as

$$h_{\text{skin}} = \sqrt{\frac{2\eta_0}{\rho \omega}}, \quad (4.123)$$

which shows that the width of the boundary layer increases for increasing zero shear-rate viscosity $\eta_0$.

### 4.3.6 Description of the configuration

The previous sections were applicable for the general case of layered media. From here on we focus on a model and measurement configuration which represents a simplified fracture model, see Figure 4.6.
We assume that the fracture can be represented by a thin horizontal layer filled with a viscoelastic or ideal fluid. Hence, in the present discussion we neglect the effects of the finite extent of the fracture as well as any interface roughness that is present in a real fractured rock specimen. The width of the fluid layer is denoted as $h$. The thin layer is embedded in a homogeneous elastic medium. The level at the top of the thin layer is denoted by $x^t_3$; the bottom of the layer with $x^s_3$. Above the layer, at level $x^s_3$ a source is present that radiates energy towards the thin layer. The distance from the source to the top of the layer is denoted as $h^S$, i.e.

$$h^S = x^t_3 - x^s_3. \quad (4.124)$$

The level of observation above the layer, where the reflected energy is observed, is denoted by $x^r_3$. We denote the level of observation underneath the layer, where the transmitted energy is observed by $x^r_3$. We introduce the distance from the top of the layer till the level of the observation on the reflection side as

$$h^R = x^t_3 - x^r_3 \quad (4.125)$$

Figure 4.6: The measurement configuration and the definitions of the layer width $h$ and the distances $h^S$, $h^T$, $h^R$ and $r$. 
and the distance from the bottom of the layer till the level of the observation on the transmitted side as

$$h^T = x_3^t - x_3^z.$$  \hspace{1cm} (4.126)

At any level at which the wave vector shows discontinuities, we can distinguish the value approached from above and below by adding, as in previous sections, a superscript \(-\) or \(+\), respectively. For example, the wave vector at the bottom of the layer at level \(x_3^z\) and approached from below is denoted by \(y(x_3^{z-})\).

### 4.3.7 The scattering operator of the layer system

When we want to determine the response of the fracture configuration, it appears convenient to introduce the scattering operator of the layer system. This scattering operator is defined on the interval \(\{x_3^{t,-} \leq x_3 \leq x_3^{t,+}\}\), from the top layer approached from above till the bottom interval approached from below, and is denoted by \(S(x_3^{t,-}; x_3^{z,+})\). The layer scattering operator can be found after successively applying Redheffer's star product to the scattering operators of the interface at level \(x_3\) between the embedding and the layer \(S(x_3^{t,-}, x_3^{t,+})\), the scattering operator of the homogeneous interval in the layer \(S(x_3^{t,+}, x_3^{z,-})\) and the scattering operator of the interface between the layer and the embedding \(S(x_3^{z,-}, x_3^{z,+})\) at level \(x_3^z\). The scattering operators of the interface can be determined with Eqs. (4.106a)–(4.106d). We introduce the local reflection and transmission operators of the interface at level \(x_3\), see Figure 4.7, with the additional shorthand notation as

$$S(x_3^{t,-}, x_3^{t,+}) = \begin{pmatrix} r \cup (x_3^{t,-}, x_3^{t,+}) & t \uparrow (x_3^{t,-}, x_3^{t,+}) \\ t \downarrow (x_3^{t,-}, x_3^{t,+}) & r \cap (x_3^{t,-}, x_3^{t,+}) \end{pmatrix} = \begin{pmatrix} r^o & t^o \\ t^i & r^i \end{pmatrix}.$$  \hspace{1cm} (4.127)

Because of the symmetry of the configuration that we investigate the transmission and reflection operators of the interface at level \(x_3^z\) show a simple relationship with those of the scattering operator of Eq. (4.127), as is also proven by Eqs. (4.108)–(4.109). Hence, we can write the scattering operator...
of the bottom interval as

$$\begin{align*}
S(x_3^{z^0}, x_3^{z^+}) &= \begin{pmatrix}
  r^i(x_3^{z^0}, x_3^{z^+}) & t^i(x_3^{z^0}, x_3^{z^+}) \\
  t^i(x_3^{z^0}, x_3^{z^+}) & r^i(x_3^{z^0}, x_3^{z^+})
\end{pmatrix} \\
&= \begin{pmatrix}
  r^i & t^i \\
  t^i & r^i
\end{pmatrix}.
\end{align*}$$

(4.128)

The scattering operator of the homogeneous interval \(\{x_3^{f^+} \leq x_3 \leq x_3^{z^0}\}\) can be determined specifically with Eq. (4.115), as

$$\begin{align*}
S(x_3^{f^+}, x_3^{z^0}) &= \begin{pmatrix}
  0 & \Theta \\
  \Theta & 0
\end{pmatrix},
\end{align*}$$

(4.129)

where we have introduced the shorthand notation \(\Theta\) for the one-way phase-delay operator of the layer as

$$\Theta = \exp(-s\Lambda^\perp_f h),$$

(4.130)

where the subscript \(s\) denotes the downgoing vertical slowness matrix inside the viscoelastic (fluid) layer. We can determine the scattering operator of
the layer by repeated application of Redheffer’s start product, Eq. (4.92), formally written as

\[
S^\| = S(x_3^+, x_3^-) = \begin{pmatrix} r^o & t^o \\ t^i & r^i \end{pmatrix}^* \begin{pmatrix} 0 & \Theta \\ \Theta & 0 \end{pmatrix}^* \begin{pmatrix} r^i & t^i \\ t^o & r^o \end{pmatrix}.
\]

(4.131)

Working out the rightmost scattering product in Eq. (4.131) with Eq. (4.92) we obtain the intermediate result

\[
S^\| = \begin{pmatrix} r^o & t^o \\ t^i & r^i \end{pmatrix}^* \begin{pmatrix} \Theta r^i \Theta & \Theta t^i \\ t^o \Theta & r^o \end{pmatrix}.
\]

(4.132)

Evaluating the remaining scattering product we obtain

\[
S^\| = \begin{pmatrix} r^o + t^o \Theta r^i \Theta [I - r^i \Theta r^i \Theta]^{-1} t^i & t^o [I - \Theta r^i \Theta r^i \Theta]^{-1} \Theta t^i \\ t^o \Theta [I - r^i \Theta r^i \Theta]^{-1} t^i & r^o + t^o \Theta r^i \Theta [I - \Theta r^i \Theta r^i \Theta]^{-1} \Theta t^i \end{pmatrix}.
\]

(4.133)

On the other hand, we can expand the triple Redheffer scattering operator product of Eq. (4.131), operating from the left to the right which results in

\[
S^\| = \begin{pmatrix} r^o + t^o \Theta r^i \Theta [I - \Theta r^i \Theta r^i \Theta]^{-1} \Theta t^i & t^o \Theta [I - r^i \Theta r^i \Theta]^{-1} t^i \\ t^o [I - \Theta r^i \Theta r^i \Theta]^{-1} \Theta t^i & r^o + t^o \Theta r^i \Theta [I - r^i \Theta r^i \Theta]^{-1} t^i \end{pmatrix}.
\]

(4.134)

Because of the symmetry relations that become evident from Eqs. (4.133) and (4.134) we can write the scattering operator of the layer system by introducing the layer reflection and transmission operators, \(\mathbf{R}^\|\) and \(\mathbf{T}^\|\) as

\[
S^\| = \begin{pmatrix} \mathbf{R}^\| & \mathbf{T}^\| \\ \mathbf{T}^\| & \mathbf{R}^\| \end{pmatrix},
\]

(4.135)
in which the reflection and transmission matrices of the layer are defined as

\[ R^\parallel = r^\circ + t^\circ \Theta r^i \left[ I - \Theta r^i \Theta r^i \right]^{-1} \Theta t^i, \quad (4.136a) \]

\[ T^\parallel = t^\circ \left[ I - \Theta r^i \Theta r^i \right]^{-1} \Theta t^i. \quad (4.136b) \]

Alternatively, we could have obtained the symmetry property of Eq. (4.135) by recognising that we can evaluate

\[ \Theta - \Theta r^i \Theta r^j \Theta = \Theta \left[ I - r^i \Theta r^j \Theta \right] = \left[ I - r^i \Theta r^j \Theta \right] \Theta, \quad (4.137) \]

which leads to the equivalence

\[ \left[ I - r^i \Theta r^j \Theta \right]^{-1} \Theta = \Theta \left[ I - r^i \Theta r^j \Theta \right]^{-1}, \quad (4.138) \]

from which we can understand the redundancy of Eqs. (4.133) and (4.134). For notational convenience we define the the *reverberation* denominator denoted with \( \Delta \) as

\[ \Delta = \left[ I - \Theta r^i \Theta r^i \right]. \quad (4.139) \]

\section*{4.3.8 The response in the embedding}

Both the transmitted and reflected response below the source are solely determined by the downgoing source, since the upgoing source term is radiated into the upper half-space. The downgoing source term is obtained with Eq. (4.33) and (4.48) as

\[ X^\downarrow = (D^{\tau \downarrow})^T F^\nu + (D^{\nu \downarrow})^T F^\tau \]

\[ = \left[ (D^{\tau \downarrow})^T \quad (D^{\nu \downarrow})^T \right] F, \quad (4.140) \]

where on the right-hand side of Eq. (4.140) the three by six dimensional matrix between square brackets transforms the source vector into the downgoing contribution. The responses in terms of the velocity and vertical traction components can be found after after application of the composition matrix in Eq. (4.29) to the response in terms of the wave vector. We introduce the three Green matrix functions \( G^\downarrow \), \( G^\uparrow \) and \( G^\perp \) as the incident, reflected and transmitted wave vector responses due to a unit downgoing wave vector.
source term as
\[ \mathbf{G}^I(x_3^I) = \mathbf{W}(h^S - h^R), \quad (4.141a) \]
\[ \mathbf{G}^\dagger(x_3^I) = \mathbf{W}(h^R) \mathbf{R} \| \mathbf{W}(h^S), \quad (4.141b) \]
\[ \mathbf{G}^\dagger(x_3^I) = \mathbf{W}(h^T) \mathbf{T} \| \mathbf{W}(h^S). \quad (4.141c) \]

Now, for the more general downgoing source of Eq. (4.140) and by using the composition back into the field vector of Eq. (4.29) the field responses, for the direct, reflected and transmitted wave are given by
\[ \mathbf{b}^I(x_3^I) = \begin{bmatrix} \mathbf{D}^v \| \mathbf{D}^r \| \mathbf{G}^I \end{bmatrix} \begin{bmatrix} (\mathbf{D}^{v \leftarrow})^T & (\mathbf{D}^{v \leftarrow})^T \end{bmatrix} \mathbf{F}, \quad (4.142a) \]
\[ \mathbf{b}^R(x_3^I) = \begin{bmatrix} \mathbf{D}^v \| \mathbf{D}^r \| \mathbf{G}^\dagger \end{bmatrix} \begin{bmatrix} (\mathbf{D}^{v \leftarrow})^T & (\mathbf{D}^{v \leftarrow})^T \end{bmatrix} \mathbf{F}, \quad (4.142b) \]
\[ \mathbf{b}(x_3^I) = \begin{bmatrix} \mathbf{D}^v \| \mathbf{D}^r \| \mathbf{G}^\dagger \end{bmatrix} \begin{bmatrix} (\mathbf{D}^{v \leftarrow})^T & (\mathbf{D}^{v \leftarrow})^T \end{bmatrix} \mathbf{F}. \quad (4.142c) \]

In the next section we will discuss the action of the source and receiver which will be representative for our piezo-electric transducers.

### 4.4 The source and receiver model

#### 4.4.1 General description

Having introduced the Green functions for the response in terms of the wave vector at specific levels we will now focus of the effect of the source and receiver transducer on the final measured signal denoted with \( S(x, t) \). Both the source as well as the receiving transducer have a finite aperture and have a specific polarization and directivity sensitivity. The effect of the finite aperture of the source and receiver transducer will act as a double spatial convolution of the spatial and temporal impulse response over their apertures. In the angular-slowness domain this can be expressed in terms of a multiplication of the Green function response with the spatial Fourier transform of the aperture functions. The specific polarization and directivity of the transducers can be conveniently expressed with an inner product as
\[ \tilde{S}(j s \alpha, x_3, s) = \mathbf{H}_J(j s \alpha, s) \mathbf{b}_J(j s \alpha, x_3, s), \quad (4.143) \]
where $\textbf{H}$ describes the receiver response. For the specific case of a receiver above the layer the transducer is directed such that it preferably detects upgoing waves, hence we write $\textbf{H}^\dagger$ for the transducer detecting the reflections. Now, the response of Eq. (4.142b) can be written as

$$
\tilde{S}^R(j\sigma \alpha, x_3^s, s) = (\textbf{H}^\dagger)^T \begin{bmatrix} \textbf{D}^v_{\uparrow\uparrow} \\ \textbf{D}^\tau_{\uparrow\uparrow} \end{bmatrix} \textbf{G}^\dagger \begin{bmatrix} (\textbf{D}^\tau_{\downarrow\downarrow})^T \\ (\textbf{D}^v_{\downarrow\downarrow})^T \end{bmatrix} \textbf{F},
$$

(4.144)

while for the receiver below the layer oriented towards detecting downgoing waves, denoted with $\textbf{H}^\dagger$ we have with Eq. (4.142c)

$$
\tilde{S}(j\sigma \alpha, x_3^s, s) = (\textbf{H}^\dagger)^T \begin{bmatrix} \textbf{D}^v_{\uparrow\downarrow} \\ \textbf{D}^\tau_{\uparrow\downarrow} \end{bmatrix} \textbf{G}^\dagger \begin{bmatrix} (\textbf{D}^\tau_{\downarrow\downarrow})^T \\ (\textbf{D}^v_{\downarrow\downarrow})^T \end{bmatrix} \textbf{F}.
$$

(4.145)

Note that we have omitted the (downgoing) direct response of the transducer $\tilde{S}^I$ of Eq. (4.142a), because we are only interested in the reflections from the layer, expressed in Eq. (4.144).

### 4.4.2 The source model

We will assume that the source in space time domain is determined by the source wavelet $W(t)$, a constant source polarization vector $\textbf{F}^0$ and an aperture function $A(x_1, x_2)$ in the horizontal plane across which we assume a constant source density distribution. Then the source vector $\textbf{F}$ reads

$$
\textbf{F}(j\sigma \alpha, x_3, s) = \textbf{F}^0 \tilde{A}(j\sigma \alpha) \tilde{W}(s) \delta(x_3^s),
$$

(4.146)

where the aperture function $A(x_1, x_2)$ in the spatial domain is defined as

$$
A(x_1, x_2) = \frac{\chi_R(r)}{\pi R^2}, \quad r = (x_1^2 + x_2^2)^{\frac{1}{2}},
$$

(4.147)

and where $\chi_R(r)$ is defined as

$$
\chi_R(r) = \{1, \frac{1}{2}, 0\} \quad \text{when} \quad \{r < R, r = R, r > R\}.
$$

(4.148)

Note that we have normalized the aperture function to unit, when integrated over its surface. For the piezo-electric source we assume a piezo-electric transducer which is impedance matched with the medium on which the transducer is clamped. More explicitly, we assume the ratio between the traction discontinuity in terms of the source vector $\textbf{F}^T$ and the velocity discontinuity in
terms of the source vector $F^v$ is given by the normal compressional or shear impedance, $Z_P$ or $Z_S$ of Eq. (4.63), for a compressional or shear transducer respectively. We split the source polarization vector as

$$F^0 = \begin{pmatrix} F^{0,v} \\ F^{0,\tau} \end{pmatrix}. \quad (4.149)$$

We postulate for the source polarization vector of a compressional source

$$F^{0,v} = (0, 0, 1)^T, \quad (4.150a)$$
$$F^{0,\tau} = Z_P F^{0,v}, \quad (4.150b)$$

while for a shear transducer polarized in the $x_1$ direction we have

$$F^{0,v} = (1, 0, 0)^T, \quad (4.151a)$$
$$F^{0,\tau} = Z_S F^{0,v}. \quad (4.151b)$$

Equations (4.150a)–(4.151b) must be interpreted as the statement that the transducers are optimal in sending a downward propagating plane wave normal to a horizontal plane, although with a limited aperture of operation.

### 4.4.3 The receiver model

In a similar manner the receiver transducers are optimally equipped by impedance matching to measure a down- or upward propagating plane wave depending on the directional orientation of the transducer. The transducer on the source side responds optimally to an upward propagating wave, while a transducer on the transmission side responds optimal to a downward propagating wave. The receiver response is described in terms of the receiver polarization vector $H^0$, and the aperture function of Eq. (4.147) as

$$H(js\alpha, s) = H^0 \hat{A}(j s \alpha), \quad (4.152)$$

We split the receiver response $H^0$ as

$$H^0 = \begin{pmatrix} H^{0,v} \\ H^{0,\tau} \end{pmatrix}. \quad (4.153)$$

The receiver response is taken in such a manner that if the source and receiver transducer would have an infinitely large aperture the response would be unit.
For the receiver response vector $\mathbf{H}^0$ of a compressional transducer directed upwards to measure mainly downgoing waves, we postulate

$$\mathbf{H}^{0,v} = \frac{1}{2}(0,0,1)^T,$$  \hspace{1cm} (4.154a)

$$\mathbf{H}^{0,r} = Z_P^{-1} \mathbf{H}^{0,v}. \hspace{1cm} (4.154b)$$

Namely, the innerproduct of the field vector for a downgoing plane wave with unit velocity amplitude with such a receiver sensitivity would give

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} 0 \\ 0 \\ \frac{1}{Z_P} \\ 0 \\ 0 \end{pmatrix} = 1,$$ \hspace{1cm} (4.155)

while an upgoing plane wave with unit velocity amplitude would give a response

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} 0 \\ 0 \\ \frac{1}{Z_P} \\ 0 \\ 0 \end{pmatrix} = 0.$$ \hspace{1cm} (4.156)

Hence, such a transducer would be insensitive to upgoing waves. For a shear transducer directed upwards and polarized in the $x_1$ direction, similar reasoning leads to

$$\mathbf{H}^{0,v} = \frac{1}{2}(1,0,0)^T,$$  \hspace{1cm} (4.157a)

$$\mathbf{H}^{0,r} = Z_S^{-1} \mathbf{H}^{v}. \hspace{1cm} (4.157b)$$

When the receiver is located above the layer, and directed towards upgoing reflected waves, we have to correct the traction component of the receiver polarization to reverse the sensitivity, leading to

$$\mathbf{H}^{0,r} = -Z_{P,S}^{-1} \mathbf{H}^{0,v}.$$  \hspace{1cm} (4.158)
Since, the aperture of real transducers are finite, the source transducer will not only generate downgoing plane waves, but generally in all directions. Also the receiver response will respond to waves coming from all directions. But because of the preference of the source and receiver transducer dictated by Eqs. (4.149)–(4.158) the detection will still be optimally adjusted to send and receive down- or upgoing normal plane waves.

4.5 Transition to cylindrical coordinates

4.5.1 The slowness transform of the aperture function

In order to benefit from the cylindrical symmetry of the aperture function, it will be more convenient to calculate the spatial Fourier transform of the aperture function by making a transition to a cylindrical coordinate system, which we define here as

\[
\begin{align*}
    x_1 &= r \cos(\phi - \psi), \\
    x_2 &= r \sin(\phi - \psi).
\end{align*}
\]

In Eq. (4.159a) the angle \( \phi \) indicates the angle between the radial angular wavenumber and its \( x_1 \)-component, i.e.

\[
\begin{align*}
    s_1 &= s_r \cos(\phi), \\
    s_2 &= s_r \sin(\phi).
\end{align*}
\]

For a fixed angular slowness we calculate the spatial integration of the slowness transform of Eq. (2.14) by keeping \( \phi \) fixed and integrating over \( \psi \), indicating the angle between the specific angular radial wave number and the position vector \( \mathbf{x} \), leading to

\[
\tilde{u}(js_\alpha r, \phi, x_3, s) = \int_{r=0}^{\infty} \int_{\psi=0}^{2\pi} \exp\left( + js_\alpha r \cos(\psi) \right) \hat{u}(r, \phi - \psi, x_3) r \, d\psi \, dr,
\]

where we have used the goniometric identity

\[
\cos(\phi) \cos(\phi - \psi) + \sin(\phi) \sin(\phi - \psi) = \cos(\psi).
\]

In the particular case of the radial aperture the function \( \hat{u}(r, \phi - \psi, x_3) \) is a function of \( r \) only, and the integration can be evaluated analytically. For
this purpose and for the continuation we use [cf. ABRAMOWITZ AND STEGUN, 1964],

\[
\int_0^{2\pi} \exp(j\omega_p r \cos(\psi)) \cos(n\psi) d\psi = 2\pi j^n J_n(\omega_p r), \quad (4.163a)
\]

\[
\int_0^{2\pi} \exp(j\omega_p r \cos(\psi)) \sin(n\psi) d\psi = 0. \quad (4.163b)
\]

Combining Eq. (4.163a) with Eq. (4.161) we obtain the slowness transform of the aperture function, defined in Eq. (4.147), as

\[
\tilde{A}(x_3, s\alpha_r) = \frac{2}{R^2} \int_{r=0}^{R} J_0(s\alpha_r r) r \, dr,
\]

\[
= \frac{2J_1(s\alpha_r R)}{s\alpha R}, \quad (4.164)
\]

where in the last step of Eq. (4.164) we have used [cf. ABRAMOWITZ AND STEGUN, 1964]

\[
\int_0^x (x')^{n+1} J_n(x') \, dx' = x^{n+1} J_{n+1}(x). \quad (4.165)
\]

Note, that in the limit of infinitely small aperture, the aperture function in the slowness domain [cf. ABRAMOWITZ AND STEGUN, 1964], becomes

\[
\lim_{s\alpha R \to 0} \frac{2J_1(s\alpha_r R)}{s\alpha R} = 1. \quad (4.166)
\]

The unit function in the slowness domain of Eq. (4.166), corresponds to a delta function in the spatial domain and therefore approaches a point source.

### 4.5.2 The inverse slowness transform in cylindrical coordinates

For the inverse transform we can exploit the symmetry of the medium by introducing the spatial cylindrical polar coordinate transformation as

\[
x_1 = r \cos(\theta), \quad (4.167a)
\]

\[
x_2 = r \sin(\theta). \quad (4.167b)
\]

We define the cylindrical angular slowness domain with respect to a fixed spatial position vector as

\[
s\alpha_1 = s\alpha_r \cos(\psi + \theta), \quad (4.168a)
\]

\[
s\alpha_2 = s\alpha_r \sin(\psi + \theta), \quad (4.168b)
\]
where we have to integrate over $\phi$, the angle of between the radial angular wavenumber and the spatial position vector and $s_{\alpha_r}$, the radial angular wavenumber itself. If we have found the solution of the field vector in the slowness domain we can find the solution in the spatial domain is found after applying the inverse slowness transformation in Eq. (2.15) written in the cylindrical coordinates resulting in

$$
\left(\frac{1}{2\pi}\right)^2 \int_{(s\alpha_r)=0}^{\infty} \int_{\psi=0}^{2\pi} \exp(-js\alpha_r r \cos(\psi)) \hat{u}(js\alpha_r, \psi + \theta, x_3, s)s_{\alpha_r} \, d\psi \, ds\alpha_r
= \chi_D \hat{u}(r, \theta, s).
$$

(4.169)

### 4.5.3 The response in cylindrical coordinates

Complications arise, stemming from the fact that we can only perform the $\psi$ integration of Eq. (4.169) for simple functions $\hat{u}(js\alpha_r, \psi + \theta, x_3, s)$. We need to rewrite our basic scattering formula to cylindrical coordinates. First, we define the radial Fourier slowness, equivalent with Eq. (2.20), as

$$
\zeta_r = j\alpha_r.
$$

(4.170)

Then, introducing the unitary matrix $J^\odot$ as

$$
J^\odot = \begin{pmatrix}
\cos(\psi + \theta) & -\sin(\psi + \theta) & 0 \\
\sin(\psi + \theta) & \cos(\psi + \theta) & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

(4.171)

we can rewrite the composition submatrices in terms of cylindrical coordinates as

$$
\begin{align*}
D_{\nu,\downarrow} &= J^\odot C_{\nu,\downarrow}, \\
D_{\nu,\uparrow} &= J^\odot C_{\nu,\uparrow}, \\
D_{\tau,\downarrow} &= J^\odot C_{\tau,\downarrow}, \\
D_{\tau,\uparrow} &= J^\odot C_{\tau,\uparrow},
\end{align*}
$$

(4.172a-d)
in which the submatrices $\mathbf{C}$, as a function of $\zeta_r$ are given by

$$
\mathbf{C}^{v,\downarrow} = \begin{pmatrix}
\zeta_r & \zeta_r \gamma_S & 0 \\
0 & 0 & \zeta_r \\
\gamma_P & -\zeta_r^2 & 0
\end{pmatrix} \mathbf{N}_v^{-1}, \quad (4.173a)
$$

$$
\mathbf{C}^{\tau,\downarrow} = \begin{pmatrix}
2\mu \zeta_r \gamma_P & 2\mu \zeta_r \chi & 0 \\
0 & 0 & \mu \zeta_r \gamma_S \\
2\mu \chi & -2\mu \zeta_r^2 \gamma_S & 0
\end{pmatrix} \mathbf{N}_v^{-1}. \quad (4.173b)
$$

Equivalent with Eqs. (4.39) and (4.42) we find the upcoming radial composition submatrices, respectively, as

$$
\mathbf{C}^{v,\uparrow} = +\mathbf{J}^{\uparrow\downarrow} \mathbf{C}^{v,\downarrow} \quad (4.173c)
$$

$$
\mathbf{C}^{\tau,\uparrow} = -\mathbf{J}^{\uparrow\downarrow} \mathbf{C}^{\tau,\downarrow}. \quad (4.173d)
$$

The similarity operator can also be written in terms of cylindrical composition operators according to

$$
(\mathbf{D}^A \otimes \mathbf{D}^B) = (\mathbf{D}^{\tau,A})^T \mathbf{D}^{v,B} + (\mathbf{D}^{v,A})^T \mathbf{D}^{\tau,B}
$$

$$
= (\mathbf{C}^{\tau,A})^T (\mathbf{J}^{\odot})^T \mathbf{J}^{\odot} \mathbf{C}^{v,B} + (\mathbf{C}^{v,A})^T (\mathbf{J}^{\odot})^T \mathbf{J}^{\odot} \mathbf{C}^{\tau,B}
$$

$$
= (\mathbf{C}^{\tau,A})^T \mathbf{C}^{v,B} + (\mathbf{C}^{v,A})^T \mathbf{C}^{\tau,B} \quad (4.174)
$$

$$
= (\mathbf{C}^A \otimes \mathbf{C}^B),
$$

where we have used the fact that the matrix $\mathbf{J}^{\odot}$ is orthogonal. Because of the vanishing components in the radial composition submatrices $\mathbf{C}$, we see that the reflection matrices at interfaces, e.g. Eqs. (4.108) and (4.108) have a block-diagonal structure according to

$$
\mathbf{R} = \begin{pmatrix}
R_{P/P} & R_{P/SV} & 0 \\
R_{SV/P} & R_{SV/SV} & 0 \\
0 & 0 & R_{SH;SH}
\end{pmatrix}. \quad (4.175)
$$

A similar block-diagonal structure is found for the transmission matrices. Since the one-way phase delay operator of Eq. (4.116) is diagonal and any wave vector response is a product of block-diagonal and diagonal matrices any response for the wave vector has a block-diagonal structure, which shows that in layered isotropic media, the scattering of SH-waves is decoupled from the P- and SV-waves.
Finally, we have to rewrite the action of the source and receiving transducers in cylindrical coordinates. Using Eqs. (4.146) and (4.150a)-(4.151b) we can rewrite the last two terms of the response of Eq. (4.144) and (4.145) for both source polarizations as

\[
\begin{bmatrix}
    (D^{\tau,\uparrow})^T & (D^{v,\downarrow})^T
\end{bmatrix} F
= \left[(D^{\tau,\downarrow})^T + Z_{PS}(D^{v,\downarrow})^T\right] F^{0,v} \tilde{A}(js\alpha) \tilde{W}(s) \delta(x_3^3)
= \left[(C^{\tau,\downarrow})^T + Z_{PS}(C^{v,\downarrow})^T\right] (J^\odot)^T F^{0,v} \tilde{A}(js\alpha) \tilde{W}(s) \delta(x_3^3),
\]  
\tag{4.176}
\]

where the last step involved the transformation to cylindrical coordinates. Applying a similar procedure to the first two terms of Eqs. (4.144) by using Eqs. (4.152) and (4.157a)-(4.158) we obtain for a reflection

\[
(H^\uparrow)^T \begin{bmatrix}
    D^{v,\uparrow} \\
    D^{\tau,\uparrow}
\end{bmatrix} = (H^{0,v})^T [D^{v,\uparrow} - Z_{PS}^{-1}D^{\tau,\uparrow}] \tilde{A}(js\alpha)
= (H^{0,v})^T J^\odot [C^{v,\uparrow} - Z_{PS}^{-1}C^{\tau,\uparrow}] \tilde{A}(js\alpha),
\]  
\tag{4.177}
\]

while for a transmission we obtain

\[
(H^\downarrow)^T \begin{bmatrix}
    D^{v,\downarrow} \\
    D^{\tau,\downarrow}
\end{bmatrix} = (H^{0,v})^T [D^{v,\downarrow} + Z_{PS}^{-1}D^{\tau,\downarrow}] \tilde{A}(js\alpha)
= (H^{0,v})^T J^\odot [C^{v,\downarrow} + Z_{PS}^{-1}C^{\tau,\downarrow}] \tilde{A}(js\alpha).
\]  
\tag{4.178}
\]

For the specific case of compressional transducers we have with Eqs. (4.150a), (4.150b), (4.154a) and (4.154b) for the source and receiver terms expressed in Eqs. (4.176)-(4.178), respectively

\[
(J^\odot)^T F^{0,v} = (0, 0, 1)^T, \quad \tag{4.179a}
\]
and

\[
(H^{0,v})^T J^\odot = \frac{1}{2} (0, 0, 1), \quad \tag{4.179b}
\]

while for a shear transducer we have with Eqs. (4.151a), (4.151b), (4.157a) and (4.157b) for the source and receiver terms

\[
(J^\odot)^T F^{0,v} = (\cos(\psi + \theta), -\sin(\psi + \theta), 0)^T, \quad \tag{4.180a}
\]
\[
(H^{0,v})^T J = \frac{1}{2} (\cos(\psi + \theta), -\sin(\psi + \theta), 0). \quad \tag{4.180b}
\]
Next, by combining Eqs. (4.176)–(4.180b), we can summarise the response of the compressional and shear transducers respectively by
\[
\tilde{S}^{PP}(js\alpha_r, \psi, x_3, s) = \frac{1}{2}(0, 0, 1)^T \Xi^{PP}(0, 0, 1)^T \tilde{A}^2(js\alpha_r) \tilde{W}(s), \tag{4.181a}
\]
\[
\tilde{S}^{SS}(js\alpha_r, \psi, x_3, s) = \frac{1}{2}(\cos(\psi + \theta), -\sin(\psi + \theta), 0)^T \Xi^{SS}
\]
\[
(\cos(\psi + \theta), -\sin(\psi + \theta), 0)^T \tilde{A}^2(js\alpha_r) \tilde{W}(s). \tag{4.181b}
\]
where the responses $\Xi$, for the compressional reflected and transmitted and the shear transmitted and reflected respectively are given by
\[
\Xi^{PP,R} = (C^{v,\uparrow} - Z^{-1}_P C^{\tau,\downarrow}) (C^{\tau,\downarrow})^T + Z_P (C^{v,\downarrow})^T, \tag{4.182a}
\]
\[
\Xi^{PP,T} = (C^{v,\uparrow} + Z^{-1}_P C^{\tau,\downarrow}) (C^{\tau,\downarrow})^T + Z_P (C^{v,\downarrow})^T, \tag{4.182b}
\]
\[
\Xi^{SS,R} = (C^{v,\uparrow} - Z^{-1}_S C^{\tau,\downarrow}) (C^{\tau,\downarrow})^T + Z_S (C^{v,\downarrow})^T, \tag{4.182c}
\]
\[
\Xi^{SS,T} = (C^{v,\uparrow} + Z^{-1}_S C^{\tau,\downarrow}) (C^{\tau,\downarrow})^T + Z_P (C^{v,\downarrow})^T. \tag{4.182d}
\]
Because of the special structure of the matrices $C$ and $G$ the matrices $\Xi$ are all described by non-full matrices
\[
\Xi = \begin{pmatrix} \Xi_{11} & 0 & \Xi_{13} \\ 0 & \Xi_{22} & 0 \\ \Xi_{31} & 0 & \Xi_{33} \end{pmatrix}. \tag{4.183}
\]
Hence we can simplify Eqs. (4.181a) and (4.181b) as
\[
\tilde{S}^{PP}(js\alpha_r, \psi, x_3, s) = \frac{1}{2}\Xi^{PP}_{33} \tilde{A}^2(js\alpha_r) \tilde{W}(s), \tag{4.184a}
\]
\[
\tilde{S}^{SS}(js\alpha_r, \psi, x_3, s) = \frac{1}{2}(\cos^2(\psi + \theta) \Xi_{11}^{SS} + \sin^2(\psi + \theta) \Xi_{22}^{SS})
\]
\[
\tilde{A}^2(js\alpha_r) \tilde{W}(s). \tag{4.184b}
\]
In order to evaluate the inverse slowness transform with respect to the integration of $\psi$ we split the terms in Eq. (4.181b) as
\[
\cos^2(\psi + \theta) = \frac{1}{2} + \frac{1}{2}(\cos(2\psi) \cos(2\theta) - \sin(2\psi) \sin(2\theta)), \tag{4.185a}
\]
\[
\sin^2(\psi + \theta) = \frac{1}{2} - \frac{1}{2}(\cos(2\psi) \cos(2\theta) - \sin(2\psi) \sin(2\theta)). \tag{4.185b}
\]
Using Eqs. (4.163a) and (4.163b) we finally obtain the compressional response in the spatial cylindrical domain as
\[
\hat{S}^{PP}(r, x_3, s) = \frac{\tilde{W}(s)}{4\pi} \int_{(s\alpha_r) = 0}^{\infty} J_0(s\alpha_r, r) \Xi^{PP}_{33} \tilde{A}^2(js\alpha_r) s\alpha_r \, ds\alpha_r, \tag{4.186}
\]
while for the shear response we have

\[
\hat{S}^{SS}(r, \theta, x_3, s) = 
\frac{\hat{W}(s)}{4\pi} \int_{(s\alpha_r)=0}^{\infty} J_0(s\alpha_r r) \frac{1}{2} (\Xi_{11}^{SS} + \Xi_{22}^{SS}) \hat{A}^2(js\alpha_r) s\alpha_r \, ds\alpha_r
\]

\[- \cos(2\theta) \frac{\hat{W}(s)}{4\pi} \int_{(s\alpha_r)=0}^{\infty} J_2(s\alpha_r r) \frac{1}{2} (\Xi_{11}^{SS} - \Xi_{22}^{SS}) \hat{A}^2(js\alpha_r) s\alpha_r \, ds\alpha_r, \]

(4.187)

From Eq. (4.186) we see that a compressional transducer contains only a monopole term, whereas the shear response of Eq. (4.187) additionally contains a quadrupole term.

Now we have finalised our theoretical description of wave propagation in layered viscoelastic media, we will use this theory in Chapter 5 to study under which conditions we can replace a thin layer by a replacing boundary condition. This helps us understand the various theoretical assumptions behind the linear-slip model. In Chapter 6 we will present a useful approximation, a convolutional model, which relates the compressional transmission response through a medium containing a layer, to the transmission response without the layer, based on the analytical result of Eq. (4.186). Further on, in that same chapter we treat the case of perpendicular incidence, which was omitted in this chapter.

Equations (4.186) and (4.187) form the starting point of the numerical modelling of wave propagation through thin layers, which is presented in Chapter 7. Furthermore, in Appendix D we investigate the occurrence of poles in the transmission and reflection responses which are associated with guided modes propagating in the horizontal direction. The starting point of this analysis is the characteristic of the response of a thin layer of Eq. (4.139).

Altogether, we have laid down a foundation to study the transmission and reflection through a hydraulic fracture.
Boundary Conditions

When a wavefield propagates across a thick layer, we observe a train of events, separated in time, associated with internal reverberations inside the layer. The time interval between these events provides direct information on the width and the velocity of the layer. When the resolution of the seismic wavefield is too limited to distinguish between the two interfaces at the bounds of a thin layer, the different scattered (reflected or transmitted) events and the direct wave interfere. This interference of different events results in a change of the shape of the wavelet, which can be labelled as an apparent dispersion. The objective of this chapter is to discuss the possibility of a replacement model, or an effective boundary condition, for a thin layer that a priori takes into account the fact that the layer width is small with respect to the seismic wavelength. For a thin layer, it is less clear what information is contained in the apparent dispersion of the wavelet. By a priori taking into account the limited resolution of the wavelet, we can deduce what parameters determine the magnitude of the dispersion and how we could use a dispersion measurement, indirectly to obtain information of the layer. We will show that the use of this a priori condition, naturally limits the validity of such a model to an input signal, which is band-limited in the frequency and slowness domain.

In literature, the standard model for describing the interaction of seismic
waves with fractures is the linear-slip model [see Jones and Whittier, 1967; Schoenberg, 1980; Rokhlin and Wang, 1991]. The linear-slip model assumes that the traction across a fracture is continuous. Related to this assumption, in this model fractures are regarded as being interfaces of vanishing width and hence its use might be troublesome for hydraulic fractures, which have a finite width. The displacement is allowed to jump across the fracture interface proportional to the traction on the interface and the fracture compliance tensor. Originally, the physical motivation behind the linear-slip model stemmed from a low-frequency approximation for a thin low-impedance layer [Jones and Whittier, 1967; Schoenberg, 1980; Rokhlin and Wang, 1991]. However, at first sight, the continuity of the traction across a thin layer and the finite width of a thin layer seem contradictory, and has often led to confusion.

In this chapter we want to question the applicability of the linear-slip model for hydraulic fractures. If the linear-slip model is applicable to hydraulic fractures, then it should at least be a valid replacement model for an idealized hydraulic fracture consisting of a thin layer of viscous, viscoelastic or non-viscous fluid in an elastic embedding.

Our approach to resolve the possible contradictions and to answer the question above, is to derive, systematically, a generalized replacement model that describes a linear relationship between the jump of the elastodynamic wavefield across the layer in terms of the average elastodynamic wavefield at both interfaces. This relationship will be shown to be valid for any layer. By making a succession of approximations, we will find a linear-slip type of condition. Hence, in this manner we will end up with a set of implicit conditions or assumptions behind the linear-slip model, in case this model should be valid for a replacement model of a thin layer.

Furthermore, in doing so, we have constructed a unified framework to characterize thin layers in terms of a generalised jump relationship of the elastodynamic wavefield quantities, which could constitute the basis for modelling schemes for thin structures. In this way we could avoid to look at the complicated internal description of the wavefield, and only look at the response in the external embedding.

### 5.1 Introduction

Fractures are known to disturb the propagation of seismic waves [Pyrak-Nolte et al., 1990; Hsu and Schoenberg, 1993]. The seismic response of a
fracture is conventionally described in terms of the linear-slip model and the related fracture compliance tensor [Jones and Whittier, 1967; Schoenberg, 1980; Rokhlin and Wang, 1991]. The linear-slip model assumes that across the fracture interface the traction \( t_i \) is continuous, while the displacement \( u_i \) is allowed to slip proportional to the traction on the interface. The traction has been defined in Chapter 3, Eq. (3.1), which for layered media reduces to the vertical traction defined in Eq. (4.4). To distinguish the field vector at both sides of a discontinuous interface, we use the superscripts + and −, denoting the field approached from above and below the interface, respectively. Introducing the symbol \([f]\) for the jump in a quantity \( f \) across the fracture, see Figure 5.1

\[
[f] \overset{\text{def}}{=} f^+ - f^-,
\]

then the linear-slip, or displacement discontinuity model, is formulated as

\[
[t_j(x, t)] = 0, \quad (5.2a)
\]
\[
[u_i(x, t)] = Y_{ij}t_j^-(x, t) = Y_{ij}t_j^+(x, t), \quad (5.2b)
\]

where the physical properties of the fracture are described by the fracture compliance tensor denoted as \( Y_{ij} \). However, no clear physical interpretation exists for this fracture compliance tensor. In principle, all fracture parameters like fracture width, interface roughness, fluid content, contact surface, should be contained in the fracture compliance. Note that the linear-slip model is very specific, because it relates the jump of the displacement at a certain position of the fracture interface solely to the traction at the fracture interface, at the same location, hence it is a \textit{local} relationship. Moreover, it relates the jump of the displacement at a certain instant, solely to the traction at the fracture interface, at the same instant, hence it is an \textit{instantaneous} relationship.
Alternatively in Pyrack-Nolte et al. [1990], Yoshioka and Kikuchi [1993] and Nihei et al. [1995] a velocity discontinuity model was suggested, which is a slightly modified version of Eqs. (5.2a) and (5.2b), expressed as

\[
\begin{align*}
[t_j(x, t)] &= 0, \quad (5.3a) \\
[v_i(x, t)] &= Y_{ij}t_j^-(x, t) = Y_{ij}t_j^+(x, t), \quad (5.3b)
\end{align*}
\]

We will show that both the displacement and velocity discontinuity model fit well into a unified framework of a linear-slip model with a frequency dependent fracture compliance. In the time domain, this results in a convolution of the fracture compliance with the interface traction\footnote{From here on we will use the linear-slip model for a general slip relation, whereas the displacement and velocity discontinuity model are used when an instantaneous relationship is found between the displacement or velocity and the traction.}

A hydraulic fracture can, at least in an idealized case, be regarded as a thin fluid-filled layer. This is especially true if the roughness of the fracture and the amplitudes of the wavefield, can be neglected compared to the width of the fracture. If the linear-slip model is valid for any hydraulic fractures, the thin-layer model and the linear-slip model should converge, under the appropriate conditions.

Alternatively, we might ask whether it is possible to derive a generalised jump condition for a more general class of thin layers, of low or high impedance. The objective of this chapter is to present such a generalization and to discuss the interpretation and validity of these jump relations, specifically in relation to the linear-slip model.

### 5.2 The jump-average field relations

In Eq. (5.1) we introduced the jump (or twice the odd part) of a field quantity across a fracture interface. In a similar manner we can define the jump across a fracture having a \textit{finite} width, see Figure 5.2. Additionally, we define an average quantity across the layer, (or the even part) according to

\[
\langle f \rangle \overset{\text{def}}{=} \frac{1}{2}(f^+ + f^-). \quad (5.4)
\]

The construction of a replacement model is most easily performed in the frequency and slowness transformed domain. With the definitions of the jump and average field, we can denote the jump and average of the field vector in
the frequency and slowness transformed domain, defined in Eq. (4.3), as

\[
\begin{align*}
[\mathbf{b}_I] &= \mathbf{b}_I^+ - \mathbf{b}_I^-; \\
\langle \mathbf{b}_I \rangle &= \frac{1}{2}(\mathbf{b}_I^+ + \mathbf{b}_I^-),
\end{align*}
\]  

(5.5a)  
(5.5b)

where the wavefield vector \( \mathbf{b}_I \), consists of all particle velocity and traction components, i.e. \( \mathbf{b} = (+\hat{v}_1, +\hat{v}_2, +\hat{v}_3, -\hat{r}_{13}, -\hat{r}_{23}, -\hat{r}_{33})^T \) as introduced in Eq. (4.3). Conversely, we can express the field vectors at both sides of the layer in terms of the jump and average field according to

\[
\begin{align*}
\mathbf{b}_I^+ &= \langle \mathbf{b}_I \rangle + \frac{1}{2} [\mathbf{b}_I], \\
\mathbf{b}_I^- &= \langle \mathbf{b}_I \rangle - \frac{1}{2} [\mathbf{b}_I].
\end{align*}
\]  

(5.6a)  
(5.6b)

We define the field propagator \( \mathbf{P}^\parallel \) of the layer system for the field vector as the matrix which relates the transformed wavefield at the bottom of the layer in terms of the wavefield at the top of the layer, which can be expressed as

\[
\mathbf{b}_I^+ = \mathbf{P}^\parallel_{IJ} \mathbf{b}_J^-.
\]  

(5.7)

Substitution of Eqs. (5.5a) and (5.5b) into (5.7) enables us to express the jump of the field vector in terms of the average field vector as

\[
[\mathbf{b}] = 2(\mathbf{I} + \mathbf{P}^\parallel)^{-1}(\mathbf{P}^\parallel - \mathbf{I})\langle \mathbf{b} \rangle.
\]  

(5.8)

The field propagator of the thin layer can be calculated from the wave propagator for a homogeneous section, see Eq. (4.112) and Figure 4.5 as

\[
\mathbf{Q}^\parallel(x_3^{\ast -}, x_3^{\ast +}) = \exp(-s\mathbf{A}_j h).
\]  

(5.9)
where \( h \) is the width of the layer. The field propagator is then found with Eq. (4.77) as
\[
\mathbf{P}^\parallel(x_3^e, x_3^t) = \mathbf{D}_f \mathbf{Q}^\parallel(x_3^e, x_3^t) \mathbf{D}_f^{-1},
\]
where the matrix \( \mathbf{D}_f \) denotes the composition matrix in the viscoelastic fluid. Evaluating the matrix term in Eq. (5.8) with the aid of Eq. (5.10) leads to
\[
\left[ \mathbf{I} + \mathbf{P}^\parallel \right]^{-1} \left[ \mathbf{P}^\parallel - \mathbf{I} \right] = \mathbf{D}_f \left( (\mathbf{I} + \mathbf{Q}^\parallel)^{-1} (\mathbf{Q}^\parallel - \mathbf{I}) \right) \mathbf{D}_f^{-1},
\]
leading to the following expression for the jump in the field vector
\[
[\mathbf{b}] = 2\mathbf{D}_f (\mathbf{I} + \mathbf{Q}^\parallel)^{-1} (\mathbf{Q}^\parallel - \mathbf{I}) \mathbf{D}_f^{-1} \langle \mathbf{b} \rangle.
\]
Hence, we have found a relationship between the jump in the field vector in terms of the average field vector. Since the matrix \( \mathbf{Q} \) is frequency and slowness dependent, the general jump-average field relationship is frequency and slowness dependent. In the time-domain this means that we cannot find an instantaneous nor a local relationship between the jump of the field and the average field. The jump of the field will also be related to the average field history and distribution. Note that to this point the analysis has been exact and no low-frequency assumption has been made yet.

### 5.3 Low-frequency approximations

The propagator of the up- and downgoing wave vector contains exponents with the one-way phase delay inside the layer as its argument. In this section we will investigate the consequences of assuming that this phase delay is small, both for the compressional and shear wave. For this purpose we continue our analysis in the real frequency domain, hence \( s \rightarrow j\omega \), and we investigate the wavefield in a band-limited domain in the Fourier-frequency transformed domain. We assume that most of the energy of the wavefield is contained within this band-limited domain. The fact that the layer is thin can then be expressed as
\[
\omega h \gamma_{PS} \ll 1,
\]
since in that case the one-way phase delay is small. Note that we can define an apparent vertical wavelength \( \lambda_3 \) inside the layer, determined by the vertical slowness and the frequency as
\[
\lambda_{3,PS} = (\omega \gamma_{PS})^{-1}.
\]
Hence Eq. (5.13) can also be formulated by assuming that the width \( h \) is small compared to the apparent vertical wavelength, i.e. \( h \ll \lambda_{3,P,S} \). On the other hand, the vertical slowness, and hence the vertical wavelength are determined by the horizontal slowness, see Eqs. (4.17a) and (4.17b) as

\[
\gamma_P = \left( \frac{\rho}{\lambda + 2\mu} - \zeta_\beta \zeta_\beta \right)^{\frac{1}{2}}, \quad \text{Re}(s\gamma_P) \geq 0 \tag{5.15a}
\]

and

\[
\gamma_S = \left( \frac{\rho}{\mu} - \zeta_\beta \zeta_\beta \right)^{\frac{1}{2}}, \quad \text{Re}(s\gamma_P) \geq 0. \tag{5.15b}
\]

As a consequence, the condition of Eq. (5.13) can only be valid in a band-limited slowness domain, since for arbitrary large horizontal slowness the vertical wavelength becomes arbitrary small. Hence for high horizontal slownesses the layer cannot be considered as thin. In practice the region for which the layer can be considered thin might extend well into the evanescent regime, which means that all propagating waves experience the layer as a thin obstruction and even so for a set of evanescent waves. Concluding, we investigate the wavefield in a band-limited domain in the frequency and slowness transformed domain, and assume that most of the energy of the wavefield is contained within this domain.

Several different low-frequency approximations can be constructed. We will limit our discussion to three kind of approximations for the exponential function where its argument, the one-way phase delay, is small. These are the first-order Taylor approximation, indicated with \( T_1(x) \), the second-order Taylor approximation, \( T_2(x) \), and finally a Padé approximation \( P_1^1(x) \). These approximations are formally written for \( x \downarrow 0 \) as

\[
\exp(x) \approx \begin{cases} 
T_1(x) = 1 + x + O(x^2), & \text{as } x \to 0 \tag{5.16a} \\
T_2(x) = 1 + x - \frac{1}{2}x^2 + O(x^3), & \text{as } x \to 0 \tag{5.16b} \\
P_1^1(x) = (1 - \frac{1}{2}x)^{-1}(1 + \frac{1}{2}x) + O(x^3), & \text{as } x \to 0. \tag{5.16c}
\end{cases}
\]

A similar definition applies when the argument of the exponential function is complex, in case of propagating waves. The Padé approximation is a rational function approximation, which is correct up to and including second order. For a discussion on Padé approximations we refer to Baker and Graves-Morris [1981].
Figure 5.3: The exponential function, its first and second Taylor approximations, $T_1(x)$ and $T_2(x)$ respectively and the Padé approximation $P^1_1(x)$, for a): the modulus of the complex exponential function, b): the phase $\theta$ of the complex exponential function, c): the exponentially damping function and d): the exponentially growing function.

In Figure (5.3) we show the exponential function, its first and second Taylor approximation and its Padé approximation both for real and complex arguments.
In Figure 5.3(a) and (b) we see that the Padé approximation is accurate, especially in the modulus and phase of the complex exponential function. The
modulus and phase are important for a correct estimation of the amplitude and delay of the propagating wave. Additionally, the Padé approximation does not diverge for large arguments, since its norm at infinity is always unity. From these observations we conclude that the Padé approximation is the most accurate and stable approximation, especially for the phase and modulus of the complex exponent. We will now show that the use of the Padé approximation leads to an elegant result for the resulting jump-average field relationship. From this point we limit ourselves to investigating the consequences of taking the Padé approximation for the exponential function.

### 5.4 The Padé approximation

The up- and downgoing wave vector propagator, in the jump-average field relationship in Eq. (5.12) is a simple diagonal matrix and each exponential term can be expressed as a Padé approximation. This procedure leads to a matrix Padé approximation of Eq. (5.9) corresponding to Eq. (5.16c) for a scalar expansion, which can be expressed as

\[
Q^\parallel = (I + \frac{1}{2} j \omega h \Lambda_f)^{-1} (I - \frac{1}{2} j \omega h \Lambda_f) + O(\omega h \Lambda_f)^3. \tag{5.17}
\]

This equation is valid up to and including second order. Collecting the important terms of Eq. (5.12) and using the Padé approximation of Eq. (5.17) we arrive at

\[
(I + Q^\parallel) = 2(I + \frac{1}{2} j \omega h \Lambda_f)^{-1}, \tag{5.18a}
\]

\[
(Q^\parallel - I) = -j \omega h (I + \frac{1}{2} j \omega h \Lambda_f)^{-1} \Lambda_f, \tag{5.18b}
\]

and hence we obtain, by combining Eqs. (5.18a) and (5.18b) the result

\[
(I + Q^\parallel)^{-1}(Q^\parallel - I) = -\frac{1}{2} j \omega h \Lambda_f. \tag{5.19}
\]

Combination of Eqs. (5.19) and (5.12) results in

\[
[b] = -j \omega h D_f \Lambda_f D_f^{-1} \langle b \rangle, \tag{5.20}
\]

which reduces finally, by using Eq. (4.24) to

\[
[b] = -j \omega h A_f \langle b \rangle. \tag{5.21}
\]
This is the equation that we are looking for, because it shows in its essence
the discretization of the system equation of Eq. (4.2) for a source-free domain,
expressed as

$$
\partial_3 b_I = -s A_{IJ} b_J.
$$

(5.22)

Another approach to derive Eq. (5.21) is to evaluate the integral of the system
equation over the width of the layer. Integration over the width $h$ of the layer
of the source-free system equation of Eq. (5.22) is expressed as

$$
\int_0^h \partial_3 b_I \, dx_3 = -s A_{IJ} \int_0^h b_J \, dx_3,
$$

(5.23)

where we have used the fact that the system matrix $A_{IJ}$ is constant in the
layer. Since the field vector $b_J$ is continuous, we can directly integrate the
term on the left-hand side of Eq. (5.23) and use the mean-value integral
theorem [RUDIN, 1953] for evaluating the term on the right-hand side, to
arrive at

$$
[b] = -s A_f b(x_3^{(0)}),
$$

(5.24)

for a certain depth level $x_3^{(0)}$ between the top and the bottom of the layer.
The Padé approximation show that when we take the mean value of the field
vector defined as

$$
b(x_3^{(0)}) = \frac{1}{2} (b^+ + b^-),
$$

(5.25)

this discretization is correct up to and including second order in the layer
thickness. The Padé approximation corresponds to using central difference
approximations and is related to the Crank-Nicholson scheme for construct-
ing stable implicit finite-difference approximations of differential equations
[RICHTMYER AND MORTON, 1957; VAN STRALEN, 1997]. Note that this ap-
proximation formally contains multiple scattering, i.e reflection interactions,
but only up to and including second order in the one-way phase delay.
Now we have found a generalized slip model, in which we have relaxed the
requirement of the continuity of the traction. We can see that generally both
the particle velocity and the traction jump by a finite amount, determined
by the system matrix $A_f$ associated with the fluid, the layer width and the
frequency.
Since the system matrix contain slowness terms up to and including second
order, we have found a relationship which prescribes that the jump in a field
quantity also contains a dependency on horizontal derivatives of the average field quantities up to and including second order. The complexity of this relationship exceeds a linear-slip model. Its applicability is not our present concern, but it clearly illustrates that generally speaking we cannot replace a thin layer by a linear-slip model. The formulation in Eq. (5.21) can be regarded as a symmetrical formulation of a linear-slip type of model. The operator that prescribes how the jump of the field vector is related to the average field vector is completely \textit{independent} on the embedding properties. Since the incident wavefield and the connected average field vector is dependent on the properties of the embedding and source excitation, the jump of the field vector itself will be dependent on both the embedding, layer and source parameters.

5.5 \textbf{Comparison with the linear-slip model}

If the linear-slip model defines a valid replacement model for an idealized fracture consisting of a thin fluid-layer, then we should need additional constraints to explain the difference between the linear-slip model of Eq. (5.2a)–(5.2b) and the generalized jump-average relationship defined in Eq. (5.21). Often fractures can be regarded as local weaknesses which in the thin-layer model is related to the low impedance or velocity of the fluid layer. Suppose that the compressional and shear velocities in the viscoelastic fluid \textit{both} are well below the shear (and hence compressional) velocity of the embedding. This can be expressed as

\begin{equation}
\hat{c}_{P,S,f} \ll \hat{c}_{S,e} \tag{5.26}
\end{equation}

and suppose we limit the discussion to propagating waves incident on the thin layer and exclude any evanescent waves. Then the horizontal Fourier-slowness is limited by

\begin{equation}
\zeta < \hat{c}_{S,e}^{-1} \ll \hat{c}_{P,S,f}^{-1}, \tag{5.27}
\end{equation}

which can be reduced to

\begin{equation}
\zeta \hat{c}_{P,S,f} \ll 1. \tag{5.28}
\end{equation}

This equation must be interpreted as an additional assumption of band-limitation in the slowness domain. Based on Snell's law, rays tend to bend
to the normal because of the low velocity of the fluid layer. The refracted angle in the fluid is given by

$$\theta = \arcsin(\xi \hat{c}_{P,S,f}) \approx 0. \quad (5.29)$$

Neglecting all first- and higher-order terms in the horizontal slowness we find that the self-coupling matrices, respectively $A^{uv}$ and $A^{\tau\tau}$, vanish, and that the cross-coupling matrices $A^{\tau v}$ and $A^{v\tau}$ become diagonal matrices containing the coefficients

$$A^{v\tau} = \begin{pmatrix}
\frac{1}{\mu} & 0 & 0 \\
0 & \frac{1}{\mu} & 0 \\
0 & 0 & \frac{1}{\lambda + 2\mu}
\end{pmatrix}, \quad (5.30a)$$

$$A^{\tau v} = \begin{pmatrix}
\rho & 0 & 0 \\
0 & \rho & 0 \\
0 & 0 & \rho
\end{pmatrix}, \quad (5.30b)$$

which in terms of the vertical traction of Eq. (4.4) leads to the simplified relations

$$[\hat{v}_i] = j\omega h A^{v\tau}_{ij} \langle \hat{i}_j \rangle, \quad (5.31a)$$

$$[\hat{t}_i] = j\omega h A^{\tau v}_{ij} \langle \hat{v}_j \rangle. \quad (5.31b)$$

Note that we have replaced the slowness transformed wave field quantities by the frequency transformed wave field quantities, because we have applied a zero-slowness approximation. In doing so, we neglect any spatial dependence and obtain a local jump-average field relationship. These two equations can be interpreted as the discrete approximations of the equation of conservation of linear momentum and the linearized deformation equation, for a ray which is refracted towards the normal inside the layer.

To assess the relative importance of the jump in the traction of Eq. (5.31b) to the jump in the particle velocity of Eq. (5.31b), we introduce a characteristic velocity amplitude $V$. Then the dimensionless particle velocities $\hat{v}_\beta^*$ and $\hat{v}_3^*$ can be written as

$$\hat{v}_\beta^* = \frac{\hat{v}_\beta}{V}, \quad (5.32a)$$

$$\hat{v}_3^* = \frac{\hat{v}_3}{V}. \quad (5.32b)$$
Since the ratio of the traction to the velocity for normal incidence is given by the normal impedance $Z_{P,S}$ of Eq. (4.63), we scale the traction according to

$$\hat{t}_3^* = \frac{\tau_3}{Z_{P,e}V}, \quad (5.32c)$$

$$\hat{t}_9^* = \frac{\tau_9}{Z_{S,e}V}. \quad (5.32d)$$

Additionally, we introduce the perpendicular one-way phase delay for the compressional and shear waves, $\theta_P$ and $\theta_S$ respectively, according to

$$\theta_P = \frac{\omega h}{\hat{c}_{P,f}}, \quad (5.33a)$$

$$\theta_S = \frac{\omega h}{\hat{c}_{S,f}}, \quad (5.33b)$$

$$z_{r,P} = \frac{Z_{P,f}}{Z_{P,e}} = \frac{\rho_f \hat{c}_{P,f}}{\rho_s \hat{c}_{P,e}}, \quad (5.33c)$$

$$z_{r,S} = \frac{Z_{S,f}}{Z_{S,e}} = \frac{\rho_f \hat{c}_{S,f}}{\rho_s \hat{c}_{S,e}}, \quad (5.33d)$$

which can be regarded as two dimensionless measures for the fracture width and the compressional and shear impedance ratio. The jump relations in Eqs. (5.31a) and (5.31b) in terms of dimensionless wavefields are then given by

$$[\hat{v}_i^*] = j \begin{pmatrix} 0 & 0 & \theta_S z_{r,S}^{-1} & 0 \\ 0 & 0 & \theta_S z_{r,S}^{-1} & 0 \\ 0 & 0 & 0 & \theta_P z_{r,P}^{-1} \end{pmatrix} \langle \hat{t}_j^* \rangle, \quad (5.34a)$$

$$[\hat{t}_i^*] = j \begin{pmatrix} 0 & \theta_S z_{r,S} & 0 \\ 0 & \theta_S z_{r,S} & 0 \\ 0 & 0 & \theta_P z_{r,P} \end{pmatrix} \langle \hat{v}_j^* \rangle. \quad (5.34b)$$

This means that the ratio between the one-way phase delay, a dimensionless measure for the width of the the layer, and the impedance ratio, in Eq. (5.34a) determines the dimensionless jump in the particle velocities. Likewise, the product of the one-way phase-delay and the impedance ratio, in Eq. (5.34b), determines the jump in the traction. Hence, the ratios $\theta_P z_{r,P}^{-1}$ and $\theta_S z_{r,S}^{-1}$ can be regarded as a dimensionless normal and transversal fracture compliance.
and the products $\theta_p z_{r,p}$ and $\theta_s z_{r,s}$ in a similar way as the dimensionless layer density. Usually, the fracture is thought to represent a local weakness; the (viscous) fluid-solid impedance ratio for a hydraulic fracture is normally quite small. In that case the jump in particle velocity is more pronounced than the jump in the traction, leading to the restricted model

$$\left[ \dot{\nu}_i \right] = j \omega h \left( \begin{array}{ccc} \frac{1}{\mu} & 0 & 0 \\ 0 & \frac{1}{\mu} & 0 \\ 0 & 0 & \frac{1}{\lambda + 2\mu} \end{array} \right) \left[ \dot{t}_j \right],$$  \hfill (5.35a)

$$\left[ t_i \right] = 0.$$  \hfill (5.35b)

By applying an inverse Fourier transforming to this equation we obtain a linear-slip model with a convolutional structure,

$$\left[ u_i \right](x, t) = h \left( \begin{array}{ccc} \frac{1}{\mu(x,t)} & 0 & 0 \\ 0 & \frac{1}{\mu(x,t)} & 0 \\ 0 & 0 & \frac{1}{\lambda(x,t) + 2\mu(x,t)} \end{array} \right) * \left[ t_j \right](x, t),$$  \hfill (5.36a)

$$\left[ t_i \right](x, t) = 0.$$  \hfill (5.36b)

where the symbol $*$ denotes a convolution in the time domain. Note that the linear-slip model is an instantaneous relationship between the jump in the particle displacement and the (average) traction, as long as the Lamé’s parameters of the layer are perfectly elastic, hence instantaneous. Because we focus on a viscoelastic fluid for the moment, the linear-slip model will have to be modified with the convolutional structure as in Eq. 5.36a. Equation (5.36a) corresponds to the (transversely-isotropic) linear-slip boundary conditions, when the frequency dependent fracture compliance is associated with the layer width and the Lamé’s parameters according to

$$Y = \left( \begin{array}{ccc} \hat{Y}_T & 0 & 0 \\ 0 & \hat{Y}_T & 0 \\ 0 & 0 & \hat{Y}_N \end{array} \right)$$  \hfill (5.37)

$$= h \left( \begin{array}{ccc} \frac{1}{\mu} & 0 & 0 \\ 0 & \frac{1}{\mu} & 0 \\ 0 & 0 & \frac{1}{\lambda + 2\mu} \end{array} \right).$$
where $\hat{Y}_N$ and $\hat{Y}_T$, are the normal and transversal fracture compliances, respectively. Additionally, when the shear deformation responds Newtonian-viscous, see Eq. (3.37) and (3.39), the shear modulus becomes imaginary, according to
\[
\hat{\mu}(x, j\omega) = j\omega\eta_0(x),
\] (5.38)
which results in a jump in the horizontal particle velocities, expressed as
\[
[u_\beta](x, t) = \frac{h}{\eta_0} (t_\beta)(x, t),
\] (5.39)
\[
[t_i](x, t) = 0.
\] (5.40)
Hence, we conclude that a velocity discontinuity model is more appropriate if the response of the fluid is viscous. The requirement that the layer must be thin, as expressed in Eq. (5.13), means that the width of the layer must be smaller than the viscous skin depth of Eq. (4.123). The modified transversal fracture compliance is then related to the width of the layer and the viscosity according to
\[
Y'_T = \frac{h}{\eta_0}.
\] (5.41)
Summarizing, the thin-layer model reduces to a linear-slip model in case we consider a band-limited domain in frequency and slowness, in the special case that the dimensionless width and dimensionless mass density of the layer are small. Since for a low-velocity layer rays bend towards the normal, approximately the normal jump-average relationship can be considered for a range of incident angles. This approximation leads to a local approximation. We have avoided the introduction of a new physical quantity. Note that even in the very simple case of an infinite flat fluid layer, the use of the linear-slip model might be too restricted for all fractures since it is an instantaneous and local jump-average relationship. Hence, care should be taken in its automatic use.

### 5.6 Equivalent model for an ideal fluid

In a similar way of reasoning we can derive a jump-average relationship for an ideal fluid. We can define the field vector, see Appendix C, as
\[
b_\beta = \begin{pmatrix} +\bar{v}_3 \\ +\bar{p} \end{pmatrix}.
\] (5.42)
With similar arguments and derivations as in previous section we find a two-by-two jump-average equation for the two-dimensional field vector in Eq. (5.42) as

\[ [b_{\alpha}] = -j\omega h A_{\alpha\beta} \langle b_{\beta} \rangle, \]

(5.43)

where the system matrix for the fluid is given by

\[
A_{\alpha\beta} = \begin{pmatrix}
0 & \rho_f^{-1}(\rho_f \kappa_f - \zeta_\alpha \zeta_\alpha) \\
\rho_f & 0
\end{pmatrix},
\]

(5.44)

These equations must be supplemented by the explicit boundary condition at both fluid-solid interfaces, that the shear stresses in the solid should vanish at these interfaces, which can be expressed as

\[ \tau_{\beta\beta} = 0. \]

(5.45)

For a low-impedance layer an approximate local relationship can be derived, in the zero-angle refracted ray approximation, similar to Eqs. (5.31a) and (5.31b, which results in

\[
[u_3](x, t) = -h\kappa \langle p \rangle(x, t),
\]

(5.46a)

\[
[p](x, t) = 0,
\]

(5.46b)

\[
\tau_{\beta\beta}(x, t) = 0,
\]

(5.46c)

which defines an instantaneous relationship for the jump in the vertical particle displacement, in terms of the average pressure. Hence, we could relate a frequency independent normal compliance of a ideal fluid-filled layer as

\[ Y_N = h\kappa. \]

(5.47)

But the standard linear-slip model as in Eq. (5.2a) does not correctly incorporate the explicit boundary condition that the shear stresses vanish at the fluid-solid interface as in Eq. (5.46c). Therefore we have found an additional limitation of the linear-slip model. Normally, for fluid-filled fractures an infinitely large shear fracture compliance is assumed, which tries to incorporate this explicit boundary condition in an implicit manner.
5.7 Conclusions

We have shown that for any layer we can derive a linear relationship between the jump in the field vector and the average field vector. In the low-frequency Padé approximation this linear relationship is determined by the layer thickness and the system matrix that appears in the frequency slowness reduced wave equation, see Eq. (4.2). The Padé approximation is a stable approximation for the phase operator of a thin layer, even in the evanescent regime and valid till second order in the small argument. When neglecting first and higher order terms in the horizontal slowness, valid for a band-limited slowness domain, the jump-average relationship becomes local. For a low density layer with elastic layer compliances, we retrieve the displacement discontinuity model which gives an instantaneous relationship between the jump in the particle displacement and the (average) traction on the fracture interface. This shows that the fracture compliance in this idealized model is determined by the layer thickness and the inverse of the longitudinal and shear moduli. For purely viscous layer parameters we find the velocity discontinuity model, which describes a linear and instantaneous relationship between the jump in the particle velocity and the (average) traction on the fracture interface, for which the modified fracture compliance is determined by the ratio of the fracture width and the viscosity. In case of an ideal fluid-filled layer the linear-slip model does not correctly incorporate the explicit boundary condition of vanishing stresses.

5.8 Transmission and reflection

In this section we will derive explicit expressions for the transmission and reflection coefficient through a linear-slip interface. Suppose we have a certain jump-average field relationship, in this section we show how to calculate the scattering matrix of such a system, in terms of the reflection and transmission matrices. We start with a jump-average field relationship

\[ [b] = -j \omega h A_f' \langle b \rangle, \]

(5.48)

where the matrix \( A_f' \) denotes some useful approximation (or exact expression), which relates the jump to the average field. We can derive an expression for the propagator matrix, according to Eq. (4.64), of the field vector for such a system by rearranging the terms in Eq. (5.48) as

\[ b^+ = (I + \frac{1}{2} j \omega h A_f')^{-1} (I - \frac{1}{2} j \omega h A_f') b^- . \]

(5.49)
In case the matrix \( A_f' \) corresponds to the system matrix \( A_f \), Eq. (5.49) can be interpreted as the Padé approximation for the propagator of the field vector \( b \).

By using Eq. (4.77) we can also derive an expression for the wave propagator of the layer system for the wave vector \( y \), resulting in

\[
Q(x_3^{Z_+}, x_3^{Z_-}) = D_e^{-1} (I + \frac{1}{2} j \omega h A_f'^{v})^{-1} (I - \frac{1}{2} j \omega h A_f'^{v}) D_e. \tag{5.50}
\]

With Eq. (4.85) we can then calculate the transmission and reflection matrices. In the static limit the wave propagator will reduce to the identity operator. The transmission operators will act as low-pass filters, causing a dispersion of the wavelet.

In the special case of the linear-slip model the matrix \( A_f' \) is given by the components

\[
A_f'^{vv} = 0, \tag{5.51a}
\]

\[
A_f'^{v\tau} = \begin{pmatrix}
\frac{1}{\mu} & 0 & 0 \\
0 & \frac{1}{\mu} & 0 \\
0 & 0 & \frac{1}{\lambda + 2\mu}
\end{pmatrix}, \tag{5.51b}
\]

\[
A_f'^{\tau v} = 0, \tag{5.51c}
\]

\[
A_f'^{\tau\tau} = 0. \tag{5.51d}
\]

In order to construct the reflection and transmission matrices for the linear-slip model, using Eq. (5.51a)–Eq. (5.51d) we have

\[
(I + \frac{1}{2} j \omega h A_f'^{v})^{-1} = \begin{pmatrix}
I & \frac{1}{2} j \omega h A_f'^{v} \\
0 & I
\end{pmatrix}^{-1} = \begin{pmatrix}
I & -\frac{1}{2} j \omega h A_f'^{v} \\
0 & I
\end{pmatrix}. \tag{5.52}
\]

Evaluating the middle term in Eq. (5.50) we find

\[
(I + \frac{1}{2} j \omega h A_f'^{v})(I - \frac{1}{2} j \omega h A_f'^{v}) = \begin{pmatrix}
I & -j \omega h A_f'^{v} \\
0 & I
\end{pmatrix}. \tag{5.53}
\]

Next, using the subdivision of the composition and decomposition matrix as given in Eqs. (4.34) and (4.48) respectively, we obtain by evaluating the
right-hand side of Eq. (5.50)

\[
Q(x_{3}^{\uparrow}, x_{3}^{\downarrow}) = \begin{pmatrix}
I + j\omega h(D_{e}^{\uparrow \uparrow})^T A_{f}^{\uparrow \uparrow} D_{e}^{\uparrow \uparrow} & j\omega h(D_{e}^{\uparrow \downarrow})^T A_{f}^{\uparrow \downarrow} D_{e}^{\uparrow \downarrow} \\
-j\omega h(D_{e}^{\uparrow \downarrow})^T A_{f}^{\uparrow \uparrow} D_{e}^{\uparrow \downarrow} & I - j\omega h(D_{e}^{\uparrow \downarrow})^T A_{f}^{\uparrow \downarrow} D_{e}^{\uparrow \downarrow}
\end{pmatrix}.
\] (5.54)

Constructing the upgoing transmission matrix \( T_{\uparrow} \) and the up- to downgoing reflection matrix \( R_{\downarrow} \) by using Eq. (4.85) we obtain

\[
T_{\uparrow} = (I + j\omega h(D_{e}^{\uparrow \uparrow})^T A_{f}^{\uparrow \uparrow} D_{e}^{\uparrow \uparrow})^{-1},
\] (5.55a)

\[
R_{\downarrow} = -j\omega h(D_{e}^{\uparrow \downarrow})^T A_{f}^{\uparrow \uparrow} D_{e}^{\uparrow \downarrow}(I + j\omega h(D_{e}^{\uparrow \uparrow})^T A_{f}^{\uparrow \uparrow} D_{e}^{\uparrow \downarrow})^{-1}.
\] (5.55b)

Since we have a configuration which is invariant for a reflection in the vertical direction we have also find the downgoing transmission matrix and the down- to upgoing reflection matrix. When we associate the system matrix \( A_{f}^{\uparrow \uparrow} \) and the layer width \( h \) with the fracture compliance as in Eq. (5.37) we obtain the transmission and reflection matrices

\[
T_{\downarrow} = (I + j\omega(D_{e}^{\uparrow \downarrow})^T Y D_{e}^{\uparrow \downarrow})^{-1},
\] (5.56a)

\[
R_{\downarrow} = -j\omega(D_{e}^{\uparrow \downarrow})^T Y D_{e}^{\uparrow \downarrow}(I + j\omega(D_{e}^{\uparrow \downarrow})^T Y D_{e}^{\uparrow \downarrow})^{-1}.
\] (5.56b)

Since the relationship between the composition matrix in the Cartesian and cylindrical coordinate system is given by Eqs. (4.172b)–(4.172c) as

\[
D^{\uparrow \uparrow} = J^{\Downarrow} C^{\uparrow \uparrow} = -J^{\Downarrow} J^{\uparrow \downarrow} C^{\uparrow \downarrow},
\] (5.57a)

\[
D^{\uparrow \downarrow} = J^{\Downarrow} C^{\uparrow \downarrow}.
\] (5.57b)

Additionally, by using the fact that the fracture compliance tensor is a diagonal tensor we have

\[
(J^{\uparrow \downarrow})^T (J^{\Downarrow})^T Y J^{\Downarrow} J^{\uparrow \downarrow} = Y,
\] (5.58)

and therefore we finally arrive at the transmission and reflection coefficients in the desired form as

\[
T_{\downarrow} = (I + j\omega(C_{e}^{\uparrow \downarrow})^T Y C_{e}^{\uparrow \downarrow})^{-1},
\] (5.59a)

\[
R_{\downarrow} = +j\omega(C_{e}^{\uparrow \downarrow})^T Y J^{\uparrow \downarrow} C_{e}^{\uparrow \downarrow}(I + j\omega(C_{e}^{\uparrow \downarrow})^T Y C_{e}^{\uparrow \downarrow})^{-1}.
\] (5.59b)

We can interpret the transmission matrix as a three-dimensional generalization of a single-pole filter, whereas the reflection response contains an
additional factor $j\omega$, corresponding with the derivative in time. In the static limit the transmission operators reduce to the unit operator and no reflection occurs. For higher frequencies, the transmission operators act as low-pass filters, causing a dispersion of the wavelet. Full expressions for all transmission and reflection coefficients can be found in Gu et al. [1996].

In this chapter we have discussed under which assumptions a thin layer reduces to a linear-slip interface. We now also understand that when we compare perpendicular transmissions through a thin fluid-filled layer of low-impedance with a transmission through a linear-slip interface, we can expect that these responses are indistinguishable. Because we have derived an expression, see Eq. (5.59a) and (5.59b) for the transmission and reflection matrices of a linear-slip interface, we can use these equation in Chapter 7 to model the transmission through a linear-slip interface. Thereby we can numerically compare the response of a thin layer with the response of a linear-slip interface and show the correspondences and differences in the response for more complicated measurement configurations than a perpendicular transmission of a plane wave.
In this chapter we present two approximations which will be of practical use later on in Chapter 11 when we will estimate the width of the fracture by using compressional transmission measurements. First, the convolutional model and second, the special case of perpendicular incidence.

For the width determination we want to use the dispersion of the compressional transmissions through the fracture, relative to the transmission measurement before fracturing. The convolutional model, provides us with a fast way to predict the amount of dispersion that we observe. Instead of having to calculate a full slowness inversion, we claim that the transmission measurement through the fracture can be predicted by convolving the transmission measurement before fracturing with the compressional transmission coefficient of a thin layer for the angle of incidence (or slowness) which connects the source and receiver transducers. The advantage of this convolutional approximation is that the width inversion is independent on the source and receiver parameters and the exact location of the fracture. Furthermore, the forward model is much faster than the full slowness inversion that is needed for an exact modelling of the waveform and hence the inversion to the width parameter can be carried out relatively fast and easy.

The most important transmission measurements that we will use in Chapter 11 use a source and receiver transducer for which the ray that connects
the source and receiver transducer is perpendicular to the layer. In Chapter 4 we omitted the case of a plane wave propagating perpendicular to the layering, because the chosen normalization matrix became singular for that special case. Here, we will treat the transmission and reflection of a perpendicular incident plane wave on a thin layer and linear-slip interface in more detail.

6.1 Motivation of the convolutional model

In the homogeneous solid embedding the propagation of compressional and shear waves is decoupled, each having a distinct propagation velocity. When we measure reflections or transmissions from the fracture interface, we can observe a set of arrivals. This is related to the fact that the travel times of the compressional and shear waves in the solid embedding differ substantially. Theoretically, the observed signal consists of events which has propagated through the embedding as compressional or as shear waves. Another possibility is that an incident compressional wave has been mode-converted at the fracture to a shear wave in the embedding or vice-versa. In the fracture itself the propagation of the compressional and shear waves is highly coupled, because of the small phase shift associated with the small width of the fracture and the large impedance contrast and hence must be treated accordingly. On the other hand, in the interpretation of our measurements it is sensible to separate the events which have propagated as compressional waves across the solid embedding from the set of shear waves and the mode-conversions, since they will arrive separated in time. In the experiments the effect of the intervening fracture for the compressional transmissions is an apparent dispersion, relative to the measurement without fracture. In the following section we derive a model which describes this apparent dispersion relative to the measurement without fracture in a simple manner.

6.2 The convolutional model

In Chapter 4 we derived the expression, see Eq. (4.145), for the measurement of the transmitted events in the slowness and Laplace transformed domain,
repeated here as

$$\tilde{S}(j s \alpha, x_3, s) = (\mathbf{H}^T)^T \left[ \begin{array}{c} \mathbf{D}^\nu \mid \downarrow \\ \mathbf{D}^\tau \mid \downarrow \end{array} \right] \mathbf{G} \left[ \begin{array}{c} (\mathbf{D}^\tau \mid \downarrow)^T \\ (\mathbf{D}^\nu \mid \downarrow)^T \end{array} \right] \mathbf{F}, \quad (6.1)$$

where the last two terms on the right-hand side of this equation, describe the source excitation in terms of amplitudes of downward propagating wave modes, the first two terms the receiver response as a result of arbitrary downgoing wave modes and the downgoing Green function, \( \mathbf{G} \mid \downarrow \) is given by Eq. (4.141c) as

$$\mathbf{G} \mid \downarrow (x_3) = \mathbf{W}(h^T)\mathbf{T}_\parallel \mathbf{W}(h^S). \quad (6.2)$$

This Green function describes the propagation of the wavemodes from the source through the embedding and layer to the receiver. When both the distance of the source to the layer \( h^S \) and the distance of the bottom of the layer to the receiving transducer \( h^T \) are large (see for an illustration Figure 4.6), the events that propagate through the embedding as compressional waves will arrive before any event that propagates through the embedding as a shear wave. If we restrict the one-way phase delay operator of Eq. (4.116) to the compressional mode in the embedding, we obtain

$$\mathbf{W}_P(h^{S,T}) = \exp(-s \gamma_{P,e} h^{S,T}) \mathbf{E}^{11}, \quad (6.3)$$

where the matrix \( \mathbf{E}^{11} \) is defined as

$$\mathbf{E}^{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (6.4)$$

By combining Eq. (6.2) with (6.3), the expression for the Green function \( \mathbf{G}_P \mid \downarrow \) of the transmitted events in the slowness and Laplace transformed domain, restricted to only the compressional mode in the embedding, can be written as

$$\mathbf{G}_P \mid \downarrow (x_3) = \exp(-s \gamma_{P,e}(h^S + h^T)) \mathbf{E}^{11} \mathbf{T}_\parallel \mathbf{E}^{11}$$

$$= \exp(-s \gamma_{P,e} H) T^{P/P}_\parallel \mathbf{E}^{11}, \quad (6.5)$$

where \( H \) is defined as the sum of \( h^S \) and \( h^T \), the total vertical travelpath in the solid embedding and \( T^{P/P}_\parallel \) is the upper-left transmission coefficient, the
compressional to compressional transmission coefficient, see Eq. (4.175) of
the layer defined in Eq. (4.136b). The measured signal of Eq. (6.1), limited
to only compressional modes in the embedding is obtained with Eq. (6.5) as

$$\tilde{S}_P(s\alpha, x_3, s) = T^{P/P}_\parallel \exp(-s\gamma_{P,\varepsilon} H) U_P,$$

(6.6)

where $U_P$ describing both the action of the source as well as the receiver
transducer on the compressional waves, has been defined as

$$U_P = (H^{i,\downarrow})^T \begin{bmatrix} D^{u,\downarrow} \\ D^{r,\downarrow} \end{bmatrix} E^{11} \begin{bmatrix} (D^{r,\downarrow})^T \\ (D^{u,\downarrow})^T \end{bmatrix} F. $$

(6.7)

Note that by restricting ourselves to compressional modes in the embedding,
we have not made an acoustic approximation since the complete elastic trans-
mission coefficient $T^{P/P}_\parallel$ does contain all multiple and mode-coupling effects
within the layer, as expressed in Eq. (4.136b).

To find the measured signal in the spatial domain we apply an inverse spatial
Fourier transform to the response of Eq. (6.7), which by using Eq. (2.15) is
expressed as

$$\tilde{S}_P(x, s) = \left(\frac{1}{2\pi}\right)^2 \int_{(s\alpha) \in \mathbb{R}^2} \exp(-js\alpha_\beta x_\beta) \tilde{S}_P(js\alpha, x_3, s) \, dA$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{(s\alpha) \in \mathbb{R}^2} \exp(-s(\gamma_{P,\varepsilon} H + j\alpha_\beta x_\beta)) T^{P/P}_\parallel (j\alpha, s) U_P(j\alpha) \, dA$$

(6.8)

If we take the limiting expression for $s \to j\omega$, and hence $s\alpha \to \omega \zeta$, see
Eq. (2.21) we obtain

$$\tilde{S}_P(x, j\omega) = \left(\frac{\omega}{2\pi}\right)^2 \int_{\zeta \in \mathbb{R}^2} \exp(-j\omega(\gamma_{P,\varepsilon} H + \zeta_\beta x_\beta)) T^{P/P}_\parallel (\zeta, j\omega) U_P(\zeta) \, dA.$$

(6.9)

When the fracture is absent, the transmission coefficient would reduce to
unity for all horizontal slownesses and hence the transmission measurement
without fracture $\tilde{S}^b_P(x, j\omega)$, is given by

$$\tilde{S}^b_P(x, j\omega) = \left(\frac{\omega}{2\pi}\right)^2 \int_{\zeta \in \mathbb{R}^2} \exp(-j\omega(\gamma_{P,\varepsilon} H + \zeta_\beta x_\beta)) U_P(\zeta) \, dA,$$

(6.10)
where again we have limited the waveform to only the waves that have propagated through the embedding as compressional waves. For reasons to become clear in Chapter 11 we denote the signal $\hat{S}_p(x, j\omega)$ as the base signal. In this section we assume that we measure in the farfield at large vertical distance from the fracture interface, which can be expressed as

$$\frac{\omega H}{c_{P,e}} \gg 1,$$

which means that even the phase delay of the fastest vertical propagating plane wave, the normal incident plane wave, is large. A small change of the horizontal Fourier-slowness will cause a small change in the vertical slowness, $\gamma_{P,e}$. But because of the large parameter of Eq. (6.11) this small change will cause a rapid fluctuation of the exponential function in Eq. (6.9), unless the phase function becomes stationary.

In the regime of the slowness where the up- and downgoing waves in the embedding are still propagating, the transmission coefficient appears to be a smooth function of the horizontal Fourier-slowness. A similar conclusion can be made for the action of the source and receiver, expressed in $U_P(\zeta)$. Beyond the last critical point of the solid embedding the transmission might not be smooth, related to the occurrence of poles, associated with strong guided waves. But since in the evanescent regime, the up- and downgoing waves damp exponentially, the energy of the guided modes in the transmission measurement for large vertical distances will be difficult to observe. Hence the influence of these singularities in the farfield is limited, and can be discarded as contributing to the measurement.

The main contribution to the integral of Eq. (6.9) will arise from points where the phase of the exponential function becomes stationary. Differentiating the phase function with respect to the horizontal Fourier slowness, we obtain

$$\frac{\partial}{\partial \zeta_\alpha} [\omega(\gamma_{P,e} H + \zeta_\beta x_\beta)] = \omega \left[ -\frac{\zeta_\alpha H}{\gamma_{P,e}} + x_\alpha \right]$$

and hence we obtain from Eq. (6.12) a single stationary phase point at the horizontal Fourier-slowness $\zeta^{SR}_\alpha$ given by the implicit expression

$$\zeta^{SR}_\alpha = \frac{x_\alpha}{H} \gamma_{P,e}.$$  

Equation (6.13) shows that the stationary point is related to the contribution from the ray which connects the source and receiver with a straight path.
An explicit expression for the Fourier-slowness $\zeta_{SR}^{\alpha}$ is obtained by squaring Eq. (6.13) resulting in

$$\zeta_{SR}^{\alpha} \cdot \zeta_{SR}^{\alpha} = \frac{x_{\alpha} x_{\alpha}}{H^2} \left( c_{P,c}^{-2} - \zeta_{\beta}^{SR} \zeta_{\beta}^{SR} \right). \tag{6.14}$$

When we now define the distance between the source and receiver transducer $d$ as

$$d = (H^2 + x_{\alpha} x_{\alpha})^{\frac{1}{2}}, \tag{6.15}$$

we can obtain an explicit expression for the horizontal Fourier-slowness, which is expressed as

$$\zeta_{\alpha}^{SR} = \frac{x_{\alpha}}{d c_{P,c}}. \tag{6.16}$$

Evaluation at the stationary point suggests that we can approximate Eq. (6.9) by

$$\hat{S}_P(x, j\omega) = \int_{\mathbb{R}^2} \exp(-j\omega(\gamma_{P,c}H + \zeta_{\beta} x_{\beta})) U_P(\zeta) \, dA, \tag{6.17}$$

where we have moved the transmission coefficient outside of the integral and replaced its slowness dependency with the slowness value at the stationary point given by Eq. (6.13).

Identifying the measurement without fracture of Eq. (6.10) in the right-hand part of Eq. (6.17), we see that, in the approximation of Eq. (6.17), we can relate the transmission measurement with fracture and by the compressional transmission measurement without fracture according to

$$\hat{S}_P(x, j\omega) = T_{\parallel}^{P/P}(-\frac{x_{\alpha}}{d c_{P,c}}, j\omega) \hat{S}_P^b(x, j\omega), \tag{6.18}$$

which in the time domain is equivalent to

$$S_P(x, t) = T_{\parallel}^{P/P}(-\frac{x_{\alpha}}{d c_{P,c}}, t) * S_P^b(x, t), \tag{6.19}$$

where the symbol * denotes a convolution in the time domain.

We conclude that in the convolutional approximation, the dispersion that we observe in the transmission measurement, relative to the measurement without fracture, is determined by a convolution with the elastodynamic layer.
transmission coefficient of the compressional to compressional wave mode. Since we measure the signal before fracturing with our time-lapse measurements, Eq. (6.18) describes a forward prediction of the amount of dispersion and can therefore advantageously be used as a fast way when inverting for fracture parameters such as the fracture width. This way we can avoid the slowness integration in every forward model, thereby reducing the necessary computation time severely. Moreover, the convolutional model will be of great value for the inversion of transmission measurements to estimates of the fracture width, because in this model the width estimate is independent on the location of the fracture and the source and receiver parameters. The measurement before fracturing provides us with a very accurate timing as well as amplitude calibration. We will use numerical modelling in Chapter 7 to check whether the accuracy offered by the convolutional model is sufficient for our purposes. An equivalent model for a reflection cannot be found because in absence of the fracture no reflection occurs and hence we lack a timing and amplitude calibration.

6.3 Perpendicular incidence

The general expressions for reflection and transmission of a layer, given by Eqs. (4.136a) and (4.136b) are too complex to grasp for a quantitative physical interpretation. The special case of normal incidence will show some main characteristics of transmission and reflection, which will also be observed in the case of a incident plane wave with an arbitrary angle of incidence. In Chapter 4, Section 4.2 we excluded the case of a plane wave propagating perpendicular to the layering since the inverse of the normalization matrix of Eq. (4.35) was ill-defined. However, if we take the limit for the Fourier-slowness \( \zeta_r \to 0 \) for the composition matrices in Eqs. (4.173a) and (4.173b) we obtain

\[
\lim_{\zeta \to 0} C^{n \to} = \frac{1}{2} \begin{pmatrix}
0 & (Z_s)^{-\frac{1}{2}} & 0 \\
0 & 0 & (Z_s)^{-\frac{1}{2}} \\
(Z_p)^{-\frac{1}{2}} & 0 & 0
\end{pmatrix},
\] (6.20a)
and

\[
\lim_{\zeta_r \to 0} C_{r^+}^{\tau} = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 & (Z_S)^{\frac{1}{2}} & 0 \\ 0 & 0 & (Z_S)^{\frac{1}{2}} \\ (Z_P)^{\frac{1}{2}} & 0 & 0 \end{pmatrix},
\]

(6.20b)

which shows that in the cylindrical domain by taking the limit of vanishing horizontal slowness \(\zeta_r \downarrow 0\), we avoid any difficulties related to the ill-defined inverse of the normalization matrix \(N_\psi\). Then we can calculate the transmission and reflection coefficients of a solid overlying a viscoelastic fluid, defined in Eq. (4.127) and Figure 4.7 by using Eqs. (4.106a)–(4.106d) resulting in

\[
\lim_{\zeta_r \to 0} t^o(x_3^-; x_3^+) = \begin{pmatrix} \frac{2\sqrt{z_r, P}}{1 + z_r, P} & 0 & 0 \\ 0 & \frac{2\sqrt{z_r, S}}{1 + z_r, S} & 0 \\ 0 & 0 & \frac{2\sqrt{z_r, S}}{1 + z_r, S} \end{pmatrix},
\]

(6.21a)

\[
\lim_{\zeta_r \to 0} t^i(x_3^-; x_3^+) = \begin{pmatrix} \frac{2\sqrt{z_r, P}}{1 + z_r, P} & 0 & 0 \\ 0 & \frac{2\sqrt{z_r, S}}{1 + z_r, S} & 0 \\ 0 & 0 & \frac{2\sqrt{z_r, S}}{1 + z_r, S} \end{pmatrix},
\]

(6.21b)

\[
\lim_{\zeta_r \to 0} r^o(x_3^-; x_3^+) = \begin{pmatrix} \frac{1 - z_r, P}{1 + z_r, P} & 0 & 0 \\ 0 & \frac{1 - z_r, S}{1 + z_r, S} & 0 \\ 0 & 0 & \frac{1 - z_r, S}{1 + z_r, S} \end{pmatrix},
\]

(6.21c)

\[
\lim_{\zeta_r \to 0} r^i(x_3^-; x_3^+) = \begin{pmatrix} \frac{1 - z_r, P}{1 + z_r, P} & 0 & 0 \\ 0 & \frac{1 - z_r, S}{1 + z_r, S} & 0 \\ 0 & 0 & \frac{1 - z_r, S}{1 + z_r, S} \end{pmatrix},
\]

(6.21d)

where \(z_r\) is defined as

\[
Z_r = \frac{Z_f}{Z_e} = \frac{\rho_f c_f}{\rho_e c_e},
\]

(6.22)

both for the compressional and shear waves, indicated with the subscripts \(P\) and \(S\), respectively. For perpendicular incidence the one-way phase delay operator \(\Theta\) can be expressed with the definitions in Eqs. (5.33a) and (5.33b)
as

$$
\lim_{\zeta \to 0} \Theta = \begin{pmatrix}
\exp(-j\theta_P) & 0 & 0 \\
0 & \exp(-j\theta_S) & 0 \\
0 & 0 & \exp(-j\theta_S)
\end{pmatrix}.
$$  (6.23)

Hence, by using Eqs. (6.21a)–(6.21d) the transmission and reflection coefficients of the layer of Eqs. (4.136a) and (4.136b) are obtained as

$$
\lim_{\zeta \to 0} T_{P,S}^\parallel = \frac{4z_{r,P,S} \exp(-j\theta_P)}{(1 + z_{r,P,S})^2 - (1 - z_{r,P,S})^2 \exp(-j\theta_P)},
$$  (6.24a)

$$
\lim_{\zeta \to 0} R_{P,S}^\parallel = \mp \frac{(1 - z_{r,P,S})(1 + z_{r,P,S})(1 - \exp(-j\theta_P))}{(1 + z_{r,P,S})^2 - (1 - z_{r,P,S})^2 \exp(-j\theta_P)},
$$  (6.24b)

where in the reflection coefficient of Eq. (6.24b) the minus sign corresponds to compressional waves and the plus sign for shear waves. Again, for the compressional and shear reflection or transmission the appropriate impedance and velocities have to be chosen.

Next, we focus on the response of a perpendicular incident plane wave for a linear-slip interface. With the composition operators in the cylindrical domain of Eqs. (6.20a) and (6.20b), the perpendicular transmission and reflection coefficients for the linear-slip model of Eq. (5.59a) and (5.59b) are obtained as

$$
\lim_{\zeta \to 0} T_{l} = \begin{pmatrix}
\frac{2}{2 + j\omega Y_N Z_{P,e}} & 0 & 0 \\
0 & \frac{2}{2 + j\omega Y_T Z_{S,e}} & 0 \\
0 & 0 & \frac{2}{2 + j\omega Y_T Z_{S,e}}
\end{pmatrix},
$$  (6.25a)

$$
\lim_{\zeta \to 0} R_{l} = \begin{pmatrix}
-\frac{j\omega Y_N Z_{P,e}}{2 + j\omega Y_N Z_{P,e}} & 0 & 0 \\
0 & \frac{j\omega Y_T Z_{S,e}}{2 + j\omega Y_T Z_{S,e}} & 0 \\
0 & 0 & \frac{j\omega Y_T Z_{S,e}}{2 + j\omega Y_T Z_{S,e}}
\end{pmatrix}.
$$  (6.25b)

We conclude that in the linear-slip approximation for a real fracture compliance, the transmission response is described by a low-pass single pole filter. In the static frequency limit the transmission response will be unity and no reflection occurs. In the time domain the low-pass single pole filter described in Eq. (6.25a) is equivalent with a convolution by a exponential relaxation function. Focusing on the reflection coefficient we notice an additional factor $j\omega$ which corresponds to a derivative in the time domain, which is also related to the fact that high frequencies are mainly reflected. When the fracture
compliance is not too large compared to the impedance of the embedding, we could simplify the reflection coefficients of Eq. (6.25b) to

$$R_l = \pm j\omega Y_{N,T} Z_{P,S},$$  \hspace{1cm} (6.26)

which is in agreement with the classical result that for the reflection the signature of the wavelet is the time-derivative of the incident wavelet [see Widess, 1973; de Voogd and den Roij, 1983].

The convolutional model will be tested on its accuracy in Chapter 7, where we numerically model the transmission through a thin fluid-filled layer. In Chapter 11 we will use both the convolutional model of Section 6.2 and the perpendicular transmission coefficients of Section 6.3 when we try to estimate the width of the hydraulic fracture. Additionally, the perpendicular transmission coefficients are useful to explain quantitatively some phenomena that we observe in the modelling results of Chapter 7.
Part II

Modelling

“When the word spirit comes out of the mouth of a Papalagi, his eyes widen, become round and staring, his chest swells up, he breathes deeply and stands tall like a warrior who has beaten his opponent. Because the 'spirit' is something he's very proud of. . . . When I'm standing here, looking at the mango tree behind the mission post, then I see the tree and not the spirit. But when seeing that it is bigger than the mission, then my spirit is working. That's why seeing is not enough for me. I also have to know something. That knowing is practiced by the Papalagi, day and night. Their spirit always behaves like a loaded firestick or a cast fishing-rod. Therefore he pities us, we the people of the many islands, because we do not practice knowing. He thinks we are stupid and deprived like the wild animals in the forest.

It may be true that we never practice knowing, or as the papalagi call it 'thinking'. But is an open question as to who the most stupid is, the one that thinks not very often or the one that thinks too much. The papalagi are constantly thinking. My hut is smaller than the palmtree. The palmtree bends over in the storm. That’s the way they think, in their particular way of course. But he also thinks about himself. I'm small. My heart is always glad when I see a girl. I enjoy very much going on a holiday etc. That may all be very nice and good and that may even bring all kinds of profits to those that like these games inside their heads. But the papalagi think so much, that for them thinking has become a habit, a necessity, and a need. He has to keep on thinking. Only after much trouble does he succeed in not thinking and instead live with his whole body at once.”

from 'The Papalagi - Speeches by Tutavii of Tiavea, a Samoan chief' on 'The Heavy Thinking Sickness'.
Numerical Implementation of the Layer Model

With the analysis of Chapter 4 we have formulated a theoretical framework to describe the scattering of elastodynamic waves by a flat fluid-filled layer of infinite lateral extent. The fluid material might behave as an ideal fluid or as any viscoelastic equivalent. The reflection and transmission for an interface, separating an ideal fluid from a solid was treated in Appendix C. Because of the complexity of the interaction it is difficult to get insight in how the thin layer expresses itself in the measurements. For that purpose we will resort to some representative numerical modelling examples. In doing so, we can answer questions which are important to keep in mind when we develop new applications for the ultrasonic measurements. Because our main objective is to assess the feasibility of monitoring the width of hydraulic fractures, we perform a sensitivity analysis of the acoustic response to the layer width. Since different wave types differ in their sensitivity to the layer parameters, different information can be obtained from them. Hence, we analyse both the scattering of compressional and shear waves. In addition, we can study the propagation of guided waves that might be excited along the fracture.
7.1 Numerical method

The theoretical analysis of the scattering by a fluid layer was performed in the slowness- and Laplace-transformed domain. Thereby we took advantage of the translation invariance of the configuration in the horizontal direction. In this chapter we are mainly interested to model the waveforms for a limited set of source and receiver positions, similar to the situation in the laboratory. For this idealized model using a planewave synthesis to model the full response is preferable to grid methods like finite-difference or finite-element, because we have more in control of the accuracy of the modelling result. Furthermore, the planewave synthesis is computationally cheap compared to finite-difference modelling. The advantage of finite-difference modelling is that snapshots of the total wavefield can be produced, which aids in the physical interpretation of the different events. In addition, finite-difference modelling can handle more complex geometries. We will show results of finite-difference modelling in Chapter 8.

In order to produce full waveforms in the space and time domain we have to carry out two inversion integrals, namely the inverse (complex) slowness and inverse Laplace transformation. Different techniques are known and being used to accomplish this inversion. For an overview we refer to Aki and Richards [1980] and Kennett [1983].

Since the main purpose of the modelling is research oriented, it is important to stress the accuracy and reliability of the method instead of the speed, or necessary CPU-time. In our experiments we have only a limited number of possible transducers positions, especially compared to surface seismic reflection surveys. Hence, we will benefit from a method which takes into account the fact that we need to invert the slowness integrals only for a small number of offsets. Performing a full Hankel-transformation of the slowness to space domain, would be computationally a disadvantage related to the fixed slowness discretization of common implementations of the numerical Hankel transformation. Fast fluctuation of the integrand may occur related to the vicinity of poles or branch-points, necessitating fine sampling at specific zones, and hence fine sampling over the whole slowness domain. The combination of the fixed sampling and varying smoothness of the integrand may result in uncontrolled behaviour of the accuracy of the numerical Hankel transform with an unacceptable large CPU-time.

Another approach to invert the slowness and frequency integral is to use the Cagniard-De Hoop method, which comprises a semi-analytical inversion of both the inverse slowness and frequency transformation at once by 'in-
separation' [see De Hoop, 1995; Van der Hilden, 1987]. The wavefield is separated in generalized-ray constituents and for each generalized ray the slowness contour is transformed in such a manner that the inversion integrals can be recognised as the one-sided Laplace transformation of the desired space-time-domain generalized-ray constituent. We discarded this method for our specific application because of two reasons. First, we wanted the modelling code to be equally well fitted for ideal fluid-filled layers as well as viscoelastic layers. The waveform dispersion of the generalized rays caused by viscoelastic loss mechanisms disturbs the possibility of a direct inversion by recognition. Second, the fluid-solid interfaces are associated with strong impedance contrasts. For some configurations, especially for large horizontal and small vertical offsets, this may lead to an exponentially increasing amount of generalized-rays within a certain time-window, see also a discussion in De Hon [1996]. Hence, the bookkeeping of all generalized-rays, including mode-conversions, becomes increasingly complex and the computation time will increase accordingly.

### 7.1.1 The slowness integration

As an alternative to the Cagniard-De Hoop method, we have chosen to perform the inverse slowness by numerical quadrature. Standard numerical integration is not suitable to integrate singularities which might occur along the real Fourier-slowness axis, when performing the inversion in the Fourier-transform domain in time.

Instead we perform our analysis in the Laplace domain for a Laplace transform parameter which has only a small fixed real component and a varying real frequency component \( \omega \), i.e.

\[
s = s_0 + j \omega, \quad s_0 > 0, \quad s_0, \omega \in \mathbb{R},
\]

where \( s_0 \) is taken to be constant. We can rewrite the product \( s \alpha_r \), the integration variable of Eqs. (4.186) and (4.187) for this specific choice of \( s \) as

\[
s \alpha_r = (s_0 + j \omega)(\text{Re}(\alpha_r) + j \text{Im}(\alpha_r))
\]

\[
= (s_0 \text{Re}(\alpha_r) - \omega \text{Im}(\alpha_r)) + j(s_0 \text{Im}(\alpha_r) + \omega \text{Re}(\alpha_r)) \quad (7.2)
\]

\[
= (s_0 \text{Im}(\zeta_r) + \omega \text{Re}(\zeta_r)) + j(-s_0 \text{Re}(\zeta_r) + \omega \text{Im}(\zeta_r)),
\]

where in the last step we have used Eq. (2.20). Since \( s \alpha_r \) is required to be real, as has been stipulated in Section 2.5, we conclude

\[
\frac{\text{Im}(\zeta_r)}{\text{Re}(\zeta_r)} = \frac{s_0}{\omega}. \quad (7.3)
\]
Hence, the Fourier-slowness moves slightly off the real axis into the complex plane, thereby avoiding the direct numerical evaluation of the integrand at singularities, that might be present associated with guided waves. Such a contour approaches the slowness inversion of the field quantities in the Fourier-frequency domain, modified by a small rotation of the integration contour to the domain where the integrand is analytical. Since in this domain the integrand can be approximated by smooth polynomials, we can use Gaussian quadrature [see e.g. Piessens et al., 1983] to compute the integrand, by evaluating it at a selected set of points. For the numerical implementation of the slowness integrals of Eqs. (4.186) and (4.187) we do not know beforehand how fast the integrand fluctuates and hence it is not clear what kind of integration rule is sufficient for our purpose. A possible approach would be to look at a sequence of Gauss quadrature rules of increasing polynomial degree of precision and look at the convergence and estimates of the error. Since generally speaking the abscissae of different Gauss integration rules do not coincide even partially, this would lead to a sequence of many function evaluations, to ensure sufficient accuracy of the integration.

Instead, we have chosen to integrate with a 15-31 Gauss-Kronrod integration rule [see Piessens et al., 1983]. First, a standard 15-point Gauss integration is calculated over a specific interval. Then a nested sequence is constructed by combining the 15 abscissae with the optimal extension with 16 additional points, see Figure 7.1. Thereby we construct a very economical pair of integration rules for the simultaneous calculation of an approximation for the integral and a related error estimate, based on the difference between the result of the 15-point Gauss and 31-point Kronrod integration rule. In that case, only 31 function evaluations are needed for the approximation of the integrand up till a polynomial degree of precision $^1$ 46 (the number of free parameters minus one) [see Piessens et al., 1983], and its error estimate.

Since for large slownesses the up- and downgoing waves become evanescent, the integrand will vanish exponentially. To approximate the analytical semi-infinite integral of Eq. (4.186) and (4.187) we determine an endpoint of the numerical integration based on this asymptotic behaviour and a certain desired numerical accuracy.

On the finite interval determined by this endpoint we use 15-31 Gauss-Kronrod integration rules as a first approximation. We continue to bisect each interval where a certain desired accuracy is not found, until over the

---

$^1$ A quadrature sum is said to be of polynomial degree of precision $n$, if it is exact for all polynomials of degree $n$ and lower, and is not exact for the class of higher order polynomials.
whole interval we have sufficient accuracy. At the end we have found a chain of Gauss-Kronrod integration rules in a set of subintervals. Since the algorithm computes an error estimate it will automatically find the difficult regions of integration related to branch points, poles, or strong oscillations. An example of the intervals which we find after subsequent bisection of a certain interval is shown in Figure 7.2. This figure shows a part of a wavefield response as a function of the horizontal slowness for a fixed frequency component, normalized to its maximum value. The slowness variable is scaled with respect to the shear slowness of the embedding. The vertical lines show the division into subintervals on which 15-31 Gauss-Kronrod integration rules are applied to ensure the desired accuracy. The relative error, estimated by the difference between the 15-point Gauss and 31-point Kronrod integration, is below $1.0 \times 10^{-5}$ in each interval. Note that we have normalized the slownesses with respect to the shear slowness of the embedding. Beyond the shear slowness of the embedding the up- and downgoing waves become evanescent in the embedding. The four difficult integration areas are related to the branch point of the compressional wave.
Figure 7.2: Intervals for the integrand normalized to unity, of the response in the slowness domain. In each subinterval the 15/31-point Gauss-Kronrod integration rule is applied. Slownesses are normalized with respect to the shear wave slowness \( \zeta_{s,e} \) of the embedding.

In the embedding, (around 0.5), the branch point of the shear wave in the embedding (around 1) and two singularities (around 1.1 and 2.6). These singularities or poles are associated with guided modes that propagate along the fracture. This function is the integrand of the response in the slowness domain for the centre frequency of the waveform that will be discussed in detail in Section 7.2.4. The complete waveform for this example is shown in Figure 7.10(a).

\subsection{7.1.2 The frequency integration}

After we have determined the response in the space and Laplace domain numerically, the space and time domain equivalent is found by using the inverse Laplace transformation of Eq. (2.6), expressed as

\[
\frac{1}{2\pi j} \int_{s_0-j\infty}^{s_0+j\infty} \exp(st)\hat{S}(x,s) \, ds = \chi_T(t)S(x,t). \tag{7.4}
\]
For our choice of the Laplace transform parameter of Eq. (7.1) with \( s_0 \) fixed, we can rewrite the \( s \)-integration to a \( \omega \)-integration, expressed as

\[
\frac{1}{2\pi j} \int_{s_0 - j\infty}^{s_0 + j\infty} \exp(st)\hat{S}(x, s) \, ds = \frac{\exp(s_0 t)}{2\pi} \int_{-\infty}^{+\infty} \exp(j\omega t)\hat{S}(x, j\omega; s_0) \, d\omega = \exp(s_0 t)\mathcal{F}^{-1}_t\{\hat{S}(j\omega; s_0)\}(x, t)
\]

(7.5)

From Eq. (7.5) we conclude that we can implement the inverse Laplace transformation with a fixed \( s_0 \) by applying an inverse Fourier transformation to the response in the Laplace domain, which is a slightly damped response compared to its Fourier domain equivalent, related to the small constant real part of \( s \). We retrieve the function in the space and time domain after correcting for the damping by multiplying with the time-varying factor \( \exp(s_0 t) \).

The actual Fourier inversion can be implemented with a standard FFT algorithm. The advantage of using a small real Laplace transform parameter is twofold. First, we avoid the direct numerical evaluation of poles. In other words, the effect of the small real Laplace transform parameter is a smoothing of the integrand. Hence, we preparing the integrand for a numerical integration with Gaussian quadrature rules. Second, to model the total duration of the waveform for some configurations we may need many samples in the time domain. The number of slowness integrations for each frequency component is equal to half the number of time samples used in the modelling, based on the symmetry properties of the FFT for real functions in the time domain. The bulk of the CPU-time consists of carrying out these slowness transformations. In order to limit the CPU-time we can choose the number of time samples less than is needed for the complete waveform, if the total waveform has a long, but weak tail. Late events arriving beyond the finite time window of the FFT, are damped in time, with the factor \( \exp(s_0 t) \). The correction factor that will be applied assumes the events to arrive within the FFT window. This factor will be much lower than the factor \( \exp(s_0 t) \) for the real arrival time \( t \). Hence, the wrap-around that normally occurs in FFT analysis is not restored to its original level but to a much weaker level.

A second technique to limit the number of time samples (and hence the amount of slowness integrations) is to skip the trailing zeros before the arrival
of the primary. If we know a priori that no energy is contained within that
time interval and we do not want to reconstruct the signal before the arrival
of the primary, we can choose to apply an additional time shift, by using

\[ t' = t + T, \tag{7.6} \]

in which case we have to modify Eq. (7.5) according to

\[
\begin{align*}
\exp\left( s_0(t' - T) \right) \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left( j\omega(t' - T) \right) \hat{S}(x, \omega; s_0) \, d\omega \\
= \exp\left( s_0(t' - T) \right) \mathcal{F}_\omega^{-1}\{ \hat{S}(\omega; s_0) \exp(-j\omega T) \}(x, t') \\
= \chi_T(t') S(x, t')
\end{align*} \tag{7.7}
\]

Hence, by applying a phase shift to the integrand before carrying out the in-
verse FFT and modifying the multiplication factor, we can reduce the number
of discretization points in the time domain.

The numerical calculations were implemented in Fortran 90 code on a Sun
Sparc10 workstation. The computation time of the following experiments
ranged between a few minutes for a short distance between the source and
receiver and on the order of 16 hours on a single CPU for a complex con-
figuration, including strong guided waves near the fracture, at large lateral
distances. Fortran 90 was chosen because of its easiness in expressing ma-
trix computations and modern features, such as user-defined data types and
operator overloading. Furthermore, Fortran 90 is the present ISO standard
since 1991 [ADAMS ET AL., 1992].

### 7.2 Modelling results

#### 7.2.1 Introduction to the numerical calculations

Since the number of parameters that can be varied in the numerical calcu-
lations is so large, we avoid redeclaring all parameters for the sequence of
configurations that follows, by defining a default set of parameters. For each
configuration we use the default parameters, unless other values are explic-
titly given. The default parameters are representative for two main laboratory
cement block experiments that we will discuss in part III. For the block sam-
ple we assume the solid to be ideally elastic. At first, we assume that the
fracture fluid that we use is an ideal fluid. The density and compressibility
7.2 Modelling results

of this fluid are representative of the silicon oil that we use in the laboratory as fracture fluid. By default, we assume a perpendicular transmission configuration in which the fracture is half-way in between the block of 0.3 m, the actual block size. The transducer aperture radius and centre frequency are representative for the actual piezo-electric transducers, which are described in part III. Since realistic fracture widths vary between 0 and 300 µm, we have taken a default value of a 100 µm. All default values are listed in Table 7.1. As input wavelet we have chosen a amplitude-modulated sinusoid, expressed as

\[ W(t) = \left( \frac{at}{b} \right)^b \exp(b - at) \sin(\omega_c t) \chi_T(t), \]  

where the constants \( a \) and \( b \) are related in such a way, that the DC component for the wavelet vanishes, which is the case when \( a \) is related to \( b \), according to

\[ a = \frac{\omega_c}{\tan\left( \frac{\pi}{b+1} \right)}. \]  

### 7.2.2 Resolution for compressional transmissions

Since our main objective is to assess the feasibility to monitor the width of hydraulic fractures, we need to consider the resolution of our measurements. Generally speaking, the ability to estimate the fracture width depends on the frequency content of the source wavelet, the layer width, the material properties and the type of measurement. For a range of fracture widths we have calculated the transmission responses of a compressional transducer sending waves to a receiving compressional transducer at the opposite side of the block. The thin solid lines in Figures 7.3(a)–(d) show the modelling results for a fracture width of 25, 100, 800 and 3200 µm, respectively. The thick solid line in Figures 7.3(a)–(d) are one and the same, since they show the modelled records without any fracture. The visual difference between these lines is caused by the need for different time-scales to capture the waveforms with a fracture, indicated with a thin solid line. We can see in Figures 7.3(a) and 7.3(b) that when the fracture is thin, the measurement with fracture will be slightly attenuated and delayed in time compared to the measurement without fracture. The combined effect can be described as an apparent dispersion of the measurement with fracture relative to the measurement without fracture. This apparent dispersion can be explained
<table>
<thead>
<tr>
<th>symbol</th>
<th>quantity</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_e$</td>
<td>density</td>
<td>$2.009 \times 10^3$ kg/m$^3$</td>
</tr>
<tr>
<td>$\lambda_e$</td>
<td>longitudinal modulus</td>
<td>$11.317 \times 10^9$ Pa</td>
</tr>
<tr>
<td>$\mu_e$</td>
<td>shear modulus</td>
<td>$9.269 \times 10^9$ Pa</td>
</tr>
</tbody>
</table>

**Embedding (ideally elastic):**

<table>
<thead>
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<th>value</th>
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</thead>
<tbody>
<tr>
<td>$\rho_f$</td>
<td>density</td>
<td>$0.970 \times 10^3$ kg/m$^3$</td>
</tr>
<tr>
<td>$\kappa_f$</td>
<td>compressibility</td>
<td>$9.857 \times 10^{-10}$ Pa$^{-1}$</td>
</tr>
<tr>
<td>$h$</td>
<td>fracture width</td>
<td>100.0 $\mu$m</td>
</tr>
</tbody>
</table>

**Layer (an ideal fluid):**

<table>
<thead>
<tr>
<th>symbol</th>
<th>quantity</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_s$</td>
<td>source/layer distance</td>
<td>0.15 m</td>
</tr>
<tr>
<td>$h_T$</td>
<td>layer/receiver distance</td>
<td>0.15 m</td>
</tr>
<tr>
<td>$r$</td>
<td>lateral distance source/receiver</td>
<td>0.0 m</td>
</tr>
<tr>
<td>$R$</td>
<td>Receiver aperture radius</td>
<td>$6.35 \times 10^{-3}$ m</td>
</tr>
<tr>
<td>$f_c$</td>
<td>centre frequency source</td>
<td>0.5 MHz</td>
</tr>
<tr>
<td>$b$</td>
<td>wavelet factor</td>
<td>2</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>sampling time</td>
<td>0.1 $\mu$s</td>
</tr>
</tbody>
</table>

**Source and receiver parameters:**

<table>
<thead>
<tr>
<th>symbol</th>
<th>quantity</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{P,e}$</td>
<td>compressional velocity</td>
<td>3.855 km/s</td>
</tr>
<tr>
<td>$c_{S,e}$</td>
<td>shear velocity</td>
<td>2.148 km/s</td>
</tr>
<tr>
<td>$c_{P,f}$</td>
<td>compressional velocity</td>
<td>1.023 km/s</td>
</tr>
<tr>
<td>$\theta_P$</td>
<td>one-way phase delay</td>
<td>$0.013 \times 2 \pi$</td>
</tr>
<tr>
<td>$z_{r,P}$</td>
<td>compressional impedance ratio</td>
<td>0.128</td>
</tr>
</tbody>
</table>

**Table 7.1:** Default parameters used for the numerical calculations, and some important values, which have been derived from the default values. The default configuration consists of a perpendicular transmission of a pair of transducers with a finite aperture, through a solid embedding and a thin layer filled with an ideal fluid.
by acknowledging that the energy contained in the multiples arrives within the time window of the direct wave, because the width of the layer is small. The energy of these multiples cannot be neglected because of the substantial impedance contrast. For a normal incident compressional plane wave, the ratio of the amplitude of the first multiple $A^m$ to the direct wave $A^0$, using Eq. (6.21d) is obtained as

$$\frac{A^m}{A^0} = \lim_{\zeta \to 0} (r^n)^2 = \left( \frac{1 - z_{r,P}}{1 + z_{r,P}} \right)^2 \approx 0.6, \quad (7.10)$$

for the default parameter set given in Table 7.1. The multiple arrivals interfere constructively with the direct wavelet to result in the apparent dispersion. We label this dispersion as apparent dispersion since it is not related to any physical intrinsic loss mechanism, but mainly transmission losses related to the fact that a part of the energy is effectively reflected from the thin layer. In Figure 7.3(d) we see that for a much larger fracture width of 3200 µm, the multiples arrive separately. The first arrival shows mainly a constant amplitude decay related to transmission losses instead of a dispersion as in Figure 7.3(a). A width determination in this case could be carried out by estimating the time delay related to the arrival of the first order (and higher order) multiple, compared to the direct arrival. The time delay between subsequent multiples, $\Delta t^m$, is directly related to the fracture width according to

$$h = \frac{1}{2} c_{P,f} \Delta t^m. \quad (7.11)$$

Hence, for a known compressional fluid velocity, we can simply estimate the fracture width by estimating the time delay $\Delta t^m$. The criterion for a successful application of this method is that the multiples arrive separately in time. This criterion is fulfilled when the width of the layer is restricted to

$$h > C \frac{c_{P,f}}{2 f}, \quad (7.12)$$

where $C$ is a factor which describes the fraction between the total duration of the wavelet and its centre period. In that case we can claim that the layer is within the resolution of the wavelet.

Similar criteria have been used in surface reflection seismology to obtain a measure of the detectability of a layer [see e.g. Widess, 1973; De Voogd and den Rooij, 1983]. Although the various criteria vary a bit depending on its specific application, all of these criteria are variations of the criterion stated
Figure 7.3: The compressional transmission response and its sensitivity to the width of the layer for a fracture width of a): 25 μm, b): 100 μm, c): 800 μm and d): 3200 μm.
in Eq. (7.12). Figure 7.3(d) shows an example for which the layer width is above the resolution of the wavelet. Figures 7.3(a) and 7.3(b) clearly shows two examples for which the layer width is below the resolution of the wavelet. Figure 7.3(c) is an example for an intermediate layer width, showing the transition between the region where the layer is resolved and unresolved. Because we perform time-lapse measurements, we have the advantage of measuring the transmitted signal before fracturing. In that case we can, alternatively to estimating the time delay \( \Delta t^m \), compare the records of the measurement without and with a fracture of a certain width. Even the small time delay shown in Figure 7.3(a) (on the order of a tenth of a \( \mu s \)) and the amplitude attenuation (of less than 10 percent) is still measurable. The amount of apparent dispersion is then again a measure of the width of the fracture for fixed material properties. Having a theory at our disposal which predicts the amount of dispersion as a function of the layer width, in principle we can estimate the width of the fracture. When the low-impedance layer is thin, a fast estimate of the amount of dispersion is given by the normal incident linear-slip transmission coefficient of Eq. (6.25a), given by

\[
\lim_{\zeta \to 0} T^{P/P}_{\zeta} = \frac{2}{2 + j\omega \kappa Z_{P,e}},
\]

where we replaced \( Y_N \) by the product of the width of the layer \( h \) and the compressibility of the fluid \( \kappa \), as in Eq. (5.47). As long as the product \( \omega h \kappa Z_{P,e} \) is sufficiently large an dispersion can be observed in a transmission measurement. Although the width of the layer in the laboratory experiments is small, the impedance contrast is large enough, resulting in a clear dispersion. Since the compressional impedance of the embedding and the fluid compressibility can be measured separately, we can determine the layer width. In Chapter 11 we will show that fluctuations of the fracture width on the order of a micrometer can be detected with an incident wavelength on the order of a cm. In this case we heavily depend on the repeatability and accurate timing of the source signature over the experiment time. Hence, for time-lapse measurements with a repeatable source wavelet, we can still claim that a layer width of 25 \( \mu m \), see Figure 7.3(a), is resolved for this type of measurement. We prefer to designate the standard resolution requirement as stated in Eqs. (7.12) as 'classical'. In the classical sense a layer with a width below a few millimetre will be unresolved, whereas with a time-lapse experiment a width of a few micrometer can still be determined. The main conclusion that can be drawn from these calculations is that in a time-lapse experiment the resolution is much higher than in the classical sense.
7.2.3 Resolution for shear-wave transmission

In the previous section we discussed the resolution of compressional transmission measurements and the possible application of a dispersion measurement as a means to estimate the fracture width. The conclusions will be different for shear waves, because of the completely different response of a fluid to shear stresses, instead of compressional stresses. Since in an ideal fluid no shear waves can propagate, any transmission in that case will be related to mode-converted energy, i.e. shear waves which are mode-converted to compressional waves in the fracture fluid and converted back to shear waves. A normal incident plane shear wave will show complete reflection, see Eq. (C.30b). Hence, for our default finite aperture transducers we expect mode conversion to be low for a perpendicular transmission and higher for increasing lateral offsets. In case of complete reflection the shear measurement will be useless for a width determination, since it contains no information about this property. We need some transmission of the shear wave, will it be be sensitive to the fracture width. On the other hand, complete shear wave reflection and the accompanying shear wave shadowing is a good indicator for detecting the moment of initiation of the fracture. Using different source and receiver combinations the progress of the fracture front can be monitored.

To investigate shear transmission for a fracture, filled with an ideal fluid we modelled the transmission for the default parameter set for three different lateral offsets, namely 0 cm, 6 cm and 15 cm, for which 6 cm and 15 cm are non-default values. The results are shown in Figures 7.4(a)–(c). The thick solid lines are the modelling results of the direct transmission without a fracture for the different lateral offsets, while the thin solid lines are the transmissions through the fracture with default values, except for the lateral offsets. For a lateral offset of 0 cm and 6 cm the shear transmission is small, as expected since mode-conversion is weak for plane waves with a small incident angle. In Figure 7.5 the transmission through the fracture for zero lateral offset of Figure 7.4(a) has been magnified to show that because of the finite aperture of the source, some weak shear-wave transmission occurs, with a maximum amplitude below a percent of the observed signal without a fracture. Compressional wave scattering for example from the sides of the block, cause a background energy level, which can be labelled as noise because no additional information is contained within the signal. Hence such a low level of shear-wave transmission is hardly useful from a practical point of view.

For the larger horizontal offset we have around 30 % of shear transmission
Figure 7.4: Shear transmissions as function of the horizontal offset $r$, (a): no offset, (b): 6.0 cm, (c): 15.0 cm and layerwidth $h$, (d): 10 $\mu$m, $r = 15.0$ cm.
Figure 7.5: Detail of the shear transmission of Figure 7.4(a), for zero angle of incidence, with the arrival times of the compressional, shear and mode-conversions, $t_{PP}$, $t_{SS}$, $t_{PS}$ and $t_{SP}$ respectively of the waves through the embedding without the layer.

compared to the direct wave, which is certainly measurable. If we compare the transmitted wave through a fracture for a non-default width of 10 μm shown in Figure 7.4(d), with the default value of 100 μm, we see that the attenuation and time-delay is slightly larger for the fracture width of 100 μm. In principle, this information can be used to estimate the width of the fracture. Nevertheless, there are few advantages in using shear transmission measurements in our experiment to estimate fracture width, compared to compressional measurements. In case of compressional transmissions the amount of dispersion is determined by the phase delay of the compressional wave in the fluid and the impedance contrast between the compressional waves in the fluid and solid. The bigger the impedance contrast, the stronger the multiples and hence the apparent dispersion. For a shear-wave transmission the mode of propagation of the wave in the fracture, filled with an ideal fluid, is also compressional. Hence, the amount of dispersion is also
determined by the phase delay of the *compressional* wave in the fluid, while it is the impedance contrast between the *shear* wave in the solid and the *compressional* wave in the fluid that determines the strength of the multiples and hence the amount of dispersion. Since in our configuration the compressional impedance of the fluid is closer to the shear impedance in the solid, the amount of dispersion for shear waves is lower than for compressional waves. This reasoning is consistent with Figures 7.6(a) and 7.6(b) which compares the compressional and shear transmission coefficients as a function of the angle of incidence for a range of values for the fracture width.

According to the theoretical transmission coefficient the bulk of the shear energy is lost when the fracture is opened, even for a very small fracture width. This is caused by the fact that when we open the fracture, the shear stresses at the fracture faces vanish immediately. For a real physical experiment the instantaneous loss of shear-wave transmission will be less extreme, since for small fracture widths the rough fracture faces will be in contact, causing some shear-wave transmission through friction and contact mechanisms. But when the fracture is mechanically open, i.e. there is no contact of the fracture faces, indeed the bulk of the shear-wave energy will be shadowed. For small angles, compressional waves do not experience any big losses related to the vanishing shear stresses.

In the previous discussion we assumed the fluid to be an ideal fluid. In our experiments we used a very viscous silicon polymer with a zero shear-rate viscosity of 600 Pa s. Since the viscosity enables the propagation of shear waves in the fluid we might expect different behaviour in that case. Although the zero shear-rate viscosity of this fluid is well-known and relatively easy to measure, much less is known about its high-frequency behaviour. In the high-frequency range the fluid might respond more elastically than viscously depending on the relaxation time or relaxation time spectrum. To investigate the shear-wave transmission for viscous fluids we will show the effect of the relaxation time of the viscous fluid, assuming a Maxwell relaxation function with a single relaxation time, see Eq. (3.33).

For a single relaxation time, the zero shear-rate viscosity is determined by the product of the limiting high frequency shear stiffness $\mu_\infty$ and the relaxation time, see Eq. (3.39). Assuming a single relaxation time for a known and fixed zero shear-rate viscosity additionally determines the limiting high frequency shear stiffness $\mu_\infty$. In Table 7.2 we show the high frequency shear stiffness and the related shear-wave velocity for a set of relaxation times, ranging from $10^{-7}$ s to 1 s.
Figure 7.6: Modulus of a): the compressional transmission coefficient $T_{P/P}$ and b): the shear transmission coefficient $T_{SV/SV}$ of a layer for various layer widths.
Zero shear-rate viscosity $\eta_0$: 600 Pa s

<table>
<thead>
<tr>
<th>$\tau_r$</th>
<th>$\mu_\infty$</th>
<th>$c_S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 s</td>
<td>$6 \times 10^2$ Pa</td>
<td>0.8 m/s</td>
</tr>
<tr>
<td>$10^{-1}$ s</td>
<td>$6 \times 10^3$ Pa</td>
<td>2.5 m/s</td>
</tr>
<tr>
<td>$10^{-2}$ s</td>
<td>$6 \times 10^4$ Pa</td>
<td>7.9 m/s</td>
</tr>
<tr>
<td>$10^{-3}$ s</td>
<td>$6 \times 10^5$ Pa</td>
<td>24.9 m/s</td>
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<td>$6 \times 10^8$ Pa</td>
<td>786.5 m/s</td>
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<tr>
<td>$10^{-7}$ s</td>
<td>$6 \times 10^9$ Pa</td>
<td>2487.1 m/s</td>
</tr>
</tbody>
</table>

Table 7.2: High frequency limit of shear stiffness and velocity for fixed zero shear-rate viscosity.

We can observe that for a fixed viscosity the high frequency shear-wave velocity decreases for increasing relaxation time. For a set of relaxation times, ranging from $10^{-7}$ to $10^{-4}$ s, we have calculated the shear transmissions for a fracture with the default values, except for the fluid properties, which are now assumed to be viscoelastic. The longitudinal modulus of the fluid is taken non-viscous and is determined by the default compressional velocity and the limiting high frequency shear stiffness $\mu_\infty$. The results are shown in Figures 7.7(a)–7.7(d). We observe that for increasing relaxation time less shear energy is transmitted. For the most viscous fluid with a relaxation time of $10^{-7}$ s, we see mainly an attenuation of 70 percent for the transmission and no substantial time delay. This can be understood as follows. Because of the viscous damping a viscous boundary layer is created in the fracture with a viscous skin depth for a normal incident plane wave with 1 MHz frequency, see Eq. (4.123) given by

$$h_{\text{skin}} = \sqrt{\frac{2\eta_0}{\rho \omega}} \approx 444 \mu m.$$  \hspace{1cm} (7.14)

Hence the layer of default width 100 $\mu$m, is thin compared to the viscous skin depth and the fracture faces will be coupled by the viscous interaction. Because the layer width is small compared to the viscous skin depth a fast estimate of the shear transmission is given by the linear-slip model. The normal shear transmission coefficient for a linear-slip model is given by Eq. (6.25a)
Figure 7.7: The shear wave transmission as a function of the relaxation time $\tau_r$ with a fixed zero shear-rate viscosity for a): $10^{-7}$ s, b): $10^{-6}$ s, c): $10^{-5}$ s and d): $10^{-4}$ s.
as

\[ \lim_{\zeta \to 0} T_1 = \frac{2}{2 + j\omega Y_T Z_{S,e}}. \]  

(7.15)

Taking the transversal fracture compliance according to Eq. (5.37), we obtain the viscous fracture compliance as

\[ Y_T = \frac{h}{\mu} \approx \frac{h}{j\omega \eta_0}, \]  

(7.16)

and hence the normal viscous transmission coefficient of Eq. (7.15) reduces to

\[ \lim_{\zeta \to 0} T_1 \approx \frac{2}{2 + \frac{h}{\eta_0} Z_{S,e}} = 0.74, \]  

(7.17)

which shows a frequency independent amplitude attenuation, corresponding with our observation in Figure 7.7(a). The imaginary fracture compliance of Eq. (7.16) is related to the velocity discontinuity model with a modified fracture compliance \( Y_T' \) according to Eq. (5.41) as the ratio between the width of the layer \( h \) and the zero shear-rate viscosity \( \eta_0 \). Hence, we conclude that for a velocity discontinuity model we observe mainly an amplitude attenuation without a strong dispersion or time delay.

Although for longer relaxation times the layer becomes more elastic, the expected shear stiffness based on the zero shear-rate is low, see Table 7.2. Hence the impedance contrast is too big to create a substantial shear transmission. In Figure 7.8 we show a detail of the shear transmission for a relaxation time of \( 10^{-3} \) s. We have omitted the direct wave, since the amplitude of the shear transmission is only about 3 percent compared to the direct wave. For this relatively long relaxation time, the fluid behaves mainly elastic in the frequency range of interest. Using a pair of imperfect shear transducers, we observe a weak compressional event around 77 \( \mu s \) and a mode conversion around 110 \( \mu s \). Then, we observe a set of events, which arrive after the arrival time of the direct shear transmission through a homogeneous medium. The first weak energy is a mode-conversion of a shear wave in the solid and a compressional wave in the fluid. After that event we observe a sequence of multiples. For a relaxation time of \( 10^{-3} \) s, we have a limiting high frequency shear velocity of approximately 25 m/s, see Table 7.2. The impedance contrast that the shear wave in the solid has to overcome to propagate through the fluid is very large and hence the direct wave is very weak. For the same
Figure 7.8: Detail of reverberations for a realistic relaxation time $\tau_r$ of $10^{-3}$ s, with the arrival times of the compressional, shear and mode-conversions, $t_{PP}$, $t_{SS}$, $t_{PS}$ and $t_{SP}$ respectively of the waves through the embedding without the layer.

reason, the strength of subsequent multiples is only slightly weaker. The ratio of the amplitude of the first multiple $A^m$ to the direct wave $A^0$, using Eq. (6.21d) is given by

$$\frac{A^m}{A^0} = \lim_{\zeta \downarrow 0} (r^{-})^2 = \left(\frac{1 - \bar{z}_{r,S}}{1 + \bar{z}_{r,S}}\right)^2 \approx 0.97,$$

which explains the strong sequence of multiples. The delay time of the multiples $\Delta t^m$ for a shear velocity of 25 m/s and a default width of 100 $\mu$m is 8 $\mu$s, which is in correspondence with the results of Figure 7.8.

Concluding, we postulate that in case of a Maxwell-fluid we could use shear waves to determine the width. In the viscous limit we can measure the frequency-independent amplitude attenuation and in the high frequency limit the multiple delay time $\Delta t^m$. The practical application in the latter case will be limited by possible background noise, which will overshadow the weak multiples.
7.2.4 Guided waves

In Appendix D we present a study on the occurrence of poles in the response of a thin ideal fluid layer in an elastic solid embedding. This resulted in an expression for pressure-symmetric and a pressure-antisymmetric guided modes. In Figure 7.9(a) and (b) we have determined the phase- and group-velocity of the guided waves for the default ideal fluid fracture. We observe two guided waves in the frequency range of interest, below 1 MHz. First, a strongly dispersive guided wave, for which both the phase- and group-velocity tend to zero in the static frequency limit. This guided mode is a pressure-symmetric mode, described by Eq. (D.30), which we will call a channel wave because it is related to the tube wave, or Stoneley wave, which propagates in a fluid-filled cylindrical borehole [PAILLET AND WHITE, 1982see ]. The fluid/solid interface response contains a strong pole, known as the Scholte pole, which for our case occurs slightly below the fluid velocity at 1018 m/s (cf. 1023 m/s for the fluid velocity), associated with the Scholte wave. For our layer configuration two fluid/solid interfaces are present, both of which could support a Scholte wave. Indeed, in the high frequency limit, we observe a pressure-symmetric mode close to the Scholte wave velocity. In the fluid these waves are evanescent and damp away exponentially with a skin depth according to Eq. (4.119) of

\[ h_{\text{skin}} = \frac{1}{\omega(c_{\text{CH}}^2 - c_{p,j}^2)^{1/2}}, \]

which for the centre frequency of 0.5 MHz results in a skin depth \( h_{\text{skin}} \) of approximately 3 mm. Hence, in the frequency range of interest the Scholte waves will be coupled at both fluid-solid interfaces, which results in the dispersive behaviour observed in Figure 7.9(a) and (b). Since the skin depth becomes larger for lower frequencies, especially the lower frequencies become highly dispersed.

The second important guided wave that is seen in Figure 7.9(a) and (b) is a weakly dispersive pressure-antisymmetric wave below 2000 m/s. For the static frequency limit the velocity of this guided mode approaches the Rayleigh wave velocity. This can be understood by acknowledging that the dimensionless layer width will tend to zero for the static limit. Since the pressure is anti-symmetric, the pressure vanishes at the centre of the layer. Hence, we observe that in the static limit this configuration approaches two solid half spaces with a zero-pressure interface. The interface wave corresponding to a solid half space with a zero pressure interface is exactly given
Figure 7.9: Existence of guided modes for the default layer, filled with an ideal fluid. In a): the phase-velocity and in b): the group-velocity, with 1: the slow channel wave, 2: The pressure anti-symmetric generalized Rayleigh wave and 3: the pressure symmetric generalized Rayleigh wave as a function of the frequency $f$. 
by the Rayleigh wave velocity. The corresponding Rayleigh velocity for our configuration is shown in Figure 7.9(a) and (b) by a dashed line, mainly as a reference. Because of the resemblance with the Rayleigh wave, we will denote this wave as a generalized Rayleigh wave. For higher frequencies the width of the low-impedance layer width becomes important and the velocity of the guided wave decreases.

In the higher frequency range, above 1 MHz, the first pressure-symmetric Rayleigh-wave appears, with a cut-off frequency around 1.3 MHz. For even higher frequencies, higher-mode generalized Rayleigh waves will be visible. But for our purposes, we conclude that the main interface waves are given by the channel wave and the generalized Rayleigh wave.

To show the excitation of both interface waves we have modelled a configuration with the non-default source and receiver parameters shown in Table 7.3. In Figure 7.10(a) we have calculated the response for the ideal fluid case. We see for the direct wave a weak shear arrival below 50 μs. The modelled waveform with the layer shows a strong arrival around 50 μs, which corresponds to the generalized Rayleigh wave arrival. At 100 μs we observe the arrival of the strongly dispersive channel wave, for which the higher frequencies arrive earlier than the lower frequencies. Since the channel wave is highly dispersive, its group velocity depends strongly on the layer width. Hence, this channel wave could be a good candidate for an additional method to estimate the width of the fracture.

Besides the difficult measurement configuration to detect such a wave, care must be taken whether such a wave exists. Namely, when we investigate the effect of the viscosity on the excitation and propagation of guided modes,
we see a strong dependence on the relaxation time of the fluid. In case of
transmissions, below a relaxation time of $10^{-2}$ seconds, no shear transmis-
sion was observed above noise level, and hence no difference could be seen with
the ideal fluid model. It appears that a small change in the shear stiffness can
cause completely different behaviour of the guided waves. In Figures 7.10(b)–
(c) and 7.11(a) and (b) we show the modelled records for a relaxation time
of 1 s, $10^{-2}$ s, $10^{-4}$ s and $10^{-6}$ s, respectively. It will be clear that for the
two longest relaxation times the fluid will be effectively elastic with a shear
modulus of $6 \times 10^2$ Pa and $6 \times 10^4$ Pa, respectively. Only for a relaxation time
of a second in 7.10(b) we see the convergence to the ideal fluid case. In our
experiments we might expect a relaxation time on the order of $10^{-2}–10^{-3}$
s[see Joseph, 1990], in which case we see a generalized Rayleigh wave, a
weakened channel wave and possibly some other higher mode guided modes.
For shorter relaxation times the fluid slowly becomes more viscous. The
guided wave that we see in case of a relaxation time of $10^{-4}$ s is even more
intricate. For an almost Newtonian viscous fluid, with a relaxation time of
$10^{-6}$ s we see that the slower guided modes have completely disappeared and
that also the generalized Rayleigh wave has damped substantially.
In Chapter 5 we concluded that a thin viscous fluid/layer in a band-limited
domain in the slowness and frequency, can be replaced approximately by a
linear-slip model. To investigate the guided modes that propagate along a
linear-slip interface, we have calculated the response of a linear-slip interface
by using the transmission response, given in Eq. (5.59a). We have taken the
default parameter set and replaced the layer by a linear-slip interface, where
the fracture compliances have been chosen according to

\begin{align}
Y_N &= h \kappa, \\
Y_T &= 100 Y_N,
\end{align}

which uses the interpretation of the normal compliance in Eq. (5.47) and
a sufficiently large shear compliance $Y_T$ to approximate the vanishing shear
stresses. We see in Figure 7.10(c) that for the linear-slip model we mainly
have a weak generalized Rayleigh wave, whereas the channel wave is com-
pletely absent. Hence, both for a thin strongly viscous layer and a linear-slip
model the slow channel wave is absent. For lower viscosity a slow channel
wave can be observed.

Note that the linear-slip model converges to a thin-layer model, only in case
of low horizontal slownesses. The channel wave occurs at high horizontal
slownesses. If we define the apparent fracture compliance as the ratio between
Figure 7.10: Modelled waveforms including the guided waves for $h^S$ and $h^T$ equal to 0.5 mm for a): an ideal fluid and a relaxation time $\tau_r$ of b): 1 s and c): 0.01 s.
Figure 7.11: Modelled waveforms including the guided waves for $h^S$ and $h^T$ equal to 0.5 mm for a relaxation time $\tau$, of a): $10^{-4}$ s, b): $10^{-6}$ s and c): a linear-slip model.
the jump in the particle displacement and the average traction, by taking the Padé approximation for the ideal fluid case, we see that the apparent fracture compliance is given by the upper-right components of the system matrix of Eq. (5.44), i.e.

$$Y_N = \rho_f^{-1}(\rho_f \kappa f - \zeta_\alpha \zeta_\alpha),$$  \hspace{1cm} (7.21)

which becomes negative for horizontal slownesses greater than the fluid slowness. The channel wave occurs at high horizontal slownesses, which violates the assumption of a limited horizontal slowness as in Eq. (5.46a). The linear-slip model is not a valid replacement model for a thin layer, if we want to include the channel wave.

In addition, to investigate the guided modes for the linear-slip model in more detail, we have calculated the poles in the responses of the linear-slip interface of Eqs. (5.59a) and (5.59b). The results are consistent with previous results on interface wave propagation along fractures [PYRAK-NOLTE ET AL., 1992; NIHEI ET AL., 1995]. For a discussion on guided SH-waves we refer to NIHEI ET AL. [1995]. In Figure 7.12 we show the velocities of the guided waves in case of a linear-slip model, for a decreasing set of transversal fracture compliances. The normal compliance was taken in correspondence with Eq. (7.20a). The velocities for the guided waves for the range of transversal fracture compliances, were all found between the shear wave velocity of the solid embedding and the Rayleigh velocity. For all fracture compliance ratios no equivalent to a channel was found. The guided waves for the range of transversal fracture compliances have in common a pressure-symmetric interface wave with a cut-off frequency of around 1.1 MHz. This wave roughly corresponds to the generalized Rayleigh pressure-symmetric wave for the thin-layer model. Apparently, this wave is only dependent on the normal fracture compliance. For increasing ratio of the normal to shear fracture compliance the pressure-antisymmetric interface wave velocity tends more and more to the Rayleigh wave velocity.

From the discussion above, we conclude that a strongly dispersive channel mode will be observed for sufficiently small shear modulus. The channel mode will be absent for strongly viscous fractures, for which the skin depth is on the order of the layer width, or a fracture which responds according to the linear-slip model.
Figure 7.12: The guided waves for the linear-slip model for increasingly smaller transversal fracture compliance $Y_T$ and constant normal fracture compliance $Y_N$. In a): the phase-velocity and in b): the group-velocity as a function of the frequency $f$. 
7.2 Modelling results

7.2.5 Convolutional model

In Chapter 6 we postulated that we can predict the amount of dispersion of the compressional transmission through the fracture, relative to the measurement before fracturing by a convolution in the time domain with the elasto-dynamic transmission coefficient for compressional waves, see Eq. (6.18). In this section we verify the accuracy of the convolutional model of Chapter 6. To accomplish that, we have both modelled the transmission through the homogeneous block as well as the transmission including a fracture. Next, we have convolved the transmission through the homogeneous block with the transmission coefficient for the appropriate angle of incidence and width of the layer.

In Figure 7.13(a) we have followed the procedure that is described above for the default parameter set given in Table 7.1. The dashed line indicates the transmission without the fracture, the thick solid line indicates the transmission with the default fracture and the thin solid line with the diamond symbols gives the result of the convolution model. We have limited the time window to the compressional arrival, which for this purpose is the time window of interest. Note that the visual agreement is very good. Closer numerical analysis shows that the error of the amplitude attenuation in the convolutional model is less than a percent, which will be sufficient for our purpose. When we try to estimate the width of the fracture we do not expect an accuracy, with an uncertainty below a percent anyway. The convolutional model is not always applicable. For example when the vertical distances $h^S$ and $h^T$ are only 5 mm, instead of the default 15 cm, the convolutional model does not correctly describe the transmission with a fracture, as can be seen in Figure 7.13(b). The shear arrivals interfere with the compressional arrivals and the source and receiver are not in the farfield. Hence, the assumptions that we made to arrive at the convolutional model are violated.

Other examples where the convolutional model is clearly not applicable have already been discussed in Section 7.2.4. The guided wave that can be detected close to the fracture interface, are not included in the convolutional model because the convolutional model assumes that the transmission response is smooth.

On the other hand, the convolutional model is not necessarily restricted to a small fracture width. In Figure 7.13(c) we show the result for a fracture width of $3200 \mu$m. The convolutional model still agrees well with the result of full waveform modelling. Also, the convolutional model is not limited to perpendicular transmission. In Figure 7.13(d) we show the result for a lateral
Figure 7.13: Comparison of the full waveform modelling result of the transmission through the layer and the convolutional approximation for a:) the default parameter set, b): source-layer and layer-receiver distances $h^S$ and $h^T$ of 5 mm c): a fracture width $h$ of 3200 $\mu$m and d): a horizontal offset of 0.15 m.
offset of 15 cm, instead of the default of no horizontal offset. In this case the convolutional model still describes the apparent dispersion accurately. Hence, we conclude that for our laboratory experiments, where the vertical distances \( h^S \) and \( h^T \) are limited to 15 cm, the horizontal offsets below 15 cm, and the fracture widths below 300 \( \mu m \), we can use the convolutional model as a fast way to predict the amount of apparent dispersion.

### 7.2.6 Low-frequency approximations

In Chapter 5 the linear-slip model was presented as a thin-layer approximation, which resulted in a low-pass transmission response described by Eq. (5.59a) and more specifically Eq. (6.25a) for perpendicular incidence. For the default parameter set and a range of fracture widths, we have modelled the thin-layer response and the corresponding linear-slip response, where we have chosen the fracture compliances similar as in Section 7.2.4, Eqs. (7.20a) and (7.20b).

In Figure 7.14(a)–(d) the results are shown for a fracture width of 25 \( \mu m \), 100 \( \mu m \), 400 \( \mu m \) and 800 \( \mu m \), respectively. For a fracture width of a 100 \( \mu m \) or below we observe the apparent dispersion, which was described in Section 7.2.2. In those cases the thin-layer and linear-slip model agree well. When the layer becomes thicker and the multiples start to arrive separately as in Figure 7.14(d), the thin-layer response shows a set of arrivals whereas the linear-slip response has vanished almost completely. Hence, the linear-slip model incorporates \textit{a priori} that the response, of a fracture or thin layer is characterised by an effective dispersion. It appears that the linear-slip response starts to deviate from the thin-layer model at a fracture width of 200 \( \mu m \), which corresponds to the scaled one-way phase delay of \( 0.026 \times 2\pi \) for the centre frequency. In fact, this a relatively narrow range since in the laboratory the width of the fracture sometimes exceeds the value of 200 \( \mu m \).

The advantage of the linear-slip model is that it is more transparent which parameters mainly influences this dispersion as discussed in Section 7.2.2, namely the product of the cycle frequency, the fracture width, the fluid compressibility and the impedance of the compressional wave in the embedding, i.e. \( \omega h k Z_{P,e} \). Also, for more complex fractures, with fracture roughness or asperities and contact-mechanism, the physical processes will become increasingly more complex. The linear-slip model in that case provides a simple effective model for transmission and reflection measurements, assuming \textit{a priori} that the fracture will manifest itself as an apparent dispersion. For the width determination we will \textit{not} use the linear-slip model, because no con-
Figure 7.14: Comparison of the modelling results for the thin-layer and linear-slip model, for a fracture width of a): 25 μm, b): 100 μm, c): 400 μm and d): 800 μm.
siderable computation time is needed for the forward model based on the convolutional model and the thin-layer transmission response.
In Figure 7.15(a) we compare the compressional transmission coefficient $T_{P/P}$ of the linear-slip and the thin-layer model as a function of the angle of incidence. We observe that even for non-normal incidence the linear-slip as a thin-layer approximation will be sufficient for most cases. Differences start to appear for higher horizontal slownesses when comparing the shear transmission coefficient $T_{SV/SV}$ of the linear-slip and the thin-layer model as a function of the angle of incidence, see Figure 7.15(b). Only for high angle of incidence, beyond 34 degrees, where total reflection occurs, noticeable differences are seen. But based on Figure 7.15(a) and (b) we conclude that for a thin layer with a width below 200 $\mu$m, pure transmission (and reflection) measurements do not provide a way to distinguish between a mechanically open thin-layer model or a linear-slip model. We have to go to responses influenced by even higher slownesses, related to guided waves, to distinguish these models.

The comparison of the linear-slip model and the thin-layer model in this chapter shows that there are two main differences between these models. First, the slow channel wave is absent for the linear-slip model. Second, in the thin-layer model the dispersion that we observe is interpreted as resulting from a true physical open fracture with a certain width, whereas in the linear-slip model the dispersion is attributed to the compliance of the fracture, which can be determined by a range of physical processes such as mechanical contact, surface roughness. The resolution analysis of the compressional and shear waves will be used in Chapter 11, where we will determine the width of hydraulic fractures.

The modelling of the guided waves will be used to understand the guided waves that are excited for a fracture of a finite lateral extent in Chapter 8. The velocities of these guided waves are used to predict the arrival times of events, which have travelled partly as guided waves along the fracture. This will aid to unravel the interpretation of the various diffracted events that we will show in Chapter 10.
Figure 7.15: Comparison of the thin-layer and linear-slip model in terms of the modulus of a): the compressional transmission coefficient $|T_{P/P}|$ and the shear transmission coefficient $|T_{SV/SV}|$. 
Scattering by a Finite Fracture

In Chapter 7 we have discussed modelling examples of transmissions through a fluid-filled layer of infinite lateral extent. Besides transmission and reflection of compressional and shear waves, also interface waves were shown to propagate along the fracture. In reality fractures will always be of finite lateral extent.


Modelling the scattering of a fracture having both a finite width as well as a finite lateral extent is a difficult numerical problem, since this problem combines a high impedance contrasts between the solid and fluid with impedance variations across the width of the fracture that occur at a scale below the wavelength. Because of the high contrast no weak contrast approximations can be made such as the Born approximation. Nevertheless this approximation has been used for modelling the scattering of waves by hydraulic fractures [Meadows AND Winterstein, 1994]. Since the size of the fracture extends several wavelengths no low-frequency approximation can be made. Furthermore, difficulties will be encountered handling the fluid-solid interface with the vanishing shear stress, the tip of the fracture which acts
as a secondary source and the interface waves that travel along the fracture interface. Nevertheless, we would like to get insight in the complex scattering phenomena by a fluid-filled fracture, to aid in the interpretation of the acoustic data that we measure in the laboratory. We will shortly review the different approaches that could be taken to tackle this numerical problem along with their advantages and disadvantages. Then, we show the possibilities of using a staggered-grid finite-difference scheme which allows for a local adjustment of the grid spacing, to conform to the impedance variations that occur across the thin fluid-filled fracture. Finally, we will interpret the results that we have obtained with a finite-difference modelling code.

8.1 Overview of numerical techniques

For a fracture of finite lateral extent we cannot use horizontal transform techniques easily, since the configuration at hand does not show translation invariance in any direction. Therefore the frequency and slowness synthesis method presented in Chapter 4 and 7 is not applicable for such a configuration. For complex configurations, probably the easiest and most flexible modelling method is the finite-difference method in the space-time domain. The continuous derivatives occurring in the wave equation are replaced by their discrete counterparts. This leads to a scheme in which the particle velocities and stresses are updated for each discrete time step. The most stable method is obtained when so called staggered grids are used [Virieux, 1986; Levander, 1988]. In the staggered grid approximation the particle velocities are ordered in a separate grid uncoupled from the stress grid, shifted half a grid in space and half a step in time. For our purpose it is important to know that the finite-difference technique with staggered grids is known to lead to stable results for the computation of derivatives across fluid-solid interfaces [Virieux, 1986], Hence, we are not forced to reformulate the numerical problem due to the presence of the explicit boundary condition that the shear stresses should vanish.

The main problem of standard finite-difference is that if we want to model a thin hydraulic fracture with a width on the order of a hundred micron, we need a grid spacing on the order of ten micron. This means that in order to model in a two-dimensional approximation a square decimetre we would already need $10000 \times 10000$ grid cells which results in a tremendous
CPU-time. As an alternative we could approximate the fracture with a linear-slip interface, representing a fracture with a vanishing width. In that case the linear-slip boundary condition must be incorporated into the finite-difference scheme. The disadvantage is that we would a priori discard any slow channel waves that can propagate along the fracture. The linear-slip condition cannot easily be implemented in a finite-difference scheme, since it is an implicit boundary condition relating the jump in the particle velocity to the interface traction. The actual implementation has been accomplished by replacing each grid cell containing a part of the fracture surface by an estimate of a homogeneous equivalent medium [Coates and Schoenberg, 1995]. Hence all grids containing the fracture with a vanishing width are replaced by a line of grid cells. This line of grid cells represents a layer, for which its width corresponds to the size of the grid cell and its elastic and fractures stiffnesses are replaced by some appropriate anisotropic stiffnesses that try to mimic the same scattering behaviour of the linear-slip interface. After this procedure, the standard finite-difference scheme can be applied. The modelling results show that at least diffractions are generated, scattered from the fracture tip and reflection and transmission losses occur due to the presence of the fracture.

A problem of the latter approach is that if we want the use the modelling scheme to understand the amplitude behaviour of the diffraction from the tip, the different assumptions that must be applied are difficult to justify at the fracture tip itself. Namely, since the excitation and propagation of the slow channel wave is coupled with the scattering at the tip of the body waves and generalized Rayleigh wave, in principle the scattering problem of a thin fluid-filled layer of finite lateral extent is not formulated exact. Although for some configurations these effects might be negligible, it is difficult to see beforehand what the order is of the errors that are made.

In addition, the interpretation of the linear-slip parameter becomes unclear in the vicinity of the tip of the fracture. Part of this problem has already been recognised in Coates and Schoenberg [1995], from which we quote:

"The second step in the validation process concerns the accuracy of the linear-slip condition itself. This can be assessed only by comparison with field data. ... similar work is needed to address the question of slip variation near the end of faults: does slip stop abruptly or taper off gently, or does the fault terminate in an extended zone of highly damaged material?"
Figure 8.1: Conceptual problem of the fracture stiffness in the vicinity of the fracture tip. Resistance against local opening of the fracture, especially in the vicinity of the tip, is controlled by the resistance against deformation of the elastic embedding.

First of all, we should assess the important physical processes at the fracture tip. Indeed, the only validation can come from real physical experiments. It is important that we consider whether it suffices to describe the local width profile at the fracture tip or whether we do need to include friction or a damage zone in front of the fluid front. But in order to assess whether friction or damage is important, we should have a reference model without friction that describes the scattering correctly. Even for such a simplified fracture model the question remains whether we can use a linear-slip model to represent the fracture. In Chapter 5 we have shown that the linear-slip model has been derived for a layer of infinite lateral extent, for which the fracture stiffness is linearly related to the fracture width. In the vicinity of the tip we expect the stiffness of the fracture to be determined not only by the width of the fracture and compressibility of the fluid but also by the presence of the tip which will effectively increase its stiffness locally, because the stiffness is controlled by resistance against deformation in the elastic embedding in the vicinity of the fracture tip. A graphical illustration of this effect is shown in Figure 8.1.

Recently, also a modified displacement discontinuity model was introduced [see Boadu, 1997], in which the fracture is mainly determined by its length
instead of its width, especially in the low-frequency regime.
In this respect it is also important to distinguish the concept of hydraulic
fracture stiffness as is being used in static stress analysis from the fracture
stiffness as is being used in the linear-slip theory. In static stress-strain theory
it is known that for a very flat two-dimensional crack under plane strain, the
shape of the fracture is described by an ellipse, where the opening of the
fracture under a constant fluid pressure $p$ in terms of the minor semi-axis $w$
is given [see Jaeger and Cook, 1969] by

$$w = \frac{2R(1 - \nu^2)}{E} p, \quad (8.1)$$

where $E$ and $\nu$ are Young’s modulus and Poisson’s ratio, respectively and $R$
is the major semi-axis. This shows that in the static limit the resistance against
opening of the very thin fracture is completely determined by the presence
of the tip and the resistance against deformation of the solid embedding.

The essential difference between the static stiffness and linear-slip fracture
stiffness is that for dynamic waves, we consider local deformations, whereas
in static stress-strain theory the opening of the fracture is global. When
the local dynamic deformations have not reached the tip of the fracture the
stiffness will be determined by the local interface properties. When the area
of local deformation contains the tip of the fracture, the effective resistance
against opening will in addition be influenced by the proximity of the tip.

Since in Eq. (8.1) we consider the opening of the complete fracture, the
resistance against opening is mainly influenced by the size of the fracture
and the stiffnesses in the solid embedding. For a direct application of the
linear-slip model we do not know beforehand how the fracture compliance
must be tapered towards the tip.

As an alternative to using the linear-slip model we could formulate the scat-
ttering by a thin fluid inclusion with a coupled set of boundary integral equa-
tions. For a configuration as shown in Figure 8.2 the scattering problem can
be formulated [Fokkema, 1979] as

$$\hat{u}_m(x', s) = \hat{v}_m(x', s) +$$

$$\int_{x \in \partial V} \left( \hat{G}_{mj}^{v, f}(x' - x)p(x)n_j + \hat{G}_{mkl}^{v, h}(x' - x)n_k\hat{v}_l(x) \right) \, dA,$$

$$x' \in \mathbb{V'} \quad (8.2a)$$
for the wavefield outside the fluid inclusion and
\[
\hat{p}(x', s) = -\int_{\partial V} (\hat{\Gamma}_i^{p,f}(x' - x)\hat{p}(x)n_i + \hat{\Gamma}_i^{p,q}(x' - x)\hat{v}_i(x)n_i) \, dA,
\]
\[x' \in V, \quad (8.2b)\]
for the wavefield internal to the fluid inclusion.

The Green states of Eqs. (8.2a) and (8.2b) [DE HOOP, 1995], are given by
\[
\hat{\Gamma}_i^{p,q}(x' - x) = s\rho_f \hat{G}_{P,f}(x' - x), \tag{8.3a}
\]
\[
\hat{\Gamma}_3^{p,f}(x' - x) = -\partial_3^i \hat{G}_{P,f}(x' - x), \tag{8.3b}
\]
\[
\hat{G}_e^{v,h}(x' - x) = -\rho_e^{-1}c_{kmij}^e \partial_m^i \hat{G}_{rk}(x' - x), \tag{8.3c}
\]
\[
\hat{G}_r^{n,f}(x' - x) = +s\rho_e^{-1}\hat{G}_{rk}(x' - x), \tag{8.3d}
\]
in which
\[
\hat{G}_{rk} = c_S^{-2}\hat{G}_{S,e}\delta_{rk} + s^{-2}\partial_k^i \partial_k^j [\hat{G}_{P,e} - \hat{G}_{S,e}], \tag{8.4}
\]
and the Green function for the Helmholtz equation is given by
\[
\hat{G}_{P;S,e;f}(x' - x) = \frac{\exp(-sc_{P;S,e;f}^{-1}|x' - x|)}{4\pi|x' - x|}, \tag{8.5}
\]
for the compressional or shear wave in the embedding or fluid medium, respectively.

When we approach the surface of the inclusion for Eqs. (8.2a) and (8.2b) these integral representations become a coupled set of integral equations, both for the pressure and particle velocities at the surface of the inclusion. The inner integral equation, Eq. (8.2b) relates the pressure and normal particle velocity for a general incident wavefield, and can therefore be interpreted as the general form of a boundary condition of the fracture. In this case the boundary condition is non-local and non-instantaneous.

From these equations we see that in principle we do not need the concept of a local fracture compliance for a hydraulic fracture determined only by the local interface parameters. This conclusion assumes that no complicated non-linear phenomena, like friction, have to be included for the tip behaviour. The solution of the pressure and particle velocity in principle depend on the geometry of the fluid inclusion and the incident field. For known pressure and particle velocity we could assign an apparent fracture compliance as a function of the lateral position as the tensor relating the jump of the particle velocity to the average traction across the interface. Hence we avoid the question how the fracture compliance should be tapered towards the tip, since the answer of this question is part of the total scattering problem.

The solution of this scattering problem is beyond the scope of this thesis and is a challenging problem in itself. But a general conclusion that can be made, is that the apparent fracture compliance is no longer only an interface parameter, but also a geometrical property dependent on the actual size (and shape) of the fracture. The integral equations indirectly express both the stiffness resulting from the interface properties, such as the width $h$ and compressibility of the fluid $\kappa_f$, as well as from the geometrical properties.

As a first guess, we can neglect the geometrical stiffness which will be most influential at the tip and replace the inner integral equation with the linear-slip condition valid for an interface of infinite lateral extent. Thereby we assume that the fracture length is large compared to the wavelength of the local deformations. The linear-slip condition can be viewed as an appropriate preconditioner of the inner integral equation. Close to the tip we can then expect the solution to become erroneous.

For the outer integral equation we could additionally assume that in first order the pressure and particle velocity at the interface of the inclusion would be given by the pressure and particle velocity if the fracture would be of infinite lateral extent as well. This approximation, known as the Kirchhoff
approximation [LIU ET AL., 1997] can be viewed as an appropriate preconditioner of the outer integral equation. Since a similar assumption has been made for the internal integral equation, this approach seems to be a natural choice. The modelling results in LIU ET AL. [1997] show that also many important features of the scattering by a hydraulic fracture are incorporated. Nevertheless, only based on a full solution of the integral equation we can judge the validity of the succession of approximations. The advantage of a boundary integral equation instead of a finite-difference technique is that the wavefield only has to be solved on the surface of the inclusion, instead of in the embedding too. The Green functions occurring in Eqs. (8.2a) and (8.2b) are all singular or hyper-singular. The singular Green functions radiate towards other locations on the fracture surface, for example on the opposite side of the fracture. Related with the fact that the width of the fracture is very thin and the strong singularity of the Green functions the correct discretization of this problem and its solution is complex.

Recent developments of the finite-difference technique has lead to a formulation which allows the grid spacing to be adjusted to the scale at which variations take place in the medium. Application of this technique was found in the modelling of the radiation and scattering of borehole experiments [see FALK ET AL., 1996]. Note that for cross-borehole or vertical seismic profiling (VSP) measurements the seismic wavelength in the embedding is usually an order larger than the size of the borehole. If we want to model the radiation of a source in the borehole or the reception of energy in the borehole, we have to refine the grid spacing locally at the borehole.

Because the borehole is a cylinder, the full solution of this scattering problem should be implemented in a three-dimensional grid. In the two-dimensional approximation the thin fluid borehole corresponds to a thin fluid layer. The similarity of the scattering problem of the fluid-filled borehole with the thin fluid-filled fracture motivated us to try this new finite-difference technique and has led to co-operation with the University of Hamburg. The finite-difference code has been developed by J. Falk [FALK ET AL., 1996] of the University of Hamburg. The medium configuration that we want to model is given in Figure 8.3.

The model includes a steel borehole with an open fluid-filled borehole in the centre and a fracture perpendicular to the wellbore with a length of 10 cm measured from the centre of the wellbore. The width of the fracture is taken 100 μm, constant over the fracture length. Hence, we do not try to simulate the complex shape of the fracture that we expect, such as a elliptical shaped
Figure 8.3: Medium and source configuration for the finite-difference modelling, with 1: the fracture, 2: the solid embedding, 3: the fluid-filled borehole, 4: the steal casing, 5: the transducer and 6: section of the model for Figure 8.4

fracture. At the top of the model we have a shear transducer with a finite aperture of half an inch in diameter, corresponding to the aperture of the transducers that we use in the laboratory. The stiffnesses and densities of the embedding and fluid are all taken similar to the default parameter set as in Chapter 7, Table 7.1 as given on page 130.

The grid spacing of the finite difference model is both determined by the scale of the structural variations and the smallest wavelength of the propagating waves. The grid staggering technique is a potential disadvantage because the location of interfaces across which the constitutive parameters show discontinuous jumps, for example the fluid-solid interface of the fracture, are not exactly defined. This problem can be overcome by using sufficiently fine spacing in the vicinity of the interfaces. The vertical grid spacing in the
embedding is 0.45 mm, whereas the horizontal grid spacing is taken as 0.15 mm. The horizontal grid spacing is taken smaller than the vertical grid spacing to accommodate to the velocity of the slow channel wave, described in Chapter 7 which propagates along the fracture. The vertical grid spacing is refined stepwise with a factor of three, which leads to a variation of vertical grid spacing according to

\[
\begin{align*}
\Delta z(1) &= 0.45 \text{ mm}, \quad (8.6a) \\
\Delta z(2) &= 0.15 \text{ mm}, \quad (8.6b) \\
\Delta z(3) &= 0.05 \text{ mm}, \quad (8.6c) \\
\Delta z(4) &= 0.016 \text{ mm} = 16 \mu\text{m}. \quad (8.6d)
\end{align*}
\]

The finest grid spacing is taken around the fluid-filled fracture with a width of 100 μm, which hence can be modelled with six grid cells in the vertical direction. The selection of the grid indicated with a square box in Figure 8.3 is shown with the details of the grid spacing in Figure 8.4 on page 167.

### 8.2 Modelling results

We will show snapshots of the wavefield in terms of the pressure \( p(\mathbf{x}, t) \) and shear strain of the displacement field, see Eq. (3.2), which in two-dimensional space are defined as

\[
\begin{align*}
p(\mathbf{x}, t) &= \frac{1}{2}(\tau_{11}(\mathbf{x}, t) + \tau_{33}(\mathbf{x}, t)), \quad (8.7a) \\
\epsilon_{13} &= \frac{1}{2}(\partial_1 u_3(\mathbf{x}, t) - \partial_3 u_1(\mathbf{x}, t)), \quad (8.7b)
\end{align*}
\]

In this way we can distinguish in the homogeneous solid embedding between the compressional wave and shear wave part of the wavefield. The snapshots are given at 70, 90, 110 and 130 μs, in Figures 8.5, 8.6, 8.7 and 8.8, respectively. The model of the medium in the snapshots is exactly as illustrated in Figure 8.3, where the origin (0, 0) is located at the centre of the borehole at the depth of the centre of the fracture. The positive vertical direction points downward, in agreement with the conventions in Chapter 4.

In the snapshot at 70 μs we mainly see the incident wavefield approaching the fracture, in Figure 8.5 indicated with a 1. Especially in this two-dimensional configuration the borehole appears to be a strong reflector which adds to a secondary incident wavefield at the fracture, in Figure 8.5 indicated with a 2. The top of the grid, despite the damping boundary conditions that have been applied still reflects some part of the energy that has been radiated upwards.
\[ \Delta x = 150\mu m \quad \Delta z(1) = 450\mu m \]

\[ \Delta z(2) = 150\mu m \]

\[ \Delta z(3) = 50\mu m \]

\[ h = 100\mu m \quad \Delta z(4) = 16.67\mu m \]

**Figure 8.4:** The grid model for the selected part in Figure 8.3 with 1: the fracture with a width indicated by the black box and 2: the solid embedding.
from the source, which subsequently is propagated downward (in Figure 8.5 indicated with a 3). Because the source transducer mainly radiates shear energy no strong events are seen yet in the pressure snapshot. Nevertheless some weak reflections and diffractions can be seen from compressional energy that has been radiated from the source. Because of the aperture of the source each events is seen with a double arrival time, from the left and right edge of the transducer aperture, respectively.

In the snapshot shown in Figure 8.6 at 90 μs, the wavefield has hit the hydraulic fracture and most energy of the shear wave is reflected. Next to the incident wavefield (indicated with a 1) behind the fracture shear-wave shadowing is observed. At larger incident angles some mode-converted shear-wave energy can be observed. A similar phenomenon is seen for the reflection of the borehole, in Figure 8.6 indicated with a 2. Accidentally, the artificial reflection from the top of the grid is just about to hit the fracture, indicated with a 3. The incident shear wave creates both a compressional diffraction from the fracture tip (indicated with a 4) and a reflection from the fracture surface as well as shear diffraction (indicated with a 5) and reflection. The borehole reflection at this instant just initiates a diffraction at the tip as well, indicated with a 6. More interesting, the interaction of the incident wavefield at the intersection of the fracture with the wellbore creates a conversion of mainly two wave modes: First, both in the pressure and shear strain we observe a fast guided mode propagating along the fracture, which is anti-symmetric with respect to the pressure (indicated with a 7). Second, we see the propagation of a much slower guided wave along the fracture, which is symmetric with respect to the pressure (indicated with an 8). For the snapshot of the shear strain the slow guided wave is overshadowed by the artificial reflection from the top of the grid. We interpret these two events as the mode-conversion to a fast pressure anti-symmetric Rayleigh wave and a slow pressure symmetric channel wave. A the fracture tip the incident wavefield is also mode-converted to a guided Rayleigh wave (indicated with a 9) which travels towards the wellbore. Along with the propagation of the compressional reflection and transmission along the borehole we see a mode-converted shear head wave propagating both upward and downward, indicated in Figure 8.6 with a 10.

In Figure 8.7 the snapshot at 110 μs shows that both the incident wavefield indicated with a 1 and the borehole reflection indicated with a 2, have reached the lower-right corner of the grid. Besides the shear reflection from the artificial reflection from the top of the grid (indicated with a 3) we can still
see the tip diffractions from the incident wavefield as well as the borehole reflection (indicated with a 5 and 6, respectively). The Rayleigh wave almost reaches the fracture tip, whereas the slow guided channel wave is now far behind the Rayleigh wave. The Rayleigh wave and slow channel wave are indicated with a 7 and 8, respectively. To the left of the slow channel wave, we can see the Rayleigh wave which has been mode-converted at the tip and propagates towards the borehole (indicated with a 9). Although this event is not clearly seen in the snapshots, additional evidence of this event will be shown later in this chapter. We still observe the strong shear head wave (indicated with a 10) along the borehole accompanying the compressional reflection and transmission.

In the snapshot at 130 μs, shown in Figure 8.8 the incident wavefield as well as the borehole reflection is no longer seen. We still see some shear energy diffracted from the fracture tip to the upper-left corner, indicated with a 5 and 6 respectively, for the incident wavefield and borehole reflection. More important, we see that the Rayleigh wave has reached the fracture tip and is re-diffracted to mainly shear energy in the forward direction, indicated in Figure 8.8 with a 12. Also, some conversion to a compressional wave is observed. When we investigate this compressional diffraction in more detail we find that the radiation pattern corresponds closely to a quadrupole. Moreover, the Rayleigh wave is partly reflected to a Rayleigh mode as well. The Rayleigh wave, which has been mode-converted at the tip is at this instant being reflected at the fluid-filled borehole (indicated with a 9). Apparently, this transition at the borehole creates a rather complex reverberating reflection.

To study the complex interaction of the guided waves that propagate along the fracture in more detail, a set of seismograms has been calculated for a line of hypothetical receivers located slightly above the fluid-filled fracture. The seismograms, again both for the pressure and shear strain are shown in Figure 8.9(a) and (b), respectively as a function of the lateral position. Again, the centre of the borehole is taken as the origin of the coordinate system. Because the borehole is fluid-filled, no energy for the shear strain is seen in the borehole with a radius of 11 mm. First we see the incident shear energy at the fracture along with the borehole reflection, indicated with a 1 and 2, respectively. The incident shear energy is diffracted to a compressional event, indicated with a 4. The shear diffraction is difficult to separate from the incident wavefield. Clearly, we see that the interaction of the wavefield at the intersection of the fracture with the borehole, creates
Figure 8.5: Snap shots of the pressure and shear strain at 70 μs, with 1: The incident shear wave, 2: the borehole reflection of the shear wave and 3: the artificial reflection of the source at the top of the grid.
Figure 8.6: Snap shots of the pressure and shear strain at 90 μs, with 1: The incident shear wave, 2: the borehole reflection of the shear wave, 3: the artificial reflection of the source, 4: the compressional diffraction, 5: the shear diffraction, 6: the shear diffraction from the borehole reflection, 7: the pressure anti-symmetric Rayleigh mode, 8: the pressure symmetric slow channel wave, 9: the tip conversion to the Rayleigh wave and 10: the headwave of the fast borehole mode.
Figure 8.7: Snap shots of the pressure and shear strain at 110 μs, with 1: The incident shear wave, 2: the borehole reflection of the shear wave, 3: the artificial reflection of the source, 5: the shear diffraction, 6: the shear diffraction from the borehole reflection, 7: the pressure anti-symmetric Rayleigh mode, 8: the pressure symmetric slow channel wave, 9: the tip conversion to the Rayleigh wave and 10: the headwave of the fast borehole mode.
Figure 8.8: Snap shots of the pressure and shear strain at 130 μs, with 3: the artificial reflection of the source, 5: the shear diffraction, 6: the shear diffraction from the borehole reflection, 8: the pressure symmetric slow channel wave, 9: the tip conversion to the Rayleigh wave, 10: the headwave of the fast borehole mode, 11: the Rayleigh to compressional rediffraction and 12: The Rayleigh to shear rediffraction.
a strong conversion to a Rayleigh mode as well as a slow channel mode, indicated with a 7 and 8, respectively. If we estimate the velocity of two guided waves, based on the slope of the arrival time curves we estimate a velocity of 1984 m/s and 1000 m/s, for the generalized Rayleigh and slow channel wave, respectively. These velocities are in good agreement with the results of Chapter 7, Figure 7.9 on page 144. Also, we see the conversion of the shear wave at the tip to a Rayleigh wave which propagates towards the borehole. Most prominently, we observe that around 115 \( \mu \text{s} \), the Rayleigh wave from the borehole is re-diffracted towards a strong shear diffraction in the forwards direction, indicated with a 12. No re-diffraction towards a compressional event is seen, mainly because of the quadrupole character of the radiation pattern of the compressional re-diffraction, which results in no energy in the forward (and backward) direction. The Rayleigh wave which has been converted at the fracture tip is reflected at the borehole, both towards a Rayleigh as well as a slow channel wave around 125 \( \mu \text{s} \). When the Rayleigh wave reflection reaches the fracture tip again, also a re-diffraction towards a shear wave is seen (indicated with a 13). Note that the pressure-symmetric slow channel wave is for the largest part reflected at the tip inside the fracture, which hence results in a relatively weak re-diffraction of this particular mode.

We want to use these finite-difference modelling results to assess the importance of all the events that are measured and discussed in Chapter 10 for a typical diffraction measurement configuration in our laboratory. For that purpose, we calculated a set of hypothetical seismograms for a line of receivers on the right side of the grid as a function of the vertical position. The origin is taken at the depth of the centre of the fracture whereas the positive vertical direction points downward, in agreement with the conventions in Chapter 4.

The strongest event that can be observed is the incident wavefield, indicated with a 1 and the borehole reflection, indicated with a 2. From a monitoring point of view, these events are important, because their arrival time is independent on the fracture dimensions. In the pressure part the strongest event is the reflection and diffraction from the fracture interface and tip, indicated with a 4, exited by the incident shear wavefield and the borehole reflection. In the shear strain we observe, especially at relatively large vertical positions, a strong shear diffraction, both from the incident wavefield (indicated with a 5) as well as from the borehole reflection (indicated with a 6). Moreover we observe the strong shear head wave from the interaction at the borehole inter-
Figure 8.9: Seismograms of pressure (a) and shear strain (b) slightly above the fracture, with 1: The incident shear wave, 2: the borehole reflection of the shear wave, 4: the compressional diffraction, 7: the Rayleigh mode, 8: the slow channel wave, 9: the tip conversion to the Rayleigh wave, 12: the Rayleigh to shear rediffraction and 13: the shear rediffraction of the Rayleigh wave after reflection at the borehole.
Figure 8.10: Seismograms for the pressure (a) and shear strain (b) on the side of the block with 1: the direct shear wave, 2: the borehole reflection, 4: the compressional reflection and diffraction, 5: the shear diffraction, 6: the shear diffraction of the borehole reflection, 10: the head wave from the fast borehole mode, 12: The rediffraction of the Rayleigh mode and 13: the shear rediffraction of the Rayleigh wave after reflection at the borehole.
face, indicated with a 10. The strong event at roughly 150 $\mu$s corresponds to the re-diffraction of the Rayleigh wave, initiated at the borehole intersection of the fracture. This event is observed at relatively small vertical offsets from the depth of the fracture. At the end of the seismograms roughly at 200 $\mu$s, we also observe the re-diffraction of the Rayleigh wave, which was initiated at the fracture tip and reflected at the borehole, before being re-diffracted again to a shear wave, mainly in the forward direction.

From the modelling example that we have discussed, we conclude that the scattering response of a hydraulic fracture is predominantly determined by the interaction of the wavefield at the fracture tip and at the intersection of the fracture with the borehole. The fracture tip generates diffractions which can be measured at the side of the block. We will use these diffractions in Chapter 10 to determine the size of the fracture during its growth. Two different guided waves can be observed, in agreement with Chapter 7, which propagate along the fracture, namely a generalized Rayleigh wave and a slow channel wave. These guided waves are also excited at the fracture tip and the intersection of the fracture with the borehole. Whereas the Rayleigh wave shows a strong re-diffraction, the slow channel wave is strongly reflected at the fracture tip. Hence, the energy reverberates in the fracture. In Chapter 10 we will show that events can be observed which indirectly prove the propagation of the generalized Rayleigh wave. We will show in Chapter 11 that the shear-wave shadowing described in this chapter is observed for our laboratory experiments.
Part III

Experiments

“When you ask the papalagi why he thinks so much, he will answer: ‘Because I don’t want to remain stupid’. A papalagi that doesn’t think is considered valea, though in reality it is better not to think often and still find your way around. But personally I am convinced it is just a pretext and the papalagi have had intentions with their thinking. Their real aim is hunting the powers of the Great Spirit. An aim they gave the fancy name of ‘research’. Research means looking at something so close up that you bump against it and even through it with your nose. That bumping and stirring around is a distasteful and lowdown habit of the papalagi. They take a beetle or butterfly, run it through with a small spear and wrench out a leg. What does it look like, such a tiny leg, apart from the body? How was it attached to the body? He breaks the leg to measure its thickness. That’s important, very important. He chips off a fragment of that leg, as small as a grain of sand, puts it under a long tube that has the magic of making everything clearly visible. They investigate everything with that big, sharp-looking eye, your tears, a piece of your skin, a hair, everything, everything. All these things are pared down until it’s impossible to chip off another fragment. Although that object has been reduced to the smallest size, it now becomes extremely important, because here starts the deepest knowledge only the Great Spirit knows about.”

from 'The Papalagi - Speeches by Tuiavii of Tiavea, a Samoan chief’ on 'The Heavy Thinking Sickness'. 
Description of the Experiments

In this chapter we will describe the experimental set-up of the laboratory experiments, where we focus on the experimental improvements which have been attained since Savić [1995], that were necessary to obtain clear dispersion measurements of compressional transmissions as well as shear-wave measurements. Some examples of the improvement of the quality of the acoustic data are shown. After that, we will shortly elaborate on two hydraulic fracture experiments performed on cement blocks. A description of the background of these fracture experiments is necessary to understand the acoustic data that will be shown in the following chapters.

9.1 Introduction

Since 1991 about a hundred small-scale hydraulic fracturing experiments have been carried in the laboratory of rock physics of the Faculty of Applied Earth Sciences. The experiments were performed on sandstone, gypsum, diatomite and cement blocks for a wide variety of borehole completions, borehole orientations, background stress states, fracture fluids and fluid injection rates. Because of the size of the tri-axial pressure machine the field conditions have to be down-scaled to simulate realistic fracture experiments. Correct scaling of experiments implies that the physical processes that are taking place
under laboratory conditions must be representative of the physics of fluid-driven fracture propagation at field scale [De Pater et al., 1992]. To obtain stable and realistic crack propagation, fracture experiments are performed with highly viscous silicon oils and low-toughness materials such as cement. The toughness of the material is a measure of the resistance of a material against fracturing [Ingraffea, 1987]. The number of field treatments which have been combined with active acoustic measurements has been limited and only partly successful because of the complex acquisition geometry and conditions at depth. Both a differential VSP-experiment [Turpening, 1984] and a cross-well experiments at the Belridge field [Wills et al., 1992] have shown the possibility to monitor fracture propagation by using the shadowing of shear waves when the fracture intervenes the source and receiver.

The advantage acoustic monitoring in the laboratory compared to the field is the flexibility to try various acquisition geometries for relatively low costs. The laboratory experiments provide a unique opportunity to gain experience in the acoustic data that can be acquired and their application. At a later stage, measurements that have proven their scientific value in the laboratory could be implemented during field treatments, if the value of such measurements would exceed the financial costs. At the same time, we must be aware that the fracturing experiments have been down-scaled to conform to the physics of fracture growth in the field and not specifically to the physics of acoustic measurements in the field. For example the high frequency behaviour of the viscous fluids in the laboratory might completely differ with the typical behaviour of fluids for lower frequency field experiments. We stress that the application of acoustic monitoring not only lies in a feasibility study for field measurements but also in the direct information that can be acquired in the laboratory on the fracture growth.

The first acoustic measurements of hydraulic fracture growth, performed in the research laboratory of Mobil [see Medlin and Massé, 1984] showed the possibility to measure fracture length by detecting transmission losses of compressional waves. Years later, a state-of-the-art acoustic monitoring system was implemented in the tri-axial pressure machine [Savić, 1995], in the laboratory of Rock Physics of the Faculty of Applied Earth Sciences. These experiments resulted in the first detection of strong diffractions scattered from the perimeter of the fracture. The arrival time of the diffractions was used to estimate the fracture length [Savić, 1995]. Both Medlin and Massé [1984] and Savić [1995] claimed the observation of a dry fracture front preceding the
fluid-filled fracture, confirming theoretical predictions [Barenblatt, 1956],
based on the interference pattern in the transmission records. No diffrac-
tions, scattered directly from both the dry tip and fluid front were measured
yet.
Even of more importance, a measurement of the width of the fracture, in the
tri-axial pressure machine could only be done at the borehole itself with a
LVDT (Linear Variable Differential Transformer). The fact that we could not
monitor the width profile of hydraulic fractures was unsatisfactory, since such
a measurement would be very practical to evaluate the theoretical models. A
more detailed description of the dimensions of the fracture, both the length
and the width, would restrict the degree of freedom of the theoretical models.
As has been described in Chapter 6, we postulate that the dispersion of com-
pressional transmissions relative to the measurement before fracturing could
be used to estimate the fracture width. Previously, no successful shear-wave
measurements have been made in the laboratory, related to the much more
critical coupling of shear contact transducers, especially combined with the
complex tri-axial pressure machine. For a complete and consistent descrip-
tion of the acoustic response of the fracture we need to include shear-wave
measurements.

9.2 Experimental set-up

A detailed description of the experimental set-up for the fracture experiments
is given in [Weijers, 1995], while the acoustic part is described in [Savić,
1995]. We will limit ourselves to those parts of the experimental set-up that
are necessary to understand the data that will be shown in the following
chapters.
A cubic block with edges of 0.3 m is mounted in the true tri-axial compression
machine. The pressure machine consists of three perpendicular compression
systems, which each can independently deliver a maximum force of 3500
kN, see Figure 9.1. This way we can apply an anisotropic stress field to the
sample. Since this is an open system, it is not possible to apply pore pressure
to the rock. For this block size, the maximum pressure that we can impose
on each side is 40 MPa, which corresponds effectively to down-hole situations
at a depth of about 3000 m.
In the cubes, a borehole with a diameter of 23 mm is made, to simulate the
well. Although different wellbore completions are used, we focus on a wellbore
which is oriented vertically in line with the smallest principal stress, whereas
in a horizontal plane the stresses are isotropic. In the borehole wall only one circumferential notch is cut, positioned in the middle. This geometry is used to initiate simple radial fractures which grow perpendicular to the borehole. Inside the borehole a Linear Variable Differential Transformer (LVDT) is clamped above and under the notch. A side view of the fracture with the borehole, borehole LVDT, transducer plates and fluid pump is shown in Figure 9.2. The LVDT consists of a core, which is free to move up and down between a primary pair of electrically charged coils. The displacement measurement is then based on the induced voltage in a secondary pair of coils [Horowitz, 1980]. When the fracture is opened the LVDT clamps will move with respect to each other. The LVDT measures both the opening of the fracture and the deformation inside the rock between the two LVDT clamps. A correction for the rock deformation is applied, although this is only on the order of a few percent of the total displacement. By this method a width measurement of the hydraulic fracture can be obtained with an accuracy on
Figure 9.2: The rock sample with 1: the fracture, 2: the lvd-clamps, 3: the lvd itself, 4: fluid pump, 5: the pressure transducer and 6: a piezo-electric transducer.

the order of a few μm.

The piezo-electric contact transducers that we use, for generating compressional and shear waves (models V103-RM and V153-RM manufactured by Panametrics) effectively generate signals in the blocks with a peak frequency of 0.5 MHz. The active element size is half an inch in diameter (12.7 mm). The present acquisition system can handle up to 48 transducers. For the experiments that we will discuss we used a combination of 24 compressional and 20 shear transducers, where the acquisition geometry is shown in Figure 9.3.

Integrated in a computer-controlled switch-box, each transducer can subsequently act as a source while the other transducers are in receiving mode. In addition, each transducer can act both as a source as well as a receiver in pulse-echo mode. The principle of physical reciprocity states that when we interchange the source and receiver state in absence of noise we obtain the same physical signal [see Fokkema and Van den Berg, 1993]. Experience with the ultrasonic data indicates that indeed the measurements show physical reciprocity, which sheds some light on the fact that the waves that we measure are still in the linear regime. Excluding reciprocal records, a complete scan with 48 transducers results in 1128 different records. Prior
to A/D conversion, we adaptively apply gain on each record. The waveform digitiser and storage device that we use, sample the signals with a frequency of 5 MHz and provides a resolution of 12 bits per word. Therefore the dynamic range is 72 dB. Each record contains 2048 samples, which results in a memory requirement of a complete scan of 4.6 MByte.

9.3 Experimental improvements

In the old experimental set-up the acoustic transducers were mounted in steel platens. On top of the transducers aluminium platens with a thickness of 2 cm and lead plates of 1 mm thick were mounted. The lead plates were greased with vaseline. This complex transducer plate assembly, also shown in Figure 9.4(a) was chosen in order to avoid shear stresses on the block, combined with reasonable coupling of the compressional transducers. Shear stresses on the sides of the block disturb the desired homogeneous stress distribution inside the block, which disturbs the propagation of the hydraulic fracture [Van Dam and De Pater, 1995].

Several disadvantages of this transducer assembly were noticed [Groenenboom, 1995]:
Figure 9.4: Comparison of the old transducer assembly (a), and the new one (b), with 1: location of rock sample, 2: piezo-electric transducer, 3: steel plate, 4: transducer cable, 5: 1 mm thick lead, greased on both sides with vaseline, 6: 2 cm thick aluminium plates, 7: 0.1 mm teflon, greased on both sides with vaseline with transducer holes, 8: 6 mm thick aluminium plates.

- During the experiment the squeezing of the vaseline between the plates and the deformation of the lead plates cause instabilities of the acoustic measurements. Slight timing changes, or drifting, resulting from these instabilities are not related directly to fracture growth and could easily be interpreted as delays related to increasing width of the fracture.

- The presence of the coupling plates causes a strong reverberation of the signal both for compressional as well as shear waves inside the plates, causing a decrease of resolution. In addition, notches are created in the spectrum and high frequency content is lost. This is undesirable for the interpretation of the diffracted events, for first arrival picking and for dispersion measurements.

- Strong attenuation of shear waves due the presence of the coupling plates and a thin vaseline layer between the different coupling plates and the rock sample.
Especially the presence of the vaseline layers\(^1\) obstructed the propagation of shear waves. Simply removing the vaseline will cause severe shear stresses on the block without substantial improved coupling, because the dry interfaces between the coupling plates and the rock sample still attenuate the shear waves severely. In addition, still strong reverberations are present in the plates. Therefore we choose to modify the transducer plate assembly. Although removal of all the coupling plates resulted in greatly improved acoustic data, unacceptably large friction on the block appeared to cause high shear stresses on the sides of the block. As a result of that, an inhomogeneous stress distribution was created inside the block and hence uncontrolled and unreliable fracture experiments would be carried out. A satisfying solution was found by direct coupling of the transducers to the block. Aluminium plates were constructed with small holes in the aluminium plates for the transducers, see Figure 9.4(b). By using thin sheets of Teflon on the remaining surface of the aluminium plate no severe disturbance of the desired stress distribution was found inside the block. In addition, the Teflon was greased with vaseline. In Figure 9.5 an illustration of the actual implementation of the new transducer assembly is shown. In the old assembly the transducer faces were not visible. The complete redesigning, implementation and testing of the transducer plates in the experimental set-up took two years. Along with the construction of the new transducer plates, we changed the locations of the transducer holes to a much denser number at the centre of the block sides, see Figure 9.3. For the old transducer plates the transducers were ordered in a less dense Cartesian grid, see [Savić, 1995]. For the construction of the width profile of the hydraulic fracture in Chapter 11 we needed a larger set of transmission records for a single wing of the hydraulic fracture. To be able to put the transducer more closely together new transducer connecters had to be made, which were smaller than the commercially available connecters. In the old experimental set-up it was not possible to align the polarization of a set of shear transducers. In the new assembly we took care of the fact that we wanted to align the polarization of the shear transducers. During the improvements it appeared that piezo-electric transducers are sensitive to temperature changes. The temperature in the transducer plates

\(^{1}\)In a sense the thin vaseline layer responds acoustically as a hydraulic fracture which slowly closes as a result of vaseline flow related to the constant pressure of the tri-axial pressure machine.
slowly increases due to the activity of the tri-axial pressure machine. Using
the small amplitude and phase changes of the received records, these tempera-
ture changes could erroneously be interpreted as fluctuations of the width
of the fracture. Therefore, we implemented a cooling system, which stabilises
the temperature during the experiment.

9.4 Data improvement

The main achievement of the new transducer assembly is that for the first
time shear waves can be recorded. An example of such a clear shear trans-
mission is shown in Figure 9.9 on page 194. Applications of these data will be
discussed in the next two chapters. In addition, the compressional transmis-
sions showed less reverberation owing to the absence of the aluminium and
lead coupling plates between the transducer and the block, shown in Fig-
ure 9.6(a) and (b). The improved resolution of the wavelet is important for
monitoring the width of hydraulic fractures with dispersion measurements. For the diffraction measurements the standard preprocessing procedure is to remove the incident wavefield, where we define the incident field as the signal that we obtain in absence of the fracture. The incident field is measured before the onset of the fracturing process. Because we are mainly interested in changes of the measurements due to the growth of the hydraulic fracture we subtract the incident field from all subsequent scans, which we refer as the difference domain. What remains is scattered energy that is in some way related to fracture growth. Small timing errors of the acoustic acquisition, also known as drift, related to squeezing of vaseline, temperature changes or instabilities in the electronics can sometimes result in a signal in a difference domain that is unrelated to fracture growth. In Figures 9.7(a) and (b) and Figures 9.8(a) and (b) we show the improvement in data quality in the diffraction measurements, both before subtracting the background field and in the difference domain, respectively. The new transducer assembly shows a much clearer diffraction and a weaker drift signal. Because the drift signal is changing very slowly, resulting in almost stationary signals, additional data quality can be obtained by filtering the data in the FK-domain, [Savić, 1995]. The data are decomposed into events with a certain constant dip for increasing scan number. Only the events with a very small dip are removed from the data, after which the data are transformed back to the original domain. A disadvantage of this technique is that any scattered energy that has a stationary arrival time is removed as well. In the subsequent chapters, without further notice, we show the diffraction data in the difference domain, after filtering in the FK-domain if necessary. With the new transducer configuration in most cases FK-filtering was not necessary anymore.

9.5 The fracturing experiments

The two main fracture experiments that we will discuss are known as experiments CNV22 and CNV21, respectively. The name indicates that we are experimenting on cement block samples (C), where the fracture initiated from a circumferential notch in the borehole wall (N). The least principal stress was chosen perpendicular to the vertical borehole (V), resulting in transverse radial fractures\(^2\). Experiment CNV22 will be mainly used to discuss

\(^2\)Since the minimum principle stress is vertical, this fracturing geometry actually corresponds to horizontal wellbores, with the fracture perpendicular to the well.
Figure 9.6: Compressional transmissions with the old transducer assembly (a) for experiment CNV01, and new transducer assembly (b) for experiment CNV22. Note the absence of strong reverberations. The fact that the dispersion is stronger for experiment CNV01 is related to the higher impedance of the cement for this experiment. The cement was mixed with sand.
Figure 9.7: Compressional diffractions for (a): the old transducer assembly for experiment CNV01, and (b): the new transducer assembly for experiment CNV22. Scattered energy is found after subtracting the first scan from each subsequent scan, see Figure 9.8.
Figure 9.8: Compressional diffractions in the difference domain for (a): the old transducer assembly for experiment CNV01 and (b): the new transducer assembly for experiment CNV22. Note the severely reduced drift and higher signal to noise ratio.
the diffracted events, while CNV22 was used mainly to clarify the transmitted measurements. The development of the fluid pressure and the width for experiment CNV22 and CNV21 is shown in Figures 9.10 and 9.11, respectively.

We observe that initially after starting the injection of fluids, the fluid pressure increases. At a certain moment the fracture is initiated, which results in a decrease of the pressurization rate, since fluids are flowing into the fracture. When the flow rate at the well head equals the flow rate into the fracture, the maximum pressure is reached. This instant is defined as the break-down time. When the pump is shut in, the fracture slowly closes.
Figure 9.10: Experiment CNV22. Fluid injection started with 0.352 cc/min and the pump was shut in at 2668 d. The initiation point of the fracture was determined to be at 718 s (around scan 55), the breakdown point at 878 s.

Figure 9.11: Experiment CNV21. Fluid injection started with 0.022 cc/min, and was increased at time 5977 s to 0.704 cc/min. The pump was shut in at 7351 s. The initiation point of the fracture was determined to be at 3444 s (around scan number 53), the breakdown point at 5069 s.
Due to the large physical dimensions of the transducers ($R = 6.35$ mm), the spacing between the transducers is large compared to the wavelength in the solid embedding. Hence, the data that we acquire are spatially aliased. Augmented by the limited number of available channels, and the three-dimensional coverage at all sides of the block, the spatial aliasing prevents us using migration algorithms to image the fracture shape. On the other hand, we are able to infer the final shape of the fracture by opening the block after the experiment.

The tip of the fracture appears to act as a strong diffractor of acoustic energy. While monitoring the fracture growth, we observe a change of the arrival time of the diffracted events related to a change of the location of the fracture tip. By combining different recordings of the compressional diffraction arrivals we can reconstruct the radius of the fracture as a function of the experiment time. For the interpretation of other diffracted events we can use this fracture-growth curve to compare predicted arrival times of more complex events with measured events.
10.1 The construction of the fracture-growth curve.

In the previous chapter in Figure 9.8(b) we showed a compressional diffraction measurement of experiment CNV22. The compressional diffracted event has been observed already in Savić [1995] and can be interpreted as a diffraction from the tip, which acts as a secondary source. The decrease of the arrival time of the diffraction is related to the growth of the fracture, which decreases the travelpath of this diffraction. The procedure to determine the fracture-growth curves starts with picking the arrival times of the diffracted event as a function of the scan number. Because we have a set of transducers, on each side we obtain a set of diffraction arrival time curves.

In Savić [1995] a grid of traveltimes was calculated first in a two-dimensional plane intersecting the fracture, for a set of relevant receiver and transducer combinations. For each source-receiver combination this resulted in a surface with the correct arrival time of the diffraction. This surface is determined by a fourth-order polynomial equation, because of the presence of the aluminium plates between the transducer and the homogeneous block. By locating the intersections of these surfaces, with the correct arrival time for each source-receiver combination, the position of the tip was determined. The determination of the equal travelt ime curves is CPU-time intensive, because ray-tracing was needed to implement the refraction at the interface separating the aluminium plate from the homogeneous block.

In this thesis a different approach has been taken to determine the position of the tip [see de Pater, 1996], which is able to determine the tip, without a two-dimensional approach. We choose to parametrize the shape of the fracture. In the simplest case, the fracture is radial and can be parametrized with the equation of a circle. More generally, we also allow for a certain tilt of the fracture off the horizontal plane. The number of independent measurements we have is enough to determine the tilt and centre of the fracture. In addition, we can inspect after the completion of the experiment (post-mortem) the final fracture surface. Although in principle we do not need to use this information, it would be unpractical not to check the input parameters with the information that we have available.

For each point on the fracture surface we can determine the hypothetical arrival time of the diffraction from that point. We assume that for a certain fracture radius and tilt, the actual diffraction point is found at the location where the arrival time is stationary, in this case a minimum. Only in that case a certain region of the perimeter around this point will interfere constructively to form the discernible diffraction. By doing so, we obtain a curve
10.1 The construction of the fracture-growth curve.

![Graph showing experiment time (s) vs. fracture radius (mm)](image)

**Figure 10.1:** Radius of the fracture as a function of scan number, or experiment time for experiment CNV22. Two lines on opposing sides of the fracture, (x-cylinder and x-door) indicate that the fracture is not completely radial.

describing the arrival time of the diffraction as a function of the fracture radius. Finally, we simply minimise the difference between the computed and observed traveltimes. This method has been successfully applied in case of radial fractures and also more complex fracture geometries. The method works equally well for experiments with or without coupling plates.

In Figure 10.1 we show the calculated fracture radius versus the scan number and the experiment time, using the first arrivals from the compressional diffraction data. We distinguish between the transducers on the door side and the cylinder side of the tri-axial pressure machine. From Figure 10.1 we can see that the fracture is not perfectly symmetrical, but grows slightly faster in the direction of the door-side on the x-direction. Since in first order the fracture shape is circular we will use the radius, keeping in mind that the approximate radius is a function of the angle.

By removing the coupling plates described in Chapter 9, the procedure for
constructing the equal travel times is now severely simplified. With the old transducer assembly the travel path and hence arrival time of the diffractions had to incorporate the refraction of the aluminium and lead plate. In Savić [1995] the refractions related to the lead plate were simply implemented by using a constant time delay.

10.2 Interpretation of the events

Previously, a lot of experience had been gained with the detection and interpretation of the diffracted compressional event. With the new transducer assembly we clearly measure other types of events which somehow are related to fracture growth.

To unravel the interpretation of all events we propose the following strategy. From finite-difference modelling as discussed in Chapter 8 we have gained insight in what events are important from an energetic point of view. By using the arrival time of the diffraction the growth of the fracture and hence the position of the tip of the fracture can be determined. The fracture-growth curve is only based on the interpretation of the first diffraction as the direct compressional diffraction. For a set of events that we might observe, we can calculate the arrival time by using ray-tracing for a known fracture size as a function of the experiment time. For the calculation of the arrival times we used the solid and fluid parameters similar to Table modpar on page 130. The velocity of the generalized Rayleigh wave was determined in Sections subsec: guided, see Figure 7.9 on page 144 as approximately 2 km/s. The location of the interaction point of the notch was taken at a radial distance of 14 mm, 3 mm outwards with respect to the radius of the borehole, 11 mm. For the source and receiver position we took the centre point of the finite aperture transducers. The calculations incorporated the three-dimensional nature of the configuration. The source and receiver locations for the following three data examples are given in Table 10.1.

In Chapter 8 we have described the interaction of the wavefield with the fracture and the borehole. The most important interaction points are the tip of the fracture and the intersection of the fracture with the borehole, i.e. the notch. To categorise the different events that we observe we will denote the interaction at the notch with a n, while the tip position closest to the source transducer is indicated with a σ. In a normal diffraction configuration this diffraction point corresponds to the minimum travel time. The point where the wavefield interacts with the fracture on the opposing side of the borehole,
<table>
<thead>
<tr>
<th>Type</th>
<th>data</th>
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<tr>
<td>P to P</td>
<td>Figure 10.4</td>
<td>(-7.21, -2.12, 15.0)</td>
<td>(-15.0, 2.55, -2.55)</td>
</tr>
<tr>
<td>S to S</td>
<td>Figure 10.6</td>
<td>(-7.21, 2.12, -15.0)</td>
<td>(-15.0, -2.55, -2.55)</td>
</tr>
<tr>
<td>P to P</td>
<td>Figure 10.8</td>
<td>(-7.21, 2.12, 15.0)</td>
<td>(15.0, 2.55, -2.55)</td>
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Table 10.1: Source and receiver input parameters for the prediction of the arrival time of the set of possible events. Source and receiver position are given as \((x, y, z)\) in cm.

distant from the source position is indicated with an \(o\)\(^1\).

The different kind of important wave modes that can propagate in our sample are a compressional wave, a shear wave and guided waves, i.e. the Rayleigh and slow channel wave. For a compressional and shear wave we use the symbols \(P\) and \(S\), respectively. Further, for the generalized Rayleigh wave propagating along the fracture we use the symbol \(R\). Based on the modelling results of Chapter 8 we postulate that the following events, which are illustrated in Figure 10.2, are important:

- The direct diffraction of the incident wavefield at the fracture tip, see Figure 10.2(a), i.e. \(PdP\), the compressional diffraction and \(SdS\), the shear diffraction.

- Diffractions that are mode converted at the tip, see Figure 10.2(a), such as \(PdS\), which denotes the compressional body wave which is mode-converted to a shear body wave at the fracture tip and vice versa the \(SdP\) diffraction.

- The strong reflection of the wavefield at the borehole tube seen in Chapter 8 is also diffracted at the fracture tip, see Figure 10.2(b). We indicate this borehole tube reflection with a \(b\). Note that this interaction is different from the interaction with the notch \(n\).

- Interaction of incident body waves at the notch generates guided waves propagating along the fracture, which can be re-diffracted at the fracture tip to body waves, see Figure 10.2(c). The most important of these

\(^1\)Note that this arbitrary distinction is based on the position of the source with respect to the borehole and not to the receiver location.
events are the PnRdP and SnRdS events. Finite-difference modelling in Chapter 8 suggests that the re-diffraction of the slow channel wave is relatively weak compared to the re-diffraction of the Rayleigh wave, which makes it less likely that we can observe any re-diffraction of the slow channel wave.

- An incident body wave at the fracture tip also causes mode-conversion to guided waves, as has been observed in Chapter 8. The body waves are partially reflected at the notch after which they propagate towards the tip again. When the Rayleigh mode is re-diffracted to a body wave, these modes can be measured at the side of the block, see Figure 10.2(d). An example of such an event is the SdRnRdS event.

- Direct and mode-converted diffractions from the opposing tip of the fracture, distant from the source transducer. The main events is the SoS event, see Figure 10.2(e).

- The guided waves that are excited at the near tip propagate towards the opposing side of the fracture. The re-diffraction to body waves can be measured, resulting in an event labelled as SdRoS, see Figure 10.2(f).

Finite-difference modelling suggests that re-diffraction of the channel wave is very weak. We will discuss three representative data sets which show the main kind of diffraction data that we measure in our experiments.

First of all, in Figure 10.3 we show the data for a standard diffraction configuration with a set of compressional transducers. The measurement configuration and the interpretation of the main events is shown in Figure 10.4. We identify for this couple of compressional transducers, the direct diffractions PdP and SdS and the mode-converted diffractions PdS and SdP, see Figure 10.2(a). We observe that the compressional transducers are also sensitive to shear waves, as has been discussed in Chapter 4 and has been observed too in the modelling of Chapter 7. In practice it is impossible to produce perfect transducers which respond only to one main mode for all directions. Because of the preference of the transducers the direct PdP event gives the strongest diffraction. Note that all the diffracted waves in principle can be used to estimate the location of the tip and hence the size of the fracture.

In Figure 10.4 in addition we observe an event which has a longer traveltime as the fracture grows, which we will denote as a receding event. Modelling with the finite-difference code as has been described in Chapter 8 has indicated that a strong guided Rayleigh wave is exited at the notch. If we
Figure 10.2: Nomenclature for six different type of travelpaths. In (a): the direct diffraction at the tip (d) close to the source transducer, (b): the borehole reflection (c): the interaction with the notch, which causes the conversion to the Rayleigh mode, (d): the reflection of the Rayleigh wave at the botch, (e): the diffraction from the tip (o) distant from the source transducer and (f): the guided wave from the tip d to opposite side, o.
calculate the arrival time of the PnRdP event, as in Figure 10.2(c) and compare it against the event that we observe, we see that the interpretation of this event as a PnRdP event is very likely, which means that we find indirect evidence of the existence of the generalized Rayleigh wave.

In Figure 10.5 we show the data for a set of shear transducers, with a similar acquisition geometry as before. This particular example is closely related to the modelling example that has been discussed in Chapter 8. The measurement configuration and the interpretation of the main events is shown in Figure 10.6. Again we see all the direct and mode-converted diffractions of which the direct SdS event is the strongest diffraction. We also observe another event which arrives earlier for a larger fracture size. The arrival time of this event corresponds to a SbSdS event, which is a shear body wave which reflected at the borehole tube and subsequently is diffracted at the fracture tip, see Figure 10.2(b).

Furthermore, we see at least two receding events. We interpret the first as a SnRdS event, which corresponds to a shear body wave which is mode-converted to a Rayleigh wave at the notch and subsequently is re-diffracted to a shear wave at the fracture tip, see Figure 10.2(c). The latter event, can be interpreted as the reflection of the generalized Rayleigh wave, which was excited at the fracture tip o, at the notch. The Rayleigh wave approaches the tip again and is then re-diffracted to shear waves, i.e. a SdRnRdS events, see Figure 10.2(d). The rather broad range of the arrival time could be related to the complex reflection at the notch combined with the fluid-filled borehole. A similar phenomenon was observed in the finite-difference modelling, see Figure 8.8.

As a final example we show the data that was recorded for a shear transducer on top sending to a shear receiver on the opposite side, compared to the previous example. The data are shown in Figure 10.7 and the interpretation of the events in and the acquisition geometry in Figure 10.8. The only clear direct diffraction that is measured is the SoS diffraction of the tip distant from the source transducer, see Figure 10.2(e). Next, we see at least three receding events. First, we have interpreted an event labelled as SnRoS, which corresponds to a incident shear body wave which is mode-converted at the notch to a Rayleigh wave, see Figure 10.2(c). The Rayleigh wave travels along the fracture and is subsequently re-diffracted to a shear wave at the tip, distant from the source receiver o. However, the bulk of the energy for this data example is found at later record times. We interpret this energy as stemming from two separate events. The late event corresponds closely
to the previous event, except that the incident wave is now a shear body wave. The slightly earlier arrival time corresponds to a direct diffraction and the tip, near to the source \( d \). This diffraction is measured at the receiver transducer at the opposite block side, i.e. a \( S_dS \) event, see Figure 10.2(e). Errors in the prediction of all arrival times in this section can be due to errors in the determination of the tip position, the neglect of the finite aperture of the transducers and some simplifications for the complicated re-diffracted events.

### 10.3 Origin of the diffraction

The fracture tip is a difficult and complex region. The physics of fracturing is poorly understood, especially the different behaviour of different materials. The theory of linear-elastic fracture mechanics predicts a static stress singularity at the fracture tip. In reality complicated phenomena at the tip will prohibit the creation of a singularity of the stresses. Many theoretical models claim the existence of a dry fracture front preceding the fluid fracture [see e.g. Khristianovic and Zheltov, 1955; Barenblatt, 1956]. Other studies claim a zone of weakened cohesion, preceding the fracture front.

Although the geometrical interpretation of the events seems to be resolved, it is less clear what actually causes the diffraction, which can be considered the dynamic equivalent of the static singularity at the fracture tip. In this section we will show some data that will shed light on this discussion.

For experiment CNV19, a fracture was created with a viscous silicon oil with a radius on the order of 0.1 m. After that, the pumps were shut in after which the fracture slowly closed. Because of the low leak-off of the viscous oil in the cement and the viscous forces, the fracture did not close completely at the wellbore. Next, the fracture was reopened, for which the pre-existing fracture does not resist to rupture. In other words, the fracture has no toughness which resists the propagation of the fluid into the fracture. In Figure 10.9 we show a stack of the direct diffractions, \( P_dP \), \( P_dS \), \( S_dP \) and \( S_dS \), respectively. The figure on the left shows the opening phase, while the figure on the right shows the reopening phase. From the direct \( P_dP \) diffraction we see that when the fracture grows, no diffraction is seen until the fracture exceeds the size of the pre-existing fracture. A similar conclusion can be drawn for the \( P_dS \) diffraction. On the contrary, for the direct shear diffraction \( S_dS \) as well as the \( S_dP \) diffraction a clear diffraction signal is seen before the fracture continues to grow. Hence the shear waves are able to pick up the reopening phase of
Figure 10.3: Data in the difference domain for a pair of compressional transducers. No important events were measured after 125 μs.
Figure 10.4: Interpretation of the data on the page on the left, along with the acquisition geometry.
Figure 10.5: Data in the difference domain for a pair of shear transducers.
Figure 10.6: Interpretation of the data on the page on the left, along with the acquisition geometry.
Figure 10.7: Data in the difference domain for a pair of compressional transducers.
Figure 10.8: Interpretation of the data on the page on the left, along with the acquisition geometry.
the fracture, contrary to the compressional waves. Even more striking, for experiment CNV20 the first opening phase was performed with water. Because of the low viscosity the fracture remained very thin which resulted in the absence of diffractions of the fracture tip. The water fracture did close almost completely, except for some fracture surface roughness effect. The reopening phase simulates a fracture with no fracture toughness. In Figure 10.10 we show a stack of the direct diffractions, $P_{dp}$ and $S_{ds}$, respectively. Again we see that the shear diffraction picks up the migration of the fluid front, whereas the compressional diffraction is observed when the fracture continues to grow beyond the pre-existing fracture.

We interpret these data as follows; compressional diffractions are weak for thin fractures. For a strong tip diffraction to be created, the gradient of the width at the tip has to be sufficient. Linear elastic fracture mechanics predicts that the width profile is proportional to the fracture toughness. In case of zero toughness the fracture becomes more wedge shaped without a strong gradient in the fracture width. Because the shear waves are much more sensitive to the tip, since its strength is mainly determined by the absence of shear stresses in case of an open fracture, independent of its width. In terms of the fracture compliances, the normal compliance based on the layer model in Chapter 5 increases linearly as a function of the width, whereas the transversal compliance is lost completely at the moment of fracturing. Apparently the compressional diffraction is just too weak in case of the zero toughness fracture, with the more wedge shaped fracture. When the fluid front reaches the fracture tip, the solid resists the rupturing again after which the width profile changes shape and develops a width gradient at the tip, such that now both the compressional and shear diffraction are observed. The fact that shear waves pick up the reopening means that the viscous fluid-filled fracture must have closed at a dominant part of the fracture. Investigation of shear transmission through the fracture shows that some energy of the shear-wave transmission is restored, which means that the fracture faces are at least partially in contact. But the strength of the shear diffractions is similar to the opening phase, which is still surprisingly strong. Another explanation could be that somehow the fluid front during reopening causes an complicated interaction including a mode conversion to compressional waves in the fluid, which effectively causes a diffraction. The sensitivity of the polarization of the shear waves to the fracture shape could be the reason why only shear waves pick up the reopening. For this moment, the most important fact is that with shear waves we can monitor the migration of
Figure 10.9: On the left the diffraction stack of the opening phase for experiment CNV19, showing the PdP, PdS, SdP and SdS diffraction respectively for later record times. In the reopening phase on the right the incident shear waves pick up the migration of the fluid front during reopening, contrary to the incident compressional waves.
the fluid front, which shows the added value of measuring shear waves. To understand the underlying mechanism, more work is required. The description is even more complicated for mode-converted diffractions. Shear transducers located above the fracture diffract some energy during the reopening phase to compressional body waves. Combining this information with the reciprocal property of our records, we expect that a hypothetical compressional source transducer located at the position of the actual receiver transducer will diffract some shear body waves to the top. Another example which supports the hypothesis that compressional diffractions are weak because of the wedge shaped tip, is given by experiment CNV21, which was designed for an optimal transmission configuration. During this experiment around scan number 193, at experiment time 5977 s, the flow rate was suddenly increased. In Figures 10.11(a) and (b) we show a measured compressional and shear diffraction, respectively. In these figures we can see that at the moment that the flow rate is increased almost instantaneously the amplitude of the diffractions increases. Based on linear fracture mechanics it can be expected that the sudden increase of the flow rate quickly changes the local width profile at the fracture tip. This expectation is supported by monitoring the width profile of this experiment, discussed in Chapter 11. The increased width at the tip apparently increases the strength of the diffractions, both for the compressional as well as the shear waves. The effect on the compressional diffractions seems to be stronger. This also means that in the amplitude of the diffraction, information is contained on the local width profile of the hydraulic fracture. Previous examples suggest that it is mainly the fluid front which is responsible for the creation of the diffraction, and not the dry tip. Although theoretical models claim the existence of a fluid lag in between the dry tip and the migration of the fluid front [BARENBLATT, 1956] a direct observation that distinguishes between the dry and fluid front had not been made in our experiments previously. In Medlin and Massé [1984] a fluid lag was observed, based on attenuation curves of a set of compressional transmission through the fracture. From a fracture mechanics point of view we know that when we decrease the confining stress, the expected length of the non-penetrated zone increases. In Figures 10.12(a) and (b) we show compressional diffraction data for a gypsum and cement experiment respectively. Both experiment where carried out with low horizontal stress (12 MPa) and vertical confining stress (8 MPa). In the gypsum experiment before the main diffraction we clearly see a weak precursor. We have interpreted this weak diffraction as stemming
Figure 10.10: The reopening phase of the direct diffraction stack showing the PdP and SdS diffraction respectively. During the opening phase with water no diffractions were observed. In the reopening phase with silicon oil on the right the incident shear waves pick up the migration of the fluid front, contrary to the incident compressional waves.
Figure 10.11: Diffraction record (a) for a pair of compressional transducers and (b) for a pair of shear transducers during experiment CNV21. When the flowrate at the pump is increased around scan number 193, experiment time 6000 s, the strength of the diffractions increase.
from the dry tip whereas the main diffraction results from the fluid front. The difference of the arrival time between these two events can be used to estimate the size of the non-penetrated zone. This estimate was in agreement with the length of the penetrated zone which was obtained by visual inspection after opening the block. In case of the cement experiment, we also see a strong indication of a change in shape of the diffracted waveform, which we interpret as resulting from the interference of the diffraction from the dry tip and the fluid front. In both the gypsum as well as the cement experiment, the fluid front seems to create a stronger diffraction than the dry front. The fluid lag increases when the size of the fracture increases. For experiment CNV22, see Figure 10.3 and 10.6 the fracture grows out of the block. Although the vertical confining stress for this experiment is higher (23 MPa), after scan number 130 the diffractions separate into different branches for a fracture which is bigger than 0.1 meter in the diameter.

The main importance of the various diffracted events is twofold. First of all, the fact that all diffractions have been observed opens the way for a more accurate determination of the fracture radius. In the field, measurement of the shear diffractions might be preferable. Whether strong compressional diffractions are generated in the field will depend on the embedding and fluid properties and the local width profile of the fracture at the tip. Second, we have observed indirect evidence of the guided generalized Rayleigh wave which travels along the fracture. This wave mode has the potential to be used as an alternative way to determine the size of the fracture, both in the laboratory as well as in the field.

We conclude that the acoustic diffraction mainly originates from the fluid front, and that the shape of the fluid front at the tip determines the diffraction strength. In some special cases we can distinguish between the fluid front and dry tip, which shows that the fluid front is the main diffractor.
Figure 10.12: Diffraction record (a) in a experiment with a gypsum rock sample and (b) with a cement experiment, both with low horizontal stresses. For the cement experiment the pump was shut in after scan number 59. Gypsum shows a clear precursor which can be interpreted as stemming from the dry tip. In case of cement block also an interference can be seen, which could have resulted from the interference from the diffraction of the dry tip.
Transmissions: Interpretation and Application

Since diffractions are scattered from the perimeter of the fracture, these measurements mainly contain information on the tip, for example on its location. Transmission measurements focus their energy on the fracture surface and hence are useful to determine properties about the interaction of the fracture interfaces. Since this interaction appears to be dependent on the hydraulic fracture width, as was shown in Figure 1.2 in Chapter 1, we aim at determining the fracture width by using compressional transmission measurements. We will show data of experiment CNV21, discussed in Chapter 9, which was optimized for measuring and interpreting the transmission measurements across the block. We will start by showing that hydraulic fractures are highly impenetrable for shear waves in our laboratory, which results in shear-wave shadowing. A similar phenomenon was observed for a differential vertical seismic profiling (VSP) experiment [Turpening, 1984] and cross-well seismic field experiments [Wills et al., 1992; Aki and Fehler, 1982]. After that, we will show that we can monitor the width profile of hydraulic fractures by using the dispersion observed in compressional transmissions.
11.1 Shear-wave transmissions and their interpretation

In Figure 11.1 we show the measured shear-wave data, the amplitude of the shear-wave events combined with the fracture width, as has been measured at the borehole and the estimated delay time of the shear wave. The first important observation that can be made, is the shadowing of the shear wave, indicating that the hydraulic fracture is highly impenetrable with respect to shear waves. We can see that when the fracture is open after scan number 200, roughly at experiment time 6200 s, no significant shear-wave amplitude is measured above the background noise level. The background amplitude level is roughly 2.5 % of the initial amplitude of the shear wave and is mainly due to weak scattered compressional events.

We postulate that the shear-wave shadowing indicates that the hydraulic fractures has no transverse fracture compliance \( Y_T \), for a relevant range of the fracture width. The large amplitude attenuation of more than 80 % and the time-delay of more than 10 \( \mu \)s, observed between scan 130 and 210, roughly experiment time 4851–6300 s, could erroneously be interpreted as resulting from a slowly increasing fracture compliance, or fracture width. But when the fracture closes the shear-wave amplitude does not restore at all, indicating that even for a fracture width below 50 \( \mu \)m the fracture has no noticeable transverse compliance. The delayed energy for the shear transmission measurement is better understood as resulting from the diffraction of the fracture tip, scattered around the fracture interface. When we interpret the delay time of the shear wave in terms of the prolonged travelpath of the diffraction, as illustrated in Figure 11.2, we can estimate the radius of the fracture. In Figure 11.3 we compare the estimate of the fracture radius based on the compressional diffraction, as in Chapter 10, with the estimate based on the delay time seen in the shear transmission. The agreement shows that the slight energy transmission that we see, mainly results from the diffraction at the fracture tip.

The combination of observations leads us to conclude that no noticeable shear-wave energy is transmitted across the fracture. More important, we conclude that apparently friction of the fracture faces is absent or negligible, indicating that the fracture is mechanically open.

In Chapter 7, see Figure 7.7(a) we modelled the shear-wave transmission through a thin layer filled with a Newtonian fluid with a zero shear-rate viscosity comparable to the fracture fluid used in the laboratory (600 Pa.s). We concluded that in that case the shear wave still propagates across the fracture for ultrasonic frequencies around 0.5 MHz with an amplitude on the
Figure 11.1: In (a) the measured shear wave transmission data, (b) the maximum amplitude of the shear wave and the measured width at the borehole. In (c) the delay time of the shear arrival shown in (a).
order of 74 percent of the transmission without fracture. On the other hand, when the relaxation time is long compared to the period of the waves, the fluid behaves elastically. The high-frequency elastic shear modulus that is related to the zero shear-rate viscosity of 600 Pa s is only very small, on the order of $6 \times 10^3$ Pa, see Table 7.2 (less than 1 promille of the shear modulus of the embedding). Modelling the shear-wave transmission through such a thin elastic layer, see Figure 7.8, results in a weak shear transmission with an amplitude on the order of 1–2 percent of the shear-wave transmission without fracture. In the laboratory no significant shear-wave transmission is seen above the background noise level in the laboratory experiments (on the order of a percent), consisting for example of compressional and shear scattering of the sides of the block. Therefore, the shear-wave shadowing that we observe in the laboratory data compels us to conclude that in the frequency range of interest of the ultrasonic waves the fluid does not respond viscously anymore. The dominant relaxation time of the fluid is therefore large compared to period of the ultrasonic waves. These conclusion are in agreement with estimates of the shear-wave speed of silicon oil using a Couette type device in Joseph [1990], which gives an estimate of the relaxation time of several ms and a shear-wave speed in the order of 15 m/s.

Because no significant shear-wave energy propagates through the fluid layer, we cannot infer the fracture width from shear-wave measurements. On the other hand, we can determine the size of the fracture by using shear waves in

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**Figure 11.2:** The shear wave shadowing and the tip diffraction traveling around the fracture interface, which can be used to estimate the extent of the fracture.
a transmission acquisition configuration, since the tip diffraction is scattered at relatively wide angles. The shear-wave measurements do indicate that a thin layer, filled with an ideal fluid, might be a reasonable approximation of the hydraulic fracture. Visual inspection of the fracture interfaces in the laboratory shows that the actual fracture faces are razor sharp with a fracture roughness on the order of the grain size of the cement ($\leq 1\mu m$).

### 11.2 Fracture initiation

For experiment CNV21, see Figure 9.11 on page 195, the initiation time of the fracture was determined at 3444 s, around scan number 53, based on a decrease of the pressurization rate in the borehole. Nevertheless, the displacement transducer (LVDT) at the borehole still gives no substantial response. The displacement transducer, described in Section 9.2, measures the
opening of the fracture. But the amplitude of the shear-wave transmission, even though this transducer is located 3.6 cm away from the centre of the borehole, start to decay at an experiment time, which agrees well with the estimated initiation time.

For experiment CNV22, the initiation time was determined at 718 s, around scan number 55. For the shear diffraction, in Figure 10.5 the earliest detection of the shear diffraction is found around scan number 60, around experiment time 754 s. The compressional diffractions are picked up even later around scan number 65, experiment time 794 s. Comparing these observations, we conclude that generally shear waves are more sensitive for determining fracture initiation, whereas the clearest data is obtained from the amplitude attenuation of the shear wave. This again shows the added value of the shear-wave measurements.

11.3 Compressional transmissions and their interpretation

For the analysis of the compressional transmissions we select a set of four compressional transducers in a row. The row of transducers both at the top and at the bottom, numbered from 1 to 4 is positioned at a distance from the centre of the block of 3.6, 6.6, 9.6 and 12.6 cm, respectively. The actual acquisition geometry is shown in Figure 11.4. We can take any combination of source and receiver to scan the fracture interface.
Since the shear-wave transmissions suggest that mechanical contact of the fracture faces can be neglected, for the compressional transmissions we postulate that we can approximate the fracture by a thin fluid-filled layer with a finite width. In that case, we can predict the amount of dispersion of compressional transmissions for a certain fracture width and fluid properties with the convolutional model as discussed in Chapter 6, Eq. (6.18),

\[ S^b_P(x, t; x^s, h) = T_{P/P}^{\alpha} \left( \frac{d c_{P,e}}{d c_{P,e}}, t; h \right) \ast S^b_P(x, t; x^s), \] (11.1)

for which \( S^b_P(x, t; x^s, h) \) denotes the theoretical prediction of the signal and \( S^b_P(x, t; x^s) \) a representative base signal before fracturing the block for the same source and receiver location, \( x^s \) and \( x \) respectively. In this model we use the compressional transmission coefficient of a thin ideal fluid-filled layer, for the horizontal slowness of the ray connecting the source transducer with the receiver transducer. The compressional thin-layer transmission coefficient depends on the width of the fracture and the fluid and embedding properties. Whereas the fracture width is an unknown property, all relevant fluid and embedding properties, such as the densities, compressibility and stiffnesses can be measured independently in the laboratory. Therefore we have explicitly added the width dependence of the transmission coefficient and the predicted signal in Eq. (11.1).

Since the measured base signal \( S^b_P(x, t; x^s) \) is discretely sampled, the Fourier transformations and convolutions are implemented with a standard FFT algorithm. The actual width estimation is based on fitting the predicted signal to the measured data. We define \( S^n_P(x, t; x^s) \) as the measured compressional transmission for a certain scan number \( n \). For the discretely sampled data we define the misfit \( E^n(x; x^s, h) \), between the predicted signal and the measured signal as a function of the scan number \( n \) as

\[ E^n(x; x^s, h) = \sum_{i=w_a}^{w_b} |S^n_{P,i}(x; x^s) - S^b_{P,i}(x; x^s, h)|^2. \] (11.2)

Since the convolutional model is only valid for the compressional arrival of the signal, and not for any shear energy that has travelled across the block we have applied an appropriate time window starting at sample \( w_a \) and ending at sample \( w_b \). Next, for each transducer combination and each scan we can estimate the width of the fracture by minimizing the misfit function for the fracture width \( h \), the only unknown parameter, expressed as

\[ E^n(x; x^s, h) = \min(h^n(x, x^s)). \] (11.3)
This procedure results in an estimated fracture width $h^n(x, x^*)$ for every scan and every source and receiver combination. The advantage of the convolutional model is that in this approximation the determination of the fracture width is independent of the location of the fracture and the source and receiver parameters. Moreover, the measurement before fracturing provides us with a very accurate timing as well as amplitude calibration. We do not need to predict the exact amplitudes of the transmissions, which are dependent on parameters which are difficult to assess, such as transducer coupling, effective aperture of the transducers and losses in the elastic embedding.

As an example of the fitting procedure we have taken scan number forty as a base scan, representative for the compressional transmission without fracture. In Figure 11.7 we show this base scan and the measured signal at the arbitrary scan number 219 for transducer one, both at the top and at the bottom. The amplitude of the transmissions are normalized to the maximum amplitude of the base scan transmission, to exploit the amplitude calibration described above. By adjusting the width of the transmission coefficient of Eq. (11.1) we find the correct amount of dispersion relative to the base signal. A width of a 100 $\mu$m results in an underestimate of the damping and delay, whereas a width of 200 $\mu$m leads to an overestimate. With a standard golden section minimum search method [PRESS ET AL., 1992], we find a fracture width of 151 $\mu$m for scan number 219.

In Figure 11.6(a) we show a selection of the compressional transmission measurement for transducer number one of this row, again for experiment CNV21. The data shown are a selection of Figure 1.2(b) in Chapter 1 om page 7, namely the range of scans 125–225.

Figure 11.6(b) shows the fitted data and the difference of the measured and fitted data, Figures 11.6(a) and (b) respectively. The difference has a relative gain of a factor 5 compared to the measured and fitted data. We can see that the difference is small and that a large part of the difference energy is due to a tip diffraction around the fracture, similar to the shear diffraction in Figure 11.1(a). Because we try to fit the complete waveform we hope that our width estimate is relatively insensitive to diffraction effects, since we try to fit that part of the signal that obeys the convolutional model. In this way we avoid to use the much more complex and CPU-time consuming forward model of a fracture of finite extent.

The transmission of transducer 1 both at the top and bottom is only 3.6 cm away from the borehole, where we measure the width directly with a displacement transducer. The width measurement in the borehole with the
displacement transducer has been described in Chapter 9, Section 9.2 and Figure 9.2 on page 185. Therefore we can compare the measured width at the borehole with the estimate based on the dispersion measurement. We have carried out the width estimate for the complete CNV21 experiment both for the phase of opening and closure, which corresponds to the scan range of 0–625. The results of this procedure are shown in Figure 11.7. First of all, we see that the width, as has been measured at the borehole, ranges from 0 to roughly 250 μm. The acoustic estimates of the width shows a similar behaviour.

During the experiment we tried to close the fracture as much as possible by applying flow back of the fluid into the wellbore. This was accomplished by opening the valves for the fluid injection. This caused the steps in the response of the displacement transducer (LVDT) in the borehole during the closure phase, see Figure 11.1(b). Around scan number 400, experiment 9815 s, the flow back was so large, that the LVDT measurement was disturbed and
Figure 11.6: At the top (a) a selection of compressional transmissions, with (b) the fitted waveforms and (c) the non-fitted part, or the difference of (a) minus (b), with a relative gain of 5.
caused a significant measurement error. We note that all small fluctuations of the width at the wellbore, related to changes in fluid injection or extraction, are reflected in the acoustic width estimates, which sheds some light on the sensitivity of the width measurement. The main systematic difference that we see, is that the acoustic estimate at 3.6 cm is smaller than the borehole measurement at the opening phase of the experiment and slightly larger at the end of the experiment at the closure phase. This difference can be understood from the difference in the position of the estimated fracture width compared to the borehole location and the dynamics of the fracture opening and closure as will be discussed later in this chapter.

We postulate that the bulk of the energy that is measured on the other side of the block has propagated through the mid-point of the source receiver combination at the fracture. For the perpendicular transmission combination of transducer number one at the top and bottom at a distance 3.6 cm from the centre this simply means that we assume that we estimate the fracture width at 3.6 cm from the borehole. To validate this concept we have estimated the fracture width for transducer number two at the top and at the bottom, with transducer one at the top and three at the bottom. Since these two transmission measurements share the same mid-point at a distance of 6.6 cm from the borehole, the width estimate should give similar results. In Figure 11.7 we show that the results are indeed in agreement, despite the fact that the acquisition geometry is different. For the transducer combination with a lateral distance of 6 cm we took the full-elastic compressional transmission coefficient, including the effect of mode-coupling inside the layer.

11.4 Width profile monitoring

The set of four compressional transducers in a row, both on the top and at the bottom, results in seven different common mid-points, as can be seen in Figure 11.4. The mid-points are at a distance of 3.6, 6.6, 9.6 and 12.6 cm from the borehole corresponding to the transducer points and 5.1, 8.1, 11.1 cm corresponding to the points right in between two adjacent transducers. The locations of these seven points in combination with the LVDT measurement at the borehole enables us to construct a width profile. By plotting the width estimates for all mid-points for a sequence of scans and interpolating in between the mid-points we obtain a clear image of the development of the fracture. The width profile for 12 scans are shown in Figures 11.8 and 11.9. Each width estimate is plotted with a black dot, from which the left-
Figure 11.7: Width comparisons of (a) the displacement transducer (LVDT) in the borehole with the acoustic determination of the nearby transducer pair and (b) a pair of transducer with zero lateral offset and an offset of 6 cm.
most represents the LVDT measurement plotted at a distance of 1.2 cm, corresponding to the wellbore radius. We combine these profiles with the location of the tip of the fracture as has been determined from compressional tip diffractions, with the technique presented in Chapter 10. The tip location is indicated with a large open circle. The dimension of the open circle, approximately 1 cm, represents a rough indication of the accuracy of the determination of the tip location. Also added to these profiles is a solid elliptical shaped set of profiles, which show the predictions of the width profile, based on a linear elastic fracture propagation model with constant fluid injection [see Barr, 1991; Van Dam and de Pater, 1997] The radius of the theoretically predicted width profile is taken in accordance with the radius determination with the diffraction measurement.

In Figure 11.8 we can see that at scan number 115, experiment time 4577 s, we start to pick up the diffraction of the fracture at approximately 2 cm. At the same time, both the LVDT and the leftmost acoustic transducers start to show the growth of the fracture width. For the first phase of the experiment, for example Figure 11.8(b) we see poor agreement of the theoretical width profile with the estimated width profile, based on the acoustic measurements. But between scan number 165 and 194, experiment time 5492 and 6023, when we have a constant low fluid injection, we observe an elliptically shaped width profile which corresponds well with the theoretical predictions. The fact that we observe some opening of the fracture before the fluid front is an indication that some damage occurs before the main fracture front. The fracture itself causes a tensile stress in front of the fracture tip. This high stress, theoretically even a stress singularity, is reduced by micro-cracking of the rock, which precedes the main fracture.

At experiment time 5977 s, shortly before scan number 195 the flow rate was suddenly increased from 0.022 to 0.704 cc/min. After scan number 195, experiment time 6042 s, the fracture accommodates the increase in flow rate by changing into a more wedge-shaped fracture instead of an elliptical shape. The wedge-shaped profile differs structurally from the theoretical predictions, since the latter assumed a constant flow injection. Theoretical modelling with discontinuities in the flow rate have not been carried out yet. At scan number 220, after more than 500 seconds of injecting with the high flow rate, the fracture gradually restores its elliptical shape. The theoretical prediction of the fracture width appears to be systematically wider, but this might be related to an error in the fracture radius estimate of the diffraction. After scan number 220, at experiment time 7351 s, the pump is shut in and the fracture
Figure 11.8: Width profiles during experiment CNV21, for six different scans, with in dark grey the borehole, in light gray the rock, in white the fluid-filled fracture.
Figure 11.9: Width profiles during experiment CNV21, for six different scans, with in dark grey the borehole, in light gray the rock, in white the fluid-filled fracture.
has grown through the block. Because the two blocks are now completely separated, the fracture shape changes from elliptical to an almost planar fracture. At scan number 390, experiment time 9631 s, we see that the acoustic width estimates all lie within a relatively small range. As we expect, the fracture becomes planar, because there is no fracture tip anymore. The acoustic estimates show a planar fracture and the acoustic estimates are in agreement with the LVDT measurement at the borehole. The combination of these facts convinces us of the validity of the method to determine the fracture width with the proposed method. To close the fracture at some moment we released all fluid pressure by opening the flow pipe-line connected to the wellbore. At later scans, Figures 11.9(k) and (l) we see that because the fluid leaves the fracture at the borehole, the fracture has a tendency to close first at the borehole, thereby hampering the fluid flow back to the borehole. In combination with the viscous forces of the fluid this obstructed closing the fracture any further, and we decided to stop scanning.

In conclusion, we have seen that shear-wave measurements can be used to determine the moment of fracture initiation. The fact that shear waves are completely shadowed means that friction between fracture faces is absent or at least negligible. This justifies that the dispersion of compressional waves is used to estimate the width of the fracture. The width measurement close to the borehole, based on the dispersion of compressional transmissions, agrees well with the LVDT measurement in the borehole. The structural differences can be explained by the dynamics of fracture growth. When the fracture has grown through the block and restores a planar shape, all width estimates at different distances from the borehole are almost equal and in agreement with the LVDT measurement at the borehole. The combination of these observations indicates that the width estimates are reliable.
Part IV

Conclusions

“That’s why it is so dangerous to throw all those thoughts, right or wrong, on the many papers immediately. They are printed, the papalagi says. That means the thoughts of many sick people are written down, even aided by a mysterious machine with a thousand hands and the strength of many chiefs. And not once or twice, no many times. Many, many times, always the same thing. Many mats covered with thoughts are pressed together in small bundles. The papalagi call those ‘books’ and they are shipped off to all parts of the country. And everybody that absorbs the thoughts is infected.”

from ‘The Papalagi - Speeches by Tuiavii of Tiavea, a Samoan chief’ on 'The Heavy Thinking Sickness'.
In the first part of this thesis we focussed on the theoretical aspects of scattering, both reflection as well as transmission, of a wavefield by a thin fluid-filled layer. In the second part we described methods for numerical modelling of the scattering, for a thin fluid-filled layers of both infinite and finite lateral extent. The theoretical and modelling results were used in the third part of this thesis to interpret the ultrasonic data from the hydraulic fracturing experiments. For the first time, we have directly measured shear waves, in addition to compressional waves in the laboratory. This was accomplished by direct coupling of the transducers to the block, which resulted in an overall significant improvement of the signal-to-noise ratio. The diffractions were used to monitor the size of the fracture, whereas the compressional transmissions were used to monitor the width profile of a fracture during its growth. The shear transmission were especially useful to determine the moment of fracture initiation. This last chapter summarises the main conclusions of this thesis.

The width of the hydraulic fractures created in our laboratory is small compared to the wavelength of the incident wavefield in the solid embedding. Nevertheless, we described scattering by a thin fluid-filled layer in Chapter 4, disregarding this particular property. We considered the hydraulic
fracture as a plane fluid-filled layer with no interface roughness or mechanical contact. Visual inspection of the fracture interfaces shows that the actual fracture faces are razor sharp and have a fracture roughness on the order of the grain size of the cement (≤ 1µm). Once the hydraulic fracture is created, the transmission of shear waves is shadowed as was observed in Chapter 11. This observation justifies the assumption that mechanical contact between the fracture faces is negligible, because mechanical contact should result in some shear-wave transmission.

In Chapter 5 we showed that any thin layer, in a band-limited domain of the slowness and frequency, can be described by a linear relationship between the jump of the elastodynamic field quantities in terms of the average field quantities. This relationship is determined by the system matrix of the viscoelastic fluid, the frequency of the wavefield and the width of the layer. The system matrix contains fluid properties, such as the density and stiffnesses, but also depends on the horizontal slowness of the plane wave in the fluid. Rays incident on a low velocity layer tend to refract towards the normal, according to Snell’s law. This enables us to approximate the jump-average relationship to the limit of vanishing horizontal slowness inside the layer. When the density of the fluid is much lower than the density of the embedding, the jump in the traction can be neglected compared to the jump in the particle displacement. In that case, the jump-average field relationship reduces to a linear-slip type of model with a convolutional structure. For an ideally elastic layer, the linear-slip model describes an instantaneous relationship between the jump in particle displacement in terms of traction, which is continuous across the interface. This model is therefore referred to as a displacement discontinuity model, a standard description of a fracture in literature. The fracture compliances of the linear-slip model are in that case determined by the ratio of fracture width and layer stiffnesses. When the fluid responds in an ideally viscous manner, the linear-slip model reduces to a linear and instantaneous relationship between particle velocity and the traction as long as the width of the fracture is below the viscous skin-depth. In the latter case, the modified fracture compliance for such a velocity discontinuity model is determined by the ratio of the width of the layer and the zero shear-rate viscosity.

We have chosen not to use the linear-slip model, because it offers no computational advantage for our purpose. In addition, we showed in Chapter 7 that, contrary to the linear-slip model, the thin-layer model includes the slow channel wave in the fluid layer, which is a physically realistic guided wave
when the fracture is mechanically open. This guided wave corresponds to the tube wave mode that propagates inside boreholes and is used in the field technique called Hydraulic Impedance Testing. Moreover, in Chapter 8 we argued that the applicability of the linear-slip model is questionable near the tip of hydraulic fractures. In the vicinity of the tip, fracture compliance will be determined by the resistance of the solid against deformation near the tip and hence might be much larger than the stiffness based on the interface properties. Hence, direct application of the linear-slip model leaves the question how to taper the fracture compliance towards the tip unanswered. Nevertheless, for compressional waves, the displacement discontinuity model shows that at first, the strength of dispersion is controlled by the product of fracture compliance, frequency and impedance in the solid embedding. For a thin layer, the fracture compliance corresponds to the ratio of fracture width and fluid stiffness. Because of the high impedance contrast between the fluid and solid, dispersion is still strong enough to be observed, although the fractures are thin.

The displacement discontinuity model is less appropriate for shear waves because this model assumes the fluid to respond elastically to shear stresses. Instead, for an ideal fluid we have to incorporate the explicit boundary condition that the shear stresses vanish on the fluid-solid interfaces. In Chapter 5 we concluded that a thin viscous layer, where the viscous skin depth is larger than the width of the layer, is better described by a velocity discontinuity model instead of displacement discontinuity model. In Chapter 7 we showed that, in relation to the transmission response of a thin viscous layer, the velocity discontinuity model predicts mainly a constant amplitude attenuation without any additional time delay.

Although we use very viscous fracture oils, we do not observe any shear-wave transmission. On the basis of the known zero shear-rate viscosity of the fluid, we expect about 75 % of the shear wave to be transmitted (see Section 7.2.3). This apparent inconsistency can be solved by assuming that the characteristic relaxation time of the fluid is long compared to the period of the ultrasonic waves. In that case, for the high-frequency waves that we excite, the fluid behaves more elastically rather than viscously. The shear velocity of the fluid associated with the fluid with a viscosity of 600 Pa·s is so small (on the order of 25 m/s), that the impedance contrast is too high to result in substantial shear-wave transmission. Note that when the velocity of the layer becomes that small, the layer can no longer be considered thin, because the wavelength of the fluid inside the layer (on the order of 50 μm)
is smaller than the width of the layer. Judgement whether a layer is thin should always be made relative to the wavelength inside the fluid layer. The fact that shear-wave transmissions are shadowed, when the fracture opens, implies that shear-wave transmission measurements are an excellent tool for determining fracture initiation.

In Chapter 11 measuring the dispersion of compressional waves is used to determine the width of hydraulic fractures. Although this dispersion is quite small, we can still accurately measure it because the transmission through the fracture is related to the transmission without fracture. In a sense, the transmission measurement without fracture supplies a calibration of the amplitude and arrival time of further measurements. For this purpose, we use the convolutional model that was presented in Chapter 6. This shows that a high accuracy of the width measurement can be attained because the changes in amplitude and arrival time are monitored, instead of absolute amplitudes and arrival times, which are much more sensitive to timing errors, source and receiver effects and errors in estimates of the properties of the solid embedding. This procedure relies heavily on the repeatability of our experiment, which includes a stable source wavelet and accurate timing of electronics of the source and receivers.

Transmission measurements might just as well be used to estimate fracture compliance instead of the width of the layer. As stipulated before, the essential difference between the thin-layer and the linear-slip model is that the linear-slip model is a conceptual model. Its fracture compliance is an effective parameter, which lumps together all fracture properties such as width, fracture roughness and contact areas. Conversely, in the thin-layer model dispersion is attributed to the actual physical fracture width. We believe that we can attribute dispersion actually to the width of the fracture, mainly for the following reason. When we create a planar fracture without a tip, the width determined with acoustic measurements at different points of the fracture, agrees well with the direct measurement of the displacement transducer (LVDT) at the borehole. Some additional confirmation is found because measured width profiles agree well with predictions that are based on linear elastic fracture mechanics with constant flow rate. The width profiles were constructed by using a limited set of 7 source/receiver combinations with different mid-points. No inconsistencies were found between perpendicular measurements and slightly oblique measurements, that share the same mid-point.

In principle the application of compressional and shear transmissions as in
the laboratory could also be used in the field. Shear transmissions could be used to estimate fracture initiation and propagation, while the compressional transmissions could be used to estimate fracture width. Besides the fact that fractures can be more complicated in the field and secondary effects such as leak-off should be incorporated in the scattering model, the main disadvantage of such a measurement is that we need at least three boreholes; one for fracture initiation and propagation, one for the source that sends the acoustic waves to the fracture and one for the receiver at the other side of the fracture. Such cross-well measurements are relatively expensive.

In Chapter 10 we observed that the tip of the fracture acts as a strong diffractor. The diffraction of compressional and shear waves is in agreement with the modelling results of Chapter 8, which was based on finite-difference techniques with locally varying grid spacings. We used the arrival times of the diffracted events for constructing the growth curve of the fracture as a function of experiment time. Moreover, we observed that the strength of the diffractions increases when the flow rate increases. The flow rate will increase the local width profile at the fracture tip, which apparently increases the strength of the tip diffraction. This also explains why, for a zero toughness fracture, we do not observe a compressional diffraction. The width profile at the tip of a zero toughness fracture is expected to be more wedge shaped, instead of showing a steep increase in width. The shear diffraction is still observed in these cases because a slight opening of the fracture already creates a huge loss of transverse fracture stiffness at the tip and hence this diffraction is much more sensitive to the presence of the fracture tip, regardless its shape. During reopening of a fracture that was not completely closed, we still observe migration of the fluid front in the shear diffractions. The reason for this is poorly understood, but is probably related to the strong sensitivity of shear waves to the angle of incidence at the fracture and the shape of the fracture interface. Shear-wave diffraction half-way an open fracture needs mode-conversion in the fluid to create a full diffraction.

Measuring clear diffractions in the field could provide a tool for estimating the size of fractures. The wide radiation pattern of the diffractions in the modelling of Chapter 8 indicates that the diffraction can be measured at various places. One could even think of using a source as well as a receiver in a single borehole to observe the diffraction, which is economically attractive, in comparison to cross-well seismic transmission measurement. On the other
hand, diffractions are mainly useful to determine the size of the fractures, whereas the combination of shear and compressional transmissions can be used to estimate the extent of a fracture as well as its width.

Of special interest in the modelling results of Chapter 8 is the existence of two types of guided waves that propagate along the fracture. In the first place, we observe a pressure-antisymmetric, generalized Rayleigh wave, which is only weakly dispersive and propagates at a velocity in the low-frequency limit that is roughly equal to the the Rayleigh velocity of a homogeneous elastic half-space. In the second place, we observe a slow pressure-symmetric channel wave, which is highly dispersive and has a velocity in the high-frequency range roughly equal to the velocity of the Scholte wave that occurs at a fluid half-space connected to a solid half-space. In the low-frequency range the group velocity of the slow channel wave tends to zero. The two types of guided waves were also predicted by a theoretical analysis of the guided waves that can propagate along a thin fluid-filled layer of infinite lateral extent, see Chapter 4 and Appendix D. In Chapter 7 these waves could only be observed for a layer of infinite lateral extent, if the distance from the source to the layer as well as the distance from the receiver to the layer are small. But modelling the interaction of the wavefield with fracture and borehole, showed that especially the tip of a fracture and its intersection with the borehole (the notch) cause strong conversion of energy to different wave modes, such as the guided modes. When guided waves reach the tip of a fracture, these waves can be partially re-diffracted to body waves. These can subsequently be measured at the sides of the block. In Chapter 10 we showed that indeed diffractions could be observed that travelled as Rayleigh waves along the fracture, before being re-diffracted into body waves. Re-diffractions of slow channel waves were not observed, which can be explained by the modelling results of Chapter 8, which show that slow channel waves are almost completely reflected inside the layer, i.e. re-diffraction is weak. The slow channel wave, as well as the generalized Rayleigh wave could possibly be observed if we succeed in incorporating a transducer inside the borehole, close to the fracture surface. Because the Rayleigh wave is relatively insensitive to fracture width, contrary to the slow channel wave, measuring tip reflections of the guided wave could be an alternative method for estimating fracture dimensions, i.e. size as well as width.

In a field situation the advantage of such a measurement over the cross-well seismic measurements is that we only need a source and receiver in the same
borehole where the fracture is initiated. If reflections are strong enough, reflections in the borehole could even be measures at the surface. For lower frequency waves, instead of picking the arrival time of these reflections, we could try to excite resonances inside the fracture. These resonances could, in principle, also be used to estimate dimensions of a fracture.

Motivated by the discussion and conclusions above, we formulate some recommendations for future work. In this thesis, diffractions were treated separately from transmissions. The reason for this, is that the description of the amplitude and radiation pattern of diffractions is much more complex than the description of the arrival times of the diffractions only. The forward model of the dispersion of the compressional transmissions was based on a simplified layer model of infinite lateral extent. The fact that an increase in flow rate results in an increase in amplitude of the diffractions, suggests that we need to describe the width profile at the tip accurately to predict the amplitude of the diffractions. The modelling approach can be based on the finite-difference modelling described in Chapter 7. A more fundamental approach would be to take the integral equations, presented in Chapter 8, as starting point.

The two guided modes that propagate in the fracture can be used to our advantage to estimate fracture dimensions as well. Since in the field such measurements could be taken in a single wellbore, this method could be preferably to cross-well seismic transmission measurements or diffraction measurements. For a feasibility study, we can start in the laboratory by implementing a piezo-electric transducer in the borehole. Thus, we can gain experience in the possibilities and complications of such measurements.
Bi-orthogonal Relations for Eigenfunctions

In this appendix we will prove the bi-orthogonal relations, see Eqs. (4.43a)–(4.43d) presented in Section 4.2. These relations are useful to construct the decomposition matrix from the composition matrix and supply an elegant formalism to express the transmission and reflection matrices of interfaces.

We start with the global reciprocity theorem for elastic waves of Eq. (3.66). As domain of application we consider a homogeneous layer bounded at two depth levels, i.e. \( \{ x_3^a \leq x_3 \leq x_3^b \} \), defining the volume \( \mathcal{V} \), where the direction \( x_3 \) is perpendicular to the layering. In state A and B we will consider different plane wave solutions for the same medium. Then the reciprocity theorem leads to

\[
\Delta^+_{ijkl} \int_{x \in \partial \mathcal{V}} (\tilde{\tau}_{kl}^A \tilde{\tau}_{ij}^B - \tilde{\tau}_{kl}^B \tilde{\tau}_{ij}^A) n_i \, dA = 0, \tag{A.1}
\]

since there are neither sources nor discrepancies in the material properties in the domain \( \mathcal{V} \), for the states A and B. We note that \( \partial \mathcal{V} \) consists of two infinite planes at the levels \( x_3^a \) and \( x_3^b \), respectively. The normal for the level \( x_3^b \) points downward similar to \( \mathbf{i}_3 \), whereas the normal for the level \( x_3^b \) points in the opposite direction, i.e. \(-\mathbf{i}_3\). Combined with the arbitrariness of the
depth levels $x_3^a$ and $x_3^b$ within the homogeneous layer, we conclude that
\[
\int_{x_T \in \mathbb{R}^2} (\tau^i_{i3} \hat{v}^B_i - \tau^B_{i3} \hat{v}^A_i) dA = C(j s \alpha^A, s), \tag{A.2}
\]
where the constant $C(j s \alpha^A, s)$ signifies that the integral on the left-hand side does not depend on $x_3$, but of course is still a function of the Laplace parameter $s$ and other wave parameters. For state $A$ we have the plane-wave solution
\[
\left( \begin{array}{c} +\hat{v}^A_i \\ -\tau^A_{i3} \end{array} \right) (x, s) = b_f(j s \alpha^A, x_3, s) \exp(-j s \alpha^A_{\beta} x_\beta), \tag{A.3}
\]
and similarly for state $B$
\[
\left( \begin{array}{c} +\hat{v}^B_i \\ -\tau^B_{i3} \end{array} \right) (x, s) = b_f(j s \alpha^B, x_3, s) \exp(-j s \alpha^B_{\beta} x_\beta). \tag{A.4}
\]
When we substitute the representations of Eqs. (A.3) and (A.4) into Eq. (A.2) we obtain the algebraic identity
\[
(2\pi)^2 b^T(j s \alpha^A, x_3, s) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} b(-j s \alpha^A, x_3, s) = C(j s \alpha^A, s), \tag{A.5}
\]
or using the decomposition in up- and downgoing waves (cf. Eq. (4.30))
\[
(2\pi)^2 y^T(j s \alpha^A, x_3, s) D^T(j s \alpha^A, x_3, s) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} y(-j s \alpha^A, x_3, s) = C(j s \alpha^A, s). \tag{A.6}
\]
From Eqs. (4.36),(4.38),(4.40) and (4.42) we obtain the composition submatrices for the sign-reversed horizontal slowness as
\[
D^{\nu,\downarrow}(-j s \alpha, s) = -D^{\nu,\downarrow}(j s \alpha, s), \tag{A.7a}
\]
\[
D^{\nu,\downarrow}(-j s \alpha, s) = -D^{\nu,\uparrow}(j s \alpha, s), \tag{A.7b}
\]
\[
D^{\tau,\downarrow}(-j s \alpha, s) = +D^{\tau,\downarrow}(j s \alpha, s), \tag{A.7c}
\]
\[
D^{\tau,\downarrow}(-j s \alpha, s) = +D^{\tau,\uparrow}(j s \alpha, s). \tag{A.7d}
\]
Replacing the composition matrix for the sign-reversed horizontal slowness in Eq. (A.6) with the submatrices in Eqs. (A.7a)–(A.7d) we obtain

\[(2\pi)^2y^T(js\alpha^A, x_3, s) \left( \begin{array}{cc} B^{\uparrow\uparrow} & B^{\uparrow\downarrow} \\ B^{\uparrow\downarrow} & B^{\downarrow\downarrow} \end{array} \right) y(-js\alpha^A, x_3, s) = C(js\alpha^A, s), \quad (A.8)\]

where the matrix $B$ has been defined as

\[
\left( \begin{array}{cc} B^{\uparrow\uparrow} & B^{\uparrow\downarrow} \\ B^{\uparrow\downarrow} & B^{\downarrow\downarrow} \end{array} \right) = \left( \begin{array}{cc} D^{\psi,\uparrow} & D^{\psi,\downarrow} \\ D^{\tau,\uparrow} & D^{\tau,\downarrow} \end{array} \right) \left( \begin{array}{cc} D^{\tau,\downarrow} & D^{\tau,\uparrow} \\ D^{\psi,\downarrow} & D^{\psi,\uparrow} \end{array} \right)^T. \quad (A.9)\]

Then, expanding the matrix multiplication of this equation we arrive at

\[
B^{\uparrow\uparrow} = (D^{\tau,\uparrow})^T D^{\psi,\downarrow} + (D^{\psi,\uparrow})^T D^{\tau,\downarrow}, \quad (A.10a) \\
B^{\uparrow\downarrow} = (D^{\tau,\uparrow})^T D^{\psi,\uparrow} + (D^{\psi,\uparrow})^T D^{\tau,\uparrow}, \quad (A.10b) \\
B^{\downarrow\uparrow} = (D^{\tau,\downarrow})^T D^{\psi,\downarrow} + (D^{\psi,\downarrow})^T D^{\tau,\downarrow}, \quad (A.10c) \\
B^{\downarrow\downarrow} = (D^{\tau,\downarrow})^T D^{\psi,\uparrow} + (D^{\psi,\downarrow})^T D^{\tau,\uparrow}. \quad (A.10d)\]

The amplitudes of the up- or downgoing waves $y$ have a $x_3$-dependence according to Eq. (4.12) of

\[
y(js\alpha, x_3, s) = y(js\alpha, x_3^a, s) \exp(-s\zeta^a_3(js\alpha, s)(x_3^b - x_3^a)). \quad (A.11)\]

From Eq. (A.8) it is clear that to avoid interaction of waves propagating in the same direction and thus a $x_3$ dependence of Eq. (A.8) we have to require

\[
B^{\uparrow\uparrow} = 0, \quad (A.12a) \\
B^{\downarrow\downarrow} = 0. \quad (A.12b)\]

Using Eqs. (4.38) and (4.42) we can show that

\[
B^{\uparrow\downarrow} = -B^{\downarrow\uparrow}. \quad (A.13)\]

The $x_3$-dependence for SV- and SH-waves, propagating in the same direction is similar, because the eigenvalues are degenerate. Hence, the matrix $B^{\uparrow\uparrow}$ is not necessarily a diagonal matrix. The coefficients that describe the interaction between the compressional and shear waves vanish, because of the condition expressed in Eq. (A.8). By an appropriate choice of the eigenvectors of the SV- and SH-waves, we can construct the matrix $B^{\uparrow\uparrow}$ and $B^{\downarrow\downarrow}$ to be diagonal, which then opens the way for a normalization of the bi-orthogonal relations in Eqs. (A.10a)–(A.10d).
Auxiliary Relations in Scattering Theory

In this appendix we will explicitly derive auxiliary relations in scattering theory that are used in Chapter 4.

B.1 Relationship between propagators and scattering matrices

We begin by deriving the various interrelations between the propagator matrices and the scattering operator. Notice that we can express the wave matrix at \( x^a_3 \) and \( x^b_3 \) in terms of the incoming waves with the aid of the scattering operator of Eq. (4.78) as

\[
\begin{pmatrix}
    w^\dagger(x^a_3) \\
    w^\dagger(x^b_3)
\end{pmatrix} =
\begin{pmatrix}
    R_{ab} & T_{ab}^\dagger \\
    I & 0
\end{pmatrix}
\begin{pmatrix}
    w^\dagger(x^a_3) \\
    w^\dagger(x^b_3)
\end{pmatrix},
\]  

(B.1)

while for \( x^b_3 \) we have

\[
\begin{pmatrix}
    w^\dagger(x^b_3) \\
    w^\dagger(x^b_3)
\end{pmatrix} =
\begin{pmatrix}
    0 & I \\
    T_{ab}^\dagger & R_{ab}^\dagger
\end{pmatrix}
\begin{pmatrix}
    w^\dagger(x^a_3) \\
    w^\dagger(x^b_3)
\end{pmatrix}.
\]  

(B.2)
We proceed by rewriting Eq. (4.75), the definition of the downward propagator with Eq. (B.1) and Eq. (B.2) as

\[
\begin{pmatrix}
0 & I \\
T_{ab}^\dagger & R_{ab}^\triangledown
\end{pmatrix}
\begin{pmatrix}
w^\dagger(x_3^a) \\
w^\dagger(x_3^b)
\end{pmatrix}
= 
\begin{pmatrix}
Q_{ba}^{\uparrow\uparrow} & Q_{ba}^{\uparrow\downarrow} \\
Q_{ba}^{\downarrow\uparrow} & Q_{ba}^{\downarrow\downarrow}
\end{pmatrix}
\begin{pmatrix}
R_{ab}^\triangledown & T_{ab}^\dagger \\
I & 0
\end{pmatrix}
\begin{pmatrix}
w^\dagger(x_3^a) \\
w^\dagger(x_3^b)
\end{pmatrix}
\]  

(B.3)

From this equation, we obtain for the downward wave matrix propagator in terms of the scattering submatrices

\[
\begin{pmatrix}
Q_{ba}^{\uparrow\uparrow} & Q_{ba}^{\uparrow\downarrow} \\
Q_{ba}^{\downarrow\uparrow} & Q_{ba}^{\downarrow\downarrow}
\end{pmatrix}
= 
\begin{pmatrix}
0 & I \\
T_{ab}^\dagger & R_{ab}^\triangledown
\end{pmatrix}
\begin{pmatrix}
R_{ab}^\triangledown & T_{ab}^\dagger \\
I & 0
\end{pmatrix}^{-1}.

(B.4)

Since the inverse on the right-hand side of this equation is found as

\[
\begin{pmatrix}
R_{ab}^\triangledown & T_{ab}^\dagger \\
I & 0
\end{pmatrix}^{-1}
= 
\begin{pmatrix}
0 & I \\
(T_{ab}^\dagger)^{-1} & -(T_{ab}^\dagger)^{-1}R_{ab}^\triangledown
\end{pmatrix}.

(B.5)

Combining Eqs. (B.4) and (B.5), the relationship between the downward wave matrix propagator and the scattering submatrices, the reflection and transmission matrices is found as

\[
\begin{pmatrix}
Q_{ba}^{\uparrow\uparrow} & Q_{ba}^{\uparrow\downarrow} \\
Q_{ba}^{\downarrow\uparrow} & Q_{ba}^{\downarrow\downarrow}
\end{pmatrix}
= 
\begin{pmatrix}
(T_{ab}^\dagger)^{-1} & -(T_{ab}^\dagger)^{-1}R_{ab}^\triangledown \\
R_{ab}^\triangledown(T_{ab}^\dagger)^{-1} & T_{ab}^\dagger - R_{ab}^\triangledown(T_{ab}^\dagger)^{-1}R_{ab}^\triangledown
\end{pmatrix}.

(B.6)

We repeat the procedure for the relationship between the scattering operator and the upward propagator \(Q_{ab}\). Using Eq. (B.1) and Eq. (B.2) we rewrite the definition of the upward propagator as

\[
\begin{pmatrix}
R_{ab}^\triangledown & T_{ab}^\dagger \\
I & 0
\end{pmatrix}
\begin{pmatrix}
w^\dagger(x_3^a) \\
w^\dagger(x_3^b)
\end{pmatrix}
= 
\begin{pmatrix}
Q_{ab}^{\uparrow\uparrow} & Q_{ab}^{\uparrow\downarrow} \\
Q_{ab}^{\downarrow\uparrow} & Q_{ab}^{\downarrow\downarrow}
\end{pmatrix}
\begin{pmatrix}
0 & I \\
T_{ab}^\dagger & R_{ab}^\triangledown
\end{pmatrix}
\begin{pmatrix}
w^\dagger(x_3^a) \\
w^\dagger(x_3^b)
\end{pmatrix}
\]  

(B.7)

Hence the upward propagator \(Q_{ba}\) in terms of the scattering submatrices is found to be

\[
\begin{pmatrix}
Q_{ab}^{\uparrow\uparrow} & Q_{ab}^{\uparrow\downarrow} \\
Q_{ab}^{\downarrow\uparrow} & Q_{ab}^{\downarrow\downarrow}
\end{pmatrix}
= 
\begin{pmatrix}
R_{ab}^\triangledown & T_{ab}^\dagger \\
I & 0
\end{pmatrix}
\begin{pmatrix}
0 & I \\
T_{ab}^\dagger & R_{ab}^\triangledown
\end{pmatrix}^{-1}.

(B.8)
The inverse on the right-hand side of this equation is found as
\[
\begin{pmatrix}
  0 & I \\
  T_{ab} & R_{ab} \\
\end{pmatrix}^{-1} = \begin{pmatrix}
  -(T_{ab})^{-1}R_{ab} & (T_{ab})^{-1} \\
  I & 0 \\
\end{pmatrix}
\tag{B.9}
\]
and the upward propagator in terms of the reflection and transmission matrices is found to be
\[
\begin{pmatrix}
  Q_{ab}^{\uparrow\uparrow} & Q_{ab}^{\uparrow\downarrow} \\
  Q_{ab}^{\downarrow\uparrow} & Q_{ab}^{\downarrow\downarrow} \\
\end{pmatrix} = \begin{pmatrix}
  T_{ab}^{\uparrow} & -R_{ab}^{\uparrow}(T_{ab}^{\downarrow})^{-1}R_{ab}^{\uparrow} & R_{ab}^{\uparrow}(T_{ab}^{\downarrow})^{-1} \\
  -(T_{ab}^{\downarrow})^{-1}R_{ab}^{\uparrow} & (T_{ab}^{\downarrow})^{-1} \\
\end{pmatrix}
\tag{B.10}
\]
For the inverse relationship, describing the scattering operator in terms of the wave propagators, we apply a similar procedure, by rewriting the definition of the scattering operator, see Eq. (4.79). First notice that the outgoing wave amplitudes can be expressed in terms of the wave matrix at \(x_3^a\) with the aid of the downward wave matrix propagator as
\[
\begin{pmatrix}
  w^{\uparrow}(x_3^a) \\
  w^{\downarrow}(x_3^b) \\
\end{pmatrix} = \begin{pmatrix}
  I & 0 \\
  Q_{ba}^{\uparrow\uparrow} & Q_{ba}^{\uparrow\downarrow} \\
\end{pmatrix} \begin{pmatrix}
  w^{\uparrow}(x_3^a) \\
  w^{\downarrow}(x_3^a) \\
\end{pmatrix}
\tag{B.11}
\]
and for the incoming wave amplitudes
\[
\begin{pmatrix}
  w^{\uparrow}(x_3^a) \\
  w^{\downarrow}(x_3^b) \\
\end{pmatrix} = \begin{pmatrix}
  0 & I \\
  Q_{ba}^{\uparrow\uparrow} & Q_{ba}^{\uparrow\downarrow} \\
\end{pmatrix} \begin{pmatrix}
  w^{\uparrow}(x_3^a) \\
  w^{\downarrow}(x_3^a) \\
\end{pmatrix}.
\tag{B.12}
\]
Similarly, we can express the outgoing wave amplitudes in terms of the upward propagator as
\[
\begin{pmatrix}
  w^{\uparrow}(x_3^a) \\
  w^{\downarrow}(x_3^b) \\
\end{pmatrix} = \begin{pmatrix}
  Q_{ab}^{\uparrow\uparrow} & Q_{ab}^{\uparrow\downarrow} \\
  0 & I \\
\end{pmatrix} \begin{pmatrix}
  w^{\uparrow}(x_3^b) \\
  w^{\downarrow}(x_3^b) \\
\end{pmatrix}
\tag{B.13}
\]
and the incoming wave amplitudes as
\[
\begin{pmatrix}
  w^{\uparrow}(x_3^a) \\
  w^{\downarrow}(x_3^b) \\
\end{pmatrix} = \begin{pmatrix}
  Q_{ab}^{\uparrow\uparrow} & Q_{ab}^{\uparrow\downarrow} \\
  I & 0 \\
\end{pmatrix} \begin{pmatrix}
  w^{\uparrow}(x_3^b) \\
  w^{\downarrow}(x_3^b) \\
\end{pmatrix}.
\tag{B.14}
\]
Therefore the definition of the scattering operator of (4.79) can be rewritten, using Eqs. (B.11)–(B.12), in terms of the downward propagator as
\[
\begin{pmatrix}
  I & 0 \\
  Q_{ba}^{\uparrow\uparrow} & Q_{ba}^{\uparrow\downarrow} \\
\end{pmatrix} = S(x_3^a; x_3^b) \begin{pmatrix}
  0 & I \\
  Q_{ba}^{\uparrow\uparrow} & Q_{ba}^{\uparrow\downarrow} \\
\end{pmatrix}
\tag{B.15}
\]
and in terms of the upward propagator with Eqs. (B.13)–(B.14) as

$$\begin{pmatrix} Q_{ab}^{\uparrow\uparrow} & Q_{ab}^{\uparrow\downarrow} \\ 0 & I \end{pmatrix} = S(x_3^a; x_3^b) \begin{pmatrix} Q_{ab}^{\downarrow\uparrow} & Q_{ab}^{\downarrow\downarrow} \\ I & 0 \end{pmatrix}.$$  \hspace{1cm} (B.16)

The scattering operator in terms of the downward wave matrix propagation submatrices can thus be expressed as

$$S(x_3^a; x_3^b) = \begin{pmatrix} I & 0 \\ Q_{ba}^{\uparrow\downarrow} & Q_{ba}^{\uparrow\uparrow} \end{pmatrix} \begin{pmatrix} 0 & I \\ Q_{ba}^{\downarrow\uparrow} & Q_{ba}^{\downarrow\downarrow} \end{pmatrix}^{-1} \hspace{1cm} (B.17)$$

and in terms of the upward wave propagator as

$$S(x_3^a; x_3^b) = \begin{pmatrix} Q_{ab}^{\uparrow\uparrow} & Q_{ab}^{\uparrow\downarrow} \\ 0 & I \end{pmatrix} \begin{pmatrix} Q_{ab}^{\downarrow\uparrow} & Q_{ab}^{\downarrow\downarrow} \\ I & 0 \end{pmatrix}^{-1}.$$  \hspace{1cm} (B.18)

Since the inverse matrices in Eq. (B.17) and (B.18) are found as

$$\begin{pmatrix} 0 & I \\ Q_{ba}^{\uparrow\uparrow} & Q_{ba}^{\uparrow\downarrow} \end{pmatrix}^{-1} = \begin{pmatrix} -(Q_{ba}^{\uparrow\uparrow})^{-1}Q_{ba}^{\uparrow\downarrow} & (Q_{ba}^{\uparrow\uparrow})^{-1} \\ I & 0 \end{pmatrix}$$  \hspace{1cm} (B.19)

and

$$\begin{pmatrix} Q_{ab}^{\uparrow\uparrow} & Q_{ab}^{\uparrow\downarrow} \\ I & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 \\ -(Q_{ab}^{\uparrow\downarrow})^{-1} \end{pmatrix}$$  \hspace{1cm} (B.20)

respectively, the scattering operator can be expressed in terms of the upward and downward propagation submatrices as

$$S(x_3^a; x_3^b) = \begin{pmatrix} R_\uparrow(x_3^a; x_3^b) & T_\uparrow(x_3^a; x_3^b) \\ T_\downarrow(x_3^a; x_3^b) & R_\downarrow(x_3^a; x_3^b) \end{pmatrix}$$

$$= \begin{pmatrix} -(Q_{ba}^{\uparrow\uparrow})^{-1}Q_{ba}^{\uparrow\downarrow} & (Q_{ba}^{\uparrow\uparrow})^{-1} \\ Q_{ba}^{\downarrow\uparrow} - Q_{ba}^{\uparrow\downarrow}(Q_{ba}^{\uparrow\uparrow})^{-1}Q_{ba}^{\downarrow\uparrow} & Q_{ba}^{\uparrow\downarrow}(Q_{ba}^{\uparrow\uparrow})^{-1} \end{pmatrix} \hspace{1cm} (B.21)$$

$$= \begin{pmatrix} Q_{ab}^{\uparrow\uparrow}(Q_{ab}^{\downarrow\downarrow})^{-1} \\ (Q_{ab}^{\downarrow\uparrow})^{-1} \end{pmatrix}$$

which finalizes the derivation of the interrelations between the wave propagator and the scattering operators.
B.2 Redheffer’s star product

In this section we will show how the scattering operator of a certain section is constructed from the scattering operators of two subsections which encompass the total section.

By using Eq. (4.91a) and (B.6) the downward propagator of the upgoing waves at $x^a_3$ to the upgoing waves at $x^c_3$ results in

$$Q_{ca}^{\uparrow\uparrow} = Q_{cb}^{\uparrow\downarrow} Q_{ba}^{\downarrow\uparrow} + Q_{cb}^{\downarrow\uparrow} Q_{ba}^{\downarrow\uparrow} = (T_{bc}^{\uparrow})^{-1} (T_{ab}^{\uparrow})^{-1} - (T_{bc}^{\uparrow})^{-1} R_{bc}^{\downarrow\uparrow} R_{ab}^{\downarrow\uparrow} (T_{ab}^{\uparrow})^{-1}$$

$$= (T_{bc}^{\uparrow})^{-1} (I - R_{bc}^{\downarrow\uparrow} R_{ab}^{\downarrow\uparrow}) (T_{ab}^{\uparrow})^{-1},$$

while for the downward propagator of the upgoing waves at $x^a_3$ to the downgoing waves at $x^c_3$, by using Eq. (B.10) we find

$$Q_{ca}^{\downarrow\uparrow} = Q_{cb}^{\downarrow\uparrow} Q_{ba}^{\downarrow\uparrow} + Q_{cb}^{\downarrow\uparrow} Q_{ba}^{\downarrow\uparrow} = R_{bc}^{\downarrow\uparrow} (T_{bc}^{\uparrow})^{-1} (T_{ab}^{\uparrow})^{-1} + (T_{bc}^{\downarrow})^{-1} R_{bc}^{\downarrow\uparrow} R_{ab}^{\downarrow\uparrow} (T_{ab}^{\uparrow})^{-1}$$

$$= T_{bc}^{\downarrow} R_{ab}^{\downarrow\uparrow} (T_{ab}^{\uparrow})^{-1} + R_{bc}^{\downarrow\uparrow} (T_{bc}^{\uparrow})^{-1} (I - R_{bc}^{\downarrow\uparrow} R_{ab}^{\downarrow\uparrow}) (T_{ab}^{\uparrow})^{-1}.$$

For the upward propagator of the downgoing waves at $x^c_3$ to the downgoing waves at $x^a_3$ we find

$$Q_{ac}^{\downarrow\downarrow} = Q_{ab}^{\downarrow\downarrow} Q_{bc}^{\downarrow\downarrow} + Q_{ab}^{\downarrow\downarrow} Q_{bc}^{\downarrow\downarrow} = (T_{ab}^{\downarrow})^{-1} (T_{bc}^{\downarrow})^{-1} - (T_{ab}^{\downarrow})^{-1} R_{ab}^{\downarrow\downarrow} R_{bc}^{\downarrow\downarrow} (T_{bc}^{\downarrow})^{-1}$$

$$= (T_{ab}^{\downarrow})^{-1} (I - R_{ab}^{\downarrow\downarrow} R_{bc}^{\downarrow\downarrow}) (T_{bc}^{\downarrow})^{-1},$$

while for the upward propagator of the downgoing waves at $x^c_3$ to the upgoing waves at $x^a_3$ we find

$$Q_{ac}^{\downarrow\uparrow} = Q_{ab}^{\downarrow\uparrow} Q_{bc}^{\downarrow\uparrow} + Q_{ab}^{\downarrow\uparrow} Q_{bc}^{\downarrow\uparrow} = R_{ab}^{\downarrow\uparrow} (T_{ab}^{\downarrow})^{-1} (T_{bc}^{\downarrow})^{-1} + (T_{ab}^{\downarrow})^{-1} R_{ab}^{\downarrow\uparrow} R_{bc}^{\downarrow\uparrow} (T_{bc}^{\downarrow})^{-1}$$

$$= T_{ab}^{\downarrow} R_{bc}^{\downarrow\uparrow} (T_{bc}^{\downarrow})^{-1} + R_{ab}^{\downarrow\uparrow} (T_{ab}^{\downarrow})^{-1} (I - R_{ab}^{\downarrow\uparrow} R_{bc}^{\downarrow\uparrow}) (T_{bc}^{\downarrow})^{-1}.$$

Hence, with Eq. (4.87), the upward transmission matrix of the section $(x^a_3; x^c_3)$ is found to be

$$T_{ac}^{\uparrow}(x^a_3; x^c_3) = (Q_{ca}^{\uparrow\uparrow})^{-1} = T_{ab}^{\uparrow} (I - R_{bc}^{\downarrow\uparrow} R_{ab}^{\downarrow\uparrow})^{-1} T_{bc}^{\uparrow},$$

(B.26)
and for the downward transmission matrix of the section \((x_3^a; x_3^c)\) we have

\[
T^\downarrow(x_3^a; x_3^c) = (Q_{ac})^{-1} \quad = T_{bc}^\dagger (I - R_{ab} \hat R_{bc})^{-1} T_{ab}^\dagger.
\] (B.27)

The down- to upward reflection matrix of the section \((x_3^a; x_3^c)\), using Eq. (4.87) is found to be

\[
R_{\uparrow}(x_3^a; x_3^c) = Q_{ac}^\dagger (Q_{ac})^{-1} \quad = Q_{ac}^\dagger T^\dagger(x_3^a; x_3^c) \quad = [T_{ab} R_{bc} (T_{bc})^{-1} + R_{ab} (T_{ab})^{-1} (I - R_{bc} R_{bc}) (T_{bc})^{-1}] \cdot [T_{bc}^\dagger (I - R_{ab} R_{bc})^{-1} T_{ab}^\dagger] \quad = R_{ab} + T_{ab}^\dagger R_{bc} (I - R_{bc} R_{bc}) T_{ab}^\dagger \quad \text{(B.28)}
\]

and the up- to downward reflection matrix of the section \((x_3^a; x_3^c)\) results in

\[
R_{\downarrow}(x_3^a; x_3^c) = Q_{ca}^\dagger (Q_{ca})^{-1} \quad = Q_{ca}^\dagger T^\dagger(x_3^a; x_3^c) \quad = [T_{bc} R_{ab} (T_{ab})^{-1} + R_{bc} (T_{bc})^{-1} (I - R_{bc} R_{bc}) (T_{bc})^{-1}] \cdot [T_{bc}^\dagger (I - R_{bc} R_{bc})^{-1} T_{bc}^\dagger] \quad = R_{bc} + T_{bc}^\dagger R_{ab} (I - R_{bc} R_{bc})^{-1} T_{bc}^\dagger \quad \text{(B.29)}
\]

Equations (B.26), (B.27), (B.28) and (B.29) together constitute, the composition of a scatterer operator out of the scatterer operator of the subdomains and is known as Redheffer's star product [Redheffer, 1961; Kennett, 1983], which is summarized as

\[
S(x_3^a; x_3^c) = S(x_3^a; x_3^b) \ast S(x_3^b; x_3^c) = \begin{pmatrix}
R_{ab} \hat R_{bc}(I - R_{ab} \hat R_{bc}) T_{ab}^\dagger & T_{ab}^\dagger (I - R_{bc} R_{bc})^{-1} T_{bc}^\dagger \\
T_{bc}^\dagger (I - R_{ab} \hat R_{bc})^{-1} T_{ab}^\dagger & R_{bc} + T_{bc}^\dagger R_{ab} (I - R_{bc} R_{bc})^{-1} T_{bc}^\dagger
\end{pmatrix}.
\] (B.30)

In the literature sometimes other forms are found, which can be obtained by using the identity

\[
(I - R_{cb} \hat R_{bc})^{-1} R_{cb} = R_{cb} (I - R_{cb} R_{cb})^{-1}.
\] (B.31)
B.3 The source and radiation condition

In this section we will implement the source and radiation condition in our recursive scattering formalism. With the determination of the scattering operator of the interface and the homogeneous region, combined with Redheffer’s star product we can construct the global scattering operator of an composite interval consisting of piecewise homogeneous layers. Special care has to be taken at the source level, which acts as an effective discontinuity in the wave vector at that level. We define the global scattering operator of the top interval, as $S_T$, denoting the scattering operator of the interval \( \{ x_3^{a,-} \leq x_3 \leq x_3^{s,-} \} \), representing the the top of the uppermost layer $x_3^{a,-}$ till the top of the source level $x_3^{s,-}$. Similarly, we will define the global scattering operator of the bottom interval, denoted as $S_B$ as the scattering operator of the interval \( \{ x_3^{s,+} \leq x_3 \leq x_3^{z,+} \} \), representing the bottom of the source level $x_3^{s,+}$ till the bottom of the lower-most layer $x_3^{z,+}$. With the definition of the scattering operator of Eq. (4.78) we obtain the scattering operator of the interval \( \{ x_3^{a,-} \leq x_3 \leq x_3^{s,-} \} \) as

\[
\begin{pmatrix}
  y^\uparrow(x_3^{a,-}) \\
  y^\downarrow(x_3^{s,-})
\end{pmatrix}
= S_T(x_3^{a,-}, x_3^{s,-}) \begin{pmatrix}
  y^\uparrow(x_3^{a,-}) \\
  y^\downarrow(x_3^{s,-})
\end{pmatrix},
\] (B.32)

while for the interval \( \{ x_3^{s,+} \leq x_3 \leq x_3^{z,+} \} \) we obtain

\[
\begin{pmatrix}
  y^\uparrow(x_3^{s,+}) \\
  y^\downarrow(x_3^{z,+})
\end{pmatrix}
= S_B(x_3^{s,+}, x_3^{z,+}) \begin{pmatrix}
  y^\uparrow(x_3^{s,+}) \\
  y^\downarrow(x_3^{z,+})
\end{pmatrix}.
\] (B.33)

Next, we can express the source condition as an effective discontinuity of the wave vector at the source level as

\[
\begin{pmatrix}
  y^\uparrow(x_3^{s,+}) \\
  y^\downarrow(x_3^{s,+})
\end{pmatrix}
- \begin{pmatrix}
  y^\uparrow(x_3^{s,-}) \\
  y^\downarrow(x_3^{s,-})
\end{pmatrix} = \begin{pmatrix}
  X^\uparrow \\
  X^\downarrow
\end{pmatrix}.
\] (B.34)

The source condition has to be supplemented with radiation conditions at infinity in order to render the solution of the scattering problem unique. The radiation condition expresses the fact that no wave can impinge on the outside of the interval, since no primary or secondary causal sources are present in both half-spaces, leading to

\[
y^\uparrow(x_3^{a,-}) = 0, \quad (B.35a)
\]

\[
y^\downarrow(x_3^{z,+}) = 0. \quad (B.35b)
\]
We will show that Eqs. (B.32)–(B.35b) together constitute a unique solution to the complex-frequency angular-slowness transformed wave equation, by constructing the unique solution for the wave vector at the source level, both approached from the top as from the bottom. Hereby we use the radiation condition, Eqs. (B.35a)–(B.35b), in the definitions of the global top and bottom scattering operators in Eqs. (B.32)–(B.33) leading to the following simplified expression for the incoming fields at the source discontinuity level intervals

\[
y^\dagger(x_3^{s-}) = R_T^\gamma(x_3^{s-}; x_3^{s-})y^\dagger(x_3^{s-}), \quad \tag{B.36a}
\]
\[
y^\dagger(x_3^{s+}) = R_B^\gamma(x_3^{s+}; x_3^{s+})y^\dagger(x_3^{s+}), \quad \tag{B.36b}
\]

where we have used the subindices \( T \) and \( B \) corresponding to the submatrices of the scattering operator of the top and bottom interval. On the other hand the outgoing fields into both half-spaces are calculated in a similar manner as

\[
y^\dagger(x_3^{a-}) = T_T^\dagger(x_3^{a-}; x_3^{s-})y^\dagger(x_3^{s-}), \quad \tag{B.37a}
\]
\[
y^\dagger(x_3^{a+}) = T_B^\dagger(x_3^{s+}; x_3^{s+})y^\dagger(x_3^{s+}). \quad \tag{B.37b}
\]

Since the subscript \( T \) is uniquely linked to the interval \( \{x_3^{a-} \leq x_3 \leq x_3^{s-}\} \) and likewise the subscript \( B \) to the interval \( \{x_3^{s-} \leq x_3 \leq x_3^{s+}\} \), we permit ourselves in the following analysis to suppress the arguments \((x_3^{s-}; x_3^{s-})\) and \((x_3^{s+}; x_3^{s+})\) in the transmission and reflection operators. We now derive an expression for the outgoing waves at the source level in terms of the source vector \( \mathbf{X} \) by combining the expressions for the incident wave vectors at the source level, Eq. (B.36a) and (B.36b) with the source condition in Eq. (B.34) leading to

\[
\begin{pmatrix}
-I & R_B^\gamma \\
-R_T^\gamma & I
\end{pmatrix}
\begin{pmatrix}
y^\dagger(x_3^{s-}) \\
y^\dagger(x_3^{s+})
\end{pmatrix} =
\begin{pmatrix}
\mathbf{X}^\dagger \\
\mathbf{X}_\downarrow
\end{pmatrix}. \quad \tag{B.38}
\]

Hence we can determine the outgoing waves at the source level by determining the inverse of the matrix on the left-hand side of Eq. (B.38). To this end we note that a multiplication of this matrix with itself results in

\[
\begin{pmatrix}
-I & R_B^\gamma \\
-R_T^\gamma & I
\end{pmatrix}
\begin{pmatrix}
-I & R_B^\gamma \\
-R_T^\gamma & I
\end{pmatrix} =
\begin{pmatrix}
I - R_B^\gamma R_T^\gamma & 0 \\
0 & I - R_T^\gamma R_B^\gamma
\end{pmatrix}, \quad \tag{B.39}
\]
which is a block-diagonal matrix. We make use of this property to calculate
the inverse of the matrix in Eq. (B.38), resulting in

\[
\begin{pmatrix}
-I & R_B \\
-R_T & I
\end{pmatrix}^{-1} =
\begin{pmatrix}
(I - R_B R_T)^{-1} & 0 \\
0 & (I - R_T R_B)^{-1}
\end{pmatrix}
\begin{pmatrix}
-I & R_B \\
-R_T & I
\end{pmatrix},
\] (B.40)

which enables us to express the outgoing waves in terms of the source and
the reflection coefficients of the top and bottom interval as

\[
\begin{pmatrix}
y^+(x_3^s, -) \\
y^+(x_3^s, +)
\end{pmatrix} =
\begin{pmatrix}
(I - R_B R_T)^{-1} & 0 \\
0 & (I - R_T R_B)^{-1}
\end{pmatrix}
\begin{pmatrix}
-I & R_B \\
-R_T & I
\end{pmatrix}
\begin{pmatrix}
X^+ \\
X^-
\end{pmatrix}.
\] (B.41)

By applying the source and radiation condition, we have constructed a unique
solution to the outgoing waves at the source level. Therefore, also the incoming
waves are determined with Eqs. (B.36a) and (B.36b) in a unique manner.
The outgoing waves at the top and bottom follow from Eqs. (B.37a) and
(B.37b). By up- or downward continuation of the wavefield with the scattering
formalism we recognize that we can uniquely determine the wavefield in the
complex-frequency angular-slowness domain at any level.

### B.4 Application for a single layer configuration

In this section we will derive expressions for the response of the configuration
of a source above a single plane layer in terms of the layer reflection and
transmission responses. To this end, we will derive an expression for the outgoing waves at the source level. Notice that no reflectors are present
above the source level, which can be expresses with Eq. (B.36a) as

\[
R_T = 0.
\] (B.42)

In this simplified case, the outgoing wave vector at the source level expressed
in Eq. (B.41) reduces to

\[
\begin{pmatrix}
y^+(x_3^s, -) \\
y^+(x_3^s, +)
\end{pmatrix} =
\begin{pmatrix}
-I & R_B \\
0 & I
\end{pmatrix}
\begin{pmatrix}
X^+ \\
X^-
\end{pmatrix}.
\] (B.43)
Hence we must determine the down- to upgoing reflection coefficient of the bottom interval, \( \{x_3^{s,+} \leq x_3 \leq x_3^{z,+}\} \) representing the bottom of the source level to the bottom of the layer. This reflection coefficient can be retrieved by application of Redheffer’s star product as a left-multiplication of the scattering operator of the layer with the scattering operator of the homogeneous interval \( \{x_3^{s,+} \leq x_3 \leq x_3^{t,-}\} \). Note that we use Eq. (4.116) for the notation of the one-way phase delay operator as a function of the difference in level solely in the embedding. The one-way phase delay operator in the layer is consistently denoted by \( \Theta \). Application of Redheffer’s star product of Eq. (4.92) together with Eq. (4.116) and (4.135) leads to

\[
S(x_3^{s,+};x_3^{z,+}) = \begin{pmatrix}
0 & W(x_3^t - x_3^s) \\
W(x_3^t - x_3^s) & 0
\end{pmatrix} \star \begin{pmatrix}
R \| T \\
T \| R
\end{pmatrix} = \begin{pmatrix}
W(x_3^t - x_3^s)R \| W(x_3^t - x_3^s) \\
T \| W(x_3^t - x_3^s)
\end{pmatrix},
\]

(B.44)

from which we can identify the down- to upgoing reflection coefficient of the bottom interval, \( \{x_3^{s,+} \leq x_3 \leq x_3^{z,+}\} \) as

\[
R_y^\perp = W(h^S)R \| W(h^S).
\]

(B.45)

From Eq. (B.43) we deduce that the downgoing wave vector just below the source level, \( y^\perp (x_3^{s,+}) \), is simply determined by the downgoing source term, expressed as

\[
y^\perp (x_3^{s,+}) = X^\perp.
\]

(B.46)

In addition we can obtain from Eq. (B.36b) together with Eq. (B.45) the upgoing wave vector just below the source level, which combined with the Eq. (B.46) results in

\[
y(x_3^{s,+}) = \begin{pmatrix}
y^\perp (x_3^{s,+}) \\
y^\perp (x_3^{s,+})
\end{pmatrix} = \begin{pmatrix}
W(h^S)R \| W(h^S) \\
I
\end{pmatrix} X^\perp.
\]

(B.47)

Hence we have determined the solution of the wave vector at the source level. We have split the response above the layer into the direct wave \( y_I \) and the reflected wave \( y_R \). At any other specific level above the layer we can
simply determine the response by downward continuation of the wave vector at the source level, by multiplication with the appropriate one-way phase delay \textit{propagator} in Eq. (4.114), expressed as

\[
y(x_3^+) = \begin{pmatrix}
y^+(x_3^+)
y^+(x_3^+)
\end{pmatrix} = \begin{pmatrix}
\exp\left(\pm \Lambda^+ (h^S - h^R)\right) & 0 \\
0 & \exp\left(-\Lambda^+ (h^S - h^R)\right)
\end{pmatrix}
y(x_3^{s,+}) \quad \text{(B.48)}
\]

\[
y^+(x_3^+) = \begin{pmatrix}
W(h^R)R_\parallel W(h^S) \\
W(h^S - h^R)
\end{pmatrix} X^+.
\]

where we have used the result of Eq. (B.47) and the fact that \(x_3^+ - x_3^- = h^S - h^R\). The wave vector below the layer can be obtained by constructing the scattering operator of the interval \(\{x_3^{s,+} \leq x_3 \leq x_3^-\}\) defined as

\[
\begin{pmatrix}
y^+(x_3^+) \\
y^+(x_3^-)
\end{pmatrix} = S(x_3^+; x_3^-) \begin{pmatrix}
y^+(x_3^+) \\
y^+(x_3^-)
\end{pmatrix} = \begin{pmatrix}
R^\gamma(x_3^+; x_3^-) & T^\dagger(x_3^+; x_3^-) \\
T^\dagger(x_3^-; x_3^+) & R^\gamma(x_3^-; x_3^+)
\end{pmatrix} \begin{pmatrix}
y^+(x_3^+) \\
y^+(x_3^-)
\end{pmatrix} \quad \text{(B.49)}
\]

This scattering operator can be obtained by application of Redheffer's start product by right-multiplication of the scattering operator of the interval \(\{x_3^{s,+} \leq x_3 \leq x_3^{s,+}\}\) in Eq. (B.44) with the scattering operator of the homogeneous interval \(\{x_3^{s,+} \leq x_3 \leq x_3^{s,-}\}\) expressed as

\[
S_T(x_3^+, x_3^-) = \begin{pmatrix}
W(h^S)R_\parallel W(h^S) & W(h^S)T_\parallel \\
T_\parallel W(h^S) & R_\parallel
\end{pmatrix}^* \begin{pmatrix}
W(x_3^+ - x_3^-) \\
W(x_3^+ - x_3^-)
\end{pmatrix} \quad \text{(B.50)}
\]

\[
= \begin{pmatrix}
W(h^S)R_\parallel W(h^S) & W(h^S)T_\parallel W(h^T) \\
W(h^T)T_\parallel W(h^S) & W(h^T)R_\parallel W(h^T)
\end{pmatrix}.
\]

For the wavefield below the layer no upgoing wavefield exist on behalf of the radiation condition in Eq. (B.35b), expressed as

\[
y^+(x_3^+) = 0. \quad \text{(B.51)}
\]
Hence, selecting the downgoing wave vector from Eq. (B.50) and combining with Eqs. (B.51) and (B.46) results in

\[
y(x^\downarrow_3) = \begin{pmatrix} y^\uparrow(x_3^\downarrow) \\ y^\downarrow(x_3^\downarrow) \end{pmatrix} = \begin{pmatrix} 0 \\ W(h^T)T_{\parallel}W(h^S) \end{pmatrix} X^\downarrow. \tag{B.52}
\]

No upgoing part is present below the layer related to the radiation condition in the lower half space. Hence, we have obtained an expression for the wave vector for a reflection and transmission measurement.
The Fluid/Solid Interface

In this appendix we will derive the reflection and transmission coefficients for the solid/liquid interface. An ideal fluid cannot sustain any shear stresses, hence the stiffness vanishes for shear deformation. Note that when the shear modulus vanishes the composition matrix (cf. Eq. (4.36), (4.40), (4.46a) and (4.46c)) become ill-defined. We must handle the ideal fluid case separately by a priori incorporating the vanishing shear stresses. For the wavefield in the ideal fluid we use the coupled linear first order acoustic wave equation in the Laplace transformed domain of Eq. (3.55a) and (3.55b), i.e.

\[ \partial_t \hat{p}(x, s) + s \rho_f(x) \hat{v}_t(x, s) = \hat{f}_i(x, s), \]  
\[ \partial_t \hat{v}_k(x, s) + s \kappa_f(x) \hat{p}(x, s) = \hat{q}(x, s). \]

where \( \kappa_f \) is the compressibility of the fluid, or the inverse of the stiffness and \( \rho_f \) is the density of the fluid. To distinguish quantities attached to the fluid from quantities attached to the solid embedding we use the subscripts \( f \) and \( e \), respectively, unless no confusion is possible. For example, since the shear velocity in the ideal fluid is not defined, at any moment \( c_s \) indicates the shear velocity in the solid embedding.

With a similar technique as in Chapter 4 (cf. Eq. (4.2)) we apply a frequency and slowness transformation to the field quantities to find a first order system,
written as
\[ \partial_3 b_\alpha = -s A_{\alpha\beta} b_\beta + F_\alpha, \]  
(C.2)
where \( b_\alpha \), the acoustic field vector in the slowness and Laplace transformed domain for the acoustic case is defined as
\[ b_\alpha = \begin{pmatrix} +\tilde{v}_3 \\ +\tilde{p} \end{pmatrix}, \]  
(C.3)
The acoustic system matrix \( A_{\alpha\beta} \) is given by
\[ A_{\alpha\beta} = \begin{pmatrix} 0 & (\rho_f)^{-1}(\rho_f \kappa_f - \zeta_\alpha \zeta_\alpha) \\ \rho_f & 0 \end{pmatrix}, \]  
(C.4)
while the source vector \( F_\alpha \) is given by
\[ F_\alpha = \begin{pmatrix} (\rho_f)^{-1} \zeta_\alpha \tilde{f}_\alpha + \tilde{q} \\ \tilde{f}_3 \end{pmatrix}, \]  
(C.5)
consisting of a rate of volume injection source \( \tilde{q} \) and a force source \( \tilde{f}_i \). We introduce the vertical slowness \( \gamma_{P,f} \) in the fluid as
\[ \gamma_{P,f} = (\rho_f \kappa_f - \zeta_\alpha \zeta_\alpha)^{\frac{1}{2}}, \quad \text{Re}(s \gamma_{P,f}) \geq 0. \]  
(C.6)
Next, we compose the field vector out of up- and downgoing waves, or inversely decompose the field vector into up- and downgoing waves by using
\[ b_\alpha = D_{\alpha\beta} w_\beta, \]  
(C.7a)
\[ w_\alpha = D_{\alpha\beta}^{-1} b_\beta, \]  
(C.7b)
where the vector \( w_\alpha \) contains the amplitudes of the up- and downgoing compressional waves, organized as
\[ w_\alpha = \begin{pmatrix} w_{\alpha}^\uparrow \\ w_{\alpha}^\downarrow \end{pmatrix}. \]  
(C.8)
The composition matrix \( D_f \) is constructed in a similar manner as in Chapter 4, cf. Eq. (4.34), resulting in
\[ D_f = \begin{pmatrix} D_{v3,\uparrow} & D_{v3,\downarrow} \\ D_{p,\uparrow} & D_{p,\downarrow} \end{pmatrix} = \begin{pmatrix} -\gamma_{P,f} & +\gamma_{P,f} \\ +\rho_f & +\rho_f \end{pmatrix} (n_{P,f})^{-1}, \]  
(C.9)
where we have chosen the normalisation \( n_{P,f} \) as

\[
n_{P,f} = (2\rho_f \gamma_{P,f})^{1/2}, \tag{C.10}
\]

in such a way, that for the decomposition matrix \( D_f^{-1} \) (cf. Eq. (4.48)) we find

\[
D_f^{-1} = \begin{pmatrix}
-D^{p,\uparrow} & -D^{v_3,\uparrow} \\
D^{p,\downarrow} & D^{v_3,\downarrow}
\end{pmatrix} = (n_{P,f})^{-1} \begin{pmatrix}
-\rho_f & +\gamma_{P,f} \\
+\rho_f & +\gamma_{P,f}
\end{pmatrix}. \tag{C.11}
\]

It can be proven, similar to the analysis in Appendix A, that the up- and downgoing waves satisfy bi-orthogonal symmetry relations, based on the acoustic reciprocity theorem. The acoustic equivalent of the Betti-Rayleigh reciprocity relation of Section 3.3 is given in Fokkema and van den Berg [1993] and de Hoop [1995].

We will assume that a solid embedding overlies a fluid halfspace. The boundary conditions that we must apply for the fluid/solid interface are:

- continuity of the vertical component of the traction \( \tilde{\tau}_{33} \) on any horizontal interface. Note that the sign convention for the isotropic pressure is the opposite for the stress. Therefore we must demand that on the horizontal interface we have \( -\tilde{\tau}_{33} = \bar{p} \).

- Since no shear stress exists in the fluid and we must satisfy the continuity of the traction on the horizontal interface, we require the shear stresses to vanish on the horizontal interface separating the solid from the fluid, i.e. \( \tilde{\tau}_{13} = \tilde{\tau}_{23} = 0 \).

These two conditions in terms of the amplitudes of the up- and downgoing waves in the fluid \( w_P \) and solid \( w \) can be combined into

\[
D_{ij}^{\tau,\uparrow}(x_3^-)w_j^{\uparrow}(x_3^-) + D_{ij}^{\tau,\downarrow}(x_3^-)w_j^{\downarrow}(x_3^-) = \begin{pmatrix}
0 \\
0 \\
D^{p,\uparrow}w_P(x_3^+) + D^{p,\downarrow}w_P(x_3^+)
\end{pmatrix}. \tag{C.12}
\]

- Since the fluid cannot penetrate into the solid we require the continuity of the vertical particle velocity \( \tilde{v}_3 \)
The continuity of the vertical particle velocity $\ddot{v}_3$ can be expressed in terms of the up- and downgoing wave amplitudes as

$$D^{\nu_3,\uparrow}_{3j}(x_3^-)w^\uparrow_j(x_3^-) + D^{\nu_3,\downarrow}_{3j}(x_3^-)w^\downarrow_j(x_3^-) = D^{\nu_3,\uparrow}_p(x_3^+)w^\uparrow_p(x_3^+) + D^{\nu_3,\downarrow}_p(x_3^+)w^\downarrow_p(x_3^+). \quad (C.13)$$

Combining the boundary conditions of Eqs. (C.12) and (C.13) and using the symmetry in the composition matrix $D$ for the fluid we obtain

$$\begin{pmatrix}
D^{\nu_3,\uparrow}_{ij} & D^{\nu_3,\downarrow}_{ij} \\
D^{\nu_3,\uparrow}_{3j} & D^{\nu_3,\downarrow}_{3j}
\end{pmatrix}
\begin{pmatrix}
w^\uparrow_j \\
w^\downarrow_j
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
0 & 0 \\
D^{p,\uparrow} & D^{p,\downarrow} \\
-D^{\nu_3,\uparrow} & D^{\nu_3,\downarrow}
\end{pmatrix}
\begin{pmatrix}
w^\uparrow_p \\
w^\downarrow_p
\end{pmatrix}. \quad (C.14)$$

To incorporate the fact that shear waves cannot propagate in the fluid we define the scattering operator for the solid/fluid interface $S_f$ having a set of vanishing components, expressed as

$$\begin{pmatrix}
w^\uparrow_i \\
w^\downarrow_i \\
0 \\
0
\end{pmatrix}
= S_f
\begin{pmatrix}
w^\uparrow_j \\
w^\downarrow_j \\
0 \\
0
\end{pmatrix},$$

$$= \begin{pmatrix}
. & . & . & . & 0 & 0 \\
. & R_{ij}^\nu & . & . & 0 & 0 \\
. & . & T_i^\uparrow & . & 0 & 0 \\
. & (T_j^\downarrow)^T & . & R^\cap & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
w^\uparrow_i \\
w^\downarrow_i \\
0 \\
0
\end{pmatrix}. \quad (C.15)$$

where $R_{ij}^\nu$ is the reflection matrix in the solid, $T_j^\downarrow$ the transmission vector of the waves from the solid to the fluid, $T_i^\uparrow$ the transmission vector from the fluid to the solid and $R^\cap$ the reflection coefficient in the fluid. In this way we can still construct the scattering matrix for a composition of fluid and solid layers, because its structure conforms to the more general elastodynamic case. The boundary conditions in Eq. (C.14) can be combined with the scattering operator in Eq. (C.15) by writing the outgoing waves in Eq. (C.14) in terms
of the incoming waves, leading to

\[
\begin{pmatrix}
D_{ij}^{\tau,\uparrow} & D_{ij}^{\tau,\downarrow} \\
D_{3j}^{\tau,\uparrow} & D_{3j}^{\tau,\downarrow}
\end{pmatrix}
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & R_{jk}^{\omega} & T_{j}^{\uparrow} & \\
\cdot & \cdot & 0 & \\
\cdot & \cdot & 0 & 0
\end{pmatrix}
\begin{pmatrix}
w_{k}^{\uparrow} \\
w_{k}^{\uparrow}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
D_{k}^{\omega} \\
D_{k}^{\omega}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 1 \\
(T_{k}^{\downarrow})^{T} & R_{k}^{\omega} & \\
R_{k}^{\omega} & \\
R_{k}^{\omega}
\end{pmatrix}
\begin{pmatrix}
w_{k}^{\uparrow} \\
w_{k}^{\uparrow}
\end{pmatrix}.
\]

(C.16)

By working out the matrix multiplication we obtain

\[
\begin{pmatrix}
D_{ij}^{\tau,\uparrow}R_{jk}^{\omega} + D_{ik}^{\tau,\downarrow} & D_{ij}^{\tau,\uparrow}T_{j}^{\uparrow} \\
D_{3j}^{\tau,\uparrow}R_{jk}^{\omega} & D_{3j}^{\tau,\uparrow}T_{j}^{\uparrow}
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
D_{k}^{\omega}(T_{k}^{\downarrow})^{T} & D_{k}^{\omega}(R_{k}^{\omega} + 1) \\
D_{k}^{\omega}(T_{k}^{\downarrow})^{T} & D_{k}^{\omega}(R_{k}^{\omega} - 1)
\end{pmatrix}.
\]

(C.17)

which describes 16 equations relating the 9 + 1 unknown reflection coefficients and the 3 + 3 unknown transmission coefficients. Note that the expressions for the reflection coefficients in the solid and the transmission coefficients from solid to fluid are decoupled from the reflection coefficient in the fluid and the transmission coefficients from the fluid into the solid. Remains us to solve Eq. (C.17). For that purpose we define the matrix \( E_{ik} \) as

\[
E_{ik} = (D_{ij}^{\tau,\uparrow})^{-1}D_{jk}^{\tau,\downarrow}.
\]

(C.18)

Multiplying the upper-left matrix and upper-right vector with \( (D^{\tau,\uparrow})^{-1} \) and subtracting the matrix \( E \) from the upper-left matrix results in

\[
\begin{pmatrix}
R_{ik}^{\omega} \\
D_{3j}^{\omega,\uparrow}R_{jk}^{\omega} + D_{3k}^{\omega,\downarrow} & D_{3j}^{\omega,\uparrow}T_{j}^{\uparrow}
\end{pmatrix}
= 
\begin{pmatrix}
D_{k}^{\omega}(D^{\tau,\uparrow})_{i3}^{-1}(T_{k}^{\downarrow})^{T} - E_{ik} \\
D_{k}^{\omega}(D^{\tau,\uparrow})_{i3}^{-1}(R_{k}^{\omega} + 1) \\
D_{k}^{\omega}(T_{k}^{\downarrow})T & D_{k}^{\omega}(R_{k}^{\omega} - 1)
\end{pmatrix}.
\]

(C.19)
Substituting the expressions for $R^\vee_{ik}$ and $T^\dagger_i$ in the lower-left vector and lower-right scalar of Eq. (C.19) and rearranging terms, we obtain

$$\left(\begin{array}{c} R^\vee_{ik} \\ (T^\dagger_i)^T \end{array} \right) = \left(\begin{array}{cc} Dp_{3j}D_{3j}^{\dagger}(D_{3j}^{\dagger})^{-1} - D_{3j}^{\dagger} & T^\dagger_i \\ Dp_{3j}D_{3j}^{\dagger}(D_{3j}^{\dagger})^{-1} & D_{3j}^{\dagger} \end{array} \right) \left(\begin{array}{c} R^\gamma \\ (T^\dagger_i)^T \end{array} \right).$$

For the moment, we define the scalar function $n_{\text{SCH}}$ as

$$n_{\text{SCH}} = [D^{\nu33} - Dp_{3j}D_{3j}^{\dagger}(D_{3j}^{\dagger})^{-1}].$$

Then, we can separate expressions for the transmission coefficients from the solid into the fluid $T^\dagger_k$ and the reflection coefficient in the fluid $R^\gamma$, resulting in

$$\left(\begin{array}{c} R^\vee_{ik} \\ (T^\dagger_i)^T \end{array} \right) = \left(\begin{array}{cc} Dp_{3j}D_{3j}^{\dagger}(D_{3j}^{\dagger})^{-1} - E_{ik} & Dp_{3j}D_{3j}^{\dagger}(D_{3j}^{\dagger})^{-1}(R^\gamma + 1) \\ n_{\text{SCH}}^{-1}(D_{3j}^{\dagger} - D_{3j})E_{ik} & n_{\text{SCH}}^{-1}(D^{\nu33} + Dp_{3j}D_{3j}^{\dagger}(D_{3j}^{\dagger})^{-1}) \end{array} \right).$$

When we find expressions for $T^\dagger_k$ and $R^\gamma$ we will also be able to find the reflection operator in the solid and the transmission operator from the fluid to the solid, as can be seen from Eq. (C.22). First, we need to determine the inverse of the matrix $D_{ij}^{\tau\dagger}$ which can be constructed by using Eq. (4.172d), expressed as

$$(D_{ij}^{\tau\dagger})^{-1} = N_v \begin{pmatrix} -2\tilde{\mu}\zeta_r\gamma_{P,e} & -2\tilde{\mu}\zeta_r\chi & 0 \\ 0 & 0 & -\tilde{\mu}\zeta_r\gamma_S \\ 2\tilde{\mu}\chi & -2\tilde{\mu}\zeta_r\zeta_r\gamma_S & 0 \end{pmatrix}^{-1} \begin{pmatrix} \cos(\psi + \theta) & \sin(\psi + \theta) & 0 \\ -\sin(\psi + \theta) & \cos(\psi + \theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (C.23)$$

We define the Rayleigh wave denominator $\Delta_R$, which is evoked by the determinant of the $P/SV$ part of the matrix $D^{\tau\dagger}$, as

$$\Delta_R = \chi^2 + \zeta\zeta_\alpha\gamma_{P,e}\gamma_S. \quad (C.24)$$
Then the inverse of $D_{ij}^{-\uparrow}$ is found as

$$(D_{ij}^{-\uparrow})^{-1} = N_v (2\mu)^{-1} \begin{pmatrix} -\zeta_1 \gamma_s \Delta_r^{-1} & -\zeta_2 \gamma_s \Delta_r^{-1} & \chi \Delta_r^{-1} \\ -\zeta_1 (\zeta_2 \gamma_s)^{-1} \chi \Delta_r^{-1} & -\zeta_2 (\zeta_2 \gamma_s)^{-1} \chi \Delta_r^{-1} & -\gamma_p \Delta_r^{-1} \\ 2\zeta_2 (\zeta_2 \gamma_s)^{-1} & -2\zeta_1 (\zeta_2 \gamma_s)^{-1} & 0 \end{pmatrix}.$$  \hspace{1cm} \text{(C.25)}

Second, we also need an expression for the matrix $E$ which can be constructed with the aid of the inverse of $D_{ij}^{-\uparrow}$ in Eq. (C.25) leading to

$$E_{ik} = (D_{ij}^{-\uparrow})^{-1} D_{jk}^{-\downarrow}\begin{pmatrix} (\chi^2 - \zeta_1 \zeta_2 \gamma_p \gamma_s \Delta_r^{-1} & -2\zeta_1 \zeta_2 \gamma_p \gamma_s \chi \Delta_r^{-1} & 0 \\ -2\chi \gamma_p \gamma_s \Delta_r^{-1} & -\chi^2 - \zeta_1 \zeta_2 \gamma_p \gamma_s \Delta_r^{-1} & 0 \\ 0 & 0 & -1 \end{pmatrix} N_v^{-1}$$

$$= \begin{pmatrix} (\chi^2 - \zeta_1 \zeta_2 \gamma_p \gamma_s \Delta_r^{-1} & -2(\zeta_1 \zeta_2 \gamma_p \gamma_s \frac{1}{2} \chi \Delta_r^{-1} & 0 \\ -2(\zeta_1 \zeta_2 \gamma_p \gamma_s \frac{1}{2} \chi \Delta_r^{-1} & -\chi^2 - \zeta_1 \zeta_2 \gamma_p \gamma_s \Delta_r^{-1} & 0 \\ 0 & 0 & -1 \end{pmatrix}. \hspace{1cm} \text{(C.26)}$$

Combining Eq. (C.21) with (C.25) we the normalisation $n_{\text{SCH}}$ is found as

$$n_{\text{SCH}} = (n_{fP})^{-1} \left[ \gamma_{P,f} + \frac{\rho_f \gamma_{P,e}}{\rho_e \Delta_r} \frac{1}{4c_s^4} \right] \hspace{1cm} \text{(C.27)}$$

$$= (n_{fP})^{-1} \gamma_{P,f} \Delta_r^{-1} \Delta_{\text{SCH}},$$

where in the last step the Scholte denominator has been defined as

$$\Delta_{\text{SCH}} = \left[ \Delta_r + \frac{\rho_f \gamma_{P,e}}{\rho_e \gamma_{P,f}} \frac{1}{4c_s^4} \right]. \hspace{1cm} \text{(C.28)}$$

The inverse of the Scholte denominator appears in all coefficients of the scattering operator and describes the characteristic of the solid/fluid interface. The vanishing of the Scholte denominator causes a pole in the transmission and reflection coefficients, associated with an interface wave known as the Scholte wave, see [DE HOOP AND VAN DER HULDEN, 1983, 1985]. Evaluating the expressions of Eq. (C.22) with Eqs. (C.25) and (C.26) results in

$$R_{PP}^\wedge = \Delta_{\text{SCH}}^{-1} \left[ \Delta_r - \frac{\rho_f \gamma_{P,e}}{\rho_e \gamma_{P,f}} \frac{1}{4c_s^4} \right]. \hspace{1cm} \text{(C.29)}$$
for the reflection of the compressional wave in the fluid. The transmission coefficients from the solid into the fluid are found to be

\[
T_{P/P}^{\downarrow} = \Delta_{\text{SCH}}^{-1} \frac{n_{P,f} \gamma_{P,e} \chi}{n_{P,e} \gamma_{P,f} c_S^2}, \tag{C.30a}
\]

\[
T_{P/SV}^{\downarrow} = -\Delta_{\text{SCH}}^{-1} \frac{n_{P,f} \gamma_{P,e} \zeta_o \zeta_s \gamma_S}{n_{SV} \gamma_{P,f} c_S^2}, \tag{C.30b}
\]

\[
T_{P/SH}^{\downarrow} = 0, \tag{C.30c}
\]

while the transmission from the fluid into the solid are given by

\[
T_{P/P}^{\uparrow} = \Delta_{\text{SCH}}^{-1} \frac{n_{P,e} \rho_f \chi}{n_{P,f} \rho_e c_S^2}, \tag{C.31a}
\]

\[
T_{P/SV}^{\uparrow} = -\Delta_{\text{SCH}}^{-1} \frac{n_{SV} \rho_f \gamma_{P,e}}{n_{P,f} \rho_e c_S^2}, \tag{C.31b}
\]

\[
T_{P/SH}^{\uparrow} = 0. \tag{C.31c}
\]

Finally, we obtain the reflection coefficients in the solid as

\[
R_{P/P}^{\omega} = -\Delta_{\text{SCH}}^{-1} \frac{n_{P,e} \rho_f \gamma_{P,e}}{n_{P,f} \rho_e c_S^2} \left[ \frac{1}{4 \epsilon_S^4} - \left( \chi^2 - \zeta_o \zeta_s \gamma_P \gamma_S \right) \right], \tag{C.32a}
\]

\[
R_{P/SV}^{\omega} = \Delta_{\text{SCH}}^{-1} \frac{n_{P,e} \gamma_{P,e}}{n_{SV}} 2 \zeta_o \zeta_s \gamma_S \chi, \tag{C.32b}
\]

\[
R_{SV/P}^{\omega} = \Delta_{\text{SCH}}^{-1} \frac{n_{SV}}{n_{P,e}} 2 \gamma_{P,e} \chi, \tag{C.32c}
\]

\[
R_{SV/SV}^{\omega} = -\Delta_{\text{SCH}}^{-1} \frac{n_{SV}}{n_{P,e}} \frac{\rho_f \gamma_{P,e}}{\rho_e \gamma_{P,f} 4 \epsilon_S^4} \left[ 1 + \left( \chi^2 - \zeta_o \zeta_s \gamma_P \gamma_S \right) \right], \tag{C.32d}
\]

\[
R_{SH/SH}^{\omega} = 1. \tag{C.32e}
\]

The expressions are in agreement with [de Hoop and van der Hulden, 1983, 1985] although slightly different expressions are found due to a different normalisation of the wave amplitudes.
Guided Waves

Analyzing the elastodynamic propagation of layered systems in the frequency and slowness transformed domain, we often see the occurrence of poles in the response functions. These poles are associated with independent propagating modes along the horizontal direction called guided waves. We have to take special care when we carry out the integration for the slowness to space transformation, especially when integrating the slowness contour for real frequencies. By using slightly complex frequencies, we avoid the direct numerical integration of poles.

More detailed analysis of the occurrence of these guided waves is important from a physical point of view. Since these independent modes propagate along a fracture, these events differ in their sensitivity to fracture parameters from radiating compressional and shear wave modes. Hence, additional information might be extracted from these events. Also these modes might be more appropriate in different measurement configuration like borehole measurements. We start our analysis by investigating the characteristic of the response functions, related to the general expression for the response of a thin viscoelastic layer embedded between two homogeneous elastic halfspaces. Then, we proceed with the special case of an ideal fluid layer as well as the interface modes for the linear-slip model.
D.1 The characteristic of the response

Recollecting Chapter 4, the transmission and reflection matrices for a parallel plane layer, Eqs. (4.136a) and (4.136b) are expressed as

\begin{align}
R^\parallel &= r^0 + t^0 \Theta r^i \Delta^{-1} \Theta t^i, \\
T^\parallel &= t^0 \Delta^{-1} \Theta t^i,
\end{align}

where the reverberation denominator is given by Eq. (4.139)

\[ \Delta = [I - \Theta r^i \Theta r^i]. \]

The inverse of the reverberation denominator becomes singular when the determinant of the denominator becomes zero, which can be expressed as

\[ \det(\Delta) = 0. \]

Since all the operators in Eq. (D.2) are decoupled for the P/SV and SH system, the denominator itself can be expressed as

\[ \Delta = \begin{pmatrix}
\Delta_1^{P/SV} & \Delta_2^{P/SV} & 0 \\
\Delta_1^{P/SV} & \Delta_2^{P/SV} & 0 \\
0 & 0 & \Delta^{SH}
\end{pmatrix}. \]

Hence, the condition for a pole in the response function for the P/SV and SH system, respectively, are

\[ \Delta_1^{P/SV} \Delta_2^{P/SV} - \Delta_2^{P/SV} \Delta_1^{P/SV} = 0, \]

and

\[ \Delta^{SH} = 0. \]

At this moment we do not give a guarantee for the existence of a pole for the general case of a (visco)-elastic layer. We are mainly interested in non-leaky guided modes, which contain solution on the real \( \zeta_r \)-axis for real frequencies, \( s \to j\omega \). Hence we take the branch-cut according to the same limiting procedure for positive frequencies, i.e. \( \omega > 0 \), as

\[ \text{Im}(\gamma_{P,S}) < 0. \]

We shall investigate the existence of a solution to Eqs. (D.5a) and (D.5b) for the special case of an ideal fluid layer.
D.2 Guided waves for an ideal fluid layer

In Appendix C we have derived the expression for the reflection (and transmission) coefficients for an interface separating an ideal fluid half space from an elastic half space. The expression for the internal reflection operator of an incident wave and reflected wave in the fluid is given by Eq. (C.15) and (C.29) as

\[
\mathbf{r}^i = \begin{pmatrix} R_{PP} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

(D.7)

where the reflection coefficient for the compressional to compressional wave in the fluid is given by

\[
R_{PP} = \Delta_{SCH}^{-1} \left[ \Delta_R - \frac{\rho_f \gamma_{P,e}}{\rho_e \gamma_{P,f} 4c_S^4} \right] \\
= \frac{\Delta_R - \Omega}{\Delta_R + \Omega},
\]

(D.8)

where the Rayleigh denominator \( \Delta_R \) has been defined in Appendix C as

\[
\Delta_R = \chi^2 + \zeta_\alpha \zeta_\alpha \gamma_{P,e} \gamma_S,
\]

(D.9)

with \( \chi \) defined in Chapter 4, Eq. (4.41) as

\[
\chi = \frac{\rho}{2\mu} - \zeta_\alpha \zeta_\alpha.
\]

(D.10)

and in the last step of Eq. (D.8) we have have defined \( \Omega \) as

\[
\Omega = \frac{\rho_f \gamma_{P,e}}{\rho_e \gamma_{P,f} 4c_S^4}.
\]

(D.11)

Since only compressional waves propagate in the ideal fluid layer the phase operator \( \Theta \) of Eq. (D.2) can be written as

\[
\Theta = \begin{pmatrix} \exp(-j\omega \gamma_{P,f} h) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

(D.12)
Evaluating the reverberating denominator in case of an ideal fluid layer by combining Eqs. (D.2), (D.7) and (D.12), we obtain

\[
\Delta = \begin{pmatrix}
\Delta_{11}^{p/sv} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\] (D.13)

where \( \Delta_{11}^{p/sv} \) is given by

\[
\Delta_{11}^{p/sv} = 1 - \exp(-2j\omega\gamma_{P,f}h)R_{PP}^2.
\] (D.14)

Hence the condition for guided modes can be expressed as

\[
1 - \exp(-2j\omega\gamma_{P,f}h)R_{PP}^2 = 0.
\] (D.15)

We can split this equation into a set of modes which are symmetric with respect to the pressure in the fluid and a set of modes which are anti-symmetric to the pressure in the fluid, according to

\[
\exp(-j\omega\gamma_{P,f}h)R_{PP} = e.
\] (D.16)

where \( e \) is defined as

\[
e = \begin{cases} 
+1 & \text{for pressure-symmetric modes,} \\
-1 & \text{for pressure-antisymmetric modes.}
\end{cases}
\] (D.17)

Using this definition, the dispersion equation of Eq. (D.16) can be rewritten as

\[
\exp(-j\omega\gamma_{P,f}h)(\Delta_R - \Omega) = e(\Delta_R + \Omega).
\] (D.18)

In the following two sections we split this equation into two domains, namely

\[
c_{P,e}^{-1}, c_{S,e}^{-1} < \zeta_r < c_{P,f}^{-1},
\] (D.19a)

\[
c_{P,f}^{-1} < \zeta_r.
\] (D.19b)

This splitting is needed to explicitly select the correct definition of the square root, related to the branch cut stated in Eq. (D.6). In order to restrict the argument of the square-root function to real and positive values we define \( \tilde{\gamma} \) as

\[
\tilde{\gamma} = (-\gamma^2)^{\frac{1}{2}},
\] (D.20)
D.3 Generalized Rayleigh waves

In the region, defined by Eq. (D.19a), $c_{P,e}^{-1}, c_{S,e}^{-1} < \zeta_r < c_{P,f}^{-1}$, the waves inside the fluid are still propagating modes, while in the solid embedding the modes are evanescent. The appropriate choice of the branch-cut in Eq. (D.6) leads to

\[
\gamma_{P,e} = -j\tilde{\gamma}_{P,e} \quad \tilde{\gamma}_{P,e} \in \mathbb{R}, \quad \text{(D.21a)}
\]
\[
\gamma_{S} = -j\tilde{\gamma}_{S} \quad \tilde{\gamma}_{S} \in \mathbb{R}, \quad \text{(D.21b)}
\]
\[
\gamma_{P,f} = \gamma_{P,f} \quad \gamma_{P,f} \in \mathbb{R}. \quad \text{(D.21c)}
\]

As a result of this the Rayleigh denominator is real and the factor $\Omega$ is imaginary,

\[
\Delta_R = \chi^2 - \zeta_r \zeta_r \tilde{\gamma}_{P,e} \tilde{\gamma}_{S} \quad \Delta_R \in \mathbb{R}, \quad \text{(D.22a)}
\]
\[
\Omega = -j\Omega', \quad \text{(D.22b)}
\]
\[
\Omega' = \frac{\rho_f \tilde{\gamma}_{P,e}}{\rho_e \gamma_{P,f}} \frac{1}{4c_s^2} \quad \Omega' \in \mathbb{R}, \quad \text{(D.22c)}
\]
\[
\exp(-j\omega_{P,f}h) = \cos(\omega_{P,f}h) - j\sin(\omega_{P,f}h). \quad \text{(D.22d)}
\]

Combining Eqs. (D.22a)-(D.22d) with the dispersion equation in Eq. (D.18) we obtain

\[
(\cos(\omega_{P,f}h) - j\sin(\omega_{P,f}h))(\Delta_R + j\Omega') = e(\Delta_R - j\Omega'), \quad \text{(D.23)}
\]

which can be split into its real and imaginary part, resulting in

\[
\cos(\omega_{P,f}h)\Delta_R + \sin(\omega_{P,f}h)\Omega' = +e\Delta_R, \quad \text{(D.24a)}
\]
\[
\sin(\omega_{P,f}h)\Delta_R + \cos(\omega_{P,f}h)\Omega' = -e\Omega'. \quad \text{(D.24b)}
\]

Multiplying Eq. (D.24a) with $\sin(\omega_{P,f}h)$, and subtracting Eq. (D.24b) after multiplying with $\sin(\omega_{P,f}h)$ leads to

\[
\Omega' = e(\Delta_R \sin(\omega_{P,f}h) - \Omega' \cos(\omega_{P,f}h)). \quad \text{(D.25)}
\]

Using standard goniometrical relations, we conclude that the dispersion relation in the symmetric case can be expressed as

\[
\Omega' \cos\left(\frac{1}{2}\omega_{P,f}h\right) = \Delta_R \sin\left(\frac{1}{2}\omega_{P,f}h\right), \quad \text{(D.26)}
\]

while for the anti-symmetric case we have

\[
\Omega' \sin\left(\frac{1}{2}\omega_{P,f}h\right) = \Delta_R \cos\left(\frac{1}{2}\omega_{P,f}h\right). \quad \text{(D.27)}
\]
D.4 Channel waves

In the domain given by Eq. (D.19b), $c_{pf}^{-1} < \zeta_r$ the up and downgoing waves in the fluid are also evanescent instead of propagating. Because of the pole in the fluid/solid reflection and transmission response also poles exist for a thin fluid layer in solid embedding. For low frequencies the Scholte waves at both interfaces are coupled and must be treated accordingly. The appropriate choice of the branch-cut in Eq. (D.6) leads to

$$\gamma_{P,e} = -j\tilde{\gamma}_{P,e} \quad \tilde{\gamma}_{P,e} \in \mathbb{R}, \quad (D.28a)$$
$$\gamma_{S} = -j\tilde{\gamma}_{S} \quad \tilde{\gamma}_{S} \in \mathbb{R}, \quad (D.28b)$$
$$\gamma_{P,f} = -j\tilde{\gamma}_{P,f} \quad \tilde{\gamma}_{P,f} \in \mathbb{R}. \quad (D.28c)$$

As a result of this we have

$$\Delta_R = \chi^2 - \zeta_r\zeta_r\tilde{\gamma}_{P,e}\tilde{\gamma}_{S} \quad \Delta_R \in \mathbb{R}, \quad (D.29a)$$
$$\Omega = \frac{\rho_f \tilde{\gamma}_{P,e}}{\rho_e \tilde{\gamma}_{P,f} 4c_s^4} \quad \Omega \in \mathbb{R}, \quad (D.29b)$$
$$\exp(-j\omega\gamma_{P,f}h) = \exp(-\omega\tilde{\gamma}_{P,f}h), \quad (D.29c)$$

which reduces Eq. (D.18) as

$$\exp(-\omega\tilde{\gamma}_{P,f}h)(\chi^2 - \zeta_r\zeta_r\tilde{\gamma}_{P,e}\tilde{\gamma}_{S} - \frac{\rho_f \tilde{\gamma}_{P,e}}{\rho_e \tilde{\gamma}_{P,f} 4c_s^4}) =$$
$$e\left(\chi^2 - \zeta_r\zeta_r\tilde{\gamma}_{P,e}\tilde{\gamma}_{S} + \frac{\rho_f \tilde{\gamma}_{P,e}}{\rho_e \tilde{\gamma}_{P,f} 4c_s^4}\right) \quad (D.30)$$

Numerical calculation show that only a symmetric mode exists for which $e = 1$. 
Bibliography


Abstract

Hydraulic fracturing is a process used in the oil and gas industry to create planar discontinuities in more or less competent rock materials through injection of fluids under high pressure, thus inducing failure. Hydraulic fracturing is used as a reservoir stimulation technique to facilitate the flow of hydrocarbons to the production wells, as well as the injectivity of water injection wells, thus greatly enhancing recovery. Despite fifty years of industry experience with hydraulic fracturing, there are still situations in which the fracture treatment fails. This is mainly due to the limited amount of control and understanding by reservoir engineers of the actual processes taking place in the deeply buried reservoir. Experience with failed fracture treatments shows that the likelihood of success increases when we improve our ability to predict or estimate the shape and size of the induced hydraulic fracture.

In the laboratory for rock physics, of the Faculty of Applied Earth Sciences, small-scale hydraulic fracture treatments under high confining stresses are carried out. These experiments are monitored using acoustic measurements, measuring the growth of the hydraulic fracture. The objective of the acoustic measurements is to visualize the dynamics of the fracture growth. When a wavefield impinges on a hydraulic fracture, the tip of the fracture acts as a strong secondary source and gives rise to strong diffractions. We can determine the position of the tip of the fracture by measuring the arrival times of these diffractions with multiple source- and receiver pairs.
The main objective of this thesis is to establish acoustic measurements as a practical technique for monitoring the width of hydraulic fractures. The width of the fracture (on the order of a 100 μm) is small compared to the wavelength of the incident wavefield (on the order of a cm). Hence, the resolution of a single transmission or reflection measurement is too low to be able to determine the width of a fracture from the difference in arrivaltime between the reflections of the two solid/fluid interfaces of the fracture.

During opening of the fracture we observe an attenuation and delay of the compressional transmissions compared to the transmission measured before fracturing. This apparent dispersion can be explained by constructive interference of the direct wave with the multiples created inside the fluid layer. This dispersion provides indirectly information on the width of the fracture. Using a detailed theoretical description of wave propagation through a thin fluid layer, we can predict the amount of dispersion of the compressional transmission as a function of the width of the fracture and the properties of the fracture fluid and solid embedding. If the properties of the fracture fluid and solid embedding are known, we can use the measured dispersion to determine the width of the fracture. The estimates of the fracture width resulting from acoustic measurements seem to agree quite well with a direct width measurement using a displacement transducer in the borehole.

By carrying out acoustic width determinations for different source and receiver combinations, an image of the development of the width profile of the hydraulic fracture can be created. The results obtained provide a detailed visualisation of the dynamics of hydraulic fracture growth, which helps to improve our understanding in the fundamental physical processes involved in hydraulic fracture growth.

In addition, we added shear wave source and receiver transducers to the acoustical measurements set-up. Unlike compressional waves, shear waves do not penetrate a hydraulic fracture. The shadowing of the shear waves as a result of the presence of a hydraulic fracture can be used to accurately determine the moment of fracture initiation.

An incident shear wavefield also generates diffractions at the tip of the fracture. These diffractions can be used to determine the size of a hydraulic fracture during its growth, with a technique similar to the one used for compressional diffractions. It was shown that, in some cases, shear waves monitor the fracture growth more accurately than compressional waves. In particular, during reopening of a pre-existing fracture, shear waves detect the migration
of the fluid front, whereas compressional diffractions are not observed.

Modelling the scattering of an incident wavefield for a configuration consisting of a hydraulic fracture perpendicular to a borehole shows that guided waves are generated, which propagate along the fracture. The existence of these guided waves is supported by theoretical predictions of wavefield propagation for a thin fluid layer. In addition, the modelling results show that these guided waves can be re-diffracted into body waves when the guided wave reaches the tip of the fracture. The observation of the re-diffracted event in the acoustic measurements in the laboratory supplies indirect evidence for the existence of these guided waves. Measuring these guided waves, for example in the borehole, could provide an alternative method for determining the sizes and the widths of hydraulic fractures.
Samenvatting

Hydraulisch scheuren is een techniek die onder andere in de olie industrie gebruikt wordt en tot doel heeft om door middel van vloeistof injectie scheuren te creëren in een gesteente. Deze techniek wordt gebruikt om de productiviteit van een reservoir te verbeteren, doordat zowel de stroming van olie en gassen naar productie putten als de injectie van water bij injectie putten vergemakkelijkt wordt.

Ondanks vijftig jaar ervaring in de industrie met deze reservoir-stimulatie techniek, zijn er nog steeds situaties waarbij de stimulatie van het reservoir door middel van hydraulisch scheuren mislukt. Dit is voornamelijk te wijten aan de beperkte controle en inzicht die de reservoir ingenieur heeft ten aanzien van de processen in het diep gelegen reservoir. Ervaring leert dat de operatie van hydraulisch scheuren meer kans van slagen heeft als we beter in staat zijn om de vorm en grootte van de scheur te voorspellen.

In het laboratorium van gesteentemechanica van de faculteit Technische Aardwetenschappen worden kleinschalige scheurexperimenten uitgevoerd onder hoge externe druk. Deze experimenten worden gecombineerd met akoestische metingsapparatuur die tijdens de scheurgroei herhaaldelijk het gesteente doormeten. Het doel van deze akoestische metingen is de dynamiek van de scheurgroei te visualiseren.

Als men een golfveld laat invallen op een hydraulische scheur blijkt dat diens scherpe uiteinden als sterke secundaire bronnen fungeren en dus sterke diffracties afstralen. De locatie van de uiteinden van de scheur kan bepaald worden door de meting van de aankomsttijden van deze diffracties voor
meerdere bron- en ontvangercombinaties.

Het voornaamste doel van dit proefschrift is om de wijdte van de scheur te kunnen bepalen aan de hand van de acoustische metingen. De wijdte van de scheur (in de orde van 100 μm) is klein ten opzichte van de golflengte van het invallende veld (in de orde van 1 cm). De resolutie van enkele transmissie- of reflectiemeting is te laag om de wijdte van de scheur te bepalen gebaseerd op het verschil in aankomsttijd tussen de reflecties van de twee vloeistof/vaste stof grensvlakken van de scheur.

Tijdens de opening van de scheur observeren we dat de transmissie metingen van de compressiegolven door de scheur een kleine amplituadedemping en fasevertraging oplopen ten opzichte van de transmissie voordat de scheur geopend was. Het blijkt dat deze schijnbare dispersie van de gemeten golfvorm het gevolg is van de interferentie van de meervoudig gereflecteerde golven met de directe golf. Indirect zit in deze dispersie informatie over de wijdte van de scheur.

Door middel van een nauwkeurige theoretische beschrijving van de golppropagatie door een dunne vloeistoflaag, kunnen we een voorspelling geven van de mate van dispersie als een functie van de wijdte van de scheur en de gesteente- en vloeistofparameters. Voor bekende vloeistof- en gesteente-eigenschappen kan de dispersiemeting vervolgens gebruikt worden om de wijdte van de scheur te bepalen. Het blijkt dat de wijdtebepaling door deze dispersiemeting in overeenstemming is met een directe wijdtebepaling van de scheur met een verplaatsingsopnemer in het boorgat.

Door de wijdtebepaling uit te voeren voor verschillende bron- en ontvangercombinaties kan een beeld worden gevormd van de ontwikkeling van het wijdteprofiel van de scheur tijdens de scheurgroei. De verkregen resultaten leveren een gedetailleerd beeld op van de dynamiek van scheurgroei, wat van belang is om het inzicht in de fundamentele processen van de scheurgroei te vergroten.

Bovendien werd het akustische experiment uitgebreid met zenders en ontvangers die voornamelijk voor schuifgolven gevoelig zijn. In tegenstelling tot de compressiegolven, blijkt de hydraulische scheur ondoorstroombaar voor schuifgolven. De schaduwwerking van de scheur voor schuifgolven kan wel gebruikt worden om nauwkeurig het moment van initiatie van de scheurgroei te bepalen. Ook een invallend veld van schuifgolven genereert diffracties aan het uiteinde
van de scheur. Deze diffracties kunnen, net als in het geval van compressiegolven, gebruikt worden om de grootte van de scheur te bepalen tijdens de groei. Het blijkt dat schuifgolven in sommige gevallen beter geschikt zijn om de scheurgroei te observeren dan compressiegolven. Tijdens heropening van een reeds bestaande scheur detecteren de schuifgolven de migratie van het vloeistoffront, dit in tegenstelling tot de compressiegolven.

Modellering van de verstrooiing van het golffeld aan een configuratie van een scheur loodrecht op een boorgat, laat zien dat er ook golven worden gegenereerd die langs de scheur lopen, zogenaamde geleide golven. Het mogelijke bestaan van zulke golven wordt ondersteund door de theorie van golfvoortplanting door een dunne vloeistoflaag in een gesteente. De modelleering laat tevens zien dat deze geleide golven als diffracties kunnen afstralen wanneer zij het uiteinde van de scheur bereiken. Waarneming van deze diffracties in de akoestische metingen uit het laboratorium geven indirect bewijs voor het bestaan van deze geleide golven. Meting van deze geleide golven in bijvoorbeeld het boorgat kan een alternatieve methode zijn om de grootte en de wijdte van scheuren te bepalen.
Jeroen Groenenboom was born at Moordrecht (near Rotterdam), the Netherlands, on October 17, 1967. He attended the Dalton Scholengemeenschap in the Hague, where he obtained the Atheneum diploma in 1986. In that year he started his study geophysics at Utrecht University. He received his bachelors degree (propedeuse) cum laude in 1987 and continued for his M.Sc. degree (doctorandus). In 1989 and 1990 he also studied Musicology at Utrecht University. In 1992 he finished his M.Sc. in solid earth geophysics, seismology, under the supervision of Professor Dr. R. Snieder. His M.Sc. thesis dealt with the multiple scattering of surface waves and the related loss of coherence of the wavelet.

For six months he worked as a software engineer for Resource Analysis, Delft, on an environmental project. In 1993, he started working at Delft University of Technology as a research assistant on the project 'Geometry of Hydraulic Fractures'.

After six months, when the second phase of the project was launched, he decided to do his Ph.D. research on the same research project, supervised by his promoter Professor Dr. ir. J.T. Fokkema and the project leader Dr. C.J. de Pater. During this Ph.D. research he presented his results twice a year at the steering meeting of the sponsors of the project, the DelFrac Consortium, consisting of eight oil- and service companies. In addition, he presented his work at five international conferences and two scientific meetings at the University of Hamburg in Germany and at the Shell research laboratory, Houston, USA. His research resulted in several international publications.
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