Performance Analysis of Advanced Third Generation Receivers

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Performance Analysis of
Advanced Third Generation Receivers

Proefschrift

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The cover portrays a sunset in Soulac-sur-Mer, France. The parachutist is Eva Hammann.
Aan Eva.
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Part I

Introduction and results
CDMA systems

1.1 Introduction and model description

Wireless communications is growing at a fabulous rate. Within ten years, the mobile phone has evolved from a primitive device, only used by business people and public services, to a small user-friendly device, accessible for anybody. With penetration rates up to 66.7% in Finland (1999) GsmGroup, the mobile phone has become one of the most popular consumer electronic products. Up to today, mobile phones are used for voice and short message services, better known as SMS. The low datarates (9.6 Kbps) do not allow the system to be used for multimedia applications, like mobile internet, video conferencing, etc. The so called third generation (3G) mobile communication system will change this. In Europe it will be called universal mobile telecommunications system (UMTS), elsewhere it is called international mobile telecommunications for the year 2000 (IMT-2000).

The 3G system will allow datarates from 144 Kbps outdoor up to 2Mbps for indoor applications. Already, the first large scale 3G trials have started. The first commercial system is already launched by NTT DoCoMo in Japan on October 1, 2001. In Europe, the first trial has started in December, 2001 by British Telecom and Manx Telecom on the British Isle of Man.

The 3G system is based on code division multiple access (CDMA), see for example Prasad (1996). CDMA is a technique that allows multiple users to transmit data at the same time. In order to do so, each user multiplies his data signal by an individual coding sequence. At the receiver, the signal of the desired user can be retrieved by taking the inner product of the transformed total signal and its corresponding coding sequence. When the coding sequences or different users are orthogonal, all data that does not originate from the desired user will be annihilated. In practice, however, the sequences are not orthogonal, so that users do interfere. In this thesis, we will investigate techniques to reduce the interference.
Below we explain the CDMA system in detail by showing how data is coded, transmitted, received and decoded.

1.1.1 Transmitting data

The data sequence of the $m^{\text{th}}$ user is $b_m = (\ldots, b_{m,-1}, b_{m0}, b_{m1}, \ldots) \in \{-1, +1\}^Z$ for $0 \leq m \leq k - 1$, where $k$ denotes the number of users. For a bit duration $T > 0$, we define the data signal $b_m(t)$ for $t \in \mathbb{R}$ of the $m^{\text{th}}$ user as

$$b_m(t) = b_m\lfloor (t/T) \rfloor, \quad 0 \leq m \leq k - 1,$$

where for $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the smallest integer larger than or equal to $x$. Thus, in a time period $T$ a bit is transmitted. In Section 1.3.2 it is explained why this system is called a binary phase shift keying (BPSK) system and why $+1$ or $-1$ is transmitted, instead of the usual $0$ or $1$. First, we will continue the description of the CDMA system. For each $m$, $0 \leq m \leq k - 1$, we have a coding sequence $a_m = (\ldots, a_{m,-1}, a_{m0}, a_{m1}, \ldots) \in \{-1, +1\}^Z$ and we put

$$a_m(t) = a_m\lfloor (t/T_c) \rfloor,$$

where $T_c = T/n$, for some integer $n$. The variable $n$ is a key variable, which controls the quality of the system. It is often called processing gain, see Section 1.3.3.

The transmitted coded signal of the $m^{\text{th}}$ user is

$$s_m(t) = (2P_m)^{1/2} b_m(t) a_m(t) \cos(\omega_c t), \quad 0 \leq m \leq k - 1,$$

(1.1)

where $P_m$ is called the power of the $m^{\text{th}}$ user and $\omega_c$ the carrier frequency. In fact, $P_m$ is the energy per transmitted bit. We allow the powers to be different, but we assume that the power changes slowly in time, so that within one bit-period, the power is fixed. This is known as coarse power control. The factor $\cos(\omega_c t)$ is used to transmit the signal in the desired frequency band. In Section 1.3 we will explain more about the background of CDMA.

1.1.2 Receiving data

The technique to generate codes $a_m(t)$ is known to the transmitter (e.g., the mobile phone of the transmitting person) and the base station. When a mobile phone user wants a connection, the base station keeps contact with the mobile phone and sends the necessary information to generate the right code.

Since all coded signals are transmitted using the same frequency and time domain, the total received signal is given by

$$r(t) = \sum_{j=0}^{k-1} s_j(t) + \eta n(t),$$

(1.2)

where $n(t)$ is a white noise process, i.e., it is the derivative of Brownian motion in distributional sense\(^1\), and $\eta \geq 0$. The white noise represents the noisy channels of the users and all

---

\(^1\)This means that the process $(B_t)_{t \geq 0}$ with $B(t) = \int_0^t n(s)ds$ is a Brownian motion.
interference of other sources that is not yet taken into account. Therefore, \( \eta \) may depend on \( k \). The white noise is called *additive white Gaussian noise* (AWGN).

The signals do not need to be synchronized, i.e., it is not necessary that all users transmit at the same time grid. However, for technical reasons we do assume so. We will comment on asynchronous systems in Section 3.3. In practice \( \omega_c T_c \) is large. For simplicity, we pick \( \omega_c T_c = \pi f_c \), where \( f_c \in \mathbb{N} \). To retrieve the data bit \( b_{m1} \), the signal \( r(t) \) is multiplied by \( a_m(t) \cos \omega_c t \) and then averaged over \([0, T]\). This results in (for a proof, we refer to Appendix A)

\[
\frac{1}{T} \int_0^T r(t) a_m(t) \cos(\omega_c t) \, dt = \left( \frac{P_m}{2} \right)^{1/2} b_{m1} + \sum_{j=0}^{k-1} \left( \frac{P_j}{2} \right)^{1/2} b_{j1} \frac{1}{n} \sum_{i=1}^n a_{ji} a_{mi} + \frac{1}{n} \sum_{i=1}^n a_{mi} \frac{\eta}{\sqrt{2T_c}} N_i,
\]

where \((N_i)_{i=1}^n\) are independent and identically distributed (i.i.d.), with \( N_i \sim \mathcal{N}(0, 1) \).

The decoded signal consists of the desired bit \( b_{m1} \), multiplied by the amplitude \((P_m/2)^{1/2}\), interference due to the other users and AWGN. In the ideal situation the vectors \((a_{m1}, \ldots, a_{mn})\) and \((a_{j1}, \ldots, a_{jn})\), \( j \neq m \), would be orthogonal, so that \( \sum_{i=1}^n a_{ji} a_{mi} = 0 \). However, it is more efficient to allow non-orthogonal codes. In practice, the \( a \)-sequences are generated by a random number generator, so that the sequences resemble random codes. To model these pseudo-random sequences \( a \), let \( A_{mi} \), \( 0 \leq m \leq k - 1 \), \( i = 1, 2, \ldots, n \), be an array of i.i.d. random variables with distribution

\[
\mathbb{P}(A_{mi} = +1) = \mathbb{P}(A_{mi} = -1) = 1/2. \tag{1.3}
\]

Assuming the coding sequences to be random, we model the signal as

\[
\left( \frac{P_m}{2} \right)^{1/2} b_{m1} + \sum_{j=0}^{k-1} \left( \frac{P_j}{2} \right)^{1/2} b_{j1} \frac{1}{n} \sum_{i=1}^n A_{ji} A_{mi} + \frac{1}{n} \sum_{i=1}^n A_{mi} \frac{\eta}{\sqrt{2T_c}} N_i. \tag{1.4}
\]

Since the received decoded signal is in general not an element of \((-1, +1)\), we estimate \( b_{m1} \) by

\[
\hat{b}_{m1}^{[1]} = \text{sgnr}_m \left\{ \left( \frac{P_m}{2} \right)^{1/2} b_{m1} + \sum_{j=0}^{k-1} \left( \frac{P_j}{2} \right)^{1/2} b_{j1} \frac{1}{n} \sum_{i=1}^n A_{ji} A_{mi} + \frac{1}{n} \sum_{i=1}^n A_{mi} \frac{\eta}{\sqrt{2T_c}} N_i \right\},
\]

where, for \( x \in \mathbb{R} \), the randomized sign function is defined as

\[
\text{sgnr}_m(x) = \begin{cases} +1, & x > 0, \\ U_m, & x = 0, \\ -1, & x < 0, \end{cases}
\]

with \( \mathbb{P}(U_m = -1) = \mathbb{P}(U_m = +1) = 1/2 \). \tag{1.5}

The random variable \( U_m \) is independent of all other random variables in the system. Thus, for every user, a coin is flipped once. This determines for that user which sign-function is used. We will comment on different choices of the sign of zero in Section 1.2.2.
The superscript \(^{(1)}\) in \(\hat{h}_{m_1}^{(1)}\) indicates that this is a tentative decision. We will see in the next section how we can improve upon these tentative estimates.

The system described so far is called the *matched filter* (MF) system. This is because the total received signal is filtered using the code that matches the user of interest.

We are interested in the bit-error probability (BEP), i.e., in \(P(\hat{h}_{m_1}^{(1)} \neq b_{m_1})\), as a function of the processing gain \(n\), the number of users \(k\), the powers \(P_j\) and the intensity \(\eta\) of the AWGN.

### 1.1.3 Improving estimates

The MF receiver is designed for unspecified random noise, and breaks down when the noise level is too high. However, interference experienced in a CDMA system is different from completely random noise, and this fact can be exploited to improve performance. Receivers that exploit information of the system (mainly the cross correlations of the codes) are often denoted by *advanced receivers* or *multiuser detection* receivers. For an overview, see MOSHAVI, BUEHNER, NICOLOSO and GOLLAMUDI (1999). We will treat some of these receivers in Chapter 3. The best known technique is a *maximum likelihood estimator*, introduced in VERDÚ (1986), which obtains jointly optimal decisions for all users using maximum likelihood detection. Unfortunately, this technique is of such high complexity that it cannot be performed real-time. A more straightforward technique is called *interference cancellation* (IC). The idea is that we try to cancel the interference due to the other users (i.e., the users with subscript \(j \neq m\)). According to PRASAD, MOHR AND KONHÄUSER (2000), IC is seen as “the most promising and the most practical technique for base-station receivers”, the so-called uplink or forward link receivers. For mobile interfaces (downlink\(^2\) or reverse link), such as mobile phones, IC is not practical because it demands that each mobile interface has access to all codes. This is clearly not desirable for security reasons. However, orthogonal codes are used downlink, so that reducing noise is of less concern. Also, blind estimation schemes exist that can improve performance, see e.g. TONG (1995). These blind schemes do not require a priori information on the structure of the interfering signals.

Interference cancellation comes in many flavours. We will restrict ourselves mainly to two versions; Hard-decision parallel interference cancellation (HD-PIC), (VARANASI AND AAZHANG (1990), JUNTTI (1998), LATVA-AHO (1999) and the references therein) and soft-decision parallel interference cancellation (SD-PIC), (BUEHNER AND WOERNER (1996), BUEHNER, KAUL, STRIGLIS AND WOERNER (1996), ELDERS-BOLL, SCHOTTEN AND BUSBOOM (1998), GUO, RASMUSSEN, SUN AND LIM (2000)). HD-PIC is seen as “the most promising IC scheme for the uplink” (cf. PRASAD (1996)). In SD-PIC, the estimation of e.g. the powers is less involved. In HD-PIC it is assumed that all powers are known, or at least can be estimated superiorly compared to the bit estimates, while in SD-PIC no such assumption is made. In Chapter 3, we will comment on different IC schemes.

---

\(^{2}\)The terms uplink and downlink are a carry-over from satellite communications.
The two procedures are described below. In case \( (P_j)_{j=0}^{k-1} \) are unknown, we estimate \( P_j \) by

\[
\hat{P}_j^{(1)} = \frac{1}{2} \left( \frac{1}{T} \int_0^T r(t) a_j(t) \cos(\omega_c t) \, dt \right)^2, \quad 0 \leq j \leq k - 1,
\]

i.e., we estimate \( P_j \) by the energy of the received signal, decoded with respect to user \( j \). This is SD-PIC. When we assume that the powers \( (P_j)_{j=0}^{k-1} \) are known, we set \( \hat{P}_j^{(1)} = P_j \). This is HD-PIC.

We estimate the data signal \( s_j(t) \) for \( t \in [0, T] \) by (recall (1.1))

\[
s_j^{(1)}(t) = (2\hat{p}_j^{(1)})^{1/2} \tilde{b}_j^{(1)}(t) a_j(t) \cos(\omega_c t).
\]

We then estimate the total interference for the \( m^{th} \) user in \( r(t) \) due to the other users by (recall (1.2))

\[
\hat{r}_m^{(1)}(t) = \sum_{j \neq m} s_j^{(1)}(t)
\]

(1.6)

Ideally, \( \hat{s}_j^{(1)}(t) = s_j(t) \) for all \( j \), so that \( r(t) - \hat{r}_m^{(1)}(t) = s_m(t) \). The interfering users have been cancelled perfectly in this case. In the general case, we expect that the interference is at least partly cancelled, so that we obtain better estimates for \( b_{m1} \).

Treating \( r(t) - \hat{r}_m^{(1)}(t) \) as an estimate for \( s_m(t) \) and using it as the received signal instead of \( r(t) \) results in

\[
\hat{b}_{m1}^{(2)} = \text{sgnr}_m \left\{ \frac{1}{T} \int_0^T (r(t) - \hat{r}_m^{(1)}(t)) a_m(t) \cos(\omega_c t) dt \right\}
\]

\[
= \text{sgnr}_m \left\{ \left( \frac{P_m}{2} \right)^{1/2} b_{m1} + \sum_{j=0}^{k-1} \left( \frac{1}{n} \sum_{i=1}^n A_{ji} A_{mi} \right) \left( \left( \frac{P_j}{2} \right)^{1/2} b_{j1} - \left( \frac{\hat{p}_j^{(1)}}{2} \right)^{1/2} \tilde{b}_{j1}^{(1)} \right) \right. 
\]

\[
\left. + \frac{1}{n} \sum_{i=1}^n A_{mi} \frac{\eta}{\sqrt{2T_c}} N_i \right\}. 
\]

(1.7)

In Figure 1.1, the above procedure is given schematically. The quantity of interest is the BEP after one stage of IC, which is \( \mathbb{P}(\hat{b}_{m1}^{(2)} \neq b_{m1}) \). In general, this probability is indeed smaller than \( \mathbb{P}(\hat{b}_{m1}^{(2)} \neq b_{m1}) \), the probability of a bit error without IC.

The above motivates an iteration of the procedure. We obtain, similarly to (1.7), the estimates \( \hat{b}_{m1}^{(s)} \), \( s \geq 2 \),

\[
\hat{b}_{m1}^{(s)} = \text{sgnr}_m \left\{ \left( \frac{P_m}{2} \right)^{1/2} b_{m1} + \sum_{j=0}^{k-1} \left( \frac{1}{n} \sum_{i=1}^n A_{ji} A_{mi} \right) \left( \left( \frac{P_j}{2} \right)^{1/2} b_{j1} - \left( \frac{\hat{p}_j^{(s-1)}}{2} \right)^{1/2} \tilde{b}_{j1}^{(s-1)} \right) \right. 
\]

\[
\left. + \frac{1}{n} \sum_{i=1}^n A_{mi} \frac{\eta}{\sqrt{2T_c}} N_i \right\}. 
\]
Chapter 1. CDMA systems

Figure 1.1: System model for 1-stage HD/SD-PIC

This is called multistage PIC, firstly proposed in Varanasi and Aazhang (1990). When we have applied $s$ stages of IC we speak of $s$-stage PIC and the corresponding bit error probability is $P(b_m^{(s+1)} \neq b_m)$. Despite the slight confusion that $s$-stage corresponds to $b_m^{(s+1)}$ instead of $b_m^{(s)}$, we have chosen to keep the notation of the existing literature (see e.g., Buehrer and Woerner (1996), Buehrer, Kaul, Striglis and Woerner (1996)). When we speak of stage $s$, we mean that $s-1$ stages of PIC are performed, e.g., at stage 1, no PIC is performed. We stress that at every stage, the same $\text{sgnr}_m$ is used, i.e., $U_m$ is only determined once.

1.2 Reformulation of the problem

It is important to observe that as $n$ tends to $\infty$, the interference due to other users will diminish. However, since $T$ is fixed and $\text{var}(\frac{1}{n}\sum A_{mi} \frac{n}{\sqrt{2}T} N_i) = \eta^2/(2nT) = \eta^2/(2T)$, the AWGN does not vanish. More precisely,

$$\text{var}\left(\left(\frac{P_m}{2}\right)^{1/2} + \sum_{j=0}^{k-1} \left(\frac{P_j}{2}\right)^{1/2} b_{j1} \frac{1}{n} \sum_{i=1}^{n} A_{ji} A_{mi}\right) = O(1/n), \quad (1.8)$$
while
\[
\text{var} \left( \frac{1}{n} \sum_{i=1}^{n} A_{mi} \frac{\eta}{\sqrt{2T_c}} N_i \right) = \mathcal{O}(1),
\]
so that for \( n \) large, the AWGN becomes dominant.

In practice, however, the powers are always adjusted in such a way that the AWGN is not dominant. In our model, we replace \( P_m \) by \( nP_m \), so that the variance in (1.8) is also \( \mathcal{O}(1) \), similarly to the approach in SADOWSKY AND BAHR (1991). This is equivalent to taking \( \sigma^2 = \eta^2/T_c \) fixed, so that the signal in (1.4) becomes
\[
n^{1/2} \left( \frac{P_m}{2} \right)^{1/2} b_{m1} + \sum_{j=x}^{k-1} \left( \frac{P_j}{2} \right)^{1/2} b_{j1} \frac{1}{n} \sum_{i=1}^{n} A_{ji} A_{mi} + \frac{\sigma}{2^{1/2} n} \sum_{i=1}^{n} A_{mi} N_i.
\]

We emphasize that the power adjustment condition above is realistic, especially in a model where IC is applied.

Since we are only interested in the sign of the above, we further divide by \( (n/2)^{1/2} \). Using \( b_{m1}^2 = 1 \) results in
\[
P_m^{1/2} b_{m1} + \sum_{j=x}^{k-1} P_j^{1/2} b_{j1} \frac{1}{n} \sum_{i=1}^{n} A_{ji} A_{mi} + \frac{1}{n} \sum_{i=1}^{n} A_{mi} N_i
\]
\[
= b_{m1} \left( P_m^{1/2} + \sum_{j=x}^{k-1} P_j^{1/2} \frac{1}{n} \sum_{i=1}^{n} A_{ji} A_{mi} + \frac{1}{n} \sum_{i=1}^{n} b_{m1} A_{mi} N_i \right).
\]

Since \( b_{j1} A_{ji} = A_{ji} \) and \( N_i = A_{0i} N_i \), we can therefore write the probability of a bit error more conveniently as
\[
\mathbb{P}(\hat{b}_{m1}^{(i)} \neq b_{m1}) = \mathbb{P} \left( \frac{\hat{b}_{m1}^{(i)}}{b_{m1}} \neq 1 \right) = \mathbb{P}(\text{sgn}_{m}(Z_{m}^{(i)}) < 0) = \mathbb{P}(Z_{m}^{(i)} < 0) + \frac{1}{2} \mathbb{P}(Z_{m}^{(i)} = 0),
\]
where \( Z_{m}^{(i)} \), for \( 0 \leq m \leq k - 1 \), is defined as
\[
Z_{m}^{(i)} = P_m^{1/2} + \sum_{j=x}^{k-1} P_j^{1/2} \frac{1}{n} \sum_{i=1}^{n} A_{ji} A_{mi} + \frac{1}{n} \sum_{i=1}^{n} A_{mi} A_{0i} N_i.
\]

We note that when \( \sigma^2 = 0 \), the event \( \{ Z_{m}^{(i)} = 0 \} \) should be taken into account, while for \( \sigma^2 > 0 \), this events has probability zero, so that \( \mathbb{P}(\hat{b}_{m1}^{(i)} \neq b_{m1}) = \mathbb{P}(Z_{m}^{(i)} \leq 0) \).

Without loss of generality, we focus on user 0. Therefore, we prefer to introduce the random variables
\[
X_{mi} = A_{0i} A_{mi}, \quad 1 \leq i \leq n, \quad m = 0, \ldots, k - 1.
\]
It is straightforward to prove that the matrix \((X_{mi})_{m=1,...,k-1,i=1,...,n}\) has independent and identically distributed elements and \(X_{0i} = A_{0i}^2 = 1\) for all \(i\), so that we obtain

\[
Z_m^{(1)} = P_m^{1/2} + \sum_{j=0}^{k-1} P_j^{1/2} \frac{1}{n} \sum_{i=1}^{n} X_{ji}X_{mi} + \frac{1}{n} \sum_{i=1}^{n} X_{mi}N_i,
\]

(1.9)

where (with a slight abuse of notation) \((N_i)_{i=1}^{n}\) are i.i.d. with \(N_i \sim \mathcal{N}(0, \sigma^2)\).

For \(s \geq 2\), we can derive a similar result. Together with

\[
(\tilde{P}_j^{(s)})^{1/2} \hat{b}_{m1}^{(s)} = \left\{ \begin{array}{ll}
P_j^{1/2} \text{sgnr}_m(Z_m^{(s)}), & \text{for HD-PIC}, \\
((Z_m^{(s)})^{2})^{1/2} \text{sgnr}_m(Z_m^{(s)}) = Z_m^{(s)}, & \text{for SD-PIC},
\end{array} \right.
\]

this results in

\[
Z_m^{(s)} = P_m^{1/2} + \sum_{j=0}^{k-1} \frac{1}{n} \sum_{i=1}^{n} X_{ji}X_{mi} \left(P_j^{1/2} - h_j(Z_j^{(s-1)})\right) + \frac{1}{n} \sum_{i=1}^{n} X_{mi}N_i,
\]

where \(h_j(x) = P_j^{1/2} \text{sgnr}_j(x)\) for HD-PIC and \(h_j(x) = x\) for SD-PIC. In particular,

\[
Z_0^{(1)} = P_0^{1/2} + \sum_{j=1}^{k-1} P_j^{1/2} \frac{1}{n} \sum_{i=1}^{n} X_{ji} + \frac{1}{n} \sum_{i=1}^{n} N_i, \tag{1.10}
\]

\[
Z_0^{(2)} = P_0^{1/2} + \sum_{j=1}^{k-1} \frac{1}{n} \sum_{i=1}^{n} X_{ji} \left(P_j^{1/2} - h_j(Z_j^{(1)})\right) + \frac{1}{n} \sum_{i=1}^{n} N_i. \tag{1.11}
\]

In order to avoid confusion, throughout this thesis we will write \(Z_0^{(s,H)}\) when we are dealing with \(s\)-stage HD-PIC. For the SD-PIC model, we will write \(Z_0^{(s,S)}\). Note the difference between the number of stages \(s\) and the estimating technique \(S\). For \(s = 1\), for which

\[
Z_0^{(1)} = Z_0^{(1,H)} = Z_0^{(1,S)},
\]

we will use \(Z_0^{(1)}, Z_0^{(1,H)}\) and \(Z_0^{(1,S)}\), depending on the system under consideration.

### 1.2.1 Matrix notation

Sometimes it is convenient to write \(Z_m^{(s)}\) in matrix notation. To do so, recall that the matrix \(X\) has elements \((X)_{ji} = X_{ji}, j = 0, \ldots, k - 1, i = 1, \ldots, n\). We stress that the rows are numbered from 0 to \(k - 1\) and the columns are numbered from 1 to \(n\). Using the definition of \(X\),

\[
\begin{bmatrix}
Z_0^{(1)} \\
Z_0^{(1)} \\
\vdots \\
Z_{k-1}^{(1)}
\end{bmatrix}
= \frac{1}{n} XX^T
\begin{bmatrix}
P_0^{1/2} \\
P_1^{1/2} \\
\vdots \\
P_{k-1}^{1/2}
\end{bmatrix}
+ \frac{1}{n} X
\begin{bmatrix}
N_1 \\
N_2 \\
\vdots \\
N_n
\end{bmatrix},
\]
and

\[
\begin{bmatrix}
Z_0^{(2)} \\
Z_1^{(2)} \\
\vdots \\
Z_{k-1}^{(2)}
\end{bmatrix} = \begin{bmatrix}
h_0(Z_0^{(1)}) \\
h_1(Z_1^{(1)}) \\
\vdots \\
h_{k-1}(Z_{k-1}^{(1)})
\end{bmatrix} + \frac{1}{n}XX^T \begin{bmatrix}
P_0^{1/2} - h_0(Z_0^{(1)}) \\
P_1^{1/2} - h_1(Z_1^{(1)}) \\
\vdots \\
P_{k-1}^{1/2} - h_{k-1}(Z_{k-1}^{(1)})
\end{bmatrix} + \frac{1}{n}X \begin{bmatrix}
N_1 \\
N_2 \\
\vdots \\
N_n
\end{bmatrix}.
\]

The matrix notation will be useful when we investigate infinite stage SD-PIC in Chapter 7.

### 1.2.2 Remarks on the sign-function

Throughout this thesis, we use the definition of the sign-function in (1.5). Several other definitions are possible. For example, we can define the sign-function to be 0 in 0 and unaltered elsewhere. Another option is to toss a coin every time a zero occurs. We have chosen for definition (1.5) for two reasons. First of all, in this case the sign-function assumes only two values, which is technically convenient. The second reason is that at stage 1 all sign-functions are determined and from that point on are deterministic. This results in consistent estimates of bits in different stages. In the case that AWGN is present, all sign-definitions give the same BEP, since \( Z_m^{(s)} \) assumes 0 with probability 0. Also in the case of unequal powers, the sign functions are the same almost everywhere, since \( Q \) is dense in \( \mathbb{R} \). Therefore, the precise details of the sign-function in 0 are somewhat academic.

### 1.3 Some background on CDMA and earlier generations mobile systems

#### 1.3.1 1G, 2G and 3G mobile systems

Already in 1946, the first mobile phone was introduced by AT&T. Even though the quality of the system was dramatic, it was the first step towards a commercial system. The first group of successful mobile systems is called the first generation (1G) systems. All 1G systems are analog and use a technique called *frequency division multiple access*. In this system, every user had its own frequency band like in common radio. Examples are the *advanced mobile phone system* (AMPS) in the USA, the *total access communication system* (TACS) in the UK and China, and the Nordic mobile telephone (NMT) system in Europe. In the Netherlands, the NMT technique is used in the *autotelefonie* systems ATF-1, ATF-2 and ATF-3.

The second generation (2G) mobile systems differs from the 1G system in two ways. The 2G is digital, which has many of advantages over analog transmission. Furthermore, instead of assigning different frequency bands to different users, each user has its own timeslot, and this allows multiple users on the same frequency without interference. This technique is called
time division multiple access (TDMA). The 2G system is better known as *global system for mobile communications* (GSM), even though this abbreviation is in fact French for *groupe speciale mobile*. In the USA, digital AMPS (D-AMPS) is similar to the GSM system.

In a CDMA system every user makes fully use of time and frequency. Ideally, orthogonal coding sequences are used to avoid interference from other users. In practice, avoiding interference at any cost is expensive and technically difficult. Therefore, using almost-orthogonal codes (like pseudo-random codes) in CDMA is very attractive. The origins of CDMA lie in the military field and navigation systems. It is developed to counteract intentional jamming, but it proved to be suitable as well for communication systems to avoid interference due to other users. Already in 1949, Shannon and Pierce introduced the basic ideas involved in CDMA, see Shannon (1984). In July 1993, the first narrowband CDMA IS-95 standard of Qualcomm was established. This standard is used in parts of the United States and Japan.

For a schematic view of FDMA, TDMA and CDMA, see Figure 1.2, (a)-(c).

![Figure 1.2: Multiple access techniques](image)

(a) FDMA  
(b) TDMA  
(c) CDMA

### 1.3.2  Binary phase shift keying transmission

The system described in the previous sections is called a *binary phase shift keying* (BPSK) CDMA system. We explain this below, while we simultaneously explain why bits ±1 are used, instead of the well-known 0,1-bits. Observe that multiplying (⋅) on the field \{-1, +1\} is equivalent with adding modulo 2 (⊕) on \{0,1\}. Indeed,

\[
\begin{align*}
(-1) \cdot (-1) &= (+1) \cdot (+1) = (+1) \iff 1 \oplus 1 = 0 \oplus 0 = 0 \\
(-1) \cdot (+1) &= (+1) \cdot (-1) = (-1) \iff 1 \oplus 0 = 0 \oplus 1 = 1.
\end{align*}
\]

Therefore, \(s_m(t)\) can be written as

\[
s_m(t) = \sqrt{2P_m} \cos(\omega_c t + (\tilde{b}_m(t) \oplus \tilde{a}_m(t))\pi), \quad 0 \leq m \leq k - 1,
\]

where \(\tilde{b}_m(t) = (1 - b_m(t))/2\) and \(\tilde{a}_m(t) = (1 - a_m(t))/2\), the \{0,1\}-versions of \(b_m(t)\) and \(a_m(t)\). Thus, the received signal is in fact a cosine with certain amplitude, distorted by a phase shift of 0 or \(\pi\) that might change every \(T/n\) time units, where the operator \(\oplus\) is used. This explains the name binary phase shift keying. Binary stands for two possible bits (±1), phase shift keying stands for the technique used to send the information ±1. In *quaternary*...
1.3 Some background on CDMA and earlier generations mobile systems

(Q)PSK. It is possible to send 4 bits. This is achieved by dividing $[0, 2\pi]$ in four parts and thus allowing the phase to be shifted by $0, \pi/2, \pi$ or $3\pi/2$. When applying BPSK, QPSK or in general M-PSK, the system is often denoted by direct-sequence (DS-)CDMA. In the current 3GPP\(^3\) proposals, both BPSK and QPSK are used for the uplink. The performance of the BPSK and QPSK is highly correlated, but the BPSK system is easier to investigate. Therefore we will only focus on BPSK in this thesis.

1.3.3 The role of the processing gain

The variable $n$ is often called processing gain. In practice, the value of $n$ ranges from $2 - 512$. The term processing gain can be understood by explaining the military origins of CDMA, see Pickholtz, Schilling and Milstein (1982). We investigate a military system where a user transmits a binary data-signal $b(t)$ and has access to energy $E_d$. However, an enemy with access to energy $E_r$ tries to jam the signal. The signal-to-noise ratio $E_d/E_r$ is an indication of the performance of the user. Suppose the user switches to CDMA, more precisely, instead of transmitting $b(t)$ directly, he now transmits $b(t)a(t)$, where $a(t)$ is a pseudo-random sequence. Both the power $P_b$, which is closely related to $E_d$ and $\cos(\omega_c t)$ are not essential and are therefore omitted. The impact of multiplying with $a(t)$ is best explained in terms of spectral analysis.

The power spectrum or energy density spectrum of a function $f$ is given by $|\hat{f}|^2$, where $\hat{f}$ is the Fourier transform of $f$, see Proakis and Manolakis (1996), Sect.4.1.4. Roughly speaking, the power spectrum describes how much energy is located at a certain frequency. A straightforward calculation gives that for $f(t) = 1_{[0,T]}(t)$, the spectrum is given by $(\sin \omega T)^2/\omega^2$. A comparison of the spectra of $1_{[0,T]}(t)$ and a specific $a(t)1_{[0,T]}(t)$ for $n = 16$ is given in Figure 1.3. Multiplying $b(t)$ with $a(t)$ results in spreading the spectrum.

![Graph](image)

**Figure 1.3:** Spectra of $1_{[0,T]}(t)$ (peaked curve) and $a(t)1_{[0,T]}(t)$ (spread curve) for $n = 16$

This has two consequences. On the one hand, the signal is more difficult to detect, since the peaks of the spectrum are much lower. In fact, if the channel contains noise, it is even possible

\(^3\)Third generation partnership project. Worldwide standard for evolution of 2G systems to a 3G mobile system, see http://www.3gpp.org
to sink under the noise level, and thus making the signal virtually invisible to enemies. On
the other hand, the main contribution of the spectrum of $b(t)$ comes from $[-2\pi/T, 2\pi/T]$, whereas after multiplying with $a(t)$ the main contribution comes from $[-2n\pi/T, 2n\pi/T]$. Thus, the best an enemy can do is to jam the whole frequency band $[-2n\pi/T, 2n\pi/T]$, while it first only had to jam $[-2\pi/T, 2\pi/T]$. This results effectively in a signal-to-noise ratio $E_d/(E_r/n) = nE_d/E_r$. Thus, the data signal has a power advantage of $n$. For the reasons above, CDMA used to be denoted by *spread spectrum multiple access* (SSMA) and the code signals $a_m(t)$ are often denoted by *spreading signals*. One drawback is the lack of frequency bandwidth. When a signal covers a frequency band of $\omega$, the coded (and therefore spreaded) signal covers a frequency band of $n\omega$. A larger processing gain thus requires more bandwidth. However, bandwidth is scarce. To avoid that different wireless applications interfere, the authorities regulate the assignment of bandwidth. For the 3G systems, bandwidths of 5MHz each have been sold to interested companies. For more information on the theory of spreading signals, see Simon, Omura, Scholtz andlevitt (1994).

The factor $\cos(\omega_c t)$ should be understood as follows. It is not desirable to transmit signals in a frequency band around zero. Multiplying by $\cos(\omega_c t)$ results in shifting the frequency to $\omega_c$ and $-\omega_c$. More mathematically, multiplying by $\cos(\omega_c t)$ in the time domain is equivalent with convoluting in the frequency domain with $\frac{1}{2}\delta(t-\omega_c) + \frac{1}{2}\delta(t+\omega_c)$. For symmetry reasons only the positive frequencies are denoted, so that we say that for the transmission of $s_m(t)$ a carrier frequency band $(\omega_c - \Delta, \omega_c + \Delta)$ is reserved, where the width $\Delta$ depends mainly on $n$. This holds for every $m$, so that indeed all users transmit at the same frequency band. In Figure 1.4, $b(t)$, $a(t)$, $\cos(\omega_c t)$ and $b(t)a(t)\cos(\omega_c t)$ are shown. The phase shifting is clearly visible.

![Figure 1.4](image.png)

Figure 1.4: Functions $b(t)$, $a(t)$, $\cos(\omega_c t)$ and $b(t)a(t)\cos(\omega_c t)$ for $n = 16$ and $\omega_c = 32\pi$

1.3.4 Coding sequences
1.4 Aim and outline of this thesis

The coding sequences are a critical component of CDMA communications. They are used to spread the message signal. These codes should be easily realizable at the transmitter and the receiver, cf. Groe and Larson (2000). Pseudo-random sequences have the following classical properties (c.f. Groe and Larson (2000)) (a chip is an \( A_{ji} \)): 1) There are near-equal events of +1 and −1 chips. 2) Run lengths of \( r \) chips with the same sign occur approximately \( 2^{-r} \) times and 3) Shifting a nonzero number of chips produces a new sequence that has an equal number of agreements and disagreements with the original sequence. Therefore, assuming perfectly random codes is a natural assumption. A typical example of pseudo-random codes are M-sequences, which are generated by an M-bit linear-feedback shift register, see Simon, Omura, Scholtz and Levitt (1994), Chpt. 5. In practice, Gold codes (Gold (1967)) and Kasami codes (Kasami (1966)) are popular to distinguish users, since they have better cross-correlation properties than M-sequences in asynchronous scenarios. Random codes have slightly worse performance than Gold or Kasami codes, but are relevant in the case of long spreading sequences.

In the downlink, it is worthwhile to use orthogonal codes, such as the Hadamard code (c.f. Rappaport (1996)), because synchronization is much easier downlink.

1.4 Aim and outline of this thesis

The basis of the existing literature on PIC receivers is either some Gaussian approximation, Monte Carlo simulations or extensive computations. Gaussian approximations provide a way to measure the performance, namely the signal-to-noise ratio (SNR), which is \( \mathbb{E} \left[ Z_n^2 \right] / \sqrt{\text{var}(Z_n^2)} \). However, it is not clear at all that PIC receivers are characterized well by the SNR. The reason to use the SNR is that it is quite common in the electrical engineering community to substitute this ratio in the so called Q-function\(^4\), which in turn is motivated by the central limit theorem (CLT).

One of the goals of this thesis is to introduce a new measure, called the exponential rate, which is based on rigorous analysis. It enables us to characterize systems by a single number. We will calculate the exponential rate for various system models. Rather than doing so for the realistic models directly, we first investigate a simple model with at most one stage of PIC, in which all powers are equal and no AWGN is present. Thus, only two key parameters \( (n \text{ and } k) \) are taken into account. This simple model characterizes the effect of PIC well, and allows for a detailed analysis. Later, we will show that the techniques developed to investigate these models can be extended to more general models, such as multistage HD-PIC or models which allow unequal powers and AWGN.

For infinite stages of IC, not much is known. For the SD-PIC receiver, results have been obtained in Elders-Boll, Schotten and Busboom (1998) and Guo, Rasmussen, Sun and Lim (2000). It is shown that the multistage SD-PIC receiver approximates the inverse of the

\[^4\]Q(x) is the probability that a standard Gaussian random variable exceeds the value \( x \), i.e., \( Q(x) = (2\pi)^{-1/2} \int_x^\infty \exp(-u^2/2)du. \)
cross correlation matrix using Jacobi iteration (see Chapter 7). To our best knowledge, no analytical results are known for the HD-PIC receiver. Only simulation results are provided in Buehrer (1999). One goal of the thesis is to gain insight in the behaviour when one applies many stages of IC. This is of practical interest, since a theoretical limit provides rules for thresholding the number of stages.

Finally, we will obtain insight in the BEP. This allows us to check the validity of the rate as a measure of performance.

The remainder of this thesis is organized as follows. Chapter 2 starts with a short introduction on large deviation theory. Using this theory, we are able to introduce the new measure of performance, which forms the basis of the thesis. We next state all important results. We have chosen to give intuitive statements of the results in this chapter only. For the full proofs and the detailed statements, we refer to the subsequent chapters. Chapter 3 deals with further research and extensions. We describe the open problems, raised in this thesis. Furthermore, we explain different models, varying from different power assumptions to completely different estimation/cancellation techniques.

In the second part of this thesis, we will give a detailed analysis of various models. In Section 4, we describe the exponential rate for a model in which all powers are equal and no AWGN is present. We will do so for the MF model, the one-stage HD-PIC and the one-stage SD-PIC model. Chapter 5 deals with the asymptotic behaviour of the rate for \( k \to \infty \). We are able to give results for the MF model and the one-stage HD-PIC. We further give asymptotic expansions of the rate for the multistage HD-PIC model. For the one-stage SD-PIC model, we investigate the behaviour for large \( k \). In Chapter 6, we deal with the more realistic model. We give techniques to calculate the exponential rate. Furthermore, we give results for the asymptotic rate for \( k \to \infty \) for both the MF and the HD-PIC model. In Chapter 7, we investigate the case where we let the number of cancellation steps increase to infinity. Chapter 8 makes an excursion from the exponential rate to the BEP. We investigate the so called second order asymptotics, which gives excellent insight in the BEP. We will do this with an analytical based approach. Finally, in Chapter 9 we investigate the second order asymptotics, using simulations.

The picture on the right indicates the relation between Chapters 4-9. Chapter 4 is the basis; in this chapter the exponential rate is explained. Furthermore, for the MF, the HD-PIC and the SD-PIC model, results are given for the model in which all powers are equal and no AWGN is present. Chapter 5 is related to Chapter 4, because various results are extended to the asymptotic case \( k \to \infty \). Chapter 6 is related to Chapter 4 and 5; it treats a more general model, using techniques of both Chapter 4 and 5. Chapter 7 deals with an essentially different system, and is therefore laterally related to results of Chapter 4. Finally, Chapter 8 and 9 deal with refinements of the results obtained in Chapter 4, using both an analytical and a simulation based approach.
Results

2.1 Introduction

In this chapter, we will describe the results obtained in this thesis. For the complete results, we refer to the chapters indicated in the sections below.

The basis of nearly all results is large deviations theory. This theory focuses on the characterization of 'rare' events, and is asymptotic in nature. The theory of large deviations consists of a set of techniques for turning hard probability problems concerning rare events into analytical problems in the calculus of variations.

Let us first give a simple example of standard large deviations behaviour. Take $k = 2$, $P_0 > 0$, $P_1 = 1$ and $\sigma^2 = 0$. Then

$$Z_0^{(1)} = P_0^{1/2} + \frac{1}{n} \sum_{i=1}^{n} X_{1i},$$

where $X_{1i}$ are i.i.d. with $P(X_{11} = 1) = P(X_{11} = -1) = 1/2$. By the law of large numbers, we have that for $n \to \infty$, $Z_0^{(1)} \to P_0^{1/2}$, so that $P(\text{sgn} r_0 (Z_0^{(1)}) < 0) \to 0$. The approximation $P(\text{sgn} r_0 (Z_0^{(1)}) < 0) = 0$ is, however, not very useful in practice. Therefore, more detailed information on how the desired probability tends to zero is necessary. It is our task to quantify at which rate this occurs. We will see that in the problems under consideration, the decay is exponentially in $n$, i.e., there exists an $I > 0$ such that

$$P(\text{sgn} r_0 (Z_0^{(1)}) < 0) = e^{-n I(1+o(1))} \quad \text{as} \quad n \to \infty.$$

The number $I$ is called the exponential rate. It quantifies the behaviour of the error probability as a function of $n$, and is therefore a good measure of performance. In the example
above, \( I \) can be calculated explicitly. It turns out that

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} (\text{sgn} \rho(Z_0^{(1)}) < 0) = I(P_0^{1/2}),
\]

where

\[
I(z) = \begin{cases} 
\frac{1 + z}{2} \log(1 + z) + \frac{1 - z}{2} \log(1 - z) & z \in [-1, 1], \\
\infty & \text{otherwise.}
\end{cases}
\]

(2.1)

A direct proof is provided in [Den Hollander (2000)], Section I.2 and is based on a reformulation using binomial random variables together with Stirling's formula \( n! = n^n e^{-n} \sqrt{2\pi n} (1 + O(1/n)) \).

The first basic result, due to Cramér ([Cramér (1938)]), identifies the exponential rate in a more general case than the one above. The proof can be found in any standard work on large deviations, for example [Den Hollander (2000)], Thm. I.4.

**Theorem 2.1 (Cramér)** Let \( X_1, X_2, \ldots \) be i.i.d. random variables and let the moment generating function \( \phi(t) = \mathbb{E} e^{tX_1} \) satisfy \( \phi(t) < \infty \) for all \( t \in \mathbb{R} \). Then for all \( a > \mathbb{E} X_1 \),

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} X_i \geq a \right) = I(a),
\]

where

\[
I(z) = \sup_{t \geq 0} \left\{ zt - \log \phi(t) \right\}.
\]

Besides identifying the rate, the proof also reveals how the random variables \( X_1, X_2, \ldots \) behave when they do make a large deviation from the mean (like \( \frac{1}{n} \sum_{i=1}^{n} X_i > a \)). It turns out that the sum becomes large because of conspiracies, not because of outliers. In other words, all the random variables \( X_1, X_2, \ldots \) behave a little bit different than usual, rather than that one \( X_i \) behaves completely different.

Whereas Cramér's theorem addresses the question "How likely is it for the sample mean to deviate from its ensemble mean", Sanov's theorem (cf. [Den Hollander (2000)], Thm. II.2) addresses the question "How likely is it for the empirical distribution to deviate from the true distribution?". It turns out that for the investigation of the MF model, Cramér's theorem gives the desired result. For the HD-PIC model, a multidimensional version of Cramér's theorem does the job, while for the SD-PIC model we need Sanov's theorem.

We next extend the previous example by allowing \( P_0 = 1/4 \) or \( P_0 = 1/16 \) each with probability 1/2. Then

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} (\text{sgn} \rho(Z_0^{(1)}) < 0) = - \lim_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{2} \mathbb{P} \left( \frac{1}{2} + \frac{1}{n} \sum_{i=1}^{n} X_{1i} \leq 0 \right) + \frac{1}{2} \mathbb{P} \left( \frac{1}{4} + \frac{1}{n} \sum_{i=1}^{n} X_{1i} \leq 0 \right) \right).
\]

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We now use an elementary fact:

\[
- \lim_{n \to \infty} \frac{1}{n} \log \left( \mathbb{P} \left( \frac{1}{2} + \frac{1}{n} \sum_{i=1}^{n} X_{1i} \leq 0 \right) + \mathbb{P} \left( \frac{1}{4} + \frac{1}{n} \sum_{i=1}^{n} X_{1i} \leq 0 \right) \right)
= \min \left\{ - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \frac{1}{2} + \frac{1}{n} \sum_{i=1}^{n} X_{1i} \leq 0 \right), - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \frac{1}{4} + \frac{1}{n} \sum_{i=1}^{n} X_{1i} \leq 0 \right) \right\}.
\]

This fact is known as "the largest-exponent-wins" principle, cf. Den Hollander (2000), Eqn.(1.2). Substituting (2.1) yields (using $\lim_{n \to \infty} \frac{1}{n} \log 2 = 0$)

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} (\text{sgn}r_0(Z^{(1)}) < 0)
= \min \left\{ \frac{3}{4} \log 3 - \log 2, \frac{5}{8} \log 5 + \frac{3}{8} \log 3 - 2 \log 2 \right\} = \frac{5}{8} \log 5 + \frac{3}{8} \log 3 - 2 \log 2.
\]

The intuitive explanation is that

\[
\mathbb{P} (\text{sgn}r_0(Z^{(1)}) < 0) = \frac{1}{2} e^{-n(0.13081\ldots + o(1))} + \frac{1}{2} e^{-n(0.033158\ldots + o(1))}
\]

and the first term is negligible compared to the second term, when $n$ is sufficiently large.

We will often use the term "typical" and "typical scenario" to denote that a certain scenario has the smallest rate. In the example above, we will typically observe a bit error when $P_0 = 1/16$, because the other scenario ($P_0 = 1/4$) makes only a tiny contribution to the BEP.

In this thesis, we will investigate the behaviour of various systems. In fact, one of the goals of this thesis is to find a good measure of performance that is easy to use. The BEP gives a good indication of the quality of the system. However, calculating or approximating this error probability is difficult. For advanced systems, often there is no theoretical base to apply a CLT. We will show that for the HD- and SD-PIC system, the SNR gives the right approximation (see Chapter 5). Luckily, it turns out that for more general systems (with more users) and more advanced systems, such as HD- or SD-PIC, the BEP is exponentially small. Therefore, the exponential rate is a natural candidate as performance measure. To a large extent, this thesis focuses on the exponential rate, depending on the number of users $k$, for the three different systems (MF, HD-PIC and SD-PIC). Especially for the HD-PIC system, the largest-exponent-wins principle will be useful.

2.2 Exponential rate for the simple model

The first goal is to understand how the interference cancellation changes the performance of the system. Let us start with one stage of interference cancellation. We focus on the key parameters $n$ and $k$, and we will assume that $P_0 = \ldots = P_{k-1} = 1$ and $\sigma^2 = 0$. This model
is known as a perfect channel with perfect power control and we will denote it by the simple model. In this model, all users are equivalent and the only noise present in the system is due to the other users. The simple model will act as a test case, to indicate the possible improvement of performance. We will argue that the simple model is characteristic, in that improvement of performance for more realistic models is comparable to the simple model.

In the simplified model, the analysis turns out to simplify significantly. The case \( k = 1 \) and \( k = 2 \) are special. We can easily show that the rate equals \( \infty \) for \( k = 1 \) and \( \log 2 \) for \( k = 2 \), regardless whether PIC is performed or not. We will focus in the remainder of this section on \( k \geq 3 \).

Since

\[
\frac{1}{2} \mathbb{P}(Z_0^{(s)} \leq 0) \leq \mathbb{P}({\text{sgnr}_0}(Z_0^{(s)} < 0)) = \mathbb{P}(Z_0^{(s)} < 0) + \frac{1}{2} \mathbb{P}(Z_0^{(s)} = 0) \leq \mathbb{P}(Z_0^{(s)} \leq 0),
\]

we have

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}({\text{sgnr}_0}(Z_0^{(s)} < 0)) = - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Z_0^{(s)} \leq 0).
\]

For the MF model, the exponential rate, denoted by \( I_k \), follows directly from Cramér's theorem. We obtain for \( k \geq 3 \),

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Z_0^{(1)} \leq 0) = I_k,
\]

where

\[
I_k = \frac{k-2}{2} \log \left( \frac{k-2}{k-1} \right) + \frac{k}{2} \log \left( \frac{k}{k-1} \right).
\]

(2.2)

For the one-stage HD-PIC system, we are able to give the exponential rate, denoted by \( H_k^{(2)} \), in two steps,

\[
H_k^{(2)} = \min_{1 \leq r \leq k-1} H_{k,r}^{(2)}, \quad \text{where} \quad H_{k,r}^{(2)} = \sup_{t \in (-\infty, 0]^2} \{- \log h_{k,r}(t)\},
\]

(2.3)

with

\[
h_{k,r}(t) = 2^{-r} \left( \sum_{j=r \text{ even}}^r \right) \left( \begin{array}{c} r \\ \frac{r}{2} \end{array} \right) e^{t_1(1+j^2)+t_2(1+2j)(\cosh t_1 j)} (k^{-1})^{k-r-1}.
\]

To obtain this, we first calculate the exponential rate of the event that \( \{\text{sgnr}_0(Z_0^{(2,r)}) < 0\} \), intersected with the events \( \{\text{sgnr}_m(Z_m^{(1)}) < 0\} \) for \( m = 1, \ldots, r \) and the events \( \{\text{sgnr}_m(Z_m^{(1)}) > 0\} \) for \( m = r+1, \ldots, k-1 \), i.e., we study

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( \max_{1 \leq m \leq r} \min_{r+1 \leq m \leq k-1} \text{sgnr}_m(Z_m^{(1)}) < 0, \min_{r+1 \leq m \leq k-1} \text{sgnr}_m(Z_m^{(1)}) > 0, \text{sgnr}_0(Z_0^{(2,r)}) < 0 \right).
\]

(2.4)

We obtain from (1.11) that, on the intersection with the event

\[
\{ \max_{1 \leq m \leq r} \text{sgnr}_m(Z_m^{(1)}) < 0, \min_{r+1 \leq m \leq k-1} \text{sgnr}_m(Z_m^{(1)}) > 0 \},
\]
2.2 Exponential rate for the simple model

$Z_0^{(2, H)}$ is equal to

$$Z_0^{(2, H)} = 1 + 2 \sum_{j=1}^{r} \sum_{i=1}^{n} \frac{1}{n} X_{ji}. \quad (2.5)$$

Note that we obtain $Z_0^{(2, H)}$ from $Z_0^{(2, H)}$ through replacing $\text{sgn}_m(Z_m^{(1)})$ by its corresponding value. We prove that the rate in (2.4) equals

$$- \lim_{n \to \infty} \frac{1}{n} \log P \left( \max_{1 \leq m \leq r} Z_m^{(1)} \leq 0, \, \min_{r+1 \leq m \leq k-1} Z_m^{(1)} \geq 0, \, Z_0^{(2, H)} \leq 0 \right). \quad (2.6)$$

The latter rate, which we will denote by $H_{k, r}^{(2)}$, proves to be the solution of the 2-dimensional minimization problem in (2.3). The desired rate $H_{k}^{(2)}$ is given by $H_{k}^{(2)} = \min_r H_{k, r}^{(2)}$, which follows from the largest-exponent-wins principle. In other words, we calculate the typical number of errors in the first stage. Let us denote

$$r_k = \arg \min_r H_{k, r}^{(2)}. \quad (2.7)$$

For $3 \leq k \leq 9$, $r_k = 1$, so that typically 1 bit error in stage 1 is observed. For $10 \leq k \leq 26$, $r_k = 2$. The intuitive explanation why $r_k$ increases with $k$ is based on two principles. Firstly, the probability of having $r$ bit-errors in the first stage is decreasing in $r$. Secondly, the probability of making an error in the second stage caused by $r$ noise terms is increasing in $r$. Intersection of the two events therefore involves a balance between a decrease (the first stage) and an increase (second stage). When $k$ becomes larger, it is easier to make errors in the first stage. This results in a change in the balance, in favour of larger $r$.

For the one-stage SD-PIC model, we define a set $X_k$ of cardinality $|X_k| = 2^{k-1}$ and, for $\rho \in [0, 1]^{||X_k||}$, the rate function

$$I_k(\rho) = (k - 1) \log 2 + \sum_{a \in X_k} \rho_a \log \rho_a.$$  

Furthermore, we represent $Z_0^{(2, s)}$ by the function $F_k$, defined on $[0, 1]^{||X_k||}$ (see Lemma 4.4). In Theorem 4.5, we give the exponential rate, denoted by $J_k^{(2)}$, as the optimum of $I_k$, restricted to $F_k = 0$ and some boundary constraints, i.e., for $k \geq 3$,

$$J_k^{(2)} = \inf_{\rho \in D} I_k(\rho), \quad (2.8)$$

where

$$D = \left\{ \rho = (\rho_a)_{a \in X_k} : \rho_a \geq 0, \, a \in X_k, \, \sum_{a \in X_k} \rho_a = 1, \, F_k(\rho) = 0 \right\}.$$

From a practical point of view, the results above can be used to generate numerical values for $I_k$, $H_{k}^{(2)}$ and $J_k^{(2)}$, see Figure 2.1. Using the numerical values, we can conclude whether HD-PIC and/or SD-PIC give an improvement in performance, i.e., whether $H_{k}^{(2)} > I_k$ and $J_k^{(2)} > I_k$. However, we prefer to make a statement concerning the rates for all $k$, rather than for the $k$ for which we solved the optimization problems. We have been able to prove the following results.
Figure 2.1: Numerical results for $I_k$ (o), $H_k^{(2)}$ (△) and $J_k^{(2)}$ (o)

Concerning the one-stage HD-PIC model, for $k = 3$,

$$H_k^{(3)} = I_k,$$

whereas for $k \geq 4$,

$$H_k^{(2)} > I_k.$$

It is striking that for $k = 3$, HD-PIC does not improve the performance. We will see in Section 2.5 that for higher $k$, a similar phenomenon occurs for multistage HD-PIC.

The SD-PIC system does not suffer from this phenomenon. We are able to prove that for $k \geq 3$,

$$J_k^{(2)} > I_k.$$

We conclude that both HD-PIC and SD-PIC have the potential to significantly improve the performance, compared to MF. With potential we mean that the statement is true, provided a sufficiently large $n$. Only for $k = 3$, HD-PIC does not improve the performance. When we compare the numerical results of $H_k^{(2)}$ and $J_k^{(2)}$ with $I_k$, we see that the increase in rate is significant for $k \geq 4$.

### 2.3 Asymptotic behaviour of exponential rate

We have shown in the previous section that for $k \geq 4$ both $H_k^{(2)}$ and $J_k^{(2)}$ are strictly larger than $I_k$. Thus, both HD-PIC and SD-PIC increase performance. However, the increase is not quantified. In Figure 2.1 it is seen that this increase is quite spectacular, especially when
$k$ is large. One wonders if the ‘speed’ at which $I_k$ tends to zero for $k \to \infty$ is higher than the speed of $H_k^{(2)}$ or $J_k^{(2)}$. In this section the behaviour of the rate is described for $k \to \infty$.

A Taylor expansion of the expression for $I_k$ (see (2.2)) results in

$$I_k = \frac{1}{2k} \left( 1 + \mathcal{O}\left(\frac{1}{k}\right) \right). \quad (2.10)$$

For the one-stage HD-PIC system, we prove that whenever $r \to \infty$ and $\frac{r}{k} \to 0$ as $k \to \infty$,

$$H_{k,r}^{(2)} = \left( \frac{1}{8r} + \frac{r}{2k} \right) \left( 1 + \mathcal{O}\left(\frac{1}{8r} + \frac{r}{2k}\right) \right).$$

From this result, the asymptotics of $H_k^{(2)}$ for $k \to \infty$ are derived:

$$H_k^{(2)} = \min_r H_{k,r}^{(2)} = \frac{1}{2\sqrt{k}} \left( 1 + \mathcal{O}\left(\frac{1}{\sqrt{k}}\right) \right)$$

and

$$r_k = \arg\min_r H_{k,r}^{(2)} = \frac{1}{2}\sqrt{k} + \mathcal{O}(1).$$

The heuristic explanation of the above result is as follows. We start with expression (2.6). Observe that $E Z_m^{(1)} = 1$, so that one expects $Z_m^{(1)}$ to be positive. Indeed, it turns out that the event \( \{ \min_{r+1} \leq m \leq k-1} Z_m^{(1)} \geq 0 \} \) in (2.6) does not contribute to the rate $H_{k,r}^{(2)}$. Independence of $\{ Z_1^{(1)}, \ldots, Z_r^{(1)}, \tilde{Z}_0^{(2,H)} \}$ is false for finite $k$, but holds asymptotically as $k \to \infty$. More precisely, we show that for $k \to \infty$

$$- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \max_{1 \leq m \leq r} Z_m^{(1)} \leq 0, \tilde{Z}_0^{(2,H)} \leq 0 \right) \approx - \lim_{n \to \infty} \frac{1}{n} \log \left( \prod_{m=1}^r \mathbb{P}(Z_m^{(1)} \leq 0) \mathbb{P}(\tilde{Z}_0^{(2,H)} \leq 0) \right),$$

and using exchangeability this yields

$$- \lim_{n \to \infty} \frac{1}{n} \log \left( \mathbb{P}(Z_1^{(1)} \leq 0)^r \mathbb{P}(\tilde{Z}_0^{(2,H)} \leq 0) \right) = r H_k^{(1)} - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\tilde{Z}_0^{(2,H)} \leq 0).$$

The first term on the right-hand side is asymptotically $r$ times $1/(2k)$ (recall (2.10)). Since $\tilde{Z}_0^{(2,H)}$ has a similar i.i.d. structure as $Z_0^{(1)}$ (see (2.5)), the second term on the right-hand side can be easily shown to be asymptotically $1/(8r)$ for $r$ large. Together, this yields

$$H_{k,r}^{(2)} \approx \frac{r}{2k} + \frac{1}{8r}.$$

Finally, minimizing over $r$ gives $H_k^{(2)} \approx 1/(2\sqrt{k})$ for $r_k \approx \frac{1}{2}\sqrt{k}$. This indicates that when we observe a bit error in the second stage, it is typically caused by $\frac{1}{2}\sqrt{k}$ errors in the first stage. This explains the dramatic increase in performance. Rather than one error for MF, now $\frac{1}{2}\sqrt{k}$ errors are made in the first stage.
Chapter 2. Results

For \((s - 1)\)-stage HD-PIC, we are able to derive similar results. When we denote the exponential rate for the \((s - 1)\)-stage HD-PIC model by \(H^{(s)}_k\), we obtain the following asymptotic result. Fix \(1 \leq s < \infty\). As \(k \to \infty\),

\[
H^{(s)}_k = \frac{s \sqrt{4}}{8 \sqrt{k}} \left( 1 + \mathcal{O} \left( \frac{1}{\sqrt{k}} \right) \right).
\]

In the above arguments, we use \(n \to \infty\) and subsequently \(k \to \infty\). We expect the results to hold also if \(k = k_n \to \infty\) sufficiently slow with \(n\). We derive an upper bound for the BEP for \(s = 1\) and \(s = 2\) that is valid for any \(n\) and \(k\). This upper bound is the Chernoff bound. Using the Chernoff bound we prove that when \(k_n \to \infty\) such that \(k_n = o \left( \frac{n}{\log n} \right)\),

\[
\mathbb{P}(\text{sgnr}_0(Z^{(s,n)}_0) < 0) \leq \exp \left( - \frac{s \sqrt{4}}{8} \frac{n}{\sqrt{k_n}} \left( 1 + o(1) \right) \right).
\]

The above result implies that when \(k_n \to \infty\) such that \(k_n = o(n/\log n)\), we have that the BEP \(\mathbb{P}(\text{sgnr}_0(Z^{(s,n)}_0) < 0)\) is to leading asymptotics bounded by \(\exp(-\frac{n}{2\sqrt{k_n}})\). The large deviation approach in which first \(n \to \infty\) and subsequently \(k \to \infty\) can be extended to cases in which \(n\) and \(k\) tend to infinity simultaneously.

For the one-stage SD-PIC model, no asymptotic results could be obtained. Therefore, we have done a least-squares analysis on the numerical results. This leads to the conclusion that \(J^{(2)}_k\) is approximately equal to \(\frac{1}{2\sqrt{k}}\). Hence, for many users, one-stage SD-PIC performs as good as one stage HD-PIC. In both systems, compared to the MF system, the improvement is significant. As a result, to obtain the same potential performance as the MF system, the processing gain could be decreased significantly (at most by a factor \(\sqrt{k}\)), or instead of \(k\) users in a MF system, many more users (at most \(k^2\)) can enter the system with HD- or SD-PIC.

Finally, we show that for the case \(k \ll n\), the signal-to-noise ratio (SNR) is in general \textit{not} a good measure of performance. In (2.10), we have seen for the MF model that \(I_k \approx 1/(2k)\). This means that

\[
\mathbb{P}(\text{sgnr}_0(Z^{(1)}_0) < 0) \sim e^{-n/(2k)}.
\]

We will show that SNR is given by \(\sqrt{n/(k-1)} \approx \sqrt{n/k}\), so that it follows, according to the CLT that \(\mathbb{P}(\text{sgnr}_0(Z^{(1)}_0) < 0) \sim e^{-n/(2k)}\). Thus, for the MF model, in which all powers are equal and no AWGN is present, the SNR is asymptotically equivalent to the exponential rate. We are able to show that for the model with unequal powers and AWGN, the same conclusion holds. Thus, for the MF model, the exponential rate is a perfect substitute for the SNR. For the HD-PIC model, our measure based on the exponential rate implies

\[
\mathbb{P}(\text{sgnr}_0(Z^{(2,n)}_0) < 0) \sim e^{-n/(2\sqrt{k})}.
\]

However, for a system with one stage of HD-PIC, using the SNR results in

\[
\mathbb{P}(\text{sgnr}_0(Z^{(2,n)}_0) < 0) \leq \exp \left( - \frac{\epsilon_k}{2(k-1)^2} \right),
\]

which is shown in Section 5.6. The latter value is far too small compared to the true asymptotics above (for \(k = o(n/\log n)\)), which clearly indicates that for the HD-PIC model.
the Gaussian approximation is no good. For the SD-PIC model, we can deduce that the
SNR is given by
\[
\frac{1 - (k - 1)/n}{\sqrt{k(k - 1)/n^2 - 2(k - 1)/n^3}} \approx \frac{n}{k}, \quad \text{if } k = o(n),
\]
which implies that
\[
P(\text{sgn} r_0(Z_0^{(2,S)}) < 0) \sim e^{-n^2/(2k^2)}.
\]
This result is clearly not the same as our large deviation result, which states
\[
P(\text{sgn} r_0(Z_0^{(2,S)}) < 0) \sim e^{-n/(2\sqrt{k})}.
\]
In Figure 2.2 and Table 2.1, numerical results for \( I_k \), \( H_k^{(2)} \) and \( J_k^{(2)} \) are shown, together with

![Figure 2.2: Rates \( I_k \) (o), \( H_k^{(2)} \) (△), \( J_k^{(2)} \) (○) and asymptotics \( \frac{1}{2k} \) (x), \( \frac{1}{2\sqrt{k}} \) (*).](image)

\( 1/(2k) \) and \( 1/(2\sqrt{k}) \). Clearly, \( I_k \) is close to the asymptotic rate \( 1/(2k) \), even for small values of \( k \). The asymptotic rate \( 1/(2\sqrt{k}) \) fits both \( H_k^{(2)} \) and \( J_k^{(2)} \) reasonable. The fit for \( H_k^{(2)} \) is worse than the fit for \( I_k \). This is because of the error terms. For MF, the error term is \( O(k^{-1}) \), while for HD-PIC the error term is \( O(k^{-1/2}) \). We remark from the numerical results that for \( k \geq 2 \),
\[
|H_k^{(2)} - J_k^{(2)}| \ll \left| H_k^{(2)} - \frac{1}{2\sqrt{k}} \right|,
\]
so that there are no indications that \( J_k^{(2)} \) does not have the asymptotic rate \( \frac{1}{2\sqrt{k}} \).
Chapter 2. Results

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<th>$J_k^{(2)}$</th>
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Table 2.1: Numerical values for $I_k$, $H_k^{(2)}$ and $J_k^{(2)}$

2.4 Extension to more realistic models

In the previous sections, we have assumed that $P_0 = P_1 = \ldots = P_{k-1} = 1$ and $\sigma^2 = 0$. This model assumption allows us to obtain good insight in the system behaviour. In practice, different users are at different geographical positions. As a result, signals of different users are received at different powers. The main result of this section is the extension of the asymptotic results for the MF and the HD-PIC system to a more realistic model. For the SD-PIC system, we did not obtain asymptotic results. Instead, a representation of the exponential rate in terms of an optimization problem, similarly to that of the simple system, will be given in Section 6.3. The obtained results, described in this section, are stated and proven in Chapter 6.

Recall that the powers are denoted by $P_0, P_1, \ldots, P_{k-1}$. We define the total power to be
\[ P = \sum_{j=0}^{k-1} P_j. \] When we investigate the quality of the system with respect to user 0, the generalization of \( k \to \infty \) is \( P_0/P \to 0 \), i.e., the relative power of user 0 tends to 0. In this situation, we are able to prove (see Chapter 6, Proposition 6.2 and Theorem 6.4) that

\[ I_k = \frac{P_0}{2P} \left( 1 + O\left( \frac{P_0}{P} \right) \right) \quad \text{and} \quad H_k^{(2)} = \frac{1}{2} \sqrt{\frac{P_0}{P}} \left( 1 + O\left( \frac{P_0}{P} \right) \right). \]

This result is identical to the results obtained for the simple system with \( k \) replaced by \( P/P_0 \). This shows that the simple model indeed is characteristic for the behaviour of the more realistic system.

Another important extension is the inclusion of noise from other (unknown) sources. In practice, it is inevitable that undesirable noise disturbs the system. For example, transmission of data from users to another base station often results in many weak signals, interfering with the signals of the base station of interest. This noise is modeled as a white noise process with intensity \( \sigma^2 \) and is denoted by AWGN.

When we take AWGN into account, we obtain (see Proposition 6.2)

\[ I_k = \frac{P_0}{2(P + \sigma^2)} \left( 1 + O\left( \frac{P_0}{P + \sigma^2} \right) \right). \quad (2.11) \]

From this result, one cannot expect that \( H_k^{(2)} \) is asymptotically equal to \( \frac{1}{2} \sqrt{\frac{P_0}{P + \sigma^2}} \), since the AWGN is not cancelled.

The result is split into two cases, depending on whether the noise is dominant or not. In the first scenario the AWGN is dominant, i.e., \( \sigma^2 \) is large. In the case where \( \sigma^2 \) is much larger than \( P \), we see that \( I_k \approx P_0/(2\sigma^2) \). We expect \( H_k^{(2)} \) to have the same form, since the expression does not depend on \( k \) or \( P_1, \ldots, P_{k-1} \). Indeed, we are able to prove that

\[ H_k^{(2)} = \frac{P_0}{2\sigma^2} \left( 1 + O\left( \frac{P_0}{\sigma^2} \right) \right). \]

When \( \sigma^2 \) is not extremely large, but \( \frac{\sigma^2}{4P_0} \geq \frac{1}{2} \sqrt{\frac{P + \sigma^2}{P_0}} \), the asymptotic result above still holds. This is an improvement over (2.11), since the contribution of the interference from the other users disappears, i.e., the term \( P_0/(P + \sigma^2) \) in (2.11) is replaced by \( P_0/\sigma^2 \).

When \( \sigma^2 \) is sufficiently small, i.e., when \( \frac{\sigma^2}{4P_0} \leq \frac{1}{2} \sqrt{\frac{P + \sigma^2}{P_0}} \), different behaviour can be observed. If the powers obey a certain technical condition\(^1\), then (see Theorem 6.4)

\[ H_k^{(2)} = \frac{1}{2} \sqrt{\frac{P_0}{P + \sigma^2}} - \frac{\sigma^2}{8(P + \sigma^2)} + O\left( \frac{P_0}{P + \sigma^2} \right). \]

The term \( \sigma^2/(8(P+\sigma^2)) \) is a correction term, due to the fact that the AWGN is not cancelled.

\(^1\)This condition is specified in Theorem 6.4.
When the powers do not obey the technical condition, it is not possible to derive the precise asymptotic form of the rate. However, we are able to derive the lower bound of the rate:

\[
H_k^{(2)} \geq \frac{1}{2} \sqrt{\frac{P_0}{P + \sigma^2}} - \frac{\sigma^2}{8(P + \sigma^2)} + \mathcal{O}\left(\frac{P_0}{P + \sigma^2}\right).
\]

This means that in all cases a certain quality can be guaranteed.

When we compare this with the result for \( I_k \), given in (2.11), we conclude that HD-PIC gives an increase in performance for both “small” and “large” \( \sigma^2 \). The condition for small \( \sigma^2 \), \( \frac{\sigma^2}{4I_k} \leq \frac{1}{4} \sqrt{\frac{P_k + \sigma^2}{P_k}} \), implies directly that

\[
H_k^{(2)} \geq \frac{1}{4} \sqrt{\frac{P_0}{P + \sigma^2}} + \mathcal{O}\left(\frac{P_0}{P + \sigma^2}\right),
\]

which is a significant improvement over \( I_k = P_0/(2(P + \sigma^2)) + \mathcal{O}(P_0^2/(P + \sigma^2)^2) \). When \( \sigma^2 \) is large, typically all interference due to other users is cancelled successfully.

We have numerical results, showing the influence of power variations on the exponential rate. We have obtained numerical results for a model with 3, 6 or 9 users, in which user 0, 2, \ldots, \( k-1 \) have power 1, and user 1 has a power varying from 0 to 6. We have furthermore obtained numerical results for a model with AWGN, where \( 0 \leq \sigma^2 \leq 10 \). The results in Figure 2.3 show that both HD-PIC and SD-PIC give a significant increase in performance over MF for \( k \geq 6 \). The SD-PIC system turns out to perform slightly better than the HD-PIC system, but for \( k \geq 6 \), the difference is negligible. For \( k = 3 \), however, HD-PIC performs worse than MF in the case that \( 0.4245 < P_1 < 1 \).

## 2.5 Optimal systems

The results so far are promising; both one-stage HD- and SD-PIC give a significant increase in performance in comparison with the MF system. We have further shown that multistage HD-PIC also gives significant increase, provided that the number of users is sufficiently large. For a fixed number of users, it is not clear whether adding another stage of PIC always increases performance. In fact, simulations show that in practice more than three stages or four of PIC is not worthwhile for moderate values of \( k \), see for example Buehler (1999), where an extensive simulation based research is performed on convergence of HD-PIC. Chapter 7 focuses on the scenario in which \( k \) is fixed and \( s \) tends to \( \infty \). In this section, we will review the results of Chapter 7.

We will call the PIC system in which we perform infinitely many cancellation steps the optimal PIC system. Throughout this section, we will assume that \( P_m > 0 \) for all \( m \) and no AWGN is present.

In the case that all powers are equal, it is easy to show that the exponential rate of \( s \) stage HD-PIC is non-decreasing in \( s \). As a consequence, \( H_k^{(\infty)} = \lim_{s \to \infty} H_k^{(s)} \) exists. In the case
2.5 Optimal systems

(a) Rates for $k = 3$ as a function of $P_1$.

(b) Rates for $k = 6$ as a function of $P_1$.

(c) Rates for $k = 9$ as a function of $P_1$.

(d) Rates for $k = 3$ as a function of $\sigma^2$.

(e) Rates for $k = 6$ as a function of $\sigma^2$.

(f) Rates for $k = 9$ as a function of $\sigma^2$.

Figure 2.3: $I_k (\diamond), H^{(2)}_k (\triangle), J^{(2)}_k (\diamond)$ and $\frac{1}{2\pi} (\times)$ for $k = 3, 6$ and 9. Figures (a)-(c) show the case in which $P_0 = P_2 = P_3 = \ldots = P_{k-1}$ and $\sigma = 0$. Figures (d)-(f) depict the case with $P_0 = \ldots = P_{k-1}$ and $\sigma \geq 0$.

of unequal powers, monotonicity of $H^{(s)}_k$ is not true, see for example Figure 2.3(d). To deal with this case, we define

$$H^{(s)}_{k;m} = - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Z^{(s)}_m \leq 0), \quad 0 \leq m \leq k - 1,$$
and

\[ H_k^{(s)} = \min_{0 \leq m \leq k-1} H_k^{(s,m)}. \]

It can be shown that this “worst case” rate is monotone in \( s \) (see Theorem 7.1).

We are able to prove that for the system with unequal powers, there exists an \( s_k \leq 2^k + 1 \), such that \( H_k^{(s)} = H_k^{(s_k)} \) for all \( s \geq s_k \). In other words, in a finite number of stages (depending on \( k \) and the powers) optimal performance is achieved.

We will next describe the performance of this optimal system. Recall that because of the hard decisions, the decisions at stage \( \sigma \) only depend on the decisions at stage \( \sigma - 1 \). We note that we use the variable \( \sigma \) to denote a stage, while in our model we reserved \( \sigma^2 \) for the intensity of the AWGN. However, throughout this section we assume that no AWGN is present, so that there is no confusion because of the double meaning of \( \sigma \). Furthermore, we will always refer to “stage \( \sigma \)”.

For illustration purposes, we will first investigate the system with 2 users. We will use that when user 1 has no bit error at stage \( \sigma \), user 2 necessarily has no bit error at stage \( \sigma + 1 \). When we denote a bit error by \( \Box \) and no bit error by \( \bullet \), in Figure 2.4, all six possible scenarios are displayed. When we denote the set of bit errors in stage \( \sigma \) by \( R_\sigma \), we see that \( (R_\sigma)_{\sigma=1}^{\infty} \) is periodic, except for at most 3 initial stages. We will denote the repeating part by periodic scenario. Once a periodic scenario is obtained, the exponential rate remains constant, so that we achieved optimal performance. We have shown that the number of initial stages is at most \( 2^2 + 1 \), which explains why the optimal system is achieved in at most 5 stages.

We will next investigate periodic scenarios in more detail. Periodic scenarios has a big “advantage” over non-periodic scenarios. As long as a scenario is not yet periodic, specifying which users have bit errors at a certain stage results in a decrease of the BEP. Indeed, it is likely to estimate a bit correctly, so that users do not tend to have a bit error. However, in a periodic scenario, specifying the initial part is sufficient. For example, to get a bit error for user 1 at stage \( s = 2002 \), it is sufficient to specify the positions of bit errors from stage 1,2 and 3 (the third and last scenario in Figure 7.1 will do). From that stage onwards, the bit errors are determined by those in stages 1,2 and 3.

Two essentially different scenarios are characterizing the behaviour of the optimal system. The first one is the so-called disjoint scenario, where at every stage user 1 has a bit error and user 2 does not, or vice versa. The other scenario, which we will call the overlapping
scenario is the scenario where at every stage both user 1 and user 2 have a bit error. Note that for both the disjoint and the overlapping scenario, the periodic behaviour kicks in at stage 1. For both scenarios, we can calculate the exponential rate. The minimum of the two exponential rates indicates which scenario typically is observed.

When \( k \geq 3 \), we extend those scenarios in the following way. For every \( r \), at stage 1, 3, 5, \ldots bit errors are made for users in some set \( R_1 \), with \( |R_1| = r \). At stage 2, 4, 6, \ldots, bit errors are made for users in the set \( R_2 \) with \( |R_2| = r \). When \( R_1 \cap R_2 = \emptyset \), we speak of the disjoint scenario. Whenever \( R_1 = R_2 \), we will call it the overlapping scenario. All other scenarios are denoted by partly overlapping. We can show (see Theorem 7.1(b)) that after at most \( 2^k + 1 \) stages, the behaviour is periodic, so that we are in either the disjoint, the partly overlapping and the overlapping scenario.

We will next investigate these scenarios. From numerical results of the exponential rate of the various scenarios, we observe the following phenomena.

1) The partly overlapping scenario is not optimal for any \( k \), i.e. both the disjoint and overlapping scenario give a smaller exponential rate. It seems that both the extremes do a better job.

2) For small \( k \), the disjoint scenario is optimal. The reason is quite simple. For \( r \) fixed, a user at stage 2 has contribution from \( r \) noise terms, while for the overlapping scenario the user has contribution from only \( r - 1 \) terms (indeed, the user does not interfere with its own signal). For higher \( k \), however, the overlapping scenario is optimal. For the case \( k \to \infty \), it is implicitly proven that the overlapping scenario is indeed optimal, and the partly overlapping scenario is never optimal.

3) For \( k \to \infty \), also \( r \to \infty \), but much slower than \( k \). The numerical results indicate that the optimal \( r \approx \sqrt{k}/2 \).

In Figure 2.5, the exponential rates \( H_k^{(s)} \) are given, together with \( H_k^{(s)} \) for \( s = 1, 2, 3 \). The results for \( s = 3 \) are obtained using similar techniques as in the proof of (2.3). However, we have not stated the result for \( s = 3 \) in this thesis. The rate \( H_k^{(s)} \) is in fact the rate corresponding to the disjoint scenario for \( r = 1 \) or \( r = 2 \). For \( k = 2, 3 \), it is seen that \( I_k = H_k^{(2)} = H_k^{(3)} \), so that \( s_2 = s_3 = 1 \). For \( 4 \leq k \leq 9 \), we see \( I_k < H_k^{(2)} = H_k^{(3)} \), so that \( s_k = 2 \). We see that one stage of HD-PIC gives an improvement in exponential rate. However, adding one more stage does not result in any improvement. For \( 10 \leq k \leq 22 \), 2-stage HD-PIC gives an improvement over one-stage HD-PIC. However, the numerical result shows that the scenario corresponding to the rate is the disjoint scenario with \( r = 1 \). Therefore \( s_k = 3 \) in this case. For \( 23 \leq k \leq 50 \), we expect \( s_k = 4 \), even though we cannot calculate \( H_k^{(4)} \).

The above results do not state anything about the behaviour of \( H_k^{(s)} \) for \( s \) large and subsequently \( k \) large. We will now state results of this type. We are able to prove that for all \( s \geq 2^k + 1 \) and all powers,

\[
H_k^{(s)} \geq \frac{1}{2} \log 2 - \frac{1}{4} = 0.09657 \ldots
\] (2.12)
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Figure 2.5: The exponential rates $H_k^{(s)}$ for $s = 1, 2, 3$ (o, △ and ○ respectively) and $H_k^{(sk)}$ (⋆).

When all powers are equal, we can prove that this statement is already true for $s \geq k + 1$.

Under some power conditions, we are able to prove that this lower bound is tight. We abbreviate $P = \sum_{m=0}^{k-1} P_m$, $P_{\min} = \min_{0 \leq j \leq k-1} P_j$ and $P_{\max} = \max_{0 \leq j \leq k-1} P_j$.

When the powers fulfill the following conditions:

$(P_1)$ There exists a $\delta > 0$ such that card($\{j : P_j \in [\delta, 1/\delta]\}$) $\to \infty$,

$(P_2)$ $\lim_{k \to \infty} kP^{-1} < \infty$,

$(P_3)$ $k^{-1}P_{\max} \to 0$,

$(P_4)$ $kP_{\min} \to \infty$,

then

$$\lim_{k \to \infty} H_k^{(sk)} = \frac{1}{2} \log 2 - \frac{1}{4}.$$  \hspace{1cm} (2.13)

We note that in the case that all powers are equal all four conditions are fulfilled.

This result is quite remarkable. For all $s$ fixed, we observed that $\lim_{k \to \infty} H_k^{(s)} = 0$. When we take $s \geq 2^k + 1$, the rate is strictly positive for all $k$, independent of the powers! This guarantees that we can obtain an arbitrarily small BEP for any number of users. Under some power conditions, we can prove that the rate $\frac{1}{2} \log 2 - \frac{1}{4}$ is sharp, i.e., for $k$ sufficiently large, we can get arbitrary close to this value. The power conditions are sufficient for the result to hold, but not necessary. To prove (2.13), we use a symmetrizing argument, where we replace different powers by the same value. A more delicate argument might lead to the same result under weaker conditions.

It is easy to see that when there exists an $M < \infty$ such that $M^{-1} < P_m < M$ for all $m$, all four power conditions are fulfilled. Another example is the case that the powers are generated according to a lognormal distribution. For almost every realization of these powers, conditions $(P_1)$ – $(P_4)$ are fulfilled. Indeed, Condition $(P_3)$ and $(P_2)$ are fulfilled
and we prove \( k \min P_j \to \infty \) and \( k^{-1} \max P_j \to 0 \) a.s. We stress that the above is true for almost every realization of the powers. We do not average the BEP over the powers.

We have obtained upper bounds for \( H_k^{(s)} \) and the lower bound \( \frac{1}{2} \log 2 - \frac{1}{4} \). In Figure 2.6, these bounds are given. The upper bound is obtained by taking the minimum over all scenarios. We observe that the convergence to the limit is very slow. We expect the rate to converge to \( \frac{1}{2} \log 2 - \frac{1}{4} \) as \( k^{-1/2} \).

![Figure 2.6: Upper and lower bound for the exponential rates \( H_k^{(s)} \).](image)

The next step is to improve our intuition on the behaviour of \( s_k \). It is clear that \( s_k \to \infty \), since otherwise it is not possible to have a strictly positive rate (recall that for \( s \) fixed, \( H_k^{(s)} \to 0 \) for \( k \to \infty \)). We have already seen that \( s_k \leq 2^k + 1 \). However, the numerical results show that \( s_k = 1 \), \( s_k = 2 \) for \( 4 \leq k \leq 9 \) and \( s_k = 3 \) for \( 10 \leq k \leq 22 \). Thus, we expect that \( s_k \to \infty \) much slower than \( 2^k + 1 \). Below we provide intuition why \( s_k \sim \log k \).

First we show the following result, which has similarities with (2.12). We are able to prove for the system with unequal powers that for all \( 0 < \varepsilon < \frac{1}{2} \log 2 - \frac{1}{4} \) and for all \( s \) such that \( s \geq \lceil \varepsilon^{-1} \log \frac{P}{P_{\text{min}}} \rceil + 1 \),

\[
H_k^{(s)} \geq \frac{1}{2} \log 2 - \frac{1}{4} - \varepsilon.
\]

Thus, for the system with equal powers, we obtain a strictly positive rate after order \( \log k \) stages.

Finally, we extend the results for the optimal HD-PIC system to the case where \( k = k_n \) depends on \( n \), using Chernoff bounds. We show that for \( k_n = o(n) \) and \( s \geq 2^{k_n + 1} \)

\[
P(\text{sgn} \eta (Z_0^{(s)}(H)) < 0) \leq e^{-nI(1+o(1))}, \quad \text{where} \quad I = \frac{1}{2} \log 2 - \frac{1}{4}.
\]

When the powers are equal, the same result holds when \( s \geq k_n + 1 \). This shows that it is not necessary to take \( n \to \infty \) first and subsequently \( k \to \infty \).
In contrast with the optimal HD-PIC system, the optimal SD-PIC system has a completely different behaviour. The soft decisions prevent that identical decisions are made twice. Therefore, the multistage SD-PIC system converges to the optimal system (when it exists), rather than that the optimal system is attained after finitely many stages as for HD-PIC. Another important difference is monotonicity. The exponential rate of the multistage HD-PIC system increases to its limit. For the multistage SD-PIC system, it is not clear that this is true. In Elders-Boll, Schotten and Busboom (1998) and Guo, Rasmussen, Sun and Lim (2000), it is shown that the multistage SD-PIC receiver approximates the inverse of the cross-correlation matrix using Jacobi iteration. The optimal SD-PIC system is shown to be related to the smallest eigenvalue $\lambda_1$ and largest eigenvalue $\lambda_k$ of the cross correlation matrix with elements $\frac{1}{n} \sum_{i=1}^n X_{ji} X_{mi}$. When $\lambda_1 > 0$ and $\lambda_k < 2$, the optimal SD-PIC system exists and no bit error is made. It is remarkable that this result is independent of the powers! However, when $\lambda_1 = 0$ or $\lambda_k > 2$, it is possible that bit errors occur for some users. For $k = 3$, we will prove that $\lambda_3 > 2$ implies bit errors. We are able to prove that

$$J_3^{(\infty)} \leq \frac{5}{8} \log 5 - \log 2 = 0.31275 \ldots$$

Furthermore, numerical investigation indicates $J_3^{(\infty)} = \frac{5}{8} \log 5 - \log 2$. This result does not depend on the powers.

For general $k$, we are able to prove for the model with unequal powers that

$$J_k^{(\infty)} \geq \frac{1}{2} - \frac{1}{2} \log 2 = 0.15343 \ldots$$

Furthermore, when all powers are equal,

$$\lim_{k \to \infty} J_k^{(\infty)} = \frac{1}{2} - \frac{1}{2} \log 2.$$

Both the optimal HD-PIC and the SD-PIC system have a strictly positive rate, independent of the powers. The limiting rate of the optimal SD-PIC system is larger than that of the optimal HD-PIC system.

2.6 Second order asymptotics

We have shown that the exponential rate is a good measure of performance. However, the exponential rate is only part of the BEP; we do not expect $e^{-n \lambda_k}$ to be a good approximation for $P(\text{sgnr}_0(Z_0^{(1)}) < 0)$. In Chapter 8, we will focus on a more detailed analysis of the BEP.

According to Bahadur and Rao (1960),

$$P(\text{sgnr}_0(Z_0^{(1)}) < 0) = \frac{\alpha_{kn}}{n^{1/2}} e^{-n \lambda_k}(1 + o(1)).$$
where

\[ \alpha_{k,n} = \sqrt{\frac{k-1}{8\pi}} \left[ \left( \frac{k-2}{k} \right)^{\frac{n_k}{2}} - \frac{n_k}{2} - \frac{1}{2} \right] + \left( \frac{k-2}{k} \right)^{1+\frac{n_k}{2}} - \frac{n_k}{2} + \frac{1}{2} \right]. \]

The factor \( \alpha_{k,n} n^{-1/2} \) is the so-called second order asymptotic. Inclusion of the second order asymptotics leads to excellent approximations for the BEP in the range of interest.

For the HD-PIC model, we have not been able to derive the second order asymptotics. Instead, we give some with conjectures. For \( k = 3 \), we conjecture that

\[ P(\text{sgnr}_0(Z_0^{(2,H)}) < 0) = \frac{\beta_{3,n}}{\sqrt{n}} e^{-nH_3^{(2)}} (1 + o(1)). \]  

(2.15)

where \( \beta_{3,n} = \alpha_{3,n} \). When we use that \( H_3^{(2)} = I_3 \) (recall (2.9)), we conclude that for a system with 3 users, one stage of HD-PIC does not result in increase in performance, neither in first nor in second order. The reason is that the atypical event \( \{ \text{sgnr}_1(Z_1^{(1)}) < 0 \} \) makes the event \( \{ \text{sgnr}_0(Z_0^{(2,H)}) < 0 \} \) typical. The first event leads to the same exponential rate and second order asymptotics as for the MF model. The second event now only gives a factor 1/2. Together with the counting factor \( \binom{k-1}{r} = 2 \) (user 1 or user 2 can have a bit error at stage 1), this leads to the desired result.

For \( k \geq 4 \), we conjecture (recall the definition of \( r_k \) in (2.7))

\[
\lim_{n \to \infty} \min \inf n^{(r_k+1)/2} e^{nH_k^{(2)}} P(\text{sgnr}_0(Z_0^{(2,H)}) < 0) > 0 \quad \text{and} \quad \lim_{n \to \infty} \max \sup n^{(r_k+1)/2} e^{nH_k^{(2)}} P(\text{sgnr}_0(Z_0^{(2,H)}) < 0) < \infty.
\]

We do not expect that the \( \lim \inf \) equals the \( \lim \sup \), just as \( \lim \sup \beta_{3,n} > \lim \inf \beta_{3,n} \). Instead, we expect a wigging behaviour, similarly to the behaviour of \( \alpha_{k,n} \). We denote the asymptotics by \( \beta_{k,n} \), similarly to \( k = 3 \), where we introduced \( \beta_{3,n} \).

The reason for the wigging behaviour is that the HD-PIC model involves, similarly to the MF model, linear combinations of lattice random variables. Furthermore, the linear combinations have rational coefficients. The factor \( n^{(r_k+1)/2} \) can be explained as follows. When we consider the probability in (2.4), we see that it is the intersection of \( k+1 \) events. The events \( \{ \text{sgnr}_m(Z_m^{(1)}) \geq 0 \} \) for \( m = r_k, \ldots, k-1 \) do not contribute to the probability, neither in first order, nor in second order. This leaves us with \( r_k + 1 \) events, i.e. \( \{ \text{sgnr}_m(Z_m^{(1)}) < 0 \} \) for \( m = 1, \ldots, r \) and \( \{ \text{sgnr}_0(Z_0^{(2,H)}) < 0 \} \). Every event has contribution of order \( n^{-1/2} \) to the second order asymptotics, just like for the MF model, one event resulted in a factor \( n^{-1/2} \).

For the SD-PIC model, we have obtained full results for the second order asymptotics for \( k = 3 \) only. For \( k = 3 \), we are able to show that

\[ \lim_{n \to \infty} \sqrt{n} e^{nJ_3^{(2)}} P(\text{sgnr}_0(Z_0^{(2,S)}) < 0) = \gamma_3. \]

where \( \gamma_3 = 0.5946 \ldots \) and where \( J_3^{(2)} = 0.3094 \ldots \) We have obtained an analytical expression for \( \gamma_3 \) in terms of the minimizer of the optimization problem in (2.8). Unlike the results for
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the MF and the HD-PIC model, the second order asymptotics depend on \( n \) only through the factor \( n^{-1/2} \); the constant \( \gamma_3 \) does not depend on \( n \).

For \( k \geq 4 \), we have no results. For \( k \geq 4 \), the number of variables makes the techniques used for \( k = 3 \) unappropriate. Instead, we confine with a conjecture. We conjecture that

\[
\lim_{n \to \infty} \sqrt{n} e^{-n f_k^{(2)}} P(\text{sgn}_0(Z_0^{(2,3)}) < 0) = \gamma_k \quad \text{exists.}
\]

2.7 Importance sampling

Chapter 9 deals with simulation techniques and results. When analysis becomes too involved, Monte Carlo simulations can often contribute to the understanding of the characteristics of a model. A Monte Carlo simulation scheme is often straightforward to implement and enables to estimate BEP’s. A disadvantage of the method is the large number of required samples. Especially when rare events are involved (which is the kind of events we are interested in), this number can be extremely large. The technique to overcome this problem is called importance sampling (IS). In IS, the number of successes (read bit errors) is increased by sampling from a different distribution function. The data is then weighted by an appropriate weighting function and this leads to unbiased estimates of the BEP. Importance sampling schemes are more efficient than ordinary Monte Carlo simulations, especially when the BEP is small. A disadvantage is the increased complexity in bookkeeping. While for ordinary Monte Carlo sampling only the number of successes is important, for IS also the weighting factors should be taken into account. We focus on tilted distribution functions, which have a special relation with the original distribution function. More precisely, when \( F(x) \) is the original distribution function, the tilted distribution is given by

\[
\hat{F}_t(x) = \frac{1}{E e^{tZ}} \int_{(-\infty,x]} e^{ty} dF(y).
\]

The variable \( t \) can be optimized, so that the required number of samples is minimal. Tilted distribution functions are useful in case of exponentially small probabilities and are closely related to large deviation theory.

For the MF model, the IS procedure is standard. For the HD-PIC and the SD-PIC model, we describe the IS procedure using the empirical measure. Finally, we give a simplified procedure for the HD-PIC model. In all cases, the procedure heavily depends on the obtained large deviation results of Section 2.2. It is therefore crucial to perform the large deviation analysis, before turning to the IS procedure.

We will next give numerical results for \( k = 3, 6 \) and 9. We treat \( k = 3 \) separately, since we have obtained analytical results or good intuition on the second order asymptotics. Furthermore, for \( k = 3 \), we can use extensive computations to obtain exact results for the BEP. For \( k = 6 \) and 9, we will use simulation results.
In Figure 2.7, the exact BEP is shown for \( k = 3 \) for the MF, the HD-PIC and the SD-PIC model, together with the large deviation approximations

\[
\frac{\alpha_{3,n}}{n^{1/2}} e^{-n \bar{s}_3} \left( = \frac{\beta_{3,n}}{n^{1/2}} e^{-n \bar{s}_5^{(2)}} \right) \quad \text{and} \quad \frac{\gamma_{3}}{n^{1/2}} e^{-n \bar{s}_6^{(2)}}.
\]

The numerical results show that the probabilities \( P(\text{sgn}_{0}(Z_0^{(1)}) < 0) \) and \( P(\text{sgn}_{0}(Z_0^{(2,R)}) < 0) \)

are almost equal, so that we expect that the second order asymptotics \( \beta_{3,n} \) are the same as \( \alpha_{3,n} \) (recall (2.15)). For all \( n \geq 10 \), our approximations for the BEP are very accurate.

We now turn to \( k = 6 \) and 9. In Figure 2.8, the estimated BEP is shown. We see that both HD-PIC and SD-PIC significantly decrease the BEP if \( n \) is not too small. For 3 and 6 users, the BEP for the SD-PIC model is smaller than that of the HD-PIC model. This is because the exponential rate is higher for SD-PIC. For \( k = 9 \) however, HD-PIC performs better than SD-PIC. The reason is that the rates are almost the same, but the second order asymptotics of HD-PIC are better than those of SD-PIC. Indeed, we have conjectured for HD-PIC a second order asymptotic of order \( n^{-(r+1)/2} \), while for SD-PIC, we have \( n^{-1/2} \). Therefore, for many users and large \( n \), we expect that the BEP for the HD-PIC model is smaller than that of the SD-PIC model.

The results also show that the IS procedures require polynomially many samples as \( n \) grows.

We next investigate the second order asymptotics into more detail using the simulation results. We have performed 30,000 simulations to estimate \( P(\text{sgn}_{0}(Z_0^{(1)}) < 0) \). In Figure 2.9(a), \( P(\text{sgn}_{0}(Z_0^{(1)}) < 0) \) is shown for \( n = 1, \ldots, 100 \) for \( k = 3, 6 \) and 9, together with 95% confidence intervals. In many cases, the confidence intervals appear at thick lines, because
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Figure 2.8: Estimated BEP for MF (○ and ×), HD-PIC (△ and ※) and SD-PIC (○ and ※), for $k = 6$ and $9$, respectively.

Both the upper and the lower confidence bound are the same within printing precision. In Figure 2.9(a) also $α_{k,n}n^{-1/2}e^{-n/4}$ is shown for $k = 3, 6, 9$, which are slightly above the estimated BEP. The parity behaviour is clearly present for $k = 3$. For $k = 6$ and $9$, it is not very clear from the simulations whether there is any parity behaviour. The large deviation approximation is again very accurate. In Figure 2.9(b), $n^{1/2}e^{-n/4}P(\text{sgnr}_0(Z_0^{(1)}) < 0)$ is shown for $n = 1, \ldots, 100$ for $k = 3, 6$ and $9$, together with 95% confidence intervals. Also $α_{k,n}$ is shown. For $k = 3$, we see that $n^{1/2}e^{-n/4}P(\text{sgnr}_0(Z_0^{(1)}) < 0)$ is quite close to $α_{3,n}$. For $k = 6$ and $9$, the convergence is much slower.

The second order asymptotics for the HD-PIC model are more involved than those for the MF model, since we first have to write the BEP as

$$P(\text{sgnr}_0(Z_0^{(2,H)}) < 0) = \sum_{r=1}^{k-1} \binom{k-1}{r} P(\text{sgnr}_0(Z_0^{(2,H)}) < 0, B_r),$$

where

$$B_r = \left\{ \max_{1 \leq m \leq r} \text{sgnr}_m(Z_m^{(1)}) < 0, \min_{r+1 \leq m \leq k-1} \text{sgnr}_m(Z_m^{(1)}) > 0 \right\}.$$ 

We will abbreviate

$$p_r = P(\text{sgnr}_0(Z_0^{(2,H)}) < 0, B_r). \quad (2.16)$$

Let us investigate the BEP and $p_r$ for $k = 3, 6, 9$ and $r = 1, 2, 3$. For $k = 3$, exact results are known. For $k = 6, 9$, we will denote the estimated $p_r$ by $\hat{p}_r$. The contribution to the BEP
(a) $\mathbb{P}(\text{sgn}_{\text{r}}(Z_0^{(1)}) < 0)$ and $\alpha_{k,n} n^{-1/2} e^{-nH_k}$.

(b) $n^{1/2} e^{-nH_k} \mathbb{P}(\text{sgn}_{\text{r}}(Z_0^{(1)}) < 0)$ and $\alpha_{k,n}$.

Figure 2.9: Second order asymptotics with 95% confidence intervals and large deviation approximations for $k = 3$ (lower curves), $k = 6$ (center curves) and $9$ (upper curves) for the MF model.

for $r \geq 4$ is negligible. We can therefore use

$$\mathbb{P}(\text{sgn}_{\text{r}}(Z_0^{(2,H)}) < 0) \approx \binom{k-1}{1} p_1 + \binom{k-1}{2} p_2 + \binom{k-1}{3} p_3.$$ 

For every $r$, we have performed 50,000 simulations to estimate $p_r$. In Figure 2.10(a), we show $\mathbb{P}(\text{sgn}_{\text{r}}(Z_0^{(2,H)}) < 0)$ for $n = 1, \ldots, 100$ for $k = 3, 6$ and 9, together with 95% confidence intervals. Also $\hat{\beta}_{k,n} n^{-\delta_k} \exp(-nH_k^{(2)})$ is shown, where $\delta_k = (r_k + 1)/2$ and $\hat{\beta}_{3,n} = \beta_{3,n}$ and where $\hat{\beta}_{k,n}$ is an estimate for $\beta_{k,n}$ for $k = 6$ and 9. Since the simulations are not accurate enough to estimate parity effects, we confine with an average value, which we obtain by averaging $(k-1)^{n^{1/2}} \exp(nH_k^{(2)}) p_1$ for $n = 75 - 100$. In Figure 2.10(b), $n^{1/2} \exp(nH_k^{(2)}) \mathbb{P}(\text{sgn}_{\text{r}}(Z_0^{(2,H)}) < 0)$ is shown for $n = 1, \ldots, 100$ for $k = 3, 6$ and 9, together with 95% confidence intervals. Also $\hat{\beta}_{k,n}$ is shown. For $k = 3$ and 6, the estimate for $\beta_{k,n}$ seems to be plausible. For $k = 9$, $\hat{\beta}_{k,n}$ seems to be too low. However, a close examination reveals that $n \exp(nH_k^{(2)}) \mathbb{P}(\text{sgn}_{\text{r}}(Z_0^{(2,H)}) < 0)$ decreases slowly for $n \geq 70$, so that even for $k = 9$ the estimate $\beta_{k,n}$ can be correct.

For $n \to \infty$, only $p_{r_k}$ is relevant. However, for small $n$, $p_r$ for $r \neq r_k$ can also contribute. Therefore, we expect that the BEP tends much slower to the asymptotic probability

$$\beta_{k,n} n^{-(r_k+1)/2} e^{-nH_k^{(2)}}$$

than the BEP tends to its limit for the MF model. Another reason why we expect the convergence towards the asymptotic result to be much slower is the following. When all events for stage 1 and 2 would be independent, the probability becomes a product of $r + 1$ probabilities, each with a slow convergence. We expect that fact that the events are not independent does not influence the convergence speed. Therefore, the convergence of HD-PIC is expected to be much slower than that of MF.
Chapter 2. Results

(a) $\mathbb{P}(\text{sgnr}_0(Z_0^{(2,3)}) < 0)$ and $\hat{\beta}_{k,n} n^{-\delta_k} e^{-n\delta_k^{(2)}}$. (b) $n^{\delta_k} e^{n\delta_k^{(2)}} \mathbb{P}(\text{sgnr}_0(Z_0^{(2,3)}) < 0)$ and $\hat{\beta}_{k,n}$.

Figure 2.10: Second order asymptotics with 95% confidence intervals and large deviation approximations for $k = 3$ (lower curves), $k = 6$ (center curves) and $9$ (upper curves) for the HD-PIC model.

In Chapter 9, we will investigate $p_*$ in more detail.

For the SD-PIC model, we have performed 150,000 simulations to estimate $\mathbb{P}(\text{sgnr}_0(Z_0^{(2,3)}) < 0)$. In Figure 2.11(a), $\mathbb{P}(\text{sgnr}_0(Z_0^{(2,3)}) < 0)$ is shown for $n = 1, \ldots, 100$ for $k = 3, 6$ and 9, together with 95% confidence intervals. Also $\gamma_k n^{-1/2} \exp(-n\delta_k^{(2)})$ is shown, where $\gamma_k$ is the estimate for $\gamma_k$. We have obtained $\hat{\gamma}_k$ by averaging $n^{1/2} \exp(n\delta_k^{(2)}) \mathbb{P}(\text{sgnr}_0(Z_0^{(2,3)}) < 0)$ for $n = 75 - 100$. This resulted in $\hat{\gamma}_6 = 2.41$ and $\hat{\gamma}_9 = 10.75$. For $k = 3$, we have taken the theoretical value $\gamma_3 = 0.5946$. We see that the large deviation approximations are accurate. In Figure 2.11(b), $n^{1/2} \exp(n\delta_k^{(2)}) \mathbb{P}(\text{sgnr}_0(Z_0^{(2,3)}) < 0)$ is shown for $n = 1, \ldots, 100$ for $k = 3, 6$

(a) $\mathbb{P}(\text{sgnr}_0(Z_0^{(2,3)}) < 0)$ and $\gamma_k n^{-1/2} e^{-n\delta_k^{(2)}}$. (b) $n^{1/2} e^{n\delta_k^{(2)}} \mathbb{P}(\text{sgnr}_0(Z_0^{(2,3)}) < 0)$ and $\gamma_k$.

Figure 2.11: Second order asymptotics with 95% confidence intervals and large deviation approximations for $k = 3$ (lower curves), $k = 6$ (center curves) and $9$ (upper curves) for the SD-PIC model.
and 9, together with 95% confidence intervals. Also $\alpha_{k,n}$ is shown. For $k = 3$, we see that $n^{1/2} \exp(n J^{(3)}_k) \mathbb{P}(\text{sgn} r_0(Z_0^{(2,3)}) < 0)$ is quite close to $\alpha_{3,n}$. For $k = 6$ and 9, the convergence is slower, but for $n \geq 50$, $\gamma_k$ is very close to $n^{1/2} \exp(n J^{(6)}_k) \mathbb{P}(\text{sgn} r_0(Z_0^{(3,5)}) < 0)$.

## 2.8 Conclusions

In this thesis, we have investigated the behaviour of three CDMA receivers, i.e., the matched filter (MF) receiver, the hard-decision parallel interference cancellation (HD-PIC) receiver and the soft-decision parallel interference cancellation (SD-PIC) receiver. The latter two receivers are seen as the most promising advanced receivers for the uplink.

We have introduced the exponential rate as a measure of performance. Using this measure, the behaviour of the receivers have been investigated for various models. We have started with the simple model in which all powers are equal and no additive white Gaussian noise (AWGN) is present. This model allows for an extensive analysis and is characteristic for the behaviour in more realistic models. We have shown that the one-stage HD-PIC as well as the one-stage SD-PIC give a significant increase in performance, compared to the MF receiver. The multistage HD-PIC receiver is shown to keep increasing performance, as long as the number of users is sufficiently large. The potential of SD-PIC is slightly higher than HD-PIC, but the difference is small. Only when the number of users is small, SD-PIC performs significantly better than HD-PIC. Since the rates are almost equal when the number of users is not too small, the second order asymptotics are of importance. The HD-PIC model shows to have better second order asymptotics. Therefore, the improved estimation powers in HD-PIC does lead to smaller bit error probabilities, even though it is only in second order. We have investigated the three systems also in a more realistic setting, in which users have different powers and AWGN is present.

Another important contribution is the investigation of the PIC receivers, where infinitely many stages of cancellation are applied. We have shown for the multistage HD-PIC system in which no AWGN is present, that after finitely many stages (but depending on the number of users), say $s_k$, the increase in worst case performance terminates. We have been able to prove that the exponential rate after $s_k$ stages remains strictly positive for $k$ large, which guarantees a certain performance, independently of the number of users or their powers. Furthermore, we have given intuition on the asymptotic behaviour of $s_k$. For the multistage SD-PIC system, we show that infinitely many stages also leads to a guaranteed performance, not depending on the powers of the users. In this thesis, also the bit error probability (BEP) itself is investigated. Using importance sampling techniques estimates of the BEP have been obtained.

The results have both theoretical and practical importance. From a theoretical point of view, the results are interesting, since they provide insight in the potential increase in performance of PIC receivers. Especially the results for the infinite stage PIC receivers contribute to the understanding of the behaviour of such systems. The results are also of practical importance. Indeed, knowing the exponential rate allows insight how to set the processing gain in order
to obtain a desired quality level. The results for the infinite stage PIC receivers gives insight in the impact of multistage PIC receivers. This could lead to rules on the optimal number of stages, by balancing complexity and increase in performance.

The contributions of this thesis do not consist of the results described above only. An equally important contribution is the developed techniques. We have introduced a new technique to measure performance. This performance measure, the exponential rate, is based on the theory of large deviations, known in probability theory. We have provided tools to calculate the exponential rate for the MF, the HD-PIC and the SD-PIC system for various models. For a simple HD-PIC model, we have developed a technique to characterize the rate for large number of users. The technique is based on a Taylor series expansion of a moment generating function. In fact, we show that the exponential rate only depends on first and second moments, so that the technique gives a CLT-like result. For more efficiency, the series expansion is performed in a non standard manner; we treat quadratic terms differently from asymmetric terms. This technique also proved to be useful for the multistage HD-PIC receiver or the model in which unequal powers and additive white Gaussian noise are incorporated. To investigate the performance of the HD-PIC model, where infinitely many stages of cancellation are applied, a Taylor expansion turned out not to suffice. Instead, bounds on moment generating functions in terms of moment generating functions of Gaussian functionals are derived. These bounds, in some cases together with a Taylor expansion, have shown to be the right tools to deal with these systems. For the infinite stage SD-PIC model, we reformulated the problem in terms of eigenvalues of cross-correlation matrices. Using this formulation, we have developed techniques to investigate the large deviation properties of these eigenvalues. The tools described above can be extended to different models and different systems. Thus, this thesis may provide a toolbox to analyze performance of various CDMA models that are not treated in this thesis.

Finally, we have investigated the second order asymptotics of the bit error probability, which leads to accurate approximations. However, the exact analysis of the bit error probability turned out to be quite involved. Instead, we have focused on simulation techniques, based on importance sampling, to get insight in the behaviour of the BEP. The main contribution is the importance sampling procedure for the HD-PIC model.
Extensions and further research

In this thesis, we have investigated several models for interference cancellation in CDMA. We have solved many interesting problems, but as always there remain open problems. Also, many different models and systems have been proposed in the literature. This chapter is dedicated to some of these extensions.

3.1 Further research based on this thesis

Further research, directly based upon this thesis, could focus on the following points.

- An asymptotic expression for the one-stage SD-PIC model. Least squares analysis indicated that $J_k^{(2)} \approx \frac{1}{2\sqrt{k}}$.

- The multistage SD-PIC model. In this thesis, we have not treated multistage SD-PIC, except for $s = \infty$. Behaviour in the direction $k$ large or $s$ large is interesting. Unlike the HD-PIC model, it is not likely that the rate for SD-PIC is monotone in $s$. The convergence to the inverse of a matrix suggests that the behaviour for $s$ odd is different from $s$ even.

- Dependence of $k$ and $n$. We have shown that many of the HD-PIC results extend to the case where $k = k_n$ depends on $n$. Often large deviation results with $k$ fixed and $n$ large give good insight in the behaviour of the system for both $k$ and $n$ large. Of special interest is the behaviour of the different models for $k = \beta n$ for $\beta \in (0,1)$ fixed.

- Extension of the optimal system to channels with AWGN. For both HD-PIC and SD-PIC, this is very interesting.
Further investigation of the second order asymptotics. Analytical results concerning
the second order asymptotics have shown to be difficult to obtain. Only for the MF
model and for the SD-PIC model with \( k = 3 \), results have been obtained. Instead,
we have focused on simulation techniques. While the exponential rate is quite robust
against perturbations of the model, the second order asymptotics are very sensitive.
Therefore, the extension of the simulation techniques to other models is likely to suc-
cceed, while extending the analytical results is, due to the complexity of the proofs,
doomed to fail. The complexity of the simulation techniques should be investigated.
Preferably, the required number of flops\(^1\) should be investigated. In this light, also
complexity reducing techniques are of interest. Using product measures as an approx-
ation for the (non-product) empirical measure may, compared to traditional Monte
Carlo sampling, increase efficiency drastically, without increasing sampling complexity
too much. For example, for the one-stage HD-PIC model, it is implicitly proven that
the measure of the first \( r \) elements of \( X_{m} \) converges to a product measure. Using this
reduces sampling complexity at the cost of a remaining exponentially many number of
required samples (with much smaller rate though).

3.2 Power models

Under the coarse power control condition, different users may have different, but fixed pow-
ers. However, wireless communication channels are unpredictable. Therefore, other propa-
gation models have been proposed. In fact, three basic models exist, cf. Groe and Larson
(2000), Section 1.4.

The first is the realistic model treated in this thesis. In this model, it is assumed that
different users have different, but fixed power and AWGN is present. This is also known as
the line-of-sight model, where line-of-sight means that there are no obstacles between the
transmitter and the receiver. The second model incorporates effects caused by reflections.
As seen in Figure 3.1, the reflections can seriously complicate the task of the receiver.
Sometimes a line-of-sight is present, so that at least one ray with substantial amplitude
is received. However, when no line-of-sight is present, the signal is only received through
reflections. When enough reflections are present, but no line-of-sight, the received power is
modeled by a Rayleigh distribution.\(^2\) Finally, the third model is a generalization of both
previous models. When a dominant line-of-sight ray and enough reflections are present, the
received power is modeled by a Rician distribution.\(^3\)

Another used model assumption involves the log-normal distribution\(^4\), cf. Simon, Omura,
Scholtz and Levitt (1994), pp.1198. When a signal is assumed to have many independent

\(^1\)Floating point operations.

\(^2\)When \( W_1, W_2 \sim N(0, \sigma^2) \) and independent, \( (W_1^2 + W_2^2)^{1/2} \) is Rayleigh distributed.

\(^3\)When \( W_1 \sim N(\mu, \sigma^2) \) and \( W_2 \sim N(0, \sigma^2) \) with \( W_1 \) and \( W_2 \) independent, \( (W_1^2 + W_2^2)^{1/2} \) is Rician
distributed. When \( \sigma^2 = 0 \), the line-of-sight model is obtained. When \( \mu = 0 \), the Rayleigh distribution arises.

\(^4\)When \( Z \sim N(0, \sigma^2) \), \( e^Z \) is log-normal distributed.
amplitude variations, with the resultant signal amplitude a product of many random terms, the log-normal distribution follows from the CLT. Long radio paths with many reflections and focussed antenna beams show this behaviour. For indoor applications, the Rayleigh and Rician model is more commonly used.

One crucial step when using random powers, is to fit the model into the large deviation framework. Let us investigate a simple example, where we take \( k = 2, P_1 = 1, \sigma^2 = 0 \) and \( P_0 \) a positive continuous random variable, independent of the vector \( X_{11}, \ldots, X_{1n} \). Then

\[
Z_{0}^{(i)} = P_0^{1/2} + \frac{1}{n} \sum_{i=1}^{n} X_{1i},
\]

where \( X_{1i} \) are i.i.d. with \( \mathbb{P}(X_{11} = 1) = \mathbb{P}(X_{11} = -1) = 1/2 \). A simple inclusion/exclusion argument yields for every \( \varepsilon > 0 \),

\[
\mathbb{P}(Z_{0}^{(i)} \leq 0) = \mathbb{P}(Z_{0}^{(i)} \leq 0|P_0 \leq \varepsilon)\mathbb{P}(P_0 \leq \varepsilon) + \mathbb{P}(Z_{0}^{(i)} \leq 0|P_0 > \varepsilon)\mathbb{P}(P_0 > \varepsilon)
\geq \mathbb{P}\left(\varepsilon^{1/2} + \frac{1}{n} \sum_{i=1}^{n} X_{1i} \leq 0\right)\mathbb{P}(P_0 \leq \varepsilon).
\]

Hence

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Z_{0}^{(i)} \leq 0) \leq - \lim_{n \to \infty} \frac{1}{n} \log \left(\mathbb{P}\left(\varepsilon^{1/2} + \frac{1}{n} \sum_{i=1}^{n} X_{1i} \leq 0\right)\mathbb{P}(P_0 \leq \varepsilon)\right).
\]

When \( P_0 \) does not depend on \( n \) and \( \mathbb{P}(P_0 \leq \varepsilon) > 0 \) for all \( \varepsilon \), it follows that

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(P_0 \leq \varepsilon) = 0.
\]

The desired rate then equals \( I(\varepsilon^{1/2}) \) (recall (2.1)). Since we can choose \( \varepsilon \) arbitrary small and \( I \) is continuous with \( I(0) = 0 \), we have that

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Z_{0}^{(i)} \leq 0) = 0.
\]
Chapter 3. Extensions and further research

A better approach that keeps us in the realms of the large deviation setting involves dependence of $P_0$ on $n$. In this case, the rates

$$-\lim_{n \to \infty} \frac{1}{n} \log P \left( \varepsilon^{1/2} + \frac{1}{n} \sum_{i=1}^{n} X_{ni} \leq 0 \right) \quad \text{and} \quad -\lim_{n \to \infty} \frac{1}{n} \log P(P_0 \leq \varepsilon)$$

are both positive. Minimization of the sum of the two rates over $\varepsilon$ identifies the rate of the BEP. This follows from Varadhan’s lemma (cf. Den Hollander (2000), Thm. III.13). The intuitive explanation why this approach works is based on two principles. Firstly, the probability of having a bit error is decreasing in $P_0$; indeed, a large power implies a small BEP. Thus, a small $P_0$ leads to a high BEP. Secondly, making $P_0$ small has a small probability. Intersection of the two events therefore involves taking $P_0$, so that the increase in error probability (due to small $P_0$) and decrease in probability (because small $P_0$ have small probability) are in balance. Using this approach, the typical behaviour of $P_0$ is thus identified.

The proposed setup is therefore as follows. For the Rician model, we take for $j = 0, \ldots, k - 1$ the random variables $W_{j1} \sim \mathcal{N}(\mu_j, \sigma^2/n)$ and $W_{j2} \sim \mathcal{N}(0, \sigma^2/n)$, where $W_{jl}$, $l = 1, 2$ are mutually independent, and independent of all other randomness in the system. Take $P_0, \ldots, P_{k-1}$, which now have a dependence on $n$, as $P_j = (W_{j1}^2 + W_{j2}^2)^{1/2}$. For the log-normal model, take $P_j = e^{W_{j1}}$.

### 3.3 Asynchronous transmission

Throughout this thesis we have assumed that all users are synchronized, so that for the total received signal $r(t)$ (recall (1.1) and (1.2)),

$$r(t) = \sum_{m=0}^{k-1} (2P_m)^{1/2} b_m(t)a_m(t) \cos(\omega_c t) + \eta n(t).$$

It is also possible that signals of users arrive at different times. In this case

$$r(t) = \sum_{m=0}^{k-1} (2P_m)^{1/2} b_m(t - \tau_m)a_m(t - \tau_m) \cos(\omega_c t + \theta_m) + \eta n(t),$$

where $\theta_m = -\omega_c \tau_m$. It is reasonable to assume that $\tau_0, \ldots, \tau_{k-1}$ are i.i.d. (and independently from all other randomness in the model) with $\tau_0 \sim \mathcal{U}(0,T)$ (see Buehrer and Woerner (1996), Buehrer, Kaul, Stelcukis and Woerner (1996), Morrow and Lernert (1989) and Sadovsky and Bahr (1991)). When $\omega_c$ is large (which is the case), it is also common to take $\theta_0, \ldots, \theta_{k-1}$ i.i.d. with $\theta_0 \sim \mathcal{U}(0,2\pi)$, independently from $\tau_0, \ldots, \tau_{k-1}$ and all other randomness in the model.

All UMTS/IMT-2000 proposals suggest asynchronous uplink transmission, so that it is crucial to include asynchronous transmission. The asynchronous transmission has two major
consequences on the system. First of all, the coding sequences should be chosen in such way that the cross correlation properties are not ruined in the asynchronous scenario. Orthogonal sequences are clearly not suitable, because orthogonality is not preserved under shifting. In practice, the cross correlations become slightly worse for well chosen sequences. However, in our model, where we assume random sequences, independent of \((\tau_m)_{m=0}^{k-1}\) and \((\theta_m)_{m=0}^{k-1}\), the cross correlations are smaller in the asynchronous case. For example,

\[
\text{Var}(X_{01}, X_{11}) = 1,
\]

while (for \(\tau \sim \mathcal{U}(0, 1)\) and \(\theta \sim \mathcal{U}(0, 2\pi)\))

\[
\text{Var}(X_{01}(\tau X_{11} + (1 - \tau) X_{12}) \cos \theta) = E(\tau^2 + (1 - \tau)^2)(\cos \theta)^2 = 1/3. \tag{3.1}
\]

The second consequence is that in an asynchronous system, we have to estimate \(\tau_m\) and \(\theta_m\). Bad estimates of the channel parameters lead to bad performance. Below we assume that \(\tau_m\) and \(\theta_m\) can be estimated with infinite precision.

Often in large deviation problems, the exponential rate is robust against small perturbations of the system. We expect that asynchronicity leads to a small perturbation, so that the rate does not change for the asynchronous model. The idea behind this is that a rare event is typically caused by a worst case scenario. As seen in (3.1), the worst case is when \(\tau\) is close to either 0 or 1, while \(\theta\) is close to 0, \(\pi\) or \(2\pi\), since this maximizes the variance. To prove this, some simple bounds on the set of possible asynchronous scenarios should be given. This will typically lead to factors of order \(n^k\), which are subexponential.

The second order asymptotics do become different. In Sadowsky and Bahr (1991), results are shown for an asynchronous MF system. Those results imply that the second order asymptotics for MF are of the form

\[
P(\text{sgn} r_0(Z_0^{(1)}) < 0) = \frac{\alpha_k}{n^{(3(k-1)+1)/2}} \exp(-n I_k(1 + o(1))),
\]

for some constant \(\alpha_k\). This means that every interfering user gives rise to a factor \(n^{-3/2}\). Furthermore, the second order asymptotics do not have a wiggling behaviour.

For the HD-PIC system, we conjecture that the BEP is of the form

\[
\frac{\beta_k}{n^{(3k+3r_k+1)/2}} \exp(-n H_{k,r_k}^{(2)}(1 + o(1)));
\]

We expect no wiggling behaviour, which explains why \(\beta_k\) only depends on \(k\). We conjecture that the BEP for the SD-PIC model has similar second order asymptotics as the MF system, i.e., of the order \(n^{-(3(k-1)+1)/2}\) and no wiggling behaviour.
3.4 Related interference cancellation systems

In this thesis we have discussed two types of parallel interference cancellation systems, namely the hard-decision and the soft-decision PIC systems. In Figure 3.2 (a), (b), the HD- and SD-PIC decisions are shown. In the literature also other decision schemes have been proposed.

Particularly interesting schemes are the so-called partial schemes, Dивсалар, Simon and Raphaeli (1998). Instead of subtracting the whole estimated signal from the total received signal, a fraction $\alpha$ is subtracted. In other words, instead of estimating $\hat{s}_m(t)$ by $r(t) - \hat{r}_m(t)$ (recall (1.6)) we consider

$$r(t) - \alpha \hat{r}_m(t), \quad 0 \leq \alpha \leq 1,$$

see Figure 3.2 (d)-(e). The effect is twofold. On the one hand, it seems reasonable not to try to cancel the whole interference, since $\hat{s}_m(t)$ is only a tentative decision. On the other hand, this reduces the bias in the second stage. For the multistage SD-PIC, partial PIC provides balancing of the inverse matrix approximation scheme, so that the approximation converges to the inverse under weaker assumptions, cf. Guo, Rasmussen, Sun and Lim (2000). According to Dивсалар, Simon and Raphaeli (1998), “A partial interference cancellation philosophy ... is in general superior to a brute force philosophy of entirely cancelling the interference at each stage”. In Buehrer (1999), simulation results show that in the case of imperfect parameter estimation, the HD-PIC is unstable. The partial HD-PIC then can improve upon stability.

Other ways to weaken the brute force approach of HD-PIC are the null-zone hard-decision (Figure 3.2(c)) and a minimum mean square error (MMSE) based decision, which turns out to involve a hyperbolic tangent function (Figure 3.2(f)), see Dивсалар, Simon and Raphaeli (1998). By scaling the decision function per user, depending on the expected reliability of the estimate, these techniques can be further optimized.

In contrast to parallel IC, also serial IC (SIC) schemes have been proposed, for example Patel and Holtzman (1994), Johansson (1998), Rasmussen, Lim and Johansson (2000). In SIC, the users are ordered, according to their power. First, the signal of the user with the highest power is estimated and subtracted from the total received signal. Next, the signal of the user with the one but largest power is estimated and subtracted, etc. An advantage of this system is the minimal amount of additional hardware (significantly less than PIC) and the fact that the SIC system performs better than the PIC system in the case that users have different powers. This is because for HD-PIC, in the case the powers are essentially different, the user with the largest power suffers hardly from the relatively small interference, so that its bit is estimated correctly with high probability. The other users then do not suffer interference from that user any more. In the PIC system, the cancellation is performed parallel, so that the system does not benefit from the power configuration. In a well-power-controlled system, however, PIC performs better (cf. Moshavi (1996)). Furthermore, there is a significant delay in the estimation scheme, because all computations are performed serial. Moshavi (1996), Juntti (1998). Finally, a good sorting algorithm should be present, since for every change in the SIC system, the powers have to be resorted (Moshavi (1996), Juntti (1998)), which is a problem especially in environments where powers are rapidly changing.
3.5 Other advanced third generation receivers

Excellent overviews of related CDMA systems have been given in Moshavi (1996) and Buehrer, Nicooso and Gollamudi (1999). Three systems are of special interest. The first receiver is the maximum likelihood detector (Verdú (1986)), which gives (binary valued) maximum likelihood estimates of the bits. The maximum likelihood sequence detector is implemented using a dynamic programming algorithm (e.g. Viterbi algorithm). The technique is unfortunately too complex to be performed real-time. A second interesting receiver is the decorrelator (Lupas and Verdú (1989)), which multiplies the received decoded signals by the inverse of $\frac{1}{n}XX^T$, so that uncorrelated signals are obtained. In absence of AWGN, this technique only fails when $XX^T$ is singular.\textsuperscript{5} However, when AWGN is present, the decorrelator enhances this noise with a factor of at most 1 over the smallest eigenvalue. When the

\textsuperscript{5}This problem is solved by the introduction of generalized inverses, for example the unique Moore-Penrose generalized inverse, see Lupas and Verdú (1989).
3.6 The rate as a measure of near-far resistance

It is well-known that the performance of the MF CDMA receiver is degraded severely when the power of some of the interfering users becomes large. This problem is known as the near-far problem. This problem is not inherent to CDMA systems. Rather, it is the inability of the MF receiver to exploit the structure of the interference. Near-far resistant receivers have a guaranteed performance level, regardless how powerful the interference is. The definition of near-far resistance requires that the bit error probability tends to zero as the AWGN level $\sigma$ tends to zero. More precisely, one compares the BEP of the system in question with a single-user system where only AWGN is present. In all the systems under investigation, i.e., the MF, the HD-PIC and SD-PIC system, the BEP does not tend to zero for $\sigma \to 0$. Therefore, according to the strict definition of near-far resistance, none of the systems described in this thesis are near-far resistant. However, we have seen that systems with one- or multistage HD-PIC have a better performance in the limiting case $P_0/P \to 0$. Furthermore, optimal HD-PIC even has a strictly positive rate.

We therefore propose two definitions, which characterizes resistance against can be used as an alternative for near-far resistance. For the model in which no AWGN is present, we define the weak near-far sensitivity as

$$\lim_{k \to \infty} \inf_{P_0, \ldots, P_{k-1}} \frac{\log \left( - \lim_{n \to \infty} \frac{1}{n} \log P(\text{sgn} r_0(Z_0^{(s)}) < 0) \right)}{\log (P_0/P)},$$

i.e., it is the power of $P_0/P$ in the rate. For $s = 1$, the weak near-far sensitivity equals 1. For $(s - 1)$-stage HD-PIC, we have proven that it equals $1/s$. For one-stage SD-PIC, we have conjectured that the weak near-far sensitivity equals $1/2$. The optimal HD- and SD-PIC models have near-far sensitivity 0. In this case, we can turn to the strong near-far sensitivity, which is simply defined as

$$\lim_{k \to \infty} \inf_{P_0, \ldots, P_{k-1}} \frac{1}{n \to \infty} \frac{1}{n} \log P(\text{sgn} r_0(Z_0^{(s)}) < 0).$$

In other words, it is the worst case rate over all possible power scenarios. We can extend these definitions to models with AWGN, by investigating the near-far sensitivity only for $\sigma \to 0$.

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6See Verdú (1986).
Part II

Results and proofs
Exponential rate for the simple model

4.1 Introduction

In this chapter, we will focus on the exponential rate for the simple model, i.e., the system where \( P_0 = P_1 = \ldots = P_{k-1} \) and \( \sigma^2 = 0 \).

We will start with the exponential rate for the special cases \( k = 1 \) and \( k = 2 \). It turns out that the analysis is simple. In the remainder of this chapter, we therefore assume that \( k \geq 3 \). In Section 4.3, the exponential rate for the MF system, denoted by \( I_k \), is given. Then, in Section 4.4, the rate for the HD-PIC system, denoted by \( H_k^{(2)} \), is identified. Section 4.5 deals with the SD-PIC system. The rate for this system is denoted by \( J_k^{(2)} \). When the rates for the three systems have been characterized, we continue with a comparison of MF, HD-PIC and SD-PIC in Section 4.6. Finally, in Section 4.7, numerical results are given. Results in Section 4.3 and 4.5 (MF and SD-PIC) have appeared in van der Hofstad, Hooghiemstra and Klok (2002), Klok, Hooghiemstra, van der Hofstad, Ojanperä and Prasad (1999) and Klok, Hooghiemstra and van der Hofstad (2002). Results for HD-PIC have appeared in van der Hofstad and Klok (2002HD).

4.2 The cases \( k = 1 \) and \( k = 2 \)

For \( k = 1 \), the analysis is trivial. Indeed, \( Z_0^{(1)} = 1 \), so that \( I_1 = H_1^{(2)} = J_1^{(2)} = \infty \). For \( k = 2 \), the analysis is also simple, since in this case \( Z_0^{(1)} = 1 + \frac{1}{n} \sum_{i=1}^{n} X_{1i} \), so that

\[
\mathbb{P}(Z_0^{(1)} = 0) = \left( \frac{1}{2} \right)^n.
\]
and \( Z_0^{(1)} > 0 \) otherwise. Therefore, the event \( \{ \text{sgnr}_0(Z_0^{(1)}) < 0 \} \) involves only one atom. Therefore \( \mathbb{P}(\text{sgnr}_0(Z_0^{(1)}) < 0) = (1/2)^{n+1} \).

For the HD-PIC system, the case \( Z_0^{(1)} > 0 \) implies \( Z_1^{(1)} > 0 \) and all bits are estimated correctly. However, when \( \frac{1}{n} \sum_{i=1}^n X_{1i} = -1 \), the following happens:

\[
\begin{align*}
Z_0^{(1)} = 0 & \quad \text{prob. } \frac{1}{2} \quad \text{sgnr}_0(Z_0^{(1)}) < 0 \quad Z_1^{(1)} = 0 \\
\text{sgnr}_0(Z_0^{(1)}) > 0 & \quad \text{prob. } \frac{1}{2} \quad Z_0^{(2,H)} = -1 \quad Z_1^{(2,H)} = +1 \\
& \quad \downarrow \quad \downarrow \\
& \quad Z_0^{(3,H)} = -1 \quad Z_1^{(3,H)} = +1 \\
& \quad \downarrow \quad \downarrow \\
& \quad Z_0^{(4,H)} = -1 \quad Z_1^{(4,H)} = +1.
\end{align*}
\]

For \( s \) odd, a bit error occurs with probability \( (1/2)^{n+1} \) (first column), and for \( s \) even, a bit error also occurs with probability \( (1/2)^{n+1} \) (third column).

Finally, observe that for the SD-PIC system with two users,

\[
Z_0^{(s,s)} = 1 - \left( -\frac{1}{n} \sum_{i=1}^n X_{1i} \right)^s. \tag{4.1}
\]

Therefore, for \( s \) odd, a bit error occurs with probability \( 1/2 \) when \( \frac{1}{n} \sum_{i=1}^n X_{1i} = -1 \). For \( s \) even, a bit error occurs with probability \( 1/2 \) when \( \frac{1}{n} \sum_{i=1}^n X_{1i} = \pm 1 \). Thus, we have that \( \mathbb{P}(\text{sgnr}_0(Z_0^{(s,s)}) < 0) = (1/2)^{n+1} \delta_{s, \text{odd}} \).

We conclude that for 2 users, both HD and SD-PIC does not give an improvement over MF. In fact, an odd number of SD cancellations (in which case \( s \) is even) increases the probability of a bit error.

### 4.3 Exponential rate for MF

The rate for the MF system is easy to determine.

**Proposition 4.1** For \( k \geq 3 \),

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Z_0^{(1)} \leq 0) = I_k,
\]

where

\[
I_k = \frac{k - 2}{2} \log \left( \frac{k - 2}{k - 1} \right) + \frac{k}{2} \log \left( \frac{k}{k - 1} \right).
\]
4.1 Exponential rate for 1-stage HD-PIC

**Proof.** Let \( Y_1, \ldots, Y_m \) be a sequence of i.i.d. random variables with \( \mathbb{P}(Y_1 = +1) = \mathbb{P}(Y_1 = -1) = 1/2 \) and let \( S_m = \sum_{i=1}^{m} Y_i \). According to Cramér's theorem, we have for \( 0 < a < 1 \)

\[
- \lim_{m \to \infty} \frac{1}{m} \log \mathbb{P}(S_m \geq am) = \inf_{t \geq 0} \{at - \log \phi(t)\},
\]

where \( \phi(t) = \mathbb{E} e^{tY_1} < \infty \), for \( t \in \mathbb{R} \). In this particular case, we have \( \phi(t) = \cosh t \). A straightforward calculation gives that the minimal value is attained at \( t^* = \frac{1}{2} \log \frac{1+a}{1-a} \) and therefore

\[
\inf_{t \geq 0} [at - \log \phi(t)] = at^* - \log \cosh t^* = \frac{1+a}{2} \log(1+a) + \frac{1-a}{2} \log(1-a). \tag{4.2}
\]

To obtain the desired result, we have to realize that \( m = (k-1)n \) and

\[
\mathbb{P}(Z_0^{(k)} \leq 0) = \mathbb{P} \left( \frac{1}{m} \sum_{i=1}^{m} Y_i \geq \frac{1}{k-1} \right).
\]

Therefore substitution of \( a = 1/(k-1) \) in

\[
(k-1) \left( \frac{1+a}{2} \log(1+a) + \frac{1-a}{2} \log(1-a) \right)
\]

gives the desired result.

\[\square\]

4.4 Exponential rate for 1-stage HD-PIC

In this section we calculate the exponential rate for the system where 1 stage of HD-PIC is applied.

**Theorem 4.2** For \( k \geq 3 \),

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Z_{0}^{(2,n)} \leq 0) = H_k^{(2)} \tag{4.3}
\]

and

\[
H_k^{(2)} = \min_{1 \leq r \leq k-1} H_{k,r}^{(2)} \quad \text{where} \quad H_{k,r}^{(2)} = \sup_{t \in (-\infty,0]^2} \{- \log h_{k,r}(t)\},
\]

with

\[
h_{k,r}(t) = 2^{-r} \sum_{j \text{ even}} \binom{r}{j} \left( \frac{r}{r+2} \right)^j e^{t_1(j+j^2)+t_2(1+2j)(\cosh t_1 j)} (\cosh t_1 j)^{k-r-1}. \tag{4.4}
\]

For all \( r \) the function \( t \mapsto h_{k,r}(t) \) is log-convex.\(^1\)

\(^1\)A function \( h(t) \) is log-convex if \( \log(h(t)) \) is convex. Log-convexity implies convexity.
Chapter 4. Exponential rate for the simple model

Remark: We will suppress the indices $k, r$ in $h_{k,r}(t)$.

Proof. Clearly $1 - \text{sgn}_{m}()$ is either 0 or 2. Thus, user $m$ contributes to $Z_{0}^{(2,H)}$ if and only if $\text{sgn}_{m}(Z_{m}^{(1)}) < 0$, see (1.11) with $h_{j}(x) = P_{j}^{1/2} \text{sgn}_{j}(x)$. Since all users are exchangeable, we write

$$
P(Z_{0}^{(2,H)} \leq 0) = \sum_{r=1}^{k-1} \binom{k-1}{r} \mathbb{P}(Z_{0}^{(2,H)} \leq 0, B_{r}),$$

where

$$B_{r} = \left\{ \text{max}_{1 \leq m \leq r} \text{sgn}_{m}(Z_{m}^{(1)}) < 0, \text{min}_{r+1 \leq m \leq k-1} \text{sgn}_{m}(Z_{m}^{(1)}) > 0 \right\}.$$ 

One verifies from (1.5) that

$$\mathbb{P}(Z_{m} < 0, \ldots) \leq \mathbb{P}(\text{sgn}_{m}(Z_{m}) < 0, \ldots) \leq \mathbb{P}(Z_{m} \leq 0, \ldots)$$

so that

$$\sum_{r=1}^{k-1} \binom{k-1}{r} \mathbb{P}(\text{max}_{1 \leq m \leq r} Z_{m}^{(1)} < 0, \text{min}_{r+1 \leq m \leq k-1} Z_{m}^{(1)} > 0, 1 + \frac{2}{n} \sum_{i=1}^{n} \sum_{j=1}^{r} X_{ji} < 0)$$

$$\leq \mathbb{P}(Z_{0}^{(2,H)} \leq 0)$$

$$\leq \sum_{r=1}^{k-1} \binom{k-1}{r} \mathbb{P}(\text{max}_{1 \leq m \leq r} Z_{m}^{(1)} \leq 0, \text{min}_{r+1 \leq m \leq k-1} Z_{m}^{(1)} \geq 0, 1 + \frac{2}{n} \sum_{i=1}^{n} \sum_{j=1}^{r} X_{ji} \leq 0).$$

(4.6)

Subsequently, we will denote $Z_{0}^{(2,H)} = 1 + \frac{2}{n} \sum_{i=1}^{n} \sum_{m=1}^{r} X_{mi}$. The bar denotes that we have knowledge of stage 1 and that we have substituted the values of the sgnr-functions. For $k \geq 3$, the exponential rate of the left-hand side of (4.6) equals the exponential rate of the right-hand side of (4.6).

It suffices to investigate

$$\sum_{r=1}^{k-1} \binom{k-1}{r} \mathbb{P}(\text{max}_{1 \leq m \leq r} Z_{m}^{(1)} \leq 0, \text{min}_{r+1 \leq m \leq k-1} Z_{m}^{(1)} \geq 0, Z_{0}^{(2,H)} \leq 0).$$

We note that $k$ is fixed, so that $\lim_{n \to \infty} \frac{1}{n} \log \binom{k-1}{r} = 0$. We next apply the "largest-exponent-wins" principle on the bounds in (4.6) and find

$$- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Z_{0}^{(2,H)} \leq 0) = \min_{1 \leq r \leq k-1} H_{k,r}^{(2)},$$

where

$$H_{k,r}^{(2)} = - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\text{max}_{1 \leq m \leq r} Z_{m}^{(1)} \leq 0, \text{min}_{r+1 \leq m \leq k-1} Z_{m}^{(1)} \geq 0, Z_{0}^{(2,H)} \leq 0).$$

(4.7)

In order to show existence of $H_{k,r}^{(2)}$ and to be able to simplify this expression, we introduce for $a \in \mathbb{R}^{r}$ and $b \in \mathbb{R}^{k-r-1}$ the rate function

$$I_{r}(a, b) = - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( \bigcap_{m=1}^{r} \{Z_{m}^{(1)} \leq a_{m}\} \cap \bigcap_{m=r+1}^{k-1} \{Z_{m}^{(1)} \geq b_{m}\} \cap \{Z_{0}^{(2,H)} \leq 0\} \right).$$

(4.8)
Clearly \( H_{k,r} = I_r(0,0) \), where \( 0 = [0, \ldots, 0]^T \).

Firstly, Cramér's theorem guarantees existence of this rate, cf. DEMBO AND ZEITOUNI (1998), Thm. 2.2.30. This is based on the i.i.d. structure and a finite moment generating function. Secondly, \( (a, b) \mapsto I_r(a, b) \) is convex, cf. DEN HOLLANDER (2000), Thm. III.27 and monotone. A third useful fact is exchangeability. Suppose \( \pi_1(a) \) is a permutation of the elements of \( a \) and \( \pi_2(b) \) is a permutation of the elements of \( b \). Then \( I_r(a, b) = I_r(\pi_1(a), \pi_2(b)) \) since within a group, the users behave identically.

We next prove that

\[
I_r(0,0) = - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \sum_{m=1}^{r} Z_m^{(1)} \leq 0, \sum_{m=r+1}^{k-1} Z_m^{(1)} \geq 0, Z_{0}^{(2,n)} \leq 0 \right) \quad (4.9)
\]

Indeed, we have that the right-hand side of (4.9), shortly denoted by RHS equals

\[
\min_{\sum a_i \leq 0, \sum b_i \geq 0} I_r(a, b)
\]

and thus, RHS \( \leq I_r(0,0) \). To prove the converse we use convexity and exchangeability. Denote the argmin by \( a^* \) and \( b^* \) and denote by \( \mathcal{P} \) the set of all permutations \( (\pi_1, \pi_2) \). Then, by exchangeability and convexity respectively

\[
I_r(\pi_1(a^*), \pi_2(b^*)) = \frac{1}{\lvert \mathcal{P} \rvert} \sum_{(\pi_1, \pi_2) \in \mathcal{P}} I_r(\pi_1(a^*), \pi_2(b^*)) \geq I_r \left( \frac{1}{\lvert \mathcal{P} \rvert} \sum_{(\pi_1, \pi_2) \in \mathcal{P}} \pi_1(a^*), \frac{1}{\lvert \mathcal{P} \rvert} \sum_{(\pi_1, \pi_2) \in \mathcal{P}} \pi_2(b^*) \right) \quad (4.10)
\]

Since \( \mathcal{P} \) is the set of all permutations, it is clear that

\[
\frac{1}{\lvert \mathcal{P} \rvert} \sum_{(\pi_1, \pi_2) \in \mathcal{P}} \pi_1(a^*) = \left( \sum a_i^* \right) [1, \ldots, 1]^T
\]

and the same obviously holds for \( b^* \). By monotonicity of \( I_r \) and the fact that \( \sum a_i^* \leq 0 \) and \( \sum b_i^* \geq 0 \), we have that the right-hand side of (4.10) equals

\[
I_r \left( \left( \sum a_i^* \right) [1, \ldots, 1]^T, \left( \sum b_i^* \right) [1, \ldots, 1]^T \right) \geq I_r(0,0)
\]

and it follows that RHS \( \geq I_r(0,0) \).

We have now proven (4.9). The next step is to show that the event \( \{ \sum_{i+1 \leq m < k-1} Z_m^{(1)} \geq 0 \} \) does not contribute to the rate. Indeed, we use \( X_{0i} = 1 \) and \( X_{mi}^2 = 1 \), for all \( m \) and \( i \), to obtain

\[
\sum_{m=1}^{k-1} Z_m^{(1)} = \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{m=1}^{k-1} \left( 1 + \sum_{j=0 \atop j \neq m}^{k-1} X_{ji} X_{mi} \right) \right) \quad (4.11)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{m=1}^{k-1} \sum_{j=0}^{k-1} X_{ji} X_{mi} \right) = \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j=1}^{k-1} X_{ji} + \left( \sum_{j=1}^{k-1} X_{ji} \right)^2 \right) \geq 0
\]
a.s., since \( x + x^2 \geq 0 \) for \( x \in \mathbb{Z} \). Hence, if \( \sum_{m=1}^{r} Z_m^{(1)} \leq 0 \), we necessarily have that \( \sum_{m=r+1}^{k-1} Z_m^{(1)} \geq 0 \) a.s. This shows that

\[
H_{k,r}^{(2)} = -\lim_{n \to \infty} \frac{1}{n} \log P \left( \sum_{m=1}^{r} Z_m^{(1)} \leq 0, \frac{Z_0^{(2, H)}}{Z_0^{(2, H)}} \leq 0 \right).
\] (4.12)

We observe that

\[
\begin{bmatrix}
\sum_{m=1}^{r} Z_m^{(1)} \\
\frac{Z_0^{(2, H)}}{Z_0^{(2, H)}}
\end{bmatrix} = \frac{1}{n} \sum_{i=1}^{n} \begin{bmatrix}
Y_{1i} \\
Y_{2i}
\end{bmatrix},
\]

where

\[
Y_{1i} = r + \sum_{m=1}^{r} \sum_{j=0 \atop j \neq m}^{k-1} X_{mi} X_{ji} \quad \text{and} \quad Y_{2i} = 1 + 2 \sum_{j=1}^{r} X_{ji}.
\] (4.13)

Similarly to (4.11) we obtain

\[
Y_{1i} = \left( \sum_{j=1}^{r} X_{ji} \right)^2 + \sum_{j=1}^{r} X_{ij} \left( 1 + \sum_{j=r+1}^{k-1} X_{ji} \right).
\] (4.14)

We abbreviate \( Y_i = [Y_{1i}, Y_{2i}]^T \). According to (4.12), we find that

\[
H_{k,r}^{(2)} = -\lim_{n \to \infty} \frac{1}{n} \log P \left( \frac{1}{n} \sum_{i=1}^{n} Y_i \leq 0 \right),
\]

where for a vector \( x \), the statement \( x \leq 0 \) implies that each entry of \( x \) is less or equal to 0. Note that \( (Y_i)_{i=1}^{n} \) are i.i.d. and therefore, according to Cramér’s theorem,

\[
H_{k,r}^{(2)} = \sup_{t \in (-\infty, 0]^2} \left\{ - \log E e^{t \cdot Y_i} \right\}.
\]

In order to show that \( E e^{t \cdot Y_i} = h(t) \), we condition on \( \sum_{i=1}^{r} X_{ii} = j \). Then

\[
\left( Y_i \left| \sum_{i=1}^{r} X_{ii} = j \right. \right) = \left[ j + j^2 + j \sum_{i=r+1}^{k-1} X_{ii}, 1 + 2j \right]^T
\]

and thus

\[
E \left( e^{t \cdot Y_i} \left| \sum_{i=1}^{r} X_{ii} = j \right. \right) = e^{t (j + j^2 + t_2(1+2j)(\cosh t_{1j})^{k-r-1}}.
\]

Furthermore, \( P(\sum_{i=1}^{r} X_{ii} = j) = 2^{-r} \binom{r}{j} \) precisely when \( j + r \) is even, so that

\[
E e^{t \cdot Y_i} = \sum_{j \text{ even}}^{r} \frac{2^{-r} \binom{r}{j}}{P(\sum_{i=1}^{r} X_{ii} = j)} = h(t).
\]

Finally, \( h(t) \) is log-convex, since it is a moment-generating function. \( \blacksquare \)

The next corollary identifies \( H_{k,1}^{(2)} \). Unfortunately, we were not able to derive an analytical expression for \( H_{k,r}^{(2)} \) for \( r \geq 2 \) explicitly.
Corollary 4.3

\[ H_{k,1}^{(2)} = \frac{3}{4} \log 3 - \log 2 + \frac{2k - 5}{4} \log \left( \frac{2k - 5}{2k - 4} \right) + \frac{2k - 3}{4} \log \left( \frac{2k - 3}{2k - 4} \right). \]

**Proof.** For \( r = 1 \), Equation (4.4) simplifies to

\[ h(t) = \frac{1}{2} \cosh(t) - 2 \left( e^{-t} + e^{2t} \right). \]

Setting the partial derivatives equal to zero gives \( t_1 = \frac{1}{4} \log \frac{2k - 3}{2k - 5} \) and \( t_2 = \frac{1}{4} \log 3 - \frac{1}{4} \log \frac{2k - 3}{2k - 5} \). Substitution leads to the desired result. \( \Box \)

**Remark:** Writing \( 2k - 5 = (2k - 4) - 1 \) and \( 2k - 3 = (2k - 4) + 1 \) and using that \((1 + x) \log(1 + x) + (1 - x) \log(1 - x) \geq 0 \) for \(|x| \leq 1\), it is easy to derive that \( H_{k,1}^{(2)} \geq 3/4 \log 3 - \log 2 \). Furthermore, expanding \( H_{k,1}^{(2)} \) for \( 1/k \to 0 \), using \( \log(1 + x) = x + O(x^2) \), leads to \( \lim_{k \to \infty} H_{k,1}^{(2)} = 3/4 \log 3 - \log 2 \).

4.5 **Exponential rate for 1-stage SD-PIC**

The main result of this section is Theorem 4.5, in which the exponential rate of \( P(Z_{0,3}^{(2,3)} \leq 0) \) is specified for arbitrary values of \( k \geq 3 \). This is more involved than the situation for HD-PIC. In fact, we need Sanov's theorem instead of Cramér's theorem.

For short notation, we introduce \( \mathcal{X}_k = \{-1, +1\}^{k-1} \) and \( M(\mathcal{X}_k) \), the set of all probability measures on \( \mathcal{X}_k \). We define the empirical measure \( L_n^a \) by

\[ L_n^a(a) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{\{a\}}(X_{i1}, \ldots, X_{i,k-1}), \quad a \in \mathcal{X}_k, \quad (4.15) \]

i.e., \( L_n^a(a) \) is the fraction of events of \( a \). Let \( \mathcal{L}_n \) denote the set of all empirical measures. Thus,

\[ \mathcal{L}_n = \{ \rho \in M(\mathcal{X}_k) : n\rho \in (\mathbb{N} \cup \{0\})^{\mathcal{X}_k} \}. \]

Note that the empirical measure \( L_n^a \) is a random element of the set \( \mathcal{L}_n \). We often abbreviate \( L_n^a(a) \) by \( (L_n)_a \).

We define

\[ \mathcal{I}_k(\rho) = (k - 1) \log 2 + \sum_{a \in \mathcal{X}} \rho_a \log \rho_a \quad (4.16) \]

The function \( \rho \mapsto \mathcal{I}_k(\rho) \) is called the rate function. It is non-negative and convex, cf. **Dembo and Zeitouni** (1998). Furthermore, \( \mathcal{I}_k(\mu_0) = 0 \), where \( \mu_0 \) is the uniform measure, i.e., \( \mu_0(a) = 2^{1-k} \) for all \( a \in \mathcal{X} \).
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Lemma 4.4 For $k \geq 3$,

$$Z_0^{(2,\xi)} = F_k(L_n),$$

where for $\rho \in M(\mathcal{X}_k)$,

$$F_k(\rho) = 1 - \sum_{m=1}^{k-1} \left( \sum_{a \in \mathcal{X}_k} a_m \rho_a \right) \left( \sum_{j=0}^{k-1} \sum_{a \in \mathcal{X}_k} a_m a_j \rho_a \right),$$

(4.17)

where $a_0 = 1$, for all $a \in \mathcal{X}_k$.

Proof. Switching over to empirical measures

$$\frac{1}{n} \sum_{i=1}^{n} X_{mi} X_{ji} = \sum_{a \in \mathcal{X}_k} a_m a_j (L_n)_a,$$

yields the lemma. ■

Remark: For $k = 3$, we write $\rho_+ = \rho(+1,+1)$, $\rho_- = \rho(+1,-1)$, $\rho_\tau = \rho(-1,+1)$ and $\rho_- = \rho(-1,-1)$. Then it is straightforward that

$$\frac{1}{n} \sum_{i=1}^{n} X_{i1} = (L_n)_+ + (L_n)_- - (L_n)_\tau - (L_n)_-,$$

$$\frac{1}{n} \sum_{i=1}^{n} X_{i2} = (L_n)_+ - (L_n)_\pm + (L_n)_\tau - (L_n)_-,$$

$$\frac{1}{n} \sum_{i=1}^{n} X_{i1} X_{i2} = (L_n)_+ - (L_n)_\pm - (L_n)_\tau + (L_n)_-.$$

Hence for $\rho \in M(\mathcal{X}_k)$, we have

$$F_3(\rho) = 1 - 2(\rho_+ - \rho_-)(2\rho_+ - \rho_\pm - \rho_\tau) - 2(\rho_\pm - \rho_\tau)^2.$$

Below follows our main result for $k \geq 3$.

Theorem 4.5 For $k \geq 3$,

$$- \lim_{n \to \infty} \frac{1}{n} \log P(Z_0^{(2,\xi)} \leq 0) = J_k^{(2)},$$

where

$$J_k^{(2)} = \inf_{\rho \in M(\mathcal{X}_k) : F_k(\rho) = 0} \mathcal{I}_k(\rho).$$
4.5 Exponential rate for 1-stage SD-PIC

**Proof.** According to Sanov’s theorem, for every set $\Gamma \subset M(\mathcal{X}_k)$,

$$
- \inf_{\rho \in \Gamma^o} \mathcal{I}_k(\rho) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(L_n \in \Gamma) \leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(L_n \in \Gamma) \leq - \inf_{\rho \in \Gamma} \mathcal{I}_k(\rho),
$$

(4.18)

where $\Gamma^o$ is the interior of $\Gamma$ considered as a subset of $\mathbb{R}^{2^{k-1}}$. Applying this theorem with $\Gamma = \{ \rho \in M(\mathcal{X}_k) : F_k(\rho) \leq 0 \}$ gives, according to Lemma 4.4, upper and lower bounds to the lim inf and lim sup of $\frac{1}{n} \log \mathbb{P}(Z_{v,2}^{(2)} \leq 0)$.

Note that $\mathcal{I}_k(\mu_0) = 0$, and that $\mu_0$ is outside the set $\{ F_k(\rho) \leq 0 \}$. It remains to prove that for each $\nu \in M(\mathcal{X}_k)$ that minimizes $\rho \mapsto \mathcal{I}_k(\rho)$ under the constraint $F_k(\rho) \leq 0$, we have $F_k(\nu) = 0$. Suppose $F_k(\nu) < 0$. Then for some $\varepsilon > 0$ small enough, we have that $F_k((1-\varepsilon)\nu + \varepsilon \mu_0) \leq 0$. Furthermore, by convexity of $\mathcal{I}_k(\rho)$,

$$
\mathcal{I}_k((1-\varepsilon)\nu + \varepsilon \mu_0) \leq (1-\varepsilon)\mathcal{I}_k(\nu) + \varepsilon \mathcal{I}_k(\mu_0) = (1-\varepsilon)\mathcal{I}_k(\nu) < \mathcal{I}_k(\nu),
$$

which is a contradiction, since $\nu$ minimizes $\rho \mapsto \mathcal{I}_k(\rho)$ over all $\rho \in M(\mathcal{X}_k)$ with $F_k(\rho) \leq 0$.

---

**Proposition 4.6** The rates $I_k$, $H_k^{(2)}$ and $J_k^{(2)}$ are monotone decreasing in $k$, $k \geq 1$.

**Proof.** We have $I_1 = H_1^{(2)} = J_1^{(2)} = \infty$, while $I_2 = H_2^{(2)} = J_2^{(2)} = \log 2$. Furthermore, $I_3 = 3/2 \log 3 - 2 \log 2$, $H_3^{(2)} \leq H_3^{(2),1} = 3/2 \log 3 - 2 \log 2 < \log 2$, so that $I_3 < I_2$ and $H_3^{(2)} < H_2^{(2)}$. For SD-PIC, we use a numerical result to show that $J_3^{(2)} < J_2^{(2)}$. For $\rho = [0.6213, 0.137, 0.137, 0.1047]$, we have $F_3(\rho) < 0$ and $\mathcal{I}_3(\rho) = 0.30967 \ldots$ Since $J_3^{(2)}$ is the infimum over all measures $\rho$ with $F_3(\rho) \leq 0$, clearly $J_3^{(2)} \leq 0.30967 \ldots$ Thus, $J_3^{(2)} < J_2^{(2)}$.

For $k \geq 3$, we can use the expressions provided in Proposition 4.1, Theorem 4.2 and Theorem 4.5.

For the MF model, we extend the domain of $k$ to $\mathbb{R}$, so that we can take the derivative of the expression for $I_k$ with respect to $k$. Observe that

$$
\frac{dI_k}{dk} = \frac{1}{2} \log \left( \frac{k-2}{k-1} \right) + \frac{1}{2} \log \left( \frac{k}{k-1} \right) = \frac{1}{2} \log \left( \frac{k(k-2)}{(k-1)^2} \right) < 0,
$$

since $k(k-2) = k^2 - 2k < k^2 - 2k + 1 = (k-1)^2$.

Concerning HD-PIC, observe that in (4.4), $k$ only shows up as the exponent of $\cosh(t_{1,j})$. Since $\cosh(t_{1,j}) \geq 1$, this directly yields $H_k^{(2)} \leq H_{k-1,r}^{(2)}$. But then $H_k^{(2)}$ is also monotone in $k$, because it is a minimum of monotone functions.

To prove the statement for the SD-PIC model, we will make use of the fact that the rate is an infimum of the rate function. Suppose $\nu$ is the minimizer, corresponding to the optimization problem in Theorem 4.5 for $k$ users. Then $F_k(\nu) = 0$. For $k+1$ users, we construct a $\rho$ as follows: $\rho(a,-1) = \rho(a,+1) = \nu_a/2$ for all $a \in \mathcal{X}_k$. When we can prove that $\mathcal{I}_{k+1}(\rho) = \mathcal{I}_k(\nu)$ and $F_{k+1}(\rho) = 0$, we have found a $\rho$ that obeys the boundary conditions $\rho \in M(\mathcal{X}_{k+1})$ and
Chapter 4. Exponential rate for the simple model

\( F_{k+1}(\rho) = 0 \). Since the \( J_{k+1}^{(2)} \) is the infimum over all \( \rho \)'s sufficing these boundary conditions, certainly \( J_{k+1}^{(2)} \leq \mathcal{I}_{k+1}(\rho) = \mathcal{I}_k(\nu) = J_k^{(2)} \).

It is easy to prove \( \mathcal{I}_{k+1}(\rho) = \mathcal{I}_k(\nu) \). Indeed,

\[
\mathcal{I}_{k+1}(\rho) = \sum_{b \in \{-1, 1\}} \sum_{a \in \chi_k} \rho(a, b) \log \rho(a, b) + k \log 2 = \sum_{b \in \{-1, 1\}} \sum_{a \in \chi_k} \frac{\nu_a}{2} \log \frac{\nu_a}{2} + k \log 2 \\
= 2 \sum_{a \in \chi_k} \frac{\nu_a}{2} (\log \nu_a - \log 2) + k \log 2 = \sum_{a \in \chi_k} \nu_a \log \nu_a + (k - 1) \log 2 = \mathcal{I}_k(\nu).
\]

It remains to prove \( F_{k+1}(\rho) = 0 \). Observe that

\[
F_{k+1}(\rho) = 1 - \sum_{m=1}^{k-1} \left( \sum_{a \in \chi_{k+1}} a_m \rho_a \right)^k \sum_{j=0}^{k-1} \sum_{a \in \chi_{k+1}} a_m a_j \rho_a \]

\[
- \sum_{m=1}^{k-1} \left( \sum_{a \in \chi_{k+1}} a_m \rho_a \right) \left( \sum_{a \in \chi_{k+1}} a_m a_k \rho_a \right) + \left( \sum_{a \in \chi_{k+1}} a_k \rho_a \right) \left( \sum_{j=0}^{k-1} \sum_{a \in \chi_{k+1}} a_k a_j \rho_a \right).
\]

Furthermore, for \( 0 \leq m, j \leq k - 1 \),

\[
\sum_{a \in \chi_{k+1}} a_m a_j \rho_a = \sum_{a \in \chi_k} a_m a_j \nu_a \quad \text{and} \quad \sum_{a \in \chi_{k+1}} a_m a_k \rho_a = 0.
\]

Thus

\[
F_{k+1}(\rho) = 1 - \sum_{m=1}^{k-1} \left( \sum_{a \in \chi_k} a_m \nu_a \right) \left( \sum_{j=0}^{k-1} \sum_{a \in \chi_k} a_m a_j \nu_a \right) = F_k(\nu) = 0.
\]

This completes the proof.

\[ \blacksquare \]

4.6 Comparison of MF, HD-PIC and SD-PIC

From a practical point of view, the results of the previous section can be used to generate numerical values for \( I_k, H_k^{(2)} \) and \( J_k^{(2)} \). Using the numerical values, we can conclude if HD-PIC and SD-PIC give an improvement in performance, i.e., whether \( H_k^{(2)} > I_k \) and \( J_k^{(2)} > I_k \). However, we prefer to make a statement concerning the rates for all \( k \), rather than for the \( k \) for which we calculated the numerical values. Theorem 4.8 and Theorem 4.9 below provide this.

For the proof of the two theorems, the following lemma is helpful. The lemma shows that the exponential rate of the probability of the rare event

\[
\{ Z_0^{(1)} \leq 0, Z_1^{(1)} \notin [7/10, 1] \}
\]

is strictly larger than \( I_k \) for \( k \geq 3 \).
Lemma 4.7 For $k \geq 3$,

$$- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Z_0^{(1)} \leq 0, Z_1^{(1)} \notin [7/10, 1)) > - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Z_0^{(1)} \leq 0) = I_k.$$

Proof. The rate of a rare event can be obtained as the infimum of the rate function $\rho \mapsto I_k(\rho)$, where $\rho \in M(\mathcal{X}_k)$ is restricted to some specified region. More precisely, we have that:

$$Z_0^{(1)} = G_{0k}(L_n), \quad Z_1^{(1)} = G_{1k}(L_n),$$

where, for $\rho \in M(\mathcal{X}_k)$,

$$G_{0k}(\rho) = 1 + \sum_{j=1}^{k-1} \sum_{a \in \mathcal{X}_k} a_j \rho_a,$$

$$G_{1k}(\rho) = 1 + \sum_{a \in \mathcal{X}_k} a_1 \rho_a + \sum_{j=2}^{k-1} \sum_{a \in \mathcal{X}_k} a_1 a_j \rho_a.$$

Let $\mu$ be the (unique) minimizer of $\rho \mapsto I_k(\rho)$, subject to $\rho \in M(\mathcal{X}_k)$ and $G_{0k}(\rho) \leq 0$, i.e., $I_k(\mu) = I_k$, the rate without interference cancellation. It is not hard to verify from Proposition 4.1 that $\mu$ is the product measure:

$$\mu_a = \prod_{j=1}^{k-1} \left[ \frac{k}{2(k-1)} \delta_{-1}(a_j) + \frac{k-2}{2(k-1)} \delta_{+1}(a_j) \right], \quad a \in \mathcal{X}_k.$$ 

Hence

$$G_{1k}(\mu) = 1 - \frac{k}{2(k-1)} + \frac{k-2}{2(k-1)} + (k-2) \left( - \frac{k}{2(k-1)} + \frac{k-2}{2(k-1)} \right)^2 = 1 - \frac{1}{k-1} + \frac{k-2}{(k-1)^2} = 1 - \frac{1}{(k-1)^2}.$$

This implies that $G_{1k}(\mu) \in [3/4, 1)$, for $k \geq 3$. Since $G_{1k}(L_n) = Z_1^{(1)}$ and $3/4 > 7/10$, the conclusion of the lemma follows.

Theorem 4.8 For $k = 3$,

$$H_k^{(2)} = I_k,$$

whereas $k \geq 4$,

$$H_k^{(2)} > I_k.$$

Proof. First of all, we have for $r \geq 2$,

$$\mathbb{P}\left( \max_{1 \leq m \leq r} Z_m^{(1)} \leq 0, \min_{r+1 \leq m \leq k-1} Z_m^{(1)} \geq 0, \bar{Z}_0^{(2,n)} \leq 0 \right) \leq \mathbb{P}(Z_0^{(1)} \leq 0, Z_1^{(1)} \leq 0),$$

$$\leq \mathbb{P}(Z_0^{(1)} \leq 0, Z_1^{(1)} \notin [7/10, 1)).$$
so that, according to Lemma 4.7, \(H_{k,r}^{(2)} > I_k\) for \(r \geq 2\). Thus, it is sufficient to prove \(H_{3,1}^{(2)} = I_3\) and \(H_{k,1}^{(2)} > I_k\) for \(k \geq 4\). For \(k = 3\) and \(4\), the statement is easy to check from Proposition 4.1 and Corollary 4.3. Corollary 4.3 gives \(H_{k,1}^{(2)} \geq 3/4 \log 3 - \log 2 = 0.1308\ldots\) and it is easy to check that \(3/4 \log 3 - \log 2 > I_5 = 0.1263\ldots\) Finally, Proposition 4.6 implies \(I_k \leq I_5\) for \(k \geq 5\), which completes the proof of the theorem.

**Theorem 4.9** For \(k \geq 3\),

\[ J_k^{(2)} > I_k. \]

**Proof.** For \(0 < \delta < 1\), we define

\[
V = \operatorname{card}\{1 \leq j \leq k - 1 : Z_j^{(1)} \in (-\infty, -\delta/4) \cup (2 + \delta/4, \infty)\},
\]

\[
T = \operatorname{card}\{1 \leq j \leq k - 1 : Z_j^{(1)} \in [\delta, 2 - \delta]\},
\]

\[
R = \operatorname{card}\{1 \leq j \leq k - 1 : Z_j^{(1)} \in [-\delta/4, \delta) \cup (2 - \delta, 2 + \delta/4)\}.
\]

Then

\[
\mathbb{P}(Z_0^{(2,5)} \leq 0) = \mathbb{P}(Z_0^{(2,5)} \leq 0, V \geq 1) + \mathbb{P}(Z_0^{(2,5)} \leq 0, T = k - 1)
\]

\[
+ \mathbb{P}(Z_0^{(2,5)} \leq 0, R = 1, T = k - 2) + \sum_{l=2}^{k-1} \mathbb{P}(Z_0^{(2,5)} \leq 0, R = l, T = k - l - 1),
\]

since \(V + T + R = k - 1\). We will show that for some fixed \(\delta\), \(0 < \delta < 1\), each of the four terms on the right-hand side of (4.19) has exponential rate strictly larger than \(I_k\).

We bound the first term in (4.19) as

\[
\mathbb{P}(Z_0^{(2,5)} \leq 0, V \geq 1) \leq 2(k - 1)\mathbb{P}(Z_0^{(1)} < -\delta/4),
\]

which has rate larger than \(I_k\), for each \(\delta > 0\).

For the second term in (4.19), we obtain

\[
\mathbb{P}(Z_0^{(2,5)} \leq 0, T = k - 1) = \mathbb{P}
\]

\[
\left(\bigcap_{j=1}^{k-1}\{Z_j^{(1)} \in [\delta, 2 - \delta]\}\bigcap\left\{1 + \frac{1}{n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} X_{ji} (1 - Z_j^{(1)}) \leq 0\right\}\right)
\]

\[
\leq \mathbb{P}
\]

\[
1 - (1 - \delta)\frac{1}{n}\sum_{j=1}^{k-1}\left|\sum_{i=1}^{n} X_{ji}\right| \leq 0
\]

\[
\leq \mathbb{P}
\]

\[
\bigcup_{(\varepsilon_1, \ldots, \varepsilon_{k-1}) \in \{-1, 1\}^{k-1}} \left\{1 - (1 - \delta)\frac{1}{n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} \varepsilon_j X_{ji} \leq 0\right\}
\]

\[
\leq 2^{k-1}\mathbb{P}
\]

\[
\delta + (1 - \delta) + (1 - \delta)\frac{1}{n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} X_{ji} \leq 0
\]

\[
= 2^{k-1}\mathbb{P}(Z_0^{(1)} \leq -\delta/(1 - \delta)).
\]
which has rate strictly larger than $I_k$, because $\delta/(1-\delta) > 0$.

For the third term in (4.19), a similar calculation gives

$$
\mathbb{P}(Z_0^{(2,S)} \leq 0, R = 1, T = k - 2) \\
\leq (k - 1) \mathbb{P}\left( \left\{ 1 + \frac{1}{n} \sum_{j=2}^{k} \sum_{i=1}^{n} X_{ji}(1 - Z_j^{(1)}) \leq 0 \right\} \right)
\cap \left\{ Z_1^{(1)} \in [-\delta/4, \delta) \cup (2 - \delta, 2 + \delta/4] \right\}^{k-1} \left\{ Z_j^{(1)} \in [\delta, 2 - \delta] \right\}
\leq (k - 1) \mathbb{P}\left( 1 - (1 + \delta/4) \frac{1}{n} \left| \sum_{i=1}^{n} X_{1i} \right| - (1 - \delta) \frac{1}{n} \sum_{j=2}^{k-1} \left| \sum_{i=1}^{n} X_{ji} \right| \leq 0 \right)
\leq (k - 1) 2^{k-1} \mathbb{P}\left( 1 + \frac{1}{n} \sum_{j=1}^{k-1} \sum_{i=1}^{n} X_{ji} \leq \frac{-\delta - \frac{5}{4} \delta \frac{1}{n} \sum_{i=1}^{n} X_{1i}}{1 - \delta} \right)
\leq (k - 1) 2^{k-1} \left( \mathbb{P}\left( Z_0^{(1)} < -\frac{\delta + \frac{5}{4} \alpha \delta}{1 - \delta} \right) + \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} X_{1i} \leq -\alpha \right) \right),
$$

by intersecting with $(\sum X_{1i})/n > -\alpha$ and its complement. Take $\alpha = 7/10$. Clearly, the first term of the righthand side in (4.22) has rate larger than $I_k$. The second probability has rate larger than $I_k$ because the exponential rate satisfies (compare the proof of Proposition 4.1),

$$
\frac{1 + 7/10}{2} \log(1 + 7/10) + \frac{1 - 7/10}{2} \log(1 - 7/10) > \frac{3}{2} \log 3 - 2 \log 2 = I_3 \geq I_k,
$$

for $k \geq 3$, by Proposition 4.6.

Finally,

$$
\sum_{l=2}^{k-1} \mathbb{P}(Z_0^{(2,S)} \leq 0, R = l, T = k - l - 1) \leq \mathbb{P}(R \geq 2) \\
\leq \left( \frac{k - 1}{2} \right) \mathbb{P}\left( Z_0^{(1)} \in [-\delta/4, \delta) \cup (2 - \delta, 2 + \delta/4], Z_1^{(1)} \in [-\delta/4, \delta) \cup (2 - \delta, 2 + \delta/4] \right)
\leq (k - 1)(k - 2) \mathbb{P}(Z_0^{(1)} < \delta, Z_1^{(1)} \notin [\delta, 2 - \delta])
\leq k^2 \mathbb{P}(Z_0^{(1)} < \delta, Z_1^{(1)} \notin [7/10, 13/10])
\leq k^2 \mathbb{P}(Z_0^{(1)} < \delta, Z_1^{(1)} \notin [7/10, 1]),
$$

by choosing $\delta \leq 7/10$. Observe from Lemma 4.7 and the continuity of the rate function, that for $\delta$ small enough

$$
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Z_0^{(1)} \leq \delta, Z_1^{(1)} \notin [7/10, 1]) > I_k.
$$

Hence for some $\delta \in (0, 7/10]$, the rate of $\mathbb{P}(R \geq 2)$ is larger than $I_k$.  \[\blacksquare\]
### 4.7 Numerical results

In this section we will give numerical results for $I_k$, $H_k^{(2)}$, $H_k^{(2)}$ and $J_k^{(2)}$. Furthermore, some issues concerning the generation of the results are described. In Table 4.1, numerical results are shown for MF, HD-PIC and SD-PIC for $k = 2, \ldots, 27$. In Figure 4.1, the same results are shown in a graphical manner.

<table>
<thead>
<tr>
<th># users</th>
<th>$I_k$</th>
<th>$H_k^{(2)}$</th>
<th>$J_k^{(2)}$</th>
<th># users</th>
<th>$I_k$</th>
<th>$H_k^{(2)}$</th>
<th>$J_k^{(2)}$</th>
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<td>0.6931</td>
<td>0.6931</td>
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</tr>
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**Table 4.1:** Numerical values for $I_k$, $H_k^{(2)}$ and $J_k^{(2)}$

The effect of PIC is clearly visible in Figure 4.1. Both HD-PIC and SD-PIC give a significant increase in exponential rate, compared to MF. For SD-PIC this effect is already present for $k = 3$, while for HD-PIC, $k$ should be larger than 3. Indeed, $I_3 = H_3^{(2)}$, so that for a system with 3 users, HD-PIC does not result in an increase in rate. It turns out that this is not a side effect like the case $k = 2$ (where events turn out to involve atoms), but an unpleasant feature of (multistage) HD-PIC. In Chapter 7, we investigate the effect that multistage HD-PIC does not necessarily result in an increase in rate.

The close resemblance between $H_k^{(2)}$ and $J_k^{(2)}$ is remarkable, taking into account that the power estimation is essentially different. The fact that knowledge of the powers (HD-PIC) results in a lower rate when $k$ is small is really unexpected. A possible explanation is that the brute force decision made in HD-PIC systems creates either a very good or a very bad scenario, while the SD-PIC approach deals more sophisticated with its information. Possibly, it is tied up with the reason behind the fact that $H_3^{(2)} = I_3$.

When $k \geq 8$, the difference between the rates is smaller than 0.01, so for higher $k$ there is virtually no difference between $H_k^{(2)}$ and $J_k^{(2)}$.

Concerning $H_k^{(2)}$, we first have to calculate $H_{k,r}$ for different values of $r$ and then we have to
minimize over $r$. We abbreviate

$$r_k = \arg\min_r H_{k,r}^{(2)}.$$  \hspace{1cm} (4.23)

In Figure 4.2 and Table 4.3, results are shown. We expect the $r_k$ to be increasing in $k$. However, we are not able to show this. The intuitive explanation why $r_k$ increases with $k$ is based on two principles. Firstly, the probability of having $r$ bit-errors in the first stage is decreasing in $r$. Secondly, the probability of making an error in the second stage caused by $r$ noise terms is increasing in $r$. Intersection of the two events therefore involves a balance between a decrease (the first stage) and an increase (second stage). When $k$ becomes larger, it is easier to make errors in the first stage. This results in a change in the balance, in favour of larger $r$.

Intuitively, for $k \leq 9$, it is sufficient to calculate $H_{k,r}^{(2)}$ for $r = 1, 2$ and $3$, because we do not expect $r_k \geq 4$ to be optimal, see Figure 4.2. However, for a complete analysis, we should calculate $H_{k,r}^{(2)}$ for all $r = 1, \ldots, k - 1$. One possible solution can be found in Lemma 5.12, which states that $H_{k,r}^{(2)}$ is bounded from below by $\frac{1}{2} \log(1 + r/k)$. This roughly implies that all $r$ higher than $k^{1/2}$ give a rate that is too high.

The behaviour of $H_{k,1}^{(2)}$ for $k \to \infty$ is clearly seen in Figure 4.2. It converges to $3/4 \log 3 - \log 2 = 0.1308 \ldots$. We observe that $H_{k,r}^{(2)}$ for $r \geq 2$ also converges to a strictly positive number. In the next chapter, those asymptotic values will be given for general $r$.

Concerning $J_k^{(2)}$, the minimization problem in Theorem 4.5 is one in $2^{k-1}$ variables. For example for $k = 11$, this results in more than 1000 variables. Going beyond $k = 11$ with a standard PENTIUM 4 computer is likely to result in a OUT OF MEMORY warning. A close examination reveals that the minimizer $\nu$ for $k \leq 11$ has a clear symmetry: $\nu_a = \nu_b$ when
Figure 4.2: Numerical results for $H_{k,r}^{(2)}$, $r = 1, \ldots, 5$ represented by $\circ, \triangle, \diamond, \times, \star$, respectively.

Table 4.3: Optimal $r$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$r_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${3, \ldots, 9}$</td>
<td>1</td>
</tr>
<tr>
<td>${10, \ldots, 26}$</td>
<td>2</td>
</tr>
<tr>
<td>${27, \ldots, 51}$</td>
<td>3</td>
</tr>
<tr>
<td>${52, \ldots, 84}$</td>
<td>4</td>
</tr>
<tr>
<td>${85, \ldots, 125}$</td>
<td>5</td>
</tr>
<tr>
<td>${126, \ldots, 174}$</td>
<td>6</td>
</tr>
<tr>
<td>${175, \ldots, 231}$</td>
<td>7</td>
</tr>
</tbody>
</table>

$\sum a_i = \sum b_i$. For $k = 3$, this reduces to $\nu_3 = \nu_{g}$. However, proving this is hard, only for $k = 3$ we are able to do this, see Chapter 8, Proposition 8.6. This symmetry corresponds to the fact that typically all users behave the same; all users have the same share in forcing a bit-error. This sounds reasonable, and for $k \leq 11$ it seems to be true. A priori implementing a symmetric empirical measure, reduces the number of variables from $2^{k-1}$ to $k$, which enables us to calculate the rate for $k \geq 12$. In the numerical results above, we have used a priori symmetric minimizer for $k \geq 12$, even though we cannot prove that an asymmetric minimizer does not result in a smaller rate. Certainly, the real rate cannot be larger, but possibly it is lower.
Asymptotic behaviour of exponential rate

The results of the previous chapter indicate that PIC indeed increases the performance significantly, whenever \( k \) is large. From the numerical results and the figure, one could wonder how the rates behave for \( k \to \infty \). It suggests that \( H_k^{(2)} \) and \( J_k^{(2)} \) converge slower to 0 than \( I_k \). In this chapter, we investigate the behaviour of \( I_k \), \( H_k^{(2)} \) and \( J_k^{(2)} \) for \( k \to \infty \). Throughout this chapter we use the notation \( h(t) \) for a moment generating function, while in fact it represents different moment generating functions in different (steps of) proofs. Results for the HD-PIC system have appeared in van der Hofstad and Klok (2002HD).

5.1 Asymptotic behaviour of \( I_k \)

From the analytical expression of \( I_k \) in Proposition 4.1, it is easy to derive the asymptotic behaviour for large \( k \). Indeed, observe that \( \log(1 + x) = x - x^2/2 + \mathcal{O}(x^3) \) and

\[
I_k = \frac{k - 2}{2} \log \left( 1 - \frac{1}{k - 1} \right) + \frac{k}{2} \log \left( 1 + \frac{1}{k - 1} \right),
\]

so that \( I_k \) equals

\[
\frac{k - 2}{2} \left( \frac{1}{k - 1} - \frac{1}{2(k - 1)^2} + \mathcal{O}\left( \frac{1}{(k - 1)^3} \right) \right) + \frac{k}{2} \left( \frac{1}{k - 1} - \frac{1}{2(k - 1)^2} + \mathcal{O}\left( \frac{1}{(k - 1)^3} \right) \right)
= \frac{1}{2(k - 1)} + \mathcal{O}\left( \frac{1}{(k - 1)^2} \right) = \frac{1}{2k} + \mathcal{O}\left( \frac{1}{k^2} \right).
\] (5.1)

The exponential rate \( I_k \) tends to 0 as \( 1/(2k) \). We note that this result more or less agrees with the central limit theorem (CLT) result. Indeed, when \( k \) is large, it is not unreasonable
Chapter 5. Asymptotic behaviour of exponential rate

to assume that the interference from the other users is Gaussian distributed. Assuming that the \(k-1\) noise terms are Gaussian, leads to

\[
\mathbb{P}(\text{sgnr}_0(Z_0^{(1)}) < 0) = Q\left( \frac{\mathbb{E} Z_0^{(1)}}{\sqrt{\text{var}(Z_0^{(1)})}} \right) = Q\left( \sqrt{\frac{n}{k-1}} \right),
\]

where \(Q(x)\) is the tail probability of a Gaussian random variable, i.e.,

\[
Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2} du.
\]

To obtain more insight in the system, we use an asymptotic result for \(Q\), cf. GRIMMETT AND STIRZAKER (1992), PROBLEM 4.11.1,

\[
\frac{1}{\sqrt{2\pi}} (x^{-1} - x^{-3}) e^{-x^2/2} \leq Q(x) \leq \frac{1}{\sqrt{2\pi}} x^{-1} e^{-x^2/2} \quad \text{for all} \quad x > 0.
\]

Thus, we see that

\[
\mathbb{P}(\text{sgnr}_0(Z_0^{(1)}) < 0) \approx Q\left( \sqrt{\frac{n}{k-1}} \right) \sim e^{-n/(2(k-1))}.
\]

The exponential rate is indeed \(1/(2(k-1)) \approx 1/(2k)\).

In the next section, we will derive asymptotic results for the rate for the HD- and SD-PIC model.

5.2 Asymptotic behaviour of \(H_k^{(2)}\)

This section deals with the asymptotic behaviour of \(H_{k,r}^{(2)}\) and \(H_k^{(2)}\) for \(k \to \infty\). We already noted in the previous chapter that \(H_{k,1}^{(2)}\) converges to \(3/4 \log 3 - \log 2\) for \(k \to \infty\). The corollary below states the behaviour of \(H_{k,r}^{(2)}\) for general \(r\) fixed and \(k \to \infty\).

Corollary 5.1 For \(r\) fixed,

\[
\lim_{k \to \infty} H_{k,r}^{(2)} = \frac{2r + 1}{4} \log \left( \frac{2r + 1}{2r} \right) + \frac{2r - 1}{4} \log \left( \frac{2r - 1}{2r} \right).
\]

Proof. Denote the right-hand side of (5.6) simply by RHS. First we note that

\[
\text{RHS} = - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( 1 + 2 \sum_{j=1}^r \frac{1}{n} \sum_{i=1}^n X_{ji} \leq 0 \right),
\]

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\]
which follows from a similar derivation as the proof of Proposition 4.1. Furthermore,

\[
\sup_{t_2 \leq 0} \{- \log h(0, t_2)\} = \sup_{t_2 \leq 0} 2^{-r} \sum_{j=0}^{r} \left( \frac{r}{j+1} \right) e^{t_2(1+2j)} = \sup_{t_2 \leq 0} e^{t_2} \mathbb{E} e^{2 \sum_{j=1}^{r} X_{ji}} \tag{5.8}
\]

\[
= - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( 1 + 2 \sum_{j=1}^{r} \frac{1}{n} \sum_{i=1}^{n} X_{ji} \leq 0 \right).
\]

We will show that for all \( \varepsilon > 0 \) and \( k \) sufficiently large, we have

\[
\text{RHS} \leq H_{k,r}^{(2)} < \text{RHS} + \varepsilon.
\]

The lower bound for \( H_{k,r}^{(2)} \) directly follows from (5.7) and (5.8). For the upper bound it is sufficient to show that for all \( \delta > 0 \), the optimal \((t_1^*, t_2^*)\) in Theorem 4.2 satisfies \( -\delta < t_1^* \leq 0 \) when \( k \) is sufficiently large. Indeed, when we assume \( -\delta < t_1^* \leq 0 \), it follows from \( \cosh x \geq 1 
\]

\[
h(t) \geq 2^{-r} \sum_{j=-r}^{r} \left( \frac{r}{j+1} \right) e^{-|t_1(j+j^2)|} e^{t_2(1+2j)} \geq h(0, t_2) e^{-2t_1|r|^2},
\]

since \(|t_1(j+j^2)| \leq |t_1|(r + r^2) \leq 2|t_1|r^2\). It follows that, assuming \( -\delta < t_1^* \leq 0 \),

\[
H_{k,r}^{(2)} = \sup_{-\delta < t_1 \leq 0, t_2 \leq 0} \{- \log h(t)\} \leq \sup_{t_2 \leq 0} \{- \log h(0, t_2)\} + \sup_{-\delta < t_1 \leq 0} 2|t_1|r^2 = \text{RHS} + 2\delta r^2.
\]

Taking \( \delta = \varepsilon/(2r^2) \) gives the desired result.

It is hence sufficient to prove that under \( t_1 \leq -\delta \) the supremum cannot be attained for \( k \) sufficiently large. Indeed,

\[
- \inf_{t_1 \leq 0} f(t_1) = \max\{- \inf_{-\delta < t_1 \leq 0} f(t_1), - \inf_{t_1 \leq -\delta} f(t_1)\},
\]

We will prove this now. Since \( \cosh(t_1r) \geq \frac{1}{2} e^{|t_1|r} \),

\[
2^{r+1} e^{t_1^* (r+2)} (\cosh(t_1r))^{r+1} \geq e^{-|t_1|(r+r^2)} e^{t_1|r(r+1)} = 1.
\]

We have, taking only the first and last term of the sum in (4.4) and using the inequality above, for \( t \in \mathbb{R}^2 \),

\[
h(t) \geq 2^{-r} \left( e^{t_1^* (-r+r^2) + t_2^* (1-2r)} + e^{t_1^* (r+r^2) + t_2^* (1+2r)} \right) (\cosh(t_1r))^{k-r-1}
\]

\[
= 2^{2r-1} (\cosh(t_1r))^{k-2r-2} \left( 2^{r+1} e^{t_1^* (-r+r^2)} (\cosh(t_1r))^{r+1} e^{t_2^* (1-2r)} + 2^{r+1} e^{t_1^* (r+r^2)} (\cosh(t_1r))^{r+1} e^{t_2^* (1+2r)} \right)
\]

\[
\geq 2^{2r-1} \left( e^{t_2^* (1-2r)} + e^{t_2^* (1+2r)} \right) (\cosh(t_1r))^{k-2r-2},
\]

\[\text{Page 71} \]
and therefore
\[
\sup_{t_1 \leq -\delta, t_2 \leq 0} \{- \log h(t)\} \leq - \log \left\{ 2^{-2r-1} \inf_{t_2 \leq 0} \left( e^{t_2(1+2r)} + e^{t_2(1-2r)} \right) \left( \cosh \delta r \right)^{k-2r-2} \right\} = - \log \left\{ C \left( \cosh \delta r \right)^k \right\} = - \log C - k \log (\cosh \delta r),
\]
where \( C = 2^{-2r-1} \inf_{t_2 \leq 0} (e^{t_2(1+2r)} + e^{t_2(1-2r)}) (\cosh \delta r)^{-2r-2} > 0 \). Thus, assuming \( t_1^* \leq -\delta \) leads to \( H^{(2)}_{k,r} < 0 \) when \( k \) is sufficiently large.

Theorem 5.2 below provides an analysis for the asymptotic behaviour of \( H^{(2)}_{k,r} \), where \( r \to \infty \) as \( k \to \infty \). The proof is based on a Taylor series expansion of a moment generating function. In fact, we show that the rate depends on first and second moments only. Implicitly, the result below is therefore a CLT statement. From this result, we can derive the asymptotics of the rate \( H^{(2)}_k \).

In order to bound the higher moments, we will frequently use two inequalities. The first is known as Hölder’s inequality, c.f. Grimmert and Stirzaker (1992), Eqn. (7.3.6). When \( p, q > 1 \) such that \( p^{-1} + q^{-1} = 1 \),
\[
|\mathbb{E} XY| \leq (\mathbb{E}|X|^p)^{1/p}(\mathbb{E}|Y|^q)^{1/q}. \tag{5.9}
\]
For \( p = q = 2 \), Hölder’s inequality reduces to Cauchy-Schwarz inequality, which states
\[
|\mathbb{E} XY| \leq (\mathbb{E} X^2)^{1/2}(\mathbb{E} Y^2)^{1/2}. \tag{5.10}
\]
The main ingredient for the asymptotic behaviour of \( H^{(2)}_k \) is the following theorem. The proof will be given in Section 5.2.1.

**Theorem 5.2** For \( k, r \to \infty \) such that \( r/k \to 0 \),
\[
H^{(2)}_{k,r} = \left( \frac{1}{8r} + \frac{r}{2k} \right) \left( 1 + \mathcal{O}\left( \frac{1}{8r} + \frac{r}{2k} \right) \right). \tag{5.11}
\]

From this result, we can prove the following theorem.

**Theorem 5.3** As \( k \to \infty \)
\[
H^{(2)}_k = \frac{1}{2\sqrt{k}} \left( 1 + \mathcal{O}\left( \frac{1}{\sqrt{k}} \right) \right)
\]
and
\[
r_k = \frac{1}{2} \sqrt{k} + \mathcal{O}(1).
\]

**Proof.** Intuitively the statement is clear, since \( H^{(2)}_k = \min_r H^{(2)}_{k,r} \) and minimizing \( \frac{1}{8r} + \frac{r}{2k} \) over \( r \) leads to the desired result. The proof is contained in the proof of Theorem 5.7 and will be omitted here. \( \blacksquare \)

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5.2.1 Proof of Theorem 5.2

**Proof.** The proof is divided into 2 steps. In step 1 we prove the lower bound. Consecutively we prove the upper bound. We abbreviate \( R_0 = \{0, \ldots, k - 1\} \), \( R_1 = \{1, \ldots, r\} \) and \( R_0^+ = R_0 \setminus R_1 \). Since \(|R_0^+| = k - r\) is used frequently, we abbreviate \( k_r = k - r\).

For any set \( A \subseteq \mathbb{N} \cup \{0\} \), let

\[
S_A = \sum_{j \in A} X_{j1}. \tag{5.12}
\]

In the proof it is useful to observe that for \( A \subseteq \mathbb{N} \cup \{0\} \)

\[
\mathbb{E} S_A^2 = |A|. 
\]

and that there exist constants \( C_m \) independent of \( A \) such that

\[
|\mathbb{E} S_A^n| \leq C_m |A|^{n/2}. \tag{5.13}
\]

Here and throughout the proof \( C \) denotes a strictly positive constant that may not depend on \( k \). \( C \) may change from line to line.

To obtain the exponential rate \( H_{k,r}^{(2)} \) for \( r \) large and \( r/k \) small, we will not use the expression obtained in Theorem 4.2. Instead, we start with expression (4.12). We write \( t_1 Y_{11} + t_2 Y_{21} = Y_q + Y_a \), where (recall (4.13) and (4.14))

\[
Y_q = t_1 \left( \sum_{j=1}^{r} X_{j1} \right)^2 + t_2 = t_1 S_{R_1}^2 + t_2, \tag{5.14}
\]

\[
Y_a = t_1 \sum_{j=1}^{r} X_{j1} \left( 1 + \sum_{j=r+1}^{k-1} X_{j1} \right) + 2t_2 \sum_{j=1}^{r} X_{j1} = t_1 S_{R_0} S_{R_1}^2 + 2t_2 S_{R_1}. \tag{5.15}
\]

We shall see that only the first moment of \( Y_q \), representing the part with the quadratic term, will contribute to the rate asymptotically. Furthermore, \( Y_a \) (asymmetric part) has mean zero and \( \mathbb{E} Y_a^3 = \mathbb{E} Y_a Y_a = \mathbb{E} Y_a Y_a^3 = 0 \).

Throughout the entire proof it is sufficient to consider the following moments:

\[
\mathbb{E} Y_q = t_1 r + t_2, \quad \mathbb{E} Y_a^2 = t_1^2 r k_r + 4t_2^2 r. \tag{5.16}
\]

Using \( e^y = 1 + y + \frac{y^2 e^y}{2} \) for some \( \zeta = \zeta_y \in [0, 1] \) and \( e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4 e^{\eta x}}{24} \) for some \( \eta = \eta_x \in [0, 1] \), respectively, we write

\[
h(t) = \mathbb{E} e^{Y_q + Y_a} = 1 + \mathbb{E} Y_q + \mathbb{E} Y_a^2/2 + e(t), \tag{5.17}
\]

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where
\[
e(t) = \mathbb{E} \left( e^{Y_a}e^{-Y_a} - 1 - Y_a - Y_a^2/2 \right) = \mathbb{E} \left( (1 + Y_a + Y_a^2e^{Y_a}/2)e^{-Y_a} - 1 - Y_a - Y_a^2/2 \right) = \mathbb{E} \left( (1 + Y_a)(1 + Y_a + Y_a^2/2 + Y_a^3/6 + Y_a^4e^{Y_a}/24 + Y_a^2e^{Y_a}e^{-Y_a}/2 - 1 - Y_a - Y_a^2/2) \right) = \mathbb{E} \left( Y_a + Y_a^3/6 + Y_a^4e^{Y_a}/24 + Y_a(Y_a + Y_a^2/2 + Y_a^3/6 + Y_a^4e^{Y_a}/24) + Y_a^2e^{Y_a}e^{-Y_a} \right).
\]

Using the symmetry of \(Y_a\), this reduces in
\[
e(t) = \mathbb{E} \left( Y_a^2e^{Y_a}/24 + Y_a(Y_a^2/2 + Y_a^4e^{Y_a}/24) + Y_a^2e^{Y_a}e^{-Y_a} \right).
\]

(5.19)

**Step 1: lower bound for \(H_{k,r}^{(2)}\)**. It is sufficient to investigate \(h(t)\) for \(t^* = (-1/k_r, -1/4)\), since every \(t\) gives a lower bound for the supremum. Hence, we will substitute \(t^*\) in (5.14).

We bound, using (5.19),
\[
e(t^*) \leq \mathbb{E} \left( Y_a^2e^{Y_a}/24 + Y_a^2e^{-Y_a} \right),
\]

since \(\eta Y_a \leq |Y_a|, Y_a \leq 0\) and \(e^{Y_a} \leq 1\) a.s. Using Hölder's inequality (5.9) with \(p = 3/2\) and \(q = 3\) yields
\[
e(t^*) \leq (\mathbb{E} Y_a^6)^{2/3}(\mathbb{E} e^{3|Y_a|})^{1/3} + (\mathbb{E} |Y_a|^3)^{2/3}(\mathbb{E} e^{3|Y_a|})^{1/3}.
\]

(5.20)

Use (5.16) to get
\[
\mathbb{E} (1 + Y_a + Y_a^2/2) = 1 - \left( \frac{1}{8r} + \frac{r}{2k_r} \right),
\]

so that \(h(t^*) = 1 - \left( \frac{1}{8r} + \frac{r}{2k_r} \right) + c(t^*)\). It is now sufficient to show that for \(t = t^*\), we have that \(\mathbb{E} e^{3|Y_a|}\) is bounded and that
\[
\mathbb{E} Y_a^6 = \mathcal{O} \left( \frac{1}{8r} + \frac{r}{2k_r} \right)^3 \quad \text{and} \quad \mathbb{E} |Y_a|^3 = \mathcal{O} \left( \frac{1}{8r} + \frac{r}{2k_r} \right)^3.
\]

Indeed, then, according to (5.20),
\[
e(t^*) \leq \mathcal{O} \left( \frac{1}{8r} + \frac{r}{2k_r} \right)^2,
\]

and it follows that
\[
H_{k,r}^{(2)} \geq -\log \left( 1 - \left( \frac{1}{8r} + \frac{r}{2k_r} \right) + \mathcal{O} \left( \frac{1}{8r} + \frac{r}{2k_r} \right)^2 \right) = \left( \frac{1}{8r} + \frac{r}{2k_r} \right) + \mathcal{O} \left( \frac{1}{8r} + \frac{r}{2k_r} \right)^2.
\]

(5.21)
which is the desired result. Thus, the remainder of this proof is focused on proving these three statements. Clearly, by symmetry, we have $\mathbb{E} e^{3|Y_0|} \leq 2\mathbb{E} e^{3|Y_0|}$. Using Cauchy-Schwarz (5.10) with $X = e^{3\zeta S_{R_0}}$ and $Y = e^{3\zeta S_{R_1}}$, results in

$$
\mathbb{E} e^{3|Y_0|} \leq \sqrt{\mathbb{E} e^{6\zeta S_{R_0}} \mathbb{E} e^{6\zeta S_{R_1}}}. 
$$

(5.22)

In order to prove that the expression above is bounded, the following lemma will be useful. The proof is not given here. The lemma follows as a special case of Lemma 6.10.

**Lemma 5.4** Suppose $A_1, A_2 \subset \mathbb{N} \cup \{0\}$ are disjoint and let $S_A = \sum_{m \in A} X_m$. Then $\mathbb{E} e^{3\zeta S_{A_1} S_{A_2}}$ is uniformly bounded whenever $\frac{\zeta^2 |A_1|}{|A_1|} \leq 1 - \varepsilon$, $0 < \varepsilon < 1$ fixed.

Since $|R_0^r| = k_r$ and $|R_1| = r$ and both $r/k_r$ and $1/r$ are $o(1)$, we conclude that $\mathbb{E} e^{3|Y_0|}$ is bounded. Indeed, $r/k_r$ and $1/r$ are clearly bounded by $1 - \varepsilon$, when $k$ is sufficiently large, so that we can apply Lemma 5.4 on the right-hand side of (5.22).

Using $(x + y)^n \leq 2^{n-1}(x^n + y^n)$, it is straightforward to show that for $t^*$,

$$
\mathbb{E} Y_a \leq C \left( \frac{1}{r^6} \mathbb{E} S_{R_0}^6 + \frac{1}{k_r^6} \mathbb{E} S_{R_1}^6 + \mathbb{E} S_{R_0}^6 \right).
$$

We obtain

$$
\mathbb{E} Y_a \leq C \left( \frac{1}{r^3} + \frac{r^3}{k_r^6} \right) \leq C \left( \frac{1}{8r} + \frac{r}{2k_r} \right)^3.
$$

Similarly,

$$
\mathbb{E} |Y_q| \leq C \left( \frac{1}{r^3} + \frac{1}{k_r^6} \mathbb{E} S_{R_1}^6 \right) \leq C \left( \frac{1}{r^3} + \frac{r^3}{k_r^6} \right) \leq C \left( \frac{1}{8r} + \frac{r}{2k_r} \right)^3.
$$

This completes the proof of the lower bound for $H_{k_r}^{(2)}$.

**Step 2: upper bound for $H_{k_r}^{(2)}$.** We will define an appropriate ellipse $\mathcal{E} \subset \mathbb{R}^2$ with $0 \notin \mathcal{E}^0$, the interior of $\mathcal{E}$. In order to show that the supremum of $-\log h(t)$ is attained in $\mathcal{E}^0$, it is sufficient to show that on the boundary of the ellipse $h(t) > 1$. Since $h(0) = 1$ and $h$ is log-convex, we can then conclude that $h(t) > 1$ outside the ellipse and thus the supremum is never attained there. Indeed, whenever $t \notin \mathcal{E}$, there exists a unique $0 < \alpha < 1$ such that $\alpha t \in \partial \mathcal{E}$. From convexity of $h$ and $h(\alpha t) > 1$ it follows that

$$
1 < h(\alpha t) = h(\alpha t + (1 - \alpha)0) \leq (1 - \alpha)h(0) + (1 - \alpha)h(\alpha t) = h(\alpha t) + (1 - \alpha).
$$

It immediately follows that $h(t) > 1$. Whenever $t \in \mathcal{E}^0$, we can prove the desired upper bound. We often minimize $h(t)$, rather than maximizing $-\log h(t)$.

We have, according to (5.17) and (5.19),

$$
h(t) = 1 + \mathbb{E} Y_a + \mathbb{E} Y_a^2 / 2 + e(t),
$$

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where \(e(t) \geq \mathbb{E} Y_a(2 Y_a^2/2 + Y_a^4 e^{Y_a})/24\). Substitution of the moments displayed in (5.16) leads to

\[
h(t) \geq 1 + t_1 r + t_2 + t^2_1 r k_r + 2 t^2_r r + \mathbb{E} Y_a Y_a^2/2
\]

\[
= 1 - \left(\frac{1}{8 r} + \frac{r}{2 k_r}\right) + \frac{1}{2} r k_r \left(t_1 + \frac{1}{k_r}\right)^2 + 2 r \left(t_2 + \frac{1}{4 r}\right)^2 + e(t).
\]

Define

\[
\mathcal{E} = \left\{ t : \frac{1}{2} r k_r \left(t_1 + \frac{1}{k_r}\right)^2 + 2 r \left(t_2 + \frac{1}{4 r}\right)^2 \leq \left(\frac{1}{8 r} + \frac{r}{2 k_r}\right)^2 \right\}.
\]

Note that the ellipse indeed contains \(t = 0\). For \(t \in \partial \mathcal{E}\) we have by the triangle inequality

\[
|t_1| \leq C \left(\frac{1}{8 r} + \frac{r}{2 k_r}\right)^{1/2} / (r k_r)^{1/2},
\]

\[
|t_2| \leq C \left(\frac{1}{8 r} + \frac{r}{2 k_r}\right)^{1/2} / r^{1/2}.
\]

Concerning \(e(t)\), we use Cauchy-Schwarz (5.10) to bound \(\mathbb{E} Y_a Y_a^2\) for \(t \in \mathcal{E}\).

\[
|\mathbb{E} Y_a Y_a^2| \leq (\mathbb{E} Y_q^2)^{1/2} (\mathbb{E} Y_a^4)^{1/2}.
\]

For convenience we further use \(\mathbb{E} |Z|^p \leq (\mathbb{E} |Z|^q)^p/q\) for \(p \leq q\) and any random variable \(Z\) to obtain\(^1\)

\[
|\mathbb{E} Y_a Y_a^2| \leq (\mathbb{E} Y_q^4)^{1/4} (\mathbb{E} Y_a^6)^{2/3}.
\]

Using \((x + y)^n \leq 2^{n-1} (|x|^n + |y|^n)\), \(x, y \in \mathbb{R}\), we arrive for \(t \in \mathcal{E}\) at

\[
\mathbb{E} Y_q^4 \leq C(t_1^4 \mathbb{E} S^8 + t_2^4) \leq C(t_1^4 r^4 + t_2^4) \leq C \left(\frac{1}{8 r} + \frac{r}{2 k_r}\right)^4 r^4 + \left(\frac{1}{8 r} + \frac{r}{2 k_r}\right)^2 r^2
\]

\[
\leq C \left(\frac{1}{8 r} + \frac{r}{2 k_r}\right)^4,
\]

\[
\mathbb{E} Y_a^6 \leq C(t_1^6 \mathbb{E} S^6 + t_2^6 \mathbb{E} S^6) \leq C(t_1^6 r^3 k_r^3 + t_2^6 r^3) \leq C \left(\frac{1}{8 r} + \frac{r}{2 k_r}\right)^3 r^3 k_r^3
\]

\[
\leq C \left(\frac{1}{8 r} + \frac{r}{2 k_r}\right)^3.
\]

This results in

\[
|\mathbb{E} Y_a Y_a^2| \leq C \left(\frac{1}{8 r} + \frac{r}{2 k_r}\right)^2.
\]

For the second term in \(e(t)\), we use \(\zeta Y_a \leq |Y_a|\) a.s. and Hölder’s inequality twice to obtain

\[
|\mathbb{E} Y_a Y_a^2 e^{Y_a}| \leq (\mathbb{E} Y_a^6)^{2/3} (\mathbb{E} Y_a^4 e^{3 |Y_a|})^{1/3} \leq (\mathbb{E} Y_q^4)^{1/4} (\mathbb{E} Y_a^6)^{2/3} (\mathbb{E} e^{12 |Y_a|})^{1/12}.
\]

The product of the first two expectations on the right-hand side are bounded by \(\left(\frac{1}{8 r} + \frac{r}{2 k_r}\right)^2\), by the bounds (5.25) and (5.26), so it suffices to show that \(\mathbb{E} e^{12 |Y_a|}\) is bounded. Following

\(^1\)This easily follows from Hölder’s inequality, applied on \(Z\) and the random variable identical to 1.
the approach in the lower bound (see (5.22)), this fact follows from Cauchy-Schwarz and Lemma 5.4, together with the bounds on \( t_1 \) and \( t_2 \).

We therefore conclude that for all \( t \in \partial \mathcal{E} \) and \( k \) sufficiently large

\[
h(t) \geq 1 + \left( \frac{1}{8r} + \frac{r}{2k_r} \right) \left( -1 + 2 - C \left( \frac{1}{8r} + \frac{r}{2k_r} \right) \right) > 1,
\]

and thus we can conclude that the infimum over \( h(t) \) is not attained on the complement of the ellipse. From (5.23), we know that for \( t \in \mathcal{E}^0 \),

\[
h(t) \geq 1 - \frac{1}{8r} - \frac{r}{2k_r} + \frac{1}{2} r_k r \left( t_1 + \frac{1}{k_r} \right)^2 + 2r \left( t_2 + \frac{1}{4r} \right)^2 - C \left( \frac{1}{8r} + \frac{r}{2k_r} \right)^2.
\]

It is clear that the minimum of the right-hand side is attained at \( t = (-\frac{1}{k_r}, -\frac{1}{4r}) \) and this leads to

\[
\inf_t h(t) \geq 1 - \left( \frac{1}{8r} + \frac{r}{2k_r} \right) \left( 1 + \mathcal{O} \left( \frac{1}{8r} + \frac{r}{2k_r} \right) \right).
\]

Finally, repeat the derivation in (5.21) to complete the proof of Theorem 5.2. \( \square \)

### 5.3 Asymptotic behaviour of \( H_k^{(s)} \) for general \( s \)

We provided a technique to deal with the asymptotic behaviour of \( H_k^{(2)} \), using Taylor series expansions and an ellipse argument. In this section, we will show that the same technique enables us to investigate multistage HD-PIC. Of course, for \( s > 2 \), the analysis becomes more involved. The main result is Theorem 5.5. We will need some additional notation. Define for \( 1 \leq \sigma \leq s-1 \), \( R_\sigma = \{ m : \operatorname{sgn} m(Z_{m, \sigma}^{(\sigma, \sigma)}) < 0 \} \). We note that we use the variable \( \sigma \) to denote a stage, while in our model we reserved \( \sigma^2 \) for the intensity of the AWGN. However, throughout this section we assume that no AWGN is present, so that there is no confusion because of the double meaning of \( \sigma \). Furthermore, we will always refer to "stage \( \sigma \)". We take by definition \( R_0 = \{ 0, \ldots, k-1 \} \), \( R_s = \{ 0 \} \) and \( R_{s+1} = \emptyset \), the empty set. For \( s = 2 \), we have \( R_1 = \{ 1, \ldots, r \} \).

Similarly to \( s = 2 \), we will investigate the situation where we include the configuration of signs of \( (Z_{j, \sigma}^{(\sigma, \sigma)})_{j=0} \), denoted by \( R = [R_1, \ldots, R_{s+1}] \). Thus, we introduce

\[
H^{(s)}_{k, B} = -\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \bigcap_{1 \leq \sigma \leq s} \left\{ \max_{m \in R_\sigma} \operatorname{sgn} m(Z_{m, \sigma}^{(\sigma, \sigma)}) < 0 \right\} \cap \bigcap_{1 \leq \sigma \leq s-1} \left\{ \min_{m \in R_\sigma} \operatorname{sgn} m(Z_{m, \sigma}^{(\sigma, \sigma)}) > 0 \right\} \right).
\]

(5.27)

Similarly to the case \( s = 2 \), we then have \( H_k^{(s)} = \min_{R} H^{(s)}_{k, B} \). We have proven in the previous section that \( H_k^{(2)} = (\frac{1}{8r} + \frac{r}{2k}) (1 + \mathcal{O}(\frac{1}{8r} + \frac{r}{2k})) \). The analog of \( \frac{1}{8r} + \frac{r}{2k} \) for general \( s \) turns out to have a similar form. When we define

\[
\mathcal{H} = \frac{1}{2} \sum_{\sigma=1}^{s} 4^{-1(\sigma \geq 2)} \frac{|R_\sigma|}{|R_{\sigma-1}|},
\]

(5.28)
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we can prove the theorem below, which is the key ingredient to the asymptotics of $H_{k}^{(s)}$. We shall prove this theorem in Section 5.3.1. We will prove it as much as possible along the same lines as Theorem 5.2.

**Theorem 5.5** Fix $1 \leq s < \infty$. When $\mathcal{H} = o(1)$ as $k \to \infty$, the two following properties for $H_{k,R}^{(s)}$ hold:

(i) 

$$H_{k,R}^{(s)} \geq \mathcal{H}(1 + O(\mathcal{H})).$$

(ii) When $R$ has the following disjoint configuration:

$$R_s = \{0\}, \quad R_\sigma = \left\{ \sum_{\sigma' = \sigma + 1}^{\sigma} |R_{\sigma'}|, \sum_{\sigma' = \sigma + 1}^{\sigma} |R_{\sigma'}| + 1, \ldots, \sum_{\sigma' = \sigma}^{n} |R_{\sigma'}| - 1 \right\}, \quad \sigma = s - 1, s - 2, \ldots, 1,$$

we have

$$H_{k,R}^{(s)} \leq \mathcal{H}(1 + O(\mathcal{H})).$$

**Remark:** For $s = 2$, the additional condition in (ii) follows from exchangeability.

Theorem 5.5 is the main ingredient to the asymptotics of $H_{k}^{(s)}$. However, we need the following additional fact, which we will prove in Section 5.3.2.

**Corollary 5.6** Let $A_1, A_2 \subset \mathbb{N} \cup \{0\}$. When $|A_1|/|A_2| \to 0$ and $\alpha = o(|A_2|^{1/2}/|A_1|^{1/2})$,

$$- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( \max_{m \in A_1} \left\{ \alpha + \frac{1}{n} \sum_{j \in A_2} \sum_{j \neq m} X_{ji}X_{mj} \right\} \leq 0 \right) \geq \frac{\alpha^2|A_1|}{2|A_2|} (1 + o(1)).$$

We are now ready to characterize the asymptotic behaviour of $H_{k}^{(s)}$.

**Theorem 5.7** Fix $1 \leq s < \infty$. As $k \to \infty$,

$$H_{k}^{(s)} = \frac{s\sqrt{4}}{8\sqrt{k}} \left( 1 + O\left( \frac{1}{\sqrt{k}} \right) \right). \quad (5.29)$$

**Proof.** Since $H_{k}^{(s)} = \min_{R} H_{k,R}^{(s)}$, substituting a specific configuration $(R_{\sigma})_{\sigma=1}^{n}$ leads to an upper bound of $H_{k}^{(s)}$. We substitute the disjoint configuration with $|R_{\sigma}| = [(k/4)^{s-\sigma/s}] = (k/4)^{s-\sigma/s}(1 + O(k^{-(s-\sigma)/s}))$ in $\mathcal{H}(1 + O(\mathcal{H}))$ (recall Theorem 5.5, (ii)) to obtain

$$H_{k}^{(s)} \leq \frac{s\sqrt{4}}{8\sqrt{k}} \left( 1 + O\left( \frac{1}{\sqrt{k}} \right) \right).$$

Hence, it is sufficient to prove that

$$H_{k}^{(s)} = \min_{R} H_{k,R}^{(s)} \geq \min_{R} \mathcal{H}(1 + O(\mathcal{H})) \geq \frac{s\sqrt{4}}{8\sqrt{k}} \left( 1 + O\left( \frac{1}{\sqrt{k}} \right) \right). \quad (5.30)$$
The first inequality follows from Theorem 5.5 (i). So let us investigate
\[ \min_{\mathcal{H}} \mathcal{H}(1 + \mathcal{O}(\mathcal{H})) = \min_{R_1, R_2, \ldots, R_{s-1}} \mathcal{H}(1 + \mathcal{O}(\mathcal{H})). \]
Since \( \mathcal{H} \) is a function only of the cardinalities of \( R_\sigma \), this equals
\[ \min_{|R_1| \in \mathbb{N}, \ldots, |R_{s-1}| \in \mathbb{N}} \mathcal{H}(1 + \mathcal{O}(\mathcal{H})). \]
We replace the condition that \( |R_\sigma| \in \mathbb{N} \) by \( |R_\sigma| \geq 1 \); this will result in a lower bound of \( H_k^{(s)} \).
It has the main advantage that it allows us to differentiate with respect to \( |R_\sigma| \).

Let us first assume that \( \frac{|R_\sigma|}{|R_{s-1}|} = o(1) \) for all \( 1 \leq \sigma \leq s \). When \( k \) is sufficiently large, there exists an \( 0 \leq M < \infty \), such that \( H_k^{(s)} \geq \mathcal{H} - M\mathcal{H}^2 \). Putting the partial derivatives of the lower bound with respect to \( |R_\sigma| \) equal to zero leads to
\[ \frac{1}{8} \left( \frac{4^{(s=1)}}{|R_{s-1}|} - \frac{|R_{s+1}|}{|R_{s-1}|^2} \right)(1 - 2M\mathcal{H}) = 0, \text{ for all } \sigma = 1, \ldots, s. \]
Since \( \mathcal{H} = o(1) \) it follows that when \( k \) is sufficiently large, the condition for a minimum is
\[ \left( \frac{4^{(s=1)}}{|R_{s-1}|} - \frac{|R_{s+1}|}{|R_{s-1}|^2} \right) = 0, \text{ for all } 1 \leq \sigma \leq s, \]
leading to \( |R_\sigma| = (k/4)^{(s-\sigma)/s} \). Substitution yields (5.30). We note that we are allowed to substitute \( |R_\sigma| = (k/4)^{(s-\sigma)/s} (1 + \mathcal{O}(k^{-1/s})) \) without changing the order of the error terms. This proves for example that for \( s = 2 \) we have \( r_k = \frac{1}{2}\sqrt{k} + \mathcal{O}(1) \).

We finally need to rule out the possibility that \( \frac{|R_\sigma|}{|R_{s-1}|} \) is not \( o(1) \) for some \( 1 \leq \sigma = \sigma^* \leq s \). Clearly \( |R_{s-1}| \to \infty \); otherwise the rate is strictly positive uniformly in \( k \). Since \( \mathbb{P}(A \leq 0, B \leq 0) \leq \mathbb{P}(A \leq 0) \), the rate of the event
\[ \bigcap_{1 \leq \sigma \leq s} \left\{ \max_{m \in R_\sigma} (Z_m^{(\sigma,H)}) < 0 \right\} \bigcap_{1 \leq \sigma \leq s} \left\{ \min_{m \in R_\sigma} (Z_m^{(\sigma,H)}) > 0 \right\} \]
is bounded from below by
\[ \lim_{n \to \infty} -\frac{1}{n} \log \mathbb{P} \left( \max_{m \in R_{\sigma^*}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left( 1 + 2^{(s-2)} \sum_{j \in R_{\sigma-1} \setminus \{m\}} X_{ji}X_{mi} \right) \leq 0 \right\} \right). \]
The event \( \{ m \in R_{\sigma^*} \{ \ldots \} \leq 0 \} \) is clearly increasing in \( |R_{\sigma^*}| \). Replacing \( R_{\sigma^*} \) by an \( R' \subset R_{\sigma^*} \) will result in a decrease of the above rate. Taking \( |R'| = \left[ 2s\sqrt{4}|R_{s-1}|/\sqrt{k} \right] = o(|R_{s-1}|) \) and using Corollary 5.6 gives
\[ \lim_{n \to \infty} -\frac{1}{n} \log \mathbb{P} \left( \max_{m \in R_{\sigma^*}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left( 1 + 2^{(s-2)} \sum_{j \in R_{\sigma-1} \setminus \{m\}} X_{ji}X_{mi} \right) \leq 0 \right\} \right) \]
\[ \geq \frac{|R'|}{8|R_{s-1}|} (1 + o(1)) \geq \frac{s\sqrt{4}}{2\sqrt{k}} (1 + o(1)) > \frac{s\sqrt{4}}{8\sqrt{k}} \left( 1 + \mathcal{O}\left( \frac{1}{\sqrt{k}} \right) \right), \]
when \( k \) is sufficiently large. In other words, when \( \frac{|R_\sigma|}{|R_{s-1}|} \) is not \( o(1) \) for some \( 1 \leq \sigma = \sigma^* \leq s \), the minimum is not attained. This completes the proof. \( \blacksquare \)
5.3.1 Proof of Theorem 5.5

Proof. Substituting $\text{sgn}_{m}(Z_{m}^{(\sigma,H)})$ in (5.27) and using similar bounds as in (4.6) yields

$$H_{k,R}^{(s)} = -\lim_{n \to \infty} \frac{1}{n} \log P\left( \bigcap_{\sigma=1}^{s} \left\{ \max_{m \in R_{\sigma}} Z_{m}^{(\sigma,H)} \leq 0 \right\}, \bigcap_{\sigma=1}^{s} \left\{ \min_{m \in R_{\sigma}} Z_{m}^{(\sigma,H)} \geq 0 \right\} \right),$$

where

$$\tilde{Z}_{m}^{(\sigma,H)} = 1 + 2^{1 \{ \sigma \geq 2 \}} \sum_{j \in R_{\sigma-1} \setminus \{ m \}} \frac{1}{n} \sum_{i=1}^{n} X_{mi}X_{ji}.$$  \hspace{1cm} (5.31)

Step 1: Proof of the lower bound (i). Since we deal with a lower bound, we are allowed to delete events. More precisely, in (5.31) we discard the events $\{ \cdot \geq 0 \}$ and we replace $\max_{m \in R_{\sigma}}$ by $\sum_{m \in R_{\sigma}}$ (because $P(\max_{m \in R_{\sigma}} \cdot \leq 0) \leq P(\sum_{m \in R_{\sigma}} \cdot \leq 0)$) to obtain

$$H_{k,R}^{(s)} \geq -\lim_{n \to \infty} \frac{1}{n} \log P\left( \bigcap_{\sigma=1}^{s} \left\{ \sum_{m \in R_{\sigma}} \tilde{Z}_{m}^{(\sigma,H)} \leq 0 \right\} \right).$$

We write this as (compare (4.12)-(4.14))

$$H_{k,R}^{(s)} \geq -\lim_{n \to \infty} \frac{1}{n} \log P\left( \frac{1}{n} \sum_{i=1}^{n} Y_{i} \leq 0 \right),$$

where $Y_{i}$ is an i.i.d random vector with coordinates

$$Y_{\sigma,i} = \sum_{m \in R_{\sigma}} \left\{ 1 + 2^{1 \{ \sigma \geq 2 \}} \sum_{j \in R_{\sigma-1} \setminus \{ m \}} X_{mi}X_{ji} \right\}$$

$$= |R_{\sigma}| - 2^{1 \{ \sigma \geq 2 \}} |R_{\sigma} \cap R_{\sigma-1}| + 2^{1 \{ \sigma \geq 2 \}} \sum_{m \in R_{\sigma}} X_{mi} \sum_{j \in R_{\sigma-1}} X_{ji}$$

and where for a vector $x$, the statement $x \leq 0$ implies that each entry of $x$ is less than or equal to zero. In the last equality, we have used that $\sum_{j \in R_{\sigma-1} \setminus \{ m \}} X_{ji}X_{mi} = \sum_{j \in R_{\sigma-1}} X_{ji}X_{mi} - 1_{\{m \in R_{\sigma-1}\}}$ and $\sum_{m \in R_{\sigma}} 1_{\{m \in R_{\sigma-1}\}} = |R_{\sigma} \cap R_{\sigma-1}|$. Cramér's theorem gives

$$H_{k,R}^{(s)} \geq \sup_{t \leq 0} \{- \log h(t)\},$$

where $h(t) = \mathbb{E} e^{<t,Y_{1}>}$ is the moment generating function of $Y_{1}$.

The domain $t \leq 0$ naturally arises from the form of the event of interest. We are dealing with events $\{ \cdot \leq 0 \}$, which implies that all $t$'s are non-positive.

We often prefer to minimize $h(t)$, instead of maximizing $-\log h(t)$. We invoke the notation $S_{A} = \sum_{j \in A} X_{j1}$, which we already introduced in (5.12). Note that $R_{\sigma} = R_{\sigma}^{+} \cup (R_{\sigma} \cap R_{\sigma-1})$ and that $R_{\sigma-1} = R_{\sigma-1}^{+} \cup (R_{\sigma} \cap R_{\sigma-1})$, where $R_{\sigma}^{+} = R_{\sigma} \setminus R_{\sigma+1}$ and $R_{\sigma}^{-} = R_{\sigma} \setminus R_{\sigma-1}$. Recall that
for \( s = 2 \), we have \( R_0 = \{0, \ldots, k-1\} \) and \( R_1 = \{1, \ldots, r\} \), so that \( R^+_0 = \{0, r+1, \ldots, k-1\} \) and \( R^+_1 = \emptyset \).

For convenience we split the inner product \( \langle t, Y_1 \rangle \) into two parts, where we use the expressions for \( R^+_\sigma \) and \( R^-_{\sigma-1} \) above:

\[
Y_q = \sum_{\sigma=1}^s t_\sigma \left( |R_\sigma| - 2^{1(\sigma \geq 2)} |R_\sigma \cap R^-_{\sigma-1}| + 2^{1(\sigma \geq 2)} \left( \sum_{j \in R_\sigma \cap R^-_{\sigma-1}} X_{j1} \right)^2 \right)
= \sum_{\sigma=1}^s t_\sigma \left( |R_\sigma| - 2^{1(\sigma \geq 2)} |R_\sigma \cap R^-_{\sigma-1}| + 2^{1(\sigma \geq 2)} S^2_{R_\sigma \cap R^-_{\sigma-1}} \right),
\]

\[
Y_a = \sum_{\sigma=1}^s t_\sigma 2^{1(\sigma \geq 2)} \left( \sum_{j \in R_\sigma} X_{j1} \right) \left( \sum_{j \in R^+_{\sigma-1}} X_{j1} \right) + \left( \sum_{j \in R^-_{\sigma-1} \cap R_\sigma} X_{j1} \right) \left( \sum_{j \in R_\sigma \cup R^+_{\sigma-1}} X_{j1} \right)
= \sum_{\sigma=1}^s t_\sigma 2^{1(\sigma \geq 2)} \left( S_{R^-_{\sigma} \cap R^+_{\sigma-1}} + S_{R^-_{\sigma-1} \cap R_\sigma} S_{R^-_{\sigma} \cup R^+_{\sigma-1}} \right).
\]

Similarly to \( s = 2 \), we have \( \mathbb{E} Y_a = 0 \). However, for general \( s \), \( \mathbb{E} Y_a e^{Y_q} \neq 0 \), \( \mathbb{E} Y^3_a e^{Y_q} \neq 0 \), so that we have to deal with additional error terms.

We substitute \( t^*_\sigma = -4^{-1(\sigma \geq 2)} / |R^-_{\sigma-1}| \), which will lead to a lower bound of the rate. We will prove that \( h(t^*) \leq 1 - \mathcal{H} + \mathcal{O}(\mathcal{H}^2) \). Since \( -\log(1 - x + \mathcal{O}(x^2)) = x(1 + \mathcal{O}(x)) \), the claim in (i) then follows.

We have (recall (5.17))

\[
h(t^*) = 1 + \mathbb{E} Y_q + \frac{1}{2} \mathbb{E} Y^2_a + e(t^*),
\]

where \( e(t^*) \) is given by (recall (5.18))

\[
e(t^*) = \mathbb{E} \left( Y^3_a/6 + Y^4_a e^{Y_\sigma} / 24 + Y_q(Y_a + Y^2_a/2 + Y^3_a/6 + Y^4_a e^{Y_\sigma} / 24) + Y^2_q e^{Y_q} e^{Y_a} \right).
\]

Similarly to (5.20) we use \( \zeta_a \leq |Y_a| \), Hölder’s inequality and the fact that \( \mathbb{E} |Z|^p \leq (\mathbb{E} |Z|^q)^{p/q} \) for \( p \leq q \) and any random variable \( Z \) to bound \( e(t^*) \). This results in

\[
e(t^*) \leq \mathbb{E} Y^3_a + \left( \mathbb{E} Y^3_a \right)^{2/3} \left( \mathbb{E} e^{3|Y_a|} \right)^{1/3} + \mathbb{E} Y_q Y_a + \left( \mathbb{E} Y^4_q \right)^{1/4} \left( \mathbb{E} Y^6_a \right)^{1/3}
+ \left( \mathbb{E} Y^4_q \right)^{1/4} \left( \mathbb{E} Y^6_a \right)^{1/3} \left( \mathbb{E} e^{12|Y_a|} \right)^{1/12} + \left( \mathbb{E} Y^4_a e^{2Y_q} \right)^{1/2} \left( \mathbb{E} e^{2Y_a} \right)^{1/2}.
\]

Since

\[
Y_q \leq \sum_{\sigma=1}^s \frac{4^{-1(\sigma \geq 2)}}{|R^-_{\sigma-1}|} (|R_\sigma| - 2^{1(\sigma \geq 2)} |R_\sigma \cap R^-_{\sigma-1}|) \leq \sum_{\sigma=1}^s \frac{4^{-1(\sigma \geq 2)}}{|R^-_{\sigma-1}|} |R_\sigma| = 2\mathcal{H},
\]
a.s. and \( \mathcal{H} \to 0 \), we have \( \mathbb{E} |Z| e^{3|2Y_q|} \leq 2 \mathbb{E} |Z| \) and \( \mathbb{E} |Z| e^{2Y_q} \leq 2 \mathbb{E} |Z| \) for any random variable \( Z \) when \( k \) is sufficiently large. As a result we can discard the factor \( e^{2Y_q} \) in (5.32).
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We will prove that $\mathbb{E} Y_q$ and $\mathbb{E} Y_a^2$ are the main contributions to the rate and that $e(t^*)$ is of lower order. We have

$$
\mathbb{E} Y_q = \sum_{\sigma=1}^{s} t^*_\sigma \left( |R_{\sigma}| - 2^{1(\sigma \geq 2)} |R_{\sigma-1} \cap R_{\sigma}| + 2^{1(\sigma \geq 2)} \mathbb{E} S_{R_{\sigma-1} \cap R_{\sigma}}^2 \right)
$$

$$
= \sum_{\sigma=1}^{s} t^*_\sigma |R_{\sigma}| - \sum_{\sigma=1}^{s} 4^{-1(\sigma \geq 2)} \frac{|R_{\sigma}|}{|R_{\sigma-1}|} = -2 \mathcal{H}.
$$

Moreover

$$
\frac{1}{2} \mathbb{E} Y_a^2 = \frac{1}{2} \sum_{\sigma=1}^{s} 4^{1(\sigma \geq 2)} (t^*_\sigma)^2 \mathbb{E} \left( S_{R_{\sigma}}^2 S_{R_{\sigma-1}}^+ + S_{R_{\sigma-1} \cap R_{\sigma}} S_{R_{\sigma}^+ \cup R_{\sigma-1}^+} \right)^2 + e_1(t^*),
$$

where

$$
e_1(t^*) = \sum_{1 \leq \sigma < \sigma' \leq s} t^*_\sigma t^*_\sigma' 2^{1(\sigma \geq 2)} 4^{1(\sigma' < 2)} \mathbb{E} \left( S_{R_{\sigma}}^2 S_{R_{\sigma-1}}^2 + S_{R_{\sigma-1} \cap R_{\sigma}} S_{R_{\sigma}^+ \cup R_{\sigma-1}^+} \right)^2.
$$

A straightforward calculation gives that

$$
\mathbb{E} S_{R_{\sigma}}^2 S_{R_{\sigma-1}}^+ S_{R_{\sigma-1} \cap R_{\sigma}} S_{R_{\sigma}^+ \cup R_{\sigma-1}^+} = \mathbb{E} S_{R_{\sigma}} S_{R_{\sigma-1}} \mathbb{E} S_{R_{\sigma}} S_{R_{\sigma-1}} S_{R_{\sigma}^+ \cup R_{\sigma-1}^+} = 0,
$$

because either the first or the second expectation equals zero. Substituting this results in

$$
\frac{1}{2} \mathbb{E} Y_a^2 = \frac{1}{2} \sum_{\sigma=1}^{s} 4^{1(\sigma \geq 2)} (t^*_\sigma)^2 \mathbb{E} \left( S_{R_{\sigma}}^2 S_{R_{\sigma-1}}^2 + S_{R_{\sigma-1} \cap R_{\sigma}} S_{R_{\sigma}^+ \cup R_{\sigma-1}^+} \right)^2 + e_1(t^*). 
$$

(5.33)

Furthermore, because

$$
\mathbb{E} \left( S_{R_{\sigma}}^2 S_{R_{\sigma-1}}^2 + S_{R_{\sigma-1} \cap R_{\sigma}} S_{R_{\sigma}^+ \cup R_{\sigma-1}^+} \right) = |R_{\sigma}| |R_{\sigma-1}^+| + (|R_{\sigma}^-| + |R_{\sigma-1}^-|) |R_{\sigma} \cap R_{\sigma-1}| = |R_{\sigma}| |R_{\sigma-1}^-| - |R_{\sigma} \cap R_{\sigma-1}|^2 \leq |R_{\sigma}| |R_{\sigma-1}|,
$$

the first term in the right-hand side of (5.33) is bounded from above by

$$
\frac{1}{2} \sum_{\sigma=1}^{s} 4^{1(\sigma \geq 2)} (t^*_\sigma)^2 |R_{\sigma}| |R_{\sigma-1}| \leq \frac{1}{2} \sum_{\sigma=1}^{s} 4^{-1(\sigma \geq 2)} |R_{\sigma}| = \mathcal{H}.
$$

Concerning $e_1(t^*)$, the following lemma will be useful.

**Lemma 5.8** Let $A_1, \ldots, A_4 \subset \mathbb{N} \cup \{0\}$. If $A_1 \cap A_2 = \emptyset$ and $A_3 \cap A_4 = \emptyset$, then

(i) \quad $0 \leq \mathbb{E} S_{A_1} S_{A_2} S_{A_3} S_{A_4} \leq |A_3| |A_4|$, 

(ii) \quad $\mathbb{E} S_{A_1} S_{A_2} S_{A_3} S_{A_4} = 2|A_1 \cap A_2||A_1 \cap A_4| \leq 2 \min\{|A_1|^2, |A_3| |A_4|\}$. 

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**Proof.** See Appendix B. 

We observe that for every $\sigma, \sigma'$ four cross terms are present in $e_1(t^*)$. Each cross term is the expectation over the product of four sums, and fits precisely in Lemma 5.8. For example, take for the first cross term $A_1 = R_{\sigma}$, $A_2 = R_{\sigma}^+$, $A_3 = R_{\sigma'}^-$, and $A_4 = R_{\sigma'-1}^+$, then from Lemma 5.8 it follows

$$0 \leq \mathbb{E S_{R_{\sigma}}^* S_{R_{\sigma}^+}^* S_{R_{\sigma}^-}^* S_{R_{\sigma'-1}^+}^* \leq |R_{\sigma'}| |R_{\sigma'-1}|.$$ 

Thus, using that $t^*_\sigma \leq 0$ for all $\sigma$ and applying Lemma 5.8 repeatedly gives

$$|e_1(t^*)| \leq C \sum_{1 \leq \sigma < \sigma' \leq s} t^*_\sigma t^*_\sigma' |R_{\sigma'}| |R_{\sigma'-1}| = C \sum_{1 \leq \sigma < \sigma' \leq s} \frac{|R_{\sigma'}|}{|R_{\sigma-1}|} \leq C \sum_{1 \leq \sigma < \sigma' \leq s} \frac{|R_{\sigma'}|}{|R_{\sigma'-1}|} \frac{|R_{\sigma}|}{|R_{\sigma-1}|} \leq C H^2,$$

since $\sigma \leq \sigma' - 1$ and thus $|R_{\sigma}| \geq |R_{\sigma'-1}|$.

We have shown that

$$h(t^*) \leq 1 - 2H + H + \mathcal{O}(H^2) + e(t^*) = 1 - H + \mathcal{O}(H^2) + e(t^*).$$

To complete the proof of (i), it is sufficient to prove that $\mathbb{E} e^{s_{X_0}}$, $\mathbb{E} e^{12 Y_0}$, and $\mathbb{E} e^{2 Y_0}$ are bounded. $\mathbb{E} Y_0^6 = \mathcal{O}(H^3)$, $\mathbb{E} Y_0 Y_0^4 = \mathcal{O}(H^2)$, $\mathbb{E} Y_0^3 \leq 0$ and $\mathbb{E} Y_0^4 = \mathcal{O}(H^4)$. Indeed, by (5.32) it then follows that $e(t) \leq \mathcal{O}(H^2)$.

The remainder of the proof of (i) is focused on the five statements above. Using $\mathbb{E} e^{s_{X_0}} \leq \mathbb{E} e^{2 Y_0} + \mathbb{E} e^{-2 Y_0}$ and following the argument of (5.22), the moment generating functions $\mathbb{E} e^{s_{Z_0}}$, $\mathbb{E} e^{12 Y_0}$, and $\mathbb{E} e^{2 Y_0}$ are uniformly bounded by Lemma 5.4.

Furthermore, by independence of $S_{R_{\sigma}}^*$ and $S_{R_{\sigma-1}^+}^*$, and of $S_{R_{\sigma}^+ \cap R_{\sigma}}$ and $S_{R_{\sigma}^+ \cup R_{\sigma-1}^+}^*$, and using $(x + y)^6 \leq 32 (x^6 + y^6)$,

$$\mathbb{E} Y_0^6 \leq C \sum_{\sigma = 1}^s \left( \mathbb{E} S_{R_{\sigma}}^* S_{R_{\sigma}^+}^* S_{R_{\sigma}^-}^* S_{R_{\sigma-1}^+}^* + \mathbb{E} S_{R_{\sigma}^+ \cap R_{\sigma}} S_{R_{\sigma}^+ \cup R_{\sigma-1}^+}^* \right) \leq C \sum_{\sigma = 1}^s \left( |R_{\sigma}^+|^3 |R_{\sigma-1}^+|^3 + |R_{\sigma}^+ \cap R_{\sigma}^+|^3 |R_{\sigma}^- \cup R_{\sigma-1}^+|^3 \right) \leq C \sum_{\sigma = 1}^s \frac{|R_{\sigma}^+|^3}{|R_{\sigma-1}|^3} \leq C H^3. $$

Similarly, $\mathbb{E} Y_0^4$ is bounded by

$$C \sum_{\sigma = 1}^s (t^*_\sigma)^4 (|R_{\sigma}^+|^4 + \mathbb{E} S_{R_{\sigma}^+ \cap R_{\sigma}}^4) \leq C \sum_{\sigma = 1}^s (t^*_\sigma)^4 (|R_{\sigma}^+|^4 + |R_{\sigma} \cap R_{\sigma-1}|^4) \leq C H^4. $$

Concerning $\mathbb{E} Y_0^3$, we have

$$\mathbb{E} Y_0^3 = \sum_{\sigma, \sigma', \sigma''} t_{\sigma}^* t_{\sigma'}^* t_{\sigma''}^* \mathbb{E} \left( S_{R_{\sigma}^+ \cap R_{\sigma}} S_{R_{\sigma}^+ \cup R_{\sigma-1}^+} + S_{R_{\sigma-1} \cap R_{\sigma}} S_{R_{\sigma}^+ \cup R_{\sigma-1}^+} + S_{R_{\sigma}^+ \cap R_{\sigma}^+} S_{R_{\sigma}^+ \cup R_{\sigma-1}^+} \right) \times \left( S_{R_{\sigma}^+ \cap R_{\sigma}} S_{R_{\sigma}^+ \cup R_{\sigma-1}^+} + S_{R_{\sigma-1} \cap R_{\sigma}^+} S_{R_{\sigma}^+ \cup R_{\sigma-1}^+} \right).$$

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Since all \( t^*_\sigma \leq 0 \) and
\[
\mathbb{E} \prod_{i=1}^l S_{A_i} \geq 0 \quad \text{for all } l, A_1, \ldots, A_l \in \mathbb{N}.
\] (5.34)
we conclude that \( \mathbb{E} Y^3_a \leq 0 \). Finally \( \mathbb{E} Y_a Y_a \) equals
\[
\mathbb{E} \left\{ \sum_{1 \leq \sigma' \leq s} t^*_\sigma \left( |R_\sigma| - 2^{(s \geq 2)} |R_\sigma \cap R_{\sigma-1}| + 2^{(s \geq 2)} S^2_{R_\sigma \cap R_{\sigma-1}} \right) \right\}
\times \sum_{1 \leq \sigma' \leq s} t^*_\sigma 2^{(s \geq 2)} \left( S^{1+1}_{R_\sigma', R_{\sigma'-1}} + S^{1+1}_{R_{\sigma'-1} \cap R_{\sigma'} R^{+1}_{\sigma-1}} \right) \}
= \sum_{1 \leq \sigma, \sigma' \leq s} t^*_\sigma t^*_\sigma 2^{(s \geq 2)} \mathbb{E} \left[ S^2_{R_\sigma \cap R_{\sigma-1}} + S^{1+1}_{R_{\sigma'-1} \cap R_{\sigma'} R^{+1}_{\sigma-1}} \right].
\]
By Lemma 5.8 (ii), we can bound this by
\[
\sum_{1 \leq \sigma, \sigma' \leq s} t^*_\sigma t^*_\sigma 2^{(s \geq 2)} \mathbb{E} \left[ S^2_{R_\sigma \cap R_{\sigma-1}} + S^{1+1}_{R_{\sigma'-1} \cap R_{\sigma'} R^{+1}_{\sigma-1}} \right] \leq C \mathcal{H}^2,
\]
since \( |R_{\sigma-1}| \geq |R_{\sigma-1}| \) whenever \( \sigma' \leq \sigma \) and \( |R_{\sigma'}| \leq |R_{\sigma+1}| \) for \( \sigma' \geq \sigma + 1 \).

**Step 2: Proof of the upper bound (ii).** An essential ingredient in the proof of the upper bound for \( s = 2 \) was exchangeability. For \( s > 2 \), the set of possible configurations \( R \) is large. Therefore we restrict ourselves to an \( \hat{R} \) for which \( R_\sigma \cap R_{\sigma'} = \emptyset \) for \( \sigma \neq \sigma' \), i.e.,
\[
R_\sigma = \{0\}, \quad R_\sigma = \left\{ \sum_{\sigma' = \sigma+1}^{s} |R_{\sigma'}|, \sum_{\sigma' = \sigma+1}^{s} |R_{\sigma'}| + 1, \ldots, \sum_{\sigma' = \sigma}^{s} |R_{\sigma'}| - 1 \right\}, \quad \sigma = s-1, s-2, \ldots, 1,
\]
By exchangeability, we have that all users within one of the sets \( R_\sigma \) or the set \( R_0^* = R_0 \setminus \{u_{s=1} R_\sigma \} \) behave the same. We first write \( H_{s,k,B}^{(s)} \) (see (5.31) as
\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( \bigcap_{1 \leq \sigma \leq s} \left\{ \max_{m \in R_\sigma} Z_{m,\sigma}^{(s,H)} \leq 0 \right\}, \bigcap_{1 \leq \sigma \leq s-1} \left\{ \min_{m \in R_{\sigma',\sigma}} Z_{m,\sigma}^{(s,H)} \geq 0 \right\}, \bigcap_{1 \leq \sigma \leq s-1, 1 \leq \sigma' \leq s, \sigma' \neq \sigma} \left\{ \min_{m \in R_0^*} Z_{m,\sigma}^{(s,H)} \geq 0 \right\} \right) .
\]
Existence of \( H_{s,k,B}^{(s)} \) follows from Cramér's theorem. Similarly to the proof of Theorem 4.2, we apply a convexity argument (recall (4.8) and below), so that \( H_{s,k,B}^{(s)} \) equals
\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( \bigcap_{1 \leq \sigma \leq s} \left\{ \sum_{m \in R_\sigma} Z_{m,\sigma}^{(s,H)} \leq 0 \right\}, \bigcap_{1 \leq \sigma \leq s-1} \left\{ \sum_{m \in R_{\sigma',\sigma}} Z_{m,\sigma}^{(s,H)} \geq 0 \right\}, \bigcap_{1 \leq \sigma \leq s-1, 1 \leq \sigma' \leq s, \sigma' \neq \sigma} \left\{ \sum_{m \in R_0^*} Z_{m,\sigma}^{(s,H)} \geq 0 \right\} \right) .
\]
The events \( \{ \sum_{m \in R_\sigma} Z_{m,\sigma}^{(s,H)} \geq 0 \}, 1 \leq \sigma' < \sigma \leq s-1 \) and \( \{ \sum_{m \in R_0^*} Z_{m,\sigma}^{(s,H)} \geq 0 \}, 1 \leq \sigma \leq s-1 \) turn out not to contribute to the rate, even though we only prove this when \( k \) is sufficiently large, as shown in Appendix C. Here we suffice with the statement of the result.
Lemma 5.9 Let

\[ \hat{H}^{(s)}_{k,R} = - \lim_{n \to \infty} \frac{1}{n} \log P \left( \bigcap_{1 \leq \sigma \leq s} \left\{ \sum_{m \in R_\sigma} \tilde{Z}_{m}^{(\sigma, R)} \leq 0 \right\} \bigcap_{1 \leq \sigma < \sigma' \leq s} \left\{ \sum_{m \in R_{\sigma'}} \tilde{Z}_{m}^{(\sigma, R)} \geq 0 \right\} \right). \]

Then for \( k \) sufficiently large

\[ H^{(s)}_{k,R} = \hat{H}^{(s)}_{k,R}. \]

We will prove that at stage \( \sigma \), only the block \( R_\sigma \) contributes to the first order of the rate. The blocks \( R_{\sigma'}, \sigma' > \sigma \) contribute only in lower order. In Figure 5.1 the situation is shown.

We next write the above as

\[ H^{(s)}_{k,R} = - \lim_{n \to \infty} \frac{1}{n} \log P \left( \bigcap_{1 \leq \sigma \leq s} \left\{ \frac{1}{n} \sum_{i=1}^{n} Y_{(\sigma),i} \leq 0 \right\} \bigcap_{1 \leq \sigma < \sigma' \leq s} \left\{ \frac{1}{n} \sum_{i=1}^{n} Y_{(\sigma',i)} \geq 0 \right\} \right), \]

where \( (Y_i)_{i=1}^{n} \) are i.i.d. and \( s(s+1)/2 \) dimensional and its components are given by

\[ Y_{(\sigma',i)} = \begin{cases} \sum_{m \in R_0} X_{mi} \sum_{m \in R_{\sigma'}} X_{mi}, & \sigma = 1, 1 \leq \sigma' \leq s, \\ |R_{\sigma'}| + 2 \sum_{m \in R_{\sigma'-1}} X_{mi} \sum_{m \in R_{\sigma'}} X_{mi}, & 2 \leq \sigma \leq \sigma' \leq s. \end{cases} \]

Let \( h(t) = \mathbb{E} e^{\Omega t} \). Cramér's theorem gives

\[ H^{(s)}_{k,R} \leq \sup_{t \in D} \{ - \log h(t) \}. \]
where
\[ D = \{ t : t_{\sigma, \sigma} \leq 0 \text{ for all } 1 \leq \sigma \leq s, \ t_{\sigma, \sigma'} \geq 0 \text{ for all } 1 \leq \sigma < \sigma' \leq s \}. \]  \hspace{1cm} (5.35)

The domain \( D \) arises from the form of the event of interest \( \frac{1}{n} \sum Y_{i,(\sigma, \sigma)} \leq 0, \ \frac{1}{n} \sum Y_{i,(\sigma, \sigma')} \geq 0 \) for all \( \sigma' \neq \sigma \).

To prove the claim in (ii), we will define an appropriate ellipse \( \mathcal{E} \), with \( 0 \in \mathcal{E}^0 \), the interior of \( \mathcal{E} \). Similarly to the proof of the upper bound on \( H_{k_0}^{(2)} \), we will show that \( h(t) > 1 \) for all \( t \in \partial \mathcal{E} \cap D \), which implies that the infimum over \( h(t) \) is attained in \( \mathcal{E}^0 \cap D \). Indeed, whenever \( t \in D \), but \( t \notin \mathcal{E} \), there exists a (unique) \( 0 < \alpha < 1 \), such that \( \alpha t \in \partial \mathcal{E} \cap D \). Convexity of \( h \) and \( h(\alpha t) > 1 \) implies
\[ 1 < h(\alpha t) = h(\alpha t + (1 - \alpha)t) \leq \alpha h(t) + (1 - \alpha)h(0) = \alpha h(t) + (1 - \alpha). \]

We write \( \langle t, Y_1 \rangle = Y_q + Y_a \), where
\[ Y_q = \sum_{\sigma = 1}^s t_{1, \sigma} S_{R_{\sigma}, R_{\sigma}}^2 + \sum_{2 \leq \sigma \leq \sigma' \leq s} t_{\sigma, \sigma'} |R_{\sigma'}|, \]  \hspace{1cm} (5.36)
\[ Y_a = \sum_{\sigma = 1}^s t_{1, \sigma} S_{R_{\sigma}, R_{\sigma}} S_{R_{\sigma'}} + 2 \sum_{2 \leq \sigma \leq \sigma' \leq s} t_{\sigma, \sigma'} S_{R_{\sigma}, R_{\sigma'}} \]  \hspace{1cm} (5.37)
\[ = \left( \sum_{\sigma = 1}^s t_{1, \sigma} S_{R_{\sigma}, R_{\sigma}} + 2 \sum_{\sigma = 2}^s t_{\sigma, \sigma} S_{R_{\sigma}, R_{\sigma}} \right) \]  \hspace{1cm} (5.37).

where we have split according to the signs of the \( t \)'s. We have (recall (5.17))
\[ h(t) = E e^{Y_q + Y_a} = 1 + E Y_q + \frac{1}{2} E Y_a^2 + r(t), \]
where, using (5.18) and \( E Y_a = 0 \),
\[ e(t) \geq E \left( Y_a^3/6 + Y_q (Y_a + Y_a^2/2 + Y_a^3/6 + Y_a^4 e^{Y_a}/24) \right). \]

We next calculate \( E Y_q \) and \( E Y_a^2 \). The former is given by
\[ E Y_q = \sum_{\sigma = 1}^s \sum_{\sigma' = \sigma}^s t_{\sigma, \sigma'} |R_{\sigma'}|. \]

For the latter, we have to do some more work. We take the square of \( Y_a \) and work out the brackets. Then we observe, by (5.34) and \( t_{\sigma, \sigma} \leq 0 \), while \( t_{\sigma, \sigma'} \geq 0 \) for all \( \sigma < \sigma' \), the expectation of the cross-terms involving \( t_{1, \sigma} t_{\sigma, \sigma'} \) and the terms with \( t_{1, \sigma} t_{\sigma, \sigma'} \) are positive. Using further that for all \( 2 \leq \sigma \leq \sigma' < \sigma'' \leq s, \)
\[ E S_{R_{\sigma}, R_{\sigma}} S_{R_{\sigma'}, R_{\sigma}} = \left| R_{1} \right| |R_{\sigma}|, \]
\[ E S_{R_{\sigma}, R_{\sigma}} S_{R_{\sigma}, R_{\sigma-1}} R_{\sigma'} = \left\{ \begin{array}{ll} |R_{1}| |R_{\sigma'}|, & \sigma = 2, \\
0, & \text{ elsewhere}, \end{array} \right. \]
\[ E S_{R_{\sigma-1}, R_{\sigma}} S_{R_{\sigma}, R_{\sigma'}} = \left\{ \begin{array}{ll} |R_{\sigma-1}| |R_{\sigma'}|, & \sigma' = \sigma - 1\ \text{ or } \sigma' = \sigma, \\
0, & \text{ elsewhere}, \end{array} \right. \]
\[ E S_{R_{\sigma-1}, R_{\sigma}} S_{R_{\sigma}, R_{\sigma'-1}} S_{R_{\sigma''}} = 0. \]
we arrive at

\[
\mathbb{E} Y^2_a \geq t_{11}^2 |R_0 \setminus R_1| |R_1| + 4 \sum_{\sigma=2}^s t_{\sigma,\sigma}^2 |R_{\sigma-1}| |R_\sigma|
\]

\[
+ \sum_{\sigma'=2}^s t_{1,\sigma'}^2 |R_0 \setminus R_{\sigma'}| |R_{\sigma'}| + 4 \sum_{2 \leq \sigma < \sigma' \leq s} t_{\sigma,\sigma'}^2 |R_{\sigma-1}| |R_{\sigma'}|
\]

\[
+ \sum_{\sigma'=2}^s t_{1,\sigma'} \left( t_{11} |R_1| |R_{\sigma'}| + 2 t_{\sigma',\sigma'} |R_{\sigma'}| |R_{\sigma'-1}| + 2 t_{\sigma'+1,\sigma'+1} |R_{\sigma'}| |R_{\sigma'+1}| \right)
\]

\[
+ \sum_{\sigma'=3}^s t_{2,\sigma'} t_{11} |R_1| |R_{\sigma'}|.
\] (5.38)

The next goal is to write \( h(t) \) as a sum of squares, similarly to (5.23). However, since the situation is more involved, we need some more abbreviations. First of all, we introduce \( t^*_{\sigma,\sigma'} \) as follows:

\[
t^*_{\sigma,\sigma'} = \begin{cases} -\frac{1}{|R_0 \setminus R_1|}, & \sigma = \sigma' = 1, \\ -\frac{1}{4|R_{\sigma-1}|}, & 2 \leq \sigma = \sigma' \leq s, \\ 0, & 1 \leq \sigma < \sigma' \leq s. \end{cases}
\]

Below we will prove that these \( t^* \)'s are asymptotically close to the optimizers of the minimization problem \( \sup_{t \in B} \{ -\log h(t) \} \). Furthermore, we introduce a slight variation on \( \mathcal{H} \) (recall (5.28)):

\[
\mathcal{H}' = \frac{1}{2} \frac{|R_1|}{|R_0 \setminus R_1|} + \frac{1}{8} \sum_{\sigma=2}^s \frac{|R_{\sigma}|}{|R_{\sigma-1}|}.
\]

Since \( 1/|R_0 \setminus R_1| = (1 + \mathcal{O}(|R_1|/|R_0|))/|R_0| \), we have that \( \mathcal{H}' = \mathcal{H}(1 + \mathcal{O}(\mathcal{H})) \).

After completing the squares (where we have incorporated all cross terms \( t_{1,\sigma} t_{\sigma,\sigma} \) in a special manner) we arrive at (the inequality is only due to the bound on \( \mathbb{E} Y^2_a \))

\[
h(t) = 1 + \mathbb{E} Y_q + \mathbb{E} Y^2_a / 2 + e(t) \geq 1 - \mathcal{H}' + Q_B(t) + e_1(t), \quad (5.39)
\]

where

\[
Q_B(t) = \frac{1}{2} |R_0 \setminus R_1| |R_1| (t_{11} - t_{11}^*)^2 + 2 \sum_{\sigma=2}^s |R_{\sigma-1}| |R_\sigma| \left( 1 - \frac{1}{2} \frac{|R_{\sigma-1}|}{|R_0 \setminus R_\sigma|} \right) (t_{\sigma,\sigma} - t_{\sigma,\sigma}^*)^2
\]

\[
+ \frac{1}{8} \sum_{\sigma'=2}^s |R_0 \setminus R_{\sigma'}| |R_{\sigma'}| (t_{1,\sigma'} - t_{1,\sigma'}^*)^2 + \sum_{2 \leq \sigma < \sigma' \leq s} |R_{\sigma-1}| |R_{\sigma'}| (t_{\sigma,\sigma'} - t_{\sigma,\sigma'}^*)^2.
\]
and

\[
e_1(t) = \frac{3}{4} \sum_{\sigma' = 2}^{s} t_{1,\sigma'}|R_{\sigma'}| + \sum_{2 \leq \sigma < \sigma' \leq s} t_{\sigma,\sigma'}|R_{\sigma'}| + \frac{1}{8} \sum_{\sigma' = 2}^{s} |R_{\sigma} \setminus R_{\sigma'}||R_{\sigma'}| t_{1,\sigma'}^2 \\
+ \frac{1}{4} \sum_{\sigma' = 2}^{s} |R_{\sigma} \setminus R_{\sigma'}||R_{\sigma'}| \left( t_{1,\sigma} + 2 \frac{|R_{\sigma' - 1}|}{|R_{\sigma} \setminus R_{\sigma'}|} (t_{\sigma',\sigma'} - t_{\sigma',\sigma''}) \right)^2 + \sum_{2 \leq \sigma < \sigma' \leq s} |R_{\sigma - 1}||R_{\sigma'}| t_{\sigma,\sigma'}^2 \\
+ \frac{1}{2} \sum_{\sigma' = 2}^{s} t_{1,\sigma'} \left( t_{11}|R_{\sigma'}| + 2t_{\sigma' + 1,s' + 1}|R_{\sigma'}||R_{\sigma' + 1}| \right) + \frac{1}{2} \sum_{\sigma' = 3}^{s} t_{2,\sigma'} t_{11}|R_{1}| |R_{\sigma'}| \\
+ \mathbb{E} Y_a^3 / 6 + \mathbb{E} Y_a (Y_a + Y_a^2 / 2 + Y_a^3 / 6 + Y_a^4 e^{Y_a}/24).
\]

We factor \( e_1(t) \) is bounded from below by (we discarded \( \sum_{3 \leq \sigma < \sigma' \leq s} t_{\sigma,\sigma'}|R_{\sigma'}| \) and the first term on the second line)

\[
e_2(t) \\
= \frac{1}{8} \sum_{\sigma = 2}^{s} |R_{\sigma} \setminus R_{\sigma'}||R_{\sigma'}| t_{1,\sigma'}^2 + \sum_{2 \leq \sigma < \sigma' \leq s} |R_{\sigma - 1}||R_{\sigma'}| t_{\sigma,\sigma'}^2 \\
+ \sum_{\sigma' = 2}^{s} t_{1,\sigma'} \left( \frac{3}{4} |R_{\sigma'}| + \frac{1}{2} t_{11}|R_{\sigma'}| + t_{\sigma' + 1,s' + 1}|R_{\sigma'}||R_{\sigma' + 1}| \right) \\
+ \sum_{\sigma' = 3}^{s} t_{2,\sigma'} \left( |R_{\sigma'}| + \frac{1}{2} t_{11}|R_{1}| |R_{\sigma'}| \right) + \mathbb{E} Y_a^3 / 6 + \mathbb{E} Y_a (Y_a + Y_a^2 / 2 + Y_a^3 / 6 + Y_a^4 e^{Y_a}/24).
\]

We next define the ellipse.

\[
\mathcal{E} = \left\{ \xi : Q_R(\xi) \leq 2H' \right\}. \tag{5.40}
\]

We can derive from (5.40) that for all \( \xi \in \mathcal{E} \),

\[
|t_{1,\sigma}| \leq 3 \left( \frac{H'}{|R_{\sigma} \setminus R_{\sigma'}||R_{\sigma}|} \right)^{1/2}, 1 \leq \sigma \leq s \tag{5.41}
\]

\[
|t_{\sigma,\sigma'}| \leq 3 \left( \frac{H'}{|R_{\sigma'}||R_{\sigma - 1}|} \right)^{1/2}, 2 \leq \sigma \leq \sigma' \leq s. \tag{5.42}
\]

Now we are in the position to bound \( e_2(t) \) for \( \xi \in \mathcal{E} \). However, this is quite involved. For this reason, we state the result in the next lemma, while we transfer the proof to Appendix D.

**Lemma 5.10** There exists a \( C \), not depending on \( k \) or \( R \), such that for \( \xi \in \mathcal{E} \),

\[
e_2(\xi) \geq -C H'^2.
\]
When $k$ is sufficiently large (and thus $H'$ sufficiently small), we now have that (recall (5.39), (5.40) and $e_1(t) \geq e_2(t)$)

$$h(t) \geq 1 + H'(-1 + 2 - C') > 1$$ for $t \in \partial \mathcal{E} \cap D$.

On the other hand, we have, according to (5.39), for $t \in \mathcal{E} \cap D$

$$h(t) \geq 1 - H' + Q_R(t) - C'H'^2.$$ 

It is clear that the infimum of the right-hand side of the equation above is attained at $t = t^*$, where $Q_R(t) = 0$, so that

$$H_{k,R}^{(s)} \leq -\log(1 - H' + O(H'^2)) = H'(1 + O(H')).$$

Since $H' = H(1 + O(H))$, this completes the proof.

\[\square\]

### 5.3.2 Proof of Corollary 5.6

**Proof.** The proof is similar to the proof of Theorem 5.5. Therefore, some details of the proof has been discarded.

We are allowed to replace max by $\sum$ (because $P(\max \cdot \leq 0) \leq P(\sum \cdot \leq 0)$). Thus, we focus on

$$- \lim_{n \to \infty} \frac{1}{n} \log P\left(\frac{1}{n} \sum_{i=1}^{n} Y_i \leq 0\right) = \max_{t \leq 0} \left\{ - \log \mathbb{E} e^{t Y_i} \right\},$$

where

$$Y_i = \alpha |A_1| - |A_1 \cap A_2| + \sum_{m \in A_1} \sum_{j \in A_2} X_{mi} X_{ji}.$$ 

Cramér's theorem gives that the rate equals

$$\sup_{t \leq 0} \left\{ - \log \mathbb{E} e^{t Y_i} \right\} \geq - \log \mathbb{E} e^{t^* Y_i},$$

where we take $t^* = -\alpha/|A_2|$. Note that it is sufficient to show

$$\mathbb{E} e^{t^* Y_i} \leq 1 - \frac{\alpha^2 |A_1|}{2|A_2|}(1 + o(1)).$$

Since $e^{t} \leq 1 + x + x^2/2 + |x|^3 e^{t} / 6$, we can suffice with calculation of the first two moments of $Y_1$ and a bound on $\mathbb{E} |Y_1|^2 e^{t^* |Y_1|}$. We have

$$\mathbb{E} Y_1 = \alpha |A_1|,$$

$$\mathbb{E} Y_1^2 = \alpha^2 |A_1|^2 + |A_1 \cap A_2|^2 + (|A_1||A_2| - 2|A_1 \cap A_2|)$$

$$- 2\alpha |A_1||A_1 \cap A_2| - 2\alpha |A_1||A_1 \cap A_2| - 2|A_1 \cap A_2|^2$$

$$\leq |A_1||A_2| + (1 + \alpha^2)|A_1|^2.$$
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Substitution yields

\[
E e^{t*Y_1} \leq 1 - \frac{\alpha^2 |A_1|}{|A_2|} + \frac{\alpha^2 |A_1|}{2 |A_2|} + \frac{1 + \alpha^2 |A_1|^2}{2 |A_2|^2} + |t^*|^3 E |Y_1|^3 e^{t^*Y_1}/6
\]

\[
= 1 - \frac{\alpha^2 |A_1|}{2 |A_2|} (1 + o(1)) + |t^*|^3 E |Y_1|^3 e^{t^*Y_1}/6.
\]

The proof is completed, if we show that \(|t^*|^3 E |Y_1|^3 e^{t^*Y_1} = o(\alpha^2 |A_1|/|A_2|)|. By Hölder’s inequality,

\[
E |Y_1|^3 e^{t^*Y_1} \leq \left( E Y_1^4 \right)^{3/4} \left( E e^{4t^*Y_1} \right)^{1/4},
\]

so that it is sufficient to prove \(|t^*|^3 (E |Y_1|^4)^{3/4} = o(\alpha^2 |A_1|/|A_2|)| and \(E e^{4t^*Y_1} < \infty\). Recall the inequality \((x + y)^4 \leq 8(x^4 + y^4)\) and (5.13) to obtain

\[
E Y_1^4 \leq 8\alpha^4 |A_1|^4 + \frac{8 E \sum S_{A_1}^4 S_{A_2}^4}{\alpha^3 |A_1|^4 + 8 \left( E \sum S_{A_1}^8 E \sum S_{A_2}^8 \right)^{1/2}} \leq C \left( \alpha^4 |A_1|^4 + |A_1|^2 |A_2|^2 \right).
\]

for some \(C\), not depending on \(A_1\) or \(A_2\). This results in

\[
|t^*|^3 \left( E Y_1^4 \right)^{3/4} \leq C \frac{\alpha^4}{|A_2|^3} \left( \alpha^3 |A_1|^3 + |A_1|^3 |A_2|^3 \right) = o \left( \frac{\alpha^2 |A_1|}{|A_2|} \right).
\]

To prove the bound on \(E e^{4t^*Y_1}\), we use

\[
4|t^*Y_1| \leq 4|t^*| |A_1| + 4|t^*| \left| \sum S_{A_1} S_{A_2} \right| = \frac{4\alpha^2 |A_1|}{|A_2|} + \frac{4\alpha |A_1|^{1/2}}{|A_2|^{1/2}} \frac{\left( S_{A_1}^2 \right)}{|A_1|^{1/2}} \frac{\left( S_{A_2}^2 \right)}{|A_2|^{1/2}} \leq 1 + \frac{1 - \varepsilon}{2} \left( \frac{S_{A_1}^2}{|A_1|} + \frac{S_{A_2}^2}{|A_2|} \right),
\]

for every \(\varepsilon > 0\), since \(\alpha |A_1|^{1/2}/|A_2|^{1/2} = o(1)\), so it is certainly bounded by \(1/2\) and \((1 - \varepsilon)/4\).

To bound \(E e^{4t^*Y_1}\), it is hence sufficient to prove

\[
E e^{1 - \frac{\varepsilon S_{A_1}^2}{|A_1|}} < \infty \quad \text{and} \quad E e^{\frac{1 - \varepsilon}{2} S_{A_2}^2} < \infty.
\]

(5.43)

Since this is the same inequality, we will prove this for \(A_1\), only. The element \(\{0\}\) plays a special role, since \(X_{01} = 1\). In this situation, we can treat \(X_{01}\) as random variable with \(P(X_{01} = +1) = P(X_{01} = +1) = 1/2\), independently from \(X_{m1}\), \(1 \leq m \leq k - 1\). Indeed, if \(\{0\} \in A_1\), we write \(S_{A_1} = S_{A_1}' + 1\) with \(A_1' = A_1 \setminus \{0\}\), and observe that

\[
S_{A_1}'^2 = (S_{A_1}'^2 + 1)^2 \overset{d}{=} (-S_{A_1}' + 1)^2 = (S_{A_1}' - 1)^2.
\]

Therefore, \(S_{A_1}^2 \overset{d}{=} (S_{A_1}' + 1)^2\). In the remainder of this proof, we will assume that \(P(X_{01} = +1) = P(X_{01} = +1) = 1/2\), independently from \(X_{m1}\), \(1 \leq m \leq k - 1\).

A simple comparison of the coefficients in the power series reveals that \(\cosh x \leq e^{x^2/2}\). Indeed, since \((2m)! \geq 2^m m!\) for all \(m \geq 0\),

\[
\cosh t = \sum_{m=0}^{\infty} \frac{t^{2m}}{(2m)!} \leq \sum_{m=0}^{\infty} \frac{(t^2)^m}{2^m m!} = e^{t^2/2}.
\]

(5.44)
Using this, together with the well-known fact that for $Z \sim \mathcal{N}(0, 1)$, $\mathbb{E}e^{tZ} = e^{t^2/2}$ for all $t$ and $\mathbb{E}e^{tZ^2/2} = (1 - t)^{-1/2}$ for $t \leq 1$, results in

$$
\mathbb{E} e^{\frac{1-\varepsilon}{2} \frac{X_k}{|A_1|}} = \mathbb{E} \mathbb{E} \left( e^{\sqrt{1-\varepsilon} \frac{X_k}{|A_1|^{1/2}}} \mid S_n \right) = \mathbb{E} \mathbb{E} \left( e^{\sqrt{1-\varepsilon} \frac{X_k}{|A_1|^{1/2}}} \mid Z \right)
$$

$$
= \mathbb{E} \left( \cosh \left( \sqrt{1-\varepsilon} \frac{Z}{|A_1|^{1/2}} \right) \right)^{|A_1|} \leq \mathbb{E} e^{(1-\varepsilon)Z^2/2} = \frac{1}{\sqrt{\varepsilon}} < \infty
$$

which is the desired result.

\[\blacksquare\]

### 5.4 Asymptotic behaviour of $J_k^{(2)}$

Unfortunately, we are not able to derive asymptotic results for $J_k^{(2)}$ similarly to those above. Instead, we do an analysis on the numerical values of $J_k^{(2)}$, obtained by standard numerical optimization methods. In analogy with (5.1) and (5.29), we assume

$$J_k^{(2)} \approx \alpha k^{-\beta}$$

for large $k$, for some $\alpha, \beta > 0$. We use the numerical values for $J_k^{(2)}$ with $k = 3, \ldots, 27$, see Table 4.1. Since we expect that the deviation between the asymptotic behaviour of $J_k^{(2)}$ and the exact values decreases as $k$ increases, we have incorporated this in our analysis. To be more precise, we do a nonlinear least squares analysis with weight $i$ proportional to $i$, i.e. we solve

$$\arg\min_{\alpha, \beta} \sum_{i=3}^{27} i \left( \alpha i^{-\beta} - J_i \right)^2.$$

The numerical solution is

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} 0.481 \\ 0.515 \end{bmatrix}.$$

This solution does not depend too much on the choice of the weight factors. For example, a weight factor proportional to $i^2$ gives $[0.44, 0.48]$, while no weight at all results in $[0.53, 0.55]$. We conclude from these results that $J_k^{(2)}$ does not tend to 0 as $k^{-1}$, but rather as $k^{-0.5}$. This indicates that for large $n$ and large, but fixed, $k$, SD-PIC works significantly better than MF. We believe that the exact values for $[\alpha, \beta]$ should be $[1/2, 1/2]$, which is very close to the least squares solutions. We see that the rates of HD- and SD-PIC are almost identical for large $k$. 

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5.5 Chernoff bounds and large $k$ depending on $n$

Cramér's theorem is proven in two steps, a lower bound and an upper bound are derived. We will use the technique of the lower bound to bound the bit error probabilities from above.

The technique that we will be using in this section is the exponential Chebycheff inequality, together with an optimization. Since $x \mapsto \exp(tx)$ is strictly decreasing for $t \leq 0$, we have for all $t \leq 0$,
\[
P(Y \leq b) = P(e^{tY} \geq e^{bt}).
\]
The Markov inequality states $P(Z \geq b) \leq \mathbb{E}|Z|/b$. Applying this on $P(e^{tY} \geq e^{bt})$ gives the exponential Chebycheff inequality
\[
P(Y \leq b) \leq e^{-bt}\mathbb{E}e^{tY}, \quad t \leq 0.
\]
This holds for any $t \leq 0$, so that
\[
P(Y \leq b) \leq \inf_{t \leq 0} e^{-bt}\mathbb{E}e^{tY}. \quad (5.45)
\]

We now take $Y = \sum_{i=1}^{n} X_i$ and $b = an$. Taking minus the logarithm of the right-hand side of (5.45), dividing by $n$ and letting $n \to \infty$ directly gives the lower bound for the exponential rate of $P(\frac{1}{n} \sum_{i=1}^{n} X_i \leq a)$. The inequality in (5.45) is denoted by the Chernoff bound, c.f. Chernoff (1952).

For the MF system, the Chernoff bound is straightforward to calculate:
\[
P(\text{sgn}_0(Z_0^{(1)}) < 0) \leq e^{-nI_k} \quad (5.46)
\]
and this holds for any $n$ and $k$. For the HD-PIC system, a similar statement can be derived. When we investigate for example the case of equal powers, we are able to prove
\[
P(\text{sgn}_0(Z_0^{(2,M)}) < 0) \leq \sum_{r=1}^{k-1} \binom{k-1}{r} e^{-nI_{k,r}^{(2)}}. \quad (5.47)
\]
The Chernoff bound is not tight in the sense that it approximates the bit error probability with any desired precision. Instead it gives an upper bound in a very simple form. Note that if $n$ is large, the Chernoff bound is dominated by the term which has the smallest rate. Take for example $k = 12$ and $n = 63$. We can show that (see numerical results) $H_{k,2}^{(2)}$ minimizes $H_{k,2}^{(2)}$. Therefore we expect that most of the sum is contributed by the second term of (5.47). Substitution of the numerical values in the Chernoff bound gives
\[
P(\text{sgn}_0(Z_0^{(2,M)}) < 0) \leq 1.32 \cdot 10^{-3} + 1.76 \cdot 10^{-2} + 5.28 \cdot 10^{-3}
+ 3.44 \cdot 10^{-4} + \ldots \approx 2.45 \cdot 10^{-2}.
\]
The main contribution clearly comes from $r = 2$ (72%), the contributions from $r = 3$ and $r = 1$ are also relevant (22% and 5%), but the remainder of the sum is negligible (1%).

We next use the Chernoff bound in order to take $k$ large with $n$. In practice, one wishes to have as many users as possible, so that the situation where $k$ is fixed and $n \to \infty$ may be violated. Instead, we now take $k = k_n \to \infty$. We prove the following result:
5.5 Chernoff bounds and large $k$ depending on $n$

**Theorem 5.11** When $k_n \to \infty$ such that $k_n = o\left(\frac{n}{\log n}\right)$,

$$
\mathbb{P}(\operatorname{sgn}_0(Z^{(s,H)}_0) < 0) \leq \exp \left( -\frac{s \sqrt{4}}{8} \frac{n}{\sqrt{k_n}} \left(1 + o(1)\right) \right).
$$

(5.48)

**Proof.** We will start by proving Theorem 5.11 for $s = 2$. The extension to $s > 2$ will follow later, and is a small adaptation of the proof for $s = 2$.

Clearly,

$$
\mathbb{P}(\operatorname{sgn}_0(Z^{(2,H)}_0) < 0) \leq \mathbb{P}(Z^{(2,H)}_0 \leq 0)
$$

$$
= \sum_{r=1}^{k_n-1} \binom{k_n-1}{r} \mathbb{P}\left(\max_{1 \leq m \leq r} Z^{(1)}_m \leq 0, \min_{r+1 \leq m \leq k_n-1} Z^{(1)}_m \geq 0, Z^{(2,H)}_0 \leq 0\right).
$$

We split the sum over $r$ in two parts: $1 \leq r \leq 4\sqrt{k_n}$ and $r > 4\sqrt{k_n}$. We start with the first term. The Chernoff bound gives

$$
\mathbb{P}\left(\max_{1 \leq m \leq r} Z^{(1)}_m \leq 0, \min_{r+1 \leq m \leq k_n-1} Z^{(1)}_m \geq 0, Z^{(2,H)}_0 \leq 0\right) \leq e^{-nH^{(2,H)}_{k_n,r}}.
$$

We bound, using that $\binom{k_n-1}{r} \leq k_n^r$ and $e^{-nH^{(2,H)}_{k_n,r}} \leq e^{-nH^{(2,H)}_{k_n}}$,

$$
\sum_{r=1}^{4\sqrt{k_n}} \binom{k_n-1}{r} e^{-nH^{(2,H)}_{k_n,r}} \leq 4k_n^{4\sqrt{k_n}+1/2} e^{-nH^{(2,H)}_{k_n}} = 4e^{4\log k_n \log 4\sqrt{k_n}} e^{-\frac{k_n \log k_n}{2\sqrt{k_n}}} e^{-nH^{(2,H)}_{k_n}}.
$$

The first factor on the right-hand side is $e^{\Theta(\sqrt{k_n})}$, since $k_n = o\left(\frac{n}{\log n}\right)$ implies $k_n \log k_n = o(n)$.

The second factor is also $e^{\Theta(\sqrt{k_n})}$, because of the same reason. Therefore,

$$
\sum_{r=1}^{4\sqrt{k_n}} \binom{k_n-1}{r} e^{-nH^{(2,H)}_{k_n,r}} \leq e^{\Theta(\sqrt{k_n})} e^{-nH^{(2,H)}_{k_n}} = e^{\Theta(\sqrt{k_n})} e^{-\frac{k_n \log k_n}{2\sqrt{k_n}}(1+o(1))}.
$$

It remains to show that the sum over $r > 4\sqrt{k_n}$ has the same asymptotics. In order to do this, we will first prove the following lemma.

**Lemma 5.12** For every $k$, $n$

$$
\mathbb{P}\left(\max_{1 \leq m \leq r} Z^{(1)}_m \leq 0, \min_{r+1 \leq m \leq k_n-1} Z^{(1)}_m \geq 0, Z^{(2,H)}_0 \leq 0\right) \leq \left(1 + \frac{r}{k}\right)^{-n/2} \leq e^{-n/4k}.
$$

**Proof.** We bound, using $\mathbb{P}(A \cap B) \leq \mathbb{P}(A)$,

$$
\mathbb{P}\left(\max_{1 \leq m \leq r} Z^{(1)}_m \leq 0, \min_{r+1 \leq m \leq k_n-1} Z^{(1)}_m \geq 0, Z^{(2,H)}_0 \leq 0\right) \leq \mathbb{P}\left(\max_{1 \leq m \leq r} Z^{(1)}_m \leq 0\right) \leq \mathbb{P}\left(\sum_{m=1}^r Z^{(1)}_m \leq 0\right).
$$
We can compute that
\[
\sum_{m=1}^{r} Z^{(1)}_m = \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{m=1}^{r} X_{mi} \sum_{j=1}^{k} X_{ji} \right).
\] (5.49)

Therefore, by (5.45), the probability of interest is bounded by
\[
\left( \inf_{t \leq s} \mathbb{E} \left( e^{\frac{t}{k} S_{R_1} S_{R_0}^{+}} \right) \right)^n \leq \left( \inf_{t \leq s} \mathbb{E} \left( e^{\frac{t}{k} S_{R_1}^{2} S_{R_0}^{+} / k} \right) \right)^n \leq \left( \mathbb{E} \left( e^{-\frac{S_{R_1}^{2}}{k} / k} e^{\frac{t}{k} S_{R_1} S_{R_0}^{+} / k} \right) \right)^n,
\]
where we invoked the notation \( R_0 = \{0, \ldots, k-1\}, \ R_1 = \{1, \ldots, r\}, \ R_0^{+} = R_0 \setminus R_1, \ S_A = \sum_{m \in A} X_{m} \) and where we have substituted \( t = -2/k \). We next bound the moment generating function from above. We first use the independence of \( S_{R_1} \) and \( S_{R_0}^{+} \) to obtain
\[
\mathbb{E} \left( e^{-\frac{S_{R_1}^{2}}{k} / k} e^{\frac{t}{k} S_{R_1} S_{R_0}^{+} / k} \right) = \mathbb{E} \left( e^{-\frac{S_{R_1}^{2}}{k} / k} \left( \cosh \left( \frac{1}{k} S_{R_1} \right) \right)^{k-1} e^{-1/k} \right).
\] (5.50)

We next use \( e^{-1/k} \leq 1 \) and \( 1 \leq \cosh t \leq e^{t^2/2} \) to bound the expression above as
\[
\mathbb{E} \left( e^{-\frac{S_{R_1}^{2}}{k} / k} \left( \cosh \left( \frac{1}{k} S_{R_1} \right) \right)^{k} \right) \leq \mathbb{E} \left( e^{-\frac{S_{R_1}^{2}}{k} / (2k)} \right).
\] (5.51)

We next wish to replace \( S_{R_1} / r \) by its Gaussian limit \( Z (Z \sim \mathcal{N}(0,1)) \). Using a result of Chapter 7 enables us to do this. Indeed, we can use Proposition 7.9(b) (use \(-1/r \leq -1/k \leq 0\)) to obtain
\[
\inf_{t \leq 0} \mathbb{E} e^{\frac{t}{k} S_{R_1}^{2} S_{R_0}^{+}} \leq \frac{1}{\sqrt{1 + r/k}}.
\] (5.52)

Using this results in
\[
\mathbb{P} \left( \max_{1 \leq m \leq r} Z^{(1)}_m \leq 0, \ \min_{r+1 \leq m \leq k-1} Z^{(1)}_m \geq 0, \ Z^{(2,H)}_0 \leq 0 \right) \leq \left( 1 + \frac{r}{k} \right)^{-n/2}.
\]

This is the first bound. Finally, observe that \( \frac{1}{2} \log(1 + x) \geq x/4 \), for all \( 0 \leq x \leq 1 \), which completes the proof of the lemma.

We now finish the proof of Theorem 5.11 for \( s = 2 \) using Lemma 5.12. Since \( k_n e^{-\frac{n}{4k_n}} = o(1) \), and thus (using \( (k_{n-1}^{-1}) \leq k_n^{-1} \))
\[
\sum_{r > 4\sqrt{k_n}} \frac{(k_n - 1)}{r} \mathbb{P} \left( \max_{1 \leq m \leq r} Z^{(1)}_m \leq 0, \ \min_{r+1 \leq m \leq k-1} Z^{(1)}_m \geq 0, \ Z^{(2,H)}_0 \leq 0 \right)
\leq \sum_{r > 4\sqrt{k_n}} k_n e^{-n \frac{r}{4k_n}} \left( e^{\log k_n - \frac{n}{4k_n}} \right)^{4\sqrt{k_n}} \sum_{r > 0} \left( e^{\log k_n - \frac{n}{4k_n}} \right)^r \leq 2 \exp \left( 4\sqrt{k_n} \log k_n - \frac{n}{\sqrt{k_n}} \right).
\]

This satisfies the required bound since \( \sqrt{k_n} \log k_n = o(\frac{n}{\sqrt{k_n}}) \), so that we have
\[
\mathbb{P}( \text{sgn}_0(Z^{(2,H)}_0) < 0) \leq e^{-\frac{n}{2\sqrt{k_n}}(1+o(1))} + 2 e^{-\frac{n}{\sqrt{k_n}}(1+o(1))} = e^{-\frac{n}{2\sqrt{k_n}}(1+o(1))}.
\]
The proof for $s > 2$ is similar, and we only point out the differences. We can use the proof of Lemma 5.12 to show that the probability that there are at least $c_1 k_n^{(s-1)/s}$ bit errors in the first stage is an error term if $c_1 > 0$ is large enough. Therefore, we only have to deal with the case where there are at most $c_1 k_n^{(s-1)/s}$ bit errors in the first stage. In this case, an easy extension of the proof of Lemma 5.12 shows that the probability that there are at least $c_2 k_n^{(s-2)/s}$ bit errors in the second stage is an error term if $c_2 > 0$ is large enough. Therefore, we may also assume that there are at most $c_2 k_n^{(s-2)/s}$ bit errors in the first stage. We can repeat this argument, so that we only have to deal with the probability that user 0 has a bit error in stage $s$, intersected by the events $|R_s| \leq c_2 k_n^{(s-2)/s}$. We now can use the Chernoff bound and show that the binomial factors are of lower order.

The above result for example implies that when $k_n \rightarrow \infty$ such that $k_n = o(\frac{n}{\log n})$, we have that $\mathbb{P}(\text{sgnr}_0(Z_0^{(1)}) < 0)$ is to leading asymptotics bounded by $\exp\left(-\frac{n}{2k_n}\right)$.

5.6 The signal-to-noise ratio as measure of performance

In this section, we will show that for the case that $k \ll n$, the signal-to-noise ratio (SNR) is in general not a good measure of performance.

In Section 5.1, we have seen for the MF model that $I_k \approx 1/(2k)$. This means that

$$\mathbb{P}(\text{sgnr}_0(Z_0^{(1)}) < 0) \sim e^{-n/(2k)}.$$  

The SNR $= \mathbb{E}Z_0^{(1)}/\sqrt{\text{var}(Z_0^{(1)})}$ is given by $\sqrt{n/(k-1)} \approx \sqrt{n/k}$ (see (5.2)), so that it follows that $\mathbb{P}(\text{sgnr}_0(Z_0^{(1)}) < 0) \sim e^{-n/(2k)}$ (recall (5.5)). Thus, for the MF model, in which all powers are equal and no AWGN is present, the SNR is asymptotically equivalent to the exponential rate. When we use some results of the next chapter, we are able to show that also for the model with unequal powers and AWGN, the same conclusion holds.

For the HD-PIC model, our measure based on the exponential rate implies

$$\mathbb{P}(\text{sgnr}_0(Z_0^{(2,\mu)}) < 0) \sim e^{-n/(2\sqrt{k})}.$$  

However, using the SNR results in

$$\mathbb{P}(\text{sgnr}_0(Z_0^{(2,\mu)}) < 0) \leq \exp\left(-\frac{\frac{n}{2k}}{2(k-1)^2}\right),$$

which we will show below. The latter value is far too small compared to the true values, which clearly indicates that the Gaussian approximation using the SNR is no good.

To prove the upper bound using the SNR, observe that

$$\mathbb{E}(Z_0^{(2,\mu)}) = 1 + 2(k-1)\mathbb{P}(\text{sgnr}_0(Z_0^{(1)}) < 0)\mathbb{E}(X_{11}\mid\text{sgnr}_0(Z_0^{(1)}) < 0) \approx 1,$$

(5.53)
when \( n \) is large compared to \( k \) since \( \mathbb{P}(\text{sgn}r_0(Z_0^{(1)}) < 0) \approx 0 \). Moreover,

\[
\text{var}(Z_0^{(2, m)}) \leq \mathbb{E} \left( (Z_0^{(2, m)} - 1)^2 \right) \leq (k - 1)^2 \mathbb{P}(\text{sgn}r_0(Z_0^{(1)}) < 0),
\]

where for the latter bound we use that \( |X_{ij}| = 1 \). From the Chernoff bound (see (5.46)), and the fact that \( I_k \geq \frac{1}{2k} \), we end up with

\[
\text{var}(Z_0^{(2, m)}) \leq (k - 1)^2 e^{-\frac{n}{2k}}.
\]

This yields that

\[
\text{SNR} \geq \frac{e^{\frac{n}{4k}}}{(k - 1)},
\]

so that

\[
e^{-\text{SNR}^2/2} \leq \exp \left(-\frac{e^{\frac{n}{2k}}}{2(k - 1)^2} \right).
\]

Finally, for the SD-PIC model, we can deduce, using the method of induction that the SNR is given by

\[
\frac{1 - (k - 1)/n}{\sqrt{k(k - 1)/n^2 - 2(k - 1)/n^3}} \approx \frac{n}{k}, \quad \text{if } k = o(n),
\]

so that, according to the Gaussian approximation (5.4),

\[
\mathbb{P}(\text{sgn}r_0(Z_0^{(2, s)}) < 0) \sim e^{-\text{SNR}^2/2} \approx e^{-n^2/(2k^2)}.
\]

This result is clearly not the same as our large deviation conjecture, which states that

\[
\mathbb{P}(\text{sgn}r_0(Z_0^{(2, s)}) < 0) \sim e^{-n/(2\sqrt{k})}.
\]

### 5.7 Numerical results

In Table 5.1 and Figure 5.2, numerical results for \( I_k, H_k^{(2)} \) and \( J_k^{(2)} \) are shown, together with \( 1/(2k) \) and \( 1/(2\sqrt{k}) \). Clearly, \( I_k \) is close to the asymptotic rate \( 1/(2k) \), even for small values of \( k \). The asymptotic rate \( 1/(2\sqrt{k}) \) is a reasonable fit for both \( H_k^{(2)} \) and \( J_k^{(2)} \). The fit for \( H_k^{(2)} \) is worse than the fit for \( I_k \). This is because for MF the error term is \( O(k^{-1}) \), while for HD-PIC the error term is \( O(k^{-1/2}) \). We remark from these numerical results that

\[
|H_k^{(2)} - J_k^{(2)}| \ll \left| H_k^{(2)} - \frac{1}{2\sqrt{k}} \right|,
\]

so that there are no indications that \( J_k^{(2)} \) does not have the asymptotic rate \( \frac{1}{2\sqrt{k}} \).

\(^2\)In this chapter, we have only proven that \( I_k \geq 1/(2k) + O(1/k^2) \). In the next chapter, we will prove that \( I_k \geq \frac{1}{2k} \).
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Table 5.1: Numerical values for $I_k$, $H_k^{(2)}$ and $J_k^{(2)}$

In Figure 5.3, simulated bit error probabilities (for $k = 3$, we obtained exact results) are compared with the Chernoff bound in (5.47) as a function of $n$. It is seen that the Chernoff bound indeed gives upper bounds for all $n$. The exponential rate appears in this figure as the slope of the bit error probability and the Chernoff bound. The difference in slopes is very small, indicating that the large deviation approach is indeed promising for performance evaluation.
Chapter 5. Asymptotic behaviour of exponential rate

Figure 5.2: Rates $I_k (\circ), H_k^{(2)} (\triangle), J_k^{(2)} (\diamond)$ and asymptotics $\frac{1}{2k} (\times), \frac{1}{2\sqrt{k}} (\ast)$.

Figure 5.3: Bit error probabilities (\circ, \triangle and \diamond) and Chernoff bounds (\times, \ast and \centerdot) for $k = 3, 6$ and 9, respectively.
6

Extension to more realistic models

In the previous chapters, the effect of PIC is investigated for the simple model in which all powers are equal and no AWGN is present. A representation of the exponential rate is given for both the HD- and the SD-PIC model. Where possible, the asymptotic behaviour of the exponential rate for large $k$ is investigated. In this chapter, we will extend many of the results for MF and one-stage PIC to the more realistic model in which unequal powers and/or AWGN are incorporated. So in this chapter, our model assumptions are that $P_0, \ldots, P_{k-1}$ are not necessarily equal and $\sigma^2 \geq 0$. Results for MF and HD-PIC have appeared in Klok and van der Hofstad (2000) and Klok (2002). Results for SD-PIC have appeared in Klok, Hooghiemstra and van der Hofstad (2002).

6.1 Exponential rate for MF

The following proposition determines $I_k$, the rate for the MF model.

**Proposition 6.1**

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Z^{(1)}_0 \leq 0) = I_k,
\]

where

\[
I_k = \sup_{t \leq 0} \left\{ -P_0^{1/2} - \sigma^2 t^2 / 2 - \sum_{j=1}^{k-1} \log \cosh(P_j^{1/2} t) \right\}.
\]

**Proof.** Since

\[
Z^{(1)}_0 = \frac{1}{n} \sum_{i=1}^n \left( P_0^{1/2} + \sum_{j=1}^{k-1} P_j X_{ji} + N_i \right),
\]
it follows from Cramér's theorem that

\[ I_k = \sup_{t \leq 0} \{- \log \mathbb{E} e^{t \mathcal{Y}_1} \} = \sup_{t \leq 0} \{- \log h(t) \}, \]

where

\[ h(t) = e^{t P_0^{1/2}} \mathbb{E} e^{t N_1} e^{t \sum_{j=1}^{k-1} P_j X_j} = e^{t P_0^{1/2}} e^{\sigma^2 t^2/2} \prod_{j=1}^k \cosh(P_j t). \quad (6.2) \]

Working out the logarithm gives the desired result. ■

6.1.1 Asymptotic behaviour of \( I_k \)

We focus on the asymptotic behaviour when the ratio between desired signal and noise tends to zero, i.e., \( P_0 /(P + \sigma^2) \to 0 \), where \( P = \sum_{m=0}^{k-1} P_m \). Note that this is quite general, since we no longer require \( k \to \infty \).

The analysis of \( I_k \) is not difficult. However, the proof is set up in such way that it carries over to the technically more involved proof regarding \( H_k^{(2)} \).

**Proposition 6.2** As \( P_0 /(P + \sigma^2) \to 0 \),

\[ \frac{P_0}{2(P + \sigma^2)} \leq I_k \leq \frac{P_0}{2(P + \sigma^2)} \left( 1 + \mathcal{O} \left( \frac{P_0^2}{(P + \sigma^2)} \right) \right). \]

**Proof.** Here and throughout this proof \( C \) denotes a strictly positive constant that may not depend on \( P_j \) and \( \sigma^2 \). The constant \( C \) may change from line to line.

Since \( I_k = \sup_{t \leq 0} \{- \log h(t) \} \), we are allowed to minimize \( h(t) \) rather than maximizing \( - \log h(t) \).

**Step 1: lower bound.** Substitution of (5.44) in (6.2) leads to a lower bound for the rate:

\[ I_k \geq \sup_{t \leq 0} \left\{ - P_0^{1/2} t - t^2 \left( \sum_{j=1}^{k-1} P_j + \sigma^2 \right)/2 \right\} = \frac{P_0}{2(P + \sigma^2 - P_0)} \geq \frac{P_0}{2(P + \sigma^2)}, \quad (6.3) \]

where the optimal \( t \) equals \(-P_0^{1/2}/(P + \sigma^2 - P_0)\). Note that substitution of any value of \( t \) would have resulted in a lower bound of the rate.

**Step 2: upper bound.** To obtain an upper bound, we will use the ellipse argument, introduced in step 2 of the proof of 5.2. This means that we will define an ellipse \( \mathcal{E} \) with \( 0 \in \mathcal{E}^0 \), the interior of \( \mathcal{E} \). In order to show that the supremum of \( - \log h(t) \) (i.e., the infimum of \( h(t) \)) is attained in \( \mathcal{E}^0 \), it is sufficient to show that on \( \partial \mathcal{E} \), the boundary of the ellipse, \( h(t) > 1 \). Since \( h(0) = 1 \) and \( h \) is convex, we can then conclude that \( h(t) > 1 \) outside the ellipse and thus the supremum is never attained there. Once we are allowed to restrict \( t \in \mathcal{E}^0 \), we can prove the desired upper bound.
We observe that $e^x \geq 1 + x + x^2/2 + x^3/6$ and $e^y \geq 1 + y$ to obtain (recall (6.2))

\[
h(t) \geq e^{tP_0^{1/2} + t^2\sigma^2/2} \mathbb{E} \left[ 1 + t \sum_{j=1}^{k-1} P_j^{1/2} X_j + \frac{t^2}{2} \left( \sum_{j=1}^{k-1} P_j^{1/2} X_j \right)^2 + \frac{t^3}{6} \left( \sum_{j=1}^{k-1} P_j^{1/2} X_j \right)^3 \right]
\]

\[= e^{tP_0^{1/2} + t^2\sigma^2/2} \left( 1 + \frac{t^2}{2} \sum_{j=1}^{k-1} P_j \right) \geq \left( 1 + tP_0^{1/2} + t^2\sigma^2/2 \right) \left( 1 + t^2 \sum_{j=1}^{k-1} P_j/2 \right)
\]

\[= 1 + tP_0^{1/2} + t^2(P + \sigma^2 - P_0)/2 + t^3P_0^{1/2}(P - P_0)/2 + t^4(P - P_0)^2/4
\]

\[= 1 - \frac{P_0}{2(P + \sigma^2 - P_0)} + \frac{P + \sigma^2 - P_0}{2} (t - t^*)^2 + \frac{P_0^{1/2}(P - P_0)}{4} t^3 + \frac{(P - P_0)^2}{4} t^4,
\]

where $t^* = -P_0^{1/2}/(P + \sigma^2 - P_0)$ and where we have used in the first inequality that the odd moments are equal to zero. We will show now that the terms with $t^3$ and $t^4$ are of smaller order. We define the ellipse as

\[\mathcal{E} = \left\{ t : \frac{P + \sigma^2 - P_0}{2} (t - t^*)^2 \leq \frac{P_0}{P + \sigma^2 - P_0} \right\}.
\]

For $t \in \mathcal{E}$, the triangle inequality yields

\[|t| \leq \left( 1 + \sqrt{2} \right) \frac{P_0^{1/2}}{(P + \sigma^2 - P_0)}.
\]

Therefore, on $\partial \mathcal{E}$, we have

\[
\left| \frac{P_0^{1/2}(P - P_0)}{2} t^3 + \frac{(P - P_0)\sigma^2}{4} t^4 \right|
\]

\[\leq C \frac{P_0^2 (P - P_0)}{(P + \sigma^2 - P_0)^3} + C \frac{P_0^2 (P - P_0)\sigma^2}{(P + \sigma^2 - P_0)^4} \leq C \frac{P_0^2}{(P + \sigma^2 - P_0)^2}.
\]

We can now conclude that on $\partial \mathcal{E}$ the minimum over $h(t)$ is never attained. Indeed, for $t \in \partial \mathcal{E}$,

\[h(t) \geq 1 - \frac{P_0}{2(P + \sigma^2 - P_0)} + \frac{P_0}{P + \sigma^2 - P_0} - C \frac{P_0^2}{(P + \sigma^2 - P_0)^2} > 1,
\]

when $P_0/(P + \sigma^2 - P_0)$ is sufficiently small.

When we restrict to $t \in \mathcal{E}$, we have

\[h(t) \geq 1 - \frac{P_0}{2(P + \sigma^2 - P_0)} + \frac{P + \sigma^2 - P_0}{2} (t - t^*)^2 - C \frac{P_0^2}{(P + \sigma^2 - P_0)^2},
\]

and the minimum of the right-hand side is attained at $t = t^*$. This results in

\[h(t) \geq 1 - \frac{P_0}{2(P + \sigma^2 - P_0)} - C \frac{P_0^2}{(P + \sigma^2 - P_0)^2}.
\]
The upper bound for the supremum of \(- \log h(t)\) is now obtained by observing that
\[
\frac{P_0}{P + \sigma^2 - P_0} = \frac{P_0}{P + \sigma^2} \left( 1 + O\left( \frac{P_0}{P + \sigma^2} \right) \right)
\]
and
\[- \log(1 - x + O(x^2)) = x + O(x^2).\]

The lower bound of \(I_k\) can also be obtained directly using the Chernoff bound. Together with (5.44) and a straightforward optimization, this yields
\[
P(Z_0^{(1)} \leq 0) \leq e^{-nP_0/(2P)}.
\]

This result is obtained in Simon, Omura, Scholtz and Levitt (1994), following the approach of Kullstam (1977). In fact, an additional factor 1/2 is obtained, so that
\[
P(Z_0^{(1)} \leq 0) \leq \frac{1}{2} e^{-nP_0/(2P)}.
\]

### 6.2 Exponential rate for HD-PIC

The proposition below characterizes the exponential rate for the model in which all powers are equal, but AWGN is present.

**Proposition 6.3** For \(P_0 = P_1 = \ldots = P_{k-1}\),
\[
- \lim_{n \to \infty} \frac{1}{n} \log P(Z_0^{(2,n)} \leq 0) = H^{(2)}_k,
\]
where
\[
H^{(2)}_k = \min_{1 \leq r \leq k-1} H^{(2)}_{k,r}, \quad \text{where} \quad H^{(2)}_{k,r} = \sup_{I = (t_1,t_2,t_3) \in D} \{ - \log h_{k,r}(t) \},
\]
with \(D = \{(s_1,s_2,s_3) : s_1 \leq 0, s_2 \leq 0, s_3 \geq 0,\}\) and \(h_{k,r}(t)\) equals
\[
2^{-r} \sum_{j=r}^{k-r-1} \sum_{l=-(k-r-1)}^{k-r-1+2l} \binom{r}{j+2l} \binom{k-r-1}{k-r-1+2l} e^{i(t_1+j^2)+(t_1+t_2)j+l+(l+1)^2+(l+1)(1+2j)+e(l+t_2+l+(l+1)^2)}.
\]

**Proof.** The proof is analogous to Theorem 4.2. The difference is that we cannot prove that the events \(\{ \cdot \geq 0 \}\) do not contribute to the rate. \(\blacksquare\)
Remark: In all numerical results, the optimal $t_3$ fulfills $t_3 = 0$, which is what we expect. Using $t_3 = 0$ a priori simplifies Proposition 6.3 significantly.

For the case with unequal powers, different users behave differently, so that we are not allowed to sum over the users. Therefore, the moment generating function of interest involves calculating

$$
E \exp \left( \sum_{m=0}^{k-1} t_m X_{m1} \left( \sum_{m=0}^{k-1} P_m^{1/2} X_{m1} \right) \right).
$$

For general $t_m$'s, the only way to do this, is by substituting the possible values for $X_{m1}$ one by one. Clearly, this is not efficient. Only in specific cases, for example when all but one users have the same power, a useable expression can be obtained. We will not go into details here.

6.2.1 Results and examples

Similarly to Section 6.1, we focus on $(P + \sigma^2)/P_0 \to \infty$, where $P = \sum_{i=0}^{k-1} P_i$. For the simple HD-PIC model in which all powers are equal and no AWGN is present, a parameter $r$ was introduced to denote the number of errors in the first stage. It has been shown that typically $r = \frac{1}{2} \sqrt{k}$ errors are made in the first stage. For the general model, the analogue for $r$ is $R \subseteq \{1, \ldots, k-1\}$, the set of users with a wrongly estimated bit. We further define for any set $A \subseteq \{0, \ldots, k-1\}$, $P_A = \sum_{m \in A} P_m$ and

$$
\rho = \frac{1}{2} \sqrt{\frac{P + \sigma^2}{P_0} - \frac{\sigma^2}{4P_0}}. \quad (6.6)
$$

As we will see below, the relevant parameter is not the set $R$ itself, but the quantity $P_R$. We will prove that when $P_R = \rho P_0$, the probability of a bit error in the second stage is dominant, i.e., $P_R = \rho P_0$ is typically observed.

The main result of this section is the following theorem, which is proven in Section 6.2.3.

**Theorem 6.4** For $P_0/(P + \sigma^2) \to 0$,

(i) If $\rho \geq 0$,

$$
H_k^{(2)} \geq \frac{1}{2} \sqrt{\frac{P_0}{P + \sigma^2} - \frac{\sigma^2}{8(P + \sigma^2)}} + O \left( \frac{P_0}{P + \sigma^2} \right).
$$

Moreover, when for $(P + \sigma^2)/P_0 \to \infty$,

$$
\min_{R \subseteq \{1, \ldots, k-1\}} \left| \frac{P_R}{P_0} - \rho \right| = O \left( \frac{P + \sigma^2}{P_0} \right)^{1/4},
$$

$$(*)$$
equality is attained:

\[ H_k^{(2)} = \frac{1}{2} \sqrt{\frac{P_0}{P + \sigma^2}} - \frac{\sigma^2}{8(P + \sigma^2)} + O\left(\frac{P_0}{P + \sigma^2}\right). \]

(ii) If \( \rho \leq 0 \),

\[ \frac{P_0}{\sigma^2} \to 0 \quad \text{and} \quad H_k^{(2)} = \frac{P_0}{2\sigma^2} + O\left(\frac{P_0^2}{\sigma^4}\right). \]

We stress that \( \rho \) is the generalization of \( r_k \). In the case of equal powers, \( P_R/P_0 = |R| \), so that it is straightforward to show that

\[ \min_{R \subseteq \{1, \ldots, k-1\}} \left| \frac{P_R}{P_0} - \rho \right| \leq 1/2. \]

For the general case, where we do not have control over the powers, (*) describes the condition under which \( \rho \) can be approximated closely enough. Due to the increased variety in possible values for \( P_0, \ldots, P_{k-1} \) and \( \sigma^2 \), it is hard to find a non-technical condition that is equivalent to (*).

When we compare Theorem 6.4 with the result for \( I_k \), given in Proposition 6.2, we conclude that HD-PIC gives a significant increase in performance for both small and large \( \sigma^2 \) (case (i) and (ii), respectively). In case (i), we see that

\[ H_k^{(2)} \geq \frac{1}{4} \sqrt{\frac{P_0}{P + \sigma^2}} + O\left(\frac{P_0}{P + \sigma^2}\right), \]

which is a significant improvement over \( I_k = P_0/(2(P + \sigma^2)) + O(P_0^2/(P + \sigma^2)^2) \). In case (ii), we observe that \( \rho \leq 0 \) implies \( R = \emptyset \). In other words, typically all interference due to other users is cancelled successfully, and only the noise remains.

We will next give some practical scenarios. We first observe that an increase in \( \sigma^2 \) should result in a decrease of \( \rho \). Indeed, since \( P + \sigma^2 \geq P \geq P_0 \),

\[ \frac{\partial \rho}{\partial \sigma^2} = \frac{\partial}{\partial \sigma^2} \left( \frac{1}{2} \sqrt{\frac{P + \sigma^2}{P_0}} - \frac{\sigma^2}{4P_0} \right) = \frac{1}{4\sqrt{P_0(P + \sigma^2)}} - \frac{1}{4P_0} \leq 0. \]

Therefore, \( \rho \leq \frac{1}{2} \sqrt{P/P_0} \).

Furthermore, it is straightforward to show that \( (P - P_0)/P_0 \geq \frac{1}{2} \sqrt{P/P_0} \) when \( P_0/(P + \sigma^2) \) is sufficiently small and \( \rho \geq 0 \). This means that \( 0 \leq \rho \leq (P - P_0)/P_0 \), hence we can always find a set \( R \) such that \( |P_R/P_0 - \rho| \leq \frac{1}{2} \max_{1 \leq j \leq k} P_j/P_0 \).

The first example is a typical scenario in the third generation mobile networks. In order to avoid that users transmit data with a relatively high or low power level, the base station assures \( P_m/P_j \leq M \) for all \( 0 \leq m, j \leq k - 1 \).
Example 6.5 Assume that there exists an \( M < \infty \) such that uniform in \( k \), \( M^{-1} < P_j < M \) for all \( 1 \leq j \leq k - 1 \). Then \( P_0/P_0 \) can approximate any value between 0 and \( (P - P_0)/P_0 \) with an error of at most \( M/2 \). Thus, once \( P_0/(P + \sigma^2) \) is sufficiently small, \( (P - P_0)/P_0 \geq \frac{1}{2} \sqrt{P/P_0} \rho \), and thus \( \rho \) can be approximated with an error of at most \( M/2 \). This means that (\(*\)) is fulfilled.

In the previous example, the error \( \min |P_0/P_0 - \rho| \) is bounded from above by the quantity \( \frac{1}{2} \max_{1 \leq j \leq k-1} P_j/P_0 \), and it is shown that this upper bound is sufficiently small. In the next proposition, a wider class of power scenarios is given.

Proposition 6.6 Denote \( \beta : \{1, \ldots, k-1\} \mapsto \{1, \ldots, k-1\} \) the function that orders \( P_1, \ldots, P_k \), e.g., \( P_{\beta(1)} \) is the smallest interfering power, \( P_{\beta(k-1)} \) the largest. Note that \( P_0 \) is discarded. Then (\(*\)) is fulfilled if there exists a constant \( 0 < C < \infty \) such that for all \( m \geq 1 \),

\[
\sum_{j=1}^{m-1} \frac{P_{\beta(j)}}{P_0} \geq \beta \left( \frac{P}{P_0} \right)^{1/4} - C \left( \frac{P}{P_0} \right)^{1/4}.
\]

(\(\ast\))

Proof. We prove the proposition by construction. We approximate \( \rho \) from below by adding one by one the largest relative power. Let \( R \) denote the obtained set. Take the unique smallest \( m = m^* \) such that \( \beta(m^*) \notin R \). By construction

\[
\sum_{m \in \beta^{-1}(R)} \frac{P_{\beta(m)}}{P_0} \leq \rho < \sum_{m \in \beta^{-1}(R)} \frac{P_{\beta(m)} + P_{\beta(m^*)}}{P_0} = \sum_{m \in \beta^{-1}(R)} \frac{P_{\beta(m)}}{P_0} + \left( \frac{P_{\beta(m^*)}}{P_0} - \sum_{m=1}^{m^*-1} \frac{P_{\beta(m)}}{P_0} \right).
\]

Condition (\(\ast\)) implies that the last term on the right-hand side is sufficiently small. Furthermore, \( \sum_{m \in \beta^{-1}(R)} P_{\beta(m)} = \sum_{m \in R} P_m \), so that (\(*\)) holds using this \( R \).

A typical example is \( P_m = 2^{m+1} \). In this case (\(\ast\)) is fulfilled (\( C = 1 \) suffices), so that (\(*\)) is fulfilled by the proposition above. Of course, since every integer can be written as a sum of powers of 2, we knew in advance that \( \rho \) could be approximated with an error of at most \( 1/2 \).

We note that the scenario in Example 6.5 is included in the proposition.

Finally, we treat two practical examples. In Example 6.7, the desired user walks away from the base station, without changing its transmitted power, i.e., the desired user has a power tending to zero. In Example 6.8, one user is present that increases its power. These phenomena are known as the near-far effect. The behaviour of the receiver in the case of a near-far scenario is considered to be extremely relevant, since it characterizes the robustness of the receiver against scenarios where extremely low or high powers cannot be avoided.

Example 6.7 We consider a system, where only the desired user is moving. To be more precise, \( P_1, \ldots, P_{k-1} \) and \( \sigma^2 \) are fixed and user 0 walks away from the base station, so that
When $P_0 \to 0$, clearly $P_0/(P + \sigma^2) \to 0$. In the case that $\sigma^2$ is small but strictly positive, we are in case (i) of Theorem 6.4 for a while, so that, when $\ast$ holds,

$$H_k^{(2)} = \frac{1}{2} \sqrt{P_0} \left( 1 + \mathcal{O}\left( \sqrt{\frac{P_0}{P + \sigma^2}} \right) \right).$$

However, it is inevitable that at a certain moment $\rho = 0$. It is easy to see that this happens when $P_0 = \sigma^4/(4(P + \sigma^2))$. When $P_0$ becomes smaller, we are in case (ii) of Theorem 6.4 and thus

$$H_k^{(2)} = \frac{P_0}{2\sigma^2} + \mathcal{O}\left( \frac{P_0}{2\sigma^2} \right) \to 0.$$

When $\sigma^2 = 0$, a different situation occurs. It is possible to derive that (but not included in this thesis)

$$H_k^{(2)} \geq \frac{P_0(1)}{2P} > 0.$$

This indicates that the inclusion of AWGN in the model is extremely relevant.

**Example 6.8** Let $P_0 = P_1 = \ldots = P_{k-2} = 1, P_{k-1} = k\gamma$ for some $\gamma \geq 0$. We further assume that $\sigma^2 = 0$, even though this is not crucial. For $k \to \infty$, clearly $P_0/P \to 0$. We have $P = 2^{(1 + 1)} k^{(1 + \gamma)}(1 + o(1))$, where $1 \vee \gamma = \max\{1, \gamma\}$. Proposition 6.2 states

$$I_k = \frac{1}{2^{(1 + 1)} + 1 k^{(1 + \gamma)}}(1 + o(1)).$$

Depending on $\gamma$, different typical behaviour can be distinguished for $H_k^{(2)}$. The proofs follow from Theorem 6.4 and Theorem 6.9, which are given in Section 6.2.2.

(i) $0 \leq \gamma < 1/2$: In this case the dominant user behaves similarly to the other users. The rate has the asymptotic form

$$H_k^{(2)} = \frac{1}{2k}(1 + o(1)).$$

Both $(\ast)$ and $(\ast\ast)$ hold.

(ii) $1/2 \leq \gamma \leq 2$: In this case it turns out that user $k - 1$ is typically cancelled successfully, while the rate has the form

$$H_k^{(2)} = \frac{1}{2^{(1 + 1)} + 1 k^{(1 + \gamma)}}2^{(1 + 1)/2}(1 + o(1)).$$

Requirement $(\ast)$ holds, but $(\ast\ast)$ does not for $\gamma > 1$.

(iii) $\gamma > 2$: The optimal value $\rho$ cannot be approximated with the desired error. It turns out that

$$H_k^{(2)} = \frac{1}{8k}(1 + o(1)).$$
whereas we would expect $H_k^{(2)} = 1/(2k^{3/2})(1+o(1))$, which is significantly smaller than $1/(8k)$. In this case, the dominant user is cancelled successfully, but all other users are estimated incorrectly (due to the large interference from the dominant user). Both $(*)$ and $(**)$ do not hold.

### 6.2.2 Asymptotic behaviour of $H_{k,R}^{(2)}$

The result of this section will be the basic ingredient for Theorem 6.4.

Recall from (1.9) and (1.11) that

$$Z_m^{(1)} = P_0^{1/2} + \sum_{j \neq m}^{k-1} P_j^{1/2} \sum_{i=1}^n X_{ji}X_{mi} + \frac{1}{n} \sum_{i=1}^n X_{mi}N_i$$

and

$$Z_0^{(2,H)} = P_0^{1/2} + \sum_{m=1}^{k-1} P_m^{1/2} \sum_{i=1}^n X_{mi} \left(1 - \text{sgnr}_m(Z_m^{(1)})\right) + \frac{1}{n} \sum_{i=1}^n N_i.$$

Clearly $1 - \text{sgnr}_m(\cdot)$ is either 0 or 2. Thus, user $m$ contributes to $Z_0^{(2,H)}$ if and only if $\text{sgnr}_m(Z_m^{(1)}) < 0$. We write

$$\mathbb{P}(Z_0^{(2,H)} \leq 0) = \sum_{R \subseteq \{1, \ldots, k-1\}} \mathbb{P}(Z_0^{(2,H)} \leq 0, B_R),$$

where

$$B_R = \left\{ \max_{m \in R} \text{sgnr}_m(Z_m^{(1)}) < 0, \min_{m \in \{1, \ldots, k-1\}\setminus R} \text{sgnr}_m(Z_m^{(1)}) > 0 \right\}.$$

One verifies from (1.5) that

$$\frac{1}{2} \mathbb{P}(Z_m \leq 0, \cdot) \leq \mathbb{P}(\text{sgnr}_m(Z_m) < 0, \cdot) \leq \mathbb{P}(Z_m \leq 0, \cdot)$$

so that

$$2^{1-k} \sum_{R \subseteq \{1, \ldots, k-1\}} \mathbb{P}\left(\max_{m \in R} Z_m^{(1)} \leq 0, \min_{m \in \{1, \ldots, k-1\}\setminus R} Z_m^{(1)} \geq 0, P_0^{1/2} + \frac{2}{n} \sum_{i=1}^n \sum_{j \in R} P_j^{1/2} X_{ji} \leq 0\right)$$

$$\leq \mathbb{P}(Z_0^{(2,H)} \leq 0)$$

$$\leq \sum_{R \subseteq \{1, \ldots, k-1\}} \mathbb{P}\left(\max_{m \in R} Z_m^{(1)} \leq 0, \min_{m \in \{1, \ldots, k-1\}\setminus R} Z_m^{(1)} \geq 0, P_0^{1/2} + \frac{2}{n} \sum_{i=1}^n \sum_{j \in R} P_j^{1/2} X_{ji} \leq 0\right).$$

(6.7)

Subsequently, we will denote $\tilde{Z}_0^{(2)} = P_0^{1/2} + 2 \sum_{j \in R} P_j^{1/2} \frac{1}{n} \sum_{i=1}^n X_{ji}$. The bar denotes that we have knowledge of stage 1 and we have substituted the values of the sgnr-functions.
We next apply the "largest-exponent-wins" principle, den Hollander (2000), Eqn.(1.2) (the probability with the smallest exponential rate will dominate all others) on the bounds in (6.7) and find (the factor $2^{1-k}$ vanishes)

$$H_{k}^{(2)} = - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Z_{0}^{(2,H)} \leq 0) = \min_{ R \subseteq \{1, \ldots, k-1\} } H_{k,R}^{(2)} \quad (6.8)$$ 

where

$$H_{k,R}^{(2)} = - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \max_{m \in R} Z_{m}^{(1)} \leq 0, \min_{m \in \{1, \ldots, k-1\} \setminus R} Z_{m}^{(1)} \geq 0, \tilde{Z}_{1}^{(2)} \leq 0 \right).$$

Existence of $H_{k,R}^{(2)}$ follows from Cramér's theorem. We note that we can also obtain the result above using the technique of open and closed sets as in (4.6).

Recall the notation $P_{A} = \sum_{j \in A} P_{j}$ and $P = \sum_{j=0}^{k-1} P_{j}$. Define for $R \subseteq \{1, \ldots, k-1\}$

$$\mathcal{H} = \frac{P_{0}}{2(4P_{R} + \sigma^{2})} + \frac{P_{R}}{2(P + \sigma^{2})}.$$ 

**Theorem 6.9** For $\mathcal{H} \to 0$,

$$H_{k,R}^{(2)} = \mathcal{H}(1 + O(\mathcal{H})).$$

**Proof.** We introduce for $A \subseteq \{0, \ldots, k-1\}$ and $(t_{10}, t_{11}, \ldots) \in \mathbb{R}^{k}$,

$$\tilde{S}_{A} = \sum_{m \in A} P_{m}^{1/2} X_{m1} \quad \text{and} \quad T_{A} = \sum_{m \in A} t_{1m} X_{m1}. \quad (6.9)$$

It is useful to observe that

$$\mathbb{E} T_{A}^{2} = \sum_{m \in A} t_{1m}^{2} \quad \text{and} \quad \mathbb{E} T_{A}^{m} \leq c_{m} \left( \sum_{m \in A} t_{1m}^{2} \right)^{m/2} \quad (6.10)$$

for some $c_{m}$ not depending on $A$. Substituting $t_{1m} = P_{m}^{1/2}$ leads to bounds of moments of $\tilde{S}_{A}$.

Similarly to the situation with equal powers we write $R_{0} = \{0, \ldots, k-1\}$ and $R^{c} = R_{0} \setminus R$. Further, we define $R^{c} = \{1, \ldots, k-1\} \setminus R$.

Recall that

$$H_{k,R}^{(2)} = - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \max_{m \in R} Z_{m}^{(1)} \leq 0, \min_{m \in R^{c}} Z_{m}^{(1)} \geq 0, \tilde{Z}_{0}^{(2,R)} \leq 0 \right).$$

**Step 1: lower bound.** Since for any events $A, B$, $\mathbb{P}(A \cap B) \leq \mathbb{P}(A)$, we can discard the event $\{ \min_{m \in R} Z_{m}^{(1)} \geq 0 \}$. This results in

$$H_{k,R}^{(2)} \geq - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \max_{m \in R} Z_{m}^{(1)} \leq 0, \tilde{Z}_{1}^{(2)} \leq 0 \right) = - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} Y_{i} \leq 0 \right), \quad (6.11)$$

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where for \( i = 1, \ldots, n \), \( Y_i \) is a \(|R| + 1\)-dimensional vector with elements

\[
Y_{1m,i} = P_{m}^{1/2} + \sum_{j \neq m} P_j^{1/2} X_{ji} X_{mi} + X_{mi} N_i, \quad m \in R,
\]

\[
Y_{20,i} = P_0^{1/2} + 2 \sum_{j \in R} P_j^{1/2} X_{ji} + N_i.
\]

Since \( (Y_i)_{i=1}^n \) is an i.i.d. sequence, we have, according to Cramér's theorem, that the rate (6.11) is given by

\[
\sup_{t \leq 0} \{-\log h(t)\}, \quad \text{where} \quad h(t) = \mathbb{E} e^{< t \Delta_i >},
\]

with

\[
Y_{1m,1} = X_{m1} \tilde{S}_{R0} + X_{m1} N_1, \quad m \in R \quad \text{and} \quad Y_{20,1} = P_0^{1/2} + 2 \tilde{S}_R + N_1.
\]

We can rewrite the inner product as

\[
< t, Y_1 > = TR \tilde{S}_{R0} + t_{20} P_0^{1/2} + 2 t_{20} \tilde{S}_R + N_1(T_R + t_{20}).
\]

so that, using that \( \mathbb{E} e^{x N_1} = e^{\sigma^2 x^2 / 2} \), and independence

\[
h(t) = \mathbb{E} e^{TR \tilde{S}_{R0} + t_{20} P_0^{1/2} + 2 t_{20} \tilde{S}_R + \sigma^2 (T_R + t_{20})^2 / 2}.
\]

For the lower bound it is sufficient to consider \( t \) in the point \( t_{1m}^* = -P_{m}^{1/2} / (P + \sigma^2), \quad m \in R \), and \( t_{20}^* = -P_0^{1/2} / (4 P_R + \sigma^2) \). Note that for this particular choice of \( t \), \( T_R = -\tilde{S}_R / (P + \sigma^2) \).

We write \( h(t) = \mathbb{E} e^{Y_{1m} + Y_a} \), where

\[
Y_q = - \frac{1}{P + \sigma^2} \tilde{S}_R^2 - \frac{P_0}{4 P_R + \sigma^2} + \frac{\sigma^2}{2} \left( \frac{\tilde{S}_R^2}{P + \sigma^2} - \frac{P_0^{1/2}}{4 P_R + \sigma^2} \right)^2,
\]

\[
Y_a = - \frac{1}{P + \sigma^2} \tilde{S}_R \tilde{R} - \frac{2 P_0^{1/2}}{4 P_R + \sigma^2} \tilde{S}_R.
\]

Using \( e^{x} = 1 + y + y^2 e^{\zeta y} / 2 \) for some \( \zeta = \zeta_y \in [0, 1] \) and \( e^x = 1 + x + x^2 / 2 + x^3 / 6 + x^4 e^{\eta x} / 24 \) for some \( \eta = \eta_x \in [0, 1] \), respectively, we write

\[
h(t) = \mathbb{E} e^{Y_q + Y_a} = 1 + \mathbb{E} Y_q + \mathbb{E} Y_a^2 / 2 + e(t),
\]

where, according to (5.18),

\[
e(t) = \mathbb{E} \left( Y_a + Y_a^3 / 6 + Y_a^4 e^{\eta y_a} / 24 + Y_q (Y_a + Y_a^2 / 2 + Y_a^3 / 6 + Y_a^4 e^{\eta y_a} / 24) + Y_q^2 e^{\eta y_a} e^{y_a} \right).
\]

We use \((x + y)^2 \leq 2(x^2 + y^2)\) to obtain

\[
Y_q \leq - \frac{\tilde{S}_R^2}{P + \sigma^2} - \frac{P_0}{4 P_R + \sigma^2} + \sigma^2 \left( \frac{\tilde{S}_R^2}{(P + \sigma^2)^2} - \frac{P_0}{(4 P_R + \sigma^2)^2} \right)
\]

\[
= - \frac{\tilde{S}_R^2}{P + \sigma^2} \left( 1 - \frac{\sigma^2}{P + \sigma^2} \right) - \frac{P_0}{4 P_R + \sigma^2} \left( 1 - \frac{\sigma^2}{4 P_R + \sigma^2} \right) \leq 0, \quad \text{a.s.}
\]
Using this, together with \( \eta Y_a \leq |Y_a| \) and \( \mathbb{E} Y_a = \mathbb{E} Y_a^3 = 0 \), results in
\[
e(t^*) \leq \mathbb{E} Y_a^4 e^{Y_a} / 24 + Y_a(Y_a + Y_a^3 / 6) + Y_a^2 e^{Y_a}.
\]
Clearly, from \( \mathbb{E} \tilde{S}_R = P_R \) and \( \mathbb{E} \tilde{S}_{Rc} = P_0^{1/2} \) (this follows since \( \{0\} \in R \) and \( X_{01} = 1 \)),
\[
\mathbb{E} Y_q = -\frac{P_R}{P + \sigma^2} - \frac{P_0}{4P_R + \sigma^2} + \frac{\sigma^2}{2} \left( \frac{P_R}{(P + \sigma^2)^2} + \frac{P_0}{(4P_R + \sigma^2)^2} \right)
\]
and
\[
\mathbb{E} Y_a^2 = \frac{P_R(P - P_R)}{(P + \sigma^2)^2} + \frac{4P_0 P_R}{(4P_R + \sigma^2)^2} + \frac{4P_0^{1/2} P_R}{(4P_R + \sigma^2)(P + \sigma^2)^2} P_0^{1/2} P_R,
\]
so that
\[
1 + \mathbb{E} Y_q + \frac{\mathbb{E} Y_a^2}{2} = 1 - \frac{P_0(4P_R + \sigma^2) - \sigma^2 P_0/2 - 2P_0 P_R}{(4P_R + \sigma^2)^2} - \frac{P_R(P + \sigma^2) - \sigma^2 P_R/2 - P_R P/2}{(P + \sigma^2)^2}
\]
\[
= \frac{P_R^2}{(P + \sigma^2)^2} + \frac{2P_0}{4P_R + \sigma^2} P_0 + \frac{P_R}{P + \sigma^2} = 1 - \mathcal{H} + \mathcal{O}(\mathcal{H}^2).
\]
(6.16)
Hölder's inequality (5.9) and the fact that \( \mathbb{E} |Z|^p \leq (\mathbb{E} \left| Z^q \right|)^{p/q} \) for \( p \leq q \) yields
\[
e(t^*) \leq (\mathbb{E} Y_a^6)^{2/3}(\mathbb{E} e^{3Y_a})^{1/3} + \mathbb{E} Y_q Y_a + (\mathbb{E} Y_a^4)^{1/4}(\mathbb{E} Y_a^6)^{1/2} + (\mathbb{E} Y_q^4)^{1/2}(\mathbb{E} e^{2Y_a})^{1/2}.
\]
(6.17)
Hence, in order to have \( e(t^*) \leq C\mathcal{H}^2 \), it is sufficient to prove that for \( t = t^* \), \( \mathbb{E} Y_q Y_a \leq 0 \), \( \mathbb{E} e^{3Y_a} \) and \( \mathbb{E} e^{2Y_a} \) are bounded and that
\[
\mathbb{E} Y_q \leq C\mathcal{H}^4 \quad \text{and} \quad \mathbb{E} Y_a^6 \leq C\mathcal{H}^3.
\]
Indeed, it then follows from (6.17),
\[
e(t^*) \leq C\mathcal{H}^2
\]
and thus, using (6.14) and (6.16), it follows that
\[
H_{3(R)}^{(2)} \geq - \log h(t^*) \geq - \log(1 - \mathcal{H} + \mathcal{O}(\mathcal{H}^2)) = \mathcal{H}(1 + \mathcal{O}(\mathcal{H})),
\]
(6.18)
which is the desired result. Thus, the remainder of this proof is focused on proving these five statements. It is clear that \( \mathbb{E} Y_q Y_a \leq 0 \), since
\[
\mathbb{E} \tilde{S}_R = \mathbb{E} \tilde{S}_{Rc} = \mathbb{E} \tilde{S}_{Rc} = 0 \quad \text{and} \quad \mathbb{E} \tilde{S}_R^2 = P_R \geq 0
\]
and \( \mathbb{E} \tilde{S}_{Rc}^2 = P_0^{1/2} P_R^{1/2} \leq 0 \).
By symmetry, we have \( \mathbb{E} e^{3Y_a} \leq 2 \mathbb{E} e^{3Y_a} \). Recall the definition of \( Y_a \) in (6.12) and use the Cauchy-Schwarz inequality on \( \exp(-3/P + \sigma^2 \tilde{S}_R \tilde{S}_{Rc}) \) and \( \exp(-3/P + \sigma^2 \tilde{S}_R \tilde{S}_{Rc}) \). This results in
\[
\mathbb{E} e^{3Y_a} \leq \left( \mathbb{E} e^{\frac{-6}{P + \sigma^2} \tilde{S}_R \tilde{S}_{Rc}} \right)^{1/2} \left( \mathbb{E} e^{\frac{-12P_0^{1/2}}{4P_R + \sigma^2} \tilde{S}_R \tilde{S}_{Rc}} \right)^{1/2}.
\]
(6.19)
In order to prove that the expression above is bounded, the following lemma will be useful.
Lemma 6.10 Suppose \( A_1, A_2 \subset \mathbb{N} \cup \{0\} \) are disjoint. Let \( \tilde{S}_A = \sum_{m \in A} P_m^{1/2} X_m \) and \( P_A = \sum_{m \in A} P_m \). Then \( \mathbb{E} e^{x \tilde{S}_A \tilde{S}_{A_2}} \) is uniformly bounded whenever \( \frac{x^2 P_{A_2}}{P_{A_1}} \leq 1 - \varepsilon \), for some fixed \( \varepsilon \in (0, 1) \).

**Proof.** Since \( A_1 \) and \( A_2 \) are disjoint, \( \tilde{S}_{A_1} \) and \( \tilde{S}_{A_2} \) are independent and \( \{0\} \in A_1 \) or \( \{0\} \in A_2 \) but not both. Suppose \( \{0\} \notin A_2 \). Then, using \( \cosh x \leq e^{x^2/2} \),

\[
\mathbb{E} e^{x \tilde{S}_{A_1} \tilde{S}_{A_2}} = \mathbb{E} \prod_{i \in A_1} \cosh \left( \frac{x P_i^{1/2}}{P_{A_1}} \tilde{S}_{A_1} \right) \leq \mathbb{E} e^{x^2 P_{A_2} / P_{A_1}} \leq \mathbb{E} e^{\frac{x^2}{2} P_{A_1}^{1/2}}.
\]

When \( \{0\} \in A_2 \), we apply \( \cosh x \leq e^{x^2/2} \) on \( \tilde{S}_{A_1} \), resulting in the same expression with \( A_1 \) replaced by \( A_2 \). We will prove now that the expressions above are finite. This proof is similar to the derivation below equation (5.43). Firstly, we may treat \( X_{01} \) as a random variable with \( \mathbb{P}(X_{01} = +1) = \mathbb{P}(X_{01} = -1) = 1/2 \), independently from \( X_m \), \( 1 \leq m \leq k - 1 \). Indeed, according to the derivation below (5.43), we are allowed to assume so.

Finally, since for \( Z \sim \mathcal{N}(0, 1) \), \( \mathbb{E} e^{tZ} = e^{t^2/2} \), we have for a \( Z \) that is independent of \( \tilde{S}_{A_1} \),

\[
\mathbb{E} e^{\frac{Z}{\sqrt{P_{A_1}}}} = \mathbb{E} e^{|Z| / \sqrt{P_{A_1}}} = \mathbb{E} \prod_{i \in A_1} \cosh \left( \frac{Z}{P_i^{1/2}} P_i^{1/2} \right) \leq \mathbb{E} e^{\frac{Z^2}{\varepsilon}} = \frac{1}{\sqrt{\varepsilon}} < \infty.
\]

Observe that the above also holds for \( P_m \equiv 1 \), in which case \( P_A = |A| \) and \( \tilde{S}_A = S_A \), so that Lemma 5.4 is proven.

We can apply Lemma 6.10 on (6.19), since both \( P_R / (P + \sigma^2) \) and \( P_0 / (4P_R + \sigma^2) \) are \( o(1) \) and thus \( \frac{\sigma P_R^4}{(P + \sigma^2)^2} \) and \( \frac{\sigma P_0^4}{(4P_R + \sigma^2)^2} \) are clearly \( \leq 1 - \varepsilon \) when \( \mathcal{H} \) is sufficiently small, so that indeed \( \mathbb{E} e^{3|Y_a|} \) is uniformly bounded. In an identical manner, \( \mathbb{E} e^{2|Y_a|} \) is also shown to be uniformly bounded.

Using \( |x + y|^l \leq 2^{l-1}(|x|^l + |y|^l) \leq 2^{l-1}(|x| + |y|)^l \) for \( x, y \in \mathbb{R} \) and \( l = 1, 2, \ldots \), together with (6.10), it is straightforward to show that for \( t = t^* \),

\[
\mathbb{E} Y_q^4 \leq C \left( \frac{1}{(P + \sigma^2)^4} \mathbb{E} \tilde{S}_R^8 + \frac{P_0^4}{(4P_R + \sigma^2)^4} + C \frac{P_0^4}{(P + \sigma^2)^6} + \frac{P_0^4}{(4P_R + \sigma^2)^8} \right)
\]

\[
\leq C \frac{P_R^4}{(P + \sigma^2)^4} + C \frac{P_0^4}{(4P_R + \sigma^2)^4} + C \frac{P_0^4}{(P + \sigma^2)^6} + \frac{P_0^4}{(4P_R + \sigma^2)^8}
\]

\[
\leq C \frac{P_R^4}{(P + \sigma^2)^4} + C \frac{P_0^4}{(4P_R + \sigma^2)^4} = \mathcal{O}(\mathcal{H}^4).
\]

Similarly,

\[
\mathbb{E} Y_a^6 \leq C \left( \frac{1}{(P + \sigma^2)^6} \mathbb{E} \tilde{S}_R^6 \mathbb{E} \tilde{S}_R^6 + C \frac{P_0^3}{(4P_R + \sigma^2)^6} \mathbb{E} \tilde{S}_R^6 \leq C \frac{P_R^3 P_0^3}{(P + \sigma^2)^6} + C \frac{P_0^3 P_0^3}{(4P_R + \sigma^2)^6}
\]

\[
\leq C \frac{P_R^3}{(P + \sigma^2)^4} + C \frac{P_0^3}{(4P_R + \sigma^2)^4} = \mathcal{O}(\mathcal{H}^3).
\]
Chapter 6. Extension to more realistic models

**Step 2: upper bound.** For the upper bound, we first observe that

$$
\mathbb{E} \left( \prod_{i=1}^{l} \tilde{S}_{A_i} \right) \geq 0 \quad \text{for all } l \in \mathbb{N}, A_i \subset \mathbb{N} \cup \{0\}. \quad (6.20)
$$

We consider

$$
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \max_{m \in R} Z_m^{(1)} \leq 0, \min_{m \in R} Z_m^{(1)} \geq 0, \tilde{Z}_1^{(2)} \leq 0 \right).
$$

This expression equals

$$
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} Y_i \in D \right),
$$

where $Y_i$ is a $k$-dimensional vector with elements

$$
Y_{1m,i} = \sum_{j=1}^{k-1} P_{j}^{1/2} X_{j1} X_{mi} + X_{mi} N_i, \quad 1 \leq m \leq k - 1,
$$

$$
Y_{20,i} = P_0^{1/2} + 2 \sum_{j \in R} P_{j}^{1/2} X_{j1} + N_i
$$

and where $D = \{ s \in (s_{11}, \ldots, s_{1,k-1}, s_{20}) : s_{1m} \leq 0, m \in R, s_{1m} \geq 0, m \in R^*, s_{20} \leq 0 \}$. According to Cramér’s Theorem,

$$
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} Y_i \in D \right) = \sup_{t \in D} \{- \log h(t)\},
$$

where

$$
h(t) = \mathbb{E} e^{<t, Y_i>}.\]

The expectation over $N_1$ is easy, since $N_1$ is independent of all other randomness in the system and $\mathbb{E} e^{x N_1} = e^{x^2 \sigma^2 / 2}$. Thus, $h(t)$ equals

$$
\mathbb{E} \exp \left\{ \sum_{m=1}^{k-1} t_{1m} \left( \sum_{j=1}^{k-1} P_{j}^{1/2} X_{j1} X_{m1} \right) \right. \left. + t_{20} \left( P_0^{1/2} + 2 \sum_{j \in R} P_{j}^{1/2} X_{j1} \right) + \left( t_{20} + \sum_{m=1}^{k-1} t_{1m} X_{m1} \right)^2 \sigma^2 / 2 \right\}.
$$

We split the exponent in $Y_q$, the quadratic part and $Y_a$, the asymmetric part. This yields

$$
Y_q = t_{20} P_0^{1/2} + T_R \tilde{S}_R + \sigma^2 (t_{20} + T_R + T_{R^*})^2 / 2, \quad (6.21)
$$

$$
Y_a = (2t_{20} \tilde{S}_R + \tilde{S}_{R^*} T_R) + \tilde{S}_{R^*} T_{R^*},
$$

where the parentheses are put in such a way to discriminate between the signs of the elements of $t$. Similarly to (6.14), we write

$$
h(t) = 1 + \mathbb{E} Y_q + \mathbb{E} Y_a^2 / 2 + e(t), \quad (6.22)
$$

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where \( c(t) \) is given in (6.15). Since we can discard all terms that are non-negative a.s., (6.15) reduces to
\[
\epsilon(t) \geq \mathbb{E} \left( Y_a + Y_a^3/6 + Y_q(Y_a + Y_a^2/2 + Y_a^3/6 + Y_a^4 e^{\gamma_q})/24 \right).
\]

A straightforward calculation gives
\[
\mathbb{E} Y_a = t_20 P_0^{1/2} + \sum_{m \in R} P_m^{1/2} t_{1m} + \frac{\sigma^2}{2} \left( t_{20}^2 + \sum_{m \in R} t_{1m}^2 + \sum_{m \in R^*} t_{1m}^2 \right)
\]
and (use \( \delta_{R_0} R_{R^*} = P \sum_{m \in R} t_{1m}^2 + \sum_{m \in R^*} \sum_{i \in R} P_m^{1/2} P_i^{1/2} t_{1m} t_{1i} \))
\[
\mathbb{E} Y_a^2 = 4P R_{20}^2 t_{20}^2 + P_{R^*} \sum_{m \in R} t_{1m}^2 + 4t_{20} P_0^{1/2} \sum_{m \in R} P_m^{1/2} t_{1m} + \sum_{m \in R} t_{1m}^2 + \sum_{m \in R^*} \sum_{i \in R} P_m^{1/2} P_i^{1/2} t_{1m} t_{1i} + \sum_{m \in R^*} P_m^{1/2} t_{1m} \sum_{m \in R^*} P_m^{1/2} t_{1m} \\
\geq 4P R_{20}^2 t_{20}^2 + P_{R^*} \sum_{m \in R} t_{1m}^2 + P \sum_{m \in R^*} t_{1m}^2 + \left( \sum_{m \in R^*} P_m^{1/2} t_{1m} \right) \left( \sum_{m \in R^*} P_m^{1/2} t_{1m} \right),
\]
where we have used \( t \in D \) to obtain the inequality. The above inequalities imply a lower bound for \( h(t) \). The next goal is to write the lower bound of \( h \) as a sum of squares. It is convenient to introduce some more abbreviations. First of all, we introduce \( t_{1m}^* \) and \( t_{20}^* \) as
\[
t_{1m}^* = \begin{cases} 
- \frac{P_m^{1/2}}{P + \sigma^2}, & m \in R, \\
0, & m \in R^*,
\end{cases}
\quad t_{20}^* = -\frac{P_0^{1/2}}{4P_{R^*} + \sigma^2}.
\]
We will prove below that the minimizers of \( \sup_{t \in D} \{-\log h(t)\} \) converge to \( t^* \) as \( \mathcal{H} \rightarrow 0 \).

In order to obtain a lower bound for \( h(t) \), we observe that (the first inequality is only due to the lower bound of \( \mathbb{E} Y_a^2 \))
\[
\mathbb{E} Y_a + \frac{1}{2} \mathbb{E} Y_a^2 \\
\geq \sum_{m \in R} P_m^{1/2} t_{1m} + \frac{P + \sigma^2}{2} - P_{R^*} \sum_{m \in R} t_{1m}^2 + \frac{P + \sigma^2}{2} \sum_{m \in R^*} t_{1m}^2 + P_{0/2} t_{20} + \frac{4P_{R^*} + \sigma^2}{2} t_{20}^2 \\
+ \left( \sum_{m \in R^*} P_m^{1/2} t_{1m} \right) \left( \sum_{m \in R^*} P_m^{1/2} t_{1m} \right) \\
= \frac{P + \sigma^2}{2} \sum_{m \in R} (t_{1m} - t_{1m}^*)^2 - \sum_{m \in R} \frac{P_m}{2(P + \sigma^2)} - \frac{P_{R^*}}{2} \sum_{m \in R^*} t_{1m}^2 + \frac{P + \sigma^2}{2} \sum_{m \in R^*} t_{1m}^2 \\
+ \frac{4P_{R^*} + \sigma^2}{2} (t_{20} - t_{20}^*)^2 - \frac{P_0}{2(4P_{R^*} + \sigma^2)} + \left( \sum_{m \in R^*} P_m^{1/2} t_{1m} \right) \left( \sum_{m \in R^*} P_m^{1/2} t_{1m} \right).
\]

When we substitute this in (6.22) and use \( \mathcal{H} = P_0/(2(4P_{R^*} + \sigma^2)) + P_{R^*}/(2(P + \sigma^2)) \), we arrive at the following lower bound for \( h(t) \)
\[
1 - \mathcal{H} + \frac{P + \sigma^2}{2} \sum_{m \in R} (t_{1m} - t_{1m}^*)^2 + \frac{P + \sigma^2}{4} \sum_{m \in R^*} (t_{1m} - t_{1m}^*)^2 + \frac{4P_{R^*} + \sigma^2}{2} (t_{20} - t_{20}^*)^2 + e_1(t), \quad (6.23)
\]
where
\[
e_1(t) = -\frac{P}{2} \sum_{m \in R} t_{1m}^2 + \frac{P + \sigma^2}{8} \sum_{m \in R^*} t_{1m}^2 + \mathbb{E} Y_a + \frac{1}{2} \left( \sum_{m \in R^*} P_m^{1/2} t_{1m} \right) \left( \sum_{m \in R} P_m^{1/2} t_{1m} \right) + \mathbb{E} Y_q Y_a \\
+ \frac{P + \sigma^2}{8} \sum_{m \in R^*} t_{1m}^2 + \mathbb{E} Y_a^3/6 \\
+ \mathbb{E} Y_q (Y_a^2/2 + Y_a^3/6 + Y_a^4 e^{-\alpha} / 24).
\] (6.24)

Note that we have split \( \frac{P + \sigma^2}{2} \sum_{m \in R} t_{1m}^2 \) into three terms.

The next step is to introduce an appropriate ellipse \( \mathcal{E} \). When we can prove that on \( (\partial \mathcal{E}) \cap D \), \( h(t) > 1 \), we can conclude that on \( \mathcal{E} \cap D \) the minimum is never attained. Indeed, for every \( x \in \mathcal{E} \cap D \), we can find a unique \( 0 < \alpha < 1 \) such that \( \alpha x \in (\partial \mathcal{E}) \cap D \). But then, since \( h(0) = 1 \) and \( h \) is convex, we can deduce that to \( h(x) > 1 \). Clearly the minimum is at most 1 \( (h(0) = 1) \), so \( x \) is never a minimizer.

We define \( \mathcal{E} \) as
\[
\mathcal{E} = \left\{ t \in \mathbb{R}^{|R^*|} : \sum_{m \in R} \frac{P + \sigma^2}{2} (t_{1m} - t_{1m}^*)^2 + \sum_{m \in R^*} \frac{P + \sigma^2}{4} (t_{1m}^* - t_{1m}^*)^2 + \frac{4P + \sigma^2}{2} (t_{20} - t_{20}^*)^2 \leq 2\mathcal{H} \right\}.
\]

It is straightforward to prove that for \( t \in \mathcal{E} \),
\[
|t_{20}| \leq C \sqrt{\frac{\mathcal{H}}{4P + \sigma^2}}, \quad \sum_{m \in R \cup R^*} t_{1m}^2 \leq C \frac{\mathcal{H}}{P + \sigma^2}. \tag{6.25}
\]

We can now bound \( e_1(t) \). Since this is quite involved, we will prove this in Appendix E.

**Lemma 6.11** There exists a constant \( C \), not depending on \( k \) or \( R \), such that for \( t \in \mathcal{E} \),
\[
e_1(t) \geq -C \mathcal{H}^2.
\]

Thus, for \( t \in \mathcal{E} \cap D \), (recall (6.23)), \( h(t) \) is bounded from below by
\[
1 - \mathcal{H} + \frac{P + \sigma^2}{2} \sum_{m \in R} (t_{1m} - t_{1m}^*)^2 + \frac{P + \sigma^2}{4} \sum_{m \in R^*} (t_{1m}^* - t_{1m}^*)^2 + \frac{4P + \sigma^2}{2} (t_{20} - t_{20}^*)^2 - C \mathcal{H}^2. \tag{6.26}
\]

This implies that for \( t \in (\partial \mathcal{E}) \cap D \), and when \( \mathcal{H} \) is small enough,
\[
h(t) \geq 1 - \mathcal{H} + 2\mathcal{H} - C \mathcal{H}^2 > 1,
\]
so that we can conclude that the minimum is never attained outside the ellipse. Finally, when \( t \in \mathcal{E} \cap D \), it is clear that the minimum of the righthand-side of (6.26) is attained at \( t = t^* \), resulting in
\[
H_{k,R}^{(2)} \leq \mathcal{H}(1 + \mathcal{O}(\mathcal{H})).
\]
6.2.3 Proof of Theorem 6.4

According to (6.8), we have to minimize $H_{k,R}^{(2)}$ over subsets $R \subseteq \{1, \ldots, k-1\}$. When we follow the naive approach, we would minimize the asymptotic rate

$$
\frac{P_0}{2(4P_R + \sigma^2)} + \frac{P_R}{2(P + \sigma^2)}
$$

over $P_R$, with the result

$$
H_{k}^{(2)} \approx \frac{1}{2} \sqrt{\frac{P_0}{P + \sigma^2}} - \frac{\sigma^2}{8(P + \sigma^2)} \quad \text{for} \quad \frac{P_R}{P_0} = \frac{1}{2} \sqrt{\frac{P + \sigma^2}{P_0}} - \frac{\sigma^2}{4P_0}.
$$

Since $P_R$ attains values in some grid, depending on the values of the individual $P_j$'s, it is not clear that $P_R/P_0$ can attain the value $\rho$. We show that under the condition in (i), $\rho$ can be attained with the right order deviation.

We will split the proof in three steps. In the first step, we show that when $\mathcal{H} = o(1)$,

$$
H_{k}^{(2)} \geq \begin{cases} 
\frac{1}{2} \sqrt{\frac{P_0}{P + \sigma^2}} - \frac{\sigma^2}{8(P + \sigma^2)} + \mathcal{O} \left( \frac{P_0}{P + \sigma^2} \right), & \rho \geq 0 \\
\frac{P_0}{2\sigma^2} + \mathcal{O} \left( \frac{P_0}{\sigma^4} \right), & \rho \leq 0.
\end{cases}
$$

(6.27)

In step 2, we prove that $\mathcal{H} \geq \varepsilon$ does not give a smaller lower bound when $P_0/(P + \sigma^2)$ is sufficiently small. Finally, we show that the asymptotic upper bound of $H_{k}^{(2)}$ equals the asymptotic lower bound, whenever we assume $\mathcal{H} = o(1)$.

**Step 1: Lower bound when $\mathcal{H} = o(1)$**. According to Theorem 6.9, there exists an $M > 0$ such that when $P_0/(P + \sigma^2)$ is sufficiently small,

$$
\min_{R \subseteq \{1, \ldots, k-1\}} H_{k,R}^{(2)} \geq \min_{R \subseteq \{1, \ldots, k-1\}} \mathcal{H} - M\mathcal{H}^2 \geq \min_{P_R \geq 0} \mathcal{H} - M\mathcal{H}^2.
$$

Taking the derivative of the right-hand side with respect to $P_R$ gives

$$
\left( \frac{-4P_0}{(4P_R + \sigma^2)^2} + \frac{1}{P + \sigma^2} \right) \left( 1 - 2M\mathcal{H} \right) = 0.
$$

When $\mathcal{H}$ is sufficiently small, $(1 - 2M\mathcal{H}) > 0$, so that the optimal $P_R$ obeys

$$
\frac{P_R}{P_0} = \max \left\{ \frac{1}{2} \sqrt{P_0(P + \sigma^2)} - \frac{\sigma^2}{4}, 0 \right\} = \max \{ \rho, 0 \}.
$$

(6.28)

The condition of Theorem 6.9 is fulfilled, since for $\rho \geq 0$,

$$
\mathcal{H} = \frac{P_0}{2(4P_R + \sigma^2)} + \frac{P_R}{2(P + \sigma^2)} = \frac{1}{2} \sqrt{\frac{P_0}{P + \sigma^2}} - \frac{\sigma^2}{8(P + \sigma^2)} = o(1),
$$

(115)
while for \( \rho \leq 0 \) we obtain from (6.28) that \( P_R = 0 \), and thus

\[
H = \frac{P_0}{2\sigma^2} = \frac{1}{2} \sqrt{\frac{P_0 (P + \sigma^2)}{\sigma^4}} \sqrt{\frac{P_0}{P + \sigma^2}} \leq \frac{1}{4} \sqrt{\frac{P_0}{P + \sigma^2}} = o(1),
\]

(6.29)

because

\[
\rho \leq 0 \Leftrightarrow \sqrt{\frac{P_0 (P + \sigma^2)}{\sigma^4}} \leq \frac{1}{2}.
\]

This results in the following lower bound

\[
H^{(2)}_k \geq \begin{cases} 
\frac{1}{2} \sqrt{\frac{P_0}{P + \sigma^2}} - \frac{\sigma^2}{8(P + \sigma^2)} + \mathcal{O}\left(\left(\sqrt{\frac{P_0}{P + \sigma^2}} - \frac{\sigma^2}{4(P + \sigma^2)}\right)^2\right), & \rho \geq 0, \\
\frac{P_0}{2\sigma^2} + \mathcal{O}\left(\frac{P_0^2}{\sigma^4}\right), & \rho \leq 0.
\end{cases}
\]

(6.30)

Step 1 is completed once we have proven that for \( \rho \geq 0 \),

\[
\mathcal{O}\left(\left(\sqrt{\frac{P_0}{P + \sigma^2}} - \frac{\sigma^2}{4(P + \sigma^2)}\right)^2\right) = \mathcal{O}\left(\frac{P_0}{P + \sigma^2}\right).
\]

This is easy, since we assumed \( \rho \geq 0 \), so that

\[
\frac{1}{2} \sqrt{\frac{P_0}{P + \sigma^2}} \leq \frac{1}{2} \sqrt{\frac{P_0}{P + \sigma^2}} + \rho \frac{P_0}{P + \sigma^2} = \sqrt{\frac{P_0}{P + \sigma^2}} - \frac{\sigma^2}{4(P + \sigma^2)} \leq \sqrt{\frac{P_0}{P + \sigma^2}}.
\]

(6.31)

We have proven the lower bounds (6.27) when \( \mathcal{H} = o(1) \).

**Step 2:** \( \mathcal{H} \geq \varepsilon \) does not give a smaller lower bound. The goal of this step is to prove that when for an arbitrary \( \varepsilon > 0 \), we have that \( \mathcal{H} \geq \varepsilon \) implies the lower bounds in (6.27). This allows us to conclude that the lower bounds are as desired. Since \( \mathcal{H} \geq \varepsilon \) implies that either \( P_0/(4P_R + \sigma^2) \geq \varepsilon \) or \( P_R/(P + \sigma^2) \geq \varepsilon \), we can focus on the two cases separately.

We first assume that \( P_0/(4P_R + \sigma^2) \geq \varepsilon \). Since \( \mathbb{P}(A \cap B) \leq \mathbb{P}(A) \), we can focus on the second stage only:

\[
H^{(2)}_{k,R} \geq - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( P_0^{1/2} + 2 \sum_{j = R} \frac{P_j^{1/2}}{n} \sum_{i=1}^{n} X_{ji} + \frac{1}{n} \sum_{i=1}^{n} N_i \leq 0 \right).
\]

(6.32)

This rate is, according to (6.3), bounded from below by

\[
\frac{P_0}{2(\sum_{j \in R}(2P_j^{1/2} + \sigma^2))} = \frac{P_0}{2(4P_R + \sigma^2)} \geq \varepsilon/2 \geq \sqrt{\frac{P_0}{P + \sigma^2}},
\]

for \( P_0/(P + \sigma^2) \) sufficiently small, because \( \varepsilon > 0 \) is fixed.

We will next treat the case \( P_R/(P + \sigma^2) \geq \varepsilon \). To do so, we split this part of the proof according to the largest power in \( P_R \). Suppose we can choose an \( \tilde{m} \in R \), such that \( P_{\tilde{m}} \geq 2\sqrt{P_0(P + \sigma^2)} \). Then, only taking the event \( \{ Z_{\tilde{m}}^{(1)} \} \) into account, the rate is bounded from below by

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( P_{\tilde{m}}^{1/2} + \sum_{j = 0}^{k-1} \frac{P_j^{1/2}}{n} \sum_{i=1}^{n} X_{ji}X_{\tilde{m}i} \leq 0 \right) \geq \frac{P_{\tilde{m}}}{2(P + \sigma^2)} \geq \sqrt{\frac{P_0}{P + \sigma^2}},
\]

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where we have used Proposition 6.2 to obtain the first inequality and \( P_m \geq 2\sqrt{P_0(P + \sigma^2)} \) to obtain the second.

When we can not choose an \( m \in R \), such that \( P_m \geq 2\sqrt{P_0(P + \sigma^2)} \), all powers \( P_m, m \in R \) must obey \( P_m \leq \sqrt{P_0(P + \sigma^2)} \). In this case, we can choose an \( R \subset R \) such that

\[
4\sqrt{P_0(P + \sigma^2)} \leq P_R \leq 6\sqrt{P_0(P + \sigma^2)}.
\]

We observe that, using \( R^* = \{1, \ldots, k - 1\} \backslash R \).

\[
\lim_{n \to \infty} \frac{1}{n} \log P(\max_{m \in R} \text{sgnr}_m(Z_m^{(1)}) < 0, \min_{m \in R^*} \text{sgnr}_m(Z_m^{(1)}) > 0, \text{sgnr}_0(Z_0^{(2, R)}) < 0) \\
\geq - \lim_{n \to \infty} \frac{1}{n} \log P(\max_{m \in R} \text{sgnr}_m(Z_m^{(1)}) < 0),
\]

We next use the lower bound in Theorem 6.9 on the scenario with users 1, \ldots, k + 1 and powers 0, \( P_2, \ldots, P_k, P_0 \) respectively (user 0 has power 0, so \( \{\text{sgnr}_0(Z_0^{(2, R)}) < 0\} \) does not contribute to the rate), with the following result:

\[
\lim_{n \to \infty} \frac{1}{n} \log P(\max_{m \in R} Z_m^{(1)} \leq 0) \geq \frac{P_R}{2(P + \sigma^2)} + O\left(\frac{P_R^2}{(P + \sigma^2)^2}\right) \geq \frac{P_R}{4(P + \sigma^2)} \\
\geq \frac{4\sqrt{P_0(P + \sigma^2)}}{4(P + \sigma^2)} = \sqrt{P_0} / (P + \sigma^2),
\]

when \( P_R/(P + \sigma^2) \) is sufficiently small. This is guaranteed for \( P_0/(P + \sigma^2) \to 0 \), because \( P_R/(P + \sigma^2) \leq 6\sqrt{P_0/(P + \sigma^2)} = o(1) \).

The result of step 2 is that when \( R \geq \varepsilon \), we have \( H_k^{(2)} \geq \sqrt{P_0/(P + \sigma^2)} \), which is larger than the lower bounds in (6.27), where we use that for \( \rho \leq 0 \), \( P_0/(2\sigma^2) \leq \frac{1}{4}\sqrt{P_0/(P + \sigma^2)} \), according to (6.29).

**Step 3: Upper bound.** For the upper bound, we can substitute the optimal \( P_R \) obtained above. It is sufficient to prove that this optimal \( P_R \) can be approximated with small enough deviation. For \( \rho \leq 0 \), the claim is trivial, since \( R = \emptyset \) suffices. Therefore, assume \( \rho \geq 0 \).

We take \( R \) such that \( P_R/P_0 - \rho = \varepsilon \), where, according to (\#), \( |\varepsilon| \leq C(P + \sigma^2)^{1/4}/P_0^{1/4} \), so that \( \varepsilon(P_0/(P + \sigma^2))^{1/2} = o(1) \). When we substitute \( P_R = \sqrt{P_0(P + \sigma^2)/2 - \sigma^2/4 + \varepsilon P_0} \) in \( R \) and
use the above fact for $\varepsilon$, we arrive at

$$
\mathcal{H} = \frac{P_0}{2(4P_R + \sigma^2)} + \frac{P_R}{2(P + \sigma^2)}
$$

$$
= \frac{1}{4} \sqrt{\frac{P_0}{P + \sigma^2}} \left( \frac{1}{1 + 2\varepsilon \sqrt{\frac{P_0}{P + \sigma^2}}} \right) + \frac{1}{4} \sqrt{\frac{P_0}{P + \sigma^2}} \left( 1 + 2\varepsilon \sqrt{\frac{P_0}{P + \sigma^2}} \right) - \frac{\sigma^2}{8(P + \sigma^2)}
$$

$$
= \frac{1}{4} \sqrt{\frac{P_0}{P + \sigma^2}} \left( 1 - 2\varepsilon \sqrt{\frac{P_0}{P + \sigma^2}} + \mathcal{O}\left( \varepsilon^2 \frac{P_0}{P + \sigma^2} \right) \right)
+ \frac{1}{4} \sqrt{\frac{P_0}{P + \sigma^2}} \left( 1 + 2\varepsilon \sqrt{\frac{P_0}{P + \sigma^2}} \right) - \frac{\sigma^2}{8(P + \sigma^2)}
$$

$$
= \frac{1}{2} \sqrt{\frac{P_0}{P + \sigma^2}} - \frac{\sigma^2}{8(P + \sigma^2)} + \sqrt{\frac{P_0}{P + \sigma^2}} \mathcal{O}\left( \varepsilon^2 \frac{P_0}{P + \sigma^2} \right)
$$

$$
= \frac{1}{2} \sqrt{\frac{P_0}{P + \sigma^2}} - \frac{\sigma^2}{8(P + \sigma^2)} + \mathcal{O}\left( \frac{P_0}{P + \sigma^2} \right).
$$

Thus, substitution of $R$ such that $P_R/P_0 = \rho + \varepsilon$ gives

$$
H_r^{(2)} \leq \mathcal{H} + \mathcal{O}(H^2) = \frac{1}{2} \sqrt{\frac{P_0}{P + \sigma^2}} - \frac{\sigma^2}{8(P + \sigma^2)} + \mathcal{O}\left( \frac{P_0}{P + \sigma^2} \right)
$$

$$
+ \mathcal{O}\left( \left( \sqrt{\frac{P_0}{P + \sigma^2}} - \frac{\sigma^2}{4(P + \sigma^2)} + \mathcal{O}\left( \frac{P_0}{P + \sigma^2} \right) \right)^2 \right).
$$

Finally, use (6.31) on the last order term to obtain the desired equality. \hfill \blacksquare

### 6.3 Exponential rate for SD-PIC

The SD-PIC model is essentially different than the MF or the HD-PIC model. The analysis of both the MF and the HD-PIC model involves an i.i.d. case. This allows Cramér’s theorem to be applied. The analysis of the SD-PIC model requires Sanov’s theorem. Therefore, we are forced to switch to the empirical measure. The rate is obtained as the optimum of the rate function over a certain area. When we include unequal powers, the area is skewed, which is a trivial extension. The inclusion of AWGN however, will turn out to be more involved. To understand this, we will first treat the case $k = 2$.

For $k = 2$, we have $X_2 = \{-1, +1\}$. We abbreviate $\rho_+ = \rho(+1)$ and $\rho_- = \rho(-1)$. Then it is straightforward from (4.15) that

$$
\frac{1}{n} \sum_{i=1}^{n} X_{1i} = (L_n)_+ - (L_n)_-.
$$
Hence, we have for the simple model (recall (4.1)) that

\[ Z_0^{(2,s)} = 1 - ((L_n)_+ - (L_n)_-)^2. \]

When we take unequal powers and AWGN into account, we have

\[ Z_0^{(2,s)} = P_0^{1/2} - \left( (L_n)_+ - (L_n)_- \right) \left( P_0^{1/2}(L_n)_+ - (L_n)_- \right) + \frac{1}{n} \sum_{i=1}^{n} X_i N_i + \frac{1}{n} \sum_{i=1}^{n} N_i. \]

We define \((M_n)_+ = \frac{1}{n(L_n)} \sum_{i=1}^{n} N_i 1_{(X_i = +1)}\) and \((M_n)_- = \frac{1}{n(L_n)} \sum_{i=1}^{n} N_i 1_{(X_i = -1)}\). Then

\[ \frac{1}{n} \sum_{i=1}^{n} X_i N_i = (L_n)_+(M_n)_+ - (L_n)_-(M_n)_- \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} N_i = (L_n)_+(M_n)_+ + (L_n)_-(M_n)_-. \]

When we introduce the extension of \(F_2(\rho)\) as

\[ F_2(\rho, \mu) = P_0^{1/2} - (\rho_+ - \rho_-) \left( P_0^{1/2}(\rho_+ - \rho_-) + (\rho_+ \mu_+ - \rho_- \mu_-) \right) + \rho_+ \mu_+ + \rho_- \mu_-, \]

then

\[ Z_0^{(2,s)} = F_2(L_n, M_n). \]

To obtain the exponential rate, we first condition on \(L_n = \rho\). The rate of the event \(\{Z_0^{(2,s)} \leq 0 | L_n = \rho\}\) is now given by

\[ \min_{\mu \in \mathbb{R}^2 : F_2(\rho, \mu) = 0} \rho_+ \mu_+^2 - \rho_- \mu_-^2 - 2\sigma^2. \]

Indeed, for \(\mu < 0\),

\[ \lim_{m \to \infty} \frac{1}{m} \log \mathbb{P} \left( \frac{1}{m} \sum_{i=1}^{m} N_i \leq \mu \right) = -\frac{\mu^2}{2\sigma^2}. \]

Finally, conditionally on \(L_n = \rho\), \((M_n)_a\) and \((M_n)_{a'}\) are independent for \(a \neq a'\). Now minimization over \(\rho\), obeying the cost of shifting \(L_n\) gives the desired result:

\[ J_2^{(2)} = \min_{\rho \in M(X_2), \mu \in \mathbb{R}^2 : F_2(\rho, \mu) = 0} \rho_+ \log \rho_+ + \rho_- \log \rho_- + \rho_+ \mu_+^2 + \rho_- \mu_-^2 + \log 2. \]

This is justified by Varadhan’s lemma (cf. Den Hollander (2000), Thm. III.13). It is interesting that the rate does not depend on \(P_1\), since \(F_2(\rho, \mu)\) does not depend on \(P_1\).

For \(k \geq 3\), we define for \(a \in X_k\)

\[ (M_n)_a = \frac{1}{n(L_n)_a} \sum_{i=1}^{n} N_i 1_{(X_i = a)}. \]

Then (recall that \(a_0 = 1\))

\[ \frac{1}{n} \sum_{i=1}^{n} X_{mi} N_i = \sum_{a \in X_k} a_n(L_n)_a(M_n)_a. \]

The following lemma identifies \(Z_0^{(2,s)}\).

\[ \text{119} \]
Lemma 6.12 For \( k \geq 3 \),
\[
Z_0^{(2,\, s)}(s) = F_k(L_n, M_n),
\]
where for \( \rho \in M(\mathcal{X}_k) \) and \( \mu \in \mathbb{R}^{2^{k-1}} \),
\[
F_k(\rho, \mu) = P_0^{1/2} - \sum_{m=1}^{k-1} \left( \sum_{a \in \mathcal{X}_k} a_m \rho_a \right) \left( \sum_{j=0}^{k-1} \sum_{a \in \mathcal{X}_k} \left( P_j^{1/2} a_m a_j \rho_a + a_j \rho_a \mu_a \right) \right) + \sum_{a \in \mathcal{X}_k} \rho_a \mu_a,
\]
where \( a_0 = 1 \), for all \( a \in \mathcal{X}_k \).

**Proof.** Switching over to empirical measures
\[
\frac{1}{n} \sum_{i=1}^{n} X_{mi}X_{ji} = \sum_{a \in \mathcal{X}_k} a_m a_j (L_n)_a
\]
and
\[
\frac{1}{n} \sum_{i=1}^{n} X_{ji}N_{i} = \sum_{a \in \mathcal{X}_k} a_j (L_n)_a (M_n)_a.
\]
yields the lemma. \( \square \)

**Theorem 6.13** For \( k \geq 2 \),
\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Z_0^{(2,\, s)} \leq 0) = J_k^{(2)},
\]
where
\[
J_k^{(2)} = \inf_{\rho \in M(\mathcal{X}_k), \mu \in \mathbb{R}^{2^{k-1}}; F_k(\rho, \mu) = 0} \mathcal{I}_k(\rho) + \sum_{a \in \mathcal{X}_k} \rho_a \frac{\mu_a^2}{2\sigma^2}.
\]

**Proof.** For \( \sigma = 0 \), the proof directly follows from the proof in Theorem 4.5. Therefore, we assume \( \sigma > 0 \). The empirical measure clearly satisfies the large deviation principle, as defined in Den Hollander (2000), Chapter II with rate \( n \) and rate function \( \mathcal{I}_k(\rho) \). Furthermore,
\[
\sup_{\mu \in \mathbb{R}^{2^{k-1}}, F_k(\rho, \mu) = 0} - \sum_{a \in \mathcal{X}_k} \rho_a \frac{\mu_a^2}{2\sigma^2}
\]
is continuous in \( \rho \) and bounded from above. Apply Varadhan's lemma (cf. Den Hollander (2000), Thm. III.13) to get the desired result. \( \square \)

**Remark:** Since all \( (M_n)_a \) are independent and Gaussian distributed, also \( F_k(L_n, M_n) \), conditional on \( L_n \), is Gaussian distributed with mean \( F(L_n, 0) \) and variance \( \sigma^2 \sum_{a \in \mathcal{X}} \rho_a^{-1} \gamma_a \), where
\[
\gamma_a = \left( \sum_{m=1}^{k} \left( \sum_{a' \in \mathcal{X}} a'_m \rho_a \right) \left( \sum_{j=0}^{k-1} \sum_{a \in \mathcal{X}} \left( P_j^{1/2} a_m a_j \rho_a + a_j \rho_a \mu_a \right) \right) + \rho_a \right)^2.
\]
This results in

\[ J_k^{(2)} = \inf_{\rho \in \mathcal{M}(X_k)} I_k(\rho) + \frac{F_k^2(\rho, 0)}{2\sigma^2 \sum_{a \in A_k} \rho_a^{-1} \gamma_a}. \]

### 6.4 Numerical results

In this section, we investigate the scenario in which users 0, 2, 3, ..., \( k - 1 \), \( k = 3, 6, 9 \) have the power 1, while the power of user 1 is varied from user from 0 to 6. We further investigate the influence of AWGN on the system.

We first turn to the in which power variations are present. We investigate a model with 3, 6 or 9 users, where user 0, 2, ..., \( k - 1 \) have power 1, and user 1 has a power varying from 0 to 6. Both \( J_k \) and \( J_k^{(2)} \) are straightforward to calculate. For \( H_k^{(2)} \), it is not clear yet how it is calculated, since no expression have been given so far. We will describe below how the results for \( H_k^{(2)} \) are generated.

Since \( H_k^{(2)} = \min_R H_k^{(2)} \), we have calculated \( H_k^{(2)} \) for every \( R \). Only five sets are relevant when \( 0 \leq P_1 \leq 6 \), namely

- \( R = \{1\} \) with \( P_R = P_1 \),
- \( R = \{2\} \) with \( P_R = 1 \),
- \( R = \{1, 2\} \) with \( P_R = 1 + P_1 \),
- \( R = \{2, 3\} \) with \( P_R = 2 \).

All other sets resulted in a rate that is too high.

We will illustrate the behaviour by the example \( k = 9 \). In Figure 6.1, we show the exponential rate for the four different cases of \( R \). Note that for \( P_1 = 0 \), \( \sqrt{P}/2 = 1.4142 \ldots \), while for \( P_1 = 6 \), \( \sqrt{P}/2 = 1.937 \ldots \). Therefore, we may think of values of \( P_R \) typically between 1.4 and 2. For different cases of \( P_1 \), we distinguish different behaviour.

For \( P_1 \) small, \( R = \{1, 2\} \) is optimal. For \( P_1 \) slightly smaller than 1, it turns out that \( R = \{2\} \) is optimal. For \( P_1 \) close to 2, but larger than 1, \( P_{(1)} = P_1 \) is optimal. Finally, for larger \( P_1 \) we see that letting user 2 and 3 have a bit error in the first stage is optimal. When \( P_1 \) becomes even larger, other scenarios are possible, because we have seen in the previous chapter that \( P_R \to \infty \) as \( P_1 \to \infty \). However, because \( P_1 > k - 1 \) implies that \( Z_1^{(1)} > 0 \) a.s., we cannot have a bit error for user 1 once \( P_1 \) is too large. In Figure 6.1, we have taken \( P_1 \) sufficiently small, so that other scenarios than the ones described above give a too high rate.

We will next use the asymptotic result for the HD-PIC model to give some intuition why this behaviour is seen. Since \( P_R = P_0 \rho \) is asymptotically optimal, we expect that the set \( R \) that minimizes \( H_k^{(2)} \) satisfies \( P_R \approx P_0 \rho \). In this case, \( P_0 = 1 \) and \( \rho = \sqrt{2 + P_1/4} \). We can approximate \( \rho \) using \( R = \{1\}, \{2\}, \{1, 2\}, \{2, 3\} \), with \( P_R = P_1, 1 + 1, 1 + P_1, 2 \), respectively. A straightforward calculation gives that for \( 0 \leq P_1 \leq 1 \), \( 1 + P_1 \) is the best approximation. For
Figure 6.1: Exponential rates $H_{0,\text{r}}^{(2)}$ for $R = \{1\}, \{2\}, \{1,2\}, \{2,3\}$, represented by $\circ, \circ, \triangle, \star$, respectively.

$1 \leq P_1 \leq 2$, $P_1$ is best, while for $2 \leq P_1 \leq 4$, 2 is optimal. This agrees with our numerical results, except for the small interval $[0.7545, 1]$, where $R = \{2\}$ is optimal. Nevertheless, the asymptotic result gives a good idea what to expect.

In Figure 6.2 (a), (b) and (c), we show results for $k = 3, 6$ and 9. For $k = 3$, we make two remarkable observations. First of all, HD-PIC performs significantly worse than MF whenever $P_1$ is between 0.4245 and 1. We have explained above why this is the case. Secondly, SD-PIC performs for small $P_1$ significantly worse than both MF and HD-PIC. The explanation is that for $P_1$ small, the soft decision introduces interference from user 0 and 2 in the decision statistic $Z_0^{(2,5)}$, even in the case $P_1 = 0$. For higher $k$, it turns out not to be of any influence. However, for $k = 3$, the additional noise influences the performance significantly.

For $k = 6$ and $k = 9$, the same effects for HD-PIC and SD-PIC are present, but not of big influence. We see that HD-PIC increases performance significantly over MF, especially when $P_1$ is large. The SD-PIC system even performs slightly better than HD-PIC, but the difference is negligible.

Another interesting issue is the influence of AWGN. In Figure 6.2 (d), (e) and (f), results are shown for $I_k$, $H_k^{(2)}$ and $J_k^{(2)}$ for $k = 3, 6$ and 9, where $0 \leq \sigma^2 \leq 10$. We have taken $P_0 = P_1 = \ldots = P_{k-1} = 1$. For comparison, $1/(2\sigma^2)$ is also plotted. This bound is the so-called single user bound, since in the absence of interfering users,

$$
P(\text{sgn} r_0(Z_0^{(k,5)}) < 0) = P\left(1 + \frac{\sigma N(0,1)}{n^{1/2}} \leq 0\right) = Q\left(\frac{n^{1/2}}{\sigma}\right) \sim \exp\left(-\frac{n P_0}{2\sigma^2}\right),
$$

according to (5.4), so that the exponential rate equals $1/(2\sigma^2)$. When we compare $H_k^{(2)}$ and $J_k^{(2)}$ to $I_k$, we see that also in the presence of AWGN, PIC significantly increases performance.
The decay of both $H_k^{(2)}$ and $J_k^{(2)}$ is for $k = 6$ and 9 comparable to the decay of $I_k$. For $k = 3$, $I_3$ decreases significantly faster than $H_k^{(2)}$ or $J_k^{(2)}$. For small $\sigma$, $H_k^{(2)}$ decreases faster than $J_k^{(2)}$. When $\sigma$ increases, $J_k^{(2)}$ follows more and more the single user bound. On the other hand, $H_k^{(2)}$ decreases until it hits the single user bound. At this point the optimal $r$ equals 0, so that the single user bound is precisely attained.

(a) Rates for $k = 3$ as a function of $P_1$.

(b) Rates for $k = 6$ as a function of $P_1$.

(c) Rates for $k = 9$ as a function of $P_1$.

(d) Rates for $k = 3$ as a function of $\sigma^2$.

(e) Rates for $k = 6$ as a function of $\sigma^2$.

(f) Rates for $k = 9$ as a function of $\sigma^2$.

Figure 6.2: $I_k$ ($\circ$), $H_k^{(2)}$ ($\triangle$), $J_k^{(2)}$ ($\diamond$) and $\frac{1}{2\sigma^2}$ ($\times$) for $k = 3, 6$ and 9. Figures (a)-(c) show the case in which $P_0 = P_2 = P_3 = \ldots = P_{k-1}$ and $\sigma = 0$. Figures (d)-(f) depict the case with $P_0 = \ldots = P_{k-1}$ and $\sigma \geq 0$. 

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Optimal systems

This chapter contains results for the multistage HD- and SD-PIC model for the case $s \to \infty$ and $k$ fixed. Simulation results (Buehrer (1999)) show that for HD-PIC, more than three stages is not worthwhile. One could wonder whether the iterative cancellation procedure converges. We will show that in many cases this is indeed the case. We call the $\infty$-stage PIC system optimal. Throughout this chapter, we will assume that users can have different powers, but we assume that $P_j > 0$ for all $j$. Furthermore, $\sigma^2 = 0$, i.e., no AWGN is present. We will start in Section 7.3 with the optimal HD-PIC model. We give results here and defer many of the proof to later sections. Results for the optimal SD-PIC model are given in Section 7.2. In Section 7.1, we will prove some useful bounds on moment generating functions. Sections 7.3, 7.4 and 7.5 are dedicated to proofs. Results for the optimal HD-PIC have appeared in van der Hofstad and Klok (2000) and van der Hofstad and Klok (2002OHD), results for optimal SD-PIC have appeared in van der Hofstad and Klok (2002OSD).

7.1 Optimal hard decision system

In this section, we study the behaviour of the infinite stage HD-PIC system, which we will denote by the optimal HD-PIC system. We have various remarkable results. First of all, we will show that after a finite number of stages of HD-PIC, the worst-case rate remains fixed. Another remarkable result is that this rate is strictly positive, independent of the powers. We have an explicit expression for the lower bound of the rate. Under certain power conditions, we will show that this lower bound is tight; when $k \to \infty$, we describe a scenario for which the rate converges to the lower bound. The required number of stages of HD-PIC as a function of $k$, denoted by $s_k$ is of importance. We derive an upper bound for $s_k$. Furthermore, we give intuition on the behaviour of $s_k$. 
We first list some easy properties of the system with unequal powers. First of all, we define the worst case rate of a bit error to be

$$H^{(s)}_k = \min_{0 \leq m \leq k-1} H^{(s)}_{k,m},$$

(7.1)

where

$$H^{(s)}_{k,m} = - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Z^{(s)}_m \leq 0), \; 0 \leq m \leq k - 1.$$

Hence, $H^{(s)}_k$ is the exponential rate of the bit error probability of that user that has the largest bit error probability. Furthermore, we recall the notation

$$R_\sigma = \{ m : \text{sgn} r_m(Z^{(\sigma,M)}_m) < 0 \}.$$  

(7.2)

We will tacitly assume that the user of interest is in the set $R_\sigma$, since otherwise that user does not have a bit error at stage $s$.

**Theorem 7.1** (a) $H^{(s)}_k$ is monotone non-decreasing in $s$.

(b) There exists an $s_k \leq 2^k + 1$, such that $H^{(s)}_k = H^{(s_k)}_k$ for all $s \geq s_k$.

The above shows that the optimal system is attained. For equal powers, we know that the rate for all users does not improve for $s \geq s_k$. For unequal powers, we can think of this system as having the optimal worse case rate, in the sense that the rate of the bit error minimized over all users is optimal.

We will use the definition of the sign-function in (1.5). However, in the proof, we will comment on other definitions of the sign-function where a similar result can be shown.

We will now prove the above theorem.

**Proof.** First of all, $s \mapsto H^{(s)}_k$ is non-decreasing. Indeed, to have a bit error, one of the interfering users has to have a bit error in the previous stage.

For the simple model, we have exchangeability of users, so that the probability of a bit error at stage $s$ is smaller than that of stage $s - 1$ and the desired statement follows. The above statement even proves that the probability of a bit error is non-decreasing.

This simple argument fails to hold when the powers are unequal. However, in this case, we can show in precisely the same way that the maximal bit error is non-decreasing. Then we take $m$ such that the bit error in stage $s$ is maximal for that user $m$. We can just repeat the argument given above, and see that we need to have a bit error from another user in the previous stage. However, the latter probability is at most the maximal probability of a bit error.

By the monotonicity above $H^{(\infty)}_k = \lim_{s \to \infty} H^{(s)}_k$ exists. To see why $H^{(\infty)}_k$ with $k$ fixed is reached in a finite number of stages $s_k$ and $s_k \leq 2^k + 1$, we fix $s \geq 2^k + 1$ and we take the scenario $(R_\sigma)'_{s=1}^{s_k}$ for which the rate $H^{(s)}_k$ is assumed. We observe that necessarily $R_{\sigma'} = R_{\sigma''}$, for some $\sigma' < \sigma'' \leq 2^k + 1$. Indeed, there are at most $2^k$ configurations $R_\sigma$, so
that after $2^k + 1$ stages, one of the configurations is assumed twice. Then $Z_m^{(\sigma'')[+1]} = Z_m^{(\sigma'+1)}$ for all $0 \leq m \leq k - 1$ and thus $R_{\sigma' + 1} = R_{\sigma'' + 1}$. It follows that $R_{\sigma' + 1} = R_{\sigma'' + 1}$ and thus $R_{\sigma + 1(\sigma'' - \sigma')} = R_\sigma$ for all $\sigma' \leq \sigma < \sigma''$, for all $i \in \mathbb{N} \cup \{0\}$. By (1.5), all stages beyond $\sigma''$ are determined by the stages $1, \ldots, \sigma''$ and do not contribute to the rate. And clearly for every stage $s' > \sigma''$, some users have bit errors. Thus, the described scenario that forces a bit error in stage $s$, also forces errors in stage $s'$. Therefore, since $H_{k}^{(s')} = 1$ is the worst-case rate over all scenarios and all users, $H_{k}^{(s')} \leq H_{k}^{(s)}$. But since $H_{k}^{(s)}$ is non-decreasing, necessarily $H_{k}^{(s')} = H_{k}^{(s)}$ for all $s' \geq \sigma''$. Since $\sigma'' \leq s \leq 2^k + 1$, the desired statement follows.

The statement of Theorem 7.1 can be extended to other definitions of the sign-function.

For example, when we let the sign of 0 to be 0, then we can copy the original proof, apart from the fact that we let $R_{\sigma,0}$ be the set of values $m$ where $Z_m^{(\sigma,0)} = 0$, and $R_{\sigma,1}$ the set where $Z_m^{(\sigma,1)} < 0$. Then, when $s \geq 3^k + 1$, we must $R_{\sigma,1,i} = R_{\sigma,i}$, for $i = 0, 1$ and some $\sigma' < \sigma'' \leq 3^k + 1$. This again implies a periodic scenario.

For the system with equal powers, we are also allowed to use a sign-function that assumes values $\pm 1$ independently every time it is used, rather than fixing it per user. Indeed, the limit determining $H_{k}^{(\infty)}$ exists, so we can reach the limit just by using the odd $n$. In this case, we cannot have $Z_m^{(\sigma,0)} = 0$, except when $\sigma = 1$. Therefore, every user draws at most one time a random sign-function, which makes the statement equivalent to the statement with our original definition of sign. For unequal powers this argument unfortunately does not hold. However, $\mathbb{Q}$ is dense in $\mathbb{R}$, so that for almost all powers $(P_1, \ldots, P_k)$ and for all integers $n$, $Z_m^{(\sigma,0)} \neq 0$. This explains that the problems arising from the definition of the sign of zero are somewhat academic.

We will next investigate this optimal system. Recall that because of the hard decisions, the decisions at stage $\sigma$ only depend on the decisions at stage $\sigma - 1$. We will first investigate the system with 2 users, for illustration purposes. We will use that when user 1 has no bit error at stage $\sigma$, user 2 necessarily has no bit error at stage $\sigma + 1$. When we denote a bit error by $\square$ and no bit error by $\bullet$, in Figure 7.1, all possible six scenarios are shown. We see

<table>
<thead>
<tr>
<th>user stage</th>
<th>1</th>
<th>2</th>
<th>1</th>
<th>2</th>
<th>1</th>
<th>2</th>
<th>1</th>
<th>2</th>
<th>1</th>
<th>2</th>
</tr>
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<tr>
<td>1</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
<td>$\square$</td>
<td>$\bullet$</td>
<td>$\square$</td>
<td>$\square$</td>
<td>$\square$</td>
<td>$\square$</td>
<td>$\square$</td>
<td>$\square$</td>
</tr>
<tr>
<td>2</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
<td>$\square$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
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</tr>
<tr>
<td>3</td>
<td>$\bullet$</td>
<td>$\square$</td>
<td>$\square$</td>
<td>$\bullet$</td>
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<td>$\bullet$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
</tr>
<tr>
<td>4</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
<td>$\square$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
</tr>
</tbody>
</table>

Figure 7.1: Possible scenarios for the HD-PIC model with 2 users.

that $(R_{\sigma})_{\sigma=1}^{\infty}$ is periodic, except for at most 3 initial stages. We will denote the repeating part by periodic scenario. Once a periodic scenario is obtained, the exponential rate remains constant, so that we achieved optimal performance. We have shown that the number of initial stages is at most $2^2 + 1$, which explains why the optimal system is achieved in at most 5 stages.
We will next investigate periodic scenarios in more detail. Periodic scenarios has a big "advantage" over non-periodic scenarios. As long as a scenario is not yet periodic, specifying which users have bit errors at a certain stage results in a decrease of the BEP. Indeed, it is likely to estimate a bit correctly, so that users do not tend to have a bit error. However, in a periodic scenario, specifying the initial part is sufficient. For example, to get a bit error for user 1 at stage $s = 2002$, it is sufficient to specify the positions of bit errors from stage 1,2 and 3 (the third and last scenario in Figure 7.1 will do). From that stage onwards, the bit errors are determined by those in stages 1,2 and 3.

Two essentially different scenarios are characterizing the behaviour of the optimal system. The first one is the so-called disjoint scenario (scenario 2 and 3 in Figure 7.1), where at every stage user 1 has a bit error and user 2 does not, or vice versa. The other scenario, which we will call the overlapping scenario is the scenario where at every stage both user 1 and user 2 have a bit error (last scenario in Figure 7.1). Note that for both the disjoint and the overlapping scenario, the periodic behaviour kicks in at stage 1. For both scenarios, we can calculate the exponential rate. The minimum of the two exponential rates indicates which scenario typically is observed.

When $k \geq 3$, we extend those scenarios in the following way. For every $r$, at stage 1, 3, 5, ... bit errors are made for users in some set $R_1$, with $|R_1| = r$. At stage 2, 4, 6, ..., bit errors are made for users in the set $R_2$ with $|R_2| = r$. When $R_1 \cap R_2 = \emptyset$, we speak of the disjoint scenario. Whenever $R_1 = R_2$, we will call it the overlapping scenario. All other scenarios are denoted by partly overlapping.

The statement of Theorem 7.1(b) is in fact that after at most $2^k + 1$ stages, the behaviour is periodic. The scenarios that we typically observe, are typically periodic already at stage 1. Indeed, specifying the bit errors at the initial non-periodic stages make the BEP smaller, while for the periodic stages we do not need to specify the positions of bit errors. The longer the number of stages where the behaviour is not yet periodic, the smaller the BEP and thus this behaviour is less typical. Therefore, the scenarios that we typically observe are either the disjoint, the partly overlapping, or the overlapping scenario. The overlapping scenario has period 1, while the other scenarios have period 2, so it is tempting to assume that this scenario gives the smallest exponential rate, and is therefore typical. However, it turns out that in the disjoint scenario, there is a strong dependence between users in the set $R_1$ experiencing interference from users in the set $R_2$ at stage $\sigma$ and users in $R_2$ experiencing interference from users in $R_1$ at stage $\sigma + 1$. This dependence makes the exponential rate smaller than we expect.

We observe the following phenomena.

1) The partly overlapping scenario is not optimal for any $k$, i.e., both the disjoint and overlapping scenario give a smaller exponential rate. It seems that both the extremes (overlapping and disjoint scenario) do a better job.

2) For small $k$, the disjoint scenario is optimal. The reason is quite simple. For $r$ fixed, a user at stage 2 has contribution from $r$ noise terms, while for the overlapping scenario the user has contribution from only $r - 1$ terms (indeed, the user does not interfere with
its own signal). For higher $k$, however, the overlapping scenario is optimal. For the case $k \to \infty$, it is implicitly proven in the proof of Theorem 7.2(a) that the overlapping scenario is indeed optimal, and the partly overlapping scenario is never optimal.

3) For $k \to \infty$, also $r \to \infty$, but much slower than $k$. The numerical results indicate that $r \approx \sqrt{k}/2$.

To illustrate 2) and 3), the optimal $r$ is shown in Table 7.1 for $k = 1 \ldots 1000$. Also, it is indicated whether the disjoint or the overlapping scenario is optimal.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$r$</th>
<th>$k$</th>
<th>$r$</th>
</tr>
</thead>
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<td>1-30</td>
<td>1(d)1</td>
<td>337-414</td>
<td>10(o)</td>
</tr>
<tr>
<td>31-73</td>
<td>2(d)</td>
<td>415-499</td>
<td>11(o)</td>
</tr>
<tr>
<td>74-83</td>
<td>3(d)</td>
<td>500-592</td>
<td>12(o)</td>
</tr>
<tr>
<td>84-107</td>
<td>5(o)2</td>
<td>593-694</td>
<td>13(o)</td>
</tr>
<tr>
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</tr>
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<td>8(o)</td>
<td>920-1000</td>
<td>16(o)</td>
</tr>
<tr>
<td>268-336</td>
<td>9(o)</td>
<td></td>
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</tr>
</tbody>
</table>

1(d) means disjoint scenario is optimal.
2(o) means overlapping scenario is optimal.

Table 7.1: Optimal scenario for optimal HD-PIC model.

In Figure 7.2, the exponential rates $H_k^{(s_k)}$ are given, together $H_k^{(s)}$ for $s = 1, 2, 3$. The results for $s = 3$ are obtained using similar techniques as in Theorem 4.2. However, we have not stated the result for $s = 3$ in this thesis. The rate $H_k^{(s_k)}$ is in fact the rate corresponding to the disjoint scenario for $r = 1$ or $r = 2$. For $k = 2, 3$, it is seen that $I_k = H_k^{(2)} = H_k^{(3)}$, so that $s_2 = s_3 = 1$. For $4 \leq k \leq 9$, we see $I_k < H_k^{(2)} = H_k^{(3)}$, so that $s_k = 2$. We see that one stage of HD-PIC gives an improvement in exponential rate. However, adding one more stage does not result in any improvement. For $10 \leq k \leq 22$, 2-stage HD-PIC gives an improvement over one-stage HD-PIC. However, the numerical result shows that the scenario corresponding to the rate is the disjoint scenario with $r = 1$. Therefore $s_k = 3$ in this case.

For $23 \leq k \leq 50$, we expect $s_k = 4$, even though we are not able to calculate $H_k^{(4)}$.

We now turn to a lower bound of the exponential rate for the optimal system. Clearly, Theorem 5.7 has no meaning for $s \to \infty$, but from Theorem 5.7 and the monotonicity of $s \mapsto H_k^{(s)}$ it follows that for the simple model for all $\varepsilon > 0$, $k^{\varepsilon}H_k^{(s_k)} \to \infty$, when $k \to \infty$. Thus, if $H_k^{(s_k)}$ converges to 0 as $k \to \infty$, it does so slower than any power of $1/k$. The theorem below states that the exponential rate of the optimal system remains strictly positive as $k \to \infty$.

In the theorem, we have abbreviated $P = \sum_{m=0}^{k-1} P_m$, $P_{min} = \min_j P_j$ and $P_{max} = \max_j P_j$.

**Theorem 7.2 (a)** For the general model in which users have different powers, for all $s \geq 2^k + 1$

$$H_k^{(s)} \geq \frac{1}{2} \log 2 - \frac{1}{4} = 0.09657 \ldots$$ (7.3)
For the system with equal powers the bound is obtained for all \( s \geq k + 1 \).

(b) When the powers fulfill the following conditions:
   
   \((P_1)\) There exists a \( \delta > 0 \) such that \( \text{card}(\{ j : P_j \in [\delta, 1/\delta]\}) \to \infty \),
   \n   \((P_2)\) \( \lim_{k \to \infty} k P_{-1} < \infty \),
   \n   \((P_3)\) \( k^{-1} P_{\text{max}} \to 0 \),
   \n   \((P_4)\) \( k P_{\text{min}} \to \infty \),
   
   then
   
   \[
   \lim_{k \to \infty} H_k^{(s)} = \frac{1}{2} \log 2 - \frac{1}{4}. \tag{7.4}
   \]

**Remark:** We note that in the case that all powers are equal all four conditions are fulfilled.

The proof of the above result will be given in Section 7.4.

**Example 7.3** We will treat three examples.

(a) When there exists an \( M < \infty \) such that \( M^{-1} < P_m < M \) for all \( m \), uniformly in \( k \), conditions \( (P_1) - (P_4) \) are fulfilled.

(b) Take \( P_m = m, m = 1, 2, \ldots, k \). Then \( P_{\text{max}} = k \), so that \( (P_3) \) is not fulfilled. However, the problem is invariant under scaling of the powers. Therefore, we are allowed to divide all powers by \( \sqrt{k} \). In this case \( (P_1) - (P_4) \) are fulfilled. Indeed, for \( k \to \infty \), there are infinitely many users with power between \( \sqrt{k}/2 \) and \( \sqrt{k} \), so that the scaled powers are between \( 1/2 \) and \( 1 \). Conditions \( (P_2), (P_3) \) and \( (P_4) \) are also easily checked to hold.

(c) The sequence of powers \( P_0, P_1, \ldots \) is generated from a log-normal distribution. We stress that this is not the same as assuming that the powers are randomly distributed. In the latter,
we have to average the exponential rate over all possible outcomes for the powers. In this example, the powers are fixed. The statement of the theorem therefore holds for almost every realization of the powers. Requirements $\mathcal{P}_1$ and $\mathcal{P}_2$ hold trivially. Furthermore, it suffices to show $\mathcal{P}_3$, since $(P^{\text{min}})^{-1} \leq P^{\text{max}}$. To show this, it is sufficient to show that there exists an $\alpha \in (0, 1)$ such that $\mathbb{P}(P^{\text{max}} > k^\alpha \text{ i.o.}) = 0$, where i.o. means infinitely often. We will show this for $\alpha = 1/2$. According to Embrechts, Kluppelberg and Mikosch (1997), Thm.3.5.1.

$$\mathbb{P}(P^{\text{max}} > \sqrt{k} \text{ i.o.}) = \mathbb{P}(P_k > \sqrt{k} \text{ i.o.}).$$

Furthermore, according to the Borel-Cantelli lemma (cf. Grimmett and Stirzaker (1992), Thm. 7.3.10) $\mathbb{P}(P_k > \sqrt{k} \text{ i.o.}) = 0$ according as

$$\sum_{k=1}^{\infty} \mathbb{P}(P_1 > \sqrt{k}) < \infty.$$

Since the tail of the log-normal distribution equals $x^{-\log x}$, it is easily checked that the condition above holds. Thus, we conclude that a.s. in the random powers, the rate for the optimal system is at least $\frac{1}{2} \log 2 - \frac{1}{4}$.

We next turn to some numerical results. We have obtained upper bounds for $H_k^{(v)}$ (the disjoint and overlapping scenarios) and the lower bound $\frac{1}{2} \log 2 - \frac{1}{4}$. In Figure 7.3, these bounds are given. The upper bound is obtained by taking the minimum over all disjoint and overlapping scenarios. We conjecture that this bound is in fact $H_k^{(v)}$. We observe that the convergence to the limit is slow. We conjecture that the rate converges to $\frac{1}{2} \log 2 - \frac{1}{4}$ as $k^{-1/2}$. This follows the fact that we believe that the overlapping scenario has asymptotic rate $\frac{1}{2} \log 2 - \frac{1}{4} + c_1 \beta + c_2 \gamma$, for some $c_1, c_2 > 0$. The factor $\frac{1}{2} \log 2 - \frac{1}{4}$ is the limit for $r \to \infty$. The factor $c_1 / r$ is the difference between the exact value and its limit, when $k = \infty$. Finally, $r/k$ accounts for the fact that $k < \infty$. Minimizing over $r$ results in $r \approx (c_1 k / c_2)^{1/2}$. This agrees with the observations made above (where we conjectured $c_1 / c_2 = 4$). Substituting $r \approx (c_1 k / c_2)^{1/2}$ results in a convergence rate of order $k^{-1/2}$.

### 7.1.1 On the number of stages to optimality

In this section, we will investigate the number of stages necessary to reach the asymptotic optimal rate $\frac{1}{2} \log 2 - \frac{1}{4}$. The main part of the theorem below will be proven in Section 7.4, some details are postponed to Appendix F.

**Theorem 7.4** For all $0 < \varepsilon < \frac{1}{2} \log 2 - \frac{1}{4}$ and for all $s$ such that $s \geq \lceil \varepsilon^{-1} \log(\frac{P}{P^{\text{min}}}) \rceil + 1$, we have

$$H_k^{(v)} \geq \frac{1}{2} \log 2 - \frac{1}{4} - \varepsilon. \tag{7.5}$$

We know that $I = \frac{1}{2} \log 2 - \frac{1}{4}$ is the asymptotically optimal rate for the system where all the powers are equal. Note that when all powers are comparable (e.g., when $M^{-1} < P_m < M$ for
some constant \( M < \infty \) uniformly in \( k \), that then \( \varepsilon^{-1} \log(P/P^{\min}) \approx \varepsilon^{-1} \log k \). Theorem 7.4 states that when we apply at least that many stages of HD-PIC, that then the limiting rate will be asymptotically bounded from below by \( I \). This suggests that \( s_k \) grows roughly like \( \log k \). It is an interesting problem to determine more precisely how \( s_k \) grows. It follows from Theorem 7.2 together with Theorem 5.7 that \( s_k \to \infty \). However, it would be of practical importance to know the precise rate, or even an upper bound on \( s_k \). We have the following conjecture:

**Conjecture 7.5**

\[
\limsup_{k \to \infty} \frac{\log s_k}{\log \log k} = 1. \tag{7.6}
\]

Conjecture 7.5 says that \( \frac{s_k}{\log k} \) cannot grow or decrease faster than any small power of \( \log k \). Note that we expect the conjecture to even hold in the case of unequal powers.

In the case of equal powers, Theorem 7.4, together with Theorem 7.2, suggests that \( s_k \leq \frac{\log k}{\epsilon_k} \) for every \( \epsilon_k \downarrow 0 \). We believe that in fact a logarithmic number of stages is required to obtain the optimal HD-PIC system. However, we have no proof for this belief. This belief stems from the fact that we expect the proof of Theorem 5.5 to hold for some \( s \) that tend to infinity with \( k \) sufficiently slowly. More precisely, we expect that the strategy described in the proof of Theorem 5.7 remains true for as long as \( s_k \leq (\log k)^{1-\epsilon} \) for any \( \epsilon > 0 \). The reason is that there is an essential change when \( s \ll \log k \) compared to \( s_k = \mathcal{O}(\log k) \), in the sense that for the former

\[
\frac{|R_\sigma|}{|R_{\sigma+1}|} \to \infty,
\]

whereas for the latter, the above converges to a constant. In this case, Theorem 5.5 has no meaning; we cannot expect it to be a good approximation. Moreover, note that when
\[ s = (\log k)^{1-\epsilon}, \text{ then substitution of } s \text{ into the right-hand side of (5.29) gives} \]
\[
\frac{s\sqrt{4}}{8\sqrt{k}} = \frac{\sqrt{4}}{8} \sigma e^{-(\log k)/s} = \frac{\sqrt{4}}{8} (\log k)^{1-\epsilon} e^{-(\log k)/s} \to 0
\]

for all positive \( \epsilon \). This is clearly far away from the rate of the optimal HD-PIC system in Theorem 7.2. Therefore, we believe \( s_k \geq (\log k)^{1-\epsilon} \) for all \( \epsilon > 0 \) as \( k \to \infty \). This explains Conjecture 7.5, since we have proven that for all \( \epsilon_k \to 0 \), \( s_k \leq \log k/\epsilon_k \).

### 7.1.2 Chernoff bound

We prove the following Chernoff bound for the optimal system:

**Theorem 7.6** When \( k_n \to \infty \) such that \( k_n = o(n) \), and \( s \geq 2^{k_n} + 1 \)

\[
\Pr(\text{sgn}_0(Z_0^{(s,H)}) < 0) \leq e^{-nI(1+o(1))}, \quad \text{where} \quad I = \frac{1}{2} \log 2 - \frac{1}{4}. \tag{7.7}
\]

When the powers are equal, the same result holds whenever \( s \geq k_n + 1 \).

In fact, we prove that for all \( n, k \), \( \Pr(\text{sgn}_0(Z_0^{(s,H)}) < 0) \leq 8^k e^{-nI} \). Therefore, for \( k = k_n \) slightly smaller than \( n \), the result still is valuable. Indeed, when \( k_n = \epsilon n \) with \( \epsilon \) sufficiently small, the BEP is still exponentially small. Therefore, the above results shows that the limit of \( k \) and \( n \) can be taken simultaneously, instead of the order, common in large deviation theory (first \( n \to \infty \), and subsequently \( k \to \infty \)).

**Proof.** The proof of Theorem 7.6 is quite easy when we use results of the proof of Theorem 7.2. In the proof, we have used the fact that for \( s \geq 2^{k_n} + 1 \), there must be a stage \( \sigma \) such that \( P_{R_{\sigma}} \geq P_{R_{\sigma-1}} \). The rate of this event is proven to be bounded from below by \( I = \frac{1}{2} \log 2 - \frac{1}{4} \).

The number of possible stages at which this can happen is \( 2^{k_n} + 1 \), and the number of possibilities for \( R_{\sigma} \) and \( R_{\sigma-1} \) are \( 2^{k_n} - 1 \) each (\( R_{\sigma} = \emptyset \) is not possible). This yields

\[
\Pr(\text{sgn}_0(Z_0^{(s,H)}) < 0) \leq \Pr\left( \bigcup_{\sigma \leq 2^{k_n} + 1} \{P_{R_{\sigma}} \geq P_{R_{\sigma-1}}\} \right)
\leq (2^{k_n} + 1)(2^{k_n} - 1)^2 \min_{A,B,P_{\sigma} \geq P_A} \Pr(R_{\sigma} = A, R_{\sigma-1} = B)
\leq 8^k e^{-In}.
\]

Since \( k_n = o(n) \), the theorem follows.
7.2 Optimal soft decision system

In this section, we will investigate the infinite stage SD-PIC system. We assume that no AWGN is present. We will first write the above system in matrix notation, and explain multistage SD-PIC as a way to approximate the inverse of $\frac{1}{n}XX^T$, the matrix of cross correlations.

When we abbreviate $Z^{(s,s)} = [Z_0^{(s,s)}, \ldots, Z_{k-1}^{(s,s)}]^T$, $P = [P_0, \ldots, P_{k-1}]^T$, we can write (recall (1.2.1))

$$Z^{(1)} = \frac{1}{n}XX^T P^{1/2}.$$

Similarly,

$$Z^{(s,s)} = Z^{(s-1,s)} + \frac{1}{n}XX^T (P^{1/2} - Z^{(s-1,s)}),$$

since in SD-PIC, $h_j(x) = x$. Working out the recursion is simple and results in

$$Z^{(s,s)} = \sum_{\sigma=0}^{s-1} \left( I - \frac{1}{n}XX^T \right)^{\sigma} \frac{1}{n}XX^T P^{1/2}.$$

For $s \rightarrow \infty$,

$$\sum_{\sigma=0}^{s-1} \left( I - \frac{1}{n}XX^T \right)^{\sigma} \rightarrow \left( I - \left( I - \frac{1}{n}XX^T \right) \right)^{-1} = \left( \frac{1}{n}XX^T \right)^{-1},$$

whenever $\lambda_1$, the smallest eigenvalue of $\frac{1}{n}XX^T$, is strictly larger than 0 and $\lambda_k$, the largest eigenvalue, is strictly smaller than 2. When this is the case,

$$\lim_{s \rightarrow \infty} Z^{(s,s)} = n(XX^T)^{-1} \frac{1}{n}XX^T P^{1/2} = P^{1/2},$$

so that all users estimate their bit correctly. Here, we use that we have assumed that $P_j > 0$ for all $j$. Therefore

$$\lim_{s \rightarrow \infty} \mathbb{P}(\text{sgn}_{m}(Z_{m}^{(s,s)}) < 0) \leq \mathbb{P}(\{\lambda_1 = 0\} \cup \{\lambda_k \geq 2\}) \leq \mathbb{P}(\lambda_1 = 0) + \mathbb{P}(\lambda_k \geq 2). \quad (7.8)$$

We will prove the following result in Section 7.5.

**Theorem 7.7** (a) For all $k$,

$$- \lim_{n \rightarrow \infty} \lim_{s \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\text{sgn}_{m}(Z_{m}^{(s,s)} < 0)) \geq \frac{1}{2} - \frac{1}{2} \log 2 = 0.15343 \ldots \quad \text{for all } m,$$

(b) When all powers are equal,

$$- \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \lim_{s \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\text{sgn}_{0}(Z_{0}^{(s,s)} < 0)) = \frac{1}{2} - \frac{1}{2} \log 2.$$
Remark: We expect Theorem 7.7(b) to hold also when powers are not equal. In the proof, we in fact show that for \( s \to \infty \), some users have errors with rate \( \frac{1}{2} - \frac{1}{2} \log 2 \). By exchangeability, this proves 7.7(b). However, a more detailed argument might prove the statement Theorem 7.7(b) for unequal powers.

The multistage SD-PIC receiver approximates the inverse of \( \frac{1}{n} X X^T \). Rather than approximating the inverse, one could also directly calculate the inverse. In Lupas and Verdú (1989), the receiver applying the inverse of \( \frac{1}{n} X X^T \) is introduced. This so-called decorrelator has a better performance than PIC receivers, but is of higher complexity; see also Section 3.5 and the references therein.

For \( k = 3 \), we are able to prove the following result

**Proposition 7.8** For \( k = 3 \),

\[
- \lim_{n \to \infty} \lim_{s \to \infty} \frac{1}{n} \log \mathbb{P}(\text{sgnr}_0(Z_0^{(s,s)}) < 0) \leq \frac{5}{8} \log 5 - \log 2 = 0.31275 \ldots
\]

Applying Sanov’s theorem and solving the corresponding optimization problem numerically indicates that this upper bound is tight. We will now prove this upper bound.

**Proof.** First of all, we will show that \( \mathbb{P}(\text{sgnr}_0(Z_0^{(s,s)}) < 0) \geq \frac{1}{4} \mathbb{P}(\lambda_3 > 2) \) when \( s \) is sufficiently large. To do so, we will use an eigenvector decomposition. We write

\[
P^{1/2} = \sum_{j=1}^{3} \alpha_j v_j, \tag{7.9}
\]

where \( (v_j)_{j=1}^{3} \) form an orthonormal basis of eigenvectors of \( \frac{1}{n} X X^T \) and \( \alpha_j = \langle P^{1/2}, v_j \rangle \). Then we immediately see that

\[
Z^{(s)} = \sum_{\sigma=0}^{s-1} \left( I - \frac{1}{n} X X^T \right) \sigma \frac{1}{n} X X^T P^{1/2} = \sum_{\sigma=0}^{s-1} \sum_{j=1}^{3} (1 - \lambda_j)^{\sigma} \lambda_j \alpha_j v_j \tag{7.10}
\]

\[
= \sum_{j=1}^{3} \left( 1 - (1 - \lambda_j)^{s} \right) \alpha_j v_j.
\]

Assume that \( \lambda_3 > 2 \). Then we can take \( s \) large, so that the sum in (7.10) is mainly determined by all terms with eigenvalue \( \lambda_3 \) > 2. The multiplicity of the eigenvalue \( \lambda_3 \) is equal to 1, since \( \lambda_1 + \lambda_2 + \lambda_3 = 3 \) (the trace equals 3). We first assume \( v_{31} \), the first element of \( v_3 \) to be strictly positive and all other elements nonzero. We will treat the other cases later. When we take \( D = \text{diag}(1, -1, 1) \text{ or diag}(1, 1, -1) \text{ or diag}(1, -1, -1) \), \( D v_3 \) is an eigenvector of \( \frac{1}{n} D X X^T D \) with the eigenvalue \( \lambda_3 \), whenever \( v_3 \) is an eigenvector with eigenvalue \( \lambda_3 \). Since every matrix \( \in \{-1, 1\}^{k \times k} \) has the same probability to be assumed, \( \lambda_3 > 2 \) implies with probability 1/4 an eigenvector with strictly positive elements. This implies \( \alpha_3 = \langle v_3, P^{1/2} \rangle > 0 \), since \( P_j > 0 \).

We can write the above sum as

\[
Z^{(s)} = (1 - (1 - \lambda_3)^{s}) \alpha_3 v_3 (1 + o(1)). \tag{7.11}
\]
Since $\lambda_3 > 2$, we have that $1 - (1 - \lambda_3)^s$ alternates when $s$ changes. Hence, the estimates of the bits all alternate. We conclude that for $s$ sufficiently large

$$- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\text{sgnr}_0(Z_0^{(s,2)}) < 0) \leq - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\lambda_3 > 2).$$

In the case that there is one element, say $j$, equal to zero, we have $\lambda_3 \leq 2$. Indeed, in this case the coding sequence of user $j$ is not interesting, so that we are in fact dealing with 2 users. In the case of 2 users, the largest eigenvalue is at most 2, since the trace equals 2, while all eigenvalues are non-negative.

We finish the proof by showing

$$- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\lambda_3 > 2) \leq \frac{5}{8} \log 5 - \log 2.$$ 

Since $v^T \frac{1}{n} X X^T v \leq \lambda_3$ for all $v$ with $\|v\|_2 = 1$, we take $v = 3^{-1/2}[1 1 1]^T$ to obtain

$$\mathbb{P}(\lambda_3 > 2) \geq \mathbb{P}\left( \frac{1}{n} [1 1 1] X X^T [1 1 1]^T > 6 \right) = \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} \left( 3 + 2(X_{1i} + X_{2i} + X_{1i}X_{2i}) \right) > 6 \right).$$

Since $\mathbb{P}(X_{1i} + X_{2i} + X_{1i}X_{2i} = -1) = 3/4$ and $\mathbb{P}(X_{1i} + X_{2i} + X_{1i}X_{2i} = 3) = 1/4$, Cramér's theorem directly gives

$$- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\lambda_3 > 2) \leq - \log \left\{ \min_{t \geq 0} e^{3t} \left( \frac{3}{4} e^{-2t} + \frac{1}{4} e^{6t} \right) \right\} = \frac{5}{8} \log 5 - \log 2 = 0.31275 \ldots$$

When we compare this rate with the numerical result for the one-stage SD-PIC model for $k = 3$, we see that the difference is small. When $k = 3$ and all powers are equal, infinite many stages of SD-PIC does not increase performance significantly, compared to one-stage SD-PIC. However, the result of the optimal SD-PIC system does not depend on the powers.

### 7.3 Bounds on moment generating functions

In this section, we will give sharp bounds on certain moment generating functions that will prove to be essential in the analysis of both the optimal HD-PIC and the optimal SD-PIC system.

For $A \subseteq \{0, \ldots, k-1\}$, we invoke

$$\tilde{S}_A = \sum_{m \in A} P_m^{1/2} X_m.$$  \hfill (7.12)

We will also use the notation $P_A = \sum_{m \in A} P_m$.

The main result of this section is
Proposition 7.9
(a) For all \( A \subset \mathbb{N} \setminus \{0\} \), \( 0 \leq u < \frac{1}{P_A} \) and all \( s \in \mathbb{R} \),
\[
\mathbb{E} \left( e^{s\hat{S}_A + \frac{u}{2} \hat{S}_A^2} \right) \leq \frac{e^{u(1-P_A)} - P_A u}{\sqrt{1 - P_A u}}.
\] (7.13)

(b) For all \( A \subset \mathbb{N} \cup \{0\} \) and all \(-\frac{1}{P_A} \leq u \leq \frac{1}{P_A}\),
\[
\mathbb{E} \left( e^{\frac{u}{2} \hat{S}_A^2} \right) \leq \frac{1}{\sqrt{1 - P_A u}}.
\] (7.14)

Note that when \( |A| \) is large, we have \( \hat{S}_A \approx P_A^{1/2} Z \), where \( Z \) has a standard normal distribution. The bounds in (7.13-7.14) show that the moment generating function of \( \hat{S}_A \) and \( \hat{S}_A^2 \), for the appropriate ranges of the variables \( s \) and \( u \) are at least bounded from above by the moment generating functions of \( P_A^{1/2} Z \) and \( P_A Z^2 \), \( e^{u/2} \) and \( 1/\sqrt{1 - P_A u} \), respectively.

Proof of Proposition 7.9(a). When \( Z \) has a standard normal distribution, we know that \( \mathbb{E}(e^{tZ}) = e^{t^2/2}. \) Hence, we get by conditioning
\[
\mathbb{E} \left( e^{s\hat{S}_A + \frac{u}{2} \hat{S}_A^2} \right) = \mathbb{E} \left( e^{(s+\sqrt{u}Z)\hat{S}_A} \right) = \mathbb{E} \left( \prod_{j \in A} \cosh(P_j^{1/2}(s + \sqrt{u}Z)) \right).
\]

We use that \( \cosh t \leq e^{t^2/2} \), to arrive at
\[
\mathbb{E} \left( e^{s\hat{S}_A + \frac{u}{2} \hat{S}_A^2} \right) \leq \mathbb{E} \left( \prod_{j \in A} e^{P_j^{1/2}(s+\sqrt{u}Z)^2} \right) = e^{\frac{P_A u^2}{2}} \mathbb{E} \left( e^{P_A s\sqrt{u}Z + \frac{uP_A}{2} Z^2} \right).
\]

We complete the proof by noting that, for \( u \leq \frac{1}{P_A} \),
\[
\mathbb{E} \left( e^{P_A s\sqrt{u}Z + \frac{uP_A}{2} Z^2} \right) = \frac{e^{\frac{P_A u^2}{2} - P_A u}}{\sqrt{1 - P_A u}},
\]
and by rearranging terms.

Proof of Proposition 7.9(b).

We first observe that
\[
\left( P_0^{1/2} + \sum_{j \in A} P_j X_{j1} \right)^2 = \left( -P_0^{1/2} - \sum_{j \in A} P_j X_{j1} \right)^2 \overset{d}{=} \left( -P_0^{1/2} + \sum_{j \in A} P_j X_{j1} \right)^2,
\]
so that we may treat \( X_{01} \) as a random variable with \( \mathbb{P}(X_{01} = +1) = \mathbb{P}(X_{01} = -1) = 1/2 \), independently from \( X_{m1}, 1 \leq m \leq k - 1 \). This assumption is identical to the one used in the derivation below equation (5.43).
The claim for \( 0 \leq u \leq \frac{1}{P_A} \) follows from Proposition 7.9(a) proved above. The claim for \( u < 0 \) is more difficult, and we will use induction in \(|A|\). Define

\[
f_A(u) = E\left(e^{\frac{n}{2}S_A^2}\right).
\]  
(7.15)

The induction hypothesis is that for \(|A| = n\),

\[
f_A(u) \leq \frac{1}{\sqrt{1 - P_Au}} \quad \text{for all} \quad -\frac{1}{P_A} \leq u \leq 0.
\]  
(7.16)

Clearly, for \(|A| = 1\), the above holds, since in this case

\[
E e^{\frac{n}{2}S_A^2} = e^{P_Au/2} = \sqrt{e^{P_Au}} \geq \sqrt{(1 + P_Au + P_A^2u^2/2)},
\]

since \( P_Au \leq 0 \). Furthermore, \( 1 + P_Au \geq 0 \), so that

\[
(1 - P_Au)^{1/2} e^{\frac{n}{2}S_A^2} \geq \sqrt{(1 - P_Au)(1 + P_Au + P_A^2u^2/2)} = \sqrt{1 - P_A^2u^2(1 + P_Au)/2} \leq 1.
\]

We next advance the induction. We write \( A = A' \cup i \) for some \( i \neq 0 \). Then

\[
f_A(u) = E\left(e^{\frac{n}{2}S_A^2}\right) = e^{P_Au/2}E\left(e^{\frac{n}{2}S_A^2 + P_i^{1/2}uS_A'X_i}\right) = e^{P_Au/2}E\left(e^{\frac{n}{2}S_A'\cosh(P_i^{1/2}uS_A')}\right).
\]

We again use that \( \cosh t \leq e^{t^2/2} \) to arrive at

\[
f_A(u) \leq e^{P_iu/2}E\left(e^{\frac{n}{2}S_A^2 + P_i^{1/2}uS_A'}\right) = e^{P_iu/2}f_A'(u + P_iu^2).
\]

To prove the claim, we first show that for \( -\frac{1}{P_A} \leq u \leq 0 \),

\[
-\frac{1}{P_A - P_i} \leq u + P_iu^2 \leq 0
\]

Indeed, since \( -\frac{1}{P_A} \leq u \leq 0 \), we have \( 0 \leq 1 + P_iu \leq 1 \), so that

\[
-1 \leq (-1 - P_iu)(1 + P_iu) \leq (P_Au - P_iu)(1 + P_iu) = (P_A - P_i)u(1 + P_iu) \leq 0,
\]

where the last inequality follows from \( P_A - P_i \geq 0 \) and \( u \leq 0 \). We therefore can substitute the induction hypothesis (7.16) for \(|A'| = n - 1\), so that it remains to show that

\[
e^{uP_i/2} \leq \frac{1}{\sqrt{1 - (P_A - P_i)(u + P_iu^2)}}.
\]

Since \( e^x \geq 1 + x + x^2/2 \) for all \( x \geq 0 \),

\[
e^{uP_i/2} = \frac{1}{e^{-uP_i}} \leq \frac{1}{\sqrt{1 - P_iu + P_i^2u^2/2}}.
\]

When we multiply and rearrange terms, we see that the inequality below is sufficient to prove the claim.

\[
(1 - P_iu + P_i^2u^2/2)(1 - (P_A - P_i)(u + P_iu^2))
\]

\[
= 1 - P_Au + P_i^2u^2/2 + P_i^2(P_A - 4P_i/3)u^3/2 - P_i^3(P_A - P_i)u^4/3
\]

\[
= 1 - P_Au + \frac{P_i^2u^2}{2} \left(1 + P_Au - P_iu(1 + (P_A - P_i)u)\right) \geq 1 - P_Au,
\]

since \( 1 + P_Au \geq 0 \), \( u \leq 0 \) and \( 1 + (P_A - P_i)u \geq 0 \). This completes the proof.

\[\square\]
7.4 The optimal rate for HD-PIC

In this section we will first prove Theorem 7.2(a) and 7.4, where we will use Proposition 7.9. Finally, we prove Theorem 7.2(b).

7.4.1 The lower bound

In this section, we will prove Theorem 7.2(a) and Theorem 7.4 simultaneously. For completeness, we will repeat the statements. For \( s \geq 2^k + 1 \),

\[
H_k^{(s)} \geq \frac{1}{2} \log 2 - \frac{1}{4}.
\]  \hspace{1cm} (7.17)

For all \( 0 < \varepsilon < \frac{1}{2} \log 2 - \frac{1}{4} \) and for \( s \geq \lfloor \varepsilon^{-1} \log \left(\frac{P}{P_{\min}}\right) \rfloor + 1 \),

\[
H_k^{(s)} \geq \frac{1}{2} \log 2 - \frac{1}{4} - \varepsilon.
\]  \hspace{1cm} (7.18)

In Section 7.1 we have shown that there is an optimal HD-PIC system, and that for \( s \geq 2^k + 1 \), the set of users with a bit error is periodic. When \( s = 2^k + 1 \), there must be a \( \sigma \leq 2^k + 1 \) such that \( P_{R_\sigma} \geq P_{R_{\sigma-1}} \). In fact, when all powers are equal, we have that \( P_{R_\sigma} = P|R_\sigma| \), and the above must happen already when \( s = k + 1 \). We focus on that level \( \sigma \) and are only interested in the event \( \{P_{R_\sigma} \geq P_{R_{\sigma-1}}\} \).

Furthermore, when \( s = \lfloor \varepsilon^{-1} \log(P/P_{\min}) \rfloor + 1 \), there must be a \( \sigma \leq \lfloor \varepsilon^{-1} \log(P/P_{\min}) \rfloor + 1 \) such that \( P_{R_\sigma} \geq (1 - \varepsilon)P_{R_{\sigma-1}} \). Indeed, when this should not be the case after \( \sigma = \lfloor \varepsilon^{-1} \log(P/P_{\min}) \rfloor \) stages, we have

\[
P_{R_\sigma} \leq (1 - \varepsilon)^{\sigma-1}P_{R_0} \leq \exp\left(\frac{\log(1-\varepsilon)}{\varepsilon} \log\left(\frac{P}{P_{\min}}\right)\right) P.
\]

Since \( 1 - \varepsilon \leq e^{-\varepsilon} \), also \( \log(1-\varepsilon)/\varepsilon \leq -1 \), so that

\[
P_{R_\sigma} \leq (1 - \varepsilon)^{\sigma-1}P \leq \exp\left(-\log\frac{P}{P_{\min}}\right) P = P_{\min}.
\]

However, then

\[
P_{R_{\sigma+1}} \geq P_{\min} \geq P_{R_\sigma} \geq (1 - \varepsilon)P_{R_\sigma},
\]

so certainly after at most \( s = \lfloor \varepsilon^{-1} \log(P/P_{\min}) \rfloor + 1 \) stages the desired event has occurred. We focus on the level \( \sigma \) at which the desired event occurs and are only interested in that level.

At this point we remark that once we obtain for all \( \varepsilon \geq 0 \) and \( P_{R_\sigma} \geq (1 - \varepsilon)P_{R_{\sigma-1}} \) that

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( P_{R_\sigma} \geq (1 - \varepsilon)P_{R_{\sigma-1}} \right) \geq \frac{1}{2} \log 2 - \frac{1}{4} - \varepsilon
\]
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holds, substitution of \( \varepsilon = 0 \) yields the statement (7.17). We will now prove the above statement.

We use that when \( A = R_{\sigma-1} \) and \( B = R_{\sigma} \), that then for all \( m \in B \), we have that

\[
P_{m}^{1/2} Z_{m}^{(\sigma,H)} = P_{m} + \frac{2}{n} \sum_{i=1}^{n} \sum_{j \in A \setminus \{m\}} (P_{j} P_{m})^{1/2} X_{ji} X_{mi} - P_{m}(1 - 2 \mathbf{1}_{\{m \in A \cap B\}}) + \frac{2}{n} \sum_{i=1}^{n} \sum_{j \in A} (P_{j} P_{m})^{1/2} X_{ji} X_{mi}
\]

As the number of configuration \( (R_{\sigma})_{\sigma=1}^{\tau} \) is just finite, we obtain that \( H_{k}^{(s)} \) is bounded from below by the minimum over \( A \) and \( B \) such that \( P_{B} \geq (1 - \varepsilon) P_{A} \) of

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \forall m \in B : \sum_{j \in A \setminus \{m\}} \sum_{i=1}^{n} (P_{j} P_{m})^{1/2} X_{ji} X_{mi} + (1 - 2 \mathbf{1}_{\{m \in A \cap B\}}) \frac{n P_{m}}{2} \leq 0 \right)
\]

\[
\geq - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \sum_{(j,m) \in A \times B} \sum_{i=1}^{n} (P_{j} P_{m})^{1/2} X_{ji} X_{mi} + (1 - 2 \mathbf{1}_{\{m \in A \cap B\}}) \frac{n P_{m}}{2} \leq 0 \right)
\]

\[
\geq - \log \left( e^{-P_{B} - 2P_{A \cap B}} \mathbb{E} \left( e^{\frac{1}{2} \sum_{j \in A, m \in B}(P_{j} P_{m})^{1/2} X_{ji} X_{mi}} \right) \right), \text{ for all } t \leq 0,
\]

where the last inequality is the exponential Chebycheff's inequality for \( t/2 \). We invoke the notation \( \tilde{S}_{A} = \sum_{j \in A} P_{j}^{1/2} X_{ji} \) to end up with

\[
H_{k}^{(s)} \geq \min_{P_{B} \geq (1 - \varepsilon) P_{A}} \left[ - \log \left( \mathbb{E} \left( e^{\frac{1}{2} \tilde{S}_{A}} \right) \right) - \frac{P_{B} - 2P_{A \cap B}}{4} t \right] \text{ for } t \leq 0. \tag{7.19}
\]

We will now bound the moment generating function from below using Proposition 7.9.

In the proof below, we have to take care when \( \{0\} \in A \setminus B \), because of Proposition 7.9(a). In all other cases, we may treat \( X_{01} \) as a random variable with \( \mathbb{P}(X_{01} = +1) = \mathbb{P}(X_{01} = -1) = 1/2 \), independently from \( X_{m1} \), \( 1 \leq m \leq k - 1 \), which is justified by the derivation below equation (5.43). When \( \{0\} \in A \setminus B \), we can exchange \( A \) and \( B \). Therefore, in the remainder of the proof we treat \( X_{01} \) as a random variable with \( \mathbb{P}(X_{01} = +1) = \mathbb{P}(X_{01} = -1) = 1/2 \), independently from \( X_{m1} \), \( 1 \leq m \leq k - 1 \).

We first write \( \tilde{S}_{A} = \tilde{S}_{A \cap B} + \tilde{S}_{A \setminus B} \), and we use the fact that \( \tilde{S}_{A \setminus B} \) is independent from \( (\tilde{S}_{B}, \tilde{S}_{A \cap B}) \) to get that \( \mathbb{E} \left( e^{\frac{1}{2} \tilde{S}_{A}} \right) \) equals

\[
\mathbb{E} \left( e^{\frac{1}{2} \tilde{S}_{B}} e^{\frac{1}{2} \tilde{S}_{A \cap B}} \prod_{j \in A \setminus B} \cosh \left( \frac{t}{2} P_{j}^{1/2} \tilde{S}_{B} \right) \right) \leq \mathbb{E} \left( e^{\frac{1}{2} \tilde{S}_{B}} e^{\frac{1}{2} \tilde{S}_{A \cap B}} e^{\frac{1}{2} P_{A \cap B} \tilde{S}_{B}} \right)
\]

We write the right-hand side of the above expression as

\[
\mathbb{E} \left( e^{\frac{1}{2} + \frac{1}{2} P_{A \cap B} \tilde{S}_{A \cap B}} e^{\frac{1}{2} + \frac{1}{2} P_{A \cap B} \tilde{S}_{B \setminus A}} \tilde{S}_{A \cap B \setminus A} + \frac{1}{2} P_{A \cap B} \tilde{S}_{B \setminus A} \right),
\]

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and use Proposition 7.9(a) with \( s = \left( \frac{t}{2} + \frac{t^2}{4} \right) P_{A \setminus B} \tilde{S}_{A \cap B} \) and \( u = \frac{t^2}{4} \) \( P_{A \setminus B} \geq 0 \) to bound the conditional expectation over \( \tilde{S}_{A \cap B} \) as

\[
\frac{1}{\sqrt{1 - P_{A \setminus B} P_{B \setminus A} t^2/4}} \mathbb{E} \left( e^{\left( \frac{t}{2} + \frac{t^2}{8} P_{A \setminus B} \tilde{S}_{A \cap B} \right) + \frac{t^2}{2} (t + P_{A \setminus B} t^2/4)^2} \right) \leq \frac{1}{\sqrt{1 - P_{A \setminus B} P_{B \setminus A} t^2/4}} \frac{1}{\sqrt{1 - P_{A \setminus B} t}}.
\]

where the last inequality is valid as long as \( |\tilde{t}| \leq \frac{1}{P_{B \setminus A}} \) and where

\[
\tilde{t} = t + \frac{t^2 P_{A \setminus B}}{4} + P_{B \setminus A} (t/2 + P_{A \setminus B} t^2/4)^2 = \frac{t}{1 - P_{A \setminus B} P_{B \setminus A} t^2/4} - \frac{1}{4} P_{A \setminus B} (P_B - P_{A \cap B}) t^2.
\]

From the first expression for \( \tilde{t} \) it is clear that \( \tilde{t} \geq t \), so that we restrict \(-1/\) \( P_{A \cap B} \leq t \leq 0 \). From the second expression for \( \tilde{t} \) above it is straightforward to prove that \( \tilde{t} \leq 0 \). We proceed by multiplying the two square roots to obtain

\[
\mathbb{E} \left( e^{\frac{t}{2} \tilde{S}_A \tilde{S}_B} \right) \leq \frac{1}{\sqrt{1 - t P_{A \setminus B} P_{B \setminus A} - \frac{t^2}{4} (P_{A P_B} - P_{A \cap B}^2)}}.
\]  

(7.20)

since

\[
P_{A \setminus B} P_{B \setminus A} + P_{A \cap B} (P_{A \setminus B} + P_{B \setminus A}) = P_{A P_B} - P_{A \cap B}^2.
\]

Substituting (7.20) into (7.19) yields that for all \(-1/P_{A \cap B} \leq t \leq 0 \) we have that \( H_k^{(s)} \) is bounded from below by

\[
\min_{P_B \geq (1 - \epsilon) P_A} \frac{1}{2} \log \left( 1 - t P_{A \cap B} - \frac{t^2}{4} (P_{A P_B} - P_{A \cap B}^2) \right) - \frac{P_B - 2 P_{A \cap B} t}{4}.
\]  

(7.21)

Since \(-1/P_{A \cap B} \leq -1/P_B \leq 0 \), substituting \( t = -1/P_B \) results in

\[
H_k^{(s)} \geq \min_{P_B \geq (1 - \epsilon) P_A} \frac{1}{2} \log \left( 1 - \frac{P_A}{4 P_B} + \frac{P_{A \cap B}}{P_B} + \frac{P_{A \cap B}^2}{4 P_B} \right) - \frac{1}{4} \left( 2 P_{A \cap B} - 1 \right).
\]  

(7.22)

It is clear that this lower bound of \( H_k^{(s)} \) is decreasing in \( P_A \) and that \( P_A \leq \frac{1}{1 - \epsilon} P_B \). Therefore substituting \( P_A = \frac{1}{1 - \epsilon} P_B \) still gives a lower bound. Clearly, \( \frac{P_{A \cap B}}{P_B} \in [0, 1] \). The above lower bound is therefore attained at

\[
\min_{0 \leq \alpha \leq 1} \frac{1}{2} \log \left( 1 - \frac{1}{4 (1 - \epsilon)} + \alpha + \frac{\alpha^2}{4} \right) - \frac{1}{4} (2 \alpha - 1) = \min_{0 \leq \alpha \leq 1} f(\alpha).
\]  

(7.23)

Differentiating \( f(\alpha) \) w.r.t. \( \alpha \) gives

\[
f'(\alpha) = -\frac{1}{2} + \frac{1}{2} \left( -\frac{1}{4 (1 - \epsilon)} + \alpha \right) = \frac{1}{4 (1 - \epsilon)} - \frac{\alpha}{2} - \frac{\alpha^2}{4}.
\]
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Hence, using \( \varepsilon < 3/4 \), \( f'(\alpha) > 0 \) for \( 0 \leq \alpha < \sqrt{1+1/(1-\varepsilon)} - 1 \) and \( f'(\alpha) < 0 \) for \( \alpha > \sqrt{1+1/(1-\varepsilon)} - 1 \). Therefore, the minimum of \( f \) is attained at either \( \alpha = 0 \) or at \( \alpha = 1 \). Note that this is the disjoint scenario and the overlapping scenario, respectively. The partly overlapping scenario is never optimal in the asymptotic sense. Substitution of \( \alpha = 0 \) and \( 1 \) yields that

\[
f(0) = \frac{1}{2} \log \left(1 - \frac{1}{4(1-\varepsilon)}\right) + \frac{1}{4} = \frac{1}{2} \log \frac{3}{4} + \frac{1}{4} + \frac{1}{2} \log \left(1 - \frac{1}{3} \left(\frac{1}{1-\varepsilon} - 1\right)\right)
\]

and

\[
f(1) = \frac{1}{2} \log \left(\frac{9}{4} - \frac{1}{4(1-\varepsilon)}\right) - \frac{1}{4} = \frac{1}{2} \log 2 - \frac{1}{4} + \frac{1}{2} \log \left(1 - \frac{1}{8} \left(\frac{1}{1-\varepsilon} - 1\right)\right).
\]

Finally, observe that for \( 0 \leq \varepsilon \leq 1/2 \), \( 1/(1-\varepsilon) \leq 1+2\varepsilon \), and for \( 0 \leq \varepsilon \leq 2/3 \),

\[
e^{-2\varepsilon} \leq 1 - 2\varepsilon + 2\varepsilon^2 \leq 1 - 2\varepsilon + 4\varepsilon/3 = 1 - 2\varepsilon/3,
\]

so that \( \log(1 - 2\varepsilon/3) \geq -2\varepsilon \). Substituting this yields

\[
\log \left(1 - \frac{1}{3} \left(\frac{1}{1-\varepsilon} - 1\right)\right) \geq \log \left(1 - \frac{2\varepsilon}{3}\right) \geq -2\varepsilon,
\]

\[
\log \left(1 - \frac{1}{8} \left(\frac{1}{1-\varepsilon} - 1\right)\right) \geq \log \left(1 - \frac{\varepsilon}{4}\right) \geq -\frac{3\varepsilon}{4} \geq -2\varepsilon.
\]

Therefore, since \( \frac{1}{2} \log \frac{3}{4} + 1/4 > \frac{1}{2} \log 2 - 1/4 \),

\[
H_k^{(s)} \geq \min\{f(0), f(1)\} \geq \frac{1}{2} \log 2 - \frac{1}{4} - \varepsilon.
\]

This completes the proof of Theorem 7.4. Substituting \( \varepsilon = 0 \) proves Theorem 7.2(a). \( \blacksquare \)

### 7.4.2 The upper bound for equal powers

In order to prove Theorem 7.2(b), we have to find a strategy that has asymptotic rate \( \frac{1}{2} \log 2 - \frac{1}{4} \). For simplicity, we will assume that all powers are equal. The proof for unequal powers is more technical and is therefore deferred to Appendix F. We note that when \( R_2 = R_1 \), that necessarily for all \( \sigma \geq 1 \), \( R_\sigma = R_1 \). This case implies bit errors at all stages, and the rate of this event is an upper bound for the rate of the optimal system. We fix \( r = |R_1| \) and for technical reasons we assume \( r \) to be odd. We will first investigate the rate

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(R_2 = R_1, |R_1| = r).
\]

Due to the fact that all users are exchangeable, and that the rate function of the vector \( (Z_m^{(1)}, Z_m^{(r,n)})_{m=0}^{k-1} \) is convex, we can follow the line of argument presented in the proof of Theorem 4.2 (see (4.9) and below) to see that

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(R_2 = R_1, |R_1| = r)
\]

\[
= - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(\sum_{m=0}^{r-1} Z_m^{(1)} \leq 0, \sum_{m=r}^{k-1} Z_m^{(1)} \geq 0, \sum_{m=0}^{r-1} \bar{Z}_m^{(2,n)} \leq 0, \sum_{m=r}^{k-1} \bar{Z}_m^{(2,n)} \geq 0\right).
\]

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In the proof of Theorem 4.2, it is also shown that $\sum_{m=r}^{k-1} Z^{(m)}_m \geq 0$, so that we can remove the event $\{\sum_{m=r}^{k-1} Z^{(m)}_m \geq 0\}$. Since $\mathbb{P}(X \leq 0, Y \leq 0) \geq \mathbb{P}(X - Y/2 \leq 0, Y \leq 0)$, we can bound the exponential rate from above by

$$-\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \sum_{m=0}^{r-1} Z^{(m)}_m - \frac{1}{2} \tilde{Z}^{(2,H)}_m \leq 0, \sum_{m=0}^{r-1} \tilde{Z}^{(2,H)}_m \leq 0, \sum_{m=r}^{k-1} Z^{(2,H)}_m \geq 0 \right),$$

where for $0 \leq m \leq r - 1$,

$$Z^{(m)}_m - \frac{1}{2} \tilde{Z}^{(2,H)}_m = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{2} + \sum_{j=r}^{k-1} X_{ji} X_{mi} \right),$$

$$\tilde{Z}^{(2,H)}_m = \frac{1}{n} \sum_{i=1}^{n} \left( 1 + 2 \sum_{j=0}^{r} X_{ji} X_{mi} \right) = \frac{1}{n} \sum_{i=1}^{n} \left( -1 + 2 \sum_{j=0}^{r} X_{ji} X_{mi} \right),$$

while for $r \leq m \leq k - 1$

$$\tilde{Z}^{(2,H)}_m = \frac{1}{n} \sum_{i=1}^{n} \left( 1 + 2 \sum_{j=0}^{r} X_{mi} X_{ji} \right).$$

We abbreviate

$$E_1 = \left\{ \frac{r}{2} + \sum_{m=r}^{k-1} \sum_{j=0}^{r-1} \frac{1}{n} \sum_{i=1}^{n} X_{mi} X_{ji} \leq 0 \right\},$$

$$E_3 = \left\{ -r + 2 \sum_{m,j=0}^{r} \frac{1}{n} \sum_{i=1}^{n} X_{mi} X_{ji} \leq 0 \right\},$$

$$E_4 = \left\{ k - r + \sum_{m=0}^{r-1} \sum_{j=0}^{k-1} \frac{1}{n} \sum_{i=1}^{n} X_{mi} X_{ji} \geq 0 \right\}.$$

To preserve a parallel with the proof of Theorem 7.2(b) in the case of unequal powers, we reserve $E_2$ for the event dealing with $Z^{(m)}_m$ for $m = r, \ldots, k - 1$. Clearly,

$$\mathbb{P}(E_1 \cap E_3) = \mathbb{P}(E_1 \cap E_3 \cap E_4) + \mathbb{P}(E_1 \cap E_3 \cap E_4^c),$$

so that, according to the largest-exponent-wins principle,

$$-\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_3) = \min \left\{ -\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_3 \cap E_4), -\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_3 \cap E_4^c) \right\}.$$

We will show that

$$\liminf_{k \to \infty} -\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_2 \cap E_4) \geq \frac{1}{2} \log \frac{3}{4} + \frac{1}{4} = 0.106158\ldots \quad (7.24)$$

for all $r$. Furthermore, we will show

$$-\lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_3) \leq -\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_3) \quad (7.25)$$
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and

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_3) \leq \frac{1}{2} \log 2 - \frac{1}{4} + o(1) = 0.096573 \ldots + o(1). \tag{7.26}
\]

for \( r \) sufficiently large. Because the three statements \( u = \min(v, w) \), \( u \leq x \) and \( w > x \) imply \( u \leq x \), this implies directly when \( r \) is large

\[
- \lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_3 \cap E_4) \leq \frac{1}{2} \log 2 - \frac{1}{4} + o(1).
\]

Equations (7.24), (7.25) and (7.26) also imply the theorem. Indeed, taking \( r \to \infty \) gives

\[
- \lim_{r \to \infty} \lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_3 \cap E_4) \leq \frac{1}{2} \log 2 - \frac{1}{4}.
\]

The remainder of this proof therefore focuses on proving (7.24), (7.25) and (7.26). We prove (7.24) in the following lemma, (7.25) in Lemma 7.11 and (7.26) in Lemma 7.12.

**Lemma 7.10** As \( k \to \infty \),

\[
\lim \inf_{k \to \infty} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_2 \cap E_4) \geq \frac{1}{2} \log 3 + \frac{1}{4}. \tag{7.27}
\]

**Proof.** We bound the rate of interest from below by

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_4) = - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( \sum_{m=1}^{k-1} Z_{m}^{(a, H)} \leq 0 \right)
\]

\[
= - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} \sum_{m=r}^{k-1} (k - r) + 2 \sum_{j=0}^{r-1} X_{ji} X_{mi} \leq 0 \right).
\]

We can follow the proof of the previous section with \( A = \{0, \ldots, r-1\} \) and \( B = \{r, \ldots, k-1\} \). This results in

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_4) \geq \frac{1}{2} \log 3 + \frac{1}{4}.
\]

The strategy of remainder of the proof of Theorem 7.2(b) is first to characterize the behaviour for \( k \to \infty \) and then showing that \( r \to \infty \) gives the desired result.

**Lemma 7.11** For \( r \) fixed,

\[
\lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_3) \leq - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_3). \tag{7.28}
\]
Proof. We focus on
\[ -\lim_{n\to\infty} \frac{1}{n} \log P \left( \frac{r}{2} + \sum_{m=r}^{k-1} \sum_{j=0}^{r-1} \sum_{i=1}^{\frac{n}{m}} X_{mi} X_{ji} \leq 0, \right. \]
\[ \left. -r + 2 \sum_{m=j=0}^{r} \sum_{i=1}^{\frac{n}{m}} X_{mi} X_{ji} \leq 0 \right) \]
Using Cramér's theorem and invoking the notation \( R = \{0, \ldots, r - 1\} \) and \( R^0 = \{r, \ldots, k - 1\} \), gives that the rate above is given by
\[ -\inf_{t_1, t_2 \leq 0} \log E \left( e^{t_1 r/2 + S_R S_{R^0} + t_2 (2S_R^2 - r)} \right) = -\inf_{t_1, t_2 \leq 0} \log \left( e^{t_1 r/2} \cosh(t_1 S_R) e^{t_2 (2S_R^2 - r)} \right), \] (7.29)
where the last equality follows since \( S_R \) and \( S_{R^0} \) are independent.

Suppose that the infimum of the right-hand side of (7.29) is attained for \(-\delta < t_1 \leq 0, \delta > 0\). Then, because \( \cosh x \geq 1 \), the rate on the right-hand side of (7.29) is bounded from above by
\[ \leq -\inf_{t_2 \leq 0} \log \left( e^{-\delta r/2} e^{t_2 (2S_R^2 - r)} \right) = \delta r/2 - \inf_{t_2 \leq 0} \log E e^{t_2 (2S_R^2 - r)} = \delta r/2 - \lim_{n\to\infty} \frac{1}{n} \log P(E_3). \]
Since
\[ -\inf_{t_2 \leq 0} f(t) = \max\{-\inf_{-\delta < t_1 \leq 0} f(t), -\inf_{t_1 \leq -\delta} f(t)\}, \]
it is sufficient to show that for all \( \delta > 0 \) and \( k \to \infty \),
\[ -\inf_{t_1 \leq -\delta, t_2 \leq 0} \log \left( e^{t_1 r/2} \cosh(t_1 S_R) e^{t_2 (2S_R^2 - r)} \right) \to -\infty. \]
We will now prove this. Since \( r \) is odd, \( \cosh(t_1 S_R) \geq \cosh(t_1) \geq e^{t_1}/2 \), so that
\[ e^{t_1 r/2} \cosh(t_1 S_R)^r \geq 2^{-r} e^{-t_1 r/2} e^{t_1 r} \geq 2^{-r}. \]
Substituting the result above gives
\[ -\inf_{t_1 \leq -\delta, t_2 \leq 0} \log \left( e^{t_1 r/2} \cosh(t_1 S_R) e^{t_2 (2S_R^2 - r)} \right) \leq -\inf_{t_1 \leq -\delta, t_2 \leq 0} \log \left( e^{-t_1 \log 2} \cosh(t_1 S_R) e^{t_2 (2S_R^2 - r)} \right) \leq -\inf_{t_2 \leq 0} \log \left( e^{-r \log 2} \delta e^{t_2 (2S_R^2 - r)} \right) \leq -\inf_{t_2 \leq 0} \log E e^{t_2 (2S_R^2 - r)} + r \log 2 - (k - 2r) \log \delta. \]
Since \( \cosh \delta > 1 \), the last term tends to \(-\infty\) for \( k \to \infty \).

\[ \boxed{} \]

To complete the proof of Theorem 7.2(b), we let \( r \to \infty \):
Lemma 7.12 \textit{For } r \to \infty, \textit{ }

\[-\lim_{n \to \infty} \frac{1}{n} \log \Pr(E_n) \leq \frac{1}{2} \log 2 - \frac{1}{4} + o(1).\]

\textbf{Proof.} We show

\[-\lim_{r \to \infty} -\log \inf_{t_2 \leq 0} \mathbb{E} e^{t_2(2S^2_n - r)} = \frac{1}{2} \log 2 - \frac{1}{4}.\]

The first step is to show that whenever \( t_2 \leq -1/r, h(t_2) = \mathbb{E} e^{t_2(2S^2_n - r)} \geq 1 \). We then can conclude that the infimum is not attained for \( t_2 \leq -1/r \), since \( h(0) = 1 < h(t_2) \). Since \( h(t_2) \) is a moment-generating function, it is log-convex, so that it suffices to show that \( h(-1/r) > 1 \). Observe that

\[e^{(-1/r)(2S^2_n - r)} = e^{1-2(\frac{S^2_n}{r})} \overset{D}{\to} e^{1-2Z^2},\]

where \( Z \sim \mathcal{N}(0,1) \). Furthermore, since \( \mathbb{E} e^{(-\frac{r}{2})(2S^2_n - r)} \leq e^\alpha < \infty \) for all \( \alpha > 1 \), it then follows from \\textsc{Grimmett and Stirzaker} (1992) \textbf{Example 7.10.15}, that as \( r \to \infty \),

\[\mathbb{E} e^{1-2(\frac{S^2_n}{r})^2} \to \mathbb{E} e^{1-2Z^2} = \frac{e}{\sqrt{5}} = 1.21565 \ldots > 1.\]

This shows that for large \( r \), \( h(t) > 1 \) for \( t \leq -1/r \). Finally, using that \( t_2 = -\beta/r \) for some \( \beta \in [0,1) \), we can again use the argument above to conclude that

\[\mathbb{E} e^{-\beta Z^2} \to \mathbb{E} e^{\beta Z^2} = \frac{e^\beta}{\sqrt{1+4\beta}}.\]

Minimizing over \( \beta \) gives \( \beta^* = 1/4 \), resulting in

\[-\lim_{r \to \infty} -\log \inf_{t_2 \leq 0} \mathbb{E} e^{t_2(2S^2_n - r)} = -\log \frac{e^{1/4}}{\sqrt{2}} = \frac{1}{2} \log 2 - \frac{1}{4},\]

which is the desired result.

This completes the proof of Theorem 7.2(b) in the case of equal powers.

\[\Box\]

7.5 \textbf{The optimal rate for SD-PIC}

\textbf{Proof of Theorem 7.7(a)}: We use (7.8), so that it suffices to prove

\[\min \left\{ -\lim_{n \to \infty} \frac{1}{n} \log \Pr(\lambda_1 = 0), -\lim_{n \to \infty} \frac{1}{n} \log \Pr(\lambda_k \geq 2) \right\} \geq \frac{1}{2} - \frac{1}{2} \log 2.\]
For any vector \( v = [v_0, v_1, \ldots, v_{k-1}] \), we define
\[
\hat{S}_v = \sum_{j=0}^{k-1} v_j X_{1j}.
\] (7.30)

Moreover, we define the function \( \mathcal{I}_k(\alpha) \) by
\[
\mathcal{I}_k(\alpha) = \min_{v : \|v\|_2 = 1} \max_t \frac{t\alpha}{2} - \log \mathbb{E} (e^{t\hat{S}_v^2}).
\] (7.31)

We will prove that
(a) For all \( \alpha \geq 1 \)
\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\lambda_k \geq \alpha) \geq \mathcal{I}_k(\alpha).
\] (7.32)
(b) For all \( 0 < \alpha \leq 1 \)
\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\lambda_1 \leq \alpha) \geq \mathcal{I}_k(\alpha).
\] (7.33)

We will prove part (a) and (b) simultaneously. We write
\[
\mathbb{P}(\lambda_1 \leq \alpha) = \mathbb{P}(\exists x : \|x\|_2 = 1, \langle x, \frac{1}{n} X X^T x \rangle \leq \alpha).
\]
\[
\mathbb{P}(\lambda_k \geq \alpha) = \mathbb{P}(\exists x : \|x\|_2 = 1, \langle x, \frac{1}{n} X X^T x \rangle \geq \alpha).
\]

We make a net of vectors \( \{x_i\}_{i=1}^{n^k} \) such that \( x_i = \bar{x}_i / \|\bar{x}_i\|_2 \), and \( \bar{x}_i \in (\mathbb{N}/n)^k \). Since the trace of \( \frac{1}{n} X X^T \) equals \( k \) and \( \lambda_1 \geq 0, \frac{1}{n} \|XX^T\| \leq k \), so that we can approximate each \( \bar{x} \) with \( \|\bar{x}_i\|_2 = 1 \) by a \( \bar{x}_i \) such that \( 0 \leq \|\bar{x}_i - \bar{x}\|_2 \leq \frac{k^2}{n^2} \) and
\[
\left| \langle \bar{x}_i, \frac{1}{n} X X^T \bar{x} \rangle - \langle \bar{x}_i, \frac{1}{n} X X^T \bar{x}_i \rangle \right| \leq \frac{k^2}{n^2}.
\]

Hence, by the triangle inequality, we get that
\[
\mathbb{P}(\lambda_1 \leq \alpha) \leq \mathbb{P}\left( \exists x_i : \|x_i\|_2 = 1, \langle x_i, \frac{1}{n} X X^T x_i \rangle \leq \alpha + \frac{k^2}{n^2} \right),
\]
\[
\mathbb{P}(\lambda_k \geq \alpha) \leq \mathbb{P}\left( \exists x_i : \|x_i\|_2 = 1, \langle x_i, \frac{1}{n} X X^T x_i \rangle \geq \alpha - \frac{k^2}{n^2} \right).
\]

The proof for \( \lambda_k \) will be identical to the one for \( \lambda_1 \), so we will focus on the latter.

The rate function is continuous on \((0, 1)\), so that
\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\lambda_1 \leq \alpha) \geq - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( \exists x_i : \|x_i\|_2 = 1, \langle x_i, \frac{1}{n} X X^T x_i \rangle \leq \alpha \right).
\]

Since the number of elements in the net is \( n^k \), which is subexponential, and since
\[
\mathbb{P}\left( \exists x_i : \|x_i\|_2 = 1, \langle x_i, \frac{1}{n} X X^T x_i \rangle \leq \alpha \right) \leq \sum_i \mathbb{P}\left( \|x_i\|_2 = 1, \langle x_i, \frac{1}{n} X X^T x_i \rangle \leq \alpha \right)
\]
\[
\leq n^k \max_{x : \|x\|_2 = 1} \mathbb{P}\left( \langle x, \frac{1}{n} X X^T x \rangle \leq \alpha \right),
\]

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we see that
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\lambda_1 \leq \alpha) \geq \lim_{n \to \infty} \frac{1}{n} \log \max_{x : \|x\|_2 = 1} \mathbb{P}\left(\frac{1}{n} X X^T x \leq \alpha\right) = \max_{x : \|x\|_2 = 1} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n} X X^T x \leq \alpha\right).
\] (7.34)

We use that for \( x \) fixed, we have that
\[
\langle x, \frac{1}{n} X X^T x \rangle = \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{m=1}^{k} x_m X_{mi}\right)^2.
\]

Hence, Cramer's theorem immediately gives that for every \( x \)
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\langle x, X X^T x \rangle \leq \alpha) = \sup_{t \in \mathbb{R}} \frac{t \alpha}{2} - \log \mathbb{E}(e^{\frac{t^2}{2}}).
\] (7.35)

Note that we minimize over \( t \leq 0 \). However, this is not crucial, so that minimization over \( t \in \mathbb{R} \) gives the same result. This proves the claims in (a) and (b).

Using (a) gives
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\lambda_1 = 0) \geq \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\lambda_1 \leq 0.02) \geq \min_{v : \|v\|_2 = 1} \sup_{t \leq 0} \frac{t}{100} - \log \mathbb{E}(e^{\frac{t^2}{2}}).
\]

Substituting \( t = -1 \) gives a lower bound. We use Proposition 7.9(b) to obtain
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\lambda_1 = 0) \geq \min_{v : \|v\|_2 = 1} \frac{-1}{100} - \log \frac{1}{\sqrt{1 + 1\|v\|_2^2}} = \frac{1}{2} \log 2 - \frac{1}{100} = 0.33657 \ldots \tag{7.36}
\]

Using (b) gives
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\lambda_k \geq 2) \geq \min_{v : \|v\|_2 = 1} \max_t \frac{t \alpha}{2} - \log \mathbb{E}(e^{\frac{t^2}{2}}).
\]

Substituting \( t = 1/2 \) gives again a lower bound. Using Proposition 7.9 now yields
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\lambda_2 \geq 0) \geq \min_{v : \|v\|_2 = 1} \frac{1}{2} - \log \frac{1}{\sqrt{1 - 1/2\|v\|_2^2}} = \frac{1}{2} - \frac{1}{2} \log 2 = 0.15343 \ldots
\]

This proves the theorem.

**Proof of Theorem 7.7(b):** In Theorem 7.7(a), we have proven that
\[
J_k^{(\infty)} \geq \frac{1}{2} - \frac{1}{2} \log 2,
\]
so that it suffices to prove
\[
J_k^{(\infty)} \leq \frac{1}{2} - \frac{1}{2} \log 2,
\]
We will do this in two steps. We will first prove that for the multistage SD-PIC model, 
\( \lambda_k > 2 \) implies a bit error with a probability of at least \((2^k k!)^{-1}\) when \( s \) is sufficiently large. Then, we will show that the exponential rate of \( \mathbb{P}(\lambda_k > 2) \) tends to \( 1/2 - (\log 2)/2 \) for 
\( k \to \infty \). Let us start with the first step, which uses an eigenvalue decomposition of \( \frac{1}{n}XX^T \).

We write for \( P = [1, \ldots, 1]^T \)

\[
P^{1/2} = \sum_{j=1}^{k} \alpha_j v_j,
\]

(7.37)

where \( (v_j)_{j=1}^{k} \) form an orthogonal normalized basis of eigenvectors of \( \frac{1}{n}XX^T \) and \( \alpha_j = \langle P^{1/2}, v_j \rangle \). Then we immediately see that

\[
Z^{(s,s)} = \sum_{\sigma=0}^{s-1} \left( I - \frac{1}{n}XX^T \right)^{\sigma} \frac{1}{n}XX^T P^{1/2} = \sum_{\sigma=0}^{s-1} \sum_{j=1}^{k} (1 - \lambda_j)^{\sigma} \lambda_j \alpha_j v_j
\]

(7.38)

\[
= \sum_{j=1}^{k} (1 - (1 - \lambda_j)^s) \alpha_j v_j.
\]

Assume that \( \lambda_k > 2 \). Then we can take \( s \) large, so that the sum in (7.38) is mainly determined by all terms with eigenvalue \( \lambda_k > 2 \). We denote the multiplicity of the eigenvalue \( \lambda_k \) by \( m_k \) and its eigenvectors by \( v_{k-m_k+1}, \ldots, v_k \). It is well-known that \( v_{k-m_k+1}, \ldots, v_k \) form an orthogonal basis. When we take \( D \) subsequently all diagonal matrices with elements \( +1 \) or \(-1\), \( Dv_k \), is an eigenvector of \( \frac{1}{n}DXX^TD \) with the eigenvalue \( \lambda_k \). Since every matrix \( \in \{-1,1\}^{k\times n} \) has the same probability to be assumed, \( \lambda_k > 2 \) implies with probability \( 2^{-k} \) an eigenvector \( v_k \) with positive elements. This implies \( \alpha_k = \langle v_k, P^{1/2} \rangle > 0 \), since \( P_j > 0 \).

We know that \( \sum_{l=m_k}^{k} \alpha_l v_l \neq [0, \ldots, 0]^T \). Indeed, this would imply all \( \alpha_l = 0 \) (the \( v_l \)'s are orthogonal), which contradicts \( \alpha_k > 0 \). Using this, we can write the above sum for every \( j \) for which \( \sum_{l=m_k+1}^{k} \alpha_l v_l \neq 0 \) as

\[
Z_j^{(s,s)} = (1 - (1 - \lambda_k)^s) \sum_{l=m_k+1}^{k} \alpha_l v_l (1 + o(1)).
\]

(7.39)

Since \( \lambda_k > 2 \), we have that \( 1 - (1 - \lambda_k)^s \) alternates when \( s \) changes. Hence, the estimates of the bits for which \( \sum_{l=m_k+1}^{k} \alpha_l v_l \neq 0 \) alternate. We complete the first part of the proof by using exchangeability. This shows that

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\text{sgn}_{0}(Z_0^{(s,s)}) < 0) \leq - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\lambda_k > 2).
\]

The second part of the proof deals with the righthand side of the equation above. Since \( v^T \frac{1}{n}XX^Tv \leq \lambda_k \) for any vector \( v \) with \( \|v\|_2^2 = 1 \), we take \( v = [1, \ldots, 1]^T/\sqrt{k} \) to obtain

\[
\mathbb{P}(\lambda_k > 2) > \mathbb{P}
\left( \frac{1}{k}[1, \ldots, 1]^T \frac{1}{n}XX^T[1, \ldots, 1]^T > 2 \right) = \mathbb{P}
\left( \frac{1}{n} \sum_{i=1}^{n} \frac{1}{k} \left( \sum_{j=0}^{k-1} X_{ji} \right)^2 > 2 \right).
\]
Chapter 7. Optimal systems

Cramér's theorem gives

\[ -\lim_{n\to\infty} \frac{1}{n} \log P(\lambda_k > 2) \leq -\log \inf_{t \geq 0} h(t), \]

where, with our usual definition \( R_0 = \{0, \ldots, k - 1\} \),

\[ h(t) = E e^{-2t + t(\sum_{j=1}^k X_j)^2 / k} = \exp e^{t(S_{R_0}^2 - 2k)}. \]

The first step is to show that whenever \( t > 7/16 \), \( h(t) > 1 \). We then can conclude that the infimum is not attained, since \( h(0) = 1 < h(t) \) for \( t > 7/16 \). Since \( h(t) \) is a moment-generating function, it is log-convex, so that it suffices to show that \( h(7/16) > 1 \). Indeed, for all \( t \geq 7/16 \), there exists an \( \alpha \in (0, 1) \), such that \( at = 7/16 \). It now follows that

\[ 1 < h(at) = h(at + (1 - \alpha) \cdot 0) \leq \alpha h(t) + (1 - \alpha) h(0) = \alpha h(t) + (1 - \alpha), \]

so that \( h(t) > 1 \). In order to prove that \( h(7/16) > 1 \), we observe that

\[ e^{\frac{7}{16k}(S_{R_0}^2 - 2k)} = e^{\frac{S_{R_0}^2}{k^2} - \frac{7}{8}} \xrightarrow{D} e^{\frac{Z^2}{16} - \frac{7}{8}}, \]

where \( Z \sim N(0, 1) \). Furthermore, \( E e^{\frac{S_{R_0}^2}{k^2} - \frac{7}{8}} \leq e^{-7/8}/\sqrt{1 - \frac{7}{8} \alpha} \), according to Proposition 7.9, we have for all \( 1 < \alpha < 8/7 \),

\[ E e^{\frac{S_{R_0}^2}{k^2} - \frac{7}{8}} < \infty. \]

It follows from Grimmett and Stirzaker (1992) Example 7.10.15, that as \( k \to \infty \),

\[ E e^{\frac{Z^2}{16} - \frac{7}{8}} \to E e^{\frac{Z^2}{16} - \frac{7}{8}} = \frac{e^{-7/8}}{\sqrt{1/8}} = 1.17906 \ldots > 1. \]

Indeed, the infimum is not attained. Finally, using that \( t = \beta \) for some \( \beta \in [0, 1) \), we can again use the argument above to conclude that

\[ E e^{\beta(S_{R_0}^2 - 2\beta)} \to E e^{\beta Z^2 - 2\beta} = \frac{e^{-2\beta}}{\sqrt{1 - 2\beta}}. \]

Minimizing over \( \beta \) gives \( \beta^* = 1/4 \), resulting in

\[ \lim_{k \to \infty} -\log \inf_{t \geq 0} E e^{t(S_{R_0}^2 / k - 2)} = -\log \frac{e^{-1/2}}{\sqrt{1/2}} = \frac{1}{2} - \frac{1}{2} \log 2, \]

which is the desired result. \( \blacksquare \)
Second order asymptotics

The main part of the thesis focuses on the exponential rate. The rate enabled us to draw qualitative conclusions concerning the system behaviour. However, the exponential rate only partly determines the BEP. In this chapter and Chapter 9, we will deal with a more detailed analysis of the BEP. We will assume \( P_0 = P_1 = \ldots = P_{k-1} \) and \( \sigma^2 = 0 \), i.e., we will investigate the simple system. In this chapter, we focus on the so-called second order asymptotics. Whereas the analysis so far (the first order asymptotics) gives (for example for MF) the asymptotic result \( \text{BEP} = e^{-n_{\xi_k} + o(n)} \), inclusion of the second order asymptotics leads to \( \text{BEP} = \alpha_{k,n} n^{-1/2} e^{-n_{\xi_k} (1 + o(1))} \). For the SD-PIC model, we derive similar results for \( k = 3 \). The results for MF and SD-PIC have appeared in van der Hofstad, Hooghiemstra and Klok (2002), Klok, Hooghiemstra, van der Hofstad, Ojanperä and Prasad (1999) and Klok, Hooghiemstra and van der Hofstad (2002).

For higher \( k \), the analysis becomes too involved. Instead, we give conjectures. For the HD-PIC model, we only have conjectures.

### 8.1 Second order asymptotics for MF

It follows from Theorem 1 of Bahadur and Rao (1960) that

**Proposition 8.1** For \( k \geq 3 \),

\[
\mathbb{P}(\text{sgn}r_0(Z^{(1)}_0) < 0) = \frac{\alpha_{k,n}}{\sqrt{n}} e^{-n_{\xi_k} (1 + o(1))},
\]

where

\[
\alpha_{k,n} = \sqrt{\frac{k-1}{8\pi}} \left[ \left( \frac{k-2}{k} \right)^{\left( \frac{n_{\xi_k}}{2} \right) - \left( \frac{\sigma^2}{2} \right) - \frac{1}{2}} + \left( \frac{k-2}{k} \right)^{\left( \frac{n_{\xi_k}}{2} \right) - \left( \frac{\sigma^2}{2} \right) + \frac{1}{2}} \right].
\]
We note that the behaviour for \( n \) odd is slightly different than the behaviour for \( n \) even, whenever \( k \) is odd. The coefficient \( \alpha_{k,n} \) only depends on the parity of \( nk \). For \( k \) even, \( \alpha_{k,n} \) therefore does not depend on \( n \), while for \( k \) odd, \( \alpha_{k,n} \) depends only on the parity of \( n \). The explanation is the lattice structure of the random variable \( Z_0^{(1)} \).

For \( k \) large, we already know that \( I_k = 1/(2k)(1+\mathcal{O}(1/k)) \). Furthermore, \( (k-2)/k \approx 1 \), so that for \( k \) large,

\[
\alpha_{k,n} \approx 2 \sqrt{\frac{k-1}{8\pi}} \approx \sqrt{\frac{k}{2\pi}}.
\]

This results in

\[
\mathbb{P}(\text{sgnr}_0(Z_0^{(1)}) < 0) \approx \sqrt{\frac{k}{2\pi n}} e^{-\frac{n}{2k}}, \quad k \to \infty. \tag{8.1}
\]

We note that this expansion is obtained by taking \( n \) large and subsequently \( k \) large. For the case that \( k = k_n \to \infty \) as \( n \to \infty \), we can use the CLT. For example, take \( k = \varepsilon n \), where \( \varepsilon \in (0,1) \). Then

\[
Z_0^{(1)} \overset{d}{=} 1 + \varepsilon \frac{1}{n\sqrt{\varepsilon}} \sum_{i=1}^{\varepsilon n^2} X_{1i},
\]

so that the CLT states that (recall the definition of \( Q \) in (5.3) and the approximation (5.4))

\[
\mathbb{P}(\text{sgnr}_0(Z_0^{(1)}) < 0) \approx Q\left(\frac{1}{\sqrt{\varepsilon}}\right) = Q\left(\sqrt{\frac{n}{k}}\right) \approx \sqrt{\frac{k}{2\pi n}} e^{-n/(2k)}.
\]

We conclude that the asymptotic analysis for \( n \) large and subsequently \( k \) large (see (8.1)), gives the same result as the asymptotic analysis for \( n \) and \( k \) coupled. This indicates that the large deviations analysis is quite flexible.

### 8.2 Second order asymptotics for HD-PIC

The second order asymptotics for the HD-PIC model are essentially more difficult to derive than for the MF model. Unlike the exponential rate, the second order asymptotics are sensitive to small perturbations of the model; hence, for every model a specific derivation is required.

We have two conjectures, dealing with \( k = 3 \) and \( k \geq 4 \). We expect that the conjecture for \( k = 3 \) can be proven with similar techniques as used in the proof of Theorem 8.4, given in the next section. We have chosen not to prove it, because the case \( k = 3 \) turns out to be special and therefore does not lead to more understanding of the general problem \( k \geq 4 \).

The conjecture, concerning \( k \geq 4 \) seems hard to prove. We only have intuition on the exponent of \( n \).
Conjecture 8.2 For $k = 3$,

$$
P(\text{sgnr}_0(Z_0^{(2,m)}) < 0) = \frac{\beta_{3,n}}{\sqrt{n}} e^{-nH_3^{(2)}} (1 + o(1)),
$$

where

$$\beta_{3,n} = \frac{1}{\sqrt{4\pi}} \left[ \left( \frac{1}{3} \right)^{\frac{1}{2} - \frac{1}{2} - \frac{1}{2}} + \left( \frac{1}{3} \right)^{\frac{1}{2} - \frac{3}{2} + \frac{1}{2}} \right].$$

According to (2.9), $H_3^{(2)} = I_3$. Furthermore, note that $\beta_{3,n} = \alpha_{3,n}$, according to Proposition 8.1. Hence, we conclude that for a system with 3 users, one stage of HD-PIC does not result in increase in performance, neither in first nor second order. The reason is that the atypical event $\{\text{sgnr}_1(Z_1^{(1)}) < 0\}$ makes the event $\{\text{sgnr}_0(Z_0^{(2,m)}) < 0\}$ typical. The first event leads to the same exponential rate and second order asymptotics as for MF. The second event now only gives a factor $1/2$. Together with the counting factor $(k-1)^2 = 2$ (user 1 or user 2 can have a bit error at stage 1), this leads to the desired result.

Conjecture 8.3 For all $k \geq 4$,

\[
\liminf_{n \to \infty} n^{(r_k+1)/2} e^{-nH_k^{(2)}} P(\text{sgnr}_0(Z_0^{(2,m)}) < 0) > 0 \quad \text{and} \quad \limsup_{n \to \infty} n^{(r_k+1)/2} e^{-nH_k^{(2)}} P(\text{sgnr}_0(Z_0^{(2,m)}) < 0) < \infty.
\]

We do not expect that the lim inf equals the lim sup. Instead, we expect a wiggling behaviour, similar to $\alpha_{k,n}$. The reason is that the HD-PIC model involves, similarly to the MF model, linear combinations of a lattice structure. Furthermore, the linear combinations only involve multiplication by integers. We will denote the wiggling function by $\beta_{k,n}$. The factor $n^{(r_k+1)/2}$ can be explained as follows. Only the bit error of user 0 at stage 2, caused by $r_k$ errors in the first stage contributes to the probability of interest. The events $\{\text{sgnr}_m(Z_m^{(1)}) \geq 0\}$ for $m = r_k \ldots, k - 1$ do not contribute to the probability, neither in first order, nor in second order. This leaves us with $r_k + 1$ events. Every event has contribution of order $n^{-1/2}$ to the second order asymptotics, just like one event resulted in $n^{-1/2}$ in the case of the MF model. Note that this is not true for $k = 3$.

### 8.3 Second order asymptotics for SD-PIC

To obtain the exponential rate for the SD-PIC model, we had to use the empirical measures. In Theorem 4.5, we obtain the rate as the infimum of the rate function $I_k$ over some area. To obtain the second order asymptotics of $P(\text{sgnr}_0(Z_0^{(2,m)}) < 0)$, we need properties of the minimizer, in particular, the uniqueness of the minimizer. This makes sense, because every minimizer will contribute to the BEP. However, we also need irrationality of two partial
derivatives. Whenever those partial derivatives are rational, the BEP will have a wiggling behaviour, similar to the MF model.

Our results are only complete for \( k = 3 \). These results are formulated in Theorem 8.4.

**Theorem 8.4** For \( k = 3 \),

\[
\lim_{n \to \infty} n^{1/2} e^{n J_3^{(2)}} \mathbb{P}(\text{sgn } r_0(Z_0^{(2, 3)} < 0) = \gamma_3,
\]

where \( \gamma_3 = 0.5946 \ldots \), and where \( J_3^{(2)} = 0.3094 \ldots \) is defined in Theorem 4.5. We have an analytical expression for \( \gamma_3 \) in terms of the minimizer \( \rho^* \) of the optimization problem in Theorem 4.5 (see (8.19)).

The proof is given in Section 8.3.2. To smoothen this proof, we will first prove in Section 8.3.1 the uniqueness of the minimizer and the irrationality of two partial derivatives.

For \( k \geq 4 \), we do not have any results. For \( k \geq 4 \), the number of variables \((2^{k-1} - 2)\) makes the techniques used for \( k = 3 \) unappropriate. Instead, we confine with a conjecture.

**Conjecture 8.5** For \( k \geq 4 \), the minimizer of the optimization problem (4.5) is unique. Furthermore,

\[
\lim_{n \to \infty} n^{1/2} e^{n J_k^{(2)}} \mathbb{P}(\text{sgn } r_0(Z_0^{(2, k)} < 0) \quad \text{exists}.
\]

We will denote this limit, under the assumption that it exists, by \( \gamma_k \). We will confirm this conjecture with simulations in Section 9.4.

### 8.3.1 Preparations

For \( k = 3 \), we can prove that the minimizer in Theorem 4.5 is unique and symmetric in the second and third coordinate. Recall our notation \( \rho_+ = \rho(+1, +1) \), \( \rho_\pm = \rho(+1, -1) \), \( \rho_\mp = \rho(-1, +1) \) and \( \rho_- = \rho(-1, -1) \) (see the remark above Theorem 4.5).

**Proposition 8.6** For \( k = 3 \), the minimizer \( \nu \) of the variational problem of Theorem 4.5 is unique. Furthermore \( \nu_\pm = \nu_\mp \).

**Proof.** We start with a lemma concerning minima of \( I_3(\rho) \), for \( \rho = (\rho_+, \rho_\pm, \rho_\mp, \rho_-) \in M(X_3) \) under linear conditions. Denote by

\[
e = \rho_\pm + \rho_\mp, \quad e \in [0, 1],
\]
\[
d = \rho_\pm - \rho_\mp, \quad d \in [-1, 1].
\]
Lemma 8.7  (a) For $|d| \geq d_0 \in (0, 1)$,

$$\mathcal{I}_3(\rho) \geq I_3 \left( \frac{1 - d_0^2}{4}, \frac{(1 - d_0)^2}{4}, \frac{(1 + d_0)^2}{4}, \frac{1 - d_0^2}{4} \right).$$

(b) For $|d| \geq d_0 \in (0, 1)$ and $\rho_+ \geq m_1 \in (\frac{(1 - d_0^2)}{4}, 1 - d_0)$,

$$\mathcal{I}_3(\rho) \geq I_3 \left( m_1, D - d_0/2, D + d_0/2, 1 - m_1 - 2D \right),$$

where

$$D = \frac{2(1 - m_1)}{3} - \frac{1}{6} \sqrt{4(1 - m_1)^2 - 3d_0^2}.$$ 

(c) For $|d| \geq d_0 \in (0, 1)$ and $\rho_+ \leq m_1 \leq (\frac{1 - d_0^2}{4})$,

$$\mathcal{I}_3(\rho) \geq I_3 \left( m_1, D - d_0/2, D + d_0/2, 1 - m_1 - 2D \right).$$

(d) For $\rho_+ \geq m_1 \in (1/4, 1)$ and $\rho_- \leq m_2 \leq (1 - m_1)/3$,

$$\mathcal{I}_3(\rho) \geq I_3 \left( m_1, (1 - m_1 - m_2)/2, (1 - m_1 - m_2)/2, m_2 \right).$$

**Proof.** (a) Minimize $\mathcal{I}_3(\rho)$ over all $\rho$ with $d \geq d_0$ (which suffices by symmetry). The infimum is attained at a $\rho$ for which $d = d_0$. Hence, we have to compute

$$\inf_{\rho+, \rho_=} \mathcal{I}_3(\rho_+, \rho_+ - d_0, \rho_-, 1 - \rho_+ - 2\rho_+ + d_0).$$

Setting the partial derivatives with respect to $\rho_+$ and $\rho_-$ equal to zero, we obtain

$$\log \rho_+ - \log (1 - \rho_+ - 2\rho_+ + d_0) = 0,$$

$$\log \rho_- + \log (\rho_- - d_0) - 2\log (1 - \rho_+ - 2\rho_+ + d_0) = 0.$$ 

Solving for $\rho_+$ and $\rho_-$ gives $\rho_+ = \frac{1 - d_0^2}{4}$ and $\rho_- = \frac{(1 + d_0)^2}{4}$.

(b) Similar as the above proof. Minimize $\mathcal{I}_3(\rho)$ over all $\rho$ with $d \geq d_0$ and $\rho_+ \geq m_1$. Since $m_1 \in (\frac{(1 - d_0^2)}{4}, 1 - d_0)$, the infimum is attained at a $\rho$ for which $d = d_0$ and $\rho_+ = m_1$. Hence, we have to compute

$$\inf_{\rho_=} \mathcal{I}_3(m_1, \rho_+ - d_0, \rho_+, 1 - m_1 - 2\rho_+ + d_0).$$

Setting the derivative with respect to $\rho_+$ equal to zero, we obtain

$$\log \rho_+ + \log (\rho_+ - d_0) - 2\log (1 - m_1 - 2\rho_+ + d_0) = 0.$$ 

Solving for $\rho_+$ gives $\rho_+ = \frac{2(1 - m_1)}{3} \pm \frac{1}{3} \sqrt{4(1 - m_1)^2 - 3d_0^2 + d_0/2}$. For the minus root $\rho_+ - d_0$ is negative, and hence only the plus root remains.

(c) Minimize $\mathcal{I}_3(\rho)$ over all $\rho$ with $d \geq d_0$ and $\rho_+ \leq m_1$. Since $m_1 \leq \min((1 - d_0^2)/4, 1 - d_0)$, the infimum is attained at a $\rho$ for which $d = d_0$ and $\rho_+ = m_1$. This gives precisely the same minimization problem as in the proof of Lemma 8.7(b).
The infimum over $I_3(\rho)$ under the boundary constraints $\rho_+ \geq m_1$ and $\rho_- \leq m_2$ is attained at a $\rho$ for which $\rho_+ = m_1$ and $\rho_- = m_2$. Minimizing over the remaining components in $\rho$ gives the desired expression.

Besides this lemma, we use the following (trivial) inequality for $x, y \in \mathbb{R}$,

$$-4d^2 \leq (4\rho_+ - 1)(4\rho_+ - 1) \leq (2e - 1)^2. \tag{8.2}$$

Also note that the solution of the minimization problem is attained at the boundary $F_3(\rho) = 0$ (cf. Theorem 4.5). Since $\rho = [0.6213, 0.137, 0.137, 0.1047]$ satisfies $F_3(\rho) < 0$ and $I_3(\rho) = 0.30967 < 0.31$, we can exclude areas where the minimal value of the rate function satisfies $I_3(\rho) \geq 0.31$.

Now assume that $\nu \in M(\mathcal{X}_3)$ minimizes $\rho \mapsto I_3(\rho)$ under the constraint $F_3(\rho) = 0$. We will show that $\nu$ is unique by proving that $\nu$ lies in a set that makes the rate function constrained to $F_3(\rho) = 0$ convex. This will be done in the following 13 steps:

1. Observe from Lemma 8.7(a) that for $|d| \geq 0.55$, the minimum of $I_3(\rho)$ exceeds 0.31. Hence $|\nu_+ - \nu_-| < 0.55$.

2. The condition $F_3(\nu) = 0$ can be written as

$$\nu_+ = \frac{1}{4} \pm \frac{1}{4} \sqrt{(4\nu_+ - 1)(4\nu_+ - 1) + 2}. \tag{8.3}$$

It follows from (8.2) that $|d| < 0.55$ implies that either $\nu_+ \leq 0.03 < 0.05$ or $\nu_+ \geq 0.47 > 0.45$.

3. For $\nu_+ > 0.45$, Lemma 8.7(b) applied with $d_0 = 0.5$ and $m_1 = 0.45$ implies that $|\nu_+ - \nu_-| < 0.5$. Indeed, assuming that $\nu_+ > 0.45$ together with $|\nu_+ - \nu_-| \geq 0.5$ gives $I_3(\rho) > 0.31$.

4. Similarly, for $\nu_+ < 0.05$, Lemma 8.7(c) implies that $|\nu_+ - \nu_-| < 0.5$.

5. $|d| < 0.5$ implies that either $\nu_+ < 0$ or $\nu_+ > 0.5$. Therefore the minus root can be excluded and $\nu_+ = r(\nu_+, \nu_-)$, where

$$r(\rho_+, \rho_+) = \frac{1}{4} + \frac{1}{4} \sqrt{(4\rho_+ - 1)(4\rho_+ - 1) + 2}. \tag{8.4}$$

6. For $\nu_+ \geq 0.5$, Lemma 8.7(b) implies that $|\nu_+ - \nu_-| < 0.32$. Otherwise the rate exceeds 0.31.

7. From (8.2), $|\nu_+ - \nu_-| \leq 0.32$ implies that $\nu_+ > 0.56$.

8. For $\nu_+ \geq 0.56$, Lemma 8.7(b) implies that $|\nu_+ - \nu_-| < 0.23$.

9. From (8.2), $|\nu_+ - \nu_-| \leq 0.23$ implies that $\nu_+ > 0.58$.

10. For $\nu_+ \geq 0.58$, Lemma 8.7(b) implies that $|\nu_+ - \nu_-| < 0.19$. 

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11. From (8.2), $|\nu_+ - \nu_\pm| \leq 0.19$ implies that $\nu_+ > 0.59$.

12. For $\nu_+ \geq 0.58$, Lemma 8.7(d) implies that $\nu_- > 0.04$.

13. The two latter statements imply that $\nu_+ + \nu_\pm \leq 1 - 0.59 - 0.04 = 0.37$.

The following two lemmas now show that $\nu$ is unique. Let

$$J(\rho_\pm, \rho_\pm) = I_3(r(\rho_\pm, \rho_\pm), \rho_\pm, \rho_\pm, 1 - r(\rho_\pm, \rho_\pm) - \rho_\pm - \rho_\pm),$$

where $r(\rho_\pm, \rho_\pm)$, defined in (8.4), represents $\rho_+$ at the boundary $F_3(\rho) = 0$ in a sufficiently small neighborhood of the minimizer $\nu$.

**Lemma 8.8** For $|d| \leq 0.19$ and $0 \leq e \leq 0.37$,

$$\rho_\pm \mapsto J(\rho_\pm, e - \rho_\pm), \quad \rho_\pm \in [\max(0, (e - 0.19)/2), (e + 0.19)/2],$$

is strictly convex and attains its minimal value at $\rho_\pm = e/2$.

**Proof.** Observe that

$$\max_{0 \leq e \leq 0.37} \frac{1}{4} + \frac{1}{4} \sqrt{(2e - 1)^2 + 2} = \frac{1}{4} + \frac{1}{4} \sqrt{3} \leq 0.69,$$

$$\min_{|d| \leq 0.19} \frac{3}{4} - e - \frac{1}{4} \sqrt{(2e - 1)^2 + 2} \geq 0.02.$$

From (8.2) and the first two inequalities we obtain that for all $\rho_+$, with $|d| \leq 0.19$ and $0 \leq e \leq 0.37$,

$$0.59 \leq r(\rho_\pm, s - \rho_\pm) \leq 0.69.$$

Using the above inequalities we will now show that

$$\frac{\partial^2 J(\rho_\pm, e - \rho_\pm)}{\partial \rho_\pm^2} > 0,$$

so that the function is strictly convex.

Observe that

$$\frac{\partial r}{\partial \rho_\pm} = \frac{2(\rho_\pm - \rho_+)}{4r - 1},$$

$$\frac{\partial^2 r}{\partial \rho_\pm^2} = -\frac{4}{4r - 1} - \frac{16(\rho_\pm - \rho_+)^2}{(4r - 1)^3},$$

where we abbreviate $r = r(\rho_\pm, e - \rho_\pm)$.
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Hence,

\[
\frac{\partial^2 J(\rho_+, \rho_-)}{\partial \rho_+^2} = -\frac{4}{4r - 1} \log \left(\frac{\rho_+}{\rho_-}\right)
+ \frac{4(e - 2\rho_+)^2}{(4r - 1)^2} \left[\left(\frac{1}{r} + \frac{1}{\rho_-}\right) - \frac{4}{4r - 1} \log \left(\frac{\rho_+}{\rho_-}\right)\right] + \frac{1}{\rho_+} + \frac{1}{e - \rho_-},
\]

where \(\rho_- = 1 - e - r\). The inequalities \(3 \log x \leq 1 + x\), for \(x > 0\), and \(\frac{4}{4r - 1} \leq 3\) (which follows from \(r \geq 0.59 \geq 7/12\)) together imply

\[
\left(\frac{1}{r} + \frac{1}{\rho_-}\right) - \frac{4}{4r - 1} \log \left(\frac{\rho_+}{\rho_-}\right) \geq \left(\frac{1}{r} + \frac{1}{\rho_-}\right) \left(1 - \frac{4r}{3(4r - 1)}\right) \geq 0.
\]

Furthermore, we use that \(1/\rho_+ + 1/(\rho_-) = e/\{\rho_+ (e - \rho_+)\} \geq 4/e\), and the obtained bounds for \(r\) and \(\rho_-\) to arrive at

\[
\frac{\partial^2 J}{\partial \rho_+^2} \geq \frac{4}{e} - 3 \log \left(\frac{0.69}{0.02}\right) > 0,
\]

for \(0 \leq e \leq 0.37\).

Since

\[
\frac{\partial J}{\partial \rho_+}(e/2, e/2) = 0,
\]

the minimum of \(J\) over \(\rho_+\) for fixed \(e \in [0, 0.37]\) is attained in \(e/2\).

**Lemma 8.9** The function

\(e \mapsto J(e/2, e/2),\)

is strictly convex and therefore has a unique minimum.

**Proof.** We have

\[
J(e/2, e/2) = I_3(\rho_+, e/2, e/2, \rho_-),
\]

where

\[
\rho_+ = \frac{1}{4} + \frac{1}{4} \sqrt{(2e - 1)^2 + 2},
\rho_- = 1 - e - \rho_+.
\]

Then

\[
\frac{d^2 J(e/2, e/2)}{de^2} = \frac{2}{4\rho_+ - 1} \log \left(\frac{\rho_+}{\rho_-}\right)
+ \frac{(2e - 1)^2}{(4\rho_+ - 1)^2} \left[\left(\frac{1}{\rho_+} + \frac{1}{\rho_-}\right) - \frac{2}{4\rho_+ - 1} \log \left(\frac{\rho_+}{\rho_-}\right)\right] + \frac{1}{e}.
\]

From \(\rho_+ > 1/2\) and \(3 \log x < 1 + x\) for \(x > 0\), we obtain that \(\frac{d^2 J}{de^2} > 0\).

Besides the uniqueness of the minimizer, we will need a convexity property in the next section. This is stated in the next lemma.
Lemma 8.10 For $0 \leq \epsilon \leq 0.37$, the $2 \times 2$-matrix
\[ Q = \nabla^2 J \bigg|_{(\rho_+, \rho_\mp) = (\epsilon/2, \epsilon/2)} \]
is strictly positive definite.

Proof. We have
\[
\frac{\partial^2 J(\epsilon/2, \epsilon/2)}{\partial \epsilon^2} = Q_{11} + 2Q_{12} + Q_{22},
\]
\[
\frac{\partial^2 J(\rho_+, \epsilon - \rho_\mp)}{\partial \rho_\mp^2} \bigg|_{\rho_\mp = \epsilon/2} = Q_{11} - 2Q_{12} + Q_{22},
\]
and by Lemmas 8.8–8.9 both quantities are positive. By symmetry, we have that $Q_{11} = Q_{22}$, so that $Q_{11} > |Q_{12}|$. This proves that $Q$ is strictly positive definite.

This proves Proposition 8.6.

Furthermore, we need an algebraic property of the minimizer.

Lemma 8.11 For $k = 3$ and for $a = \pm$ or $0$, $\frac{\partial}{\partial \rho_a}(\bar{v})$ is irrational.

Proof. Verify that $\frac{\partial}{\partial \rho_a}(\bar{v}) = \frac{4\rho_a - 1}{2(4\rho_a - 1)}$, for $a = \pm$ and $a = 0$. We will prove that for $a = \pm$ or $0$, $\frac{4\rho_a - 1}{2(4\rho_a - 1)}$ is irrational.

For $\rho \in [0, 1]$, let
\[ v_\rho = \frac{1}{4} + \frac{1}{4} \sqrt{(4\rho - 1)^2 + 2} \]
be the symmetric version of (8.13). We want to minimize with respect to $\rho$
\[ v_\rho \log v_\rho + 2\rho \log \rho + (1 - 2\rho - v_\rho) \log(1 - 2\rho - v_\rho), \tag{8.5} \]
and show that for the minimizer $\rho^*$, we have that $\frac{4\rho^* - 1}{2(4\rho^* - 1)} \notin \mathbb{Q}$. Since the minimizer $\rho^*$ is equal to $v_\pm = v_0$, this proves the claim. The minimizer $\rho^*$ has to satisfy
\[ v'_{\rho^*}(\log v_{\rho^*} - \log(1 - 2\rho^* - v_{\rho^*})) + 2(\log \rho^* - \log(1 - 2\rho^* - v_{\rho^*})) = 0, \tag{8.6} \]
where
\[ v'_{\rho^*} = \frac{4\rho^* - 1}{4v_{\rho^*} - 1}. \]
Suppose that $v'_{\rho^*} = -p/q$, with $p, q \in \mathbb{N}$, such that $p < q$. Then
\[
\left( \frac{p}{q} \right)^2 = \frac{(4\rho^* - 1)^2}{(4v_{\rho^*} - 1)^2} = \frac{(4\rho^* - 1)^2}{(4\rho^* - 1)^2 + 2},
\]
so that
\[ \rho^* = \frac{1}{4} - \frac{1}{4} \sqrt{\frac{2p^2}{q^2 - p^2}}. \]
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Here we used that $2\rho^* = \nu_\pm + \nu_\mp \leq (1 - \nu_\pm) < 1/2$. We can rewrite (8.6) as

$$\left( \frac{v_{\rho^*}}{1 - 2\rho^* - v_{\rho^*}} \right)^{-p/q} \left( \frac{\rho^*}{1 - 2\rho^* - v_{\rho^*}} \right)^2 = 1 \tag{8.7}$$

From $-p/q = (4\rho^* - 1)/(4v_{\rho^*} - 1)$, it follows that

$$v_{\rho^*} = \frac{1}{4} + \frac{q}{4} \sqrt{\frac{2}{q^2 - p^2}}.$$

Hence, (8.7) is equivalent to

$$(\sqrt{2(q^2 - p^2)} + 2q)^p(\sqrt{2(q^2 - p^2)} + (4p - 2q))^{2q-p} = (\sqrt{2(q^2 - p^2)} - 2p)^{2q}. \tag{8.8}$$

In the remainder of the proof we will show that (8.8) has no integer solutions $p < q$.

Choose $p$ and $q$ having no common factor. There are two cases, depending on whether $2(q^2 - p^2)$ is a square or not.

**Case 1:** $2(q^2 - p^2)$ is not a square. From (8.8) and Newton’s binomial we also obtain that

$$(-\sqrt{2(q^2 - p^2)} + 2q)^p(-\sqrt{2(q^2 - p^2)} + (4p - 2q))^{2q-p} = (-\sqrt{2(q^2 - p^2)} - 2p)^{2q}. \tag{8.9}$$

Combining (8.8) and (8.9), we get

$$(p^2 + q^2)^p(q^2 - 8pq + 9p^2)^{2q-p} = (3p^2 - q^2)^{2q}. \tag{8.10}$$

Now, $3p^2 - q^2$ must contain a prime factor, because if we suppose that $3p^2 - q^2 = 1$ or $-1$, then it follows from (8.10) that $p^2 + q^2 = 1$. This gives $p = 0, q = 1$, which is not a solution of (8.8).

Let $j$ be a prime factor of $3p^2 - q^2$. Then, from (8.10), $j$ must be a prime factor of $(p^2 + q^2)^p$, $(q^2 - 8pq + 9p^2)^{2q-p}$ or of both. We will first show that $j$ cannot be a prime factor of both $(p^2 + q^2)^p$ and $(q^2 - 8pq + 9p^2)^{2q-p}$. Indeed, if $j$ is a prime factor of both terms, then it is also a prime factor of $p^2 + q^2$, and of $q^2 - 8pq + 9p^2$. Now, if $j$ is odd then $j$ cannot be a prime factor of $p^2 + q^2$, since then it would also be a prime factor of $3p^2 - q^2 + p^2 + q^2 = 4p^2$ and of $3(p^2 + q^2) - (3p^2 - q^2) = 4q^2$, which would imply that $p$ and $q$ have a common (odd) factor. However, if $3p^2 - q^2$ is even, then $p$ and $q$ are both odd, since they cannot both be even. Furthermore, $p^2 + q^2$ is 2 modulo 4. Hence, by (8.10) and the fact that $3p^2 - q^2$ is a power of 2, we have that $p^2 + q^2 = 2$, which contradicts $p < q$.

At this stage, we know that $j^{2q}$ is a prime factor of $(p^2 + q^2)^p$, $(q^2 - 8pq + 9p^2)^{2q-p}$, but not of both. From $gcd(p, q) = 1$ it follows that $gcd(2q, p) = 1$ or 2. Hence, $j^{2q}$ is a prime factor of $p^2 + q^2$ or of $q^2 - 8pq + 9p^2$. In the first case this implies that

$$2^q \leq j^q \leq p^2 + q^2 \leq 2q^2,$$
so that \( q \leq 6 \). Similarly, in the second case

\[
2^q \leq j^q \leq |q^2 - 8pq + 9p^2| = |(q - 3p)^2 - 2pq| \leq 2q^2,
\]

so that again \( q \leq 6 \). We see that there are no solutions of (8.8) with \( p < q \).

**Case 2:** \( 2(q^2 - p^2) \) is a square. Let \( 2(q^2 - p^2) = t^2 \). First of all, \( t \) is even and \( p, q \) are odd. But then \( t^2 = 2(q + p)(q - p) \) is an 8-fold, and therefore \( t \) is a 4-fold. We arrive at \( t = 4 \prod p_i^{2a_i} \), where \( p_i \) are prime. \( 2(q^2 - p^2) = t^2 \) now goes over in

\[
\frac{(q + p)(q - p)}{2} = 2 \prod p_i^{2a_i}.
\]

Since \( (q + p)/2 \) and \( (q - p)/2 \) are relative prime, it follows that both terms are quadratic and one of them is multiplied by 2, i.e., we have that either \( (q - p)/2 = 2a^2 \) and \( (q + p)/2 = (b - a)^2 \), or that \( (q - p)/2 = a^2 \) and \( (q + p)/2 = 2(b - a)^2 \). This gives that either

\[
q = 2a^2 + (b - a)^2, \quad p = (b - a)^2 - 2a^2, \quad \text{or} \quad q = a^2 + 2(b - a)^2, \quad p = 2(b - a)^2 - a^2,
\]

where \( a \) and \( b \) are integers having no common prime factors. Suppose that we are in the first case. Then we have that \( 2(q^2 - p^2) = 16a^2(b - a)^2 \), so that (8.8) becomes

\[
(a^2 + b^2)^p(b^2 - 7a^2)^{2a - p} = (a^2 - 4ab + b^2)^{2q}.
\]

Let \( j \) be a prime factor of \( a^2 - 4ab + b^2 \). Then, similarly as above, if \( j \) is odd, \( j \) cannot be a prime factor of both \((a^2 + b^2)^p\) and \((b^2 - 7a^2)^{2a - p}\), since then it would also be a prime factor of \((a^2 + b^2) - (b^2 - 7a^2) = 8a^2\) and \(7(a^2 + b^2) + (b^2 - 7a^2) = 8b^2\), which implies that \( a \) and \( b \) have common factors. The latter is not possible by construction. Hence, we are left with the case where \( a^2 - 4ab + b^2 \) is a power of 2. This again implies that \( a^2 + b^2 \) is a power of 2, so that \( a^2 + b^2 \) must be equal to 2. This leads to \( a = b = 1 \), which is not a solution to (8.11). Hence, \( j^2q \) is a prime factor of \((a^2 + b^2)^p\) or of \((b^2 - 7a^2)^{2a - p}\), but not of both. Again, since \( \gcd(2q, p) = 1 \) or 2, we have that \( j^q \) is a prime factor of \( a^2 + b^2 \) or of \( b^2 - 7a^2 \). In the first case, we estimate

\[
2^q \leq j^q \leq a^2 + b^2 = (b - a)^2 + 2a(b - a) + 2a^2 \leq 2(b - a)^2 + 3a^2 \leq 2q.
\]

Hence, \( q \leq 2 \).

Similarly, in the second case

\[
2^q \leq j^q \leq |b^2 - 7a^2| = |(b - a)^2 + 2a(b - a) - 5a^2| \leq (b - a)^2 + a^2 + (b - a)^2 + 5a^2 \leq 4q,
\]

so that \( q \leq 4 \). Again, there are no solutions with \( p < q \).

The case where \( q = a^2 + 2(b - a)^2, p = 2(b - a)^2 - a^2 \) is proved the same way, using that in this case (8.8) turns into

\[
((b - a)^2 + b^2)^p((b - a)^2 + b^2 - 4a^2)^{2a - p} = (3(b - a)^2 - b^2)^{2q}.
\]

We again see that there are no solutions of (8.8) with \( p < q \).
8.3.2 Proof of Theorem 8.4

We will prove the statement for $\mathbb{P}(Z_0^{(2,3)} \leq 0)$ rather than $\mathbb{P}({\text{sgnr}}_0(Z_0^{(2,3)}) < 0)$. However, a close examination of the proof reveals that the same result holds for $\mathbb{P}(Z_0^{(2,3)} < 0)$, so that, because

$$\mathbb{P}(\text{sgnr}_0(Z_0^{(2,3)}) < 0) = \frac{1}{2}(\mathbb{P}(Z_0^{(2,3)} < 0) + \mathbb{P}(Z_0^{(2,3)} \leq 0)).$$

the desired statement directly follows.

The proof will be divided into 5 steps.

**Step 1: Multinomial probabilities.** Use that the empirical measure has a multinomial distribution to get $\mathbb{P}(L_n = \rho) = \left(\frac{1}{2}\right)^{2n} n! \prod_{a \in {\mathcal{X}_3}} \frac{1}{(n a)}!$, $\rho \in {\mathcal{L}} \subset M({\mathcal{X}_3})$. By Stirling’s formula

$$\sqrt{2\pi m} e^{-m} m^m \leq m! \leq \sqrt{2\pi m} e^{-m} m^m (1 + \frac{1}{12m}),$$

we arrive at

$$\mathbb{P}(L_n = \rho) = \frac{(2\pi n)^{-3/2}}{\prod_{a \in {\mathcal{X}_3}} \rho_a^{1/2}} e^{-nI_3(\rho)} (1 + O([n \min_{a \in {\mathcal{X}_3}} \rho_a]^{-1}),$$

where

$$I_3(\rho) = \log 4 + \sum_{a \in {\mathcal{X}_3}} \rho_a \log \rho_a.$$ 

As before we denote by $\nu$ the unique minimizer (recall Proposition 8.6) of the variational problem. Then for every $\epsilon > 0$, there exists a $\delta > 0$, such that for $n$ large enough

$$\mathbb{P}(F_3(L_n) \leq 0, ||L_n - \nu||_1 > \epsilon) \leq e^{-n[I_3(\nu)+\delta]},$$

where $||\cdot||_1$ is the $\ell_1$-norm. Hence, since $\nu$ is strictly positive,

$$\mathbb{P}(F_3(L_n) \leq 0) = \frac{(2\pi n)^{-3/2}}{\prod_{a \in {\mathcal{X}_3}} \nu_a^{1/2}} \sum_{\rho : F_3(\rho) \leq 0} e^{-nI_3(\rho)} (1 + o(1)).$$

**Step 2: The sum over $\rho_+$.** To obtain compact notation we define $\tilde{\rho} = (\rho_+, \rho_\tau)^T$, and

$$J(\tilde{\rho}) = I_3(r(\tilde{\rho}), \tilde{\rho}_1, \tilde{\rho}_2, 1 - r(\tilde{\rho}) - \tilde{\rho}_1 - \tilde{\rho}_2),$$

where

$$r(\tilde{\rho}) = \frac{1}{4} + \frac{1}{4} \sqrt{(4\tilde{\rho}_1 - 1)(4\tilde{\rho}_2 - 1) + 2}. \quad (8.13)$$

We write $\rho_- = 1 - \sum_{a \in {\mathcal{X}_3}} \rho_a$, fix $\rho_-$ and $\rho_\tau$ and invoke the notation $\tilde{\rho}$. For $||\rho - \nu||_1 < \epsilon$, the condition $F_3(\rho) \leq 0$ is now equivalent to $\rho_+ \geq r(\tilde{\rho})$.

If we make a Taylor expansion in $\rho_+ = r(\tilde{\rho})$, we obtain

$$I_3(\rho) = J(\tilde{\rho}) + (\rho_+ - r(\tilde{\rho})) \frac{\partial}{\partial \rho_+} I_3(\rho)|_{\rho_+ = \epsilon},$$

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where $\xi$ is in between $r(\tilde{\rho})$ and $\rho_+$. Since $||\rho - \nu||_1 < \epsilon$, we have that $\xi - r(\tilde{\rho}) = O(\epsilon)$. Define
\[
\alpha_{\tilde{\rho}} = e^{-\frac{\rho}{\rho_+}T_3(\rho)}|_{\rho_+ = r(\tilde{\rho})}.
\]
Then
\[
\sum_{\rho, \rho_+ \geq r(\tilde{\rho})} e^{-nT_3(\rho)} = \sum_{\rho} e^{-nJ(\tilde{\rho})} \sum_{\rho, \rho_+ \geq nr(\tilde{\rho})} \alpha_{\tilde{\rho}}^{nr(\tilde{\rho}) - nr(\tilde{\rho})} (1 + o(1)).
\]
Furthermore, since $||\rho - \nu||_1 < \epsilon$, we have that $\alpha_{\tilde{\rho}} = \alpha_{\tilde{\rho}} + O(\epsilon) = \frac{\nu_{\epsilon}}{\nu_{\epsilon}} + O(\epsilon) < 1$. This proves that with $\alpha = \alpha_{\tilde{\rho}}$
\[
\sum_{\rho, \rho_+ \geq r(\tilde{\rho})} e^{-nT_3(\rho)} = \sum_{\rho} e^{-nJ(\tilde{\rho})} \frac{\alpha^{nr(\tilde{\rho}) - nr(\tilde{\rho})}}{1 - \alpha} (1 + o(1)),
\]
where $\lfloor x \rfloor$ is the smallest integer larger than or equal to $x$. Hence, using that $T_3(\nu) = J_3^{(2)}$, we arrive at
\[
\sqrt{n}e^{-nJ_3^{(2)}} P(F_3(L_n) \leq 0) = \frac{(2\pi)^{-3/2}}{\prod_{a \in \chi_3} \nu_a^{1/2} n} \sum_{\rho} \frac{\alpha^{nr(\tilde{\rho}) - nr(\tilde{\rho})}}{1 - \alpha} e^{-n(J(\tilde{\rho}) - J(\tilde{\nu}))} (1 + o(1)).
\]
(8.14)

**Step 3: Taylor expansion of the exponential rate.** Since $F_3(\nu) = 0$, $\tilde{\nu}$ minimizes $\tilde{\rho} \mapsto J(\tilde{\rho})$. Therefore, we have that $\nabla J(\tilde{\nu}) = 0$ and Taylor expansion of $J$ leads to
\[
J(\tilde{\rho}) - J(\tilde{\nu}) = (\tilde{\rho} - \tilde{\nu})^T \nabla^2 J(\tau)(\tilde{\rho} - \tilde{\nu})/2,
\]
where $\tau$ is some interpolation point in between $\tilde{\rho}$ and $\tilde{\nu}$. Since we can restrict ourselves to $\rho$’s with $||\rho - \nu||_1 < \epsilon$, we also have that $\nabla^2 J(\tau) = M + O(\epsilon)$, where $M = \nabla^2 J(\tilde{\nu})$. This gives
\[
\sqrt{n}e^{-nJ_3^{(2)}} P(F_3(L_n) \leq 0) = \frac{(2\pi)^{-3/2}}{\prod_{a \in \chi_3} \nu_a^{1/2} n} \sum_{\rho} \frac{\alpha^{nr(\tilde{\rho}) - nr(\tilde{\rho})}}{1 - \alpha} e^{-n(\tilde{\rho} - \tilde{\nu})^T M(\tilde{\rho} - \tilde{\nu})/2} (1 + o(1)).
\]
(8.15)

**Step 4: Strategy of the proof.** It is time to reveal how we intend to prove the theorem. Introduce a sequence of distribution functions $G_n$ on $[0, 1]$,
\[
G_n(x) = \frac{1}{Z_n} \sum_{\tilde{\rho}} 1_{\{x, x\}}([nr(\tilde{\rho})] - nr(\tilde{\rho})) e^{-n(\tilde{\rho} - \tilde{\nu})^T M(\tilde{\rho} - \tilde{\nu})/2},
\]
where $Z_n$ is defined by $G_n(1) = 1$:
\[
Z_n = \sum_{\tilde{\rho}} e^{-n(\tilde{\rho} - \tilde{\nu})^T M(\tilde{\rho} - \tilde{\nu})/2}.
\]
(8.16)
Chapter 8. Second order asymptotics

Observe that
\[ \frac{1}{n} \sum_{\tilde{\rho}} \alpha^{[n r(\tilde{\rho}) - n r(\tilde{\nu})]} e^{-n(\tilde{\rho} - \tilde{\nu})^T M(\tilde{\rho} - \tilde{\nu})/2} = \frac{Z_n}{n} \int_0^1 \alpha^x dG_n(x), \]
and hence according to (8.15),
\[ \sqrt{n} e^{n \gamma_3} \mathbb{P}(F_3(L_n) \leq 0) = \frac{(2\pi)^{-3/2}}{(1 - \alpha) \prod \nu_a^{1/2}} \frac{Z_n}{n} \int_0^1 \alpha^x dG_n(x)(1 + o(1)). \tag{8.17} \]
We will show that for \( m = 1, 2, \ldots , \)
\[ \lim_{n \to \infty} \int_0^1 e^{2\pi i m x} dG_n(x) = \int_0^1 e^{2\pi i m x} dx = 0. \tag{8.18} \]
By the selection principle (Feller (1971), p. 267), each subsequence \( \{n'\} \), has a further subsequence \( \{n''\} \) such that \( G_{n''} \) converges weakly to a proper distribution function \( G \) on \([0, 1]\). This implies in particular that for every continuous function \( u \) on \([0, 1]\), which is periodic \((u(0) = u(1))\),
\[ \lim_{n' \to \infty} \int_0^1 u(x) dG_{n''}(x) = \int_0^1 u(x) dG(x). \]
In turn, (8.18) will then imply that the Fourier coefficients \( \int_0^1 e^{2\pi i m x} dG(x) \) are those of the uniform distribution and this pinpoints the limit \( G \), so that in fact each convergent subsequence has the same weak limit which is the uniform distribution function, if (8.18) holds. This implies that the sequence \( G_n \) converges weakly to the uniform distribution on \([0, 1]\).

Since \( u(x) = \alpha^x \), \( x \in [0, 1] \), is continuous, we conclude that
\[ \lim_{n \to \infty} \int_0^1 \alpha^x dG_n(x) = \int_0^1 \alpha^x dx = \frac{1 - \alpha}{\log(1/\alpha)}, \]
and hence this implies that for \( n \to \infty \),
\[ n^{1/2} e^{n \gamma_3} \mathbb{P}(F_3(L_n) \leq 0) \rightarrow \gamma_3 = |\log \alpha|^{-1} \left( 2\pi |M| \prod_{a \in \partial \lambda_3} \nu_a \right)^{-1/2}, \tag{8.19} \]
because obviously
\[ \frac{Z_n}{n} \rightarrow \int \int e^{-(s,t)M(s,t)/2} ds dt = \frac{2\pi}{\sqrt{|M|}}. \]
The determinant \(|M|\) is strictly positive by Lemma 8.10. In the final step we will show (8.18).

**Step 5: The Fourier coefficients.** The last step, in which we deal with the Fourier coefficients for \( m > 0 \), is the most delicate one. Fix \( m > 0 \). By a Taylor expansion of \( r(\tilde{\rho}) \) around \( \tilde{\rho} = \tilde{\nu} \), we get that
\[ r(\tilde{\rho}) = r(\tilde{\nu}) + (\tilde{\rho} - \tilde{\nu})^T \nabla r(\tilde{\nu}) + (\tilde{\rho} - \tilde{\nu})^T \nabla^2 r(\tilde{\nu})(\tilde{\rho} - \tilde{\nu})/2 \]
\[ = \nu + (\tilde{\rho}_1 - \tilde{\nu}_1 + \tilde{\rho}_2 - \tilde{\nu}_2)r'(\tilde{\nu}) + (\tilde{\rho} - \tilde{\nu})^T \nabla^2 r(\tilde{\nu})(\tilde{\rho} - \tilde{\nu})/2, \]

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where \( \tau \) is in between \( \tilde{\rho} \) and \( \tilde{\nu} \) and where \( r'(\tilde{\nu}) \) is defined as

\[
\frac{\partial r}{\partial \tilde{\rho}_1}(\tilde{\nu}) = \frac{\partial r}{\partial \tilde{\rho}_2}(\tilde{\nu}),
\]

which are equal by symmetry of the minimizer. Hence, with \( j = (n\tilde{\rho}_1 - [n\tilde{\nu}_1], n\tilde{\rho}_2 - [n\tilde{\nu}_2])^T \in \mathbb{Z}^2 \) and using that \( \tilde{\nu}_1 = \tilde{\nu}_2 \), we can write

\[
nr(\tilde{\rho}) = n\nu_+ + r'(\tilde{\nu})(j_1 + j_2) + 2r'(\tilde{\nu})([n\tilde{\nu}_1] - n\tilde{\nu}_1) + j^T \nabla^2 r(\tilde{\nu}) j / (2n) + O \left( \frac{\|j\|^3}{n^2} \right).
\]

(8.20)

Substitution of (8.20) gives, since \( M \) is strictly positive definite,

\[
\sum_{\tilde{\rho}} e^{-2\pi imr(\tilde{\rho})} e^{-n(\tilde{\rho} - \tilde{\nu})^T M(\tilde{\rho} - \tilde{\nu}) / 2} = e^{-2\pi im[n\nu_+ + 2r'(\tilde{\nu})([n\tilde{\nu}_1] - n\tilde{\nu}_1)]} \times \sum_{j \in \mathbb{Z}^2} e^{-2\pi im(j_1 + j_2)r'(\tilde{\nu})} e^{-j^T (M + 2\pi im \nabla^2 r(\tilde{\nu})) j / (2n)} (1 + o(1)).
\]

(8.21)

Because \( mr'(\tilde{\nu}) \) is not an integer (which we know from Lemma 8.11, we have

\[
e^{-2\pi ir(\tilde{\nu}) m \ell} = c_m \int_{\ell}^{\ell + 1} e^{-2\pi ir(\tilde{\nu}) mx} dx \quad \text{with} \quad c_m = \frac{2\pi imr'(\tilde{\nu})}{1 - e^{-2\pi imr'(\tilde{\nu})}}.
\]

Use this result and the fact that for \( \|x - j\|_1 \leq 2 \)

\[
e^{-j^T (M + 2\pi im \nabla^2 r(\tilde{\nu})) j / (2n)} = e^{-2\pi ir(\tilde{\nu}) x / (2n) (1 + O \left( \frac{\|x\|}{n} \right))},
\]

to obtain

\[
\frac{1}{n} \sum_{j \in \mathbb{Z}^2} e^{-2\pi im(j_1 + j_2)r'(\tilde{\nu})} e^{-j^T (M + 2\pi im \nabla^2 r(\tilde{\nu})) j / (2n)}
\]

\[
= c_m^2 \frac{1}{n} \int_{\mathbb{R}^2} e^{-2\pi im(x_1 + x_2)r'(\tilde{\nu})} e^{-x^T (M + 2\pi im \nabla^2 r(\tilde{\nu})) x / (2n)} dx (1 + o(1))
\]

\[
= c_m^2 \int_{\mathbb{R}^2} e^{-2\pi im \sqrt{\sqrt{n}(x_1 + x_2)} r'(\tilde{\nu})} e^{-x^T (M + 2\pi im \nabla^2 r(\tilde{\nu})) x / 2} dx (1 + o(1)).
\]

The resulting integral gives zero by the Riemann-Lebesgue lemma, which shows that for \( m > 0 \),

\[
\lim_{n \to \infty} \int_0^1 e^{2\pi imx} dG_n(x) = \lim_{n \to \infty} \frac{1}{Z_n} \sum_{\tilde{\rho}} e^{-2\pi imr(\tilde{\rho})} e^{-n(\tilde{\rho} - \tilde{\nu})^T M(\tilde{\rho} - \tilde{\nu}) / 2} = 0,
\]

because \( Z_n / n \to \frac{2\pi}{\sqrt{|M|}} \).

\[\square\]
Importance sampling

In the previous chapter, we have seen that the second order asymptotics for HD-PIC and the SD-PIC system were difficult to derive, only for the SD-PIC model with 3 users, we have obtained a result. In those cases, where analytical methods fail, Monte Carlo simulation methods can contribute to the understanding of the characteristics of the system. However, in practice, the Monte Carlo simulation often requires a large amount of computer time. For example, estimating a BEP of the order $10^{-5}$ with 10% accuracy requires more than 10 million independent trials. A well-known technique to overcome this problem is called importance sampling (IS), Fishman (1995), Sect. 4.1. In telecommunications, this method is used in various applications, for an excellent review, see Smith, Shafi and Gao (1997). In IS, the number of successes (read bit errors) is increased by sampling from a different density function. The data is then weighted by an appropriate weighting function and this leads to unbiased estimates of the BEP. For a (more realistic) MF system, an IS procedure is described in Sadowsky and Bahr (1991).

In mathematical terms, suppose we have a random variable $Z$ with density $f$ and we are interested in the probability $p = P(Z \leq 0) \ll 1$. When we have performed $N$ standard Monte Carlo simulations $Z_1, \ldots, Z_N$, we approximate $p$ by

$$\hat{p} = \frac{1}{N} \sum_{l=1}^{N} 1_{\{Z_l \leq 0\}}.$$ 

The IS technique is based on forcing $\{Z \leq 0\}$ to occur more often. This is done with a different density $\tilde{f}$ and we denote by $\tilde{Z}$, the random variable of interest under the new density. In the optimal case, $\tilde{f}$ is chosen in such way that $E \tilde{Z} = 0$. Under the new density, the approximation above does not give the right approximation for $p$. Instead, we have to use

$$\hat{p} = \frac{1}{N} \sum_{l=1}^{N} \frac{f(\tilde{Z}_l)}{\tilde{f}(\tilde{Z}_l)} 1_{\{\tilde{Z}_l \leq 0\}},$$

where $\tilde{Z}_l$ are the simulated $\tilde{Z}$'s. The factor $f(\tilde{Z}_l)/\tilde{f}(\tilde{Z}_l)$ is called the weight.
When \( \tilde{f}(x) = \tilde{f}_t(x) \) is chosen to have the form \( \tilde{f}_t(x) = e^{tx}f(x)/\mathbb{E}e^{tZ} \) for some \( t \in \mathbb{R} \), \( \tilde{f}_t \) is called a tilted or twisted density. We define the tilted distribution function by

\[
\tilde{F}_t(x) = \frac{1}{\mathbb{E}e^{tZ}} \int_{(-\infty,x]} e^{ty}dF(y),
\]

where \( F \) is the distribution function of \( Z \). Tilted distribution functions are very useful in large deviations analysis. In fact, tilted distribution functions are used to prove Cramér's theorem.

### 9.1 Importance sampling for MF

The IS procedure for the simple MF model is straightforward. Recall that

\[
Z_0^{(1)} = 1 + \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{k-1} X_{ji},
\]

with \( X_{ji} \) i.i.d. When we define by \( F(x) \) the distribution function of \( X_{11} \) and \( z_0 = 1 + (x_{11} + \ldots + x_{k-1,n})/n \),

\[
\mathbb{P}(\text{sgn}_0(Z_0^{(1)}) < 0) = \int_{\text{sgn}_0(z_0)<0} dF(x_{11}) \cdots dF(x_{k-1,n}).
\]

Substituting the tilted distribution function in the right-hand side above yields

\[
\int_{\text{sgn}_0(z_0)<0} \left( (\mathbb{E}e^{tx_{11}})e^{-tx_{11}}d\tilde{F}_t(x_{11}) \right) \cdots \left( (\mathbb{E}e^{tx_{k-1,n}})e^{-tx_{k-1,n}}d\tilde{F}_t(x_{k-1,n}) \right)
\]

\[
e^{-tn} \left( \mathbb{E}e^{tx_{11}} \right)^{(k-1)n} \int_{\text{sgn}_0(z_0)<0} e^{-t(n+\sum_{i=1}^{n} \sum_{j=1}^{k-1} x_{ji})}d\tilde{F}_t(x_{11}) \cdots d\tilde{F}_t(x_{k-1,n})
\]

\[
e^{-tn} \left( \mathbb{E}e^{tx_{11}} \right)^{(k-1)n} \int_{\text{sgn}_0(z_0)<0} e^{-tnz_0}d\tilde{F}_t(x_{11}) \cdots d\tilde{F}_t(x_{k-1,n}).
\]

So far, we did not specify \( t \). We choose \( t \) in a special manner; namely, we choose \( t = t^* \), the minimizer of \( e^{-tn}(\mathbb{E}e^{tx_{11}})^{(k-1)n} \). Note that we already calculated this optimizer to calculate \( I_k \). Indeed, taking minus the logarithm of the expression above and and dividing by \( n \) gives the optimization problem

\[
\sup_{t \in \mathbb{R}} \{ t - (k - 1) \log \mathbb{E}e^{tx_{11}} \},
\]

which is the standard form of Cramér's theorem. Therefore, we can easily derive from Proposition 4.1 that \( t^* = \frac{1}{2} \log \frac{k-2}{k} < 0 \), and

\[
t^* - (k - 1) \log \mathbb{E}e^{tx_{11}} = I_k.
\]
Observe that in the proof of Proposition 4.1, a different approach has been followed. Of course the result is the same. From the substitution \( t = t^* \), we obtain
\[
\mathbb{P}(\text{sgnr}_0(Z^{(1)}_0) < 0) = \mathbb{E} \mathbf{1}_{\{\text{sgnr}_0(Z^{(1)}_0) < 0\}} = e^{-n I_k} \mathbb{E} e^{-t^* n Z^{(1)}_0} \mathbf{1}_{\{\text{sgnr}_0(Z^{(1)}_0) < 0\}}.
\] (9.1)
where \( Z^{(1)}_0 \) is drawn from the tilted measure. Note that under the tilted distribution \( \tilde{F}_{t^*} \),
\[
\mathbb{P}(\tilde{X}_{ji} = +1) = 1 - \mathbb{P}(\tilde{X}_{ji} = -1) = \frac{1}{2} e^{t^*} = \frac{k - 2}{2(k - 1)}.
\]
We now describe the complete IS procedure. We repeat the following trial \( N \) times:

- Draw \((\tilde{X}_{ji})_{1 \leq j \leq k, 1 \leq i \leq n}\) i.i.d. according to the tilted measure, i.e.,
\[
\mathbb{P}(\tilde{X}_{ji} = +1) = 1 - \mathbb{P}(\tilde{X}_{ji} = -1) = \frac{k - 2}{2(k - 1)}.
\]

- Calculate \( e^{-t^* n Z^{(1)}_0} \mathbf{1}_{\{\text{sgnr}_0(Z^{(1)}_0) < 0\}} \), where the sgnr-function is taken randomly for every trial.

- Store the result.

We now approximate \( \mathbb{P}(\text{sgnr}_0(Z^{(1)}_0) < 0) e^{n I_k} \) by the average of the obtained samples. Confidence intervals can also be built using the obtained information.

The profit of this procedure is the reduction of the required number of samples. We have seen in Proposition 8.1 that \( \mathbb{P}(\text{sgnr}_0(Z^{(1)}_0) < 0) = \mathcal{O}(n^{-1/2} e^{-n I_k}) \). Therefore,

\[
\frac{\text{var} \mathbf{1}_{\{\text{sgnr}_0(Z^{(1)}_0) < 0\}}}{\mathbb{P}(\text{sgnr}_0(Z^{(1)}_0) < 0)^2} = \mathcal{O}(n^{1/2} e^{n I_k}).
\] (9.2)

However, for the IS procedure with the tilted measure,

\[
\frac{\text{var}(e^{-n I_k} e^{-t^* n Z^{(1)}_0} \mathbf{1}_{\{\text{sgnr}_0(Z^{(1)}_0) < 0\}})}{\mathbb{P}(\text{sgnr}_0(Z^{(1)}_0) < 0)^2} \leq \frac{e^{-2n I_k} \text{var} \mathbf{1}_{\{\text{sgnr}_0(Z^{(1)}_0) < 0\}}}{\mathbb{P}(\text{sgnr}_0(Z^{(1)}_0) < 0)^2} \leq \frac{e^{-2n I_k}}{2} = \mathcal{O}(n). \] (9.3)

Therefore, when we estimate \( \mathbb{P}(\text{sgnr}_0(Z^{(1)}_0) < 0) \) for growing \( n \), we need for the ordinary Monte Carlo sampling exponentially many samples \( (n^{1/2} e^{n I_k}) \), while for the IS procedure, we need at most \( \mathcal{O}(n) \) samples (a more detailed analysis gives \( \mathcal{O}(n^{1/2}) \)). The disadvantage of the IS method is the more involved bookkeeping. While for ordinary Monte Carlo sampling only the number of successes is relevant, for the IS procedure also the weights should be taken into account. Moreover, knowledge of large deviation properties are required.

As shown above, IS can only be done efficiently, when the large deviation properties are already known. Without knowing the large deviation properties (like the minimizer) of the problem, it is not possible to perform IS in a sensible manner.
9.2 Importance sampling for HD-PIC and SD-PIC

We have seen in the previous section that IS can decrease simulation time. We will now describe an importance sampling procedure for both the simple HD-PIC and SD-PIC model. We will start with the description for the SD-PIC model, since it is less involved than that of the HD-PIC model.

Recall that the exponential rate for the SD-PIC model is obtained by translating the random variables \( X_{ji} \) into the (random) empirical measure \( L_n \), see Section 4.5. The decision statistic \( Z^{(2)}_0 \) is translated in \( F_k(L_n) \). The rate then followed as the infimum of the rate function \( I_k(\rho) \) over the area \( \{ F_k(\rho) \leq 0 \} \), see Theorem 4.5.

The optimization of the rate function over some area is in fact determining the optimal tilted empirical measure. Therefore, the minimizer \( \rho^* \) is the tilted measure which we will use for the importance sampling procedure. Suppose that we generate \( \rho_l, l = 1, \ldots, N \) according to this tilted measure, i.e., \( \tilde{L}_n \) is multinomial(\( \rho^* \)) distributed. It is easy to show that the following expression is a unbiased estimate of \( \mathbb{P}(\text{sgnr}_0(Z^{(2,S)}_m) \leq 0) \):

\[
\frac{1}{N} \sum_{l=1}^{N} \left( \prod_{a} (2^{k-1} \rho^*_a)^{n(\rho_l)_a} \right) 1_{\{\text{sgnr}_0(F_k(\rho_l)) \leq 0\}}.
\]

However, this is only true in the case the minimizer is unique. For \( k = 3 \), we have proven in Proposition 8.6 that this is the case. For \( k \geq 4 \), we expect this to hold, but we do not have any proof. In this section, we will assume that this is the case. Again, the large deviation properties of the problem are essential. It is crucial to fully understand the problem, before turning to an importance sampling procedure.

To estimate \( \mathbb{P}(\text{sgnr}_0(Z^{(2,S)}_m) \leq 0) \), we need to generate \( \tilde{L}_n \), as seen above. It is not straightforward how to simulate a multinomial distribution. One option is to partition the interval \([0,1]\) into intervals of lengths \( \rho^*_a \) for \( a \in \mathcal{X}_k \). A standard uniform sample then decides the sampling. The disadvantage is that \( |\mathcal{X}_k| = 2^{k-1} \), so that for large \( k \), this procedure is involved. For the HD-PIC model, a simplified procedure is proposed below. Unfortunately, for the SD-PIC model, we have not been able to derive a less involved procedure.

We next give a complete description of the IS procedure for the SD-PIC model. We repeat the following trial \( N \) times:

- Draw \( \tilde{L}_n \) according to the tilted measure \( \rho^* \), i.e., we partition the interval \([0,1]\) in pieces of lengths \( \rho^*_a \). We draw \( n \) standard uniform trials and count how many are in each interval. This gives the vector \( n\tilde{L}_n \).
- Calculate \( \left( \prod_{a \in \mathcal{X}_k} (2^{k-1} \rho^*_a)^{n(\tilde{L}_n)_a} \right) 1_{\{\text{sgnr}_0(F_k(\tilde{L}_n)) \leq 0\}} \) where the sgnr-function is taken randomly for every trial.
- Store the result.
Then average the outcomes to get an estimate for \( \mathbb{P}(\text{sgrn}_{0}(Z_0^{(2,H)}) < 0) e^{n_{2k}} \).

For the HD-PIC model, we recall from (4.5) that

\[
\mathbb{P}(\text{sgrn}_{0}(Z_0^{(2,H)}) < 0) = \sum_{r=1}^{k-1} \binom{k-1}{r} \mathbb{P}(\text{sgrn}_{0}(Z_0^{(2,H)}) < 0, B_r),
\]

where

\[
B_r = \left\{ \max_{1 \leq m \leq r} \text{sgrn}_m(Z_m^{(1)}) < 0, \min_{r+1 \leq m \leq k-1} \text{sgrn}_m(Z_m^{(1)}) > 0 \right\}.
\]

We will abbreviate

\[
p_r = \mathbb{P}(\text{sgrn}_{0}(Z_0^{(2,H)}) < 0, B_r).
\]

We can solve \( H_{k,r}^{(2)} \) using empirical measures. Indeed, similar to the expression in Lemma 4.4, there exists functions of the empirical measure, which describe \( Z_m^{(1)} \) and \( Z_0^{(2,H)} = 1 + 2 \sum_{m=1}^{n} \sum_{i=1}^{n} X_{mi} \). We will denote these functions by \( G_m(L_n) \) and \( H_{rk}(L_n) \), respectively. Then from (4.12), it follows that

\[
H_{k,r}^{(2)} = -\lim_{n \to \infty} \frac{1}{n} \log p_r = \inf_{\rho \in M(\mathcal{X}_k) : \sum_{j=1}^{r} G_m(\rho) = 0, H_{rk}(\rho) = 0} \mathcal{I}_k(\rho).
\]

We know that the minimizer corresponding to the optimization problem of \( p_r \) is unique. Therefore, for large \( n \), the BEP will consist mainly of \( \binom{k-1}{r} p_r \). However, \( \binom{k-1}{r} p_r \) for \( r \) close to \( r_k \) may also contribute to the BEP. Instead of simulating \( \mathbb{P}(\text{sgrn}_{0}(Z_0^{(2,H)}) < 0) \), one should simulate \( p_r \) for \( r \) close to \( r_k \). Once \( \binom{k-1}{r} p_r \) is too small, compared to \( \binom{k-1}{r} p_r \), the estimate of \( \mathbb{P}(\text{sgrn}_{0}(Z_0^{(2,H)}) < 0) \) has been identified. We see that is it crucial to understand the large deviation properties of the system.

For the HD-PIC model, the procedure is identical to the procedure for SD-PIC, except that we calculate for every trial (for a fixed \( r \))

\[
\left( \prod_{a \in \mathcal{X}_k} (2^{k-1} \rho_a^{(2)} n(L_n)) \right) \mathbb{1}_{\{\max_{1 \leq m \leq r} \text{sgrn}_m(G_m(L_n)) < 0, \min_{r+1 \leq m \leq k-1} \text{sgrn}_m(G_m(L_n)) > 0, \text{sgrn}_0(H_{rk}(L_n)) < 0\}}.
\]

### 9.3 Simplified procedure for HD-PIC

One disadvantage of the procedure described above is that we simulate \( L_n \), rather than \( X_{jt} \). Luckily, it turns out that \( \rho^* = \rho^*_r \) has a specific structure, which significantly reduces complexity. To see this, we will first prove the following result.

**Proposition 9.1** For \( k \geq 3 \), we have

\[
H_{k,r}^{(2)} = \inf_{\rho \in M(\mathcal{X}_k) : \sum_{j=1}^{r} G_m(\rho) = 0, H_{rk}(\rho) = 0} \mathcal{I}_k(\rho).
\]
Furthermore, \( \rho^* \), the optimizer of this problem, has the following form:
\[
\rho^*_a = \nu_a \left( \frac{1}{1 + e^{-\lambda u_a}} \right)^{k-r-1} \left( \frac{e^{-\lambda u_a}}{1 + e^{-\lambda u_a}} \right)^{k-r-1},
\]
(9.6)

where \( u_a = \sum_{j=1}^r a_j \), \( v_a = \sum_{j=r+1}^{k-1} a_j \), \( \nu = [\nu_{-r}, \nu_{-r+2}, \ldots, \nu_r] \) is a vector of length \( r + 1 \) and \( \lambda < 0 \). Furthermore, \( \lambda \) and \( \mu \) follow from
\[
H_{k,r}^{(2)} = \max_{\lambda \geq 0} \min_{\nu \in D} L(\nu, \lambda),
\]
where
\[
L(\nu, \lambda) = (k-1) \log 2 + \sum_{u=-r-r+2}^r \left\{ \left( \frac{r}{u+u} \right) \nu_u \right. \\
\left. \times \left( \log \nu_u + \frac{\lambda}{2} (u^2 + u) + (k-r-1) \left( \log(1+e^{\lambda u}) + \lambda u \left( 1 - \frac{4}{1+e^{-\lambda u}} \right) \right) \right) \right\},
\]
and
\[
D = \left\{ \nu : \nu_u \geq 0, \sum_{u=-r-r+2}^r \left( \frac{r}{u+u} \right) \nu_u = 1, 1 + 2 \sum_{u=-r-r+2}^r \left( \frac{r}{u+u} \right) u \nu_u = 0 \right\}
\]

**Proof.** The first statement directly follows from Sanov’s theorem.

For the second and third statement, we start from expression (4.12), i.e.,
\[
H_{k,r}^{(2)} = -\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \sum_{m=1}^r Z_m^{(1)} \leq 0, \tilde{Z}_0^{(2,H)} \leq 0 \right),
\]
where
\[
\sum_{m=1}^r Z_m^{(1)} = \frac{1}{n} \sum_{i=1}^n Y_{1i} = \frac{1}{n} \sum_{i=1}^n \left( \left( \sum_{j=1}^r X_{ji} \right)^2 + \sum_{j=1}^r X_{ji} \left( 1 + \sum_{j=r+1}^{k-1} X_{ji} \right) \right),
\]
\[
\tilde{Z}_0^{(2,H)} = \frac{1}{n} \sum_{i=1}^n Y_{2i} = \frac{1}{n} \sum_{i=1}^n \left( 1 + 2 \sum_{j=1}^r X_{ji} \right).
\]

We first partition \( \{1, \ldots, n\} \) in sets \( N_u = \{ i \in \{1, \ldots, n\} : \sum_{j=1}^r X_{ji} = u \} \), \( u = -r, -r + 2, \ldots, r \). Then
\[
Y_{1i} = \sum_{u=-r}^{u+even} \left( (u^2 + u) |N_u| + u \sum_{i \in N_u} \sum_{j=r+1}^{k-1} X_{ji} \right),
\]
\[
Y_{2i} = 1 + 2 \sum_{u=-r}^{u+even} u |N_u|.
\]
Conditionally on the sets \( N_u \), \( \sum_{i \in N_u} \sum_{j=r+1}^{k-1} X_{ji} \) are mutually independent in \( u \). Therefore, conditionally on \( N_u \), the rate of the probability that \( \sum Y_{1u} \leq 0 \) equals, according to Cramér’s theorem,

\[
\min_{q \in D_N} (k - r - 1) \sum_{u=-r,-r+2}^{r} \frac{|N_u|}{n} (\log 2 + q_u \log q_u + (1 - q_u) \log(1 - q_u)),
\]

where \( q = [q_{-r}, q_{-r+2}, \ldots, q_r] \) and

\[
D_N = \left\{ q = (q_{-r}, q_{-r+2}, \ldots, q_r) : \sum_{u=-r,-r+2}^{r} \left( u^2 + u \right) \frac{|N_u|}{n} + (k - r - 1) u \frac{|N_u|}{n} (2q_u - 1) \leq 0 \right\}.
\]

We next observe that \( (N_u)_{u=-r,-r+2} \) is multinomial distributed with probabilities \( (1/2)^r \left( \frac{r}{2} \right) \). Therefore, using Varadhan’s lemma (cf. DEN HOLLANDER (2000), THM. III.13) results in

\[
H_{k,r}^{(3)} = \min_{(x,u) \in D} \sum_{u=-r}^{r} \nu_u \log \left( \frac{\nu_u}{(1/2)^r \left( \frac{r}{2} \right)} \right) + (k - r - 1) \nu_u \log 2 + q_u \log q_u + (1 - q_u) \log(1 - q_u)),
\]

where \( \nu = [\nu_{-r}, \nu_{-r+2}, \ldots, \nu_r] \) and

\[
D = \left\{ (\nu, q) : \nu_u \geq 0, \sum_{u=-r,-r+2}^{r} \nu_u = 1, 0 \leq q_u \leq 1, 1 + 2 \sum_{u=-r,-r+2}^{r} u \nu_u \leq 0, \sum_{u=-r}^{r} \left( u^2 + u \right) \nu_u + (k - r - 1) u \nu_u (2q_u - 1) \leq 0 \right\}.
\]

We next replace \( \nu_u \) by \( \left( \frac{r}{2} \right) \nu_u \) and collect the terms with \( \log 2 \). This results in

\[
H_{k,r}^{(3)} = \min_{(x,u) \in D'} \left( k - r - 1 \right) \log 2 + \sum_{u=-r}^{r} \left( \frac{r}{2} \right) \nu_u \left( \log \nu_u + \log(1 - q_u) \right),
\]

where

\[
D' = \left\{ (\nu, q) : \nu_u \geq 0, \sum_{u=-r}^{r} \left( \frac{r}{2} \right) \nu_u = 1, 1 + 2 \sum_{u=-r}^{r} \left( \frac{r}{2} \right) u \nu_u \leq 0, \sum_{u=-r}^{r} \left( \frac{r}{2} \right) \nu_u \left( u^2 + u \right) + (k - r - 1) u (2q_u - 1) \leq 0 \right\}.
\]

When we apply the Lagrangian dual method (c.f. JAHN (1996), LEMMA 6.5) on the last constraint in \( D' \), we arrive at

\[
H_{k,r}^{(3)} = \max_{\lambda \geq 0} \min_{(x,u) \in D} \left( k - r - 1 \right) \log 2 + \sum_{u=-r}^{r} \left( \frac{r}{2} \right) \nu_u \left( \log \nu_u + \lambda (u^2 + u) + \log(1 - q_u) \log(1 - q_u) + \lambda u (2q_u - 1) \right),
\]

\[
\frac{\lambda}{2} \sum_{u=-r}^{r} \nu_u (u^2 + u) + \sum_{u=-r}^{r} \left( \frac{r}{2} \right) \nu_u \left( u^2 + u \right) + \lambda u (2q_u - 1) \leq 0.
\]
where

$$D = \left\{ (\nu, q) : \nu_u \geq 0, \sum_{u \text{ even}}^{r} \left( \frac{r}{r+u} \right) \nu_u = 1, 1 + 2 \sum_{u \text{ odd}}^{r} \left( \frac{r}{r+u} \right) \nu_u \leq 0 \right\}.$$ 

Taking the partial derivatives of the rate with respect to $q_u$ gives

$$\left( \frac{r}{r+u} \right) \nu_u \left( \log q_u - \log(1 - q_u) + 2\lambda u \right) = 0 \Rightarrow q_u = \frac{1}{1 + e^{2\lambda u}}.$$ 

We substitute this in the rate function. Since $1 - q_u = e^{2\lambda u}/(1 + e^{2\lambda u})$, we obtain

$$q_u \log q_u + (1 - q_u) \log(1 - q_u) + \lambda u (2q_u - 1)$$

$$= -\frac{1}{1 + e^{2\lambda u}} \log(1 + e^{2\lambda u}) - \frac{e^{2\lambda u}}{1 + e^{2\lambda u}} 2\lambda u - \frac{e^{2\lambda u}}{1 + e^{2\lambda u}} \log(1 + e^{2\lambda u}) + \frac{2\lambda u}{1 + e^{2\lambda u}} - \lambda u$$

$$= -\log(1 + e^{2\lambda u}) + \lambda u \left( 1 - \frac{4}{1 + e^{-2\lambda u}} \right).$$

We next replace $\lambda$ by $\lambda/2$ and observe that in $D$, we can replace the inequality $1 + 2\sum_{u=-r,-r+2}^{r} (\frac{r}{r+u}) \nu_u \leq 0$ by an equality, by convexity of the rate function.

The above procedure directly gives the form for $\rho^*$. 

The advantage of this structure for $\rho^*$ will become clear below.

### 9.3.1 $r = 1$

We will first describe how we simulate $p_1$, see (9.4). First, we solve the optimization problem in Proposition 9.1 for $r = 1$. For example for $k = 6$, this gives the numerical results $\nu_{-1} = 3/4$, $\nu_{1} = 1/4$ and $\lambda^* = 0.2513$. To force a bit error at stage 2 for 6 users, we therefore simulate $X_i$, according to $\nu_{-1} = 3/4$ and $\nu_{1} = 1/4$, i.e., we take $P(X_i = +1) = 1 - P(X_i = -1) = 1/4$. We next partition $\{1, \ldots, n\}$ in subsets $N_+$ and $N_-$, where

$$N_+ = \{i : X_i = +1\}, \quad \text{and} \quad N_- = \{i : X_i = -1\}.$$ 

Substitution in $\tilde{Z}^{(i)}_1$ gives

$$\tilde{Z}^{(i)}_1 = 1 + \frac{1}{n} \sum_{i=1}^{n} X_{1i} + \sum_{j=2}^{\lfloor k \rfloor} \left( \sum_{i \in N_+} X_{ji} - \sum_{i \in N_-} X_{ji} \right).$$

We expect the first sum on the right-hand side to be approximately $-1/2$, due to the measure $\nu$. In order to force $\text{sgn}_1(\tilde{Z}^{(i)}_1) < 0$, $\sum_{i \in N_+} \tilde{X}_{ji}, 2 \leq j \leq k - 1$ should have a high probability to be negative. Similarly, $\sum_{i \in N_-} \tilde{X}_{ji}, 2 \leq j \leq k - 1$ should be positive with high probability.

The correct procedure follows from (9.6). We take for $j = 2, \ldots, k - 1$,

$$P(\tilde{X}_{ji} = +1) = 1 - P(\tilde{X}_{ji} = -1) = \frac{1}{\lambda + \exp(\lambda t)} < 1/2, \quad i \in N_+ (u_a = +1),$$

$$P(\tilde{X}_{ji} = +1) = 1 - P(\tilde{X}_{ji} = -1) = \frac{1}{\lambda + \exp(-\lambda t)} > 1/2, \quad i \in N_- (u_a = -1).$$

Finally, the remaining $\tilde{Z}^{(i)}_m$'s are typically positive, so that indeed the desired event is simulated.
9.3.2 \( r = 2 \)

For \( r = 2 \), we first solve the optimization problem op Proposition 9.1. For example, for \( k = 6 \), this gives the numerical results \([\nu_{-2}, \nu_{0}, \nu_{2}] = [0.3263, 0.5973, 0.0763]\) and \( \lambda^* = 0.4970 \).

We next generate \( \tilde{X}_{1i} \) and \( \tilde{X}_{2i} \) according to \([\nu_{-2}, \nu_{0}, \nu_{2}]\). We partition \( \{1, \ldots, n\} \) in the sets \( N_+, N_\pm, N_\pi \) and \( N_- \), where

\[
N_+ = \{ i : \tilde{X}_{1i} = +1, \tilde{X}_{2i} = +1 \},
\]

\[
N_\pm = \{ i : \tilde{X}_{1i} = +1, \tilde{X}_{2i} = -1 \},
\]

\[
N_\pi = \{ i : \tilde{X}_{1i} = -1, \tilde{X}_{2i} = +1 \},
\]

\[
N_- = \{ i : \tilde{X}_{1i} = -1, \tilde{X}_{2i} = -1 \}.
\]

Then substitution in \( \tilde{Z}_{1}^{(1)} \) and \( \tilde{Z}_{2}^{(1)} \) gives

\[
\tilde{Z}_{1}^{(1)} = 1 + \frac{1}{n} \sum_{i=1}^{n} \tilde{X}_{1i} + \sum_{i=1}^{n} \sum_{j=3}^{n} \tilde{X}_{1i} \tilde{X}_{ji} + \frac{1}{n} \left[ \sum_{j=3}^{n} \tilde{X}_{ji} + \sum_{i \in N_+} \tilde{X}_{ji} - \sum_{i \in N_\pi} \tilde{X}_{ji} - \sum_{i \in N_-} \tilde{X}_{ji} \right], \quad (9.7)
\]

\[
\tilde{Z}_{2}^{(1)} = 1 + \frac{1}{n} \sum_{i=1}^{n} \tilde{X}_{2i} + \sum_{i=1}^{n} \sum_{j=3}^{n} \tilde{X}_{1i} \tilde{X}_{ji} + \frac{1}{n} \left[ \sum_{j=3}^{n} \tilde{X}_{ji} - \sum_{i \in N_+} \tilde{X}_{ji} + \sum_{i \in N_\pi} \tilde{X}_{ji} - \sum_{i \in N_-} \tilde{X}_{ji} \right]. \quad (9.8)
\]

It is easy to check that \( \mathbb{E} \tilde{Z}_{0}^{(r, n)} = 0 \), as desired. For \( k = 6 \), substituting \( \nu \) indeed gives

\[
\mathbb{E} \tilde{Z}_{0}^{(r, n)} = 1 + \sum_{u=-r+2}^{r} \left( \frac{r}{r+u} \right) \nu u = 1 + 2 \left( -2 \cdot 0.3263 + 0 \cdot 2 \cdot 0.5973 + 2 \cdot 0.0763 \right) = 0.
\]

We further observe from (9.7) and (9.8) that whenever \( \sum_{i \in N_+} \tilde{X}_{ji} \) for \( j = 3, \ldots, k-1 \) is typically negative, it helps both \( \tilde{Z}_{1}^{(1)} \) and \( \tilde{Z}_{2}^{(1)} \) to be more negative (and thus force a bit error). Using the same argument, whenever \( \sum_{i \in N_-} \tilde{X}_{ji} \) for \( j = 3, \ldots, k-1 \) is typically positive, it helps both \( \tilde{Z}_{1}^{(1)} \) and \( \tilde{Z}_{2}^{(1)} \) to be more negative. On the other hand, with \( \sum_{i \in N_\pi} \tilde{X}_{ji} \) we can either decrease \( \tilde{Z}_{1}^{(1)} \) or \( \tilde{Z}_{2}^{(1)} \) (with probability 1), but as an undesired consequence \( \tilde{Z}_{2}^{(1)} \) or \( \tilde{Z}_{1}^{(1)} \), respectively, is increased. It turns out that it is optimal not to change the measure of \( \sum_{i \in N_\pi} \tilde{X}_{ji} \). Similarly, the measure of \( \sum_{i \in N_\pi} \tilde{X}_{ji} \) is not changed.

We combine this in the generation of \( \tilde{X}_{ji} \), \( 3 \leq j \leq k-1 \). We simulate according to

\[
\mathbb{P}(\tilde{X}_{ji} = +1) = 1 - \mathbb{P}(\tilde{X}_{ji} = -1) = \frac{1}{1+\exp(2\lambda^*)} < 1/2, \quad i \in N_+ (u_a = 2),
\]

\[
\mathbb{P}(\tilde{X}_{ji} = +1) = 1 - \mathbb{P}(\tilde{X}_{ji} = -1) = \frac{1}{1+\exp(0\lambda^*)} = 1/2, \quad i \in N_\pm \cup N_\pi (u_a = 0),
\]

\[
\mathbb{P}(\tilde{X}_{ji} = +1) = 1 - \mathbb{P}(\tilde{X}_{ji} = -1) = \frac{1}{1+\exp(-2\lambda^*)} > 1/2, \quad i \in N_- (u_a = -2).
\]

In this way, we make optimal use of the structure of the system. We note that the procedure above follows from (9.6).
9.3.3 $r \geq 3$

The case $r \geq 3$ goes similar. First, we simulate $\tilde{X}_{i_1}, \ldots, \tilde{X}_{i_r}, 1 \leq i \leq n$, according to the optimal measure $\nu$. We then partition $\{1, \ldots, n\}$ in subsets $N_a$. For every subset, we generate $\tilde{X}_{ji}, r + 1 \leq j \leq k - 1$ independently, according to
\[
\mathbb{P}(\tilde{X}_{ji} = +1) = 1 - \mathbb{P}(\tilde{X}_{ji} = -1) = \frac{1}{1 + e^{\lambda u_a}}.
\]

Similarly to the case $r = 2$, we are allowed to take the subsets $N_a$ in which $u_a$ is the same together.

We conclude with the complete IS procedure. We repeat the following trial $N$ times:

- Fix $r$, e.g., take $r = r_k$.
- Draw the empirical measure of $\tilde{X}_1, \ldots, \tilde{X}_r$ according to the tilted measure $\nu$, i.e., we partition the interval $[0,1]$ in pieces of lengths $\nu_i^*$. We draw $n$ standard uniform trials and count for every interval how many are in that interval.
- Calculate $\tilde{X}_{i_1}, \ldots, \tilde{X}_{i_r}$ for $1 \leq i \leq n$ from the empirical measure above.
- When $\sgnr_0(\tilde{Z}_0^{(2,1)}) > 0$, store 0. Else continue.
- For every $u = -r, -r + 2, \ldots, r$, collect the $i$'s such that $\sum_{j=1}^r \tilde{X}_{ji} = u$ in the set $N_u$. For every $i$ in $N_u$, simulate $\tilde{X}_{r+1,i}, \ldots, \tilde{X}_{k-1,i}$ according to $\mathbb{P}(X_{r+1,i} = +1) = 1/(1 + e^{\lambda u})$.
- Calculate $\sgnr_m(Z_m^{(1)})$ for all $m$, using the matrix notation. Calculate furthermore $\tilde{L}_n$, which easily follows from the partial measure in the first step, together with $\sum_{j=r+1}^{k-1} \tilde{X}_{ji}$.
- Calculate
\[
\left( \prod_a (2^{k-1} \rho_a^*)^{n(\ell_m)_a} \right) \mathbb{1}_{\{\max_{1 \leq m \leq r} \sgnr_m(Z_m^{(1)}) < 0, \min_{r+1 \leq m \leq k-1} \sgnr_m(Z_m^{(1)}) > 0\}}.
\]
- Store the result.

In Klok (2001), a brief overview is given of the procedure above, together with simulation results.
9.4 Numerical results

In this section, we will give numerical results for \( k = 3, 6 \) and 9. We treat \( k = 3 \) separately, since we have obtained analytical results or good intuition on the second order asymptotics. Furthermore, for \( k = 3 \), we can use extensive calculations to obtain exact results for the BEP. For \( k = 6 \) and 9, we will use simulation results.

Recall that \( \alpha_{k,n} \) is the function for which

\[
\mathbb{P}(\text{sgn} r_0(Z_0^{(1)}) < 0) = \frac{\alpha_{k,n}}{\sqrt{n}} e^{-n I_k(1 + o(1))}.
\]

Similarly, \( \beta_{k,n} \) is the function for which we have conjectured that

\[
\mathbb{P}(\text{sgn} r_0(Z_0^{(2,H)}) < 0) = \frac{\beta_{k,n}}{\sqrt{n}} e^{-n H_k^{(2)}} (1 + o(1))
\]

and \( \gamma_k \) is the function for which we have conjectured

\[
\mathbb{P}(\text{sgn} r_0(Z_0^{(2,S)}) < 0) = \frac{\gamma_k}{\sqrt{n}} e^{-n J_k^{(2)}} (1 + o(1)).
\]

In Figure 9.1, the exact BEP is shown for \( k = 3 \) for the MF, the HD-PIC and the SD-PIC model, together with the large deviation approximations

\[
\frac{\alpha_{3,n}}{n^{1/2}} e^{-n I_3} = \frac{\beta_{3,n}}{n^{1/2}} e^{-n H_3^{(2)}} \quad \text{and} \quad \frac{\gamma_3}{n^{1/2}} e^{-n J_3^{(2)}}.
\]

The probabilities \( \mathbb{P}(\text{sgn} r_0(Z_0^{(1)}) < 0) \) and \( \mathbb{P}(\text{sgn} r_0(Z_0^{(2,H)}) < 0) \) are very close, so that we expect that the second order asymptotics \( \beta_{3,n} \) are the same as \( \alpha_{3,n} \). For all \( n \geq 10 \), our approximations for the BEP are very accurate.

We have obtained simulation results for \( k = 6 \) and 9. We have used 30,000 simulations for MF, 150,000 for HD-PIC and 150,000 for SD-PIC. In Figure 9.2, the estimated BEP is shown. We see that both HD-PIC and SD-PIC significantly decrease the BEP if \( n \) is not too small. For 3 and 6 users, the BEP for the SD-PIC model is smaller than that of the HD-PIC model. This is because the exponential rate is higher for SD-PIC. For \( k = 9 \) however, HD-PIC performs better than SD-PIC. The reason is that the rates are almost the same, but the second order asymptotics of HD-PIC are better than those of SD-PIC. Indeed, we expect for HD-PIC order \( n^{-(r_k+1)/2} \), while for SD-PIC, we expect order \( n^{-1/2} \). Therefore, for high number of users, we expect the BEP for the HD-PIC model to be smaller than that of the SD-PIC model.

From the numerical results, we also see that the importance sampling procedures require polynomial many samples. Especially for large \( n \), this will result in a significant increase in efficiency.

We next investigate the second order asymptotics into more detail using the simulation results.
Figure 9.1: Exact BEP for MF (○), HD-PIC (△) and SD-PIC (○) and large deviation approximation for MF/HD-PIC (×) and SD-PIC (♦).

### 9.4.1 Comparison with second order asymptotics of MF

We have performed 30,000 simulations to estimate $\Pr(\text{sgnr}_0(Z_0^{(1)}) < 0)$. In Figure 9.3(a), $\Pr(\text{sgnr}_0(Z_0^{(1)}) < 0)$ is shown for $n = 1, \ldots, 100$ for $k = 3, 6$ and 9, together with 95% confidence intervals. Also $\alpha_{k,n} n^{-1/2} \exp(-n I_k)$ is shown for $k = 3, 6, 9$, which are slightly above the estimated BEP. The parity behaviour is clearly present for $k = 3$. For $k = 6$ and 9, it is not very clear from the simulations whether there is any parity behaviour. The large deviation approximation is accurate. In Figure 9.3(b), $n^{1/2} \exp(n I_k) \Pr(\text{sgnr}_0(Z_0^{(1)}) < 0)$ is shown for $n = 1, \ldots, 100$ for $k = 3, 6$ and 9, together with 95% confidence intervals. Also $\alpha_{k,n}$ is shown. For $k = 3$, we see that $n^{1/2} \exp(n I_k) \Pr(\text{sgnr}_0(Z_0^{(1)}) < 0)$ converges quite fast to $\alpha_{3,n}$. For $k = 6$ and 9, the convergence is much slower.

### 9.4.2 Comparison with second order asymptotics of HD-PIC

The second order asymptotics for the HD-PIC model are more involved than those for the MF model, since the BEP is the sum of $p_r$ for $r = 1, \ldots, k - 1$. Let us investigate the BEP and $p_r$ for $k = 3, 6, 9$ and $r = 1, 2, 3$. For $k = 3$, exact results are known. For $k = 6, 9$, we will denote the estimated $p_r$ by $\hat{p}_r$. The contribution to the BEP for higher $r$ is negligible and therefore not interesting. We can therefore use

$$
\Pr(\text{sgnr}_0(Z_0^{(2,\mathcal{H})}) < 0) \approx \binom{k-1}{1} p_1 + \binom{k-1}{2} p_2 + \binom{k-1}{3} p_3.
$$

For every $r$, we have performed 50,000 simulations to estimate $p_r$. In Figure 9.4(a), we show $\Pr(\text{sgnr}_0(Z_0^{(2,\mathcal{H})}) < 0)$ for $n = 1, \ldots, 100$ for $k = 3, 6$ and 9, together with 95% confidence
Figure 9.2: Estimated BEP for MF (○ and ×), HD-PIC (△ and ●) and SD-PIC (○ and ●), for \( k = 6 \) and 9, respectively.

intervals. Also \( \hat{\beta}_{k,n} n^{-\delta_k} \exp(-nH^{(2)}_k) \) is shown, where \( \delta_3 = 1/2 \) and \( \hat{\beta}_{3,n} = \beta_{3,n} \) and where \( \delta_6 = \delta_9 = 1 \) and \( \hat{\beta}_{k,n} \) is an estimate for \( \beta_{k,n} \). Since the simulations are not accurate enough to estimate parity effects, we confine with an average value, which we obtain by averaging \( n^{1/2} \exp(nH^{(2)}_{k,1}) P_{1}(k=1) p_1 \) for \( n = 75 - 100 \). Note that we have used the results for \( p_1 \) only, since we expect \( p_1 \) to give the main contribution to the BEP, so that the second order asymptotics only depend on \( p_1 \). In Figure 9.4(b), \( n^{6k} \exp(nH^{(3)}_k) P(\text{sgn}_0(Z^{(2),0}_k) < 0) \) is shown for \( n = 1, \ldots, 100 \) for \( k = 3, 6 \) and 9, together with 95% confidence intervals. Also \( \hat{\beta}_{k,n} \) is shown. For \( k = 3 \) and 6, the estimate for \( \beta_{k,n} \) seems to be plausible. For \( k = 9 \), \( \hat{\beta}_{k,n} \) seems to be too low. However, a close examination reveals that \( n \exp(nH^{(2)}_k) P(\text{sgn}_0(Z^{(2),0}_k) < 0) \) decreases slowly for \( n \geq 70 \), so that for larger \( n \), \( \hat{\beta}_{k,n} \) may not be unlikely.

For \( n \rightarrow \infty \), only \( p_{r_k} \) is relevant. However, for small \( n \), \( p_r \) for \( r \neq r_k \) can also give a contribution. Therefore, we expect convergence of the BEP to the asymptotic expression

\[
\beta_{k,n} n^{-(r_k+1)/2} e^{-nH^{(2)}_k}
\]

to be much slower than the convergence of the BEP for the MF model. Another reason why we expect the convergence towards the asymptotic result to be slow is the following. When all events for stage 1 and 2 would be independent, the probability becomes a product of \( r+1 \) probabilities, each with a slow convergence. We expect that the fact that the events are not independent does not influence the convergence speed too much. Therefore, the convergence of HD-PIC is expected to be much slower than that of MF.

For a more detailed analysis of \( p_r \), we have given \( \hat{p}_1, \hat{p}_2, \hat{p}_3 \) and \((^{(k-1)}\hat{p}_1 + ^{(k-1)}\hat{p}_2 + ^{(k-1)}\hat{p}_3)\) for \( k = 6 \) and 9, see Figure 9.5(a) and (b). We expect that the main contribution to the BEP comes from \( r = 1 \), since \( r = 1 \) minimizes \( H^{(2)}_{k,r} \). We see that this is indeed the case. Only for
(a) $\mathbb{P}(\text{sgn} r_0(Z_0^{(1)}) < 0)$ and $\alpha_{k,n} n^{-1/2} e^{-nH_k}$.  
(b) $n^{1/2} e^{nH_k} \mathbb{P}(\text{sgn} r_0(Z_0^{(1)}) < 0)$ and $\alpha_{k,n}$.

Figure 9.3: Second order asymptotics with 95% confidence intervals and large deviation approximations for $k = 3$ (lower curves), $k = 6$ (center curves) and $9$ (upper curves) for the MF model.

(a) $\mathbb{P}(\text{sgn} r_0(Z_0^{(2,H)}) < 0)$ and $\hat{\beta}_{k,n} n^{-\delta_k} e^{-nH_k^{(2)}}$.  
(b) $n^{\delta_k} e^{nH_k^{(2)}} \mathbb{P}(\text{sgn} r_0(Z_0^{(2,H)}) < 0)$ and $\hat{\beta}_{k,n}$.

Figure 9.4: Second order asymptotics with 95% confidence intervals and large deviation approximations for $k = 3$ (lower curves), $k = 6$ (center curves) and $9$ (upper curves) for the HD-PIC model.

small $n$, also $r = 2$ contributes to the BEP. For $k = 9$, the contribution to the BEP from $r = 1$ and $r = 2$ are more equivalent, since $H_{91}^{(2)}$ and $H_{92}^{(2)}$ are close. Due to second order ($n^{-1}$ and $n^{-3/2}$), the contribution from $r = 1$ is dominant.

For all $n$, $p_3$ does not contribute. For large $n$, this could be expected, since $H_{6}^{(2)} = 0.16214 \ldots$ and $H_{9}^{(2)} = 0.14868 \ldots$, while $H_{6,3}^{(2)} = 0.39222 \ldots$ and $H_{9,3}^{(2)} = 0.22403 \ldots$

9.4.3 Comparison with second order asymptotics of SD-PIC
Figure 9.5: Estimated BEP of the HD-PIC model (o) and $(k-1)\hat{p}_r$ together with 95% confidence intervals for $k = 6, 9$ and $r = 1, 2, 3, \triangle, \diamond, \times$, respectively.

We have performed 150,000 simulations to estimate $\mathbb{P}(\text{sgnr}_0(Z_0^{(2,5)}) < 0)$. In Figure 9.6(a), $\mathbb{P}(\text{sgnr}_0(Z_0^{(2,5)}) < 0)$ is shown for $n = 1, \ldots, 100$ for $k = 3, 6$ and 9, together with 95% confidence intervals. Also $\hat{\gamma}_kn^{-1/2}\exp\left(-nJ_k^{(2)}\right)$ is shown, where $\hat{\gamma}_k$ is the estimate for $\gamma_k$. We have obtained $\hat{\gamma}_k$ by averaging $n^{1/2}\exp(nJ_k^{(2)})\mathbb{P}(\text{sgnr}_0(Z_0^{(2,5)}) < 0)$ for $n = 75-100$. This resulted in $\hat{\gamma}_6 = 2.41$ and $\hat{\gamma}_9 = 10.75$. For $k = 3$, we have taken $\gamma_3 = 0.5946$. We see that the large deviation approximations are very accurate. In Figure 9.6(b), $n^{1/2}\exp(nJ_k^{(2)})\mathbb{P}(\text{sgnr}_0(Z_0^{(2,5)}) < 0)$ is shown for $n = 1, \ldots, 100$ for $k = 3, 6$ and 9, together with 95% confidence intervals. Also $\alpha_{k,n}$ is shown. For $k = 3$, we see that $n^{1/2}\exp(nJ_k^{(2)})\mathbb{P}(\text{sgnr}_0(Z_0^{(2,5)}) < 0)$ converges quite fast to $\alpha_{3,n}$. For $k = 6$ and 9, the convergence is slower, but for $n \geq 50$, $\hat{\gamma}_k$ is very close to $n^{1/2}\exp(nJ_k^{(2)})\mathbb{P}(\text{sgnr}_0(Z_0^{(2,5)}) < 0)$. 

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Appendices

In this appendix, we will prove theorems and lemmas for which the proof is too technical or distracting for treatment in the thesis.

Appendix A: Expression for decoded bit

We multiply the signal \( r(t) = \sum_{j=0}^{k-1} s_j(t) + \eta n(t) \) by \( a_m(t) \cos(\omega_c t) \) and average over \([0,T]\):

\[
\frac{1}{T} \int_0^T r(t) a_m(t) \cos(\omega_c t) \, dt
\]

\[
= \frac{1}{T} \int_0^T s_m(t) a_m(t) \cos(\omega_c t) \, dt + \sum_{j \neq m} \frac{1}{T} \int_0^T s_j(t) a_m(t) \cos(\omega_c t) \, dt
\]

\[
+ \eta \frac{1}{T} \int_0^T n(t) a_m(t) \cos(\omega_c t) \, dt
\]

\[
= \frac{(2P_m)^{1/2}}{T} \int_0^T b_m(t) a_m^2(t) \cos^2(\omega_c t) \, dt + \sum_{j \neq m} \frac{(2P_j)^{1/2}}{T} \int_0^T b_j(t) a_j(t) a_m(t) \cos^2(\omega_c t) \, dt
\]

\[
+ \eta \frac{1}{T} \int_0^T n(t) a_m,\lfloor t/T_c \rfloor \cos(\omega_c t) \, dt
\]

\[
= \frac{(2P_m)^{1/2}}{T} \int_0^T b_m,\lfloor t/T \rfloor a_m^2,\lfloor t/T \rfloor \cos^2(\omega_c t) \, dt + \sum_{j \neq m} \frac{(2P_j)^{1/2}}{T} \int_0^T b_j,\lfloor t/T \rfloor a_j,\lfloor t/T \rfloor a_m,\lfloor t/T \rfloor \cos^2(\omega_c t) \, dt
\]

\[
+ \frac{1}{n} \sum_{i=0}^{n-1} \eta \frac{1}{T_c} a_{mi} \int_{iT_c}^{(i+1)T_c} n(t) \cos(\omega_c t) \, dt
\]

\[
= \frac{(2P_m)^{1/2}}{T} b_{m1} \int_0^T \cos^2(\omega_c t) \, dt + \frac{1}{n} \sum_{j \neq m} b_{j1} \sum_{i=1}^{n} a_{ji} a_{mi} \frac{(2P_j)^{1/2}}{T_c} \int_{iT_c}^{(i+1)T_c} \cos^2(\omega_c t) \, dt
\]

\[
+ \frac{1}{n} \sum_{i=1}^{n} a_{mi} \eta \frac{f^{(i+1)T_c}}{T_c} \cos^2(\omega_c t) \, dt^{1/2} N_i,
\]
where \((N_i)_{i=1}^n\) are independent and identically distributed, with \(N_i \sim \mathcal{N}(0, 1)\). Since \(\omega_c T_c = \pi f_c\), where \(f_c \in \mathbb{N}\),

\[
\int_{iT_c}^{(i+1)T_c} \cos^2(\omega_c t) \, dt = \frac{T_c}{2}.
\]

Therefore

\[
\frac{1}{T} \int_0^T r(t) a_m(t) \cos(\omega_c t) \, dt = \left(\frac{P_m}{2}\right)^{1/2} b_{m,1} + \sum_{j=0}^{k-1} \left(\frac{P_j}{2}\right)^{1/2} b_{j,1} - \frac{1}{n} \sum_{i=1}^n a_{j,i} a_{m,i} + \frac{1}{n} \sum_{i=1}^n a_{m,i} \frac{\eta}{\sqrt{2T_c}} N_i,
\]

which is the desired statement.

**Appendix B: Proof of Lemma 5.8**

**Proof of Lemma 5.8 (i).**

When \(S_A = \sum_{j \in A} X_j\) for \(A \subset \mathbb{N} \cup \{0\}\), we claim that for \(A_1 \cap A_2 = A_3 \cap A_4 = \emptyset\),

\[
\mathbb{E} S_{A_1} S_{A_2} S_{A_3} S_{A_4} \leq |A_3||A_4|.
\]

Note that \(\mathbb{E} S_{A_1} \leq 1\) and that \(\mathbb{E} S_{A_1} S_{A_2} = |A_1 \cap A_2|\), regardless whether \(\{0\} \in A_1 \cup A_3\) or not.

We perform induction in \(|A_4|\). When \(A_4 = \{0\}\), we have \(S_{A_4} = 1\), so that

\[
\mathbb{E} S_{A_1} S_{A_2} S_{A_3} S_{A_4} = \mathbb{E} S_{A_1} S_{A_2} S_{A_3} = |A_1 \cap A_3| \mathbb{E} S_{A_2} + |A_2 \cap A_3| \mathbb{E} S_{A_1 \setminus A_3} \leq |A_3| = |A_3||A_4|.
\]

When \(A_4 = \{i\} \neq \{0\}\), we have

\[
\mathbb{E} S_{A_1} S_{A_2} S_{A_3} U_i = \begin{cases} 0 & \{i\} \notin A_1 \cup A_2 \cup A_3, \\ \mathbb{E} S_{A_1} S_{A_2} S_{A_3} \leq |A_2 \cap A_3| \leq |A_3| & \{i\} \in A_1, \{i\} \notin A_2 \cup A_3, \\ \mathbb{E} S_{A_2} S_{A_3} \leq |A_1 \cap A_3| \leq |A_3| & \{i\} \in A_2, \{i\} \notin A_1 \cup A_3, \end{cases}
\]

so that for \(|A_4| = 1\) the claim is true. Next, we write \(A_4 = A'_i \cup \{i\}\) for some \(i \neq 0\). By construction, \(i\) cannot be in \(A_3\), but it can be in \(A_1, A_2\) but not in both. Suppose the claim is true for \(|A_4| = n - 1\). Then for \(|A_4| = n\), we have three cases.

Case 1: \(\{i\} \notin A_1 \cup A_2 \cup A_3\). Then

\[
\mathbb{E} S_{A_1} S_{A_2} S_{A_3} S_{A_4} = \mathbb{E} S_{A_1} S_{A_2} S_{A_3} S_{A'_i} \leq |A_3||A'_i| \leq |A_3||A_4|.
\]

Case 2: \(\{i\} \in A_1, \{i\} \notin A_2 \cup A_3\). Then \(S_{A_1} = S_{A'_i} + U_i\) and \(S_{A_4} = S_{A'_i} + U_i\), so that

\[
\mathbb{E} S_{A_1} S_{A_2} S_{A_3} S_{A_4} = \mathbb{E} S_{A_2} S_{A_3} + \mathbb{E} S_{A'_i} S_{A_2} S_{A_3} S_{A'_i} \leq |A_2 \cap A_3| + |A_3||A'_i| \leq |A_3||A_4|.
\]

Case 3: \(\{i\} \in A_2, \{i\} \notin A_1 \cup A_3\). This is identical to case 2, where \(A_1\) and \(A_2\) are interchanged. We conclude by induction that the claim holds for all \(A_4\). ■

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Proof of Lemma 5.8 (ii).

For \( A_1 \cap A_2 = A_3 \cap A_4 = \emptyset \),

\[
\mathbb{E} S_{A_1}^2 S_{A_3} S_{A_4} = \mathbb{E} S_{A_1}^2 S_{A_3 \setminus A_1} S_{A_4 \setminus A_4} + \mathbb{E} S_{A_1}^2 S_{A_3 \setminus A_3} S_{A_4 \setminus A_4} + \mathbb{E} S_{A_1}^2 S_{A_3 \cap A_1} S_{A_4 \cap A_4} + \mathbb{E} S_{A_1}^2 S_{A_3 \cap A_1} S_{A_4 \cap A_4} = 2 \mathbb{E} S_{A_1} S_{A_3 \cap A_1} S_{A_4 \cap A_4} = 2 |A_1 \cap A_3||A_1 \cap A_4|.
\]

It is clear that

\[2 |A_1 \cap A_3||A_1 \cap A_4| \leq 2 \min\{|A_1|^2, |A_2||A_4|\},\]

which proves the lemma.

\[\blacksquare\]

Appendix C: Proof of Lemma 5.9

For convenience, we introduce

\[B_n = \bigcap_{1 \leq \sigma \leq s} \left\{ \sum_{m \in R_{\sigma}} \tilde{Z}_m^{(\sigma, H)} \leq 0 \right\} \bigcap_{1 \leq \sigma' < \sigma \leq s - 1} \left\{ \sum_{m \in R_{\sigma'}} \tilde{Z}_m^{(\sigma, H)} \geq 0 \right\} \bigcap_{1 \leq \sigma \leq s - 1} \left\{ \sum_{m \in R_{\sigma}} \tilde{Z}_m^{(\sigma, H)} \geq 0 \right\} \]

By definition \( \tilde{H}_{k, R}^{(s)} = - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(B_n) \). Clearly

\[\mathbb{P}(B_n) = \mathbb{P}\left( B_n \cap \bigcup_{1 \leq \sigma' < \sigma \leq s - 1} \left\{ \sum_{m \in R_{\sigma'}} \tilde{Z}_m^{(\sigma, H)} \leq 0 \right\} \bigcup_{1 \leq \sigma \leq s - 1} \left\{ \sum_{m \in R_{\sigma}} \tilde{Z}_m^{(\sigma, H)} \leq 0 \right\} \right) \quad (C.1)\]

According to the "largest-exponent-wins" principle we have that \( \tilde{H}_{k, R}^{(s)} \) is bounded from above by the minimum of \( H_{k, R}^{(s)} \) (the rate of the first term on the right-hand side and the rate of second term on the right-hand side). The proof is complete when we can show that \( H_{k, R}^{(s)} \to 0 \) for \( k \to \infty \) and that the second rate is bounded from below by some \( \delta > 0 \) independent of \( k \).

The first fact follows from Theorem 5.5, even though we omitted the tilde. Concerning the rate of the second term on the right-hand side of (C.1), we have

\[\mathbb{P}\left( B_n \bigcup_{1 \leq \sigma' < \sigma \leq s - 1} \left\{ \sum_{m \in R_{\sigma'}} \tilde{Z}_m^{(\sigma, H)} \leq 0 \right\} \bigcup_{1 \leq \sigma \leq s - 1} \left\{ \sum_{m \in R_{\sigma}} \tilde{Z}_m^{(\sigma, H)} \leq 0 \right\} \right) \leq \sum_{1 \leq \sigma' < \sigma \leq s - 1} \mathbb{P}\left( \sum_{m \in R_{\sigma'}} \tilde{Z}_m^{(\sigma, H)} \leq 0 \right) + \sum_{1 \leq \sigma \leq s - 1} \mathbb{P}\left( \sum_{m \in R_{\sigma}} \tilde{Z}_m^{(\sigma, H)} \leq 0 \right).\]
Again we can apply the ‘largest exponent wins’ principle, so that it is sufficient to show that all rates corresponding to the probabilities on the right-hand side are larger than some \( \delta > 0 \). We distinguish four cases:

\[
(i) \quad 1 \leq \sigma' = \sigma - 1 \leq s - 2 : \quad \sum_{m \in R_{\sigma - 1}} \tilde{Z}_{m}^{(\sigma, \nu)} = \frac{1}{n} \sum_{i=1}^{n} 2 \left( \sum_{j \in R_{\sigma - 1}} X_{ji} \right)^{2} - |R_{\sigma - 1}|
\]

\[
(ii) \quad 1 \leq \sigma' < \sigma - 1 \leq s - 2 : \quad \sum_{m \in R_{\sigma'}} \tilde{Z}_{m}^{(\sigma, \nu)} = \frac{1}{n} \sum_{i=1}^{n} 2 \left( \sum_{j \in R_{\sigma'}} X_{ji} \right) \left( \sum_{j \in R_{\sigma - 1}} X_{ji} \right) + |R_{\sigma'}|
\]

\[
(iii) \quad \sigma = 1 : \quad \sum_{m \in R_{0}^{*}} \tilde{Z}_{m}^{(1)} = \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j \in R_{0}^{*}} X_{ji} \right)^{2} + \left( \sum_{j \in R_{0} \setminus R_{0}^{*}} X_{ji} \right) \left( \sum_{j \in R_{0}^{*}} X_{ji} \right)
\]

\[
(iv) \quad 2 \leq \sigma \leq s - 1 : \quad \sum_{m \in R_{0}^{*}} \tilde{Z}_{m}^{(\sigma, \nu)} = \frac{1}{n} \sum_{i=1}^{n} 2 \left( \sum_{j \in R_{0}^{*}} X_{ji} \right) \left( \sum_{j \in R_{\sigma - 1}} X_{ji} \right) + |R_{0}^{*}|
\]

Note that case (ii) and (iv) are essentially the same by replacing \( R_{\sigma'} \) with \( R_{0}^{*} \), so we have to deal with three cases only. In Van der Hofstad and Klok (2002OHD), we have proven the lemma using weak convergence arguments. However, in Chapter 7, we have proven bounds on moment generating functions that makes it easier to prove the lemma. Therefore, instead of using the weak convergence arguments, we have chosen to use the bounds on the moment generating functions.

The proposition that we intend to use, is Proposition 7.9(b). For completeness, the result of this proposition is repeated below. For the proof, we refer to the proof of Proposition 7.9(b).

Let \( S_{A} = \sum_{j \in A} X_{ji} \). For \( |t| \leq 1/|A| \),

\[
\mathbb{E} e^{tS_{A}/2} \leq \frac{1}{\sqrt{1 - |A| |t|}}. 
\]  \( \text{(C.2)} \)

We are now ready to deal with the four cases.

Case (i):

\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \sum_{m \in R_{\sigma - 1}} \tilde{Z}_{m}^{(\sigma, \nu)} \leq 0 \right)
\]

\[
= \sup_{t \leq 0} \left\{ - \log \mathbb{E} e^{2tS_{R_{\sigma - 1}} - t|R_{\sigma - 1}|} \right\} \geq \log \mathbb{E} e^{2tS_{R_{\sigma - 1}} - t|R_{\sigma - 1}|} \bigg|_{t = -1/(4|R_{\sigma - 1}|)}
\]

\[
= \frac{1}{4} - \log \mathbb{E} e^{2tS_{R_{\sigma - 1}} - t|R_{\sigma - 1}|} \geq \frac{1}{4} - \log \frac{1}{\sqrt{1 + 1}} = \frac{1}{2} \log 2 - \frac{1}{4} = 0.09657 \ldots > 0,
\]

by (C.2).
Case (ii) and (iv) (we use $|R_{\sigma'}| \geq |R_{\sigma-1}|$ in the last inequality):

$$- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \sum_{m \in R_{\sigma'}} \bar{Z}_m^{(u)} \leq 0 \right)$$

$$= \sup_{t \leq 0} \left\{ - \log \mathbb{E} e^{2tS_{R_{\sigma'}}^{2} |1+1/R_{\sigma'}|} \right\} \geq \sup_{t \leq 0} \left\{ - \log \mathbb{E} e^{2tS_{R_{\sigma'}}^{2} |1+1/R_{\sigma'}|} \right\}$$

$$\geq - \log \mathbb{E} e^{2tS_{R_{\sigma'}}^{2} |1+1/R_{\sigma'}|} = 1 + \frac{1}{4} \log \frac{3}{4} = 0.1062 \ldots > 0.$$

Case (iii):

$$- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \sum_{m \in R_{0,5}} \bar{Z}_m^{(i)} \leq 0 \right)$$

$$= \sup_{t \leq 0} \left\{ - \log \mathbb{E} e^{tS_{R_{0,5}}^{2} + tS_{R_{0,5}}^{2} |R_0/R_{0,5}|/2} \right\} \geq \sup_{t \leq 0} \left\{ - \log \mathbb{E} e^{tS_{R_{0,5}}^{2} + tS_{R_{0,5}}^{2} |R_0/R_{0,5}|/2} \right\}$$

$$\geq - \log \mathbb{E} e^{tS_{R_{0,5}}^{2} + tS_{R_{0,5}}^{2} |R_0/R_{0,5}|/2} = - \log \mathbb{E} e^{tS_{R_{0,5}}^{2} \left( \frac{1-|R_0/R_{0,5}|}{2|R_0/R_{0,5}|} \right)}.$$

We bound $|R_0/R_{0,5}|$ by 1, so that a similar derivation as in case (i) results in a lower bound for this rate of $\frac{1}{2} \log(1+1/2) = \frac{1}{2} \log \frac{3}{2} = 0.3466 \ldots > 0$.

We conclude that if $k$ is sufficiently large we have $\hat{H}_{k,l}^{(s)} = H_{k,l}^{(s)}$. 

**Appendix D: Proof of Lemma 5.10**

We repeat the inequalities (5.41) and (5.42):

$$|t_{1,\sigma}| \leq 3 \left( \frac{\mathcal{H}'}{|R_0/R_{\sigma}||R_{\sigma}|} \right)^{1/2}, \quad 1 \leq \sigma \leq s \quad \text{(D.1)}$$

$$|t_{\sigma,\sigma'}| \leq 3 \left( \frac{\mathcal{H}'}{|R_{\sigma}/|R_{\sigma-1}|} \right)^{1/2}, \quad 2 \leq \sigma \leq \sigma' \leq s. \quad \text{(D.2)}$$
We rearrange the terms of $e_2(t)$ to get

$$e_2(t) = \sum_{\sigma' = 2}^{s} t_{1,\sigma'} \left( \frac{3}{4} |R_{\sigma'}| + \frac{1}{2} t_{11} |R_1||R_{\sigma'}| + t_{\sigma'+1,\sigma'+1} |R_{\sigma'}||R_{\sigma'+1}| \right)$$

$$+ \sum_{\sigma' = 3}^{s} t_{2,\sigma'} \left( |R_{\sigma'}| + \frac{1}{2} t_{11} |R_1||R_{\sigma'}| \right) + \mathbb{E} Y_q Y_a$$

$$+ \frac{1}{8} \sum_{\sigma' = 2}^{s} |R_0 \setminus R_{\sigma'}||R_{\sigma'}|^2 t_{2,\sigma'}^2 + \sum_{2 \leq \sigma < s' \leq s} |R_{\sigma-1}||R_{\sigma'}| t_{\sigma,\sigma'}^2 + \mathbb{E} Y_a^3 / 6$$

$$+ \mathbb{E} Y_q (Y_a^2 / 2 + Y_a^3 / 6 + Y_a^4 e^{\mathbb{E} Y_a} / 24).$$

We will treat the terms on the different lines of (D.3) separately.

**Step a: First line.** This term is easy to treat. Indeed, for $t \in \mathcal{E} \cap D$ (recall (5.35)), it follows from (D.1) and (D.2) that

$$t_{1,\sigma'} |R_{\sigma'}| \left( \frac{3}{4} + \frac{1}{2} t_{11} |R_1| + t_{\sigma'+1,\sigma'+1} |R_{\sigma'+1}| \right)$$

$$\geq t_{1,\sigma'} |R_{\sigma'}| \left( \frac{3}{4} - \frac{3}{2} \mathcal{H}'^{1/2} \frac{|R_1|^{1/2}}{|R_0 \setminus R_1|^{1/2}} - 3 \mathcal{H}^{1/2} \frac{|R_{\sigma'+1}|}{|R_{\sigma'}|^{1/2} |R_{\sigma'+1}|^{1/2}} \right)$$

$$\geq t_{1,\sigma'} |R_{\sigma'}| \left( \frac{3}{4} - C \mathcal{H}' \right) \geq 0,$$

when $k$ is sufficiently large, since $t_{1,\sigma'} \geq 0$ for $\sigma' \geq 2$. This immediately implies that all terms on the first line are positive.

**Step b: Second line.** A similar derivation as in step a gives that

$$\sum_{\sigma' = 3}^{s} t_{2,\sigma'} \left( |R_{\sigma'}| + \frac{1}{2} t_{11} |R_1||R_{\sigma'}| \right) \geq 0.$$

Concerning $\mathbb{E} Y_q Y_a$ (recall the definitions of $Y_q$ and $Y_a$ in (5.36) and (5.37)), since $\mathbb{E} Y_a = 0$, the constant term $\sum_{2 \leq \sigma < s'} t_{\sigma,\sigma'} |R_{\sigma'}|$ in $Y_q$ gives no contribution. Furthermore, when we use (5.34) and

$$\mathbb{E} S_{R_0 \setminus R_1}^2 S_{R_0 \setminus R_1} S_{R_1} = 0, \quad 1 \leq \sigma \leq s,$$

$$\mathbb{E} S_{R_1}^2 S_{R_{\sigma-1}} S_{R_\sigma} = 0, \quad 2 \leq \sigma \leq s,$$

we arrive at

$$\mathbb{E} Y_q Y_a \geq \mathbb{E} \left\{ \left( \sum_{\sigma' = 2}^{s} t_{1,\sigma'} S_{R_0 \setminus R_1}^2 \right) \left( t_{11} S_{R_0 \setminus R_1} S_{R_1} + 2 \sum_{\sigma = 2}^{s} t_{\sigma,\sigma} S_{R_{\sigma-1}} S_{R_\sigma} \right) \right\}$$

$$+ \mathbb{E} \left\{ \left( t_{1,1} S_{R_1}^2 \right) \left( \sum_{\sigma = 2}^{s} t_{1,\sigma} S_{R_0 \setminus R_1} S_{R_\sigma} + 2 \sum_{2 \leq \sigma < s'} t_{\sigma,\sigma'} S_{R_{\sigma-1}} S_{R_{\sigma'}} \right) \right\} = 0.$$
Step c: Third line. As seen in (5.37), \( Y_a \) consists of a term with negative factors \( (t_{\sigma,\sigma}) \) and a term with positive factors \( (t_{\sigma,\sigma'}) \). Writing \( (x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \) and using (5.34),

\[
EY_a^3 \geq E \left( t_{11}S_{R_0 \setminus R_1}S_{R_1} + 2 \sum_{\sigma = 2}^{s} t_{\sigma,\sigma}S_{R_{\sigma-1}}S_{R_{\sigma}} \right)^3 \\
+ 3E \left( t_{11}S_{R_0 \setminus R_1}S_{R_1} + 2 \sum_{\sigma = 2}^{s} t_{\sigma,\sigma}S_{R_{\sigma-1}}S_{R_{\sigma}} \right) \left( \sum_{\sigma = 2}^{s} t_{1\sigma}S_{R_{\sigma}}S_{R_0 \setminus R_{\sigma}} + 2 \sum_{2 \leq \sigma < \sigma' \leq s} t_{\sigma,\sigma'}S_{R_{\sigma-1}}S_{R_{\sigma'}} \right)^2.
\]

Since

\[
E \left( t_{11}S_{R_0 \setminus R_1}S_{R_1} \right) = \begin{cases} 0, & \text{if } \sigma = 2, \\
2|R_1||R_{\sigma-1}||R_\sigma|, & 3 \leq \sigma \leq s, \\
|R_1||R_2||R_3|, & \sigma = 2, \sigma' = 3 \text{ or } \sigma = 3, \sigma' = 2, \\
0, & 4 \leq \sigma \leq s \text{ or } 4 \leq \sigma' \leq s, \\
0, & 2 \leq \sigma, \sigma', \sigma'' \leq s,
\end{cases}
\]

the first term on the right-hand side of (D.4) equals

\[
12t_{11}t_{22}t_{33}|R_1||R_2||R_3| + 12t_{11}^2 \sum_{\sigma = 3}^{s} t_{\sigma,\sigma}|R_1||R_{\sigma-1}||R_\sigma|.
\]

We use (D.1) and (D.2) to obtain that the expression in (D.5) is \( O(H^2) \).

For the second term of (D.4) we use Cauchy-Schwarz:

\[
\left| E \left( \left( t_{11}S_{R_0 \setminus R_1}S_{R_1} + 2 \sum_{\sigma = 2}^{s} t_{\sigma,\sigma}S_{R_{\sigma-1}}S_{R_{\sigma}} \right) \left( \sum_{\sigma = 2}^{s} t_{1\sigma}S_{R_{\sigma}}S_{R_0 \setminus R_{\sigma}} + 2 \sum_{2 \leq \sigma < \sigma' \leq s} t_{\sigma,\sigma'}S_{R_{\sigma-1}}S_{R_{\sigma'}} \right)^2 \right) \right|
\leq \left( E \left( t_{11}S_{R_0 \setminus R_1}S_{R_1} + 2 \sum_{\sigma = 2}^{s} t_{\sigma,\sigma}S_{R_{\sigma-1}}S_{R_{\sigma}} \right)^2 \right)^{1/2}
\times \left( E \left( \sum_{\sigma = 2}^{s} t_{1\sigma}S_{R_{\sigma}}S_{R_0 \setminus R_{\sigma}} + 2 \sum_{2 \leq \sigma < \sigma' \leq s} t_{\sigma,\sigma'}S_{R_{\sigma-1}}S_{R_{\sigma'}} \right)^4 \right)^{1/2}.
\]

Using (D.1), (D.2) and \((x + y)^2 \leq 2(x^2 + y^2)\), it is straightforward to show that

\[
\left( \left( t_{11}S_{R_0 \setminus R_1}S_{R_1} + 2 \sum_{\sigma = 2}^{s} t_{\sigma,\sigma}S_{R_{\sigma-1}}S_{R_{\sigma}} \right)^2 \right)^{1/2}
\leq \left( 2E \left( t_{11}^2S_{R_0 \setminus R_1}^2S_{R_1}^2 + 4 \sum_{\sigma = 2}^{s} t_{\sigma,\sigma}^2S_{R_{\sigma-1}}^2S_{R_{\sigma}}^2 \right) \right)^{1/2}
\leq 2^{1/2} \left( t_{11}^2|R_0 \setminus R_1||R_1| + 4 \sum_{\sigma = 2}^{s} t_{\sigma,\sigma}^2|R_{\sigma-1}||R_\sigma| \right)^{1/2} \leq C\mathcal{H}^{1/2}.
\]

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and, using \((x + y)^4 \leq 8(x^4 + y^4)\) and \((x^2 + y^2)^{1/2} \leq (|x| + |y|)\) for \(x, y \in \mathbb{R}\),

\[
\left(\mathbb{E} \left( \sum_{\sigma = 1}^{s} t_{1, \sigma} R_{\sigma} S_{R_0 \setminus R_{\sigma}} + 2 \sum_{2 \leq \sigma < \sigma' \leq s} t_{\sigma, \sigma'} S_{R_{\sigma - 1} \setminus R_{\sigma'}} \right) \right)^{1/2} \\
\leq \left( 8 \left( \sum_{\sigma = 2}^{s} t_{1, \sigma}^4 |R_{\sigma}|^2 |R_0 \setminus R_{\sigma}|^2 + 16 \sum_{2 \leq \sigma < \sigma' \leq s} t_{\sigma, \sigma'}^4 |R_{\sigma - 1}|^2 |R_{\sigma'}|^2 \right) \right)^{1/2} \\
\leq C \left( \sum_{\sigma = 2}^{s} t_{\sigma}^2 |R_{\sigma}||R_0 \setminus R_{\sigma}| + 4 \sum_{2 \leq \sigma < \sigma' \leq s} t_{\sigma, \sigma'}^2 |R_{\sigma - 1}||R_{\sigma'}| \right),
\]

so that for the third line of (D.3) we obtain

\[
\frac{1}{8} \sum_{\sigma' = 2}^{s} |R_0 \setminus R_{\sigma'}||R_{\sigma'}|^2 \|t_{1, \sigma'}^2 + \sum_{2 \leq \sigma < \sigma' \leq s} |R_{\sigma - 1}||R_{\sigma'}|^2 t_{\sigma, \sigma'}^2 + \mathbb{E} Y_a^4 / 6 \\
\geq \left( \frac{1}{8} - C\mathcal{H}^{1/2} \right) \sum_{\sigma = 2}^{s} |R_0 \setminus R_{\sigma}| |R_{\sigma}| t_{1, \sigma}^2 + (1 - C\mathcal{H}^{1/2}) \sum_{2 \leq \sigma < \sigma' \leq s} |R_{\sigma - 1}||R_{\sigma'}|^2 t_{\sigma, \sigma'}^2 - C\mathcal{H}^2 \geq 0,
\]

when \(k\) is sufficiently large.

**Step d: Fourth line.** Concerning \(\mathbb{E} Y_q(Y_a^2/2 + Y_a^3/6 + Y_a^4 e^{Y_a}/24)\), clearly

\[
|\mathbb{E} Y_q Y_a^2| \leq (\mathbb{E} Y_q^4)^{1/4}(\mathbb{E} Y_a^6)^{1/6}, \\
|\mathbb{E} Y_q Y_a^3| \leq (\mathbb{E} Y_q^4)^{1/4}(\mathbb{E} Y_a^6)^{1/2} \\
|\mathbb{E} Y_q Y_a^4 e^{Y_a}| \leq (\mathbb{E} Y_q^4)^{1/4}(\mathbb{E} Y_a^6)^{2/3}(e^{12|Y_a|})^{1/12}.
\]

Thus, in order to prove that \(\mathbb{E} Y_q(Y_a^2/2 + Y_a^3/6 + \mathbb{E} Y_q Y_a^4 e^{Y_a}) = O(\mathcal{H}^2)\) for \(q \in \mathcal{E}\), it is sufficient to prove that \(\mathbb{E} Y_q^4 \leq C\mathcal{H}^4\), \(\mathbb{E} Y_a^6 \leq C\mathcal{H}^3\) and \(\mathbb{E} e^{12|Y_a|}\) is bounded. We show that now, using (D.1) and (D.2).

\[
\mathbb{E} Y_q^4 \leq C \sum_{\sigma = 1}^{s} t_{1, \sigma}^4 |R_{\sigma}|^4 + \sum_{2 \leq \sigma < \sigma' \leq s} t_{\sigma, \sigma'}^4 |R_{\sigma'}|^4 \\
\leq C \sum_{\sigma = 1}^{s} \mathcal{H}^2 |R_{\sigma - 1}|^2 |R_{\sigma}|^4 + \sum_{2 \leq \sigma < \sigma' \leq s} \mathcal{H}^2 |R_{\sigma - 1}|^2 |R_{\sigma'}|^2 |R_{\sigma'}|^4 \\
= C \sum_{\sigma = 1}^{s} \mathcal{H}^2 |R_{\sigma}|^2 |R_{\sigma - 1}|^2 + \sum_{2 \leq \sigma < \sigma' \leq s} \mathcal{H}^2 |R_{\sigma - 1}|^2 |R_{\sigma'}|^2 \leq C\mathcal{H}^2.
\]

Using similar techniques, we see that \(\mathbb{E} Y_a^6 \leq C\mathcal{H}^3\). Finally, \(\mathbb{E} e^{12|Y_a|}\) can be bounded, using similar techniques as in the lower bound, together with (D.1) and (D.2). We conclude that indeed \(|\mathbb{E} Y_q(Y_a^2/2 + Y_a^3/6 + Y_a e^{Y_a})| \leq C\mathcal{H}^2\) for all \(q \in \mathcal{E}\).

The four steps together imply that \(c_2(t) \geq -C\mathcal{H}^2\), which is the statement of the lemma. ■
Appendix E: Proof of Lemma 6.11

Recall (6.24),

\[
e_1(t) = -\frac{P_R}{2} \sum_{m \in R} t_{1m}^2 + \frac{P + \sigma^2}{8} \sum_{m \in R^*} t_{1m}^2 + \mathbb{E} Y_a + \frac{1}{2} \left( \sum_{m \in R} P_{m}^{1/2} t_{1m} \right) \left( \sum_{m \in R} P_{m}^{1/2} t_{1m} \right) + \mathbb{E} Y_q Y_a
\]

\[
+ \frac{P + \sigma^2}{8} \sum_{m \in R^*} t_{1m}^2 + \mathbb{E} Y_a^3 / 6
\]

\[
+ \mathbb{E} Y_q (Y_a^2 / 2 + Y_a^3 / 6 + Y_a^4 e^{\gamma Y_a} / 24),
\]

and (6.25)

\[
|t_{20}| \leq C \sqrt{\frac{\mathcal{H}}{4 P_R + \sigma^2}}, \quad \sum_{m \in R \cup R^*} t_{1m}^2 \leq C \frac{\mathcal{H}}{P + \sigma^2}.
\]

The proof is split in 4 steps. In every step one line of the right-hand side of (E.1) is treated and is proved to be $\geq 0$, $\mathcal{O}(\mathcal{H}^2)$ or a combination of those two.

**Step a: First line.** This term is easily bounded as

\[
-\frac{P_R}{2} \sum_{m \in R} t_{1m}^2 \leq C \frac{P_R}{P + \sigma^2} \mathcal{H} \leq C \mathcal{H}^2, \quad t \in \mathcal{E}
\]

by (E.2) and $P_R/(P + \sigma^2) \leq 2 \mathcal{H}$.

**Step b: Second line.** Since

\[
\mathbb{E} Y_a = \sum_{m \in R^*} P_m^{1/2} t_{1m},
\]

and from (6.21)

\[
\mathbb{E} Y_q Y_a = t_{20} P_0^{1/2} \mathbb{E} Y_a + \mathbb{E} T_R \tilde{S}_R Y_a + \frac{\sigma^2}{2} \mathbb{E} (t_{20} + T_R + T_{R^*})^2 Y_a,
\]

it is straightforward that

\[
t_{20} P_0^{1/2} \mathbb{E} Y_a + \mathbb{E} T_R \tilde{S}_R Y_a = t_{20} P_0^{1/2} \left( \sum_{m \in R^*} P_m^{1/2} t_{1m} \right) + \left( \sum_{m \in R} P_m^{1/2} t_{1m} \right) \left( \sum_{m \in R^*} P_m^{1/2} t_{1m} \right)
\]

and

\[
\frac{\sigma^2}{2} \mathbb{E} (t_{20} + T_R + T_{R^*})^2 Y_a = \frac{\sigma^2}{2} \mathbb{E} (t_{20} + T_R)^2 (2 t_{20} \tilde{S}_R + \tilde{S}_R T_R + \tilde{S}_R T_{R^*})
\]

\[
+ \frac{\sigma^2}{2} \mathbb{E} (t_{20} + T_R) T_{R^*} (2 t_{20} \tilde{S}_R + \tilde{S}_R T_R + \tilde{S}_R T_{R^*})
\]

\[
+ \frac{\sigma^2}{2} \mathbb{E} T_{R^*}^2 (2 t_{20} \tilde{S}_R + \tilde{S}_R T_R + \tilde{S}_R T_{R^*}).
\]

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Using $t \in D$ and (6.20), it follows that $\mathbb{E} (t_{20} + T_R)^2 \tilde{S}_R T_{R^*} \geq 0$. Similarly, other terms with such combinations of $t$’s can be shown to be $\geq 0$. This leads to

$$\frac{\sigma^2}{2} \mathbb{E} (t_{20} + T_R + T_{R^*})^2 Y_a$$

(E.6)

$$\geq \frac{\sigma^2}{2} \mathbb{E} (t_{20} + T_R)^2 (2t_{20} \tilde{S}_R + \tilde{S}_R T_R) + \sigma^2 \mathbb{E} (t_{20} + T_R) T_{R^*} \tilde{S}_R T_{R^*}$$

$$+ \frac{\sigma^2}{2} \mathbb{E} T_R^2 (2t_{20} \tilde{S}_R + \tilde{S}_R T_R)$$

$$= 2\sigma^2 t_{20}^2 \sum_{m \in R} P_{m/n}^{1/2} t_{1m} + \sigma^2 t_{20} P_0^{1/2} \sum_{m \in R} t_{1m}^2 + \sigma^2 t_{20} P_0^{1/2} \sum_{m \in R^*} t_{1m}^2$$

$$+ \frac{\sigma^2}{2} \left( \sum_{m \in R} P_{m/n}^{1/2} t_{1m} \right) \left( \sum_{m \in R^*} t_{1m}^2 \right).$$

The first two terms in the line above are of order $\mathcal{H}^2$. Indeed, by (E.2), and $|\langle x, y \rangle| \leq \|x\| \|y\|$, we have

$$\left| 2\sigma^2 t_{20}^2 \sum_{m \in R} P_{m/n}^{1/2} t_{1m} + \sigma^2 t_{20} P_0^{1/2} \sum_{m \in R} t_{1m}^2 \right| \leq \left| 2\sigma^2 t_{20}^2 \left( P_R \sum_{m \in R} t_{1m}^2 \right)^{1/2} + \sigma^2 t_{20} P_0^{1/2} \sum_{m \in R} t_{1m}^2 \right|$$

$$\leq C \left\{ \frac{\mathcal{H}}{4 P_R + \sigma^2} P_R \frac{\mathcal{H}}{P + \sigma^2} \right\}^{1/2} + \sigma^2 \left( \frac{\mathcal{H}^{1/2}}{(4 P_R + \sigma^2)^{1/2}} P_0^{1/2} \frac{\mathcal{H}}{P + \sigma^2} \right) \leq C \mathcal{H}^2,$$

since $\sigma^2/(4 P_R + \sigma^2)$ and $\sigma^2/(P + \sigma^2)$ are bounded by 1. Thus, according to (E.3), (E.4), (E.5) and (E.6), $\mathbb{E} Y_a + \mathbb{E} Y_q Y_a$ is bounded from above by

$$\left( \sum_{m \in R^*} t_{1m}^2 \right) \left( \sigma^2 t_{20} P_0^{1/2} + \frac{\sigma^2}{2} \sum_{m \in R} t_{1m}^2 \right) + \left( \sum_{m \in R^*} P_{m/n}^{1/2} t_{1m} \right) \left( 1 + t_{20} P_0^{1/2} + \sum_{m \in R} P_{m/n}^{1/2} t_{1m} \right) - C \mathcal{H}^2.$$

When $\mathcal{H}$ is sufficiently small, we now arrive at

$$\frac{P + \sigma^2}{8} \sum_{m \in R^*} t_{1m}^2 + \left( \sum_{m \in R^*} P_{m/n}^{1/2} t_{1m} \right) \left( \sum_{m \in R} P_{m/n}^{1/2} t_{1m} \right) + \mathbb{E} Y_a + \mathbb{E} Y_q Y_a$$

$$\geq (P + \sigma^2) \left( \sum_{m \in R^*} t_{1m}^2 \right) \left( \frac{1}{8} + \frac{\sigma^2 t_{20} P_0^{1/2}}{P + \sigma^2} + \frac{\sigma^2}{2(P + \sigma^2)} \sum_{m \in R} P_{m/n}^{1/2} t_{1m} \right)$$

$$+ \left( \sum_{m \in R^*} P_{m/n}^{1/2} t_{1m} \right) \left( 1 + t_{20} P_0^{1/2} + 2 \sum_{m \in R} P_{m/n}^{1/2} t_{1m} \right) - C \mathcal{H}^2$$

$$\geq (P + \sigma^2) \left( \sum_{m \in R^*} t_{1m}^2 \right) \left( \frac{1}{8} - C \frac{\sigma^2}{P + \sigma^2} \mathcal{H}^{1/2} P_0^{1/2} \frac{\mathcal{H}}{(4 P_R + \sigma^2)^{1/2}} - \frac{\sigma^2}{2(P + \sigma^2)} \left( P_R \sum_{m \in R} t_{1m}^2 \right)^{1/2} \right)$$

$$+ \left( \sum_{m \in R^*} P_{m/n}^{1/2} t_{1m} \right) \left( 1 - C \mathcal{H} - 2 \left( P_R \sum_{m \in R} t_{1m}^2 \right)^{1/2} \right) - C \mathcal{H}^2$$

$$\geq (P + \sigma^2) \left( \sum_{m \in R^*} t_{1m}^2 \right) \left( \frac{1}{8} - C \mathcal{H} \right) + \left( \sum_{m \in R^*} P_{m/n}^{1/2} t_{1m} \right) \left( 1 - C \mathcal{H} \right) - C \mathcal{H}^2 \geq -C \mathcal{H}^2,$$

where we have used again $|\langle x, y \rangle| \leq \|x\| \|y\|$ and $t_{1m} \geq 0$ for $m \in R^*$.  

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Step c: Third line. We use (6.20), (6.21) and the fact that \( t \in D \) to obtain
\[
\mathbb{E} Y^3_a \geq \mathbb{E} \left( 2t_{20}\tilde{S}_R + \tilde{S}_{R signalling T_R} \right)^3 + 3\mathbb{E} \left( 2t_{20}\tilde{S}_R + \tilde{S}_{R signalling T_R} \right) \tilde{S}^2_{R_0} T^2_{R*}.
\]
because \( \mathbb{E} (2t_{20}\tilde{S}_R + \tilde{S}_{R signalling T_R})^2 \tilde{S}^2_{R_0} T^2_{R*} \geq 0 \) and \( \mathbb{E} \tilde{S}^2_{R_0} T^2_{R*} \geq 0 \). By symmetry arguments, the first term on the right-hand side above equals 0. When \( \mathcal{H} \) is sufficiently small, the second term, together with the remaining term on the third line of (E.1), is bounded using Cauchy-Schwarz, yielding
\[
\frac{P + \sigma^2}{8} \sum_{m \in R*} t^2_{1m} + \frac{1}{2} \mathbb{E} \left( 2t_{20}\tilde{S}_R + \tilde{S}_{R signalling T_R} \right) \tilde{S}^2_{R_0} T^2_{R*},
\]
\[
\geq \frac{P + \sigma^2}{8} \sum_{m \in R*} t^2_{1m} - \frac{1}{2} \left( \mathbb{E} (2t_{20}\tilde{S}_R + \tilde{S}_{R signalling T_R})^2 \right)^{1/2} \left( \mathbb{E} \tilde{S}^2_{R_0} \right)^{1/4} \left( \mathbb{E} T^2_{R*} \right)^{1/4},
\]
\[
\geq \frac{P + \sigma^2}{8} \sum_{m \in R*} t^2_{1m} - C \left( 4P t_{20} + P_{R*} \sum_{m \in R} t^2_{1m} + 4t_{20} P_{R*}^{1/2} P_{R*}^{1/2} \right) \left( \sum_{m \in R} t^2_{1m} \right)^{1/2} \left( \sum_{m \in R*} t^2_{1m} \right)^{1/2}
\]
\[
\geq \frac{P + \sigma^2}{8} \sum_{m \in R*} t^2_{1m} - C \mathcal{H}^{1/2} P \sum_{m \in R*} t^2_{1m} \geq 0
\]
where we have used (E.2).

Step d: Fourth line. Finally, whenever we can prove \( \mathbb{E} Y^2_q \leq C \mathcal{H}^2 \), \( \mathbb{E} Y^6_a \leq C \mathcal{H}^3 \) and \( \mathbb{E} e^{12 \gamma a} \) is bounded, it follows from Hölder’s inequality that \( \mathbb{E} Y^2_q Y^2_a = \mathcal{O}(\mathcal{H}^2) \), \( \mathbb{E} Y^6_a Y^4_a = \mathcal{O}(\mathcal{H}^2) \) and \( \mathbb{E} Y^4_a Y^4_a = \mathcal{O}(\mathcal{H}^2) \). Recall (6.21) and use \( (x + y)^l \leq 2^{l-1}(x^l + y^l) \) for \( l = 2, 4 \) to obtain
\[
\mathbb{E} Y^2_a \leq C P_0 t^2_{20} + C \mathbb{E} \tilde{S}^2_{R} T^2_{R*} + C \sigma^4 \left( t^4_{20} + \mathbb{E} T^4_{R*} + \mathbb{E} T^4_{R*} \right)
\]
\[
\leq C P_0 t^2_{20} + C \left( \mathbb{E} \tilde{S}^4_{R} + \mathbb{E} T^4_{R*} \right)^{1/2} + C \sigma^4 \left( \mathbb{E} T^4_{R*} + \mathbb{E} T^4_{R*} \right)^{1/2}
\]
\[
\leq C P_0 t^2_{20} + C \left( \mathbb{E} \tilde{S}^2_{R} \right)^{1/2} + C \sigma^4 \left( \mathbb{E} T^2_{R*} + \mathbb{E} T^2_{R*} \right)^{1/2} + C \sigma^4 \left( \mathbb{E} T^2_{R*} + \mathbb{E} T^2_{R*} \right)^{1/2}
\]
\[
\leq C P_0 \frac{\mathcal{H}}{4 P_{R*} + \sigma^2} + C P_{R*} \frac{\mathcal{H}}{P + \sigma^2} + C \sigma^4 \left( \frac{\mathcal{H}^2}{(4 P_{R*} + \sigma^2)^2} + \frac{\mathcal{H}^2}{(P + \sigma^2)^2} + \frac{\mathcal{H}^2}{(P + \sigma^2)^2} \right)
\]
\[
= C \mathcal{H} \frac{P_0}{4 P_{R*} + \sigma^2} + C \mathcal{H} \frac{P_{R*}}{P + \sigma^2} + C \mathcal{H}^2 \left( \frac{\sigma^4}{(4 P_{R*} + \sigma^2)^2} + \frac{\sigma^4}{(P + \sigma^2)^2} \right) \leq C \mathcal{H}^2.
\]
Similarly, we bound \( \mathbb{E} Y^6_a \), resulting in \( \mathbb{E} Y^6_a \leq C \mathcal{H}^3 \). To bound \( \mathbb{E} e^{12 \gamma a} \), we use
\[
\mathbb{E} e^{12 \gamma a} \leq \mathbb{E} e^{12 \gamma a} + \mathbb{E} e^{-12 \gamma a}
\]
and apply Lemma 6.10, together with the bounds (E.2).
Appendix F: Proof of Theorem 7.2(b)

The powers fulfill the following conditions:

\((P_1)\) There exists a \(\delta > 0\) such that \(\text{card}(\{j : P_j \in [\delta, 1/\delta]\}) \to \infty\),

\((P_2)\) \(\lim_{k \to \infty} kP^{-1} < \infty\),

\((P_3)\) \(k^{-1}P^\text{max} \to 0\),

\((P_4)\) \(kP^\text{min} \to \infty\).

Theorem 7.2(a) states that \(H_k^{(g_k)} \geq \frac{1}{2} \log 2 - \frac{1}{4}\), so that it is sufficient to prove

\[
\lim_{k \to \infty} H_k^{(g_k)} \leq \frac{1}{2} \log 2 - \frac{1}{4}.
\]

Similar to the case in which all powers are equal, we will focus on the event \(\{R_2 = R_1 = R\}\). This specifies \(R_\sigma\) for all \(\sigma\). We will show the theorem, using a symmetrizing argument. More precisely, we replace different powers by the same value. In this case, we can use exchangeability, together with convexity arguments in order to simplify the analysis.

We take \(R \subset \{j : P_j \in [\delta, 1/\delta]\}\) and we assume that \(|R|\) is fixed and odd and \(k\) is large.

Our starting point is the probability

\[
P(R_2 = R_1 = R) = P\left(\max_{m \in R} Z_m^{(1)} \leq 0, \min_{m \in R} Z_m^{(1)} \geq 0, \max_{m \in R} \tilde{Z}_m^{(2,H)} \leq 0, \min_{m \in R} \tilde{Z}_m^{(2,H)} \geq 0\right),
\]

where

\[
Z_m^{(1)} = \frac{1}{n} \sum_{i=1}^{n} \left( P_m^{1/2} X_{j_i} X_{m_i} \right), \quad \tilde{Z}_m^{(2,H)} = \frac{1}{n} \sum_{i=1}^{n} \left( P_m^{1/2} + \sum_{j \in R} P_j^{1/2} X_{j_i} X_{m_i} \right).
\]

Since \(P(X \leq 0, Y \leq 0) \geq P(X - Y/2 \leq 0, Y \leq 0)\), we can bound the probability from below by

\[
P\left(\max_{m \in R} Z_m^{(1)} - \frac{1}{2} \tilde{Z}_m^{(2,H)} \leq 0, \min_{m \in R} Z_m^{(1)} - \frac{1}{2} \tilde{Z}_m^{(2,H)} \geq 0, \max_{m \in R} \tilde{Z}_m^{(2,H)} \leq 0, \min_{m \in R} \tilde{Z}_m^{(2,H)} \geq 0\right),
\]

where for \(m \in R\),

\[
Z_m^{(1)} - \frac{1}{2} \tilde{Z}_m^{(2,H)} = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{2} P_m^{1/2} + \sum_{j \in R} P_j^{1/2} X_{j_i} X_{m_i} \right),
\]

\[
\tilde{Z}_m^{(2,H)} = \frac{1}{n} \sum_{i=1}^{n} \left( P_m^{1/2} + 2 \sum_{j \in R} P_j^{1/2} X_{j_i} X_{m_i} \right),
\]

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while for \( m \in R^c \)

\[
Z_m^{(1)} - \frac{1}{2} \tilde{Z}_m^{(2,H)} = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{2} P_m^{1/2} + \sum_{j \in R^c \atop j \neq m} P_j^{1/2} X_{ji} X_{mi} \right)
\]

\[
\tilde{Z}_m^{(2,H)} = \frac{1}{n} \sum_{i=1}^{n} \left( P_m^{1/2} + 2 \sum_{j \in R} P_j^{1/2} X_{ji} X_{mi} \right).
\]

We can further bound the probability of interest from below by the probability of the event

\[
\left\{ \max_{m \in R, j \in R^c} \frac{1}{n} \sum_{i=1}^{n} X_{mi} X_{ji} \leq -\frac{(P_{max}^{R^c} P_{max}^{R})^{1/2}}{2P_{R^c}}, \min_{m, j \in R^c \atop j \neq m} \frac{1}{n} \sum_{i=1}^{n} X_{mi} X_{ji} \geq 0, \right\}
\]

\[
\max_{j \in R^c \atop j \neq m} \frac{1}{n} \sum_{i=1}^{n} X_{mi} X_{ji} \leq -\frac{(P_{max}^{R^c})^{1/2}}{2(|R| - 1)(P_{min}^{R})^{1/2}}, \min_{m \in R, j \in R^c} \frac{1}{n} \sum_{i=1}^{n} X_{mi} X_{ji} \geq -\frac{(P_{max}^{R})^{1/2}}{2|R|(P_{min}^{R})^{1/2}} \right\}.
\]

Indeed, using \( \sum_{m \in R^c} P_m^{1/2} \geq \frac{P_{R^c}}{(P_{max}^{R^c})^{1/2}} \)

\[
Z_m^{(1)} - \frac{1}{2} \tilde{Z}_m^{(2,H)} = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{2} P_m^{1/2} + \sum_{j \in R^c} P_j^{1/2} X_{ji} X_{mi} \right)
\]

\[
\leq \frac{1}{2} (P_{max}^{R})^{1/2} - 2 \sum_{j \in R^c \atop j \neq m} \left( P_j^{1/2} \frac{P_{max}^{R^c} P_{max}^{R}}{2P_{R^c}} \right) \leq \frac{1}{2} (P_{max}^{R})^{1/2} - \frac{1}{2} (P_{max}^{R})^{1/2} = 0.
\]

Furthermore,

\[
\tilde{Z}_m^{(2,H)} = \frac{1}{n} \sum_{i=1}^{n} \left( P_m^{1/2} + 2 \sum_{j \in R^c \atop j \neq m} P_j^{1/2} X_{ji} X_{mi} \right)
\]

\[
\leq (P_{max}^{R})^{1/2} - 2 \sum_{j \in R^c \atop j \neq m} (P_{min}^{R})^{1/2} \frac{(P_{max}^{R})^{1/2}}{2(|R| - 1)(P_{min}^{R})^{1/2}} \leq (P_{max}^{R})^{1/2} - \frac{(P_{max}^{R})^{1/2}}{(P_{min}^{R})^{1/2}} = 0.
\]

Similarly, for \( m \in R^c \), the event in (F.3) implies

\[
Z_m^{(1)} - \frac{1}{2} \tilde{Z}_m^{(2,H)} = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{2} P_m^{1/2} + \sum_{j \in R^c} P_j^{1/2} X_{ji} X_{mi} \right) \geq \frac{1}{2} P_m^{1/2} \geq 0,
\]

\[
\tilde{Z}_m^{(2,H)} = \frac{1}{n} \sum_{i=1}^{n} \left( P_m^{1/2} + 2 \sum_{j \in R} P_j^{1/2} X_{ji} X_{mi} \right)
\]

\[
\geq (P_{min}^{R})^{1/2} - 2 \sum_{j \in R} (P_{max}^{R})^{1/2} \frac{(P_{min}^{R})^{1/2}}{2|R|(P_{max}^{R})^{1/2}} = 0.
\]

In order to assure that the event in (F.3) is not empty, we must assure that

\[
- \frac{(P_{max}^{R})^{1/2}}{2P_{R^c}} \geq - \frac{(P_{min}^{R})^{1/2}}{2|R|(P_{max}^{R})^{1/2}} \iff \frac{(P_{max}^{R})^{2} P_{max}^{R}}{P_{min}^{R} P_{R^c}^2} |R|^2 \leq 1.
\]

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Since $P_R^{\max} \leq P_{Rc}^{\max}, P_{Rc}^{\min} \geq P_{Rc}^{\min}$ and $(\mathcal{P}_4)$ together with our choice of $R$ implies $P_R^{\max} \leq 1/\delta$, it follows from $(\mathcal{P}_2), (\mathcal{P}_3)$ and $(\mathcal{P}_4)$ that

$$\frac{(P_R^{\max})^2 P_{Rc}^{\max} |R|^2}{P_{Rc}^{\min} P_{Rc}^2} \leq \delta^{-2} |R|^2 \frac{P_{Rc}^{\max}}{P_{Rc}^2 P_{Rc}^{\min}} = \delta^{-2} |R|^2 \frac{k^2}{k} \frac{P_{Rc}^{\max}}{k P_{Rc}^{\min}} \to 0,$$

for $k \to \infty$, so that the inequality holds, since $|R|$ is fixed and $k$ is sufficiently large.

We have now symmetrized the problem. The next step is to observe that the events are exchangeable for $m, j$. Together with convexity of the rate function, we can replace the max and min by $\sum$, as we will now show. This is a standard large deviation argument.

We write $\vec{a} \mapsto J_k(\vec{a})$ to be the rate function of the cross correlations $\vec{W} = (W_{jm})_{1 \leq j < m \leq k}$, where

$$W_{jm} = \frac{1}{n} \sum_{i=1}^{n} A_{ij} A_{im}. \quad (F.4)$$

This means that, for any set $A$, we let

$$J_k(A) = - \lim_{n \to \infty} \frac{1}{n} \log P(\vec{W} \in A), \quad (F.5)$$

and we can compute $J_k(A)$ using

$$J_k(A) = \min_{\vec{a} \in A} J_k(\vec{a}). \quad (F.6)$$

This rate function $\vec{a} \mapsto J_k(\vec{a})$ exists by Cramér’s theorem, is convex by Den Hollander (2000), Thm. III.27, and is minimal for $\vec{a} = \vec{0}$, where it takes the value $J_k(\vec{0}) = 0$. We note that

$$H_k^{(s_k)}(\vec{a}) \leq \min_{\vec{a} \in A} J_k(\vec{a}), \quad (F.7)$$

where $A$ is defined to be the set of vectors such that the coordinates satisfy the inequalities in $(F.3)$. We now let

$$E_1 = \left\{ \sum_{m \in R, j \in Rc} \frac{1}{n} \sum_{i=1}^{n} X_{mi} X_{ji} \leq -|R||Rc| \frac{(P_{Rc}^{\max})^{1/2}}{2 P_{Rc}^{\max}} \right\},$$

$$E_2 = \left\{ \sum_{m,j \in R, m \neq j} \frac{1}{n} \sum_{i=1}^{n} X_{mi} X_{ji} \geq 0 \right\}, \quad (F.8)$$

$$E_3 = \left\{ \sum_{m,j \in R, m \neq j} \frac{1}{n} \sum_{i=1}^{n} X_{mi} X_{ji} \leq -|R| \frac{(P_{Rc}^{\max})^{1/2}}{2 P_{Rc}^{\min}(P_{Rc}^{\max})^{1/2}} \right\},$$

$$E_4 = \left\{ \sum_{m \in R, j \in Rc} \frac{1}{n} \sum_{i=1}^{n} X_{mi} X_{ji} \geq -|R||Rc| \frac{(P_{Rc}^{\min})^{1/2}}{2 |R||Rc|(P_{Rc}^{\max})^{1/2}} \right\}.$$

Then we see that

$$- \lim_{n \to \infty} \frac{1}{n} \log P(E_1 \cap E_2 \cap E_3 \cap E_4) = \min_{\vec{b} \in B} J_k(\vec{b}). \quad (F.9)$$

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where $B$ is the set such that the inequalities in (F.8) hold. Note that $A \subseteq B$, and we wish to show that the minimum in (F.9) is attained for a $\tilde{b}^* \in A$. That proves that the rate on the r.h.s. of (F.7) equals the rate of $\mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4)$.

Let $\tilde{b}^*$ be the minimizer of (F.9). Note that by exchangeability, if we interchange the elements in $R$ and in $R^c$, we obtain the same rate. We write $\Pi_R$ for the set of permutations of the elements in $R$ and $\Pi_{R^c}$ the set of permutations of the elements in $R^c$, and we define $\Pi = \Pi_R \times \Pi_{R^c}$. Hence, for a $\pi \in \Pi$, we have $\pi = (\pi_1, \pi_2)$ where $\pi_1 \in \Pi_R, \pi_2 \in \Pi_{R^c}$.

Then we obtain that
\[
\min_{\tilde{b} \in B} J_k(\tilde{b}) = \frac{1}{|\Pi|} \sum_{\pi \in \Pi} J_k(\pi(\tilde{b}^*)).
\]  
(F.10)

By convexity, we obtain that
\[
\min_{\tilde{b} \in B} J_k(\tilde{b}) \geq J_k\left(\frac{1}{|\Pi|} \sum_{\pi \in \Pi} \pi(\tilde{b}^*)\right).
\]  
(F.11)

We have ended up with the vector $\tilde{\sigma}^* = \frac{1}{|\Pi|} \sum_{\pi \in \Pi} \pi(\tilde{b}^*)$. This vector takes only three different values, as we will explain now. Indeed, let $j \neq j'$ be any two indices in $R$, and $m \neq m'$ any two indices in $R^c$. Then $\sigma_{nm}^*$ takes the value $a_{j}^* l$ for $n, l \in R$, the value $a_{jm}^*$ for $n \in R, l \in R^c$ and $a_{nm}^*$ for any $n, l \in R^c$. Hence, we must have that each of the elements satisfies the inequalities in (F.3), so that $\tilde{\sigma}^* \in A$. This completes the proof.

We next write
\[\mathbb{P}(E_1 \cap E_2 \cap E_3) = \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4) + \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4^c).\]

By the largest-exponent-wins principle, we have
\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_2 \cap E_3)
= \min \left\{ - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4), - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4^c) \right\}.
\]

We will show that
\[
\lim \inf_{k \to \infty} - \frac{1}{n} \log \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4) \geq \frac{1}{2} \log \frac{3}{4} + \frac{1}{4} = 0.106158 \ldots
\]  
(F.12)

for all $R$, when $k$ is sufficiently large. Furthermore, we will show that
\[
- \lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_2 \cap E_3) \leq - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_3)
\]  
(F.13)

and
\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_3) \leq \frac{1}{2} \log 2 - \frac{1}{4} + o(1) = 0.096573 \ldots + o(1),
\]  
(F.14)

for $|R|$ sufficiently large. Because the three statements $u = \min(v, w), u \leq x$ and $w > x$ imply $u \leq x$, this implies directly when $|R|$ is large
\[
- \lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4) \leq \frac{1}{2} \log 2 - \frac{1}{4} + o(1).
\]
This proves the theorem, since indeed
\[ -\lim_{|R| \to \infty} \lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4) \leq \frac{1}{2} \log 2 - \frac{1}{4}. \]

The remainder of this proof therefore focuses on proving (F.12), (F.13) and (F.14). We prove (F.12) in the following lemma, (F.13) in Lemma F.2 and (F.14) in Lemma F.3.

**Lemma F.1** For \( k \) sufficiently large, we have, under power conditions \((P_1)\) and \((P_4)\),
\[ -\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4') \geq \frac{1}{2} \log \frac{3}{4} + \frac{1}{4}. \]

**Proof.** We have
\[ -\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4') \geq -\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_4') = \sup_{t_4 \leq 0} -\log \left\{ \mathbb{E} e^{t_4 |R||R'| + t_4 \sum_{m \in R, j \in R'} X_{m \cdot j}} \right\}. \]
Recall the definition of \( S_R \), which is \( S_R = \sum_{m \in R} X_{m \cdot j} \). Since \( R^c \) and \( R \) are by definition disjoint, and \( \cosh t \leq e^{t^2/2} \), it follows that
\[ \mathbb{E} e^{t_4 S_R S_{R'}} = \mathbb{E} (\cosh(t_4 S_R))^{1/2} \leq \mathbb{E} e^{t_4^2/2} e^{S_{R'}^2/2} \cdot \]
Furthermore, by (7.14), this is further bounded from above by \((1 - t_4^2 |R||R'|)^{-1/2} \), as long as \( t_4^2 |R||R'| \leq 1 \). Note that conditions \((P_1)\) and \((P_4)\) imply that \( P_{\text{min}}^{|R'|} \geq P_{\text{max}}^{|R|} \) when \( k \) is sufficiently large, because \( P_{\text{max}}^{|R|} \) is bounded by our choice for \( R \) and \( |R'| = k(1 - o(1)) \). Substituting this yields
\[ -\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4') \]
\[ \geq \sup_{t_4 \leq 0} -t_4 \frac{|R| (P_{\text{min}}^{|R|})^{1/2}}{2 (P_{\text{max}}^{|R'}|R'|)^{1/2}} + \frac{1}{2} \log(1 - t_4^2 |R||R'|) \]
\[ \geq \sup_{t_4 \leq 0} -t_4 \frac{|R|^{1/2} |R'|^{1/2}}{2} + \frac{1}{2} \log(1 - t_4^2 |R||R'|) \]
\[ \geq -t_4 \frac{|R|^{1/2} |R'|^{1/2}}{2} + \frac{1}{2} \log(1 - t_4^2 |R||R'|) \bigg|_{t_4 = -\frac{1}{2} |R|^{1/2} |R'|^{1/2}}. \]
Observe that the substituted \( t_4 \) indeed fulfills \( t_4^2 |R||R'| \leq 1 \). Substitution of the above result yields
\[ -\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4') \geq \frac{1}{2} \log \frac{3}{4} + \frac{1}{4}. \]

The second lemma is more involved. It uses the Taylor series expansions techniques developed for deriving the asymptotic behaviour for \( k \to \infty \) of \( H_{k^*}^{(3)} \) in Theorem 5.2.
Lemma F.2 For $|R|$ fixed and odd, we have under condition $(\mathcal{P}_1) - (\mathcal{P}_4)$,
\[
- \lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_2 \cap E_3) \leq - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_3).
\]

Proof. Cramér’s theorem gives that
\[
- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_1 \cap E_2 \cap E_3) = \sup_{t \in D} - \log \mathbb{E} e^{Y_t + Y_2 + Y_3},
\]
where $t = (t_1, t_2, t_3)$, $D = \{ t : t_1, t_3 \leq 0, t_2 \geq 0 \}$ and
\[
Y_1 = t_1 \left( -|R||R^c| \left( \frac{P_{\max} R_{\max}^{1/2}}{2 P_R} \right) \right) + \sum_{m \in R, j \in R^c} X_{m_1} X_{j_1},
\]
\[
Y_2 = t_2 \left( \sum_{m,j \in R, m \neq j} X_{m_1} X_{j_1} \right),
\]
\[
Y_3 = t_3 \left( \frac{|R|(P_{\max}^{1/2})}{2(P_R^{1/2})} \right) + \sum_{m,j \in R, m \neq j} X_{m_1} X_{j_1}.
\]
We rewrite this, invoking $S_A = \sum_{j \in A} X_{j_1}$ as
\[
Y_1 = t_1 \left( -|R||R^c| \left( \frac{P_{\max} R_{\max}^{1/2}}{2 P_R} \right) + S_R S_{R^c} \right),
\]
\[
Y_2 = t_2 \left( S_{R^c}^2 - |R^c| \right),
\]
\[
Y_3 = t_3 \left( |R| \left( \frac{(P_{\max}^{1/2})}{2(P_R^{1/2})} - 1 \right) + S_R^2 \right).
\]

It is sufficient to prove that for $k \to \infty$, for all $t_1, t_2$,

\[
\mathbb{E} e^{Y_1 + Y_2 + Y_3} \geq \mathbb{E} e^{Y_3}(1 - o(1)).
\]

Indeed, then
\[
\sup_{t \in D} \{ - \log \mathbb{E} e^{Y_1 + Y_2 + Y_3} \} \leq \sup_{t_3 \leq 0} \{ - \log (\mathbb{E} e^{Y_3}(1 - o(1))) \} = \sup_{t_3 \leq 0} \{ - \log \mathbb{E} e^{Y_3} + o(1) \}
\]
\[
= - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_3) + o(1),
\]
so that for $k \to \infty$, the statement of the lemma follows. We abbreviate
\[
h(t_1, t_2, t_3) = \mathbb{E} e^{Y_1 + Y_2 + Y_3}
\]
We will prove that indeed
\[
h(t_1, t_2, t_3) \geq \mathbb{E} e^{Y_3}(1 - o(1)), \quad k \to \infty.
\]
To do this, we will define an appropriate ellipse \( E \subset \mathbb{R}^2 \) with elements \( t_1, t_2 \) with \((0,0) \in E^0\), the interior of \( E \). In order to show that the supremum of \(- \log h(t_1, t_2, t_3)\) is attained in \( E^0 \), it is sufficient to show that on the boundary of the ellipse \( h(t_1, t_2, t_3) > \mathbb{E} e^{Y_3} \). Since \( h(0,0, t_3) = \mathbb{E} e^{Y_3} \) and \( h \) is log-convex (since it is a moment-generating function), we can then conclude that \( h(t_1, t_2, t_3) > \mathbb{E} e^{Y_3} \) outside the ellipse. Whenever \((t_1, t_2) \in E^0\), we can prove \( h(t_1, t_2, t_3) \geq \mathbb{E} e^{Y_3}(1 - o(1)) \) for \( k \to \infty \).

Since \( e^x \geq 1 + x + x^2/2 + x^3/6 \), we have

\[
\mathbb{E} e^{Y_1 + Y_2 + Y_3} \geq \mathbb{E} \left( e^{Y_3} (1 + Y_1 + Y_2 + (Y_1 + Y_2)^2/2 + (Y_1 + Y_2)^3/6) \right). \tag{F.17}
\]

We will first calculate the required moments. Since \(|R|\) is odd, \( S_R \geq 1 \),

\[
\begin{align*}
\mathbb{E} e^{Y_3} Y_1 & = \mathbb{E} e^{Y_3} \left( -t_1 |R||R^c| \frac{(P_{R^c}^{\max} P_R^{\max})^{1/2}}{2P_{R^c}} \right), \\
\mathbb{E} e^{Y_3} Y_2 & = 0, \\
\mathbb{E} e^{Y_3} Y_1^2 & = \mathbb{E} e^{Y_3} \left( t_1^2 |R|^2 |R^c|^2 \frac{P_{R^c}^{\max} P_R^{\max}}{4P_{R^c}^2} + t_1^2 S_1^2 |R^c| \right) \geq \mathbb{E} e^{Y_3} t_1^2 |R^c|, \tag{F.18} \\
\mathbb{E} e^{Y_3} Y_1 Y_2 & = 0, \\
\mathbb{E} e^{Y_3} Y_2^2 & = \mathbb{E} e^{Y_3} t_2^2 (|R|^2 |R^c|^2 + 2|R^c|) \geq \mathbb{E} e^{Y_3} 3t_2^2 |R^c|^2.
\end{align*}
\]

Concerning \( \mathbb{E} e^{Y_3} (Y_1 + Y_2)^3 \),

\[
\mathbb{E} e^{Y_3} (Y_1 + Y_2)^3 \geq -4 \mathbb{E} e^{Y_3} (|Y_1|^3 + |Y_2|^3).
\]

Using (F.15) and \(|S_R| \leq |R|\), a.s., results in

\[
|Y_1^3| \leq 4 |t_1|^3 \left( |R|^3 |R^c|^3 \frac{(P_{R^c}^{\max} P_R^{\max})^{3/2}}{8P_{R^c}^3} + |R|^3 |S_1^3| \right) \quad \text{and} \quad |Y_2^3| \leq 4 |t_2|^3 \left( |R|^3 |R^c|^3 + |S_1^3| \right),
\]

so that (using \( E |S_R^{\max}| \leq C |R^c|^{m/2} \) for some \( C < \infty \))

\[
-\mathbb{E} e^{Y_3}(|Y_1^3| + |Y_2^3|) \geq -\mathbb{E} e^{Y_3} \left( C |t_1|^3 |R|^3 |R^c|^3 \frac{(P_{R^c}^{\max} P_R^{\max})^{3/2}}{P_{R^c}^3} + C |t_1|^3 |R|^3 |R^c|^3 + C |t_2|^3 |R^c|^3 \right).
\]

We next substitute the moments, displayed in (F.18), together with the bound above in the expression in (F.17) to obtain

\[
\begin{align*}
h(t_1, t_2, t_3) & \geq \mathbb{E} e^{Y_3} \left( 1 - t_1 |R||R^c| \frac{(P_{R^c}^{\max} P_R^{\max})^{1/2}}{2P_{R^c}} + \frac{t_2^2 |R^c|}{2} + 2 |R^c|^2 t_2^2 + e(t_1, t_2) \right) \\
& = \mathbb{E} e^{Y_3} \left( 1 - \mathcal{H} + \frac{|R|}{2} (t_1 - t_1^2 + \frac{3 |R|^2 t_2^2}{2} + e(t_1, t_2), \tag{F.19}
\right)
\]
where
\[ H = \frac{|R|^2 |R^e| P_{R^e}^{\max} P_{R}^{\max}}{8P_{R^e}^2}, \]
\[ t_1^* = -\frac{|R|(P_{R^e}^{\max} P_{R}^{\max})^{1/2}}{2P_{R^e}}. \]
\[ e(t_1, t_2) = -C|t_1|^3 |R|^3 |R^e|^3 \left( \frac{P_{R^e}^{\max} P_{R}^{\max}}{P_{R^e}^3} \right)^{3/2} - C|t_1|^3 |R|^3 |R^e|^3 - C't_2^3 |R^e|^3. \]

According to (P_1), \( P_R \leq P_{R^e}^{\max}|R| \leq \delta^{-1}|R| \) is bounded, so that \( P = P_{R^e} + P_R = P_{R^e}(1 - o(1)) \). This implies, using (P_1) - (P_4),
\[ H \leq \frac{|R|^2 k P_{R^e}^{\max} \delta^{-1}}{8P^2(1 - o(1))} = \frac{|R|^2 \delta^{-1} P_{R^e}^{\max} k^2}{8} \frac{k}{P^2} (1 + o(1)) \rightarrow 0, \quad k \rightarrow \infty. \] (F.20)

We define the ellipse \( \mathcal{E} \) as
\[ \mathcal{E} = \left\{ (t_1, t_2) : \frac{|R|^2}{2} (t_1 - t_1^*)^2 + \frac{3|R^e|^2}{2} t_2^2 \leq 2H \right\}. \]

To show that on \( \partial \mathcal{E} \), \( h(t_1, t_2, t_3) > \mathbb{E} e^{Y_3} \), it is sufficient to show that for every \( |R| \) fixed and \( k \rightarrow \infty \),
\[ |e(t_1, t_2)| = o(H). \] (F.21)

Indeed, on \( \partial \mathcal{E} \), according to (F.19)
\[ h(t_1, t_2, t_3) = \mathbb{E} e^{Y_3} \left( 1 - H + 2H + e(t_1, t_2) \right) > \mathbb{E} e^{Y_3}. \]

Therefore, we focus on \( e(t_1, t_2) \) for \( (t_1, t_2) \in \partial \mathcal{E} \). Note that for \( (t_1, t_2) \in \mathcal{E} \) by the triangle inequality,
\[ |t_1| \leq \frac{|R|(P_{R^e}^{\max} P_{R}^{\max})^{1/2}}{P_{R^e}}, \quad |t_2| \leq \frac{1}{\sqrt{6}} \frac{|R|(P_{R^e}^{\max} P_{R}^{\max})^{1/2}}{|R^e|^2 P_{R^e}.} \] (F.22)

Substituting this in \( e(t_1, t_2) \) gives
\[ |e(t_1, t_2)| \leq C \frac{|R|^3 (P_{R^e}^{\max} P_{R}^{\max})^{3/2}}{P_{R^e}^3} \left( \frac{|R|^3 |R^e|^3 (P_{R^e}^{\max} P_{R}^{\max})^{3/2}}{P_{R^e}^3} + |R|^3 |R^e|^3 \right) + C \frac{|R|^3 (P_{R^e}^{\max} P_{R}^{\max})^{3/2}}{|R^e|^3 P_{R^e}^3} \leq C(H^3 + |R|^3 H^{3/2}) + C' H^{3/2}. \]

Since \( H = o(1) \) according to (F.20) and \( |R| \) is fixed, indeed \( |e(t_1, t_2)| = o(H) \). This proves that for \( k \) large,
\[ h(t_1, t_2, t_3) > \mathbb{E} e^{Y_3}, \quad (t_1, t_2) \notin \mathcal{E}. \] (F.23)

Finally, for \( (t_1, t_2) \in \mathcal{E}^0 \), we use the results above to obtain
\[ h(t_1, t_2, t_3) \geq \mathbb{E} e^{Y_3} \left( 1 - H + \frac{|R|^2}{2} (t_1 - t_1^*)^2 + \frac{3|R^e|^2}{2} t_2^2 - o(H) \right). \]
Minimization over $t_1$ and $t_2$ clearly gives $t_1 = t_1^*$ and $t_2 = 0$, so that for all $t_1, t_2 \in \mathcal{E}$

$$h(t_1, t_2, t_3) \geq \mathbb{E} e^{t_3} \left(1 - \mathcal{H} - o(\mathcal{H})\right).$$

Finally, since $\mathcal{H} = o(1)$, $1 - \mathcal{H} - o(\mathcal{H}) = 1 - o(1)$. Together with (F.23), this proves claim (F.16).

**Lemma F.3** For $|R| \to \infty$,

$$- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_3) \leq \frac{1}{2} \log 2 - \frac{1}{4} + o(1).$$

**Proof.** Clearly

$$- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_3) = \sup_{t_3 \leq 0} \left\{- \log \mathbb{E} e^{t_3}\right\}, \quad \text{where} \quad Y_3 = t_3 \left(|R| \left(\frac{(P_{\max})^{1/2}}{2(P_{\min})^{1/2}} - 1\right) + S_R^2\right).$$

When we prove

$$\lim_{|R| \to \infty} \sup_{t_3 \leq 0} \left\{- \log \mathbb{E} e^{t_3}\right\} \leq \frac{1}{2} \log 2 - \frac{1}{4},$$

we conclude that for $|R|$ sufficiently large,

$$- \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(E_3) \leq \frac{1}{2} \log 2 - \frac{1}{4} + o(1).$$

Thus, it remains to show (F.24). For short notation, we denote

$$j_R = \frac{(P_{\max})^{1/2}}{2(P_{\min})^{1/2}} - 1/2.$$

The crucial step in this part of the proof is that we still have some freedom to choose $R$. We already chose $R \subset \{j : P_j \in [\delta, 1/\delta]\}$ and $|R|$ is odd. Because of (P$_1$), we can choose $R$ such that whenever $|R| = r$, there exists a number $\bar{P}$ with $\max_{m \in R} |P_m - \bar{P}| \leq \bar{P}/r$. This implies $P_{\max} \leq \bar{P}(1 + 1/|R|)$ and $P_{\min} \geq \bar{P}(1 - 1/|R|)$, so that for $|R| \to \infty$,

$$0 \leq j_R \leq \frac{1}{2} \sqrt{\frac{1 + 1/|R|}{(1 - 1/|R|)}} - \frac{1}{2} \to 0.$$  

Now write

$$- \inf_{t_3 \leq 0} \log \mathbb{E} \exp\left(t_3 \left(\frac{P_{\max}}{2P_{\min}} - \frac{1}{2}\right) + t_3 S_R^2\right) = - \inf_{t_3 \leq 0} \log \mathbb{E} \exp\left(t_3 j_R + t_3 (S_R^2 - |R|/2)\right).$$

We can next follow the argument provided in Lemma 7.12, together with the observation that whenever $X_n \mathcal{P} \sim X$ and $c_n \to c$ that also $c_nX_n \mathcal{P} \sim cX$. Indeed, in all distributional convergence arguments $-1/|R| \leq t_3 \leq 0$, so that we are allowed to replace $e^{t_3 j_R}$ by its limit 1. This proves that indeed for $|R|$ large

$$- \inf_{t_3 \leq 0} \log \mathbb{E} e^{t_3} \leq \frac{1}{2} \log 2 - \frac{1}{4} + o(1),$$

The lemmas together complete the proof of Theorem 7.2(b) in the case of unequal powers.
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KLOK, M.J. (2001), Importance sampling for DS-CDMA systems with 1-stage HD-PIC, Proceedings of WPMC'01, Aalborg, Denmark, 1635-1640.


Bibliography


Summary/Samenvatting

Performance Analysis of Advanced Third Generation Receivers by M.J. Klok

In this thesis, we consider a wireless telecommunications system, the so-called universal mobile telecommunications system (UMTS). This system is based on the code-division multiple-access (CDMA) technique. This technique enables multiple users to transmit data, without sharing bandwidth or time. Instead, data is separated using codes. Since in practice, these codes are not perfect, users will interfere, so that not all databits are received correctly. The probability of such a bit error is of practical interest. Because it is quite complex to calculate the exact bit error probability, we introduce a quality measure, the exponential rate, based upon the speed at which the bit error probability tends to zero as the bandwidth increases. The exponential rate is calculated with the so-called ‘theory of large deviations’. The exponential rate is investigated for various models.

In this thesis, we investigate two techniques to reduce interference, the so-called hard-decision parallel interference cancellation (HD-PIC) and soft-decision parallel interference cancellation (SD-PIC), which are iterative methods to reduce interference. These methods are seen as the most promising techniques for implementation. For a simple model, the exponential rate is investigated. The main conclusion is that both HD-PIC and SD-PIC give a significant increase in performance compared to the standard receiver. The potential increase of SD-PIC is slightly larger than HD-PIC, but the difference is negligible.

The behaviour of the exponential rate is investigated for a large number of users. A systematic method is described to do so for the HD-PIC system. A simple expression for the exponential rate is obtained, also for the system in which multiple iterations are performed to cancel the interference. This shows that increasing the number of iterations results in an increase in performance, when the number of users is sufficiently large. The method is extended to a more general model, in which different users can have different powers, and where noise from external sources interfere. Again the conclusion is that the HD-PIC system improves the performance, compared to the standard receiver. For the SD-PIC system, a method is described to calculate the exponential rate for the general model. Numerical results show that the performance of the SD-PIC receiver is virtually the same as that of the HD-PIC receiver, when the number of users is larger than 5.

The behaviour of the exponential rate is also investigated in the case that the number of users is fixed, while infinitely many iterations are performed. We assume that users can have different powers, but that there is no additive noise present. For the HD-PIC system, we show that after finitely many iterations (depending on the number of users and individual powers), the exponential rate remains constant. Furthermore, this rate is strictly positive, meaning that a certain minimal quality can be guaranteed. An expression is given for this minimal rate, and it is shown that under some conditions on the powers, the exponential rate indeed converges to this minimal rate. Furthermore, intuition is given on the required number of iterations. For the SD-PIC system with infinitely many iterations, an lower bound for the exponential rate is given. Both the HD-PIC and the SD-PIC system with infinitely many iteration have a guaranteed minimal quality.

Finally, the probability of an incorrect reception of a databit is investigated, using both analytical techniques as simulation methods. Efficient methods, named importance sampling methods, are introduced to estimate these bit error probabilities.
Performance Analysis of Advanced Third Generation Receivers door M.J. Klok

In dit proefschrift beschouwen we een draadloos telecommunicatie systeem, genaamd universal mobile telecommunications system (UMTS). Dit systeem is gebaseerd op een techniek genaamd code-division multiple-access (CDMA). Met deze techniek is het mogelijk dat meerdere gebruikers data verzenden, waarbij ze de volledige bandbreedte en tijd gebruiken. De data wordt gescheiden door middel van codes. Aangezien in de praktijk deze codes niet perfect zijn, zullen gebruikers elkaar storen, wat kan leiden tot verkeerd ontvangst van databits. Interessant is de kans op zo’n verkeerd ontvangst. In de praktijk zal deze kans erg klein zijn. Omdat de exacte berekening erg complex kan zijn, wordt een kwaliteitsmaat geïntroduceerd, de exponentiële snelheid, die eenvoudig te gebruiken is. Deze kwaliteitsmaat is gebaseerd op de snelheid waarmee de kans op een fout naar nul gaat als de bandbreedte toenemt, en wordt onderzocht met de ‘theorie der grote afwijkingen’. De exponentiële snelheid wordt voor verschillende modellen berekend.

Deze thesis beschouwt twee technieken om de storing te verminderen, de zogenaamde hard-decision parallel interference cancellation (HD-PIC) en soft-decision parallel interference cancellation (SD-PIC). Beiden zijn iterative methoden zijn om de ruis te verminderen. Deze methoden worden gezien als de meest veelbelovende technieken. Voor een simpel model worden de exponentiële snelheden onderzocht. Voornaamstec conclusie is dat zowel HD-PIC als SD-PIC een grote kwaliteitsverbetering bewerkstelligen ten opzichte van het standaard systeem. SD-PIC heeft iets meer potentie dan HD-PIC, maar het verschil is klein.

Het gedrag van de exponentiële snelheid wordt verder onderzocht voor grote aantallen gebruikers. Systematische methoden worden beschreven om dit te doen voor het HD-PIC systeem. Een eenvoudige uitdrukking voor het HD-PIC systeem wordt verkregen, ook voor het geval dat meerdere malen de ruis gefilterd wordt. Dit toont aan dat vergroting van het aantal iteraties een kwaliteitsverbetering blijven geven, mits het aantal gebruikers voldoende groot is. De methode wordt uitgebreid naar een algemener model, waarin verschillende gebruikers verschillende vermogens hebben, en waarbij ook ruimte van externe bronnen aanwezig is. Resultaat is dat ook voor dit model het HD-PIC systeem meer kwaliteit geeft dan het standaard systeem. Voor het SD-PIC systeem wordt een methode beschreven om de exponentiële snelheid uit te rekenen in het geval van het algemene model. Numerieke resultaten laten zien dat de kwaliteit van het SD-PIC systeem vrijwel hetzelfde is als die van het HD-PIC systeem, mits het aantal gebruikers groter dan 5 is.

Het gedrag van de exponentiële snelheid wordt ook onderzocht in het geval dat het aantal gebruikers van het systeem vast is, en er een oneindig aantal iteraties toegepast wordt. Hierbij wordt aangenomen dat gebruikers verschillende vermogens mogen hebben, maar dat er geen externe ruiskrommen aanwezig zijn. Voor het HD-PIC systeem wordt aangetoond dat na een eindig aantal iteraties (afhankelijk van het aantal gebruikers en hun vermogens) de exponentiële snelheid niet meer toenemt, wat betekent dat de kwaliteit niet meer toeneemt. De verkregen exponentiële snelheid is strikt positief, wat betekent dat een minimale kwaliteit gegaarandeerd kan worden. Er wordt een uitdrukking gegeven voor deze minimale rate en er wordt aangetoond dat onder bepaalde voorwaarden van de vermogens deze minimale rate wordt aangenomen. Verder wordt er intuition gegeven over het benodigde aantal iteraties. Voor het SD-PIC systeem met een oneindig aantal iteraties, wordt ook een ondergrond voor de exponentiële snelheid gegeven. Zowel het HD-PIC als het SD-PIC systeem met oneindig veel iteraties heeft dus een gegaarandeerde minimale kwaliteit.

Tenslotte wordt de kans op bitfouten onderzocht met behulp van zowel analytische technieken als simulatiemethoden. Efficiënte methoden, genaamd importance sampling methoden, worden geïntroduceerd om deze kleine foutenkansen te kunnen schatten via simulatie.

Keywords: CDMA, parallel interference cancellation, bit error probability, large deviations, exponential rate, second order asymptotics, importance sampling.

M.J. Klok, April 2002.
Abbreviations

1G  First generation, 11
2G  Second generation, 11
3G  Third generation, 3
3GPP Third generation partnership project, 13
AMPS Advanced mobile phone system, 11
ATF Autotelefonie, 11
AWGN Additive white Gaussian noise, 5
BEP Bit-error probability, 6
BPSK Binary phase shift keying, 12
CDMA Code division multiple access, 3
CLT Central limit theorem, 15, 69
D-AMPS Digital advanced mobile phone system, 12
DS Direct sequence, 13
GSM Global system for mobile communications, 12
GSM Groupe speciale mobile, 12
HD-PIC Hard-decision parallel interference cancellation, 6
IC Interference cancellation, 6
i.i.d. Independent and identically distributed, 5
IMT-2000 International mobile telecommunications for the year 2000, 3
IS Importance sampling, 36
NMT Nordic mobile telephone, 11
PIC Parallel interference cancellation, 6
QPSK Quaternary phase shift keying, 13
SD-PIC Soft-decision parallel interference cancellation, 6
SIC Serial interference cancellation, 48
SNR Signal-to-noise ratio, 13
SSMA Spread spectrum multiple access, 14
TACS Total access communication system, 11
TDMA Time division multiple access, 12
UMTS Universal mobile telecommunications system, 3
Notation

$A_{mi}$ Coding bit (chip) $i$ of $m$-th user modeled as a random variable, 5
$a_{mi}$ Coding bit (chip) $i$ of $m$-th user, 4
$a_n(t)$ Coding signal of $m$-th user, 4
$b_{mi}^{(1)}$ Tentative estimation of data bit $i$ of the $m$-th user, 5
$b_{mi}^{(s)}$ Estimated data bit $i$ of $m$-th user after $s-1$ iterations, 7
$b_n(t)$ Data signal for $m$-th user, 4
$B_r$ $\{\max_{1 \leq m \leq R} \sgn_{m}(Z_{m}^{(1)}) < 0, \min_{1 \leq m \leq k-1} \sgn_{m}(Z_{m}^{(1)}) > 0\}$, 38, 56
$B_R$ $\{\max_{m \in \mathbb{R}} \sgn_{m}(Z_{m}^{(1)}) < 0, \min_{m \in \{1, \ldots, k-1\} \setminus \mathbb{R}} \sgn_{m}(Z_{m}^{(1)}) > 0\}$, 107
$E_1, E_2, E_3, E_4$ Events used to prove the upper bound for optimal system, 143, 196
$E$ Ellipse used to prove the asymptotics of $H_{k,t}^{(2)}, H_{k,R}^{(s)}$ and $H_{k,R}^{(2)}$, 76, 88, 114
$F_k(\rho)$ Representation of $Z_0^{(2,s)}$ in terms of empirical measure, 60
$F_k(\rho, \mu)$ Representation of $Z_0^{(2,s)}$ in terms of empirical measure and measure of AWGN, 120
$G_{mk}(\rho)$ Representation of $Z_{m}^{(1)}$ in terms of empirical measure, 63, 171
$h_j(x)$ Estimation function for $P_j^{1/2}b_j$, as a function of the decision statistic, 10
$H_{k}^{(2)}$ Exponential rate for 1-stage HD-PIC model, 20, 53, 55, 102
$H_{k,R}^{(2)}$ Interim expression of exponential rate for 1-stage HD-PIC model, 20, 56, 102
$H_{k,R}^{(3)}$ Interim expression of exponential rate for 1-stage HD-PIC model (unequal powers), 108
$H_{k,R}^{(s)}$ Exponential rate of user $m$ for $(s-1)$-stage HD-PIC, 30, 126
$H_{k,R}^{(s)}$ Exponential rate for $(s-1)$-stage HD-PIC model, 24
$H_{rk}(\rho)$ Representation of $Z_{m}^{(2,s)}$ in terms of empirical measure, 171
$h(t)$ Moment generating function, 69
$\mathcal{H}$ Abbreviation for asymptotics of $H_{k,R}^{(s)}$, 77
$I_k(\rho)$ Rate function $(k-1) \log 2 + \sum_{a \in X_k} \rho_a \log \rho_a$, 21, 59
$I_k(\alpha)$ $\min_{|\alpha|_{\ell_2} = 1} \max_{1 \leq i \leq 2} - \log \mathbb{E}(e^{1/2} \alpha_i^2)$, 147
$I_k$ Exponential rate for matched filter model, 20, 53, 54, 99
$J_k^{(2)}$ Exponential rate for 1-stage SD-PIC model, 21, 53, 60, 120
$k$ Number of users, 4
$k_r$ $k - r$, 73
$L_n^{X}, J_n$ Empirical measure of $(X_{mj})_{m=1, \ldots, n, j=1, \ldots, k-1}$, 59
\( \mathcal{L} \)  
Set of all empirical measures on \((\mathbb{N} \cup \{0\}) \setminus X_n \), 59

\( M(X_k) \)  
Set of all probability measures on \(X_k\), 59

\( N_t \)  
Random variable representing additive white Gaussian noise, 10

\( n(t) \)  
White noise process, 4

\( n \)  
Processing gain, 4

\( \omega_c \)  
Carrier frequency, 4

\( P_m \)  
Power of \( m \)-th user, 4

\( \hat{P}_m^{(1)} \)  
Tentative estimation of \( P_m \), 7

\( p_r \)  
\( \mathbb{P}(\text{sgn}_0(Z_0^{(2,H)}) < 0, B_r) \), 38, 171

\( P_{\max}, P_{\min} \)  
\( \max_{0 \leq j \leq k-1} P_j \) and \( \min_{0 \leq j \leq k-1} P_j \), respectively, 32, 129

\( P_A \)  
\( \sum_{m \in A} P_m \), 103

\( Q(x) \)  
Tail probability of a standard Gaussian random variable, 70

\( r(t) \)  
Total received signal, 4

\( r_k \)  
argmin\( R_k^{(2)} \), 21, 67

\( r \)  
Number of bit errors in stage 1 for the simple system, 20, 56

\( R \)  
Set of users with a bit errors in stage 1, 103

\( R_0 \)  
The set \( \{0, \ldots, k-1\} \), 73

\( R_r \)  
The set \( \{m : \text{sgn}_m(Z_m^{(s,H)}) < 0\} \), 77

\( R \)  
\( \{R_1, \ldots, R_{s-1}\} \), 77

\( \rho \)  
Asymptotic expression for optimal \( P_R/P_0 \), corresponding with \( R = \arg\max_R H_k^{(2)} \), 103

\( \rho_+, \rho_- \)  
Short notation for \( \rho(1,1) \) and \( \rho(-1,1) \) for \( k = 2 \) and \( \rho(+1,1) \) and \( \rho(-1,1) \) for \( k = 3 \), 60, 118

\( \rho_+, \rho_- \)  
Short notation for \( \rho(1,1) \) and \( \rho(-1,1) \) for \( k = 3 \), 60

\( S_A \)  
\( \sum_{j \in A} X_j \), 73

\( \hat{S}_A \)  
\( \sum_{j \in A} P_j^{1/2} X_{j1} \), 108

\( \sigma^2 \)  
Intensity of scaled white noise process, 9

\( \delta_m(t) \)  
Transmitted coded signal of \( m \)-th user, 4

\( \hat{S}_v \)  
\( \sum_{k=0}^{k-1} v_i X_{j1} \), 147

\( T_A \)  
\( \sum_{j \in A} t_{1j} X_{j1} \), 108

\( T \)  
Data bit duration, 4

\( t \)  
Vector \( t \). Variable of moment generating functions, 69

\( X_{mi} \)  
Coding bit of \( m \)-th user after reformulation, 9

\( X \)  
Matrix with elements \( X_{mi} \), \( 0 \leq m \leq k-1, 1 \leq i \leq n \), 10

\( X_k \)  
\( \{-1,+1\}^{k-1} \), 59

\( Y_a \)  
Part of \( < t, Y_1 > \) representing (mainly) the asymmetric part, 73

\( Y_0 \)  
Part of \( < t, Y_1 > \) representing (mainly) the quadratic terms, 73

\( Y_{\sigma,i} \)  
Random variable, representing the (sum of) decision statistic(s) at stage \( \sigma \), 58

\( Y_{(\sigma',\sigma,i)} \)  
Random variable, representing decision statistic \( \sigma' \) at stage \( \sigma \), 85

\( Z_{m}^{(1)} \)  
Tentative decision statistic for \( m \)-th user, 9

\( Z_{m}^{(2)} \)  
Decision statistic for \( m \)-th user at stage \( s \), 10

\( Z_{m}^{(3)} \)  
Decision statistic for \( m \)-th user at stage \( s \) (HD-PIC), 10

\( Z_{m}^{(3,2)} \)  
Decision statistic for \( m \)-th user at stage \( s \) (HD-PIC), 21, 56, 80, 107

\( Z_{m}^{(4)} \)  
Decision statistic for \( m \)-th user at stage \( s \) (SD-PIC), 10

\( Z_{m}^{(5,6)} \)  
\( [Z_{m}^{(5,6)}, \ldots, Z_{k-1}^{(5,6)}] \), 134
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Dankwoord

Toen ik, na vier jaar studie begon met afstuderen, had ik nooit gedacht nog vijf jaar op de TU Delft te verblijven. Het begon met een interessant project over benaderingsmethoden voor CDMA systemen, opgezet door Gerard Hooghiemstra, Ramjee Prasad en Tero Ojanperä. Het project omhelsde voornamelijk systemen zonder interference cancellation, maar er werd mij gevraagd om 'even' te onderzoeken hoe de benaderingsmethoden aangepast konden worden aan geavanceerde systemen, zoals het SD-PIC en HD-PIC systeem. Dit bleek niet zo eenvoudig! Ik ben, samen met anderen, vier jaar bezig geweest om verscheidene aspecten van deze systemen te onderzoeken ... en er zijn nog diverse andere interessante aspecten. Het onderzoek was niet zo succesvol geweest als ik niet de steun had gehad van familie, vrienden en collega's. Ik ben ze veel dank verschuldigd. Een aantal mensen wil ik specifiek bedanken.

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The following note is incorrect.

Note 1: Although the central limit theorem is frequently applied to \( k \) identically distributed interferences, it can also be applied to \( n \) identically distributed chips from a single interferer. As a result, the Gaussian approximation can be quite accurate even for a small value of \( k \) provided that \( n \) is sufficiently large.


Consider Theorem 5.7 of this thesis. This result directly implies

\[
H_k^{(s)} = \frac{n}{8} \sqrt{\frac{3}{k-1}} \left( 1 + \mathcal{O}\left( \frac{1}{\sqrt{k-1}} \right) \right). \tag{1}
\]

The techniques used to prove Theorem 5.7 cannot give any improvement on the order of the error term. However, for \( s = 1 \), a Taylor series expansion gives

\[
I_k = \frac{1}{2(k-1)} \left( 1 + \mathcal{O}\left( \frac{1}{(k-1)^2} \right) \right).
\]

which is better than the result in (1) for \( s = 1 \). In Table 5.1 on pagina 97 of this thesis we see this, since \( \frac{1}{2(k-1)} \) approximates \( I_k \) much better than \( \frac{1}{2k} \).

You see, wire telegraph is a kind of a very, very long cat. You pull his tail in New York and his head is moowing in Los Angeles. Do you understand this? And radio operates exactly the same way: You send signals here, they receive them there. The only difference is that there is no cat.

Einstein when he described the radio

In fact, Einstein does not describe the radio (in which case the transmission is from one to many), but mobile telephones (in which case the transmission is from one to one).

Consider a dartboard with radius 1. We assume that if a player aims at a certain point, the place it hits is (two-dimensional) Gaussian distributed with mean the bull and variance \( \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). The player maximizes its score by aiming at triple 20 when \( \sigma < 0.03 \) and aiming on the bull otherwise (a player which hits the bull with probability 0.99999 has \( \sigma = 0.035 \)).

Experienced skydivers tend to misjudge the risk of skydiving. They have a relatively small probability to get into an accident, because of their experience: they typically deal with a large deviation event. However, experienced skydivers often use small parachutes, so that the safety margins are small. Thus, even if an accident, the injury is often very severe. A smart skydiver balances both aspects... most skydivers do not take the kind of injury into account.
Beschouw de onafhankelijke stochasten \( X_1, X_2, \ldots \) met \( P(X_j = 1) = 1 - P(X_j = 0) = 1/j \). Het is welbekend dat
\[
Z_n = \sum_{j=1}^{n} X_j X_{j+1}
\]
het aantal vaste punten is in een random permutatie van \( \{1, \ldots, n + 1\} \), en dat \( Z_n \) in verdeling convergeert naar een standaard Poisson verdeelde stochaat, zie bijvoorbeeld ARRATIA, BARBOUR AND TAVARÉ (1992). Een eenvoudig bewijs is als volgt.

We definieren voor \( n \geq 1 \), de stochast \( Y_n = \sum_{i=1}^{n} X_i X_{i+1} + X_{n+1} X_1 \). De volgende recursieve vergelijking en zijn oplossing gelden:
\[
\begin{bmatrix}
\mathbb{E} Z_n \\
\mathbb{E} Y_n
\end{bmatrix}
= \frac{1}{(n+1)!} \prod_{j=1}^{n} \begin{bmatrix} j & 1 \\ j & s \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix}
= \begin{bmatrix}
\mathbb{E} Z_{n+1} \\
\mathbb{E} Y_{n+1}
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
\sum_{j=0}^{n+1} \frac{1 - j}{n+1} (s-1)^j \\
\sum_{j=0}^{n+1} (s-1)^j
\end{bmatrix}
\]

Omdat \( \mathbb{E} Z_{\infty} \rightarrow e^s - 1 \), convergeert \( Z_n \) in verdeling naar een standaard Poisson verdeelde stochaat.


Laat \( \alpha_0, \alpha_1, \ldots, \alpha_k \in \mathbb{R} \). We definiëren de matrix \( A^{(k)} \) door
\[
\begin{align*}
A^{(k)}_{i,i-1} &= \alpha_{i-1} + \alpha_i, & 1 \leq i \leq k, \\
A^{(k)}_{i,i} &= -\alpha_i, & 1 \leq i \leq k - 1, \\
A^{(k)}_{i+1,i} &= -\alpha_{i-1}, & 2 \leq i \leq k, \\
A^{(k)}_{i,i} &= 0 & \text{elders,}
\end{align*}
\]
i.e.,
\[
A^{(k)} = \begin{bmatrix}
\alpha_0 + \alpha_1 & -\alpha_1 & 0 & 0 & \cdots & 0 \\
-\alpha_1 & \alpha_1 + \alpha_2 & -\alpha_2 & 0 & \cdots & 0 \\
0 & -\alpha_2 & \alpha_2 + \alpha_3 & -\alpha_3 & \cdots & \cdots \\
0 & 0 & -\alpha_3 & \alpha_3 + \alpha_4 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & 0 & -\alpha_{k-1} + \alpha_k
\end{bmatrix}.
\]

Dan geldt
\[
\det(A^{(k)}) = \prod_{i=0}^{k} \alpha_j.
\]
Met behulp van dit resultaat kunnen we eenvoudig aantonen dat de functie
\[
f(r_1, r_2, \ldots, r_k) = a_0 r_1 + \frac{a_1 r_2}{r_1} + \ldots + \frac{a_{k-1} r_k}{r_{k-1}} + \frac{a_k}{r_k}, \quad r_1, \ldots, r_k > 0
\]
voor \( a_1, \ldots, a_k > 0 \) een uniek minimum heeft. Transformatie van \( p_j = \log r_j \) levert namelijk een functie op waarvan de matrix van dubbele afgeleiden precies de vorm van \( A^{(k)} \) heeft.
De volgende noot is niet correct.

Noot 1: Alhoewel de centrale limiet stelling veelvuldig toegepast wordt op $k$ identiek verdeelde stoorders, kan het ook toegepast worden op $n$ identiek verdeelde chips van één enkele stoorer. Als gevolg kan de normale benadering behoorlijk nauwkeurig zijn, zelfs voor kleine $k$ mits $n$ voldoende groot is.


We beschouwen Stelling 5.7 van dit proefschrift. Dit resultaat impliceert direct

$$ H_k^{(s)} = \frac{s}{8} \sqrt{\frac{4}{k-1}} \left( 1 + \mathcal{O} \left( \frac{1}{\sqrt{k-1}} \right) \right). \quad (1) $$

De technieken die gebruikt zijn om Stelling 5.7 te bewijzen, kunnen geen verbetering geven op de orde van de foutterm. Echter, voor $s = 1$ geeft een Taylor expansie

$$ I_k = \frac{1}{2(k-1)} \left( 1 + \mathcal{O} \left( \frac{1}{(k-1)^2} \right) \right), $$

wat beter is dan het resultaat in (1) voor $s = 1$. Dit is duidelijk te zien in Tabel 5.1 op pagina 97 van dit proefschrift, waar $\frac{1}{2(k-1)}$ de rate $I_k$ veel beter benadert dan $\frac{1}{2k}$.


Einstein die desgevraagd de radio beschrijft

Einstein beschrijft hier niet de radio (waarbij de transmissie van één naar velen is), maar mobiele telefoons (waarbij de transmissie van één naar één is).

Beschouw een dartbord met straal 1. We nemen aan dat als een speler mik op een bepaald punt, dat dan de uiteindelijke plaats (twee-dimensionaal) normaal verdeeld is met gemiddelde het mikpunt en variantie $\sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. De speler maximaliseert zijn uitkomst door op de triple 20 te mikken als $\sigma < 0.03 \ldots$ en anders op de bull te mikken (ter vergelijking, een speler zou met kans 0.99999 de bull moeten raken om $\sigma = 0.035$ te hebben).

Ervaren parachutisten neigen het risico van parachutespringen te onderschatten. Ze hebben relatief weinig kans op een ongeluk, vanwege hun ervaring; we hebben typisch te maken met een grote afwijkingen kans. Echter, ze gebruiken vaak kleine parachutes, zodat de veiligheidsmarges klein zijn. Dus, gegeven een ongeluk, hebben ze vaak zeer zwaar letsels. Een slimme parachutist balanceert beide aspecten ... de meesten vergeten de aard van het letsel mee te nemen in hun beschouwing.
Consider the independent random variables \(X_1, X_2, \ldots\) with \(\mathbb{P}(X_j = 1) = 1 - \mathbb{P}(X_j = 0) = 1/j\). It is well-known that
\[
Z_n = \sum_{j=1}^{n} X_j X_{j+1}
\]
represents the number of fixed points in a random permutation of \(\{1, \ldots, n + 1\}\), and that \(Z_n\) converges in distribution to a standard Poisson distributed random variable, see for example Arratia, Barbour and Tavaré (1992). A simple proof of this fact is the following argument.

We define for \(n \geq 1\), the random variable \(Y_n = \sum_{i=1}^{n} X_i X_{i+1} + X_n X_1\). The following recursive expression and its solution hold:
\[
\begin{bmatrix}
\mathbb{E} S^{Z_n} \\
\mathbb{E} S^{Y_n}
\end{bmatrix} = \frac{1}{(n+1)!} \prod_{j=1}^{n} \begin{bmatrix}
\frac{1}{j} & 1 \\
\frac{1}{s} & 1
\end{bmatrix},
\Rightarrow
\begin{bmatrix}
\mathbb{E} S^{Z_n} \\
\mathbb{E} S^{Y_n}
\end{bmatrix} = \left[ \begin{array}{c}
\sum_{j=0}^{n+1} \frac{(1 - \frac{j}{n+1}) (s - 1)^j}{j!} \\
\sum_{j=0}^{n+1} (s - 1)^j \\
\sum_{j=0}^{n+1} (s - 1)^j
\end{array} \right].
\]
Since \(\mathbb{E} S^{Z_n} \to e^{s-1}\), the random variable \(Z_n\) converges in distribution to a standard Poisson distributed random variable.


Let \(\alpha_0, \alpha_1, \ldots, \alpha_k \in \mathbb{R}\). We define the matrix \(A^{(k)}\) by
\[
A^{(k)} = \begin{cases}
\alpha_0 + \alpha_1, & 1 \leq i \leq k, \\
-\alpha_1, & 1 \leq i \leq k-1, \\
-\alpha_i, & 2 \leq i \leq k, \\
0, & \text{elsewhere},
\end{cases}
\]
i.e.,
\[
A^{(k)} = \begin{bmatrix}
\alpha_0 + \alpha_1 & -\alpha_1 & 0 & 0 & \cdots & 0 \\
-\alpha_1 & \alpha_1 + \alpha_2 & -\alpha_2 & 0 & \cdots & 0 \\
0 & -\alpha_2 & \alpha_2 + \alpha_3 & -\alpha_3 & \cdots & \vdots \\
0 & 0 & -\alpha_3 & \alpha_3 + \alpha_4 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & -\alpha_{k-1} & \alpha_{k-1} + \alpha_k & 0
\end{bmatrix}.
\]
Then
\[
\det(A^{(k)}) = \sum_{j=0}^{k} \prod_{j \neq 0} \alpha_j.
\]
This result easily shows that the function
\[
f(r_1, r_2, \ldots, r_k) = a_0 r_1 + a_1 \frac{r_2}{r_1} + \cdots + a_{k-1} \frac{r_k}{r_{k-1}} + a_k \frac{1}{r_k}, \quad r_1, \ldots, r_k > 0
\]
for \(a_1, \ldots, a_k > 0\) has a unique minimizer. Indeed, transforming \(\rho_j = \log r_j\) gives a function for which the matrix of second derivatives has the form of \(A^{(k)}\).
J. Kick was born on 16 September 1973 in Utrecht. In 1991, he went to the secondary school 'RSG Broekede' in Breukelen, where he graduated in 1993. He moved to Delft, where he studied applied mathematics at Delft University of Technology. In 1998, he graduated cum laude on his MSc thesis 'Comparison of CDMA-approximation techniques'. After his graduation, he started to work at the faculty of Information Technology and Systems in a research position, where he investigated the performance of third generation wireless systems. In 2001, he became a PhD-student in the same faculty at the section Applied Probability Theory, where he finished this thesis.