Abstract: For the experimental investigation of the static liquefaction of a slope of water-saturated fine sand first a uniform submerged sand layer is composed in a large liquefaction tank by applying fluidisation by upward flowing pore water from the bottom of the tank. Subsequently a submerged slope is created by applying hydraulic suction-dredging. Finally this submerged slope of uniform loose fine sand is loaded by inclining the tank and/or applying an impact load on the crest, inducing static liquefaction of the slope and a subsequent liquefied flow slide.

In this paper the composition of the fluidization system is investigated, involving the required upward water flow, the internal diameter of the horizontal fluidization tubes for supplying the water, the diameter and distance between the lateral holes in the tubes for injecting the water into the sand bed and the water pressure of the supplied water to the fluidization tubes. For achieving a uniform sand bed, the fluidizing upward water flow must be practically constant along the full length of the horizontal fluidization tubes.

To quantify the effects of the affecting parameters of the fluidization system, the water flow in the tubes is formulated mathematically, implemented numerically by applying the finite element method and quantified using a parametric numerical study.

It is concluded that a set of horizontal parallel plastic tubes on the bottom of the tank at relative distances of 0.2 [m] in combination with the application of an input water pressure of only $p_{in}=50$ [kPa] is suitable if tubes with internal diameters of $D=40$ [mm] are selected.

Keywords: Fluidization system, uniform loose fine sand bed, liquefaction testing, scale model, FEM

1 INTRODUCTION

Fluidization of sand has been employed in experiments to investigate the liquefaction phenomenon (e.g. Vardoulakis, 2004, Haigh et al., 2012). Besides it provides an appropriate method to prepare a uniform, loosely packed sand bed, which will be susceptible to liquefaction after deposition (Hryciw et al., 1985). In this paper the composition of a fluidization system for sand bed preparation purposes is investigated, involving the required upward water flow, the internal diameter of the horizontal fluidization tubes for supplying the water, the diameter and distance between the lateral holes in the tubes for injecting the water into the sand bed and the water pressure of the supplied water to the fluidization tubes.

The static liquefaction tank has a length of $l = 5$ m and a width of $w = 2$ m, while the maximum water depth in the initially horizontal tank is $h_w = 2$ m from the bottom to the lower edge (overflow) of the tank.

In the tank a water saturated bed of uniform fine sand with a mean grain size of 100 ȝm is placed. This sand bed is expected to have an initial thickness of about $h_0 = 1.5$ m at an initial porosity of about $n_0 = 0.4$, which corresponds to a void ratio $e_0 = n_0/(1-n_0) = 0.4/0.6 \approx 0.667$.

With a specific density of the sand particles $\rho' = 2700$ kg/m$^3$, for the dry mass $M$ of the sand bed in the tank follows

$$M' = (1-n_0)h_0 \rho'wl = (1-0.4) \times 1.5 \times 2700 \times 2 \times 5 = 24317 \text{ kg}$$ (1)
To create a uniform and very loose sand bed, a fluidizing upward pore water flow is applied from a fluidization system placed at the bottom of the tank. During the fluidization the sand expands to an estimated maximum thickness of $h_t = 2.00 \text{ m}$ at a porosity $n_m$, which follows from the mass conservation of the solid particles, namely

$$(1-n_0)h_0 = (1-n_m)h_m \Rightarrow n_m = 1 - (1-n_0)\frac{h_t}{h_m} = 1 - (1-0.4)\frac{1.5}{2.0} \approx 0.55$$ (2)

which corresponds to a void ratio $e_m = n_m/(1-n_m) = 0.55/0.45 \approx 1.222$. It should be noted that this value follows from an estimated expansion during fluidization, which needs to be verified experimentally. For the bulk saturated density $\rho^{sat}$ follows

$$\rho^{sat} = (1-n_m)\rho_s + n_m\rho^w = (1-0.55)\times2700 + 0.55\times1000 \approx 1765 \text{ kg/m}^3$$ (3)

During fluidization the corresponding dimensionless vertical gradient $\Delta\varphi/\Delta z$ of the steady hydraulic potential $\varphi$ must balance the total volumetric weight of the saturated soil (Terzaghi, 1925, Vardoulakis, 2004), namely

$$\frac{\Delta\varphi}{\Delta z} = -\frac{\rho^{sat}}{\rho^w\Delta z} = -\frac{1765}{1000} = -1.765\left[-\right]$$ (4)

According to Darcy (1856) the corresponding vertical discharge $q_z$ is expressed by

$$q_z = -k_z\frac{\Delta\varphi}{\Delta z}$$ (5)

in which the vertical permeability $k_z$ is expressed by

$$k_z = \kappa_z \frac{\rho^w g}{\mu^w} \approx \frac{C n_m^3 d_z^2 \rho^w g}{(1-n_m)^2 \mu^w} \approx \frac{0.0072\times0.55^3\times10^{-8}}{(1-0.55)^2} \frac{10^4}{0.001002} \approx 5.9\times10^{-4} \left[\frac{\text{m}}{\text{s}}\right]$$ (6)

in which $\kappa_z = C n_m^3 d_z^2/(1-n_m)^2 \approx 0.0072\times0.55^3\times10^{-8}/(1-0.55)^2 \approx 5.915\times10^{-11} \left[\text{m}^2\right]$ is the vertical intrinsic permeability component (Taylor, 1948) and $\rho^w g \approx 10^4 \left[\text{N/m}^3\right]$ is volumetric weight of water. Furthermore, at a temperature $20^\circ\text{C}$ the viscosity of pure water equals $\mu^w \approx 0.001002 \left[\text{Ns/m}^2\right]$ (Batchelor, 1990) and the density of water is $\rho^w \approx 998.2 \left[\text{kg/m}^3\right]$ (Batchelor, 1990), from which follows for the corresponding kinematical viscosity of water $\nu^w = \mu^w/\rho^w \approx 0.001002/998.2 \approx 1.004\times10^{-6} \left[\text{m}^2/\text{s}\right]$.

Substituting (6) in (5) gives for the vertical discharge $q_z$ of the pore water per unit of cross-sectional area

$$q_z = -k_z\frac{\Delta\varphi}{\Delta z} \approx 5.9\times10^{-4}\times1.765 \approx 1.04\times10^{-3} \left[\frac{\text{m}}{\text{s}}\right]$$ (7)

The fluidization system is considered to be composed of $N=10$ horizontal plastic tubes put at the bottom of the tank and directed in the longitudinal direction of the tank, thus with a length of $l = 5 \text{ m}$ and at relative distances of $w/N = 2/10 = 0.2 \text{ m}$. 

2 STRENGTH OF PLASTIC TUBES

The tubes for the fluidization system need to be durable and sufficiently resistant to wearing by the fluidization water jetting out of the holes in the sides of the tubes, which makes PVC tubes attractive.

Considering the position of the fluidization tubes at the bottom of the tank, they are loaded by the weight of the sand and the induced changes by the fluidization and the subsequent experiments. Consequently the tubes need to be sufficiently stiff and strong. Considering the length of the tank the tubes also need to have the same length of 5 metres.

In table 1 the characteristics of the available tubes for the internal pressure classes of 10 and 16 [bar] and a length of 5 [m] are summarized. The strength of PVC pressure tubes is indicated in terms of classes of maximum internal pressure in [bar].

Table 1. Internal diameter $D$ and wall thickness of PVC pressure tubes for the internal pressure classes of 10 and 16 [bar] and a length of 5 [m].

<table>
<thead>
<tr>
<th>Internal diameter $D$ [mm]</th>
<th>Wall thickness [mm]</th>
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<tbody>
<tr>
<td></td>
<td>Pressure class 10 [bar]</td>
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<tr>
<td>25</td>
<td>-</td>
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<tr>
<td>32</td>
<td>1.6</td>
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<tr>
<td>40</td>
<td>1.9</td>
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<td>63</td>
<td>2.4</td>
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<tr>
<td>75</td>
<td>2.9</td>
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3 HOLES IN PLASTIC TUBES

Small horizontal holes are drilled in the fluidization tubes opposite to each other and at distances of provisionally $l_{hole} = 0.2$ m. For the discharge of water from the plastic tubes through each hole follows

$$q_{hole} = q_z \times l_{hole} \times \frac{w}{2N} = 1.04 \times 10^{-3} \times 0.2 \times 0.1 \approx 2.08 \times 10^{-5} \left[ \frac{\text{m}^3}{\text{s}} \right]$$

The discharge $q_{hole}$ through a horizontal hole with diameter $d_{hole}$ in a plastic tube depends on the difference of the water pressures $p_{int}$ inside and $p_{ext}$ outside the tube. This relation follows from the conservation of energy, where the loss of potential energy equals the gain of kinetic energy, namely

$$(p_{int} - p_{ext}) = \rho \omega g (h_{int} - h_{ext}) = \frac{\rho \omega v_w^2}{2} \Rightarrow v_w = \sqrt{\frac{2(p_{int} - p_{ext})}{\rho \omega}}$$

The maximum internal water pressure in the plastic tube is provisionally taken at $p_{int}=10^5$ [Pa] and for the external pore water pressure $p_{ext}$ at the bottom of the tank the magnitude follows from the case with maximum flow during fluidization, as expressed by (3), leading to

$$p_{ext} = \rho \omega g h^m = 1765 \times 10 \times 2 \text{ [Pa]} = 3.53 \times 10^4 \text{ [Pa]}$$

Then for the velocity $v_w$ of the water flowing through the hole follows

$$v_w = \sqrt{\frac{2(p_{int} - p_{ext})}{\rho \omega}} = \sqrt{\frac{2(10^5 - 3.53 \times 10^4)}{10^3}} \approx 11.3 \left[ \frac{\text{m}}{\text{s}} \right]$$
The cross-sectional area \( A_{\text{hole}} \) of the hole follows from the discharge \( q_c \) (7) per hole, the velocity \( v_w \) (11) of the water through the hole and the dimensionless contraction coefficient \( c \approx 0.62 \), namely

\[
A_{\text{hole}} = \frac{q_{\text{hole}}}{cv_w} \approx \frac{2.08 \times 10^{-5}}{0.62 \times 11.3} \approx 2.97 \times 10^{-6} \left[ \text{m}^2 \right]
\]  

(12)

Then for the corresponding diameter \( d_{\text{hole}} \) of the hole follows

\[
d_{\text{hole}} = \sqrt{\frac{4A_{\text{hole}}}{\pi}} \approx \sqrt{\frac{4 \times 2.97 \times 10^{-6}}{\pi}} \approx 1.94 \times 10^{-3} \left[ \text{m} \right]
\]  

(13)

From (8), (11), (12) and (13) the following parameter concerning the holes in the fluidization tubes can be derived, namely

\[
\left( \frac{d_{\text{hole}}}{l_{\text{hole}}} \right)^2 = \frac{q_c w \sqrt{2 \rho_w}}{\pi c N \sqrt{p_{\text{in}} - p_{\text{ext}}}} = \frac{1.04 \times 10^{-3} \times 2 \sqrt{2 \times 998.2}}{\pi \times 0.62 \times 10 \sqrt{p_{\text{in}} - p_{\text{ext}}}} \approx \frac{4.77 \times 10^{-3}}{\sqrt{p_{\text{in}} - p_{\text{ext}}}} \left[ \text{m} \right]
\]  

(14)

which is quantified by substituting (7) and the other parameters, namely \( w = 2 \left[ \text{m} \right] \), \( \rho_w \approx 998.2 \left[ \text{kg/m}^3 \right] \), \( c \approx 0.62 \), \( N = 10 \), and is applicable if the water pressures are expressed in terms of \( \left[ \text{N/m}^2 \right] \).

4 DIAMETER OF PLASTIC TUBES

The flux of water \( q_{\text{tube}} \) distributed by each tube reads

\[
q_{\text{tube, in}} = q_c \times l \times \frac{w}{N} = 1.04 \times 10^{-3} \times 5 \times 0.2 \approx 1.04 \times 10^{-3} \left[ \text{m}^3 \right]
\]  

(15)

Then the corresponding average velocity \( V_{\text{tube}} \) of the inflowing water of each tube is expressed by

\[
V_{\text{tube, in}} = \frac{q_{\text{tube, in}}}{A_{\text{tube}}} = \frac{4q_{\text{tube, in}}}{\pi D^2} = \frac{4 \times 1.04 \times 10^{-3}}{\pi D^2} \approx \frac{1.32 \times 10^{-3}}{D^2} \left[ \frac{\text{m}}{\text{s}} \right]
\]  

(16)

The diameter \( D \) of the plastic tubes and the pressure \( \bar{p} \) of the water entering the tubes need to be chosen such that the outflow through all the holes is practically constant along their length and the amount of influx corresponds to the required outflow for fluidization.

The average velocity \( V \) of horizontal water flow through a cross-section at any location \( x \) along a smooth circular tube with length \( l = 5 \left[ \text{m} \right] \) is for linear viscosity \( \mu_w \) of the pore water and laminar flow according to Poiseuille (Batchelor, 1990) described by

\[
V = \frac{D^2 \ dp}{32 \mu_w \ dx}
\]  

(17)

in which \( dp/dx \) is the horizontal water pressure gradient locally in the tube. Laminar flow in smooth tubes is limited to the small mean velocity \( V \) range, specified by \( 0 \leq R_e \leq \sim 2000 \), which involves the dimensionless Reynolds number \( R_e \), namely

\[
R_e = \frac{D \rho_w V}{\mu_w} = \frac{DV}{\nu_w} \approx \frac{DV}{1.004 \times 10^{-6}}
\]  

(18)
in which \( v^w \) is the kinematic viscosity of water in \( \text{[m}^2/\text{s}] \), the tube diameter \( D \) is in [m] and mean velocity \( V \) is in \( \text{[m/s]} \). For larger Reynolds number \( R_e > 2000 \) (Batchelor, 1990) the flow is expected to be turbulent.

Considering the loss of water flowing through the holes along both sides of the tube per unit of length, namely \( 2q_{\text{hole}}/l_{\text{hole}} \) according to (8), (11) and (12), and the negative gradient \( dV/dx \) of the average water velocity in the tube, the conservation of mass of water can be expressed by

\[
A_{\text{tube}} \frac{dV}{dx} + 2 \frac{q_{\text{hole}}}{l_{\text{hole}}} = 0
\]

which after rearrangement and substitution of (11) and (12) gives

\[
\frac{dV}{dx} = -\frac{2}{A_{\text{tube}}} \frac{q_{\text{hole}}}{l_{\text{hole}}} = -\frac{2}{A_{\text{tube}}} \frac{A_{\text{hole}} c v^w}{l_{\text{hole}}} = -\frac{2 c d^2_{\text{hole}}}{D^2 l_{\text{hole}}} \sqrt{\frac{2(\rho_{\text{int}} - \rho_{\text{ext}})}{\rho^w}}
\]

in which \( A_{\text{tube}} = \pi D^2/4 \) is the cross-sectional area of a plastic tube. Differentiation of (17) and substituting this in (20) leads to the following expression for the distribution of the water pressure in the fluidization tube, namely

\[
\frac{d^2 p}{dx^2} = \frac{64 \mu^w c d^2_{\text{hole}}}{D^4 l_{\text{hole}}} \sqrt{\frac{2(\rho_{\text{int}} - \rho_{\text{ext}})}{\rho^w}} = 0
\]

This expression can be simplified to

\[
\frac{d^2 s}{dx^2} - B \sqrt{s} = 0
\]

in which \( s \) is a dimensionless stress measure, defined by

\[
s = \left( \frac{p - p_{\text{ext}}}{p_{\text{ext}}} \right) \Rightarrow p - p_{\text{ext}} = s p_{\text{ext}}
\]

and the parameter \( B \) in (22) is expressed by

\[
B = \frac{64 \mu^w c d^2_{\text{hole}}}{D^4 l_{\text{hole}}} \sqrt{\frac{2}{\rho^w p_{\text{ext}}}}
\]

After substituting (14) in (24) the expression of parameter \( B \) in (22) and (24) reduces to

\[
B = \frac{128 \mu^w q^w}{\pi D^4 N} \frac{1}{p_{\text{ext}}} \sqrt{\frac{p_{\text{ext}}}{\rho - p_{\text{ext}}}}
\]

in which occur as the only unknown variables the internal diameter of the tubes \( D \) and the pressure \( p = \rho \) of the water flowing into the tube at \( x = 0 \).

Further simplification of field equation (22) is achieved by introducing the dimensionless quantity

\[
t = \sqrt{s} \Rightarrow s = t^2
\]

Then after differentiation of \( s \) with respect to \( x \) follows
\[
\frac{ds}{dx} = \frac{dt^2}{dx} = 2t \frac{dt}{dx} = \frac{d^2s}{dx^2} = 2 \frac{d}{dx} \left( t \frac{dt}{dx} \right) = 2 \left( \frac{dt}{dx} \right)^2 + t \frac{d^2t}{dx^2}
\]  
\text{(27)}

Substituting (26) and (27) in (22) leads to the following non-linear differential equation for the distribution of the water pressure in the fluidization tube in terms of the dimensionless variable \( t \) (26), namely

\[
\frac{d^2t}{dx^2} + \frac{1}{t} \left( \frac{dt}{dx} \right)^2 - \frac{B}{2} = 0
\]  
\text{(28)}

This expression (28) can be solved numerically for a range of values of parameter \( B \), leading to relations of the distribution of the internal water pressure \( p(x) \) over the length \( l = 5 \text{ m} \) of the tube.

The boundary conditions of the water flow through the tube are

At begin of tube: \( x = 0 \Rightarrow p = \bar{p} = 10^5 \text{ [Pa]} \)  
\text{(29)}

At end of tube: \( x = l \Rightarrow V = \frac{D^2}{32 \mu_w} \frac{dp}{dx} = 0 \left[ \text{m/s} \right] \Rightarrow \frac{dp}{dx} = 0 \left[ \text{N/m}^2 \right] \)  
\text{(30)}

It is noted that boundary condition (29) enables to estimate the corresponding magnitude of the diameter \( d_{hole} \) by applying (11)~(13) for any input pressure of the pore water \( \bar{p} \) before calculating the internal pressure distribution within the tube. However, this implies that the resulting average velocity of the inflowing water \( V_{in} \) of each tube will differ from the required value \( V_{tube, in} \) according to (16). Consequently for any given water pressure \( \bar{p} \) at the beginning of the tube the remaining variables of the problem are:

\[ D - \text{internal diameter of plastic tube} \]
\[ q(x) - \text{resulting influx of water at begin of tube} \]
\[ p(x) - \text{resulting internal water pressure distribution} \]

In terms of dimensionless variable \( t = \sqrt{s} = \sqrt{(p-p_{ext})/p_{ext}} \), defined by (23) and (26), the boundary conditions (29) and (30), which involve the water pressure \( p \), can be transformed and expressed in terms of quantity \( t \), which occurs in (28), namely by respectively

At begin of tube: \( x = 0 \Rightarrow t = \bar{t} \bigg|_{x=0} = \sqrt{\bar{p}/p_{ext}} - 1 = \sqrt{10^5/p_{ext}} - 1 \)  
\text{(31)}

At end of tube: \( x = l \Rightarrow \frac{dt}{dx} \bigg|_{x=l} = \frac{1}{2t} \frac{dp}{dx} = 0 \)  
\text{(32)}

For estimating the relevant range of parameter \( B \) (24) the following industrial internal tube diameters \( D \) are considered, namely 0.025, 0.032, 0.040, 0.050, 0.063 and 0.075 [m], which are available at lengths of 5 [m] and are classified for maximum pressures of 10 and 16 [bar], summarized in table 1. At a temperature of \( 20^\circ \text{C} \) for pure water the viscosity \( \mu_w \approx 0.001002 \text{ [Ns/m}^2] \). For the vertical water flux \( q_z \), inducing fluidization of the fine sand in the tank has been found \( q_z \approx 1.04 \times 10^{-3} \text{ [m/s]} \) (7). The width \( w \) of the tank is \( w = 2 \text{ [m]} \) and the number \( N \) (8) of tubes is \( N = 10 \). The water pressure at fluidisation at the bottom of the tank \( p_{ext} = 3.53 \times 10^4 \text{ [Pa]} \) (10).

After substituting all above-mentioned known quantities in (25), the lower and higher limits of parameter \( B \) can be calculated by applying the following expression
lower limit \( B = \frac{1}{D^2} \cdot \frac{0.0241}{10^8} \sqrt[3]{\frac{3.53 \times 10^4}{(\bar{p} - 3.53 \times 10^4)}} \) \( \geq \) higher limit \( \left[ \frac{1}{m^2} \right] \) \( (33) \)

For the controlled input pressure \( \bar{p} = 10^5 \) [Pa] of the water at the beginning of the tube and the internal tube diameter \( D \) in table 1, the range of parameter \( B \) (25) is:

\[
\bar{p} = 10^5 \text{[Pa]} \quad \Rightarrow \quad 2.71 \times 10^{-6} \geq B \geq 4.54 \times 10^{-4} \left[ \frac{1}{m^2} \right]
\]

(34)

4.1 Composing integral expression for finite element analysis

The integral expression for finite element analysis is obtained by applying the weak formulation. To this end first the differential field expression (28) is multiplied by dimensionless weight function \( N^p(x) \) and integrated over length \( l \) of the tube, namely

\[
\int _{l} N^p \left( \frac{d^2t}{dx^2} + \frac{1}{t} \left( \frac{dt}{dx} \right)^2 - \frac{B}{2} \right) dx = 0
\]

(35)

Each term of (35) gets a dimension of \([1/m]\).

Next the boundary conditions (31) and (32) are multiplied by other suitable weight functions, integrated over the boundary and added to (35). The choice of the other suitable weight functions for the boundary conditions is such that the dimensions of all terms in the resulting weak expression are equal. It is noted that in this specific case the dimensions of the terms of (35) are already known to be \([1/m]\).

First boundary condition (31) is considered, which for this 1-dimensional case only occurs for \( x = 0 \), namely \( t - \bar{t} \rvert_{x=0}=0 \). After introducing dimensionless weight function \( M^p(x) \) and noting that the terms of this boundary condition are dimensionless, an additional unknown scalar quantity \( \eta \) is introduced to arrive at the following expression for its contribution to the weak formulation, namely

\[
-M^p \eta \left( t - \bar{t} \right)_{x=0} = 0
\]

(36)

In this 1-dimensional case the boundary integral is replaced by summation, which for this type of boundary condition is limited to only one occurrence. For making the dimension of (36) equal to that of (35) with dimension \([1/m]\), the unknown scalar quantity \( \eta \) must also have dimension \([1/m]\) as both the weight function \( M^p(x) \) and scalar quantity \( t \) are dimensionless. Finally the negative sign of (36) should be noticed, which is due to the direction of the outward normal unit vector on the left side boundary at \( x = 0 \), which formally occurs in this term as could be seen more easily by considering 2- and 3-dimensional formulations.

Finally boundary condition (32) at \( x = l \) is considered, reading \( dt/dx-(dt/dx)_{x=l} = 0 \). The dimension of this term equals that of (35), consequently multiplication by the dimensionless weight function \( +M^p(x) \) suffices, while also accounting for the positive sign of the outward normal unit vector on the boundary at \( x = l \). Consequently the resulting term for the weak formulation becomes

\[
+M^p \left( \frac{dt}{dx} - \frac{dt}{dx} \right)_{x=l} = 0
\]

(37)

Next adding (35), (36) and (37) gives for the expression of the weak formulation

\[
\int _{l} N^p \left( \frac{d^2t}{dx^2} + \frac{1}{t} \left( \frac{dt}{dx} \right)^2 \frac{B}{2} \right) dx - M^p \eta \left( t - \bar{t} \right)_{x=0} + M^p \left( \frac{dt}{dx} - \frac{dt}{dx} \right)_{x=l} = 0
\]

(38)
Next the term with the higher order of spatial differentiation \( d^2 t / dx^2 \) is considered. Then applying first the divergence theorem and subsequently partial differentiation to the integral term of the above-mentioned higher order spatial differential term \( d^2 t / dx^2 \), thus \( dt / dx \), gives

\[
\int \frac{d}{dx} \left( N_p \frac{dt}{dx} \right) \, dx = \left( N_p \frac{dt}{dx} \right)_{x=0} - \left( N_p \frac{dt}{dx} \right)_{x=1} + \int \frac{dN_p}{dx} \frac{dt}{dx} \, dx + \int N_p \frac{d^2 t}{dx^2} \, dx
\]  

(39)

Then from (39) for the term with the higher order spatial differential in (38) the following expression results

\[
\int N_p \frac{d^2 t}{dx^2} \, dx = -\int \frac{dN_p}{dx} \frac{dt}{dx} \, dx + \left( N_p \frac{dt}{dx} \right)_{x=0} - \left( N_p \frac{dt}{dx} \right)_{x=1}
\]  

(40)

This mathematical procedure enables the reduction of the maximum order of spatial differentiation. At this stage of the elaboration (40) can be substituted in (38), leading to

\[
-\int \frac{dN_p}{dx} \frac{dt}{dx} \, dx + \left( N_p \frac{dt}{dx} \right)_{x=0} - \left( N_p \frac{dt}{dx} \right)_{x=1} + \int N_p \left( \frac{1}{t} \left( \frac{dt}{dx} \right) - \frac{B}{2} \right) \, dx + \\
-M^p \eta \left( t - \bar{t} \right)_{x=0} + M^p \left( \frac{dt}{dx} - \frac{dt}{dx} \right)_{x=1} = 0
\]  

(41)

Next changing the sign of (41) and replacing the dimensional weight function \( M^p (x) \) by

\[
M^p (x) = -N^p (x)
\]  

(42)

gives

\[
\int \frac{dN_p}{dx} \frac{dt}{dx} \, dx - \left( N_p \frac{dt}{dx} \right)_{x=0} - \left( N_p \frac{dt}{dx} \right)_{x=1} + \int N_p \left( \frac{1}{t} \left( \frac{dt}{dx} \right)^2 - \frac{B}{2} \right) \, dx + \\
-M^p \eta \left( t - \bar{t} \right)_{x=0} + M^p \left( \frac{dt}{dx} - \frac{dt}{dx} \right)_{x=1} = 0
\]  

(43)

Further simplification of (43) is possible by cancelling some boundary terms, leading to

\[
\int \frac{dN_p}{dx} \frac{dt}{dx} \, dx - \left[ \frac{N_p}{t} \left( \frac{dt}{dx} \right)^2 - \frac{B}{2} \right]_{x=0} + \left( N_p \frac{dt}{dx} \right)_{x=1} = 0
\]  

(44)

When using the dimensionless scalar quantity \( t \) as the nodal scalar variable and taking the shape functions equal to the previously introduced dimensionless weight functions \( N^p (x) \) the local quantity \( t \) is the sum of the weighted nodal quantities \( \hat{t}^p \), namely

\[
t = N^p \hat{t}^p
\]  

(45)

in which the nodal quantity of node \( p \) is indicated by \( \hat{t}^p \) and summation is assumed for the repeated node number \( p \).
In this case the boundary condition at \( x = 0 \), with prescribed quantity \( \bar{T} \big|_{x=0} \), thus \( t \cdot \bar{T} \big|_{x=0} = 0 \), will be satisfied automatically, by which the term with the unknown scalar \( \eta \) can be omitted. Consequently on this boundary at \( x = 0 \) for the solution the term \( (dt/dx) \big|_{x=0} \) is not needed anymore, thus can be omitted.

Furthermore, considering the other boundary condition at \( x = l \), involving a zero term, namely \( (dt/dx) \big|_{x=l} = 0 \) \( (32) \), also that term can be omitted.

Substituting of these consequences in \( (44) \) leads to the following weak integral expression for finite element analysis of steady field expression \( (28) \) and steady boundary conditions \( (31) \) and \( (32) \), namely in index format

\[
\int_l n^p_j t_j \, dx - \int_l n^p_j \left( \frac{t_j t_j}{t} - \frac{B}{2} \right) \, dx = 0
\]  

under the condition that the steady boundary condition at \( x = 0 \), thus \( t \cdot \bar{T} \big|_{x=0} = 0 \), is applied directly on the corresponding nodal value.

In numerical analysis expression \( (46) \) can approach zero by applying the iterative Newton-Raphson process of the following residual, involving the non-zero quantities of node \( p \), namely

\[
f^p = \int_l n^p_j t_j \, dx - \int_l n^p_j \left( \frac{t_j t_j}{t} - \frac{B}{2} \right) \, dx
\]  

After substituting the nodal values \( (47) \) becomes

\[
f^p = \int_l n^p_j n^q_j \, dx \, \hat{t}^q - \int_l n^p_j \frac{N^q_j \hat{t}^q N^i_j \hat{t}^i}{N^k \hat{t}^k} \, dx + \frac{B}{2} \int_l n^p_j \, dx
\]  

For the Newton-Raphson iteration process the derivative of the residual \( f^p \) \( (48) \) with respect to nodal variable \( \hat{t} \) is needed. When applying partial differentiation of both the first and second integral terms of \( (48) \) leads to a non-symmetric matrix \( K^{pq} \), namely

\[
df^p = \frac{\partial f^p}{\partial \hat{t}^q} \, d\hat{t}^q = \int l \left\{ n^p_j n^q_j - 2 n^p_j \frac{\hat{t}^j \hat{N}_j^q N^i_j \hat{t}^i}{N^k \hat{t}^k} + n^p_j \frac{N^m _j \hat{t}^m \hat{N}_j^q \hat{t}^i}{N^k \hat{t}^k} \right\} \, dx \, d\hat{t}^q = K^{pq} \, d\hat{t}^q
\]  

Based on \( (49) \) the \( (n+1) \)-th Newton-Raphson iteration step can be defined by

\[
K^{pq} \left( \hat{t}^q \big|_{n+1} - \hat{t}^q \big|_n \right) = f^p \big|_n
\]  

in which for the residual \( f^p \big|_n \) at the end of the \( n \)-th iteration step the expression \( (48) \) can be applied.

In the first Newton-Raphson iteration step for the magnitude of the residual \( f^p \big|_{n=0} \) it is assumed that the pressure in terms of stress measure \( t \) equals the prescribed pressure at \( x = 0 \) and decreases linearly along the length of the tube by about 10%. Before it had been found that for an initially constant pressure distribution the matrix \( K^{pq} \) of the Newton-Raphson iteration scheme was singular and consequently for that initial assumption the first Newton-Raphson iteration step could not be achieved.

After the finite element calculation of the converged solution, thus when all nodal stress measures \( \hat{t}^q \big|_{n} \) at convergence are known, the resulting gradient \( dt/dx \big|_{x=0} \) of stress measure \( t \) at \( x = 0 \) can be calculated using \( (44) \) for \( x = 0 \). For the first element with length \( l \) follows from \( (44) \) and \( (48) \) for the first node \( p = 1 \) that
\[
\left. \frac{dt}{dx} \right|_{x=0} = \left( N^{p=1} \frac{dt}{dx} \right)_{x=0} = -\int \limits_{i}^{j} N^{p=1}_i N_i^q \, dx \, \hat{t}^i + \int \limits_{i}^{j} N^{p=1}_i \frac{N_i^q \hat{t}^i \hat{t}^j}{N^i \hat{t}^i} \, dx - \frac{B}{2} \int \limits_{i}^{j} N^{p=1} \, dx = -f^{p=1} \tag{51}
\]

which indicates that the gradient \( \left. \frac{dt}{dx} \right|_{x=0} \) is equal to the residual value \(-f^{p=1}\) of the first node, which is non-zero due to the boundary conditions of the prescribed stress measure \(32\), thus \( t^i|_{x=0}=0 \).

Subsequently from \((17), (23), (26) and (27)\) follows for the average velocity \( V|_{x=0} \) at the entrance of the tube at \( x=0 \) that

\[
V|_{x=0} = -\frac{D^2}{32\mu} \frac{dp}{dx}|_{x=0} = -\frac{D^2}{32\mu} p_{ext} \frac{ds}{dx}|_{x=0} = -\frac{D^2}{16\mu} p_{ext} t|_{x=0} \frac{dt}{dx}|_{x=0} \tag{52}
\]

### 4.2 Composing finite element program

The finite element program for executing the above-mentioned mathematical expressions has been composed while taking advantage of both the example programs and finite element libraries by (Smith, Griffiths, 2004) and some previously composed libraries by the second author, all in f95.

The composition of the program has been started using both the example program p41.f95 for 1-dimensional finite element analysis, and program p91.f95, involving global non-symmetric matrices. This implies also the application of the subroutine library main.f95 (Smith, Griffiths, 2004). Furthermore, the second author has applied two home-made subroutine libraries for achieving alternative iterative solutions of the global non-symmetric matrices. The developed program involves a structure of 3 nested loops, namely:

- **Pressures-loop**: for considering various values of prescribed pressures.
- **Diameters-loop**: for considering various internal tube diameters
- **Newton-Raphson iteration-loop**, involving:
  - Composition of global tangent non-symmetric band matrix
  - Calculation of global residual
  - Composition of convergence criterion
  - Solution of global equations by means of Newton-Raphson iteration step with prescribed boundary pressure at \( x=0 \)
  - Application of convergence criterion

Just after the start of the first loop for the initial distribution of the nodal stress measure \( \hat{t}^i \) a linearly decreasing value is assumed, starting at the prescribed input magnitude at \( x=0 \) and decreasing by 10% over the full length of 5 \([m]\) of the tube. This is done to obtain a non-singular matrix \( K^{pq} \) (50), necessary to perform the first Newton-Raphson iteration step.

Finally both within this nested loop structure and following this structure, some output quantities are calculated and stored in arrays, which are subsequently written in specific structured ways to some output files, which can be used by other software to generate graphical output to facilitate the interpretations. This additional software generates Postscript output files, to be read finally by the graphics Corel-Designer software for composing graphs and drawings.

### 4.3 Interpretation of calculations

In figure 1 the calculated distributions of the internal water pressure \( p(x) \) along the length \( l = 5 \,[m] \) of the fluidization tube are illustrated for internal tube diameters \( D = 25, 32, 40, 50, 63, 75 \,[mm] \) (see table 1) and input water pressures \( p_{wa} = 10^3, 9\times10^3, 8\times10^3, 7\times10^3, 6\times10^3, 5\times10^3 \,[N/m^2] \) for a fluidization system composed \( N=10 \) fluidization tubes. For all considered internal diameters the internal pressure
decreases only marginally, with the maximum decrease occurring for the tubes with the smaller internal diameter \( D = 25 \text{ [mm]} \) as expected, for which the relative decrease \( \Delta p / p_{in} \) has been collected in table 2.

The corresponding influx of water \( q_{tube} \text{ [m}^3\text{/s]} \) per fluidization tube is illustrated in figure 2 as a function of the internal tube diameter \( D \text{ [m]} \) and for the same input water pressures, \( p_{in} \). All calculated results are of the same order of magnitude, namely about \( q_{tube} \approx 1.034 - 1.040 \times 10^{-3} \text{ [m}^3\text{/s]} \), which is of the same order of magnitude as the required magnitude of \( q_{tube, in} \approx 1.04 \times 10^{-3} \text{ [m}^3\text{/s]} \) (15).

![Figure 1. Calculated distribution of water pressure along length of fluidization tube for internal tube diameters \( D=25,32,40,50,63,75 \text{ [mm]} \) and input water pressure \( p_{in}=10^5, 9 \times 10^4, 8 \times 10^4, 7 \times 10^4, 6 \times 10^4, 5 \times 10^4 \text{ [N/m}^2\text{]} \).](image1)

![Figure 2. Influx of water \( q_{tube} \text{ [ltr/s]} \) in a fluidization tube versus internal tube diameter \( D \text{ [cm]} \) for input water pressures \( p_{in}=105, 9 \times 10^4, 8 \times 10^4, 7 \times 10^4, 6 \times 10^4, 5 \times 10^4 \text{ [N/m}^2\text{]} \).](image2)

In table 3 from the calculated results the calculated diameters \( d_{hole} \) of the holes have been collected for a range of distances \( l_{hole} \) between the holes in the range \( 50 \geq l_{hole} \geq 10 \text{ [mm]} \) and for both limits of the considered range of input pressures, namely \( p_{in}=10^5 \text{ [N/m}^2\text{]} \) and \( p_{in}=5 \times 10^4 \text{ [N/m}^2\text{]} \).

### Table 2. Relative decrease of internal pressure \( \Delta p / p_{in} \) along length of fluidization tube with smaller internal diameter \( D=25 \text{ [mm]} \) and a range of input pressure \( p_{in} \).

<table>
<thead>
<tr>
<th>( p_{in} \text{ [N/m}^2\text{]} )</th>
<th>( \Delta p / p_{in} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^5 )</td>
<td>0.0027</td>
</tr>
<tr>
<td>( 9 \times 10^4 )</td>
<td>0.0030</td>
</tr>
<tr>
<td>( 8 \times 10^4 )</td>
<td>0.0034</td>
</tr>
<tr>
<td>( 7 \times 10^4 )</td>
<td>0.0039</td>
</tr>
<tr>
<td>( 6 \times 10^4 )</td>
<td>0.0045</td>
</tr>
<tr>
<td>( 5 \times 10^4 )</td>
<td>0.0054</td>
</tr>
</tbody>
</table>

### Table 3. Calculated hole diameter \( d_{hole} \) for a range of distances \( l_{hole} \) and two entrance pressures \( p_{in} \) in a fluidization tube.

<table>
<thead>
<tr>
<th>( l_{hole} \text{ [mm]} )</th>
<th>( d_{hole} \text{ [mm]} )</th>
<th>( p_{in} \text{ [N/m}^2\text{]} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^5 )</td>
<td>( 5 \times 10^4 )</td>
<td>( 50 )</td>
</tr>
<tr>
<td>( 10 )</td>
<td>( 5 \times 10^4 )</td>
<td>( 40 )</td>
</tr>
<tr>
<td>( 20 )</td>
<td>( 5 \times 10^4 )</td>
<td>( 30 )</td>
</tr>
<tr>
<td>( 30 )</td>
<td>( 5 \times 10^4 )</td>
<td>( 20 )</td>
</tr>
<tr>
<td>( 40 )</td>
<td>( 5 \times 10^4 )</td>
<td>( 10 )</td>
</tr>
</tbody>
</table>

On this basis it could be concluded that in principle even the tube with the smaller internal diameter \( D = 25 \text{ [mm]} \) in combination with the application of the smaller input pressure \( p_{in} = 5 \times 10^4 \text{ [N/m}^2\text{]} \) would be suitable. However, the calculated results also indicate that the initial assumption of laminar...
flow (17) in the tubes was not justified, namely from the parametric study the calculated Reynolds number (18) is found to be in the range $17000 < Re < 52000$.

To account for both the actual turbulent water flow in the fluidization tubes and the stability of the granular filter bed at the bottom of the tank, fluidization tubes with an internal diameter of 40 [mm] are selected. Furthermore, the holes in both sides of the tubes for injecting water horizontally into the granular filter bed are located at relative distances along the length of the tubes of $l_{\text{hole}}=50$ [mm] and are given hole diameters of $d_{\text{hole}}=1.4$ [mm].

5 CONCLUSIONS

In this paper the fluidization system for creating a homogeneous layer of loose fine sand is investigated. This uniform loose fine sand layer will form the basis of subsequent static liquefaction testing of submerged slopes. The required upward fluidizing water flow is supplied through a set of parallel fluidization tubes at the bottom of a tank and injected into the base of the sand layer through small lateral holes in these fluidization tubes.

First the required upward water flow for fluidization is quantified. Then the process of the water flow through the fluidization tubes and the water injection into the base of the sand layer is described, the corresponding integral expression for a finite element analysis is developed and implemented numerically. Finally the resulting computer program is used to perform a parametric study of the uniformity of the water supply for fluidization.

The computational results clarify the effects on the uniformity of the resulting upward fluidizing flow by the water pressure as supplied to the fluidization tubes, the internal diameter of the fluidization tubes and the diameters and respective distances of the lateral holes in these fluidization tubes. Despite the actual occurrence of turbulent flow in the tubes, the investigation has enabled the selection of an appropriate fluidization system.

REFERENCES


