THE IMPULSE RESPONSE METHOD FOR THE
CALCULATION OF STATISTICAL PROPERTIES
OF AIRCRAFT FLYING IN RANDOM
ATMOSPHERIC TURBULENCE

by

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DELFT - THE NETHERLANDS

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SUMMARY

A fast and simple calculation of ensemble properties of output signals of a linear system perturbed by gaussian stochastic input signals, can be performed by the so called impulse response method. This method applied to aircraft motions due to atmospheric turbulence, has been derived, from properties in the frequency-domain, in Report VTH-138 (Ref. 3 of this Report).

The present Report shows that this method can be very directly derived, using simple basic concepts of modern system theory.

For the sake of completeness, a short recapitulation of the derivation, using frequency-domain techniques is given in this Report. It is shown that some minor corrections should be applied to the method as given in Ref. 3. A numerical example gives an impression of the effect of these corrections on the results. A comparison has been made with results of a rather different, digital calculation of the covariance matrix, and moreover, with results obtained by a Monte Carlo simulation.
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The initial conditions on page 40 and 41 of the Appendix should read:

\[
\begin{bmatrix}
\hat{u}(0) \\
\alpha(0) \\
\theta(0) \\
\frac{q_c}{V}(0) \\
\hat{u}_g(0) \\
\alpha_g(0) \\
\alpha_g^\pi(0)
\end{bmatrix}
= \begin{bmatrix}
x\hat{\alpha}_g \cdot \sigma_{\alpha_g} \cdot \sqrt{\frac{2V}{L_g}} \\
z\hat{\alpha}_g \cdot \sigma_{\alpha_g} \cdot \sqrt{\frac{2V}{L_g}} \\
0 \\
m\hat{\alpha}_g \cdot \sigma_{\alpha_g} \cdot \sqrt{\frac{2V}{L_g}} \\
\sigma_{\alpha_g} \cdot \sqrt{\frac{2V}{L_g}} \\
0 \\
0
\end{bmatrix}
\]

and:

\[
\begin{bmatrix}
\hat{u}(0) \\
\alpha(0) \\
\theta(0) \\
\frac{q_c}{V}(0) \\
\hat{u}_g(0) \\
\alpha_g(0) \\
\alpha_g^\pi(0)
\end{bmatrix}
= \begin{bmatrix}
x\hat{\alpha}_g \cdot \sigma_{\alpha_g} \cdot \sqrt{\frac{3V}{L_g}} \\
z\hat{\alpha}_g \cdot \sigma_{\alpha_g} \cdot \sqrt{\frac{3V}{L_g}} \\
0 \\
m\hat{\alpha}_g \cdot \sigma_{\alpha_g} \cdot \sqrt{\frac{3V}{L_g}} \\
\sigma_{\alpha_g} \cdot \sqrt{\frac{3V}{L_g}} \\
0 \\
(1-2\sqrt{3}) \frac{c}{L_g} \cdot \sigma_{\alpha_g} \cdot \sqrt{\frac{V}{L_g}}
\end{bmatrix}
\]
Erratum

Equation (A-3) of the Appendix, page 40 should read:

\[
\begin{pmatrix}
\dot{u} \\
\dot{\alpha} \\
\dot{\theta} \\
\ddot{\theta} \\
\dddot{z} \\
\end{pmatrix} = 
\begin{pmatrix}
x_u & x_\alpha & x_\theta & 0 & x_{ug} & -\frac{c}{L_g} & x_{ug} & x_{ag} & x_{ag} \\
\end{pmatrix} \\
\begin{pmatrix}
z_u & z_\alpha & z_\theta & z_q & z_{ug} & -\frac{c}{L_g} & z_{ug} & z_{ag} & z_{ag} \\
\end{pmatrix} \\
\begin{pmatrix}
m_u & m_\alpha & m_\theta & m_q & m_{ug} & -\frac{c}{L_g} & m_{ug} & m_{ag} & m_{ag} \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \\
\end{pmatrix} + 
\begin{pmatrix}
\dddot{u} \\
\dddot{\alpha} \\
\dddot{\theta} \\
\dddot{\theta} \\
\dddot{z} \\
\end{pmatrix} = 
\begin{pmatrix}
x_{ug} & m_{ug} & -\frac{c}{L_g} & m_{ug} & m_{ag} & m_{ag} & \frac{a_c}{V} \\
\end{pmatrix} \\
\begin{pmatrix}
z_{ug} & z_{ag} & z_{ag} \end{pmatrix} \\
\begin{pmatrix}
m_{ug} & m_{ag} & m_{ag} \end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 \\
\end{pmatrix} \\
\end{pmatrix} 
\]

\[
\begin{pmatrix}
x_{ug} & m_{ug} & -\frac{c}{L_g} & m_{ug} & m_{ag} & m_{ag} & \frac{a_c}{V} \\
\end{pmatrix} \\
\begin{pmatrix}
z_{ug} & z_{ag} & z_{ag} \end{pmatrix} \\
\begin{pmatrix}
m_{ug} & m_{ag} & m_{ag} \end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 \\
\end{pmatrix} \\
\begin{pmatrix}
0 & 0 & 0 \\
\end{pmatrix} \\
\end{pmatrix} = 
\begin{pmatrix}
\frac{1}{w_1} \cdot k_1 \\
\frac{1}{w_2} \cdot k_2 \\
\end{pmatrix} 
\]

(A-3)
LIST OF SYMBOLS

[A] \( n \times n \) symmetric matrix, system matrix

\( b \) wing span

[B] forcing matrix

c element of the covariance matrix \([C]\)

c.g. aircraft's centre of gravity

\( \bar{c} \) aerodynamic mean chord

\([C] = [c_{ij}] = \sigma_{\dot{x}_i \dot{x}_j} \) covariance matrix of the state vector \( \ddot{x} \)

\( C_L \) lift coefficient

\( C_m \) pitching moment coefficient

\( C_{m_q} = \frac{\partial C_m}{\partial q} \)

\( C_{m_u} = \frac{1}{\frac{1}{2} \rho v^2 \bar{c}^2} \frac{\partial M}{\partial u} \)

\( C_{m_u, g} = \frac{1}{\frac{1}{2} \rho v^2 \bar{c}^2} \frac{\partial M}{\partial \dot{g}} \)

\( C_{m_g} = \frac{3c_m}{\dot{u} \bar{c}} \)

\( C_{m_{\alpha}} = \frac{\partial C_m}{\partial \alpha} \)

\( C_{m_{\alpha, g}} = \frac{\partial C_m}{\partial \dot{g}} \)
\[ c_{m4} = \frac{\partial c_m}{\partial \alpha} \frac{\partial \alpha}{\partial V} \]

\[ c_{m4g} = \frac{\partial c_m}{\partial g} \frac{\partial g}{\partial V} \]

\[ c_x \quad \text{coefficient of aerodynamic force along the aircraft's X-axis} \]

\[ c_{xu} = \frac{1}{\frac{1}{\rho} V^2 S} \cdot \frac{\partial x}{\partial u} \]

\[ c_{xug} = \frac{1}{\frac{1}{\rho} V^2 S} \cdot \frac{\partial x}{\partial g} \]

\[ c_{xdg} = \frac{\partial c_x}{\partial \alpha} \frac{\partial \alpha}{\partial g} \]

\[ c_{x0} \quad c_x \text{ in the steady flight condition} \]

\[ c_{x\alpha} = \frac{\partial c_x}{\partial \alpha} \]

\[ c_{x\alpha g} = \frac{\partial c_x}{\partial \alpha} \frac{\partial g}{\partial g} \]

\[ c_{x\dot{\alpha}g} = \frac{\partial c_x}{\partial \alpha} \frac{\partial \dot{\alpha}}{\partial g} \]

\[ c_z \quad \text{coefficient of aerodynamic force along the aircraft's Z-axis} \]

\[ c_{zq} = \frac{\partial c_z}{\partial q} \frac{\partial q}{\partial V} \]

\[ c_{zu} = \frac{1}{\frac{1}{\rho} V^2 S} \cdot \frac{\partial z}{\partial u} \]
\[ c_{z\alpha} = \frac{1}{\frac{1}{\rho v^2 S}} \cdot \frac{\partial z}{\partial \alpha} \]

\[ c_{z\dot{\alpha}} = \frac{\partial c_z}{\partial \dot{\alpha}} \]

\[ c_{z\dot{\alpha}_g} = \frac{\partial c_z}{\partial \dot{\alpha}_g} \]

\[ c_{z\gamma} = \frac{\partial c_z}{\partial \gamma} \]

\[ \frac{d}{dt} \text{ differential operator} \]

\[ D_c = \frac{\dot{c}}{V} \cdot \frac{d}{dt} \text{ dimensionless differential operator} \]

\[ \text{E}\{x\} \text{ expected value of the stochastic variable } x \]

\[ \mathcal{F} \text{ Fourier operator, } \mathcal{F}\{x(t)\} = X(\omega) \]

\[ [F(t)] \text{ transition matrix} \]

\[ I_y \text{ aircraft's moment of inertia around the Y-axis} \]

\[ g \text{ acceleration due to gravity} \]

\[ h \text{ altitude deviation from a datum altitude} \]
\( h(t) \) impulse response

\( H(\omega) \) transfer function

\( \text{Im} \) imaginary part of a complex variable or function

\( j = \sqrt{-1} \)

\( K \) feedback gain factor

\( K_y \) dimensionless radius of gyration, related to \( I_y \) by \( \mu \frac{K_y^2}{c} = \frac{I_y}{\rho S c^3} \)

\( \mathcal{L} \) Laplace operator, \( \mathcal{L}\{x(t)\} = X(P) \)

\( L_g \) integral scale of turbulence

\( m = \frac{W}{g} \) aircraft mass

\[
\begin{align*}
\{m_q, m_u, m_{ug}, m_{ug}, m_{\alpha}, m_{\alpha g}, m_{\delta}, m_{\theta}\}
\end{align*}
\]

Stability-, gust- and input-derivatives in abbreviated notation, see the Appendix.

\( M \) aerodynamic moment about the aircraft's Y-axis

\( n \) order of equations
\[ n_{\text{c.g.}} = \frac{a_z}{g} \]  \( \text{normal acceleration factor at the aircraft's c.g.} \)

\( P \) \( \text{complex variable} \)

\( q \) \( \text{pitching velocity about the aircraft's Y-axis} \)

\( \text{Re} \) \( \text{real part of a complex variable or function} \)

\( S \) \( \text{wing area} \)

\( t \) \( \text{time} \)

\( T \) \( \text{engine thrust, deviation of the thrust from that needed in the steady flight condition} \)

\[ T_{c'} = \frac{T}{\frac{1}{2} \rho V^2 S} \]  \( \text{thrust coefficient} \)

\( u \) \( \text{change in } \dot{V} \text{ along the aircraft's X-axis} \)

\[ \dot{G} = \frac{u}{V} \]

\( u_g \) \( \text{horizontal gust velocity, parallel to the aircraft's plane of symmetry} \)

\[ \dot{u}_g = \frac{u_g}{V} \]

\( \dot{V} \) \( \text{velocity of the aircraft's c.g. relative to the earth (assuming no steady wind velocity)} \)

\( [V] \) \( \text{intensity matrix of white noise input-signals} \)

\( w \) \( \text{white noise signal} \)
\( \bar{w}_g \) \hspace{1cm} \text{vertical gust velocity} \\
\( W \) \hspace{1cm} \text{aircraft weight} \\
\( x(t) \) \hspace{1cm} \text{(stochastic) time variable} \\
\( \bar{x} \) \hspace{1cm} \text{state vector, vector of the motion variables} \\
\begin{align*} 
\begin{cases} 
\dot{x}_u \\
 x_{ug} \\
 x_{\dot{u}g} \\
 x_{\alpha} \\
 x_{\dot{\alpha}g} \\
 x_{\alpha g} \\
 x_{\theta} \\
 x_{q} \\
 X \\
y(t) \\
 Y(\omega) = \mathcal{F}\{y(t)\} \hspace{1cm} \text{Fourier transform of } y(t) \\
\begin{cases} 
z_q \\
z_u \\
z_{ug} \\
z_{\alpha} \\
z_{\dot{\alpha}g} \\
z_{\alpha g} \\
z_{\delta} \\
z_{\theta} \\
\end{cases} \hspace{1cm} \text{stability-, gust- and input-derivatives in abbreviated notation, see the Appendix} \\
\end{align*}
\( Z \) aerodynamic force along the aircraft's Z-axis

\( \alpha \) angle of attack, dimensionless change in \( \dot{V} \) along the aircraft's Z-axis

\( \alpha_g = \frac{w_g}{V} \) gust angle of attack

\( \alpha_g^* \) auxiliary variable in the filter equations

\( \delta_e \) elevator angle

\( \delta(t) \) impulse function at \( t \)

\( \gamma \) flight path angle, angle between the aircraft's velocity vector \( \dot{V} \) and the horizontal plane

\( \theta \) angle of pitch

\( \mu \) average of a stochastic variable

\( \mu_c = \frac{m}{\rho S \bar{c}} \) relative aircraft mass

\( \rho \) air density

\( \sigma_x^2 \) variance of \( x \)

\( \sigma_{x_1x_2} \) covariance of \( x_1 \) and \( x_2 \)

\( \tau \) time constant, correlation time interval

\( \Phi(\omega) \) power spectral density (power spectrum)

\( \omega \) angular frequency
Subscripts and superscripts

c.g.  centre of gravity

e  engine

g  gust, turbulence

i  input

u  output

\( u^, g \)  horizontal turbulence

\( \bar{g} \)  vertical turbulence

\( \alpha^, g \)  conjugate of a complex variable or function

\( * \)  auxiliary variable in filter equations

T  transpose of a vector or matrix

Frame of reference

All stability derivatives are defined relative to a frame of reference having its origin 0 at the aircraft's centre of gravity c.g., the X-axis being in the plane of symmetry, parallel with the velocity vector \( \vec{V} \) in the steady flight condition, and taken positive in the forward direction. The Y-axis is perpendicular to the plane of symmetry and taken positive to starboard while the Z-axis is perpendicular to the X-O-Y-plane and positive downwards.
1. INTRODUCTION

In the design of control systems perturbed by stochastic signals it is often necessary to know the probability of a stochastic variable $x(t)$ exceeding a certain value. Thus the probability distribution or the probability density distribution of $x(t)$ is to be known. As the method by which these distributions can be obtained depends on the stochastic processes under consideration, it is useful to define the following classes:

1. Random stationary gaussian processes.
2. Random gaussian processes with time-dependent statistical properties.

As a treatment of the third class (non-gaussian processes) is beyond the scope of this report, it may be sufficient to state that distributions in this case are usually to be obtained by the sampling of a simulation in the time-domain (Monte Carlo method), although recently a theoretical calculation of the distribution density function of non-gaussian atmospheric turbulence was introduced by Reeves et al (Ref. 13).

In the case of gaussian processes (classes 1 and 2), where the so called first and second central moments completely determine the probability distribution as well as the probability density distribution, it is sufficient to obtain these two central moments i.e. the average of $x(t)$ and the variance $\sigma_x^2$ of $x$. When dealing with deviations of $x$ from a certain datum, as often is the case, the average $\bar{x}$ is zero and only $\sigma_x^2$ need to be obtained. This can be done in a number of ways.

For class 1 (stationary processes) the variance can be obtained by a Monte Carlo simulation. Under the assumption of ergodicity it may be determined either as an ensemble average at one instant in time from a relatively large number of replications of $x(t)$, or, as a time average from a large number of samples taken, at certain intervals, from one realisation of $x(t)$. For time-varying gaussian processes (class 2) the
variance of $x$ will be a function of time and can only be determined as an ensemble average.

Due to the large number of replications or samples needed for an accurate estimation of the variance, a Monte Carlo simulation is a rather time consuming method.

Another possible method, for stationary gaussian processes only (class 1), is integrating the (auto) power spectral density, or more briefly power spectrum of $x$, $\phi_{xx}(\omega)$ according to (see Ref. 1):

$$\sigma_x^2 = \int_0^\infty \phi_{xx}(\omega) \cdot d\omega$$

(1)

Although this method may be useful for a quick estimation of $\sigma_x^2$ in the case of a simple low order system with a single random input signal, allowing the integral (1) to be obtained in a simple analytical form, calculation of the variance will be rather cumbersome for more complicated systems of a higher order.

The subject of this report is still another and rather different method of calculating the variance of a stochastic variable $x(t)$. It enables the variance of stochastic output signals of complicated linear systems if a higher order, perturbed by gaussian noise signals, to be obtained in a fast and simple manner.

Stated very briefly, this calculation, in the case of stationary gaussian signals, is based on obtaining the variance of a stochastic output signal $x(t)$ by an integration in the time-domain:

$$\sigma_x^2 = \int_0^\infty y^2(t) \cdot dt$$

(2)
where $y(t)$ is a particular, suitably chosen deterministic time function. The relation between the deterministic time function $y(t)$ and the stochastic process $x(t)$ can be established by using Parseval's theorem (Ref. 11).

It can be shown (see Refs. 3 and 4) that if $x(t)$ is the stochastic output signal of a system having a single white noise input signal, the time function $y(t)$ is identical with the impulse response $h(t)$ of that system. The way in which the time function $y(t)$ may thus be obtained for an aircraft perturbed by atmospheric turbulence is illustrated in Fig. 1 in which the noise filter, having a white noise input signal and yielding coloured noise representing the gust velocities as an output signal, is joined with the aircraft into one dynamical element. In this case $y(t)$ is identical with the impulse response $h(t)$ of the noise filter cum aircraft.

The principle of obtaining the variance as the integral with respect to time of the impulse response squared was given by Laning and Battin in 1956 (Ref. 1). Etkin (Ref. 2) in 1960 published an adaption of this impulse response method to aircraft perturbed by vertical turbulence, yielding however, only an approximate representation of the Dryden power spectral density of vertical turbulence. This spectrum appears in fact to be that of horizontal turbulence.

In Ref. 3 it was shown by a mathematically rather more rigorous derivation that by using the impulse response method, the Dryden power spectra for horizontal and vertical turbulence could exactly be represented, while in Ref. 4 the method was extended to the case of gaussian random processes with (known) time varying characteristics. In Ref. 14 the impulse response method was applied to the case of asymmetric aircraft motions.

In practical calculations, use is made of the fact that impulse responses are identical with responses to properly chosen initial conditions.
Using an analogue computer the "impulse response method" allows variances (and covariances) very easily and quickly to be computed, once the necessary initial conditions have been obtained.

Due to the fact that the impulse response method has hitherto only been derived from properties in the frequency-domain, the derivation of the differential equations for \( y(t) \) and the pertaining initial conditions by inverse Laplace transforms is not a straightforward matter.

It is shown in this Report (Chapter 2) that the impulse response method can more clearly and directly be derived in the time-domain by using basic concepts of modern system theory. In this derivation it is simultaneously proved that the method holds for variances as well as covariances and that the integral according to eq. (2), if taken from 0 to \( t \) represents the growth of the variance with time. Probably most important is the fact that the initial conditions referred to before are obtained explicitly once the system equations have been properly defined.

For the sake of completeness a short recapitulation of the derivation of the impulse response method is given in Chapter 3. The time functions \( y(t) \) for the gust velocities have not explicitly been derived but it is shown, using frequency-domain techniques, that they are identical with impulse responses of the turbulence filters. The equations of the turbulence filters and the necessary initial conditions are derived in Chapter 4, using the time-domain derivation given in Chapter 2.

It appeared that in the initial conditions as given in Refs. 3 and 4 some terms are missing, notably in the initial conditions necessary to obtain the signals \( y(t) \) for the time derivatives of the gust velocities. As a consequence the calculated variances of Refs. 3 and 4 will show an inaccuracy. Chapter 5 gives some numerical results to quantitatively show the influence of the necessary corrections to the initial conditions.
2. THE DERIVATION OF THE "IMPULSE RESPONSE METHOD" FROM AN EXPRESSION FOR THE COVARIANCE MATRIX AS A FUNCTION OF TIME

For a given constant linear system, the state equation is, see Refs. 7 and 8:

\[ \dot{\bar{x}}(t) = [A] \cdot \bar{x}(t) + [B] \bar{x}_1(t) \]  \hspace{1cm} (3)

where \( \bar{x}(t) \) is the state vector, the elements of which are the state variables, while \( \bar{x}_1 \) is the vector valued driving function. In the case of an aeroplane the state variables are the aircraft's motion variables while the vector \( \bar{x}_1(t) \) may consist of input signals like control surface deflections or gust velocities.

If the system is of the \( n \)-th order, \([A]\) is an \( n \times n \) matrix, whereas if \( \bar{x}_1 \) consists of \( m \) input signals, \([B]\) is an \( n \times m \) matrix.

First the general solution to eq. (3) for an arbitrary \( \bar{x}_1(t) \) and arbitrary initial conditions \( \bar{x}(0) \) is considered. This solution is usually written using the transition matrix \([F(t)]\), defined by:

\[ [F(t)] = \mathcal{L}^{-1} \left[ \{P[I] - [A]\}^{-1} \right] \]  \hspace{1cm} (4)

where \( \mathcal{L}^{-1} \) denotes the inverse Laplace transform, \( P \) is the complex Laplace variable and \([I]\) is the unit matrix. The solution of the state equation (3), using the transition matrix defined by (4) becomes, see also Refs. 7 and 8:

\[ \bar{x}(t) = [F(t)] \cdot \bar{x}(0) + \int_{0}^{t} [F(t - \tau)] [B] \bar{x}_1(\tau) \, d\tau \]  \hspace{1cm} (5)

The first term of the right hand side of eq. (5) is the system's response to a set of initial conditions \( \bar{x}(0) \), the second is the response to a set of input signals \( \bar{x}_1(t) \).

To demonstrate that the impulse response of the system is equivalent to the response to certain initial conditions, a property to be used in the following derivations, consider the case where all of the \( m \)
elements of $\bar{x}_i$ are zero except one, for instance the $j$-th element $x_{ij}$ of $\bar{x}_i(t)$. The matrix $[B]$ then reduces to the one pertaining vector $\bar{b}_j$. If in addition the initial conditions $\bar{x}(0)$ are zero, then $\bar{x}(t)$ is given by:

$$\bar{x}(t) = \int_0^t [F(t - \tau)] \bar{b}_j . x_{ij}(\tau) \, d\tau$$  \hspace{1cm} (6)

Now let $x_{ij}(\tau)$ be a unit impulse function at $\tau = 0$. Substitution of

$$x_{ij}(\tau) = \delta(0)$$

into eq. (6) yields the impulse response of the system:

$$\bar{x}(t) = \bar{h}_j(t) = [F(t)] . \bar{b}_j$$  \hspace{1cm} (7)

The subscript $j$ in $h_j(t)$ denotes the impulse response to $x_{ij} = \delta(0)$. By comparing eqs. (7) and (5) it is apparent that the impulse response is identical with a response to initial conditions (first term of the right hand side of eq. (5)) if:

$$\bar{x}(0) = \bar{b}_j$$

Next a similar system is considered, now driven by a number of uncorrelated zero mean white noise input signals, having a vector $\bar{w}(t)$. In the case of an aircraft perturbed by atmospheric turbulence such a system may be obtained by joining the noise filters, shaping white noise into the coloured noise representing the gust velocities, and the aircraft itself into one dynamical system as illustrated in Fig. 1.

The white noise signals and their characteristics need some explanation. Being zero mean processes, the first central moments i.e. the averages are zero:
\[ \bar{\mu} = \mathbb{E}\{\bar{w}(t)\} = 0 \]

As to the second central moment, or the variance of white noise, this cannot be used as a characteristic as it would theoretically be infinite. Therefore, the intensity \( v \) of a constant white noise process is introduced.

According to Ref. 7 the covariance function of white noise can be idealized as:

\[ C_w(\tau) = v . \delta(\tau) \]  \hspace{1cm} (8)

where \( \delta \) is a delta unit (impulse) function and \( v \) is called the intensity of a white noise process. The power spectral density of white noise can be obtained from (8) as the Fourier transform of \( v . \delta(\tau) \):

\[ \phi_{ww}(\omega) = \frac{v}{\pi} \]

Accordingly the intensity matrix \([V]\) of a vector valued white noise process contains the intensities \( v_{pp} \) of the corresponding white noise signals \( w_p(t) \) as the diagonal elements, the off-diagonal elements being zero as the white noise processes are assumed to be uncorrelated.

In Ref. 7 it is shown that the covariance matrix \([C_x(t)]\) of the state vector \( \vec{x} \) of a constant linear system driven by a number of gaussian white noise signals is given by:

\[ [C_x(t)] = [F(t)] [C_x(0)] [F(t)]^T + \]

\[ + \int_{0}^{t} [F(t - \tau)] [B] [V] [B]^T [F(t - \tau)]^T d\tau \]  \hspace{1cm} (8)

where \([C_x(0)]\) is the covariance matrix at time \( t = 0 \), \([F]\) is the transition matrix mentioned before and \( ^T \) denotes the transpose of a
matrix. If all variances and covariances of the system's state vector are zero at \( t = 0 \), so \( [C_x(0)] = 0 \), then the first right hand term disappears.

Now suppose that all white noise input signals are zero except one element \( w_j(t) \) of \( \tilde{w}(t) \), the intensity of which is taken to be:

\[
v_{jj} = 1
\]

The matrix \([B]\) now again reduces to \( \tilde{b}_j \) and eq. (8) can be written as:

\[
[C_x(t)] = \int_0^t [F(t - \tau)] \tilde{b}_j \cdot \tilde{b}_j^T \cdot [F(t - \tau)]^T d\tau
\]

\[
= \int_0^t [F(t - \tau)] \tilde{b}_j \left[[F(t - \tau)] \tilde{b}_j\right]^T d\tau
\]

Next a new variable \( \nu = t - \tau \) is introduced. Then the covariance matrix is written as:

\[
[C_x(t)] = -\int_0^t [F(\nu)] \tilde{b}_j \cdot [F(\nu) \tilde{b}_j]^T d\nu
\]

After changing the boundaries of integration, and changing the argument of integration from \( \nu \) to \( t \), eq. (9) now becomes:

\[
[C_x(t)] = \int_0^t [F(t)] \tilde{b}_j \cdot \left[[F(t)] \tilde{b}_j\right]^T dt
\]

Comparing eqs. (10) and (7) is appears that:

\[
[C_x(t)] = \int_0^t \tilde{h}_j(t) \cdot \tilde{h}_j(t)^T dt
\]

In particular, a diagonal element \( c_{pp}(t) \) of \( [C_x(t)] \), being the variance \( \sigma_{x_p}^2 \) of the \( p \)-th element of \( \tilde{x}(t) \), is, according to eq. (11):
\[ c_{pp}(t) = \sigma_{x_p}^2(t) = \int_0^t h_{jp}(t) \, dt \]  
\[ (12a) \]

An off-diagonal element is then:

\[ c_{pk}(t) = \sigma_{x_p x_k}(t) = \int_0^t h_{jp}(t) \cdot h_{jk}(t) \cdot dt \]
\[ (12b) \]

According to equations (12a) and (12b) the growth in time of the variance of an output signal \( x_p(t) \) of a system perturbed from \( t = 0 \) onwards, by a single white noise signal \( w_j(t) \) can be obtained by integrating the square of the system's impulse response with respect to time, while the covariance of two output signals is obtained by integrating the product of the impulse response functions.

For practical calculations the impulse response is usually obtained as the response to properly chosen initial conditions i.e.:

\[ \bar{x}(0) = \bar{b}_j \]

see eq. (7).

The variances and covariances caused by all \( m \) white noise input signals acting simultaneously from \( t = 0 \) onwards, can, still assuming uncorrelated white noise signals and system linearity, be obtained by summing the results of eqs. (12a) and (12b) for all \( m \) input signals:

\[ \sigma_{x_p}^2 = \sum_{j=1}^m \int_0^t h_{jp}(t) \cdot dt \]

\[ \sigma_{x_p x_k}(t) = \sum_{j=1}^m \int_0^t h_{jp}(t) \cdot h_{jk}(t) \cdot dt \]

The way in which, in the case of an aircraft perturbed by uncorrelated horizontal and vertical turbulence, the total variance of an output
signal is obtained as the sum of the separate variances, is schematically given in the diagram of Fig. 2.

One of the advantages of the derivation given here, apart from the obvious benefit of avoiding any inverse Laplace transforms from the frequency-domain to the time-domain, is that the necessary initial conditions are obtained explicitly once the state equation (3) is known. This equations and the resulting initial conditions have been derived in the Appendix for the symmetric motions of an aircraft flying through turbulence.
3. THE IMPULSE RESPONSE METHOD DERIVED FROM PROPERTIES IN THE FREQUENCY-DOMAIN

Suppose a stochastic variable \( x(t) \) is given, the power spectral density of which is \( \Phi_{xx}(\omega) \). Then the variance of \( x(t) \) can be obtained according to eq. (1):

\[
\sigma^2_x = \int_{0}^{\infty} \Phi_{xx}(\omega) \, d\omega
\] (12)

Next consider a new variable \( y(t) \) being deterministic instead of stochastic. The Fourier transform of \( y(t) \) is \( Y(\omega) \) defined by:

\[
Y(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} y(t) \cdot e^{-j\omega t} \, dt = \mathcal{F} \{ y(t) \}
\] (13)

whereas \( y(t) \) follows from \( Y(\omega) \) by the inverse Fourier transform:

\[
y(t) = \frac{1}{2} \int_{-\infty}^{+\infty} Y(\omega) \cdot e^{j\omega t} \, d\omega = \mathcal{F}^{-1} \{ Y(\omega) \}
\] (14)

\( \mathcal{F} \) and \( \mathcal{F}^{-1} \) denote the Fourier and inverse Fourier transforms respectively.

Now according to Parseval's theorem, see Ref. 11, the following relation between \( y_1(t) \) and \( y_2(t) \) on the one side and their Fourier transforms \( Y_1(\omega) \) and \( Y_2(\omega) \) on the other side holds:

\[
\int_{-\infty}^{+\infty} y_1(t) \cdot y_2(t) \, dt = \frac{\pi}{2} \int_{-\infty}^{+\infty} Y_1(\omega) \cdot Y_2^*(\omega) \, d\omega
\] (15)

where \( ^* \) denotes the complex conjugate.

If \( y_1(t) = y_2(t) = y(t) \) and \( y(t) = 0 \) for \( t < 0 \) then, as \( |Y(\omega)|^2 \) is an even function in \( \omega \), it follows that:

\[
\int_{0}^{\infty} y^2(t) \, dt = \pi \int_{0}^{\infty} |Y(\omega)|^2 \, d\omega
\] (16)
If finally $y(t)$ is chosen such that:

$$\pi |Y(\omega)|^2 = \Phi_{xx}(\omega)$$

(17)

equations (1), (15) and (16) result in:

$$\int_0^\infty y^2(t) \, dt = \int_0^\infty \Phi_{xx}(\omega) \, d\omega = \sigma_x^2$$

According to the foregoing it can be concluded that the variance $\sigma_x^2$ of a stochastic variable $x(t)$ can be determined by integrating the square of a deterministic time function $y(t)$ with respect to time from $t = 0$ to $t = \infty$. The conditions to be fulfilled by $y(t)$ are:

$$y(t) = 0 \quad \text{for } t < 0$$

(18a)

$$\pi |Y(\omega)|^2 = \Phi_{xx}(\omega)$$

(18b)

Now let $x(t)$ be the input signal to a system having a transfer function $H(\omega)$ and the stochastic output signal $x_u(t)$, the power spectrum of which is, see Ref. 6:

$$\Phi_{x_u x_u}(\omega) = |H(\omega)|^2 \cdot \Phi_{xx}(\omega)$$

(19)

It can easily be shown by using eq. (19) and the way in which input and output signals are related by the transfer function, that the variance $\sigma_u^2$ of a stochastic output signal $x_u(t)$ due to a stochastic input signal $x(t)$, can be found by integrating the square of the deterministic output signal $y_u(t)$ caused by the deterministic input signal $y(t)$.

From equations (18a) and (18b) the deterministic time function for a given stochastic signal $x(t)$ can be explicitly derived if the power spectrum $\Phi_{xx}(\omega)$ is known.
This has been done in Refs. 3 and 4 for the horizontal and vertical gust velocities characterized by the Dryden turbulence spectra. Obtaining the time functions \( y(t) \) by the inverse Fourier transform from the power spectra, forming the differential equations governing \( y(t) \) and finally finding the initial conditions necessary for practical calculations is a rather cumbersome procedure. As such a procedure may be circumvented by the derivation in the time-domain as given in the former Chapter, the time functions \( y(t) \) will not be derived explicitly in this Report. Suffice it to show finally, using the equations (18a) and (18b) that if \( x(t) \) is the stochastic output signal of a system driven by white noise, \( y(t) \) is identical with the impulse response \( h(t) \) of that system.

Let the white noise input signal \( w(t) \) have a power spectrum (see Chapter 2):

\[
\Phi_{ww}(\omega) = \frac{v}{\pi}
\]

The power spectrum of the output signal \( x(t) \), is then, if the intensity \( v \) is taken to be \( v = 1 \):

\[
\Phi_{xx}(\omega) = |H(\omega)|^2, \quad \Phi_{ww}(\omega) = \frac{1}{\pi} |H(\omega)|^2
\]

(20)

If the result of eq. (20) is substituted in eq. (18b) the condition for \( Y(\omega) \) becomes:

\[
|Y(\omega)|^2 = \left|\frac{1}{\pi} H(\omega)\right|^2
\]

Then, as \( y(t) \) and \( Y(\omega) \) are related by the inverse Fourier transform, so will \( y(t) \) and \( \frac{1}{\pi} H(\omega) \):

\[
y(t) = \mathcal{F}^{-1}\left\{\frac{1}{\pi} H(\omega)\right\}
\]

(21)

Now consider the Laplace transform and the inverse Laplace transform, defined as (see Ref. 11):
\[ Y(P) = \int_{0}^{\infty} y(t) e^{-Pt} \, dt = \mathcal{L}\{y(t)\} \quad (22) \]

\[ y(t) = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} Y(P) e^{P t} \, dP = \mathcal{L}^{-1}\{Y(P)\} \quad (23) \]

where \( \mathcal{L} \) and \( \mathcal{L}^{-1} \) denote the Laplace and inverse Laplace transforms respectively and \( P \) is the complex Laplace variable.

Considering the definitions of the Fourier and Laplace transforms according to eqs. (13), (14), (22) and (23) it is apparent that:

\[ y(t) = \mathcal{F}^{-1}\{Y(\omega)\} = \pi \mathcal{L}^{-1}\{Y(P)\} \quad \text{for} \quad P = j\omega \quad (24) \]

As the relation of the impulse response \( h(t) \) and the transfer function \( H(P) \) is given by (see Ref. 8):

\[ h(t) = \mathcal{L}^{-1}\{H(P)\} \quad (25) \]

it follows from eqs. (21), (24) and (25) that:

\[ y(t) = h(t) \]
4. THE SYSTEM EQUATIONS OF AIRCRAFT WITH TURBULENCE FILTERS

4.1. The differential equations for the turbulence filters

As mentioned in Chapters 1 and 2, the gust velocities driving an aircraft flying in turbulence can be obtained by filtering white noise signals. The differential equations governing these filters are derived from the power spectra of horizontal and vertical turbulence. These spectra are, according to Dryden, Ref. 10:

\[
\Phi_{\hat{g}_h \hat{g}_h}(\omega) = \frac{2}{\pi} \sigma_{\hat{g}_h}^2 \frac{L}{V} \frac{1}{1 + \left(\frac{\omega L}{V}\right)^2} \tag{26}
\]

\[
\Phi_{\alpha_g \alpha_g}(\omega) = \frac{1}{\pi} \sigma_{\alpha_g}^2 \frac{L}{V} \frac{1 + 3 \left(\frac{\omega L}{V}\right)^2}{\left(1 + \left(\frac{\omega L}{V}\right)^2\right)^2} \tag{27}
\]

First consider the stochastic horizontal gust velocity \(\hat{g}(t)\) only. This gust velocity is obtained by the filtering of a white noise signal \(w(t)\) by a dynamical element having a transfer function \(H_{\hat{g}}(\omega)\). Let the power spectrum of the white noise signal \(w(t)\) be \(\Phi_{w_1 w_1}(\omega)\), then:

\[
\Phi_{\hat{g}_h \hat{g}_h}(\omega) = \left|H_{\hat{g}}(\omega)\right|^2 \cdot \Phi_{w_1 w_1}(\omega) \tag{28}
\]

The spectrum of the white noise signal is defined as (see also Chapter 2 of this Report):

\[
\Phi_{w_1 w_1}(\omega) = \frac{v_1}{\pi} \tag{29}
\]

where \(v_1\) is the intensity of the white noise signal. Here \(v_1\) is taken to be \(v_1 = k^2\). Then it follows from eqs. (26), (28) and (29) that:

\[
\frac{2}{\pi} \sigma_{\hat{g}_h}^2 \frac{L}{V} \cdot \frac{1}{1 + \left(\frac{\omega L}{V}\right)^2} = \left|H_{\hat{g}}(\omega)\right|^2 \frac{k^2}{\pi} \tag{30}
\]
The transfer function $H_{G_g}(\omega)$ can now be derived from:

$$|H_{G_g}(\omega)| = \frac{1}{k_1} \cdot \sigma_{G_g} \cdot \sqrt{\frac{2L_g}{V}} \cdot \frac{1}{\sqrt{1 + \omega L_g \frac{G_g}{V} \frac{2}{V}}}$$

which can also be written as:

$$|H_{G_g}(\omega)| = \frac{1}{k_1} \cdot \sigma_{G_g} \cdot \sqrt{\frac{2L_g}{V}} \cdot \frac{1}{\sqrt{1 + j \omega L_g \frac{G_g}{V}}}$$

or:

$$H_{G_g}(\omega) = \frac{1}{k_1} \cdot \sigma_{G_g} \cdot \sqrt{\frac{2L_g}{V}} \cdot \frac{1}{1 + j \omega L_g \frac{G_g}{V}}$$  \hspace{1cm} (31)$$

It should be remarked that mathematically a solution of eq. (30) leading to an expression for $H_{G_g}(\omega)$ as in eq. (31) with $1 - j \omega \frac{L_g G_g}{V}$ as the denominator is also possible.

This, however, would lead to unstable differential equations for the turbulence filter and hence would result in an infinite value of $\sigma_{G_g}^2$. Such a solution can thus be rejected for physical reasons.

Bearing in mind that:

$$H_{G_g}(\omega) = \frac{\hat{G}_g(\omega)}{\omega_1(\omega)}$$

and replacing the imaginary variable $j\omega$ by $\frac{d}{dt}$, the differential equation for the horizontal turbulence filter is found to be:

$$\ddot{\hat{G}}_g \frac{\hat{G}}{V} = - \frac{c}{L_g} \hat{G}_g + \sigma_{G_g} \cdot \frac{c}{V} \cdot \sqrt{\frac{2V}{L_g \cdot \omega_1 \frac{L_g}{V}}}$$ \hspace{1cm} (32)$$

Taking the power spectrum for vertical turbulence according to eq. (27) a similar derivation results in the differential equation:
\[ \dot{\alpha}_g \left( \frac{L_g}{V} \right)^2 + \frac{2L_g}{V} \dot{\alpha}_g + \alpha_g = \sigma_{\alpha_g} \sqrt{\frac{L_g}{V}} \cdot \dot{w}_2 \cdot \frac{1}{k_2} + \]

\[ + \sigma_{\alpha_g} \sqrt{\frac{3L_g}{V}} \cdot \dot{w}_2 \frac{1}{k_2} \]

(33)

where \( w_2 \) is the white noise signal driving the vertical turbulence filter, the intensity being \( v_2 = k_2^2 \).

To obtain a more suitable form of eq. (33), an auxiliary variable \( \alpha^* \) is introduced:

\[ \alpha^* = \frac{\alpha_g}{V} - \sigma_{\alpha_g} \frac{c}{V} \cdot \sqrt{\frac{3V}{L_g}} \cdot \frac{1}{k_2} \]

After some elaboration eq. (33) can be written as:

\[ \begin{bmatrix}
\dot{\alpha}_g \\
\dot{\alpha}_g^*
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-\frac{c^2}{L_g} & -2c \frac{c}{L_g}
\end{bmatrix} \begin{bmatrix}
\alpha_g \\
\alpha_g^*
\end{bmatrix} + \begin{bmatrix}
\sigma_{\alpha_g} \frac{c}{V} \cdot \sqrt{\frac{3V}{L_g}} \\
(1-2\sqrt{3}) \frac{c}{L_g} \sigma_{\alpha_g} \frac{c}{V} \cdot \sqrt{\frac{V}{L_g}}
\end{bmatrix} \cdot \dot{w}_2 \frac{1}{k_2}
\]

(34)

Equations (33) and (34) are now joined:

\[ \begin{bmatrix}
\dot{\alpha}_g \\
\dot{\alpha}_g \\
\dot{\alpha}_g^*
\end{bmatrix} = \begin{bmatrix}
-\frac{c}{L_g} & 0 & 0 \\
0 & 0 & 1 \\
0 & -\frac{c^2}{L_g} & -2c \frac{c}{L_g}
\end{bmatrix} \begin{bmatrix}
\alpha_g \\
\alpha_g \\
\alpha_g^*
\end{bmatrix} + \begin{bmatrix}
\dot{G}_g \\
\dot{G}_g \\
\dot{G}_g^*
\end{bmatrix}
\]
\[
\begin{bmatrix}
\sigma_{g} \frac{c}{V} \sqrt{\frac{2V}{L_g}} & 0 \\
0 & \sigma_{\alpha} \frac{c}{V} \sqrt{\frac{3V}{L_g}} \\
0 & (1-2\sqrt{3}) \frac{c}{L_g} \sigma_{\alpha} \frac{c}{V} \sqrt{\frac{V}{L_g}}
\end{bmatrix}
\begin{bmatrix}
w_1 \\
\frac{1}{k_2}
\end{bmatrix}
+ \begin{bmatrix}
w_2 \\
\frac{1}{k_2}
\end{bmatrix}
\]
(35)

Eq. (35) can be written more consisely:

\[
\begin{bmatrix}
\dot{\bar{v}}_g \\
\dot{\bar{w}}
\end{bmatrix} = [A_g] \begin{bmatrix}
\bar{v}_g \\
\bar{w}
\end{bmatrix} + [B_g] \begin{bmatrix}
\bar{w}
\end{bmatrix} + \begin{bmatrix}
\bar{w}_1 \\
\bar{w}_2
\end{bmatrix} \begin{bmatrix}
w_1 \\
\frac{1}{k_2}
\end{bmatrix} \begin{bmatrix}
w_1 \\
\frac{1}{k_2}
\end{bmatrix}
\]
(36)

4.2. The initial conditions for the differential equations of the aircraft with turbulence filters

The differential equations for the aircraft have been derived in the Appendix. They slightly differ in form from those given in Ref. 3 as it is strictly not necessary to eliminate the variables $D_{c g}$ and $D_{c \alpha g}$. The aircraft's system equation has been derived in vector-matrix notation:

\[
\begin{bmatrix}
\dot{\bar{x}}_a \\
\dot{\bar{x}}_i
\end{bmatrix} = [A_a] \begin{bmatrix}
\bar{x}_a \\
\bar{x}_i
\end{bmatrix} + [B_a] \begin{bmatrix}
\bar{x}_i
\end{bmatrix}
\]
(37)

where $\bar{x}_a$ is the aircraft's state vector containing the motion variables and $\bar{x}_i$ consists of the dimensionless gust velocities $\bar{g}_g$ and $\bar{\alpha}_g$ and their (dimensionless) time derivatives $D_{c g}$ and $D_{c \alpha g}$.

Eqs. (36) and (37) have in the Appendix been compounded to the system equations of aircraft with turbulence filters:

\[
\begin{bmatrix}
\dot{\bar{x}} \\
\dot{\bar{w}}
\end{bmatrix} = [A] \begin{bmatrix}
\bar{x} \\
\bar{w}
\end{bmatrix} + [B] \begin{bmatrix}
\bar{w}
\end{bmatrix} + \begin{bmatrix}
\bar{w}_1 \\
\bar{w}_2
\end{bmatrix} \begin{bmatrix}
w_1 \\
\frac{1}{k_2}
\end{bmatrix} \begin{bmatrix}
w_1 \\
\frac{1}{k_2}
\end{bmatrix}
\]
(38)

Here $\bar{x}$ contains the motion variables and the variables $\bar{g}_g$, $\bar{\alpha}_g$ and $\bar{\alpha}_g$; $\bar{w}$ is the vector of the (uncorrelated) white noise signals $w_1$ and $w_2$. 
The separate sets of initial conditions to obtain the desired responses $\bar{y}(t)$ for the calculation of variances due to horizontal and vertical turbulence have been obtained in the Appendix by setting:

$$\bar{x}(0) = \bar{b}_1$$

and:

$$\bar{x}(0) = \bar{b}_2$$

It appears that the terms in $\bar{b}_1$ and $\bar{b}_2$ due to the time derivatives $D_c \bar{U}_g$ and $D_c \bar{a}_g$ of the gust velocities $\bar{U}_g$ and $\bar{a}_g$ are missing in the initial conditions as given in Refs. 3 and 4.

That the time function $y(t)$, for example in the case of $D_c \bar{U}_g$, leads to initial conditions of a number of aircraft variables can be seen as follows.

The stochastic variable $D_c \bar{U}_g(t)$ is given by eq. (32), setting $k_1 = 1$, as:

$$\dot{\bar{U}}_g \frac{c}{V}(t) = D_c \bar{U}_g(t) = -\frac{c}{L_g} \bar{U}_g(t) + \sigma_{\bar{U}_g} \frac{c}{V} \sqrt{\frac{2V}{L_g}} \cdot w(t)$$

As the deterministic time function for $D_c \bar{U}_g$, denoted by $y_{D_c \bar{U}_g}(t)$ is identical with the impulse response of the filter, it will be given by:

$$y_{Dc \bar{U}_g}(t) = -\frac{c}{L_g} y_{\bar{U}_g}(t) + \sigma_{\bar{U}_g} \frac{c}{V} \sqrt{\frac{2V}{L_g}} \cdot \delta(0) \quad (39)$$

where $\delta(0)$ is the impulse function at $t = 0$.

When the aircraft's response $\bar{y}(t)$ is to be obtained for the calculation of variances due to horizontal turbulence, $y_{Dc \bar{U}_g}(t)$ is in fact one of the input signals to the aircraft proper. According to eq. (39) this
input signal contains an impulse function which should be taken into account by properly chosen initial conditions for a number of aircraft motion variables.

A similar argument applies to the vertical turbulence velocity.
5. NUMERICAL EXAMPLE

Using the differential equations as described in the Appendix and the initial conditions as given in this report and Ref. 3, the variances of a number of motion-variables have been calculated of a four-engined jet transport aircraft flying through turbulence in the approach configuration. For comparison the same variances have also been calculated with a digital method as described in Refs. 5 and 6 and also with the results of Monte Carlo simulation.

The aircraft, the data of which are given in Table 1, is stabilized on the I.L.S. glide path by an autopilot, characteristics of which are also given in Table 1. Airspeed is controlled by an autothrottle. There have been no efforts to optimize the autopilot according to any mathematically defined criterion. The feedback constants of the autopilot and autothrottle have been chosen such that, without these constants attaining too large absolute values, reasonably fast and well damped responses occur after a deviation from the glide path.

The equations of the autopilot and autothrottle have of course been added to the equations of motion of the unstabilized aircraft as given in the Appendix.

The aircraft is considered to fly through a stationary, homogeneous field of turbulence, characterized by:

\[ \sigma_{u_g} = 0.297 \text{ m/sec} \quad L_{u_g} = 150 \text{ m} \]
\[ \sigma_{\omega_g} = 0.282 \text{ m/sec} \quad L_{\omega_g} = 150 \text{ m} \]

These values correspond with those at a height of 265 m in the atmospheric model according to Pritchard (Ref. 12) for a neutral atmosphere. The same model was used in Ref. 4. The above-mentioned values of \( \sigma_{u_g} \) and \( \sigma_{\omega_g} \) are those for an average windspeed at the reference height of 9.05 m of 0.5 m/sec (approx. 1 knot).
The numerical results are summarized in Table 2. In the first place it appears that for horizontal turbulence, the influence of the corrected initial conditions is small and approximately in the order of the limited accuracy of analogue computation. This is not amazing bearing in mind the fact that the influence of $D_c \hat{g}$ on the aircraft's motions is generally small anyhow.

The variances due to vertical turbulence, however, all become considerably larger for the case of the corrected initial conditions and show a much better agreement with the results of the digital calculation and those of the Monte Carlo simulation.

Figures 3 through 7 give the growth in time of a number of variances due to vertical turbulence. The agreement, in the case of corrected initial conditions, with the results of the digital calculation, is again quite satisfactory.
6. CONCLUSIONS

In this Report it was shown that the impulse response method to calculate variances of output signals of linear systems perturbed by gaussian noise signals can be derived fast and straightforwardly using concepts of modern system theory. The impulse response method can thus be shown to hold for the calculation of the covariance matrix as a function of time as well as for the stationary case ($t \rightarrow \infty$).

The greatest advantage, especially for large and complicated systems, lies in obtaining explicitly the initial conditions needed to start the calculation, once the system equations are known.

Some corrections appeared to be necessary to the method as given in Ref. 3. These corrections particularly appear to have a quantitative influence on the variances due to vertical turbulence, as shown in the numerical example. The numerical results of the - corrected - impulse response method as obtained by an analogue computation are in very good agreement with results obtained by a different, digital method and by a Monte Carlo simulation.
7. REFERENCES

1. J.H. Laning
   R.H. Battin
   - Random processes in automatic control.

2. B. Etkin
   - A simple method for the analogue computation of the mean-square response of airplanes to atmospheric turbulence.
   UTIA Technical Note no. 32, 1960.

3. O.H. Gerlach
   - Calculation of the response of an aircraft to random atmospheric turbulence. Part I.

4. O.H. Gerlach
   G.A.J. van de Moesdijk
   J.C. van der Vaart
   - Progress in the mathematical modelling of flight in turbulence.
   A.G.A.R.D. - C.P. 140.

5. H.L. Jonkers
   - Application of the Kalman filter to flight path reconstruction from flight test data including estimation of instrumental bias errors.

6. H.L. Jonkers et al
   - Digital calculation of the propagation
in time of the aircraft gust response covariance matrix. Report VTH. Delft University of Technology, Department of Aerospace Engineering (to be published).

7. H. Kwakernaak  
   R. Sivan  
   - Linear optimal control systems.  

8. L.A. Zadeh  
   C.A. Desoer  
   - Linear system theory.  

9. A. Papoulis  
   - Probability, random variables, and stochastic processes.  

10. H.L. Dryden  
    S.K. Friedlander  
    L. Topper  
    - A review of the statistical theory of turbulence.  
    Classic papers on statistical theory.  

11. G.A. Korn  
    T.M. Korn  
    - Handbook of mathematics for scientists and engineers.  

12. F.E. Pritchard  
    - A statistical model of atmospheric turbulence and a review of the

13. P.M. Reeves  
   R.G. Joppa  
   V.M. Ganzer


14. M. Baarspul  
   O.H. Gerlach

APPENDIX. THE DIFFERENTIAL EQUATIONS FOR THE AIRCRAFT WITH TURBULENCE FILTERS

The linearized differential equations for the symmetrical motions due to turbulence are, if \( \delta_e = 0 \):

\[
\begin{bmatrix}
C_{X_u} - 2\mu_c D_c & C_{X_\alpha} & C_{Z_\alpha} & 0 \\
C_{Z_u} & C_{Z_\alpha} - (2\mu_c - C_{Z_\alpha}) D_c & -C_{X_\alpha} & 2\mu_c + C_{Z_\alpha} \\
0 & 0 & -D_c & 1 \\
C_{mu} & C_{m_\alpha} D_c & 0 & C_{m_q} - 2\mu_c K Y^2 D_c \\
\end{bmatrix}
\begin{bmatrix}
\dot{\alpha} \\
\dot{\theta} \\
\dot{\gamma} \\
\dot{\psi} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
C_{X_{ug}} & C_{X_{\alpha g}} & C_{X_{\alpha g}} & C_{X_{\alpha g}} \\
C_{Z_{ug}} & C_{Z_{\alpha g}} & C_{Z_{\alpha g}} & C_{Z_{\alpha g}} \\
0 & 0 & 0 & 0 \\
C_{m_{ug}} & C_{m_{\alpha g}} & C_{m_{\alpha g}} & C_{m_{\alpha g}} \\
\end{bmatrix}
\begin{bmatrix}
\dot{\alpha}_g \\
\dot{\theta}_g \\
\dot{\gamma}_g \\
\dot{\psi}_g \\
\end{bmatrix}
\]

(A-1)

The desired form of the system equation:

\[
\ddot{x} = [A] \dot{x} + [B] \ddot{x}_i
\]

can be obtained by eliminating the term \( C_{m_\alpha D_c} \alpha \) in (A-1). This is done by multiplying the \( Z \)-equation of (A-1) with \( \frac{C_{m_\alpha}}{2\mu_c - C_{Z_\alpha}} \), summing the result and the \( M \)-equation of (A-1) and finally dividing the \( X \)-equation.
by $2\mu_c$, the $Z$-equation by $2\mu_c - CZ_\alpha$ and the $M$-equation by $2\mu_c K_Y^2$. The result is, in an abbreviated notation:

\[
\begin{bmatrix}
\dot{\alpha} \\
\dot{\theta} \\
\dot{q}_c 
\end{bmatrix}
- \frac{c}{V}
= \begin{bmatrix}
x_w & x_\alpha & x_\theta & 0 \\
z_w & z_\alpha & z_\theta & z_q \\
0 & 0 & 0 & 1 \\
m_w & m_\alpha & m_\theta & m_q 
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\theta \\
q_c 
\end{bmatrix}
+ \begin{bmatrix}
x_{ug} & x_{ug} & x_{\alpha g} & x_{\theta g} \\
z_{ug} & z_{ug} & x_{\alpha g} & x_{\theta g} \\
0 & 0 & 0 & 0 \\
m_{ug} & m_{ug} & m_{\alpha g} & m_{\theta g} 
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_g \\
\ddot{D}_c \dot{u}_g \\
\alpha_g \\
D_c \alpha_g 
\end{bmatrix}
\tag{A-2}
\]

The coefficients in (A-2) have the following meaning:

\[
\begin{align*}
x_u &= \frac{C_{X_u}}{2\mu_c} \\
z_w &= \frac{C_{Z_u}}{2\mu_c - CZ_\alpha} \\
\frac{C_{X_\alpha}}{2\mu_c} \\
z_\alpha &= \frac{C_{Z_\alpha}}{2\mu_c - CZ_\alpha} \\
x_\theta &= \frac{C_{X_\theta}}{2\mu_c - CZ_\alpha} \\
z_\theta &= \frac{C_{X_\theta}}{2\mu_c - CZ_\alpha} \\
m_u &= \frac{C_{m_u} + C_{Z_u} \cdot \frac{C_{m_\alpha}}{2\mu_c - CZ_\alpha}}{2\mu_c K_Y^2} \\
m_\alpha &= \frac{C_{m_\alpha} + C_{Z_\alpha} \cdot \frac{C_{m_\alpha}}{2\mu_c - CZ_\alpha}}{2\mu_c K_Y^2} \\
m_\theta &= \frac{-C_{X_\theta} \cdot \frac{C_{m_\alpha}}{2\mu_c - CZ_\alpha}}{2\mu_c K_Y}
\end{align*}
\]
\[ \begin{align*}
 z_q &= \frac{2\mu_c + Cz_q}{2\mu_c - Cz_\alpha} \\
 m_q &= \frac{C_m q + \frac{2\mu_c + Cz_q}{2\mu_c - Cz_\alpha} \cdot C_m \alpha}{2\mu_c K_Y^2} \\
 x_{u_g} &= \frac{Cx_{u_g}}{2\mu_c} \\
 z_{u_g} &= \frac{Cz_{u_g}}{2\mu_c - Cz_\alpha} \\
 m_{u_g} &= \frac{C_m u_g + Cz_{u_g} \cdot C_m \alpha}{2\mu_c K_Y^2} \\
 x_{\dot{u}_g} &= \frac{Cx_{\dot{u}_g}}{2\mu_c} \\
 z_{\dot{u}_g} &= \frac{Cz_{\dot{u}_g}}{2\mu_c - Cz_\alpha} \\
 m_{\dot{u}_g} &= \frac{C_m \dot{u}_g + Cz_{\dot{u}_g} \cdot C_m \alpha}{2\mu_c K_Y^2} \\
 x_{\alpha_g} &= \frac{Cx_{\alpha_g}}{2\mu_c} \\
 z_{\alpha_g} &= \frac{Cz_{\alpha_g}}{2\mu_c - Cz_\alpha} \\
 m_{\alpha_g} &= \frac{C_m \alpha_g + Cz_{\alpha_g} \cdot C_m \alpha}{2\mu_c K_Y^2} \\
 x_{\dot{\alpha}_g} &= \frac{Cx_{\dot{\alpha}_g}}{2\mu_c} \\
 z_{\dot{\alpha}_g} &= \frac{Cz_{\dot{\alpha}_g}}{2\mu_c - Cz_\alpha} \\
 m_{\dot{\alpha}_g} &= \frac{C_m \dot{\alpha}_g + Cz_{\dot{\alpha}_g} \cdot C_m \alpha}{2\mu_c K_Y^2}
\end{align*} \]

It should be remarked that the above definitions of the derivatives with respect to the gust velocities and their time derivatives differ from the ones given in Ref. 3, where \( D_{\alpha_g} \) and \( D_{\dot{\alpha}_g} \) were eliminated from the equations. Eliminating these variables, however, is not strictly necessary, as they may be obtained by a linear combination of certain elements of the state vector \( \bar{x} \) of the system consisting of aircraft and turbulence filter. The state equations of this system are obtained by adding the turbulence filter equation (35) of Chapter 4 to (A-2).

The result is:
\[
\begin{bmatrix}
\dot{\hat{u}} \\
\dot{\hat{\alpha}} \\
\dot{\theta} \\
\dot{\frac{c}{V}} \\
\dot{\hat{u}}_g \\
\dot{\hat{\alpha}}_g \\
\dot{\alpha}_g \\
\ddot{\alpha}_g \\
\end{bmatrix}
= \begin{bmatrix}
x_u & x_\alpha & x_\theta & 0 & x_{ug} & -\frac{V}{L_g} & x_{g\hat{u}} & x_{g\alpha} & x_{g\alpha} \\
zu & z_\alpha & z_\theta & z_q & z_{ug} & -\frac{V}{L_g} & z_{g\hat{u}} & z_{g\alpha} & z_{g\alpha} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
m_u & m_\alpha & m_\theta & m_q & m_{ug} & -\frac{V}{L_g} & m_{g\hat{u}} & m_{g\alpha} & m_{g\alpha} \\
0 & 0 & 0 & 0 & -\frac{c}{L_g} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{c^2}{L_g^2} & -\frac{2c}{L_g} & \alpha_g \\
\end{bmatrix}
\begin{bmatrix}
\hat{u} \\
\hat{\alpha} \\
\theta \\
\frac{c}{V} \\
\hat{u}_g \\
\hat{\alpha}_g \\
\alpha_g \\
\ddot{\alpha}_g \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_{ug} \sigma_{ug} \cdot \sqrt{\frac{2V}{L_g}} \\
z_{ug} \sigma_{ug} \cdot \sqrt{\frac{2V}{L_g}} \\
0 \\
m_{ug} \sigma_{ug} \cdot \sqrt{\frac{2V}{L_g}} \\
\sigma \frac{c}{V} \cdot \sqrt{\frac{2V}{L_g}} \\
0 \\
0 \\
(1-2\sqrt{3}) \frac{c}{L_g} \sigma_{ug} \frac{c}{V} \sqrt{\frac{V}{L_g}} \\
\end{bmatrix}
\begin{bmatrix}
w_1 \cdot \frac{1}{k_1} \\
w_2 \cdot \frac{1}{k_2} \\
\end{bmatrix}
\]

Eq. (A-3) can, taking \( k_1 = k_2 = 1 \) in an abbreviated matrix-vector notation been written as:
\[
\dot{x} = [A] \; x + [B] \; \bar{w} = [A] \; \bar{x} + \bar{b}_1 \; w_1 + \bar{b}_2 \; w_2
\] (A-4)

The desired response \( \bar{y}(t) \) yielding the deterministic equivalent variables can, according to Chapter 2 be obtained by responses to initial conditions:

\[
\bar{x}(0) = \bar{b}_1 \quad \text{(horizontal turbulence only)} \quad (A-5)
\]

and:

\[
\bar{x}(0) = \bar{b}_2 \quad \text{(vertical turbulence only)} \quad (A-6)
\]

The set of initial conditions for the case of horizontal turbulence follows from (A-3), (A-4) and (A-5), bearing in mind the factor \( \frac{C}{V} \) in the left hand side of (A-3):

\[
\begin{bmatrix}
G(0) \\
\alpha(0) \\
\theta(0) \\
\frac{\bar{q}}{V}(0) \\
\bar{u}_g(0) \\
\alpha_g(0) \\
\alpha_g^*(0)
\end{bmatrix} =
\begin{bmatrix}
x_g \frac{V}{c} \cdot \sigma_g \cdot \sqrt{\frac{2V}{L_g}} \\
z_g \frac{V}{c} \cdot \sigma_g \cdot \sqrt{\frac{2V}{L_g}} \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]
For the case of vertical turbulence, the set of initial conditions becomes:

\[
\begin{bmatrix}
\mathbb{u}(o) \\
\alpha(o) \\
\theta(o) \\
\frac{q}{V}(o) \\
\tilde{u}_g(o) \\
\alpha_g(o) \\
\alpha_{g^*}(o)
\end{bmatrix} =
\begin{bmatrix}
x_{\alpha_g} \frac{V}{c} \sigma_{\alpha_g} \sqrt{\frac{3V}{L_g}} \\
\frac{z_{\alpha_g}}{c} \sigma_{\alpha_g} \sqrt{\frac{3V}{L_g}} \\
0 \\
\frac{m_{\alpha_g}}{c} \sigma_{\alpha_g} \sqrt{\frac{3V}{L_g}} \\
0 \\
\sigma_{\alpha_g} \sqrt{\frac{3V}{L_g}} \\
(1-2\sqrt{3}) \frac{c}{L_g} \sigma_{\alpha_g} \sqrt{\frac{V}{L_g}}
\end{bmatrix}
\]
Table 1: Aircraft data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W )</td>
<td>96160 kg</td>
</tr>
<tr>
<td>( V )</td>
<td>71.24 m/sec</td>
</tr>
<tr>
<td>( S )</td>
<td>260.68 m²</td>
</tr>
<tr>
<td>( \bar{c} )</td>
<td>6.10 m</td>
</tr>
<tr>
<td>( \rho )</td>
<td>0.125 kg sec²/m⁴</td>
</tr>
<tr>
<td>( x_{c,g} )</td>
<td>0.36 ( \bar{c} )</td>
</tr>
</tbody>
</table>

| \( C_{X_o} \) | -0.0507 |
| \( C_{X_d} \) | 0.655 |
| \( C_{X_{\alpha}} \) | 0.655 |
| \( C_{X_{\delta}} \) | 0 |
| \( C_{\Delta_{\alpha}} \) | 0.370 |
| \( C_{\Delta_{d}} \) | 0 |
| \( C_{\Delta_{\alpha}} \) | 0.370 |
| \( C_{\Delta_{d}} \) | 0 |
| \( C_{Z_0} \) | -1.163 |
| \( C_{Z_u} \) | -2.326 |
| \( C_{Z_{\alpha}} \) | -5.04 |
| \( C_{Z_{\delta}} \) | -0.342 |
| \( C_{Z_{\alpha_{\delta}}} \) | -5.04 |
| \( C_{Z_{\delta_{\alpha}}} \) | 4.255 |
| \( C_{Z_{\alpha_{\delta}}} \) | -2.326 |
| \( C_{Z_{\delta_{\alpha}}} \) | -0.957 |
| \( C_{m_u} \) | 0 |
| \( C_{m_{\alpha}} \) | -0.72 |
| \( C_{m_{\alpha}} \) | -0.72 |
| \( C_{m_{\delta}} \) | -1.218 |
| \( C_{m_{\delta}} \) | -1.055 |
| \( C_{m_{\alpha_{\delta}}} \) | 7.400 |
| \( C_{m_{\delta_{\alpha}}} \) | 0 |
| \( C_{m_{\delta_{\alpha}}} \) | -0.584 |

The aircraft is stabilised by an autopilot controlling the elevator angle according to:

\[
\delta_e = -K_h \cdot h - K_H \cdot \int h dt - K_\theta \cdot \theta - K_{\delta_e} \cdot \delta_e
\]

Airspeed is controlled by an autothrottle. The transfer function of the dimensionless change of the thrust \( \Delta T'_c \) and the dimensionless deviation \( \delta \) of the airspeed is:

\[
\frac{\Delta T'_c(P)}{\delta(P)} = -K_\delta \cdot \frac{1}{1 + \tau_e P}
\]

The feedback constants and time constants are:

\[
K_h = -0.020 \quad K_\theta = -2.5 \quad K_{\delta_e} = 1.5 \\
k_H = -0.002 \quad K_{\delta} = -1.0 \quad \tau_e = 1 \text{ sec.}
\]
Table 2: Calculated variances of a number of motion variables

Analogue 1: Initial conditions according to Ref. 3
Analogue 2: Initial conditions according to this report

Horizontal turbulence, $\sigma_{u_g} = 0.297 \text{ m/sec } L_{u_g} = 150 \text{ m}$

<table>
<thead>
<tr>
<th></th>
<th>$\sigma_u^2$ $\times 10^6$</th>
<th>$\sigma_\alpha^2$ $\times 10^6$ (rad$^2$)</th>
<th>$\sigma_\theta^2$ $\times 10^6$ (rad$^2$)</th>
<th>$\sigma_{\frac{q\bar{c}}{V}}^2$ $\times 10^8$ (rad$^2$)</th>
<th>$\sigma_h^2$ (m$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analogue 1</td>
<td>0.560</td>
<td>1.55</td>
<td>6.99</td>
<td>1.040</td>
<td>0.1158</td>
</tr>
<tr>
<td>Analogue 2</td>
<td>0.551</td>
<td>4.56</td>
<td>6.90</td>
<td>1.005</td>
<td>0.1145</td>
</tr>
<tr>
<td>Digital</td>
<td>0.570</td>
<td>4.74</td>
<td>7.17</td>
<td>1.035</td>
<td>0.118</td>
</tr>
<tr>
<td>Monte Carlo</td>
<td>0.603</td>
<td>4.96</td>
<td>7.48</td>
<td>0.895</td>
<td>0.124</td>
</tr>
</tbody>
</table>

Vertical turbulence, $\sigma_{\alpha_g} = 0.282 \text{ m/sec } L_{u_g} = 150 \text{ m}$

<table>
<thead>
<tr>
<th></th>
<th>$\sigma_u^2$ $\times 10^6$</th>
<th>$\sigma_\alpha^2$ $\times 10^6$ (rad$^2$)</th>
<th>$\sigma_\theta^2$ $\times 10^6$ (rad$^2$)</th>
<th>$\sigma_{\frac{q\bar{c}}{V}}^2$ $\times 10^8$ (rad$^2$)</th>
<th>$\sigma_h^2$ (m$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analogue 1</td>
<td>2.17</td>
<td>17.75</td>
<td>17.95</td>
<td>1.94</td>
<td>0.245</td>
</tr>
<tr>
<td>Analogue 2</td>
<td>2.48</td>
<td>20.1</td>
<td>24.06</td>
<td>3.74</td>
<td>0.346</td>
</tr>
<tr>
<td>Digital</td>
<td>2.51</td>
<td>20.0</td>
<td>24.17</td>
<td>3.70</td>
<td>0.349</td>
</tr>
<tr>
<td>Monte Carlo</td>
<td>2.36</td>
<td>20.87</td>
<td>29.6</td>
<td>4.03</td>
<td>0.370</td>
</tr>
</tbody>
</table>
Fig. 1. The calculation of the variance of an output signal.
\[ \sigma_{x_u}^2 = \sigma_{x_{u1}}^2 + \sigma_{x_{u2}}^2 \]

Fig. 2. The calculation of the variance of an output signal caused by uncorrelated horizontal and vertical turbulence.
Fig. 3. The growth in time of $\sigma_u^2$ due to vertical turbulence.
Fig. 4. The growth in time of $\sigma^2_\alpha$ due to vertical turbulence.
Fig. 5. The growth in time of $\sigma_\theta^2$ due to vertical turbulence.
Fig. 6. The growth in time of $\frac{\sigma^2}{q_c}$ due to vertical turbulence.
Fig. 7. The growth in time of $\sigma_h^2$ due to vertical turbulence.