Drop Patterns in Rayleigh-Taylor Instabilities of a Thin Layer

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Summary

The statistic properties of a two-dimensional lattice of pendant drops formed by Rayleigh-Taylor instabilities of a thin layer have been investigated using image treatment techniques. The program used to analyse the images is called "Mouse", a fortran based program developed locally. Two additional programs were written for the analysis of the images, one calculating the Voronoi construction and one determining the Minimal Spanning Tree-graph.

Based on a Voronoi construction of lattices of drops, various properties have been determined. Distribution of the number of closest neighbors of a central drop, angles between two closest neighbors and a central drop and the evolution of the number of drops have shown that a lattice of drops evolves in time towards a disordered, triangular pattern.

The statistical method Minimal Spanning Tree has been applied on evolving drop patterns with success. Nevertheless, it has been difficult to discern a disordered regular pattern due to the lack of quantitative information obtained from the Minimal Spanning Tree-graph.

The analogy between drop coalescence and wall breakage in soap froths have been discussed. Although a qualitative analogy was evident, a quantitative analogy could not be concluded decisively.
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1. Introduction

The goal of the presented work is to study the statistical properties of two-dimensional pendant drop lattices and to quantify their evolution.

The drops are formed by a Rayleigh-Taylor instability of a thin homogeneous layer of fluid. The Rayleigh-Taylor instability is a gravitational instability which occurs when there is an adverse stratification in a fluid. An example of this kind of instability is the surface between two fluids with a different density, the upper fluid being the heaviest one.

The experimental procedure is represented schematically in figure (1). At the beginning of the experiment, the thin homogeneous fluid layer will transform in 'rolls' initiated by small perturbations (in this case the sides of the fluid layer), 'rolls' which in turn will break up into drops. These phenomena are described by Fermigier. The drops form a lattice on the area originally covered with the oil layer. The lattice is often ordered in a triangular manner, but with many defects. The lattice of drops evolves by coalescence mechanisms in time.

At the Laboratoire de Physique et de Mécanique des Milieux Hétérogènes, Ecole de Physique et de Chimie Industrielles in Paris, additional research concerning the propagation of fronts ('rolls') on the homogeneous surface is being done at this moment by A. Wajs, a French DEA-student.

Most publications until now treat investigations concerning the Rayleigh-Taylor instability of thick layers. In that field the practical applications are ample, among them the rising of magma or salt domes in geophysics and laser implosion of fusion targets. Applications of Rayleigh-Taylor instabilities in a thin layer are also known, for example instabilities of vapor films formed at the surface of a hot solid.

* A triangular drop network is the same as a network of hexagonal cells, constructed around triangular ordered drops. One often uses the term hexagonal lattice when actually a triangular lattice is meant.
2. Theoretical Background

2.1. Evolution equation and linear stability

In the experiments we study the Rayleigh-Taylor instability in a thin layer of silicon oil. The heavier phase is the thin layer at the lower side of a glass plate. The light phase is the air underneath the oil phase. The development of the thickness of the thin oil layer $h=h_0+\zeta(x,y,t)$, as represented in figure (2) is determined by the following equation of partial derivatives:

$$\frac{\partial \zeta}{\partial t} + \frac{1}{3\eta} \cdot \nabla((h_0 + \zeta)^3 \cdot \nabla(\rho \cdot g \cdot \zeta + \gamma \cdot \nabla^2 \zeta)) = 0$$

(1)

$\eta$ is the viscosity of the oil, $\gamma$ the surface tension and $\rho$ is approximated by the oil density. This equation is a mass balance of the thin layer. After linearisation one obtains the following relation:

$$\frac{\partial \zeta}{\partial t} + \frac{h_0^3}{3\eta} \cdot (\rho \cdot g \cdot \frac{\partial^2 \zeta}{\partial x^2} + \gamma \cdot \frac{\partial^2 \zeta}{\partial x^4}) = 0$$

(2)

This relation contains a stabilizing term related to the surface tension and a destabilizing term related to the gravitational force. The nonlinear term is neglected for the establishment of the dispersion relation. It is important though to consider the nonlinear term, because it explains the formation of hexagonal structures from the rolls. After the substitution of $\zeta = \exp(iqx - \sigma t)$ into equation (2), one obtains the following expression:

$$\sigma(q) = \frac{h_0^3}{3\eta} \cdot (\rho \cdot g \cdot q^2 - \gamma \cdot q^4)$$

(3)

One can draw the dispersion curve (figure (3)) and the following groups can be derived:

$$\sigma_M = \frac{h_0^3}{12\eta} \cdot \frac{\rho^2 \cdot g^2}{\gamma} \quad \text{and} \quad q_M = \sqrt{\frac{\rho \cdot g}{2\gamma}}$$

(4)

It appears that for values of $q$ superior to $\sqrt{2} \cdot q_M$, the front is stable, and for values of $q$ below that value the front is unstable.
2.2. "Roll" formation

According to Dee, Langer\textsuperscript{3} and van Saarloos\textsuperscript{4,5,6,7}, the Swift-Hohenberg equation is a general equation that permits to observe front propagation with the formation of periodic patterns:

\[ \frac{\partial u}{\partial t} = (\varepsilon - (\frac{\partial^2}{\partial x^2} + 1)^2)u - u^3 \]  \hspace{1cm} (5)

In this equation \( \varepsilon \) is a control parameter such that for \( \varepsilon > 0 \), the state \( u=0 \) is unstable. It has been shown\textsuperscript{3,4,6} that from general equation (5), the front propagation dependence is proportional to the following groups:

\[ u = f(x - ct) = \exp(ik^* \cdot x + w(k^*) \cdot t) \]  \hspace{1cm} (6)

with \( k^* \) being the point of stationary phase in the complex k-plane.

Dee and Langer have established the following conditions for the selection of the quantity \( k^* \) (figure (4)):

\[ \text{Re}(ic^* \cdot k^* + w(k^*)) = 0 \quad \text{and} \quad ic^* + \frac{dw(k^*)}{dk^*} = 0 \]  \hspace{1cm} (7)

These equations permit to calculate \( c^* \) and \( k^* \) with the linearized dispersion relation \( w(k) = \varepsilon - (1 - q^2)^2 \) derived from (5). The wavelength resulting behind the propagating front \( \lambda_L = 2\pi/q_L \) can be deduced from the conservation condition of the number of "rolls" as seen by two observers moving at a speed \( c^* \), one behind the front and one ahead of the front:

\[ c^* \cdot q_L = c^* \cdot k^* - \text{Im}(w(k^*)) \]  \hspace{1cm} (8)

If we assume that the front is governed by a marginal linear stability mechanism, and that close to the front \( \zeta(x) = \exp(iqx + q^*(q)t) \), the speed \( v^* \) of the front propagation is given by the equations (8).
2. Theoretical Background

One can deduce the following relations:

\[ v' = 0.54 \frac{h_0^3 (\rho \cdot g)^{3/2}}{\eta \cdot \gamma^{1/2}} \]  

(9)

and

\[ \lambda_L = 9.55 \sqrt{\frac{\gamma}{\rho \cdot g}} = 0.92 \frac{2\pi}{q_M} \]  

(10)

Equation (10) provides a theoretical estimation of the wave length, about 1.3 cm, which appeared to be in good order with the experiments so far. However, it has not yet been possible to distinguish experimentally between the wavelength resulting behind the front, \( \lambda_L \), and the wavelength of the first "roll", \( \lambda_M \). At this moment experiments are done to verify the theoretical relation between speed and thickness of the fluid layer.

2.3. Drop formation by non-linear effects

If one expands equation (1) over the Fourier modes, one obtains the evolution equation of the amplitude \( A_q(t) \) of a mode \( q \):

\[ \frac{dA_q}{dt} = (2q^2 - q^4) \cdot A_q + 3 \sum_{q, q_0} (q_s \cdot q_b)(2 - q_s^2)A_{q_s} \cdot A_{q_b} \cdot \delta(q - q_s - q_b) + \\
+ 3 \sum_{q, q_0, q_c} (q_s \cdot q)(2 - q_s^2)A_{q_s} \cdot A_{q_b} \cdot A_{q_c} \cdot \delta(q - q_s - q_b - q_c) + \ldots \]  

(11)

where the wave numbers \( q \) and the time \( t \) have been nondimensionalized, using \( 1/q_M \) and \( 1/\sigma(q_M) \) as units of length (horizontal) and of time, with \( A_q \) normalized by the initial thickness \( h_0 \). The wave number \( q \) equals in the second term of equation (11) \( q_a + q_b \) and in the third term \( q_a + q_b + q_c \). The symbol \( \delta \) is a sort of Dirac function that is either 0 or 1. The presence of a second order term can be noted, meaning that the system is not invariant by amplitude reflection. This property usually tends to favor the occurrence of an hexagonal symmetry in the physics of interfacial instabilities. This tendency can be better understood by considering the growth of a pair of modes \( \pm q_1 \), with a perturbation of small amplitude on the two other pairs of modes.
±q₂ and ±q₃ (figure (5)). Basically, this system is determined by a competition between a "roll" pattern \( AR(t) = A_q - A_{q₁} = A_{q₁} - A_{q₃} \) and a hexagonal pattern \( AH(t) = A_{q₂} = A_{q₃} \). Developing the equation assuming real amplitudes and considering the most unstable wavelength \( q_M = 1 \), gives the following result:

\[
\frac{dA_R}{dt} = A_R - 3A_R \cdot A_H \quad \text{and} \quad \frac{dA_H}{dt} = A_H + 3A_H^2 + 3A_H \cdot A_R
\]  

(12)

From these equations it is clear that the second order interaction tends to amplify the growth of the hexagonal pattern and to damp the roll perturbation. In figure (6) the evolution of the amplitudes for some different modes is given (not only the here derived roll and hexagonal patterns, but also a square and an inverse hexagonal mode is given^8).

2.4. Structure and stability of the pendant drop lattices

After the evolution of all the rolls into a network of drops, the drop network itself will evolve in time. The main phenomenon causing the evolution of the pattern is drop coalescence.

The best way to study network development caused by drop coalescence is to assure that no drops will spontaneously fall. Therefore, the initial layer thickness has to be chosen with care. According to Myshkis^11, who determined the critical volume for the stability of a pendant drop as a function of the contact angle of the fluid interface with the solid interface, the critical volume of a pendant drop can be calculated. In the case of a contact angle of zero, the resulting volume is \( V^* = 19\lambda_C^3 \). In this equation \( \lambda_C \) is the capillary length. Taking into account the surface of a hexagonal cell the critical thickness is given by \( h^* = (19\sqrt{3}/16\pi^2)\lambda_C \). Experience has shown that one should take an initial layer thickness well below the calculated value.

In order to obtain parameters that can describe the statistics of the network, it is necessary to construct a cell around each drop. The best way to construct a cell around each drop is to calculate the Voronoi network based on the central points of gravity of the drops. The Voronoi construction is also known as the Wigner-Seitz construction. The cellular array is constructed by assigning to each center of gravity of a drop all points which are closest to it.
Theoretical Background

The approach used to incorporate this theorem in a computer algorithm, is to construct all mid perpendiculars of the segments of two neighboring points. The resulting convex area inside these mid perpendiculars is considered to be the area assigned to each point, forming a cell (figure (7)). This construction is unique and fills space. Variables resulting from the Voronoi network such as cell surface, distribution of number of closest neighbors (i.e. number of cell sides) and angles between the central points and their closest neighbors are evaluated in the section Results and Discussion. In the appendix the synopsis of the algorithm written to calculate the Voronoi construction is included.

The Minimal Spanning Tree is a graph obtained by connecting the centers of gravity of the drops in such a way that all the drops are included in the tree and no closed loops are included. The resulting tree contains all edge lengths (center to center distances of point pairs), the resulting weight (sum of edge lengths) being minimal. From the total tree weight one deduces the average edge length (dividing the total tree length by (N-1)) and its standard deviation. For comparison with known data the average edge length and the standard deviation are normalized by dividing by the square root of the average cell surface:

\[
m = \frac{\text{total\_tree\_length}}{\text{number\_of\_points} - 1} . \frac{1}{\sqrt{\text{cell\_surface}}}\]  

\[
\sigma = \text{standard\_deviation} \cdot \frac{1}{\sqrt{\text{cell\_surface}}}\]  

The resulting data are evaluated in histogram and the normalized average edge length \(m\) and its standard deviation \(\sigma\) are placed in the \((m,\sigma)\)-plane, where it can be compared with other known two dimensional distributions. The \((m,\sigma)\)-lines in the diagram connect the regular or semiregular mosaical distributions (composed of a set of regular polygons) in two dimensions from the perfect arrangement toward a "perfect" random distribution. The lines are calculated by starting at the perfect arrangement (\(\sigma=0\)) and giving each point a new position deduced from its previous position using a Gaussian distribution with a standard deviation \(\omega\). For values of \(\omega\) of several times \(m\) the random distribution is reached. In the analysis of \(m\) and \(\sigma\) of a distribution of points one has to pay attention to the dependence of the number of points \(N\) considered. This is explained by the following: The Minimal
Spanning Tree will search for every closest next point from the collection of points. If only a few points are considered, a greater percentage of the points of the collection can only be approached from one side, because they are at the side of the considered area. This means that the distance added to the tree does not have to be the shortest distance theoretically available to reach this point. The resulting effect is that with a decreasing amount of points considered, the total tree length will increase. By constructing a graph showing the dependence of the normalized average edge length of the amount of points considered, it is possible to determine the minimal amount of points needed for a correct result. A synopsis of the algorithm written to construct the Minimal Spanning Tree is included in the appendix.

An empirical relation considering the dependence of the average cell surface $y$ of an $x$-sided cell of the number of sides $x$ is the following:

$$a + b \cdot x = y$$

(15)

The parameters $a$ and $b$ determine the straight line. This relation is known as the Lewis Law. This law was found by Lewis in the twenties, analyzing the cells of cucumbers. Later, it was mathematically shown that this relation is true for infinite structures. Investigation has shown that this law is approximately true for finite lattices and that it is applicated best for 4- to 8-sided cells.

2.5. The analogy between wall breakage in a soap froth and coalescence in a drop lattice

The effect of drop coalescence on the distribution of the drops (change in the Voronoi network) can approximately be compared with wall breakage between two bubbles in foam, see figure (8). If the wall breaks between an $i$-sided and an $j$-sided bubble the resulting bubble will have $k$ sides:

$$i + j - 4 = k$$

(16)

In the drop patterns, one considers closest neighbors (number of sides in the Voronoi network) instead of walls of a cell.
3. Experimental Part

The experimental set-up is represented in figure (9).

In the experiment an amount of approximately 20 g. oil was put on a horizontal glass plate. The plate was placed on three adjustable standards, to ensure a horizontal position. The oil spread in two days to a homogeneous layer of a thickness of approximately 0.24 mm. The spreading was stopped by lines drawn with a marker. During the two days the oil was protected against dust by putting a second glass plate above the first one. The second plate rested on two horizontal bars and the plates were closed with cellotape. At the beginning of the experiment the protecting glass plate and the cellotape were removed and the glass plate with the oil was turned quickly up-side-down. If a regular network of drops was wanted at the beginning of the experiment, a prepared glass plate with pins glued on it following a regular pattern was put on the oil layer before turning the plate up side down. The pins caused a regular pattern of disturbances on the oil layer, which resulted in a regular drop pattern after turning the layer. Close to the oil layer two oil containing objects were placed, allowing a calibration of thickness measurements with the transmitted light intensity. These glass objects are made of two microscope slides glued together under a certain angle. The resulting fluid gradient in the objects allows the calibration according the Beer-Lambert law (however, for the statistical analysis the thickness of the layer was of minor importance and was not used). From the time at which the glass plate was turned the evolution of the experiment was filmed with a camera and recorded on a high resolution videotape. After the experiment the images were taken from the videotape to analyze.

The oil used for the experiment was composed of 100 gram silicon oil Rhodorsil 47 V 500 PROLABO and 2 gram blue dye Organol PROLABO. After heating and stirring the oil was filtered. The oil had a density $\rho=0.97$ g/cm$^3$, a surface tension $\gamma=21$ dynes/cm and a viscosity $\eta=485$ cP at 20 degrees Celsius.

Since the wave and drop dynamics are very sensitive to dust and other sorts of pollution, it is necessary to clean the glass plate thoroughly before preparing the fluid pancake. First, the oil left over from an earlier experiment was removed with cleaning paper and water. Then the plate was rubbed with paper and acetone (purity 99%) and put in a acetone bath, each side down for about twenty minutes. Finally both sides of the plate were put in a
distilled water bath during half a minute. Removed from the water bath, the plate was dried with paper and put in place in the experimental set-up. Before adding the oil, the dust particles remaining on the glass plates were removed with special optical tissue.

The equipment used for the recording was as follows. The camera was a COHU RS-170 black and white camera of the type CCD, with a Avenir TV Zoom lens of 12.5-75 mm and a yellow CFrance filter 49 mm (yellow is the complementary color of blue, blue being the color of the dye used). The timer was a FOR.9 video timer and the video recorder was a JVC CR-6650E, using videotapes type SONY KCA-60BRS (broadcast standard). The images were treated with the program Image, release 1.53 on a Apple Macintosh Iici and stored on optical disk. After image treatment the images were analyzed on a Iris Indigo computer of Silicon Graphics computer systems in a UNIX environment. The algorithms made for analysis of the images were written as an extension of Mouse (a locally made image treatment program by M. Fermigier et al.), in the language FORTRAN. The algorithms used the library of the Silicon Graphics.

The procedure for the image treatment was as follows. A selected amount of images of the videotape was imported in the Apple Macintosh (sized 676 x 511 pixels), each pixel containing a value ranging from 1 to 256, representing the gray level (figure (10)). Of the images a selection of 512 x 480 pixels was made (i.e. the maximal image format of the program Mouse). The same selection was used on each image. The 512 x 480 pixel images were thresholded, using for each experiment a constant value, usually about 144 (figure (11)). The binarized images were send to the Silicon Graphics, and analyzed with the Voronoi (figure (12)) and the Minimal Spanning Tree (figure (13)) algorithm. Finally, the data were plotted in programs such as Cricket Graph and Igor on the Apple Macintosh.

Three experiments were realized, one with a drop pattern nucleated by front propagation, one with an imposed triangular network and one with an imposed square network. The shape of the fluid layer was rectangular in all experiments. In table 1. the relevant information is given. In the case of a forced drop pattern, the distance expected to result from front propagation was taken. This means for a triangular network d=1.5 cm (the distance between drops resulting from not disturbed ‘rolls’) and d=1.3 cm for the square network (the distance of drops resulting from two perpendicular meeting ‘rolls’).
Table 1. Realized experiments.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Layer Thickness (mm)</td>
<td>0.24</td>
<td>0.24</td>
<td>0.24</td>
</tr>
<tr>
<td>Initial Drop Pattern</td>
<td>Caused by Front Propagation</td>
<td>Imposed Triangular</td>
<td>Imposed Square</td>
</tr>
<tr>
<td>Initial Drop Distance (cm)</td>
<td>1.4 ± 0.2</td>
<td>1.5</td>
<td>1.3</td>
</tr>
<tr>
<td>Pancake Dimensions (cm)</td>
<td>34.0 x 34.0</td>
<td>29.3 x 29.8</td>
<td>27.3 x 27.3</td>
</tr>
</tbody>
</table>
4. Results and Discussion

4.1. Properties resulting from the constructed Voronoi cells

For the statistical analysis of the two dimensional drop patterns, the Voronoi theorem has been used to construct cells around the drops. In the next part, the results of this analysis will be discussed.

4.1.1. Distribution of the number of sides

The histogram of the number of sides of the cells is given in figure (14) for experiment 4. The figures (15),(16),(17) represent the normalized distributions for the experiments 4, 5 and 6. The long time behavior of experiment 5 is missing because of a slight non-horizontality of the glass plate.

Comparing the three experiments, one notices a regular distribution around the 6-sided cell, a tendency to develop 6 sides and a broadening of the distribution after longer periods of time. After one day the resulting distribution for the experiments has about 35% 6-sided cells and 24% 5- and 7-sided cells.

4.1.2. Number of drops and average cell surface versus time

The number of drops versus time of the three experiments is given in figures (18) to (20). The relative decrease of the number of drops for all three experiments is given in figure (21). The related average cell surface versus time is given in the figures (22),(23),(24). Since the considered area of the image in this analysis is taken constant, the graphs are approximately their inverses.

Concerning the number of drops, it appears that the dependence of the number of drops on time is exponential. The main reason will be that a drop coalescence is proportional to the (average) drop distance. This means that each time a drop falls, the (average) drop distance will increase, resulting in a decreasing frequency of drop coalescence, etc.
At the fifth experiment, one sees even a slight increase of the number of drops, resulting from the non-horizontallity of the glass plate and the apparent stable drop distribution (drops started to fall after 10 minutes, in the other experiments right from the beginning). This slight increase can be explained by the competition of the speed of coalescence and the moving of independent drops. Because for the experiment a constant image-area selection was used, apart from disappearing drops caused by coalescences, there were incoming drops. At the beginning of the experiment the speed of disappearance was in the case of experiment 5 less than the moving speed of the incoming drops.

If one compares the speed of drop disappearance between the experiments, it is obvious that the triangular pattern is the most stable. The most unstable pattern is the square pattern, since it has the fastest normalized decrease.

4.1.3. Distribution of angles between two closest neighbors

The development of the distribution of angles between the closest neighbors of a central drop, the angle being the angle between two neighboring closest neighbors and the central drop, are given in the figures (25) to (33). Considering the results of the fourth experiment, one sees a constant maximum at 60 degrees. In time, the distribution broadens slightly. For experiment 5 there is a narrow maximum around 60 degrees, which broadens in time*.

For experiment 6, after 7 minutes a local maximum can be seen around 90 degrees, and the overall maximum is situated at 50 degrees. These two maxima come together after some time at about 60 degrees. The two maxima can be explained by the fact that the starting network was almost square. Every small movement of a drop results in one or two extra sides, giving cells with 2 pairs of sides under an angle of approximately 90 degrees and two small sides under angles of approximately 45 degrees. In time this pattern evaluates toward a more regular distribution, characterized by a maximum at 60 degrees. Evident from this experiment is that the preferred distribution of points is a disturbed triangular pattern (meaning hexagonal cells).

* For a correct representation of the results of experiment 5, all values on the horizontal axis of the figures (27),(28),(29),(30) should be reduced with 6 degrees.
4.1.4. Verification of Lewis Law

In figure (34) the average surface of x-sided cells versus the number of sides is fitted to a straight line. Since the dependence of the experimentally measured average surface of the cells depends linear of the number of sides, the Lewis Law is satisfied.

4.1.5. Filling up open areas in an ordered pattern

At the beginning of experiment 5 and 6, not all the pins of the glass plate used to impose a regular pattern touched the oil layer. If one compares the images taken just after turning the oil layer and those taken when the pattern has set in (figures (35) to (38)), i.e. after some minutes, it is obvious that the untouched areas of experiment 5 filled up with exactly the same pattern of drops as the pattern already imposed on the rest of the layer. For experiment 6 though, the left open areas filled up spontaneously with a pattern resembling the triangular one, and not the square one. These observations indicate that a triangular pattern is favored compared to a square pattern.

4.2. The Minimal Spanning Tree analysis

The Minimal Spanning Tree graph is a statistical method from which characteristic parameters can be deduced. The main parameters are the normalized average edge length (m) and its standard deviation (σ). Also, the histogram of m gives information concerning the existence of a gradient in the distribution of drops. The results of this analysis will be discussed in the next part.

4.2.1. Justifying the use of the Minimal Spanning Tree

To justify the use of the Minimal Spanning Tree for analysis of a arrangement of points, a minimum amount of points is required. Therefore, the dependence of m and σ versus N is plotted in the figures (39) and (40). In these figures, the values of m and σ are calculated for a part of the area of the total selection (N/2, N/4, N/8 and N/16). A regression is calculated for the points of figure (40), being m=0.919+0.228*N−0.904. Based on this calculation and a similar calculation performed by Billia\textsuperscript{15} it is decided to use a
minimum amount of 210 points for the Minimal Spanning Tree calculation.

The histogram of the distribution of edge lengths (figure (41)) is difficult to interpret. The maximum observed can be seen as one peak, meaning no occurrence of a gradient, or can be interpreted as two neighboring maxima in the middle of the distribution. In general, two maxima in the distribution indicate a gradient in the distribution of drops. This would mean that a slight gradient exists in the arrangement of drops in the fourth experiment. This can be caused by the fact that the drops at the sides of the region result merely of front propagation and the drops at the center were formed already before the 'rolls' could arrive. As can be seen in figure (11), the drops in the center appear to have a slightly bigger center to center distance.

4.2.2. The \((m,\sigma)\)-plane

The resulting values of \(m\) and \(\sigma\) are placed in the \((m,\sigma)\)-plane (figure(42)). The values for the initially square and triangular pattern stay close to the lines calculated by Dussert\textsuperscript{12,13} for these patterns. The points resulting from experiment 4 appear to start close to the triangular line, but do not follow that trajectory toward greater disorder. For the three experiments, the development in time is toward a situation of greater disorder, as expected.

4.3. An indication on coalescence analogy

In figure (43) 5 coalescences are observed and their effect on the Voronoi network. Only for 2 of the 5 cases (b and c), formula (16) can be applied. It appears that the coalescence of two drops results in a change of location of the center of gravity of the resulting drop compared to the locations of the two separated drops. The center of gravity is the parameter which determines the cell in the Voronoi construction. This is different compared to the breakage of a bubble wall, were the resulting pressure (being the determining force) will be approximately the same as in the two separated bubbles. Since the resulting pressure does not change significantly, the remaining cell walls will not change from position.
5. Conclusion

A novel technique has been applied to a new type of problem with success.

The Minimal Spanning Tree can be applied on fluid dynamics like a drop pattern. If initially an ordered drop pattern is imposed, the Minimal Spanning Tree analysis indicates an evolution of the pattern towards greater disorder. A disadvantage of this statistical technique however is the lack of quantitative information obtained. One should not use the Minimal Spanning Tree method for less than approximately 200 drops.

Based on the analysis of data obtained by the Voronoi construction on various patterns of pending drops, the following can be concluded:

The distribution of the number of closest neighbors of a central drop is normal and is centered around 6.

The distribution of angles between closest neighbors and a central drop indicates the formation of a disordered, triangular lattice.

An imposed triangular lattice is the more stable than a drop lattice originated by propagating fronts. A square lattice is the most unstable lattice. The number of drops left decreases exponentially in time.

The Lewis Law can be applied to the cells formed around the drops.

Qualitatively, drop coalescence can be compared with wall breakage in soap froths. Quantitatively, one should take into consideration how the position of the center of gravity of the cell effects the position of the cell walls.
## Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>parameter of Lewis Law</td>
<td>-</td>
</tr>
<tr>
<td>AH</td>
<td>amplitude causing a hexagonal pattern</td>
<td>-</td>
</tr>
<tr>
<td>AQ</td>
<td>amplitude</td>
<td>-</td>
</tr>
<tr>
<td>AR</td>
<td>amplitude causing a &quot;roll&quot; pattern</td>
<td>-</td>
</tr>
<tr>
<td>b</td>
<td>parameter of Lewis Law</td>
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</tr>
<tr>
<td>c*</td>
<td>general velocity of the propagating front</td>
<td>-</td>
</tr>
<tr>
<td>g</td>
<td>gravitational acceleration</td>
<td>m/s^2</td>
</tr>
<tr>
<td>h</td>
<td>thickness of the oil layer</td>
<td>m</td>
</tr>
<tr>
<td>h*</td>
<td>critical thickness</td>
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<tr>
<td>h0</td>
<td>initial thickness of the oil layer</td>
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</tr>
<tr>
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</tr>
<tr>
<td>j</td>
<td>number of sides</td>
<td>-</td>
</tr>
<tr>
<td>k</td>
<td>number of sides</td>
<td>-</td>
</tr>
<tr>
<td>k*</td>
<td>wave number of the propagating front</td>
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</tr>
<tr>
<td>m</td>
<td>normalized average edge length</td>
<td>-</td>
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<tr>
<td>N</td>
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<tr>
<td>t</td>
<td>time</td>
<td>s</td>
</tr>
<tr>
<td>q</td>
<td>wave number</td>
<td>1/m</td>
</tr>
<tr>
<td>qsubscript</td>
<td>wave number</td>
<td>1/m</td>
</tr>
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<td>1/m</td>
</tr>
<tr>
<td>qM</td>
<td>wave number associated with the pattern caused by the most unstable situation</td>
<td>1/m</td>
</tr>
<tr>
<td>u</td>
<td>general front propagation velocity</td>
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</tr>
<tr>
<td>v*</td>
<td>velocity of the propagating front</td>
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<tr>
<td>V*</td>
<td>critical volume of a pendant drop</td>
<td>m^3</td>
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<td>w(k)</td>
<td>amplification rate</td>
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<tr>
<td>δ</td>
<td>Dirac-function that is either 0 or 1</td>
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<tr>
<td>ε</td>
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<tr>
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<td>wavelength resulting from the most unstable situation</td>
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<tr>
<td>Symbol</td>
<td>Description</td>
<td>Unit</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------</td>
<td>---------</td>
</tr>
<tr>
<td>( \rho )</td>
<td>density</td>
<td>kg/m³</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>normalized standard deviation</td>
<td>-</td>
</tr>
<tr>
<td>( \sigma(q) )</td>
<td>growth rate</td>
<td>1/s</td>
</tr>
<tr>
<td>( \sigma_M )</td>
<td>maximum growth rate</td>
<td>1/s</td>
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<tr>
<td>( \zeta )</td>
<td>deviation of the initial thickness</td>
<td>m</td>
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- caméra CCD
- plaque de verre (fine)
- crêpe d'huile
- table lumineuse
- retournement
- plaque de verre (épaisse)
- timer
- écran vidéo
- magnétoscope
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Number of Cells

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Occurrence (%)

![Graph showing the occurrence of number of sides in time for experiment 4.]
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Occurence (%)

![Graph showing the number of sides in time for experiment 5.](image1)

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Occurence (%)

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Number of Drops

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Number of Drops
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Number of Drops

Time (mn)

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Decrease

Time (mn)

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- Initially triangular lattice
- Lattice formed by front propagation
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Average Cell Surface (cm²)

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Average Cell Surface (cm²)
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Average Cell Surface (cm²)
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Average Surface (cm²)

\[ y = 0.57893 + 0.29575x \quad R^2 = 0.890 \]
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Appendix: Computer Algorithms

The major steps of the Voronoi and Minimal Spanning Tree algorithms are given in the following lines. Just the important steps or actions are mentioned, which served as a guideline for the writer.

Schematic representation of the Voronoi algorithm (The program 'Voronoi' treats a binarized image):

- Setting all counting or summing variables zero.

- Calling a subroutine which calculates the centers of gravity of the objects in the image and stores them in an array A.

- Asking the area in pixel coordinates and selecting the points within the area to be future 'central points'.

- Taking the first central point.

- Selecting the twenty closest neighbors of this central point in array B.

- Defining the mid perpendiculars of the central cell and its twenty closest neighbors and storing them in array C.

- Taking the mid perpendicular of the central point and its closest neighbor and searching for the closest intersection of this line with one of the other mid perpendiculars. The point belonging to this next line is the second neighbor.

- Once the second neighbor is found, the search continues in such a way that the neighbors are found each time next to the old neighbor. This search continues until the next neighbor equals the first neighbor.

- These points and the parameters are stored in the arrays D and E.

- The counting variable of number of sides of a cell is adjusted, the surface of the cell is calculated by means of
pixel counting within the cell area and the angle between central point and its closest neighbors is calculated.

- The wanted variables are written to a file.

- Based on array E containing the parameters of the mid perpendiculars constructing the central cells, the Voronoi network is drawn on the screen.

**Schematic representation of the Minimal Spanning Tree (MST) algorithm (the algorithm treats a binarized image):**

- Calling a subroutine which calculates the centers of gravity of the objects.

- Selecting the area to consider for the MST calculation and the points within this area.

- Taking a start point and searching its closest neighbor.

- Searching among the points not yet included in the tree for the closest neighbor of the points already included in the tree, and assigning this point to the tree. Repeating this until all points are included in the tree.

- Writing all point to point distances to an array A and the parameters of the connecting lines to an array B.

- Calculating the Normalized Average Edge Length and its Normalized Standard Deviation.

- Drawing the tree on the screen with the data stored in the array B.