PRECONDITIONING A FINITE ELEMENT SOLVER OF THE EXTERIOR HELMHOLTZ EQUATION

E. Turkel*, Y. A. Erlangga†

* Tel-Aviv University, Department of Mathematics, Israel
e-mail: turkel@post.tau.ac.il
† Technische Universitat Berlin, Institut fur Mathematik, D-10623 Berlin, Germany
e-mail: erlangga@math.tu-berlin.de

Key words: Helmholtz equation, scattering, preconditioning, BICGSTAB

Abstract. We consider acoustic scattering about a general body. This is described by the Helmholtz equation exterior to the body. In order to truncate the infinite domain we use the BGT absorbing boundary condition. The resultant problem in a finite domain is solved by a finite element procedure. This yields a large sparse system of linear equations which is neither symmetric nor positive definite. We solve the system by an iterative Krylov space type method. To increase the rate of convergence a preconditioner is introduced. This preconditioner is based on a different Helmholtz equation with complex coefficients. This preconditioned system is again solved by a Krylov space method with an ILU preconditioner. Computations are presented to show the efficiency of this technique.

1 INTRODUCTION

We consider scattering about general bodies based on the Helmholtz equation

\[ \Delta u + k^2 u = 0 \] (1)

The equation is discretized using linear finite elements. Several recent surveys discuss the difficulties in numerically solving the Helmholtz equation exterior to a body, see for example [8, 11, 12]. Since we consider the exterior of a body it is necessary to truncate the infinite domain and introduce an artificial outer surface. On this surface we introduce boundary conditions that reduce reflections back into the physical domain. The conditions we choose are the second order Bayliss-Gunzburger-Turkel (BGT) [1] absorbing boundary conditions. With this boundary condition the solution is complex. The resultant discretized system is not positive definite and is also not symmetric. For large values of the wave number \( k \) it becomes difficult to directly invert the matrix. Hence, we shall only consider Krylov space methods for an iterative solution of the system. In particular we shall use the BICGSTAB [14] iterative method. It is well known that Krylov methods converge slowly unless the discrete system is preconditioned.
Bayliss, Goldstein and Turkel [2] introduced a preconditioner based on the Laplace equation. It was found that this worked well for low wave numbers but became less efficient as \( k \) increased. Erlangga, Oosterlee and Vuik [5] introduced a generalization where the preconditioner is the solution to the full Helmholtz equation but with a complex \( k \). Nevertheless, one still has to solve a system the same size as the original system. So, the modified Helmholtz equation needs to be also solved by an inner iterative method. In a later paper Erlangga, Oosterlee and Vuik [6] consider a multigrid (MG) method. However, for a general unstructured mesh, multigrid is difficult to implement. Hence, we shall also solve the inner preconditioned problem by BICGSTAB. A complex \( k \) introduces dissipation which makes the preconditioned system easier to solve than the original system with a real \( k \). We precondition the modified Helmholtz equation by ILU(0). Furthermore, we have a reasonable initial guess for the inner iteration. We typically reduce the residual for the outer (physical) problem by 6 orders of magnitude while the inner iteration is only reduced by three orders of magnitude with a maximum of 15 iterations. Hence, one now needs fewer iterations for the inner system.

2 Equations

We consider two dimensional scattering about a general boundary. The total field is given by \( u^{\text{tot}} = u^{\text{scat}} + u^{\text{inc}} \). We consider \( u^{\text{inc}} \) as given by a plane wave. Then [4]

\[
\Delta u + k^2 u = 0 \text{ exterior to } D \\
on \partial D \quad u = u^{\text{scat}} + e^{ik \cdot x} \quad \text{(soft body)} \\
or \quad \frac{\partial u}{\partial n} = \frac{\partial}{\partial n}(u^{\text{scat}} + e^{ik \cdot x}) \quad \text{(hard body)}
\]

\[
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u^{\text{scat}}(x)}{\partial r} - ik u^{\text{scat}}(x) \right) = 0 \quad \text{Sommerfeld radiation condition}
\]

We discretize the equation in the interior of \( D \) by a standard linear finite element method. To approximate the Sommerfeld radiation condition we first truncate the domain by introducing an artificial surface at \( R_{\text{max}} = 2 \). Hence, the outer surface is a circle independent of the shape of the scatterer (assumed to be within the unit circle). On this artificial surface we impose the second order BGT radiation condition [1]

\[
B_2 u = 2 \left( ik - \frac{1}{r} \right) \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \left( 2k^2 + \frac{3ik}{r} - \frac{3}{4r^2} \right) u = 0
\]

(3)

To improve accuracy at lower wave numbers we replace this by

\[
B_2 u = \frac{\partial u}{\partial r} - k \left[ \frac{H_0'(kr)}{H_0(kr)} u + \left( \frac{H_0'(kr)}{H_0(kr)} - \frac{H_1'(kr)}{H_1(kr)} \right) \frac{\partial^2 u}{\partial \theta^2} \right] = 0
\]

(4)

where \( H_0'(kr) = -H_1(kr) \) and \( H_1'(kr) = 0.5(H_0(kr) - H_2(kr)) \).

The resultant linear system of equations is sparse but has complex quantities, is not self adjoint and its positive part is not positive definite. We shall concentrate on iterative methods
since direct solvers are limited to coarse grids especially in three dimensions. Furthermore, for linear finite elements the grid grows like $k^{3/2}$ [3] in order to maintain a constant accuracy. Hence, for high wave numbers there is a need for very fine grids. In particular we shall consider Krylov space methods.

Numerous authors have found that BICGSTAB is an effective iterative method to solve the linear system arising from the Helmholtz equation. However, this method, along with other Krylov space methods, converges reasonably well only if the equations are preconditioned. Hence, we replace

$$ Au = b $$

by

$$ AP^{-1}v = b \quad Pu = v $$

For the preconditioner to be efficient we require two conditions. The first is that $P$ is an approximation to $A$ and second that $Pu = v$ is easier to solve than $Au = b$.

Over the years two types of preconditioners have been developed [13]. In the first kind the preconditioner utilizes algebraic properties of the matrix. A typical example is $ILU$ where lower and upper factors are derived which are sparse and $LU$ agrees with $A$ for some set of elements of $A$. A different class of preconditioners is based on properties of the differential system, i.e. an operator based preconditioner. An example of this is basing the preconditioner on the solution of the Laplace equation [2]. A similar idea was used by Singer and Turkel [10] in studying the Helmholtz equation in a waveguide with a PML to absorb outgoing waves. Here a Helmholtz equation with a small wave number was used as the preconditioner. Erlangga, Vuik and Oosterlee [5, 6, 7] extended this idea by considering a Helmholtz equation with a complex wave number as the preconditioner. They originally considered a purely imaginary value for $k^2$ [5, 7] but later considered a complex valued $k^2$ coupled with multigrid [6]. Hence, we consider a preconditioner which is an approximate solution to

$$ \Delta u + k^2_{\text{prec}} u = 0 $$

In this study we combine elements of the previous papers. As in [5, 6, 7] we consider a complex value for $k^2_{\text{prec}}$ rather than a purely imaginary value. We are interested in finite element methods and unstructured meshes and so the implementation of geometric multigrid is difficult. The use of an algebraic multigrid for general nonsymmetric problems is still not well developed. Hence, we solve the preconditioned problem by BICGSTAB rather than multigrid. Using BICGSTAB on the preconditioned problem is still fairly slow and so we precondition this problem using ILU(0). All runs were done using MATLAB on a PC with a 3.4Ghz processor, 2GB of memory and with Microsoft Windows.
3 RESULTS

As a model case we consider scattering about a hard ellipse (natural boundary condition at the scatterer) with an additional tower to resemble a submarine. The ellipse has a major axis of 1 and a minor axis of $\frac{1}{5}$. The outer surface is at $R=2$. A typical grid is shown in figure 1.

![Figure 1: Submarine and grid](image)

In summary, we have several layers of iterations. The outer loop is based on solving the original scattering problem with BICGSTAB. This is preconditioned by the approximate solution of a modified Helmholtz problem with a complex wave number. The modified Helmholtz problem is again solved by BICGSTAB. In order to make it more effective this modified Helmholtz problem is also preconditioned but now by ILU(0). Within the inner iterations one needs an initial guess of the solution. This is given by the final solution of the previous inner iteration. Erlangga et. al. found that when using multigrid for this problem an optimal modified $k^2$ was $k^2_{\text{prec}} = (1 + 0.5i)k^2$. The lower the imaginary component the faster the outer iterations will converge. If one could choose $k^2_{\text{prec}} = k$ then the inner iteration would converge in one cycle. However, in that case we could solve the original problem easily. When using BICGSTAB we found that we could choose $k^2_{\text{prec}} = (1 + 0.3i)k^2$. We surmise that this is because BICGSTAB is more robust than MG which strongly relies on the properties of the smoother. $k^2_{\text{prec}}$ is also used in the far field boundary condition. We note that if the imaginary part is zero then BICGSTAB does not converge. So we need BICGSTAB with an ILU(0) preconditioner on the original finite element matrix. If we use the MATLAB command luinc with a drop tolerance of $10^{-5}$ then the inner iteration converges in 1 iteration, i.e. we are getting the exact LU decomposition.

To examine the effectiveness of the inner iteration we solve the preconditioned problem by Gaussian elimination. This, of course, is for demonstration purposes only. If we could invert the preconditioned problem by Gaussian elimination we could invert the original matrix by Gaussian elimination. We first check the effect of $k$ and the mesh on the convergence rate of the outer BICGSTAB. Hence, we solve the modified Helmholtz equation with $k^2_{\text{prec}} = (1 + 0.3i)k^2$ by Gaussian elimination to see the best we can do. We see from table 1 that the number of BICGSTAB steps needed with the new preconditioning essentially depends only on $k$ and not
on the number of elements. We see in figure 2 that for $k = 20$ much of the convergence history difficulties occur at the beginning. The dependence on $k$ is roughly proportional to $k^{1/2}$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>mesh</th>
<th># outer iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8 × 48</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>64 × 384</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>16 × 96</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>64 × 384</td>
<td>11</td>
</tr>
<tr>
<td>10</td>
<td>16 × 96</td>
<td>15</td>
</tr>
<tr>
<td>10</td>
<td>64 × 384</td>
<td>16</td>
</tr>
<tr>
<td>20</td>
<td>32 × 192</td>
<td>26</td>
</tr>
<tr>
<td>20</td>
<td>96 × 576</td>
<td>25</td>
</tr>
</tbody>
</table>

Table 1: Convergence rate for BICGSTAB solving exactly the modified problem

We now use BICGSTAB with an ILU(0) preconditioning to solve the outer preconditioned problem (5). In table 2 we present some results for lower wave numbers. In the first three cases there are about 20 points per wave length. Since for linear elements we wish $k^3 h^2$ to be constant we also solved the case $k = 10$ with a finer mesh. The outer iteration is considered solved when we reduce the residual by six orders of magnitude. Since the inner iteration is only a preconditioner there is no purpose in making it too accurate. It was found that if the inner iteration reduces the initial residual by three orders of magnitude that this was sufficient for an efficient algorithm. We also allowed a maximum of 15 inner iterations. In practice the reduction in the residual rarely reached three orders of magnitude (except for small $k$) and the inner iterations were stopped after 15 iterations. For larger $k$ than studied here it may require more than 15 inner iterations to allow the total algorithm to converge. Allowing the preconditioner to vary may bring difficulties in conjuction with BICGSTAB but none were observed in this study.
Comparing table 1 with table 2 we see that with these parameters we do not require more outer iterations when the inner problem is solved by BICGSTAB than when it is inverted exactly. We compare the time (in seconds) to solve the linear system with the time needed for the assembly of the finite matrix mass and stiffness matrices. We again plot the convergence rate for $k = 30$ in Fig. 3. We now see, even clearer, that almost half the iterations are required to smooth the initial guess.

<table>
<thead>
<tr>
<th>$k$</th>
<th>mesh</th>
<th># outer iter</th>
<th>total # iter</th>
<th>time sol</th>
<th>time assem</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$8 \times 48$</td>
<td>5</td>
<td>55</td>
<td>1.5</td>
<td>0.4</td>
</tr>
<tr>
<td>5</td>
<td>$16 \times 96$</td>
<td>11</td>
<td>157</td>
<td>5.8</td>
<td>1.3</td>
</tr>
<tr>
<td>10</td>
<td>$32 \times 192$</td>
<td>16</td>
<td>256</td>
<td>31.4</td>
<td>9.1</td>
</tr>
<tr>
<td>10</td>
<td>$64 \times 384$</td>
<td>16</td>
<td>350</td>
<td>194</td>
<td>90.1</td>
</tr>
<tr>
<td>20</td>
<td>$64 \times 384$</td>
<td>26</td>
<td>416</td>
<td>226</td>
<td>92.9</td>
</tr>
<tr>
<td>25</td>
<td>$96 \times 576$</td>
<td>43</td>
<td>688</td>
<td>1006</td>
<td>392.6</td>
</tr>
<tr>
<td>30</td>
<td>$96 \times 576$</td>
<td>64</td>
<td>1024</td>
<td>1220</td>
<td>399</td>
</tr>
</tbody>
</table>

Table 2: Convergence rate for BICGSTAB for various wave numbers

![The iterations using BICGSTAB + Right Pre](image)

Figure 3: Convergence for $k = 30$

For the final set of results we consider only the higher wave number case with $k = 20$ with a $64 \times 384$ mesh. We now see the effect of various changes in parameters on this case. Consider $k^2_{\text{prec}} = (\alpha + i\beta)k^2$. As before we use 15 iterations of BICGSTAB for the modified Helmholtz equation coupled with an ILU(0) preconditioner. The modified Helmholtz equation uses the complex $k$ not only in the equation by also in the farfield boundary condition. If we use the original $k$ in the farfield boundary condition (4) we require 34 outer iterations instead of 26 for $\beta = 0.3$. Without the ILU(0) preconditioner for the modified Helmholtz equation the BICGSTAB does not converge even with $\beta = 0.7$. 

6
<table>
<thead>
<tr>
<th>$(\alpha, \beta)$</th>
<th># outer iter</th>
<th>total # iter</th>
<th>time sol</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 0.25)$</td>
<td>27</td>
<td>432</td>
<td>241</td>
</tr>
<tr>
<td>$(1, 0.30)$</td>
<td>26</td>
<td>416</td>
<td>223</td>
</tr>
<tr>
<td>$(1, 0.35)$</td>
<td>34</td>
<td>544</td>
<td>280</td>
</tr>
</tbody>
</table>

Table 3: Convergence rate for BICGSTAB for $k = 20$

4 CONCLUSIONS

We solve the Helmholtz equation exterior to a two dimensional body by a finite element method with an absorbing boundary condition in the far field. The resultant nonsymmetric linear system is solved by BICGSTAB. To accelerate the convergence the system is right preconditioned by a modified Helmholtz equation with a complex wave number. This preconditioned system is again solved by BICGSTAB with an ILU(0) preconditioning. With this combined system we achieve fast convergence even for high wave numbers.

REFERENCES


