SOLUTION OF THE BOUSSINESQ EQUATIONS BY MEANS OF THE FINITE ELEMENT METHOD

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Abstract—A finite element method is presented for the computation of flows that are influenced by buoyancy forces. The accuracy of several finite elements is studied by solving the Bénard problem and determining the critical Rayleigh number. It is found that the accuracy is greatly enhanced if the shape functions satisfy a certain requirement that arises from the physical nature of the problem.

1. INTRODUCTION

The finite element method is rapidly gaining popularity in computational fluid mechanics. Finite element solutions for a variety of problems for the Navier–Stokes equations have been obtained by a number of authors, for example [1–5]. In the present paper finite element solutions are obtained for a more general set of equations, namely the Boussinesq equations. Specifically, internal flows driven by buoyancy forces that are caused by temperature differences are considered. It is found that the shape functions must not only satisfy the usual requirements, but that the accuracy is greatly enhanced if an additional requirement is satisfied, that arises from the physical nature of the problem.

2. THE PHYSICAL PROBLEM

Let Ω be a two-dimensional region, possibly multiply connected. The boundary $\partial\Omega$ is kept at a prescribed temperature, which may vary along $\partial\Omega$. Let Ω contain a fluid with positive thermal expansion coefficient. In general the density variations caused by the variable temperature field will set the fluid in motion. If the density differences are not too large, the steady state flow is governed by the Boussinesq equations [6, p. 507]

$$(u_i u_j)_{,j} = -p_{,i} + \sigma^{1/2} R^{-1/2} u_{i,jj} + \theta \delta_{i2}, \quad (2.1)$$

$$(\theta u_i)_{,i} = \sigma^{-1/2} R^{-1/2} \theta_{,jj}, \quad (2.2)$$

$$u_{i,j} = 0. \quad (2.3)$$

Cartesian tensor notation has been adopted. The range of Latin indices is 1,2. The direction of gravity is along the negative x_2 -axis. The Prandtl number $\sigma = (\nu/\kappa)$ equals the ratio of dynamic viscosity and thermal diffusivity, and the Rayleigh number R is defined as

$$R = g\beta\Delta T L^3/(\kappa\nu), \quad (2.4)$$

with g the acceleration of gravity, β the thermal expansion coefficient, ΔT a measure of the temperature variation and L a typical length scale for Ω . The variables in (2.1)–(2.3) have been non-dimensionalized with length unit L , temperature unit ΔT and velocity unit $\{g\beta L \Delta T\}^{1/2}$.

The boundary conditions are

$$\theta = \text{prescribed}, \quad u_i = 0 \text{ on } \partial\Omega. \quad (2.5)$$

3. LINEARIZATION AND FINITE ELEMENT APPROXIMATIONS

In [1] a finite element formulation is given for a large class of flow problems that includes the Boussinesq eqns (2.1)–(2.3) as a special class. The same method has been considered by a

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number of authors for the Navier-Stokes equations, i.e. eqns (2.1)–(2.3) with $\theta = 0$; see for example [2–5]. If this method to obtain a finite element approximation is combined with the method of Newton to solve the resulting system of nonlinear equations, then the method used in this paper is obtained. For completeness a brief derivation is given below, together with the introduction of a convenient notation.

We prefer to apply Newton-linearization directly to eqns (2.1)–(2.3).

One obtains

$$(u_i \delta u_j)_{,j} + (\delta u_i u_j)_{,j} + \delta p_{,i} - \sigma^{1/2} R^{-1/2} \delta u_{i,jj} - \delta \theta \delta_{i2} = -f_i, \quad (3.1)$$

$$(\theta \delta u_j)_{,j} + (\delta \theta u_j)_{,j} - \sigma^{-1/2} R^{-1/2} \delta \theta_{,jj} = -f_3, \quad (3.2)$$

$$\delta u_{i,j} = -f_4, \quad (3.3)$$

with

$$f_i = (u_i u_j)_{,j} + p_{,i} - \sigma^{1/2} R^{-1/2} u_{i,jj} - \delta_{i2} \theta, \quad (3.4)$$

$$f_3 = (\theta u_j)_{,j} - \sigma^{-1/2} R^{-1/2} \theta_{,jj}, \quad (3.5)$$

$$f_4 = u_{i,j}. \quad (3.6)$$

Here u_i , θ , p denote the previous iterand; the new iterand is given by $u_i + \delta u_i$, $\theta + \delta \theta$, $p + \delta p$. (Note that δu_i , $\delta \theta$, δp do not denote variations in the sense of variational calculus).

It is convenient to introduce a more concise notation. The unknowns will be denoted by v_α : $v_1 = u_1$, $v_2 = u_2$, $v_3 = \theta$, $v_4 = p$. Greek indices have the range 1(1)4. The domain Ω is divided in finite elements with Q_α the set of nodal points for each unknown v_α ; v_α and v_β , $\alpha \neq \beta$ may or may not have all their nodal points in common. The unknowns v_α are approximated by $\hat{v}_\alpha$, defined by

$$\hat{v}_\alpha(x_i) = N_\alpha^q(x_i) v_\alpha^q. \quad (3.7)$$

The summation convention applies to all indices, except to Greek lower indices. A Latin superscript is always accompanied by a subscript; its range is Q_α with s the accompanying subscript. In other words, eqn (3.7) stands for

$$\hat{v}_\alpha(x_i) = \sum_{q \in Q_\alpha} N_\alpha^q(x_i) v_\alpha^q; \quad \text{no sum over } \alpha. \quad (3.8)$$

The shapefunctions N_α^q have as usual the following property

$$N_\alpha^q(x_{i\alpha}^r) = \delta^{qr}, \quad (3.9)$$

with $x_{i\alpha}^r$ the location of nodal point $r \in Q_\alpha$, and δ^{qr} the Kronecker delta.

The nodal values v_α^q are determined by means of the Galerkin method. A function R_α called the residue is obtained by substitution of (3.7) in (3.1)–(3.3). Second derivatives of u_i , δu_i , θ , $\delta \theta$ are integrated by means of the divergence theorem of Gauss. The nodal values v_α^q then follow from

$$\int_\Omega N_\alpha^q R_\alpha d\Omega = 0, \quad \alpha = 1(1)4, \quad q \in Q_\alpha, \quad (3.10)$$

and application of the boundary conditions. At some nodal point the pressure is put equal to zero.

The structure of the system (3.10) is as follows. The integration in (3.10) is carried out element by element. To be computed are the integrals

$$J_{e\alpha}^q = \int_{\Omega_e} N_\alpha^q R_\alpha d\Omega_e, \quad (3.11)$$

for the nodal points $q \in Q_\alpha \cap \Omega_e$, where Ω_e is an element.

One finds, omitting the subscript e

$$J_\alpha^q = \delta v_j^s v_\alpha^r a_{j\alpha}^{qrs} + \delta v_\alpha^s v_j^r a_{j\alpha}^{qrs} + \delta v_4^r b_\alpha^{qr} + \sigma^{1/2} R^{-1/2} \delta v_\alpha^r c_\alpha^{qr} - \delta_{\alpha 2} \delta v_3^r d_\alpha^{qr} + f_\alpha^q, \quad \alpha = 1, 2; \quad (3.12)$$

$$J_3^q = \delta v_j^s v_3^r e_j^{qrs} + \delta v_3^s v_j^r e_j^{qrs} + \sigma^{-1/2} R^{-1/2} \delta v_3^r g^{qr} + f_3^q, \quad (3.13)$$

$$J_4^q = v_j^r h_j^{qr} + f_4^q, \quad (3.14)$$

where

$$\left. \begin{aligned} f_\alpha^q &= v_j^s v_\alpha^r a_{j\alpha}^{qrs} - v_4^r b_\alpha^{qr} + \sigma^{1/2} R^{-1/2} v_\alpha^r c_\alpha^{qr} + v_3^r d_\alpha^{qr} \delta_{\alpha 2}, \quad \alpha = 1, 2, \\ f_3^q &= v_j^s v_3^r e_j^{qrs} + \sigma^{-1/2} R^{-1/2} v_3^r g^{qr}, \\ f_4^q &= v_j^r h_j^{qr}. \end{aligned} \right\} \quad (3.15)$$

$$\left. \begin{aligned} a_{\beta\alpha}^{qrs} &= \langle N_\alpha^q, (N_\alpha^r N_\beta^s)_{,\beta} \rangle \\ b_\alpha^{qr} &= \langle N_\alpha^q, N_4^r{}_{,\alpha} \rangle \\ c_\alpha^{qr} &= \langle N_{\alpha,j}^q, N_{\alpha,j}^r \rangle - \int_{\partial\Omega \cap \partial\Omega_e} N_\alpha^q (\partial N_\alpha^r / \partial n) d\Omega_e \\ d_\alpha^{qr} &= \langle N_\alpha^q, N_3^r \rangle \\ e_\alpha^{qrs} &= \langle N_3^q, (N_3^r N_\alpha^s)_{,\alpha} \rangle \\ g^{qr} &= \langle N_{3,j}^q, N_{3,j}^r \rangle - \int_{\partial\Omega \cap \partial\Omega_e} N_3^q (\partial N_3^r / \partial n) d\Omega_e \\ h_\alpha^{qr} &= \langle N_4^q, N_{\alpha,\alpha}^r \rangle \end{aligned} \right\} \quad (3.16)$$

with

$$\langle u, v \rangle = \int_{\Omega_e} uv d\Omega_e.$$

The integrals in (3.16) have been computed by means of Gauss quadrature of the order of the polynomial to be integrated.

4. THE BÉNARD PROBLEM

Our objective in this section is not to make extensive and accurate computations of the Bénard problem, but to use this problem as a test to discriminate between various finite elements with respect to their accuracy for the Boussinesq equations.

The finite elements of the Lagrange simplex type which are shown in Fig. 1 are considered. Not all of these elements can be used to solve the Boussinesq equations, or even the Navier-Stokes equations. Element 1 is bound to give very inaccurate results, because exact satisfaction of the continuity eqns (2.3) is possible only if v_i is identically zero in the case $v_i = 0$ on $\partial\Omega$. This is easily seen by considering an element with two nodes on $\partial\Omega$; from (2.3) it follows that in the third node $v_i = 0$. Hence, element 1 will no longer be considered. Element 2 has been used for solving the Navier-Stokes equations in [2 and 4]. In order to test its accuracy for the Boussinesq equations we have used it for solving the Bénard problem (Fig. 2). Two infinite parallel walls are kept at temperatures T_1 and T_2 , with T_1, T_2 the temperature of lower and upper wall, respectively, and $T_1 \geq T_2$. Define

$$\theta = (T - 1/2(T_1 + T_2)) / (T_1 - T_2). \quad (4.1)$$

The unit of length is L . It is well known [6] that for $R \leq R_c \cong 1707$ the fluid remains at rest,

Fig. 1. Four finite elements for the Boussinesq equations. $i = 1, 2$; $\alpha = 1, 2, 3, 4$.

Fig. 2. The Bénard problem.

whereas for $R > R_c$ motion sets in. We have tried to determine R_c numerically. It is known that the most unstable flow pattern consists of a periodic array of square cells. Therefore we have taken Ω to be the square region $ABCD$, see Fig. 2, with periodic boundary conditions along AC and BD . The element configurations of Fig. 3 have been used. The Newton iteration process was started either with the solution for a fluid at rest, i.e.

$$v_i = 0, \quad v_3 = \theta = 1/2 - x_2, \quad v_4 = p = -1/2(1/2 - x_2)^2, \quad (4.2)$$

with the following perturbation for the velocity

$$v_2 = 1 \text{ along } AC, \quad v_2 = -1 \text{ along } BD, \quad (4.3)$$

Fig. 3. Finite element configurations for the Bénard problem.

or else with the solution for another Rayleigh number. At the centre of the cell $v_4 = p = 0$ is prescribed if there is a node at the centre, else $v_4 = p = 0$ in the lower left hand corner.

The combination (element 2, mesh 1) must be rejected out of hand, because the boundary conditions completely determine the temperature. For the combinations (element 2, mesh 2 and mesh 3) the fluid was found to be in motion in the range $1000 < R < 3000$, so that a reasonable approximation to R_c is not obtained.

The reason that element 2 is unsuitable for the Bénard problem is, that the solution for $R \leq R_c$, as given by eqn (4.2), cannot be represented. Whereas in reality the buoyancy force and the hydrostatic pressure balance each other exactly, this is impossible for element 2. The finite element method only allows an exact balance if p is piecewise a second or higher degree polynomial.

Elements 3 and 4 remain. Both are able to represent (4.2) exactly. However, it was found that the combination (element 3, mesh 2) gave $v_i \equiv 0$ for a Rayleigh number as high as $R = 3000$, so that element 3 must be considered inaccurate. Of course, it is possible that for a fine enough mesh a reasonable approximation for R_c would be obtained, but the results produced with element 4 are so much better, that element 3 also does not warrant further attention. Apparently, for element 3 the temperature field is too much constrained by the linear shape functions.

Table 1. R_c^* computed with element 4

Mesh	1	2	3
R_c^*	2130	1727	1710

With element 4 the approximations R_c^* to $R_c \approx 1707$ listed in Table 1 were obtained.

R_c^* was determined by monitoring the velocity in the point $(1/3, 1/3)$. This quantity is given for a number of cases in Fig. 4. The results of Fig. 4 were obtained for all meshes by starting the Newton iteration process at $R = 3000$ with the flow field defined by eqn (4.2) and (4.3). The flows for the lower values of R were obtained by using the solution for the next value of R as starting iterand. The iterations were terminated when the maximum norm between two successive iterands divided by the maximum norm of the last iterand was less than ϵ , with $\epsilon = 10^{-5}$, with the following exceptions mesh 1, $R = 3000$: $\epsilon = 3 \cdot 10^{-5}$ and mesh 3, $R = 1715$:

Fig. 4. Velocity in $(1/3, 1/3)$ with element 4, x, mesh 1; o, mesh 2; •, mesh 3.

Fig. 5. Two finite element configurations.

$\epsilon = 3 \cdot 10^{-4}$. At most 7 Newton iterations were needed to achieve convergence; convergence was found to be quadratic.

Clearly, for the Bénard problem the accuracy of element 4 is much better than the accuracy of the other elements considered. It seems likely, that this will also be true for other solutions of the Boussinesq equations.

5. AN INTERNAL FLOW PROBLEM

To show the feasibility of the method, the solution of a more complicated problem will be described. Consider the flow in a region Ω consisting of the unit square with on the boundary $\theta = 0$ and within the centre a square with sides of length $1/3$ with temperature kept at $\theta = 1$. The two finite element configurations of Fig. 5 were used, together with element 4 of the preceding section. The calculations were carried out in one half of Ω , with symmetry conditions prescribed on the axis of symmetry. Figure 6 gives $-v_2$ in $(1/9, 2/3)$ for several values of the Rayleigh number. The results represented in Fig. 6 indicate that for $R < 30,000$ mesh 2 is fine enough to approximate the solution to about 10%.

Apart from calculating with different mesh configurations, an indication of the accuracy may

Fig. 6. $-v_2$ in $(1/9, 2/3)$ as functions of R . O, mesh 1; ●, mesh 2.

Fig. 7. Nusselt numbers N_1 and N_2 . O, mesh 1; ●, mesh 2; ---, N_1 ; - · - ·, N_2 .

Fig. 8. Velocity vector field and isotherms, $R = 25,000$, mesh 2. $\circ$, $\theta = 0.1$; Δ , $\theta = 0.3$; $+$, $\theta = 0.5$; $\times$, $\theta = 0.7$; $\bullet$, $\theta = 0.9$.

be obtained by comparing the heat transfer on the inner boundary $\partial\Omega_1$ and the outer boundary $\partial\Omega_2$. The dimensionless heat transfer, also called the Nusselt number N , is given by

$$N_i = (-1)^i \int_{\partial\Omega_i} \frac{\partial\theta}{\partial n} d\Omega_i, \quad i = 1, 2, \quad (5.1)$$

with $\partial\theta/\partial n$ the temperature derivative along the outward normal. Since there are no heat sources in the interior, one should have

$$N_1 = N_2. \quad (5.2)$$

Due to the truncation error of the finite element method, eqn (5.2) is not satisfied. Figure 7 gives N_i for a number of cases. Figure 7 suggests that as the mesh is refined the solution converges and $N_1 - N_2 \rightarrow 0$. In a practical situation N_i would be of engineering interest. The computations show that already for moderate values of R the heat transfer increases appreciably due to the motion of the fluid in the interior.

As a further illustration of the solution of this problem, Fig. 8 gives the velocity vector field and the isotherms for $R = 25,000$, mesh 2.

We do not claim great accuracy for the calculations just presented; Figs. 6 and 7 suggest that the mesh should be refined further if one wants 1% accuracy, say. For large R one would have to place many elements near the walls in order to resolve the boundary layers.

Again, the Newton iterations converged quadratically. For the results represented in Figs. 7 and 8, $\epsilon < 5 \times 10^{-5}$, with ϵ as defined in the preceding section.

At most 6 Newton iterations were necessary with the starting iterand defined by eqns (4.2) and (4.3).

6. CONCLUSIONS

It has been shown that the finite element method lends itself quite well for the solution of the Boussinesq equations, provided the finite element is chosen with care. In particular it is necessary that the finite element method allows an exact balance between the buoyancy force

and the hydrostatic pressure. It is recommended that the accuracy of finite elements for the Boussinesq equations is tested by application to a problem for which much is known about the exact solution, such as for example the Bénard problem. We have demonstrated that element 4 as defined in Section 3 fulfills the requirements.

The fact that the Boussinesq equations are nonlinear does not cause difficulties if one uses Newton's method. This method was found to converge very fast for the problems considered.

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