THE COMPUTATION OF AERODYNAMIC LOADS ON HELICOPTER BLADES IN FORWARD FLIGHT, USING THE METHOD OF THE ACCELERATIONPOTENTIAL

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAPPEN AAN DE TECHNISCHE HOGESCHOOL DELFT, OP GEZAG VAN DE RECTOR MAGNIFICUS IR. H.B. BOEREMA, HOOGLERAAR IN DE AFDELING DER ELEKTRO-TECHNIEK, VOOR EEN COMMISSIE AANGEWZEZEN DOOR HET COLLEGE VAN DEKANEN TE VERDE-DIGEN OP WOENSDAG 5 MAART 1975 DES MIDDAGS OM 16.00 UUR

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Summary

The analysis of aerodynamic loads on helicopter blades is usually based on some form of lifting line analysis, either Prandtl's classical theory, Weissinger's extended lifting line theory (3/4-chord method), or more or less intuitively derived modifications thereof. However, doubts have been expressed as to whether these methods are really applicable to the unsteady, sheared flow encountered by the blades of a helicopter rotor in forward flight. In the present study this question has been investigated for the case of incompressible flow with small perturbations. It is concluded that indeed several errors are introduced when existing lifting line theories are applied to helicopter blades without any further precautions.

In this thesis the description of the flowfield is based on the acceleration potential instead of the more usual velocity potential. The use of the acceleration potential allows a relatively easy derivation of lifting line theory using a "matched asymptotic expansion" technique. The systematic rather than intuitive treatment of lifting line theory afforded by this approach enables one to derive the form which lifting line theory should assume in order to be applicable to the case of the helicopter rotor.

Two theories are developed, both fully applicable to the helicopter blade. The first one involves errors of relative order of magnitude $A^{-2}$ (where $A$ is the aspect ratio of the blades). The second one is a more elaborate higher order method, involving relative errors of the order $A^{-3}$. If applied to the simpler case of the unswept wing in steady flow, these methods would reduce to Prandtl's classical method and to Weissinger's 3/4-chord method respectively.

The matched asymptotic expansion analysis yields at the same time the complete pressure distribution over the blade's surface, which is a great advantage over the existing lifting line methods.
Finally, the derived theory offers the means to cut down computing times considerably in actual numerical computations, compared with the usual methods. The evaluation of the induced velocity in points on the blade, which requires normally a two-dimensional integration over the skewed helical vortex sheets forming the rotorwake, is reduced to a one-dimensional integration using the acceleration potential.
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Symbols

\[ a_0 \] coning angle
\[ a_1 \] first harmonic, longitudinal flapping angle
\[ a_n \] general constant in series expressions
\[ \tilde{a}_\eta \] unit vector normal to \( \eta = \text{constant surface} \)
\[ \tilde{a}_\varphi \] unit vector normal to \( \varphi = \text{constant surface} \)
\[ A \] aspect ratio
\[ A^* \text{ (indexed)} \] constant in series expression
\[ A^* \text{ (indexed)} \] non-dimensional constant in series expression
\[ b \] span
\[ b \] index indicating blade number
\[ b_1 \] first harmonic, lateral flapping angle
\[ b_n \] general constant in series expression
\[ B \] total number of blades
\[ B \text{ (indexed)} \] constant in series expression
\[ c \] chord
\[ c_n \] general constant in series expression
\[ C \] constant
\[ D \] constant
\[ f \text{ (indexed)} \] general function symbol
\[ F \] function
\[ F \text{ (indexed)} \] specific functions defined in chapter VIII
\[ g \] strength of singular, two-dimensional pressure field
\[ g^* \] non-dimensional form of \( g \)
\[ G, G^* \] functions defined in (4-35) and (4-36)
\[ G \] general function symbol
\[ h \] strength of singular, two-dimensional pressure field
\[ H \] Heaviside's unit function
\[ i \] integer
\[ I \] moment of inertia with respect to blade root
\[ I \] integer
\[ I_{1,2,3} \] integrals defined in chapter V
integrals defined in chapter IX
modified Bessel function of the first kind
integer
integer
integer
modified Bessel function of the second kind
lift per unit span
moment with respect to mid-chord point, per unit span
separation constant
aerodynamic moment of lift w.r. to blade root
$\psi_b$-independent part of $M_a$
separation constant, integer
integer
order of magnitude
pressure or pressure perturbation
separation constant
unperturbed pressure
pressure perturbation on upper or lower surface
pressure fields used in chapter IV
near or far pressure field
pressure field of a dipole- or quadrupole line
pressure field of a dipole-line situated at $c/2$ or $c/4$ position
pressure field of a dipole line with specific dipole-strength, defined in chapter V.
Legendre polynomial
derivative of Legendre polynomial
associated Legendre function of the first kind
separation constant
Legendre functions of the second kind
position vector
r cylindrical coordinate
r' same as r, but with shifted origin
ro specific value of r
r integer
rmax integer
rb spanwise coordinate along blade, with hub as origin
ro specific value of rb
R blade length
R function of the coordinate r
s integer
S integer
t time
t(indexed) specific value of t
u, u' x-component of velocity or velocity-perturbation
U unperturbed velocity
v, v' y-component of velocity or velocity-perturbation
V velocity vector
vi "induced" velocity component
v(indexed) part of velocity perturbation in Y-direction
w, w' z-component of velocity or velocity perturbation
wi "induced" velocity component
w(indexed) part of velocity perturbation in z-direction
W_n polynomial of order n
x Cartesian coordinate
x general variable
xw, xb, xr x-coordinate in wing-, blade- or rotor-fixed system
xo specific value of x
y Cartesian coordinate
yw, yb, yr y-coordinate in wing-, blade- or rotor-fixed system
yo specific value of y
z Cartesian coordinate
zw, zb, zr z-coordinate in wing-, blade- or rotor-fixed system
zo specific value of z
Z function of z-coordinate
α angle of attack, or flow angle v/U
α₀ angle of incidence
αᵣ angle of attack of rotordisc
β flapping angle, or auxiliary variable
γ Euler's number or Lock's number
e total blade twist, or auxiliary variable
ξ auxiliary variable in z-direction
η elliptical coordinate
η' same, but with shifted origin
η(indexed) specific value of η
H function of η
θ spheroidal coordinate
θ' same as θ, but with shifted origin
θᵣ specific value of θ used in chapter IX
θ function of θ
θᵣ angle of incidence of blade root w.r. to tip path plane
θₒ azimuth-independent component of θᵣ
Λ sweep angle
μ advance ratio U/ωᵣ, or auxiliary variable
ν reduced frequency ωb/U
ξ auxiliary variable
ρ air density
ρ(indexed) distance
τ non-dimensional time
τ(indexed) specific value of τ
ϕ elliptical coordinate
ϕ' same as ϕ, but with shifted origin
ϕ function of ϕ
χ cylindrical or spheroidal coordinate
χ' same as χ, but with shifted origin
χ function of χ
\( \psi \)  
\( \psi' \)  
\( \psi^* \)  
\( \psi_\beta \)  
\( \psi_\beta (\text{indexed}) \)  
\( \Delta \psi_\beta \)  
\( \Psi \)  
\( \omega \)  
\( \Omega \)  
spheroidal coordinate  
same as \( \psi \), but with shifted origin  
specific value of \( \psi \) used in chapter IX  
blade azimuth angle  
specific value of \( \psi_\beta \)  
small angle measured azimuth-wise  
function of \( \psi \)  
angular pitching velocity  
angular velocity of blade rotation
I. Introduction

Prandtl's classical model of a lifting surface represented by a lifting line may seem to be something of the past. Lifting surface methods, whose practical use has been made feasible by the advent of fast computers, have largely superseded the lifting line theories with their inherent short-comings. There is, however, one area of aerodynamics where lifting line theories are still in general use: the area of helicopter rotor flow analysis. Good reviews of the analytical methods used for calculating the load distribution over helicopter blades may be found in references 1 and 2, and no attempt will be done here to describe the numerous approaches to the problem. All these approaches have in common that they are based ultimately on lifting line theory in one form or the other. This is, no doubt, due to the great complexity of the flow around a helicopter rotor in forward flight.

Fig. 1 shows schematically the system of skewed helical vortex sheets trailed by a helicopter rotor. The vorticity in the wake consists of the so-called trailing vorticity, resulting from the spanwise variations of circulation along the blades, as well as the so-called shed vorticity, resulting from the time-variations of the blade circulation. In order to determine the induced velocity in points of the blade surface ("collocation points") it is usual to apply Biot and Savart's law, which requires a two-dimensional numerical integration over the skewed helical wake.

To limit the amount of computational effort needed for a complete analysis of the time- and spanwise loading of a rotor blade, it is then essential to limit the number of collocation points to an absolute minimum. One tends to minimize especially the number of points along the blade-chords, since it is well known that the spanwise variations of the loading along a rotor blade can be very rapid. One is thus forced by practical considerations to use, what might be called some type of "one point" lifting surface method, in other words Prandtl's lifting line theory or the 3/4-chord point method due to Weissinger.
concerning the use of lifting line models in rotor analysis will remain applicable to the non-linear methods in existence. These methods allow for deformations of the vortex sheets in the rotorwake, but do not depart from the more fundamental concepts of lifting line theory.

It will be shown that the description of an inviscid flow field by the method of the acceleration potential is almost ideally suited to the stated purpose of the study. The acceleration potential, being proportional to the pressure in incompressible flows, does not show any discontinuities in the flowfield. This is in direct contrast with the more usual method of the velocity potential where the discontinuities (vortex sheets) play an essential role. When the blades are modelled into lifting lines (as far as their far field effect is concerned), the absence of discontinuities in the field permits the complete pressure field of the rotor to be expressed analytically as the field due to a set of pressure-dipole lines. The evaluation of the velocity in some point of the flow at a certain instant of time is equivalent to the computation of the velocity acquired by a particle of air travelling through the known pressure field and passing the considered point at the required time. The computation of the induced velocities along the rotorblade thus requires only a one-dimensional integration of the equations of motion with respect to time instead of the two-dimensional spatial integration over the helical vortex sheets needed in the velocity method. This results in considerable savings in computing time. The absence of sheets of discontinuity also facilitates the derivation of the blade's near field, by means of a "matched asymptotic expansion" procedure. Such a systematic derivation instead of the usual intuitive one has the advantage of leading almost automatically to the form which the near field and chordwise load distribution should take under the special circumstances met in a rotorblade analysis. The asymptotic procedure may furthermore be used to derive a higher order lifting line theory. The latter may be of special importance in relation to helicopter analysis: although in general the blades have
large geometrical aspect ratios, the flow is aerodynamically more comparable to a relatively small aspect ratio case, because of the rapid spanwise variations of the loading.

A brief review of the study presented in this thesis will now be given. Chapter II introduces the theory of the acceleration potential, as compared with the method of the velocity potential. Although the theory of the acceleration potential is well known in the field of unsteady aerodynamics, it has, with the one exception of ref. 4, not been used before in the field of rotor aerodynamics, so that a rather extensive introduction was considered necessary.

Chapter III shows an application of this theory to a classical problem, viz. the thin, two-dimensional aerofoil. Many of the expressions derived in this chapter are needed in later chapters.

Chapter IV discusses classical lifting line theory, as considered from the point of view of the acceleration potential. The lifting line theory is developed systematically, by using a matched asymptotic expansion technique.

A higher-order lifting line theory, taking one more term of the asymptotic expansion into account, is developed in chapter V for the uncambered rectangular wing in steady parallel flow.

In chapter VI expressions are developed for the far pressure field of a lifting line in terms of series of associated Legendre-functions. These closed form expressions are useful for the actual numerical implementation of the lifting line theories as formulated in the chapters IV and V.

The modifications of the theory necessitated by unsteady and sheared flow are treated in chapter VII. The case considered there is an uncambered, rectangular wing with sweep, subject to a harmonic pitching motion. The reduced frequency of the pitching motion has been chosen such, that many of the obtained results are directly applicable to helicopter blades.

The case of the helicopter rotor is treated in chapters VIII and IX.
Chapter VIII contains the derivation and formulation of the boundary value problem, and chapter IX its solution. The method to obtain actual numerical results for the pressure distribution over the blades is also briefly discussed in chapter IX.

Having obtained the full higher order lifting line expressions for the case of the helicopter blade, including all relevant unsteady phenomena of inviscid theory, chapter X contains a comparison with existing methods based upon (velocity) vortex theory.

The final conclusions are summarized once more in chapter XI.
II. The governing equations

2.1 The method of the velocity potential

The basic governing equations for incompressible inviscid flows have been established already in the 18th century by Leonhard Euler, who developed the well known Euler equations expressing the conservation of momentum in the fluid, as well as the equation of continuity expressing the conservation of mass. These equations take the form:

\[
\text{div } \mathbf{V} = 0 \quad \text{(continuity)}
\]

or

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\] (2-1)

and

\[
\rho \frac{DV}{Dt} = - \text{grad } p \quad \text{(Euler equation)}
\] (2-2)

or

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial x}
\] (2-3)

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial y}
\] (2-4)

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial z}
\] (2-5)

Both Lagrange and Laplace have developed the method of solving this set of equations, by introducing the concept of the velocity potential. The velocity potential \( \phi(x,y,z) \) is defined as a scalar function such that

\[
\mathbf{V} = \text{grad } \phi
\] (2-7)

or

\[
u = \frac{\partial \phi}{\partial x}
\] (2-8)

\[
v = \frac{\partial \phi}{\partial y}
\] (2-9)

\[
w = \frac{\partial \phi}{\partial z}
\] (2-10)
It is easily shown that such a scalar function $\phi(x,y,z)$ always exists in flow regions where the rotation of the fluid particles is zero:

$$\text{rot } \mathbf{V} = 0$$ (2-11)

or

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$ (2-12)

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$ (2-13)

$$\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = 0$$ (2-14)

Writing the equation of continuity in terms of the velocity potential by substituting (2-7) into (2-1) results in the equation of Laplace:

$$\text{div } \text{grad } \phi = 0$$ (2-15)

or

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$ (2-16)

2.2 The method of the acceleration potential

Prandtl indicated in 1936 an alternative way to treat three-dimensional flows in which the velocity perturbations are small compared with the undisturbed velocity $U$. The assumption that all higher order perturbations may be neglected is directly used to simplify the basic flow equations (2-1) and (2-3). For incompressible, inviscid flows this simplification works out as follows. Substituting the equations (2-12) to (2-14) expressing $\text{rot } \mathbf{V} = 0$ into the Euler equations leads to:

$$\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} (u^2 + v^2 + w^2) = - \frac{1}{\rho} \frac{\partial p}{\partial x}$$ (2-19)

$$\frac{\partial v}{\partial t} + \frac{1}{2} \frac{\partial}{\partial y} (u^2 + v^2 + w^2) = - \frac{1}{\rho} \frac{\partial p}{\partial y}$$ (2-20)

$$\frac{\partial w}{\partial t} + \frac{1}{2} \frac{\partial}{\partial z} (u^2 + v^2 + w^2) = - \frac{1}{\rho} \frac{\partial p}{\partial z}$$ (2-21)
Now writing \( u = U + u' \) (2-22)
\[ v = v' \] (2-23)
\[ w = w' \] (2-24)

where \( u' \), \( v' \) and \( w' \) are small perturbations, and substituting this into (2-19) to (2-21) leads after linearization to:

\[
\frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (2-25)
\]

\[
\frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} \quad (2-26)
\]

\[
\frac{\partial w'}{\partial t} + U \frac{\partial w'}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} \quad (2-27)
\]

or, again using the linearized relations expressing rot \( \nabla = 0 \):

\[
\frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (2-28)
\]

\[
\frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} \quad (2-29)
\]

\[
\frac{\partial w'}{\partial t} + U \frac{\partial w'}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} \quad (2-30)
\]

The first of these equations is partially differentiated with respect to \( x \), the second with respect to \( y \), and the third with respect to \( z \). Summing and using the linearized continuity equation

\[
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \quad (2-31)
\]

then yields:

\[
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0 \quad (2-32)
\]
which is Laplace's equation, where the Laplace operator div grad works upon the pressure \( p \). Eq. (2-32) is valid in steady as well as unsteady flowfields if, as assumed during the derivation, the undisturbed velocity \( U \) is a constant, independent of the space- and time coordinates. Instead of the pressure \( p \), one can also put the pressure perturbation \( p' = p - p_\infty \) in (2-32). For simplicity the notation \( p(x,y,z) \) will in the following always be used for the field of pressure perturbations. The field \( p(x,y,z) \) is a characteristic scalar function for the flow field, just like the velocity potential \( \phi(x,y,z) \). The essential difference between \( p \) and \( \phi \) is the fact that in incompressible flows the function \( p(x,y,z) \) can never exhibit discontinuities in the free flow away from the physical boundaries, in contrast to the function \( \phi(x,y,z) \) that does show discontinuities.

In chapter III the relation between the pressure perturbation and the velocity potential will be needed. This may be derived from (2-28), writing the latter equ. like:

\[
\frac{\partial}{\partial x} \left\{ \frac{\partial \phi'}{\partial t} + U \frac{\partial \phi'}{\partial x} + \frac{p}{\rho} \right\} = 0 \quad (2-33)
\]

where \( p \) denotes the pressure perturbation. Integrating (2-33) yields

\[
p = -\rho \frac{\partial \phi'}{\partial t} - f U \frac{\partial \phi'}{\partial x} \quad (2-34)
\]

which is valid for incompressible, linearized flow where the undisturbed velocity \( U \) far upstream has been assumed to be independent of both the space- and time coordinates.

2.3 Some basic flow phenomena, as seen from the point of view of the pressure method

Lifting forces are described within the pressure method as pressure dipole distributions over the surfaces supporting the external forces. As a simple example, fig. 2 shows schematically the flux lines of the pressure gradients associated with a two-dimensional thin aerofoil.
When air particles move through this field, the direction of their acceleration is opposed to the direction of the pressure gradient. It is interesting to note that this way of describing the flow around an aerofoil is almost identical to the model developed by Lanchester, which in his case was based almost entirely on physical intuition (ref. 4): "The fluid particles, which are gradually influenced by the plane while passing through the field of force established around it, will receive an upward acceleration as they approach the aerofoil, and will have an upward velocity as they encounter its leading edge. While passing instead under or over the aerofoil, the field of force is in the opposite direction, viz., downward, and thus the upward motion is converted into a downward motion. Then, after the passage of the aerofoil, the air is again in an upwardly directed field, and the downward velocity imparted by the aerofoil is absorbed".

The magnitude of the pressure gradients on the surface of the aerofoil itself is thus dictated by the requirement that air particles arriving at the nose of the aerofoil shall be turned exactly along the aerofoil. From this we can immediately see the consequences of an increase of angle of attack or camber: the intensity of the pressure gradients at the surface will have to be greater, which requires a greater strength of the distributed pressure dipoles, and hence leads to a greater pressure difference between upside and bottom of the aerofoil.

As a second example, in fig. 3 the flow about a three-dimensional lifting surface is pictured schematically. A description can again be given by referring to Lanchester (ref. 5): "... the lines of force being no more constrained to lie in parallel planes would diverge, some portion of them escaping, as it were, and passing around the tips of the aerofoil laterally. The fluid traversing these lateral regions will then have upward momentum communicated to it during the whole time that it is in these regions, and will be finally left in a state of upward motion. The fluid traversing instead, the middle region, crossed by the aerofoil, will receive as in the preceding case, an upward acceleration before encountering the leading edge of the wing, a downward acceler-
ation while passing under or over the aerofoil, and again an upward acceleration after the passing of the aerofoil. But here the upward and downward momentum will no longer balance each other, as owing to the lateral spread of the ascending field forward of the aerofoil, the upward velocity communicated to the fluid before and after the passage of the wing is less than the downward velocity imparted to it during the passage of the wing. Consequently the portion of the fluid traversing the middle region will be ultimately left with some residual downward momentum; while, as noted, the fluid passing laterally around the wing on both sides has received an upward momentum". This model explains induced drag as follows. The decreased upwash along the leading edge of the wing (that is, the downwash superimposed upon the two-dimensional flow around an aerofoil of the same shape as the wing-section) causes a smaller dipole-strength in the nose region of the wing sections (fig. 2), and hence a decreased nose suction. The nose suction will then no longer balance the component of the pressure force acting on the rest of the wing section in drag direction.

2.4 The interpretation of the process of linearization
In a given pressure field, the motion of a particle of air can be calculated from the equation (see eq. 2-3):

\[
\vec{\dot{r}} = - \frac{1}{\rho} \text{grad } p(r,t)
\]  

(2-35)

Here \( \vec{r} \) is the position vector of a given particle (the Lagrangian point of view), while the forces acting between the considered particle and its neighbouring particles are replaced by the expression grad \( p \), which is a function of time and of the position of the particle. For a further development it is convenient to break down (2-35) in its components, at the same time changing to a set of first-order differential equations:

\[
\frac{du}{dt} = - \frac{1}{\rho} \frac{\partial p}{\partial x} (x,y,z,t)
\]  

(2-36)
Linearization of this set of equations can be effected by assuming that the velocity perturbations are small, so that the path of the particle deviates little from the path which would be described in the absence of pressure disturbances. In this case, eqs. (2-39) to (2-41) are simplified to:

\[
\begin{align*}
\frac{dx}{dt} &= U \\
\frac{dy}{dt} &= 0 \\
\frac{dz}{dt} &= 0
\end{align*}
\]

where \( U \) is the undisturbed velocity far upstream, directed along the \( X \)-axis.

It may be shown easily that the linearization process described here is equivalent to the linearization of the Euler equations effected in chapter 2.2. To show this, it is necessary to change from the Lagrangian point of view (given particle) to the Eulerian (given position) by means of expressions of the form:

\[
\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt}, \text{ etc.}
\]
Substituting then eqs. (2-42) to (2-44) into eqs. (2-36) to (2-38) yields:

\[
\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial x} (x,y,z,t) \tag{2-45}
\]

\[
\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial y} (x,y,z,t) \tag{2-46}
\]

\[
\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial z} (x,y,z,t) \tag{2-47}
\]

Comparing the resulting expressions with (2-28) through (2-30) shows the equivalence of the two ways of linearization explained in chapter 2.2 and in the present chapter respectively. The discussion of the present chapter has now led to the following slightly more "physical" interpretation of the linearization process.

In linear theory the pressure gradients are determined that would have acted upon a particle of air, if it had been forced to follow a straight path, coinciding far upstream with the unperturbed path. After this, the assumption is introduced that the particle "feels" the same pressure gradients along its real, perturbed path. Using this approximation to the real pressure gradients along its real path, the perturbation velocities obtained by the particle can be calculated easily. All this may seem rather trivial, were it not that from this interpretation a self evident method follows to obtain an approximation for non-linearized flows. The way to obtain an approximation to the velocity field with relatively great perturbations is: calculate the paths and velocities of the air particles under the influence of the pressure gradients along their perturbed path, i.e. integrate directly the equation of motion (2-35).

Since the pressure field is still determined by the Laplace equation (2-32) which was derived by linearization, this method does not lead to an exact delinearization. In fact, it is rather more a convenient engineering procedure to deform the linearized flowfield spatially, in order to correct for some of the most important (but not all) of the
real non-linear effects. From sample calculations it appears furthermore that it is often better to correct for spatial distortion of the flowfield only in the direction of the main external forces acting in the flowfield. If for example the lift is acting in y-direction, only eq. (2-43) is replaced by (2-40), while (2-42) and (2-44) are retained.

The above explained view of the linearization process of the incompressible flow equations leads to an important conclusion. Since in linearized analyses the particle paths (and hence the streamlines in steady flow) are approximated by straight lines parallel to the oncoming flow, the boundary conditions for the pressure field must accordingly be applied along straight boundaries parallel to the undisturbed flow.

For instance, at the surface of a thin aerofoil a certain magnitude of the normal component of the pressure gradient is required, in order to change the particle velocities in such a way that they will follow exactly the aerofoil contour. These boundary conditions must be replaced by conditions, requiring the normal component of the pressure gradient to assume the same value along a straight line, which is being substituted for the actual curved mean line of the aerofoil. This of course does not deviate from the practice in the linearized velocity potential theory.

Here is also found the justification of Prandtl's "straightening out" of the vortex sheet behind a lifting line.
III. The calculation of steady two-dimensional aerofoil characteristics, using the linearized pressure method

3.1 The basic equations for the linearized pressure field in elliptic-cylinder coordinates

Elliptic cylinder coordinates are obtained by taking an orthogonal family of confocal ellipses and hyperbolas in a plane and translating them in the z-direction (fig. 4). The relation between the elliptic coordinates and rectangular coordinates may be written as:

\[
X = \frac{c}{2} \cosh \eta \cos \varphi \quad (3-1)
\]

\[
y = \frac{c}{2} \sinh \eta \sin \varphi \quad (3-2)
\]

\[
z = z \quad (3-3)
\]

It is convenient to let the variables cover the following ranges:

\[
0 \leq \eta < +\infty \quad (3-4)
\]

\[
-\pi \leq \varphi \leq +\pi
\]

\[
-\infty < z < +\infty
\]

so that a unique coordinization of the points of space results.

The surfaces \( \eta = \) constant are elliptic cylinders, given by

\[
\frac{x^2}{(c/2 \cosh \eta)^2} + \frac{y^2}{(c/2 \sinh \eta)^2} = 1 \quad (3-5)
\]

The surfaces \( \varphi = \) constant are the hyperbolic cylinders

\[
\frac{x^2}{(c/2 \cos \varphi)^2} - \frac{y^2}{(c/2 \sin \varphi)^2} = 1 \quad (3-6)
\]
The distance between the foci of the ellipses is $c$. The elliptic cylinders degenerate for $n = 0$ to the flat surface between the lines $x = -\frac{c}{2}$ and $x = +\frac{c}{2}$.

On this surface the coordinate transformation reads

$$x = \frac{c}{2} \cos \varphi$$

$$z = z$$

(3-7)

(3-8)

The remainder of the XOZ-plane is given by $\varphi = 0$ (positive values of $x$) or $\varphi = \pi$ (negative values), where the coordinates are given by the relation

$$x = \pm \frac{c}{2} \cosh \eta$$

(3-9)

Here $\cosh \eta$ ranges from $+1$ to $+\infty$ for $\eta$ varying between 0 and $\infty$.

Laplace's equation is in elliptic-cylinder coordinates transformed into (ref. (6)):

$$\nabla^2 p = \frac{4}{c^2 (\cosh^2 \eta - \cos^2 \varphi)} \left( \frac{\partial^2 p}{\partial \eta^2} + \frac{\partial^2 p}{\partial \varphi^2} \right) + \frac{\partial^2 p}{\partial z^2} = 0$$

(3-10)

which reduces for $p$ independent of $z$ (two-dimensional problems) to:

$$\frac{\partial^2 p}{\partial \eta^2} + \frac{\partial^2 p}{\partial \varphi^2} = 0$$

(3-11)

Solutions of this equation may be easily found for the case that $p$ consists of the product of two functions, one dependent only upon $\eta$, the other only on $\varphi$:

$$p = H(\eta) \phi(\varphi)$$

(3-12)
The separation of variables leads in the present case to the following form of Laplace's equation:

$$\frac{1}{H} \frac{d^2 H}{d\eta^2} + \frac{1}{\phi} \frac{d^2 \phi}{d\phi^2} = 0$$  \hspace{1cm} (3-13)

Because the first part in this equation is a function of $\eta$ only and the second part of $\phi$, both parts must be constants. This leads to the two separation equations:

$$\frac{d^2 H}{d\eta^2} + n^2 H = 0 \hspace{1cm} (3-14)$$

$$\frac{d^2 \phi}{d\phi^2} + n^2 \phi = 0 \hspace{1cm} (3-15)$$

solutions are

$$H = e^{\pm n\eta} \text{ or } H = \sinh n\eta$$  \hspace{1cm} (3-16)

and

$$\phi = \frac{\sin}{\cos} n\phi$$  \hspace{1cm} (3-17)

Because the solution must be periodic, i.e. must assume the same value when $\phi$ is increased by $2\pi$, the separation constant $n$ is restricted to the integer values $n = 1, 2, \ldots$, while the other possible solutions $H = \sin(n\eta)$, etc. and $\phi = \sinh (n\phi)$ etc. are not applicable.

The linearity of the equation of Laplace allows us to build up the general solution as the sum of particular solutions, e.g.:

$$\phi = \sum_{n=1}^{\infty} \left(a_n \sinh(n\eta) \cdot \sin(n\phi) + b_n e^{-n\eta} \cdot \cos(n\phi) + \ldots\right)$$  \hspace{1cm} (3-18)

An even simpler solution will be needed later, resulting from Laplace's equation for the case that the solution is also independent of $\eta$: 
\[
\frac{d^2 \phi}{d\phi^2} = 0 \tag{3-19}
\]

with the solution

\[
\phi = c\phi + D \tag{3-20}
\]

The next step will be to adjust the constants in the general solution to fit the particular boundary conditions of the problem under consideration.

3.2 The thin aerofoil of arbitrary mean line, at the ideal angle of attack

We will consider a two-dimensional aerofoil with leading edge on the X-axis in the point \( x = -c/2 \), and trailing edge at \( x = +c/2 \). In linearized theory the boundary conditions on the aerofoil may be replaced by the same conditions applied to the part of the X-axis between \( x = -c/2 \) and \( +c/2 \). In our coordinate system the aerofoil is thus given by the coordinate surface \( \eta = 0 \). In this section the aerofoil is assumed to be placed at the ideal angle of attack, i.e. at such an incidence w.r. to the flow that no streamline kink occurs at the leading edge.

The following linearized form of the equation of motion for an air-particle applies (chapter 2):

\[
\frac{DV}{Dt} = U, \quad \frac{\partial V}{\partial x} = -\frac{1}{\rho} \text{ grad } p \tag{3-21}
\]

where \( U \) is the undisturbed velocity of the flow along the X-axis, \( V \) indicates the vector of the velocity perturbations \( u, v \) and \( w \), while in this case by \( p \) is meant the pressure disturbance with respect to the static pressure in the oncoming flow. Indicating the angle between the disturbed velocity and the undisturbed flow by \( \alpha = \frac{v}{U} \), equation (3-21) gives:
This expression is valid throughout the flow, in the free field as well as on the surface of the aerofoil. Thus (3-22) requires the component \( \frac{\partial p}{\partial y} \) of the pressure gradient to assume a given value along the surface of the aerofoil, proportional to the curvature \( \frac{d\alpha}{dx} \) of its surface. The value of \( \frac{\partial p}{\partial y} \) specified by (3-22) is equal in magnitude for the top and bottom of the aerofoil although its direction is antisymmetric w.r. to the X-axis, from which it follows that the pressure on the surface must correspond to a dipole-distribution over the surface. This implies a jump of pressure between the two sides of the aerofoil, and hence a lifting force.

Looking back at the general solution (3-18) of the pressure field in elliptic-cylinder coordinates, and considering that the pressure will also be distributed antisymmetrically w.r. to the X-axis, the pressure field of the aerofoil must take the form:

\[
p = -\frac{1}{2\pi} \sum_{n=1}^{\infty} a_n e^{-n\eta} \sin(n\phi)
\] (3-23)

This form indeed assures an antisymmetric pressure field with respect to the X-axis, which approaches zero in the far field (\( \eta \to \infty \)) as well as along the edges of the aerofoil (\( \eta = 0, \phi = 0 \) or \( \pi \)).

When the notation \( a_\eta \) is used for the unit vector perpendicular to the surface \( \eta = \) constant in a point \( (\eta,\phi) \), and \( a_\phi \) indicates the unit vector perpendicular to the surface \( \phi = \) constant, the gradient of a function is given in elliptic cylinder coordinates as (ref. 6):

\[
\text{grad } p = \frac{1}{\frac{c}{2}(\cosh^2 \eta - \cos^2 \phi)^{\frac{1}{2}}} \left\{ a_\eta \frac{\partial p}{\partial \eta} + a_\phi \frac{\partial p}{\partial \phi} \right\}
\] (3-24)

Combining this expression with equation (3-23) shows that the value of \( \frac{\partial p}{\partial y} \) on the surface of the aerofoil is given by:
\[ \frac{\partial p}{\partial y} = \frac{1}{\pi c \sin \phi} \sum_{n=1}^{\infty} a_n \sin(n \phi) \] (3-25)

which must, in order to satisfy the boundary conditions, be identical to \(-\rho \frac{U^2}{2} \frac{d\alpha}{dx}\) along the surface.

Therefore the unknown coefficients \(a_n\) are given by the Fourier expression:

\[ a_n = -\frac{2 \rho U^2}{n} \int_0^{\pi} \frac{d\alpha}{dx} \sin \phi \sin(n \phi) \, d\phi \] (3-26)

The lift \(l\) acting on the aerofoil follows from equation (3-23) by integrating the pressure \(p\) over the surface:

\[ l = -\int_{-c/2}^{+c/2} p_{\text{upper}} \, dx + \int_{-c/2}^{+c/2} p_{\text{lower}} \, dx = \]

\[ = -c/2 \int_0^{\pi} p(\phi) \sin \phi \, d\phi - c/2 \int_0^{2\pi} p(\phi) \sin \phi \, d\phi = \]

\[ = -c/2 \int_0^{2\pi} p(\phi) \sin \phi \, d\phi = \]

\[ = \frac{c}{4} a_1 \] (3-27)

At large distances from the origin, the elliptic-cylinder coordinates approach circular-cylinder coordinates, because \(\sinh \eta \approx \cosh \eta\) for \(\eta \to \infty\). The polar coordinate \(r\) becomes approximately equal to \(c/2. \sinh \eta\) or to \(1/4 \, c \, e^{+\eta}\). Also, for \(\eta \to \infty\) the expression for the pressure (3-23) becomes dominated by the term \(n = 1\). This means that
\[
p \propto - \frac{a_1}{2\pi} e^{-\eta \sin \phi} - \frac{\sin \phi}{2\pi r^3} 1/4 \frac{c}{a_1} =
\]

\[
= - \frac{\sin \phi}{2\pi r^3} \lambda \quad \text{for large } \eta, \text{i.e. large } r \quad (3-28)
\]

The limiting case thus shows that at large distances from the aerofoil the pressure field approaches the field of a discrete pressure-dipole of strength equal to the total lift on the aerofoil, and with dipole-orientation (positive from the source-side to the sink-side of the dipole) opposed to the direction of the lift. The moment of the lift distribution around the origin is given by the following integration:

\[
m = \int_{-c/2}^{+c/2} p_{\text{upper}} \cdot x \, dx - \int_{-c/2}^{+c/2} p_{\text{lower}} \cdot x \, dx =
\]

\[
= 1/8 \frac{c^2}{2\pi} \int_{0}^{2\pi} p(\phi) \sin 2\phi \, d\phi =
\]

\[
= - \frac{1}{16} \frac{c^2}{4} \cdot a_2 \quad (3-29)
\]

Finally, it has to be known what the geometrical angle of incidence should be, in order to realize the assumed condition of a continuous velocity distribution near the leading edge (ideal angle of attack). For this condition, the angle of upflow just in front of the leading edge should equal the slope of the aerofoil mean line at the leading edge. The upwash angle follows by integrating eq. (3-22) along the X-axis from \( x = -\infty \) to \( x = -c/2 \), that is an integration in the plane \( \phi = \pi \), from \( \eta = +\infty \) to \( \eta = 0 \). In the plane \( \phi = \pi \) the gradient component \( \frac{\partial p}{\partial y} \) follows from (3-23) and (3-24) as:

\[
\frac{\partial p}{\partial y} = - \frac{1}{c/2 \sinh \eta} \left( \frac{\partial p}{\partial \phi} \right)_{\phi=\pi} =
\]
so that

\[ \alpha_{x=-c/2} = -\frac{1}{\rho U^2} \int_{-\infty}^{\infty} \frac{\partial p}{\partial y} \, dx = \quad (3-31) \]

\[ = -\frac{1}{2\pi \rho U^2} \sum_{n=1}^{\infty} (-1)^n a_n \]

3.3 The thin aerofoil at arbitrary angles of attack

At angles of attack differing from the ideal, the boundary conditions for the pressure gradient are identical to the boundary conditions at the ideal angle of attack over the greatest part of the aerofoil. This is because a pure rotation of the aerofoil does not change the curvature of the mean-line. Things are very different however in the vicinity of the leading edge. In the physical flow around an aerofoil with finite thickness, the result of a change of angle of attack is a shift of the stagnation point along the nose of the aerofoil. One can look upon this phenomenon, as an effective change of the curvature of the mean-line near the nose, so that the leading edge of the mean line still coincides with the stagnation point, and the slope of the mean line still corresponds to the angle of upflow. One could consider the physical behaviour of an aerofoil of finite thickness as a case of variable geometry, such that the ideal angle of attack is being kept identical with the real angle of attack. As the thickness of the aerofoil decreases at a given angle of attack, the length of the "variable geometry" part of the mean line decreases. So does the radius of curvature of this part, with the resulting effect that the angle over which the flow is being turned remains constant. In the limiting case, an infinitely thin aerofoil at arbitrary angle of attack will therefore exhibit a pressure distribution that is singular at the leading edge, but has \( \frac{\partial p}{\partial y} \) -values over the rest of the aerofoil equal to the values at the ideal angle of attack. In other words, the pressure distribution at arbitrary angles of attack may be considered as the sum of the distribution at the ideal angle of
attack and the pressure distribution of a flat plate at an angle of attack. To sum up, this additional pressure field will have to satisfy the following conditions: $p$ must be a harmonic function approaching zero for $\eta \to \infty$. $\frac{\partial p}{\partial y} = 0$ on the surface $\eta = 0$, except in the point $\varphi = \pi$, where $p \to -\infty$.

It is difficult to satisfy these boundary conditions by using series solutions of the type given by eq. (3-18). This is the same as saying that in this case the method of separation of variables fails. However, one can easily construct the non-separated solution of the present boundary value problem, by making use of the relation $p = -\rho U \frac{\partial \phi}{\partial x}$, derived in chapter II. The field of the velocity potential around a flat-plate aerofoil placed at an angle of incidence $\alpha$ with respect to the undisturbed velocity $U$, and having smooth flow at its trailing edge (Kutta-condition satisfied in $x = \pm c/2$), is given by:

$$\phi = U \alpha \frac{c}{2} (e^{-\eta} \sin \varphi - \varphi) \quad (3-32)$$

as may be checked easily. The corresponding pressure field may be obtained from (3-32) by applying the relation

$$\frac{\partial \phi}{\partial x} = \frac{1}{c/2 \left( \cosh \eta \cos^2 \varphi \right)} \left( \sinh \eta \cos \varphi \frac{\partial \phi}{\partial \eta} - \cosh \eta \sin \varphi \frac{\partial \phi}{\partial \varphi} \right) \quad (3-33)$$

which yields:

$$p = -\rho U^2 \alpha \frac{\sin \varphi}{\cosh \eta \cos \varphi} \quad (3-34)$$

The lift on the flat-plate aerofoil is obtained from:

$$\mathcal{L} = -\int_{-c/2}^{+c/2} p_{\text{upper}} \, dx + \int_{-c/2}^{+c/2} p_{\text{lower}} \, dx =$$
Several results needed in later chapters will now be derived. First of all, it is attempted to calculate directly from (3-34) the value of the velocity-component \( v = \frac{\partial \phi}{\partial y} \) on the surface of the aerofoil. The value of \( \frac{\partial p}{\partial y} \) on the X-axis for \( x < -c/2 \) follows from:

\[
\frac{\partial p}{\partial y} = \frac{(-\text{grad } p)_{\phi=\pi}}{c/2} = \frac{\rho U^2 \alpha_o}{\sinh(\cosh^{-1})} \quad \text{for } y=0 \quad x<-c/2
\]  

Since \( \frac{\partial p}{\partial y} \) vanishes on the surface of the aerofoil itself, \( v/U \) on the aerofoil is, according to (3-22):

\[
\frac{(v/U)}{y=0} = \frac{-1}{\rho U^2} \int_{x=0}^{c/2} \left( \frac{\partial p}{\partial y} \right)_{y=0} \ dx = -\frac{\rho U^2 \alpha_o}{c/2} \frac{1}{\sinh(\cosh^{-1})} \quad \text{for } x<-c/2
\]  

The integral in the right hand side is divergent as it stands. It must be recognized however that the integral should be interpreted as an integration up to a point behind the leading edge, i.e. past the leading edge singularity. Eq. (3-37) should really be interpreted like:

\[
\frac{(v/U)}{y=0} = \frac{-1}{\rho U^2} \lim_{\varepsilon \to 0} \int_{x=0}^{0} \left( \frac{\partial p}{\partial y} \right)_{y=\varepsilon} \ dx.
\]
Interpreted in this way, the integral in (3-37) must have a finite value which is, in order to obtain $$(v/U)_{y=0} = \alpha$$:

$$\int_{x=0}^{\infty} \frac{d\eta}{\cosh \eta - 1} = 1 \quad (3-38)$$

on condition that the special meaning indicated above is attached to the integral. The integral (3-38) will occur several times again in later chapters.

What is also very important for subsequent developments, is a consideration of the behaviour of the pressure field at large distances from the aerofoil. For this purpose the inverse transformation for the elliptic-cylinder coordinates is used:

$$\eta = \text{arccosh} \left( \frac{\rho_2 + \rho_1}{c} \right) \quad (3-39)$$

$$\varphi = \text{arccos} \left( \frac{\rho_2 - \rho_1}{c} \right) \quad (3-40)$$

where

$$\rho_1 = \{(x-c/2)^2 + y^2\}^{\frac{1}{2}} \quad (3-41)$$

$$\rho_2 = \{(x+c/2)^2 + y^2\}^{\frac{1}{2}} \quad (3-42)$$

Using (3-41) and (3-42), $\rho_1/c$ and $\rho_2/c$ may be written in the form of infinite series in terms of the circular cylinder coordinates $r$ and $\chi$:

$$\rho_1/c = \frac{r}{c} \left( 1 + \frac{1}{2} \frac{c}{r} \cos \chi + \frac{1}{8} \left( \frac{c}{r} \right)^2 \sin^2 \chi + \frac{1}{16} \left( \frac{c}{r} \right)^3 \cos \chi \sin^2 \chi + \ldots \right)$$

for $r/c \to \infty \quad (3-43)$

$$\rho_2/c = \frac{r}{c} \left( 1 - \frac{1}{2} \frac{c}{r} \cos \chi + \frac{1}{8} \left( \frac{c}{r} \right)^2 \sin^2 \chi - \frac{1}{16} \left( \frac{c}{r} \right)^3 \cos \chi \sin^2 \chi + \ldots \right)$$

for $r/c \to \infty \quad (3-44)$
so that, substituting into (3-39) and (3-40), the following expansion may be derived:

\[
p = -\frac{\ell}{\pi c} \frac{\sin \phi}{\cosh \eta + \cos \phi}
\]

\[
= -\frac{\sin \chi}{2\pi r} \frac{\ell}{4} + \frac{\sin 2\chi}{2\pi r^2} \frac{\ell}{4} \frac{c}{8} - \frac{\sin 3\chi}{2\pi r^3} \frac{\ell}{8} \frac{c^2}{8} + \ldots
\]

for \( r/c \to \infty \) \hspace{1cm} (3-45)
IV. Classical lifting line theory

4.1 The near- and far field boundary value problem for the rectangular wing in steady parallel flow

The linearized boundary value problem for an uncambered wing with span b and chord c (fig. 5) may be formulated as follows:

\[ \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0 \]  \hspace{1cm} (4-1)

\[ p \rightarrow 0 \text{ for } x^2 + y^2 + z^2 \rightarrow \infty \]  \hspace{1cm} (4-2)

\[ \frac{\partial p}{\partial y} = 0 \text{ on the wing surface} \]  \hspace{1cm} (4-3)

\[ p \rightarrow -\infty \text{ along the leading edge, such that} \]

\[ \frac{v}{U(z)} = -\alpha_0(z) \text{ on the wing} \]  \hspace{1cm} (4-4)

Instead of trying to find an exact solution for this problem, we will try to determine an approximation for the pressure field around the wing, to a predetermined order of accuracy. The idea of this (asymptotic) approximation is due to van Dyke (ref. 7). In the present chapter all effects of a higher order than \( O(A^{-1}) \) are neglected, which procedure will be shown to lead to the classical lifting line theory. Here A is the aspect ratio b/c of the wing.

On physical grounds the assumption seems justified that in the immediate vicinity of the lifting surface, staying away from the tip regions, the characteristic length scale for spanwise pressure variations is the wingspan b, whereas the characteristic length of chordwise pressure variations is the chordlength c. In mathematical form this may be translated into the assumption that

\[ \frac{\partial^2 p}{\partial (\frac{x}{c/2})^2}, \frac{\partial^2 p}{\partial (\frac{y}{c/2})^2} \text{ and } \frac{\partial^2 p}{\partial (\frac{z}{b/2})^2} \text{ are of the same order of magnitude.} \]
Substitution of the non-dimensional "characteristic" coordinates $\frac{x}{c/2}$, $\frac{y}{c/2}$ and $\frac{z}{b/2}$ into the Laplace equation (4-1) leads to:

$$\frac{\partial^2 p}{\partial (\frac{x}{c/2})^2} + \frac{\partial^2 p}{\partial (\frac{y}{c/2})^2} = -\frac{1}{A^2} \frac{\partial^2 p}{\partial (\frac{z}{b/2})^2}$$  \hspace{1cm} (4-5)$$

showing that in the limit $A \to \infty$ p satisfies a two-dimensional Laplace-equation. Now it is attempted to describe the rate at which the pressure field becomes two-dimensional for $A \to \infty$ by an asymptotic series:

$$p(x,y,z) = p_0(x,y,z) + \frac{1}{A} p_1(x,y,z) + \frac{\ln A}{A^2} p_2(x,y,z) + \frac{1}{A^2} p_3(x,y,z) + \ldots \text{ for } A \to \infty$$  \hspace{1cm} (4-6)$$

This particular form of asymptotic behaviour will be substantiated later. Substituting the series (4-6) into (4-5) and multiplying successively by $A$, $A^2 \ln^{-1}(A)$, $A^3$, etc., one finds on taking each time the limit $A \to \infty$ the following equations to be satisfied by $p_k$ ($k=0,1,2,\ldots$):

$$\frac{\partial^2 p_k}{\partial (\frac{x}{c/2})^2} + \frac{\partial^2 p_k}{\partial (\frac{y}{c/2})^2} = 0 \quad (k=0,1,2)$$  \hspace{1cm} (4-7)$$

$$\frac{\partial^2 p_3}{\partial (\frac{x}{c/2})^2} + \frac{\partial^2 p_3}{\partial (\frac{y}{c/2})^2} = -\frac{\partial^2 p_0}{\partial (\frac{z}{b/2})^2}$$  \hspace{1cm} (4-8)$$

etc.

In a theory accurate up to the order $A^{-1}$, the boundary value problem for the near field of the lifting surface thus simplifies to

$$\frac{\partial^2 p_{\text{near}}}{\partial (\frac{x}{c/2})^2} + \frac{\partial^2 p_{\text{near}}}{\partial (\frac{y}{c/2})^2} = 0$$  \hspace{1cm} (4-9)$$
\[
\frac{\partial p_{\text{near}}}{\partial y} = 0 \text{ on the wingsurface} \quad (4-10)
\]
\[
p + \infty \text{ along the leading edge, such that } v/U = -c_0 \text{ on the wingsurface} \quad (4-11)
\]

The physical assumption that \( \frac{\partial^2 p}{\partial (x/c^2)^2}, \frac{\partial^2 p}{\partial (y/c^2)^2} \text{ and } \frac{\partial^2 p}{\partial (z/b^2)^2} \) are of the same order of magnitude can be valid only near the wing surface, not too close to the wingtips. At larger distances from the wing the characteristic length scale for the pressure field may be assumed to be equal to \( b \) in all directions, so that the so-called "far pressure field" will not satisfy eq. (4-9). For this reason the boundary value problem (4-9) to (4-11) cannot contain a boundary condition at infinity, which causes the problem to be undetermined as yet. The general solution satisfying eqs. (4-9) through (4-11) may be taken from chapter III as:

\[
p(\eta, \varphi, z) = -g(z) \frac{\sin \varphi}{\cosh \eta + \cos \varphi} + \sum_{n=1}^{\infty} a(n) \cosh(n\eta) \sin(n\varphi) \quad (4-12)
\]

where the occurrence of the second term in the right hand side cannot be ruled out by a condition at infinity as in the purely two-dimensional case. Neither can \( g(z) \) be determined by the velocity integration

\[
v(0,0,z) = -\frac{1}{pU} \int_{-\infty}^{0} \frac{\partial p}{\partial y}(x,0,z) \, dx \quad (4-13)
\]

since \( \partial p/\partial y \) is, at large values of \( x \), not correctly given by a differentiation of the solution \( p_{\text{near}} \).

The far pressure field of the wing can also be simplified when tolerating errors of order \( a^{-2} \). It will be shown now that the far field may be approximated in this case by the field of a line of pressure singularities. In order to show this, it is necessary to refer to the elliptic-
cylinder system as used in chapter III. Such a system degenerates at large distances asymptotically into a circular system, so that the singularities distributed along the chord of an aerofoil seem to shrink into a discrete singularity. In paragraph 3.2 the cylindrical coordinate \( r \) was therefore at large distances equated to:

\[
\frac{r}{c/2} \sinh \eta \frac{c/2}{c/4} \cosh \eta \frac{c/4}{c/4} e^\eta
\]  

Eq. (4-14) means that the factor \( e^{-\eta \frac{c/4}{c/4}} \) is neglected with respect to the factor \( e^{\eta \frac{r}{c/4}} \). In other words, if approximation (4-14) is used, one must accept relative errors of the order \( (\frac{r}{c/4})^{-2} \). If a certain given accuracy is required, the factor \( (\frac{r}{c/4})^{-2} \) in the two-dimensional case thus determines the distance where the so-called "far field" begins and the "near field" ends.

On the other hand, if in the three-dimensional case it is stated that the far field of a wing can be represented by a line singularity, we specify by this the accuracy of our analysis. For, what is meant by "far field" in this case, is the part of space at distances \( r \) from the wing that are of the same order as the span \( b \). In points of this far field we "see" the wing at its correct span, while the wing chords seem to have shrunk into points carrying discrete singularities. The lifting line model is therefore equivalent to expressing our willingness to accept relative errors in the analysis of the far field of the order \((c/b)^2 = A^{-2}\). As will be seen later, the first term of the far field is of the order \( O(A^{-1}) \) with respect to the leading term of the complete pressure field. Therefore, the lifting line approximation for the far field will remain useful in a theory which is accurate up to and including terms of order \( O(A^{-2}) \).

It should be noted that in the asymptotic limit \( A \to \infty \) the far pressure field of the wing to any order simplifies to the field of a lifting line. Therefore, the asymptotic expansion might formally be continued up to terms of a higher order than \( O(A^{-2}) \). However, since the asymptotic solution for the lifting surface problem is in practice applied only to cases
of finite aspect ratio, the formal solution may be expected to begin to diverge after some terms. The considerations given above indicate that the last meaningful term will be the one of order \(O(A^{-2})\).

Using the non-dimensional "characteristic" coordinates \(\frac{x}{b/2}, \frac{y}{b/2}, \frac{z}{b/2}\), the boundary value problem for the far field thus simplifies to:

\[
\frac{\partial^2 p_{\text{far}}}{\partial (\frac{x}{b/2})^2} + \frac{\partial^2 p_{\text{far}}}{\partial (\frac{y}{b/2})^2} + \frac{\partial^2 p_{\text{far}}}{\partial (\frac{z}{b/2})^2} = 0 \quad (4-15)
\]

\[p_{\text{far}} \rightarrow 0 \text{ for } x^2 + y^2 + z^2 \rightarrow \infty \quad (4-16)\]

\(p_{\text{far}}\) singular along the line \(x = y = 0, -b/2 \leq z \leq b/2\) and antisymmetrical with respect to the plane \(y = 0\).

This problem is again undetermined.

Both the near- and far field problem can be completed by the so-called "matching condition". This condition may be derived by requiring that it shall be possible to build up from the near- and far field a composite field, which is uniformly valid throughout the flow. Such a composite field may be formed by summing the near- and far pressure field, and subtracting a field which at large distances from the wing is identical (to the required order of accuracy) to the near field and which is close to the wing surface identical to the far field. Without going into the proof of existence of such a "common" field, it is clear that if such a field exists, then we must require that the functions \(p_{\text{far}}\) and \(p_{\text{near}}\) become equal to the same function, so that also:

\[
(p_{\text{far}})_{r \rightarrow 0(c)} \sim (p_{\text{near}})_{r \rightarrow 0(b)} \quad (4-17)
\]
to the required order of accuracy.

Writing condition (4-17) in the respective characteristic coordinates and taking the asymptotic limit $A \to \infty$:

$$\lim_{r \to 0} p_{\text{far}} \sim \lim_{r/c/2 \to \infty} p_{\text{near}}$$  \hspace{1cm} (4-18)

This "matching" condition will be seen to be sufficient to complete the set of lifting line equations.

4.2 The field of a lifting line in circular-cylinder coordinates

In order to derive the far pressure field, it is convenient to introduce circular-cylinder coordinates $(r, \chi, z)$ (fig. 6), in which Laplace's equation reads (ref. 6):

$$\frac{\partial^2 p}{\partial (\frac{r}{b/2})^2} + \frac{1}{r} \frac{\partial p}{\partial (\frac{r}{b/2})} + \frac{1}{(\frac{r}{b/2})^2} \frac{\partial^2 p}{\partial \chi^2} + \frac{\partial^2 p}{\partial (\frac{z}{b/2})^2} = 0$$  \hspace{1cm} (4-19)

Let

$$p = R\left(\frac{r}{b/2}\right)X(\chi)Z\left(\frac{z}{b/2}\right)$$  \hspace{1cm} (4-20)

then the following separated equations are obtained:

$$Z'' + q^2 Z = 0$$  \hspace{1cm} (4-21)

having as solutions $\sin(q \frac{Z}{b/2})$ or $\cos(q \frac{Z}{b/2})$,

$$X'' + p^2 X = 0$$  \hspace{1cm} (4-22)

having as solutions $\sin(p\chi)$ or $\cos(p\chi)$,

and

$$\left(\frac{r}{b/2}\right)^2 R'' + \frac{r}{b/2} R' - \{p^2 + q^2 \left(\frac{r}{b/2}\right)^2\} R = 0$$  \hspace{1cm} (4-23)
having as solutions $I_p\left(\frac{r}{b/2}\right)$ and $K_p\left(\frac{r}{b/2}\right)$, the modified Bessel-
functions of the first and second kind respectively.

In the following, use will be made only of the functions $K_p\left(\frac{r}{b/2}\right)$,
because these functions approach zero for $r \to \infty$, and become singular
for $r \to 0$. Also, because of the required periodicity in $\chi$, only integer
values of $p$ are allowed. In view of the desired antisymmetry with re-
spect to $\chi = 0$, the possible solutions of (4-22) are furthermore re-
stricted to the sine-solutions. The general solution for the far pressure
field $p(r,\chi,z)$ must thus be built up from solutions of the form

$$p(r,\chi,z) = \sin(n\chi) K_n\left(\frac{r}{b/2}\right) \{ A(q) \cos\left(\frac{z}{b/2}\right) + \\
+ B(q) \sin\left(\frac{z}{b/2}\right) \} \quad (4-24)$$

As will be shown, a line distribution of dipoles with dipole-strength
$f_1\left(\frac{z}{b/2}\right)$ is obtained by choosing

$$n = 1 \quad (4-25)$$

$$A(q) = q \int_{-\infty}^{+\infty} f_1\left(\frac{\zeta}{b/2}\right) \cos\left(\frac{\zeta}{b/2}\right) d\left(\frac{\zeta}{b/2}\right) \quad (4-26)$$

$$B(q) = q \int_{-\infty}^{+\infty} f_1\left(\frac{\zeta}{b/2}\right) \sin\left(\frac{\zeta}{b/2}\right) d\left(\frac{\zeta}{b/2}\right) \quad (4-27)$$

and integrating (4-24) over all values of $q$ between 0 and $\infty$:

$$p_{dip}(r,\chi,z) = \frac{\sin\chi}{2\pi} \frac{1}{\pi} \int_{-\infty}^{+\infty} f_1\left(\frac{\zeta}{b/2}\right) d\left(\frac{\zeta}{b/2}\right) \int_{0}^{\infty} q K_1\left(\frac{r}{b/2}\right) \cdot \\
\cdot \cos\left(\frac{\zeta-z}{b/2}\right) dq \quad (4-28)$$
Similarly, a line distribution of quadrupoles with strength \( f_2 \left( \frac{z}{b/2} \right) \) is given by:

\[
 p_{\text{quad}} \left( r, \chi, z \right) = \frac{\sin(2\chi)}{4\pi} \frac{1}{\pi} \int_{-\infty}^{+\infty} f_2 \left( \frac{\zeta}{b/2} \right) d \left( \frac{\zeta}{b/2} \right) \int_{0}^{\infty} q^2 K_2 \left( q \frac{r}{b/2} \right) \cdot \cos \left( q \frac{\zeta - z}{b/2} \right) dq
\]  

(4-29)

In chapter IV and V we will have need of both the fields (4-28) and (4-29) and it is useful to study their behaviour for \( \frac{r}{b/2} \rightarrow 0 \) in some detail. From the theory of Bessel functions it follows that the functions \( K_1(x) \) and \( K_2(x) \) can be represented by the ascending series (ref. 8):

\[
K_1(x) = \frac{1}{x} + \frac{x}{2} \left\{ \ln \left( \frac{x}{2} \right) + (\gamma - \frac{1}{2}) \right\} + \frac{3}{16} \left\{ \ln \left( \frac{x}{2} \right) + (\gamma - \frac{5}{4}) \right\} + \ldots
\]  

(4-30)

\[
K_2(x) = \frac{2}{x} - \frac{1}{2} - \frac{x^2}{8} \left\{ \ln \left( \frac{x}{2} \right) + (\gamma - \frac{3}{4}) \right\} + \ldots
\]  

(4-31)

where \( \gamma \) is Euler's constant. The expansions converge in the interval \( 0 < x < \infty \).

From the expansions (4-30) and (4-31), together with the asymptotic expression (ref. 9)

\[
K_n(x) \sim \left( \frac{\pi}{2x} \right)^{\frac{1}{2}} e^{-x} \left\{ 1 + \frac{4n^2 - 1}{8x} + \ldots \right\}
\]  

(4-32)

it may be seen that the integrands in (4-28) and (4-29) are absolutely integrable with respect to \( q \).

Substituting (4-30) into (4-28) yields on expanding asymptotically:

\[
p_{\text{dip}} \left( r, \chi, z \right) = \frac{\sin \chi}{2\pi} \frac{1}{\pi} \int_{-\infty}^{+\infty} f_1 \left( \frac{\zeta}{b/2} \right) d \left( \frac{\zeta}{b/2} \right) \int_{0}^{\infty} \cos \left( q \frac{\zeta - z}{b/2} \right) dq + \ldots
\]
\[ + \frac{\sin x}{4\pi} \int_{-\infty}^{\infty} \ln\left(\frac{r}{b/2}\right) \frac{1}{\pi} \int_{-\infty}^{\infty} f_1\left(\frac{r}{b/2}\right) d\left(\frac{r}{b/2}\right) \int \frac{q^2 \cos(q \frac{z-z'}{b/2}) dq}{0} + \]

\[ + \frac{\sin x}{4\pi} \int_{-\infty}^{\infty} \ln\left(\frac{r}{b/2}\right) \frac{1}{\pi} \int_{-\infty}^{\infty} f_1\left(\frac{r}{b/2}\right) d\left(\frac{r}{b/2}\right) \int \frac{q^2 \ln(q/2) + \gamma - 1/2}{0} . \]

where one has to define the integrands in the q-integrations as "generalised functions" of q. It is shown by Lighthill (ref. 10) that expressions like (4-33) should be interpreted as:

\[ p_{\text{dip}} = \lim_{\beta \to 0} \left\{ \frac{\sin x}{2\pi \ln\left(\frac{r}{b/2}\right)} \frac{1}{\pi} \int_{-\infty}^{\infty} f_1\left(\frac{r}{b/2}\right) d\left(\frac{r}{b/2}\right) \int e^{-\beta q} . \]

\[ \cdot \cos(q \frac{z-z'}{b/2}) dq + \ldots \right\} \]

This interpretation of (4-33) will be tacitly assumed in the following, although the actual limiting process is avoided by performing the integration with respect to \( \zeta \) before the integration with respect to q, which procedure will be seen to lead to meaningful integrals even in the usual sense.

Now the first term in the right hand side of (4-33) is simply the Fourier-integral representation of a two-dimensional field due to a dipole singularity whose strength is given by the function \( f_1\left(\frac{z}{b/2}\right) \). Under the assumption that \( f_1\left(\frac{z}{b/2}\right) \) and \( f_1'\left(\frac{z}{b/2}\right) \) are continuous functions, whereas \( f_1\left(\frac{z}{b/2}\right) = 0 \) for \( \left(\frac{z}{b/2}\right) \geq 1 \), the double integral in the second term can be reduced by an integration by parts, to obtain the Fourier-integral representation of the function \(-f_1''\left(\frac{z}{b/2}\right)\). The result becomes:
There is no need to express the function $G^{*}(z_{b/2})$ in terms of $f_{1}(z_{b/2})$ as will be seen later. For convenience, the logarithmic expression in the second term is rewritten as $\ln\left(\frac{r_{c/4}}{r}\right) = \ln\left(\frac{r}{c/4}\right) - \ln(2A)$ and (4-35) is written like:

$$p_{dip}(r,\chi,z) = \frac{\sin\chi}{2\pi} f_{1}(z_{b/2}) - \frac{\sin\chi}{4\pi} f'_{1}(z_{b/2}) \frac{r}{b/2} \ln\left(\frac{r}{b/2}\right) +$$

$$+ \frac{\sin\chi}{4\pi} \frac{r}{b/2} G^{*}(z_{b/2}) + 0 \left\{ \left(\frac{r}{b/2}\right)^{3} \ln\left(\frac{r}{b/2}\right) \right\} \ (4-36)$$

In the more general case, when the continuity conditions are not satisfied by $f_{1}$ and $f'_{1}$ in the points $z_{b/2} = \pm 1$, it is useful to approximate $f_{1}$ by a function $F(z_{b/2})$ which is continuous and has a continuous first derivative, and which furthermore possesses the following characteristics:

$$F\left(\frac{z}{b/2}\right) = f_{1}\left(\frac{z}{b/2}\right) \text{ for } -1 + \varepsilon < \frac{z}{b/2} < 1 - \varepsilon \quad (4-37)$$

$$F\left(\frac{z}{b/2}\right) = 0 \text{ for } \left|\frac{z}{b/2}\right| > 1 \quad (4-38)$$

Expression (4-36) is valid for the dipole-distribution whose strength is given by $F\left(\frac{z}{b/2}\right)$. On taking the limit $\varepsilon \to 0$, it is clear that (4-36) remains valid for discontinuous functions $f_{1}$, except in the planes $z_{b/2} = \pm 1$. The latter restriction is not important for practical purposes however.
As remarked earlier, there is no need to derive the exact form of the function $G\left(\frac{z}{b/2}\right)$ in terms of $f_1\left(\frac{z}{b/2}\right)$. From (4-36) one can derive the expression

$$G\left(\frac{z}{b/2}\right)=8\pi A \text{p}_{\text{dip}}\left(\frac{c/4,\pi/2,z}{b/2}\right)-8 A^2 f_1\left(\frac{z}{b/2}\right)+0(A^{-2} \ln A) \quad (4-39)$$

In chapter VI an expression will be derived to evaluate the function $p_{\text{dip}}(r,\chi,z)$ efficiently. Expression (4-39) then suffices to readily evaluate the function $G\left(\frac{z}{b/2}\right)$ to the required order of accuracy.

Similarly, the quadrupole field can be expanded to read:

$$p_{\text{quad}}(r,\chi,z) = \frac{\sin 2\chi}{2\pi} f_2\left(\frac{z}{b/2}\right) + \frac{\sin 2\chi}{8\pi} f''_2\left(\frac{z}{b/2}\right) +$$

$$+ 0 \left\{ (\frac{r}{b/2})^2 \ln\left(\frac{r}{b/2}\right) \right\} (\frac{r}{b/2} \to 0) \quad (4-40)$$

4.3 Application of the matching condition

The behaviour of the near field (4-12) at distances of the order $r \to 0(b)$ may be investigated by application of the relations (4-14) and the expansion (3-45). In the present approximation in which terms of order $0(A^{-2})$ with respect to the two-dimensional near field are neglected, the expansion yields:

$$p_{\text{near}} = g\left(\frac{z}{b/2}\right) \frac{\sin \chi}{\frac{r}{c/2}} + a_1\left(\frac{z}{b/2}\right) \sin \chi \frac{r}{c/2} + \ldots$$

for $r \to \text{order } b \quad (4-41)$

where the $a_n$-series must be retained fully, since there is a priori nothing known about the order of magnitude of the coefficients $a_n$. Apparently, (4-41) can only be matched to a dipole far field. From (4-36) it follows, neglecting terms of relative order $0(A^{-2})$:
\[ P_{\text{far}} \sim f \left( \frac{z}{b/2} \right) \frac{\sin \chi}{2\pi r b/2} \text{ for } r \to \text{order } c \] (4-42)

which shows that matching is achieved by choosing

\[ f \left( \frac{z}{b/2} \right) = -\frac{2\pi}{A} g \left( \frac{z}{b/2} \right) \] (4-43)

\[ a_n \left( \frac{z}{b/2} \right) = 0 \ (n = 1, 2, \ldots) \] (4-44)

4.4 The composite field and its integration

The composite pressure field is formed by summing the near- and far pressure field and subtracting their "common part", which is

\[ (p_{\text{far}}) \to \text{order } c \quad \text{or, what is the same, } (p_{\text{near}}) \to \text{order } b \]

\[ p = \frac{-g \left( \frac{z}{b/2} \right) \sin \varphi}{\cosh \eta + \cos \varphi} + p_{\text{dip}}(r, \chi, z) + g \left( \frac{z}{b/2} \right) \frac{\sin \chi}{r c/2} \] (4-45)

At large distances the common part cancels the near field except for the negligible terms of \( O(A^{-2}) \) and the far field, which is of order \( O(A^{-1}) \) remains. Close to the lifting surface, the common part cancels the first term of the far field expansion (4-36). What remains is the near field and the negligible terms of \( O(A^{-2}) \) of the far field.

In order to satisfy the boundary condition \( v/U = -\varphi_0 \) on the wingsurface, we calculate \( v/U \) along the mid-chord line \( x = 0 \) by integrating:

\[ v/U(0,0,z) = -\frac{1}{\rho U^2} \int_{-\infty}^{0} \frac{\partial p}{\partial y} (x,0,z) \, dx \] (4-46)

If \( v/U(0,0,z) \) attains the required value, the value of \( v/U \) will be correct to \( O(A^{-1}) \) over the whole wingsurface, because of the already
satisfied boundary condition (4-10).

Now the quantity \( v_i \) is introduced, the so-called "induced downwash", defined by:

\[
\frac{v_i}{U} (0,0,z) = \frac{1}{\rho U^2} \int_{-\infty}^{0} \frac{\partial}{\partial y} \left\{ p_{\text{dip}}(r,X,z) + g\left(\frac{z}{b/2}\right) \frac{\sin \chi}{r} \right\} \mathrm{d}x \quad (4-47)
\]

This is the part of \( v/U(0,0,z) \) associated with the dipole-terms in (4-45), counted positive in negative y-direction (i.e. it is considered as a real downwash). The near field term in (4-45) leads immediately to \( v/U(0,0,z) \) of the two-dimensional aerofoil, which is:

\[
v/U(0,0,z)_{\text{two-dim}} = -\frac{g}{\rho U^2} \quad (4-48)
\]

Substituting into (4-46) and equating \( v/U(0,0,z) \) to \(-\alpha \circ(z)\), the following integral equation for the function \( g\left(\frac{z}{b/2}\right) \) is obtained:

\[
\alpha \circ\left(\frac{z}{b/2}\right) = \frac{1}{\rho U^2} g\left(\frac{z}{b/2}\right) + \frac{v_i}{U} (0,0,\frac{z}{b/2}) \quad (4-49)
\]

which transforms by the relation \( \ell = g \text{nc} \) into the well-known classical result

\[
\ell = 2\pi (\alpha \circ - \frac{v_i}{U}) \frac{1}{2} \rho U^2 c \quad (4-50)
\]
V. Higher-order lifting line theory, including terms of order $A^{-2}$

5.1 The near- and far field boundary value problem

If terms of order $A^{-2}$ are included, i.e. if $p_{\text{near}}(x,y,z)$ is assumed to be given by the form

$$p_{\text{near}} = p_0 + \frac{1}{A} p_1 + \frac{n A}{A^2} p_2 + \frac{1}{A^2} p_3$$  \hspace{1cm} (5-1)

then, according to the relations (4-7) and (4-8), the following boundary value problem results for the near pressure field:

$$\frac{\partial^2 p}{\partial \left(\frac{x}{c/2}\right)^2} + \frac{\partial^2 p}{\partial \left(\frac{y}{c/2}\right)^2} = -\frac{1}{A^2} \frac{\partial^2}{\partial \left(\frac{z}{b/2}\right)^2} \left( p_{\text{two-dim}} \right)$$ \hspace{1cm} (5-2)

$$\frac{\partial p}{\partial y} = 0 \text{ on the wingsurface}$$  \hspace{1cm} (5-3)

$$p \rightarrow -\infty \text{ along the leading edge, such that } \frac{V}{U} = -\alpha_o(z) \text{ on the wing surface}$$ \hspace{1cm} (5-4)

$$\left( p_{\text{far}} \right) \sim \left( p_{\text{near}} \right) \text{ (r+order 0(c)), (r+order 0(b))}$$ \hspace{1cm} (5-5)

where, comparing with the near field boundary value problem of classical lifting line theory, the two-dimensional Laplace equation has been changed into the two-dimensional Poisson equation (5-2).

According to section 4.1, the far pressure field corresponds to a field of line singularities, if relative errors of order $A^{-2}$ are allowed. Now from the theory of chapter 4 it appears that the leading term in the far field expression (i.e. the dipole singularity) is of the order $A^{-1}$ with respect to the leading term in the composite pressure field. This shows, that for the purpose of building up a composite field accurate to order $A^{-2}$, the far field representation by a distribution of line singularities...
may still be used. In that case the boundary value problem for the far field remains unchanged compared with classical lifting line theory.

5.2 General solution of the near field problem

Using

\[ p_{\text{two-dim}} = -g\left(\frac{z}{b/2}\right) \frac{\sin \varphi}{\cosh \eta + \cos \varphi} \] (5-9)

the two-dimensional Poisson equation (5-2) may be written in elliptic-cylinder coordinates as

\[ \frac{1}{\cosh \eta \cos^2 \varphi} \left( \frac{\partial^2 p}{\partial \eta^2} + \frac{\partial^2 p}{\partial \varphi^2} \right) = \frac{1}{A^2} g''\left(\frac{z}{b/2}\right) \frac{\sin \varphi}{\cosh \eta + \cos \varphi} \] (5-10)

or:

\[ \frac{\partial^2 p}{\partial \eta^2} + \frac{\partial^2 p}{\partial \varphi^2} = \frac{1}{A^2} g''\left(\frac{z}{b/2}\right) (\cosh \eta \sin \varphi - \frac{1}{2} \sin 2\varphi) \] (5-11)

having the particular solution

\[ p(\eta, \varphi, z) = \frac{1}{A^2} g''\left(\frac{z}{b/2}\right) \left( \frac{1}{2} \eta \sinh \eta \sin \varphi + \frac{1}{8} \sin 2\varphi \right) \] (5-12)

as may be checked by substitution.

The general solution of (5-10) contains (5-12) as well as the general solution of the two-dimensional Laplace equation. Since the field (5-12) does not contribute to the value of \( \frac{\partial p}{\partial \eta} \) at \( \eta = 0 \) we may in accordance with the boundary condition (5-3) write the near field in the general form:

\[ p(\eta, \varphi, z) = -h\left(\frac{z}{b/2}\right) \frac{\sin \varphi}{\cosh \eta + \cos \varphi} + \sum_{n=1}^{\infty} a_n \left(\frac{z}{b/2}\right) \cosh(n \eta) \sin(n \varphi) + \frac{1}{A^2} g''\left(\frac{z}{b/2}\right) \left( \frac{1}{2} \eta \sinh \eta \sin \varphi + \frac{1}{8} \sin 2\varphi \right) \] (5-13)
5.3 Application of the matching condition

Again using the relation

\[
r = \frac{c}{2} \cosh \frac{\eta}{2} \cosh \frac{\eta}{2} \sinh \frac{\eta}{2} \cosh \frac{\eta}{2} \sinh \frac{\eta}{2} = \frac{c}{4} \exp \eta
\]

and the expansion (3-45) derived in chapter III, the near field expansion becomes to the order \( A^{-2} \):

\[
P_{\text{near}} \sim \frac{\sin \chi}{2\pi r} h\left(\frac{z}{b/2}\right) \frac{\pi c}{2} + \frac{\sin 2\chi}{2\pi r^2} h\left(\frac{z}{b/2}\right) \frac{\pi c^2}{4} +
\]

\[
+ a_1 \left(\frac{z}{b/2}\right) \frac{r}{c/2} \sin \chi + \ldots +
\]

\[
+ \frac{1}{2A^2} g''\left(\frac{z}{b/2}\right) \ln\left(\frac{r}{c/4}\right) r \sin \chi + \frac{1}{8A^2} g''\left(\frac{z}{b/2}\right) \sin 2\chi
\]

for \( r \to \) order \( b \) \n
(5-14)

This expansion must be matched to the far field expansion derived in section 4.2, now consisting of the field of a dipole-line \( p_{\text{dip}} \), as well as the field of a quadrupole-line \( p_{\text{quad}} \). Thus, according to (4-36) and (4-40):

\[
P_{\text{far}} \sim \frac{\sin \chi}{2\pi (b/2)} f_1\left(\frac{z}{b/2}\right) - \frac{\sin \chi}{4\pi} f''\left(\frac{z}{b/2}\right) \frac{r}{b/2} \ln\left(\frac{r}{c/4}\right) +
\]

\[
+ \frac{\sin \chi}{4\pi} \frac{r}{b/2} G\left(\frac{z}{b/2}\right) + \frac{\sin 2\chi}{2\pi b/2} f_2\left(\frac{z}{b/2}\right) + \frac{\sin 2\chi}{8\pi} \frac{r}{b/2} f''\left(\frac{z}{b/2}\right)
\]

for \( r \to \) order \( c \) \n
(5-15)

Equating the corresponding terms in the two expansions (5-14) and (5-15) shows that matching is achieved when choosing:

\[
h\left(\frac{z}{b/2}\right) = g\left(\frac{z}{b/2}\right)
\]

(5-16)
\[ a_1\left(\frac{z}{b/2}\right) = \frac{1}{4\pi A} \, G\left(\frac{z}{b/2}\right) = \frac{2}{2} \{ p_{\text{dip}}(c/4, \pi/2, \frac{z}{b/2}) + 2 \, g\left(\frac{z}{b/2}\right) \} \quad (5-17) \]

\[ a_n\left(\frac{z}{b/2}\right) = 0 \quad (n > 2) \quad (5-18) \]

\[ f_1\left(\frac{z}{b/2}\right) = -\frac{2\pi}{A} \, g\left(\frac{z}{b/2}\right) \quad (5-19) \]

\[ f_2\left(\frac{z}{b/2}\right) = \frac{\pi}{A} \, g\left(\frac{z}{b/2}\right) \quad (5-20) \]

By matching we can also find now the near field outside the wingspan. For \( |\frac{z}{b/2}| > 1 \) the antisymmetry of the lift problem evidently requires \( p = 0 \) in the plane \( y = 0 \), thus leading to solutions of the type

\[ p = \sum_{n=1}^{\infty} c_n \, \sinh(n\eta) \, \sin(n\varphi) \quad (5-21) \]

so that

\[ p_{\text{near}} \sim c_1 \, \frac{r}{c/2} \, \sin\chi \, \ldots \, \text{for} \quad |\frac{z}{b/2}| > 1 \quad (5-22) \]

and \( r \rightarrow \text{order } b \)

which is matched to the far field expansion

\[ p_{\text{far}} = \frac{\sin\chi}{4\pi} \, \frac{r}{b/2} \, G\left(\frac{z}{b/2}\right) \quad \text{for} \quad |\frac{z}{b/2}| > 1 \quad (5-23) \]

and \( r \rightarrow \text{order } c \)

showing

\[ c_1\left(\frac{z}{b/2}\right) = \frac{1}{4\pi A} \, G\left(\frac{z}{b/2}\right) = \frac{2}{2} \{ p_{\text{dip}}(c/4, \pi/2, \frac{z}{b/2}) + 2 \, g\left(\frac{z}{b/2}\right) \} \quad (5-24) \]

\[ c_n\left(\frac{z}{b/2}\right) = 0 \quad (n \geq 2) \quad (5-25) \]
5.4 The composite field

The composite field is again formed by summing the near- and far pressure field, and subtracting their common part:

\[
p = \left[ -g\left(\frac{z}{b/2}\right) \frac{\sin\phi}{\cosh\eta + \cos\phi} + g\left(\frac{z}{b/2}\right) \frac{\pi}{c} \frac{\sin\chi}{2\pi r} + p_{\text{dip}}(r, \chi, z) \right] + \\
+ \left[ 2(\cosh\eta \sin\phi - \frac{r}{c/2} \sin\chi)\{p_{\text{dip}}(c/4, \pi/2, \frac{z}{b/2}) + 2g(\frac{z}{b/2}) \} + \\
+ \frac{1}{2A^2} \ln\left(\frac{r}{c/4} \frac{r}{c/2} \sin\chi\right) \right] + \\
+ \left[ p_{\text{quad}}(r, \chi, z) - g\left(\frac{z}{b/2}\right) \frac{\pi}{4} \frac{\sin 2\chi}{2\pi r^2} + \\
+ \frac{1}{8A^2} g''\left(\frac{z}{b/2}\right) (\sin 2\phi - \sin 2\chi) \right] \quad |\frac{z}{b/2}| < 1 \quad (5-26)
\]

where the terms have been grouped such that the first part between square brackets contains the composite field of classical lifting line theory, the second term between square brackets contains the additional field components of order $A^{-2}$ associated with the lift, and the last bracketed part consists of the terms arising from the pitching moment.

Outside the wingspan the near field (5-21) and the common part (5-22) cancel each other, since $\sinh\eta \sin\phi = \frac{r}{c/2} \sin\chi = \frac{y}{c/2}$, so that

\[
p = p_{\text{dip}}(r, \chi, z) + p_{\text{quad}}(r, \chi, z) \quad \frac{|z|}{b/2} > 1 \quad (5-27)
\]

The expressions may be written in a considerably simpler form, by noting that the far field is up to order $A^{-2}$ equal to the field of a single dipole line, placed along the quarter-chord line of the wing, as will be shown now.

If $p_{\text{dip}}_{c/4}(x, y, z)$ denotes the pressure field due to a dipole distribution along the quarter-chord line of the wing, and $p_{\text{dip}}_{c/2}(x, y, z)$...
denotes the pressure of a dipole distribution along the mid-chord line, then:

\[
\begin{align*}
\mathbf{P}_{\text{dip}}(x,y,z) &= \mathbf{P}_{\text{dip}}(x+c/4,y,z) = \\
&= \mathbf{P}_{\text{dip}}(x,y,z) + \frac{\partial \mathbf{P}_{\text{dip}}(c/4,y,z)}{\partial x} + \frac{\partial^2 \mathbf{P}_{\text{dip}}(c/2) c^2}{32} + \ldots = \\
&= \mathbf{P}_{\text{dip}}(x,y,z) + \frac{\partial \mathbf{P}_{\text{dip}}(c/2) c}{8} + \frac{\partial^2 \mathbf{P}_{\text{dip}}(c/2) c^2}{32} + \ldots
\end{align*}
\]

(5-28)

According to eqs. (4-28) and (5-19):

\[
\begin{align*}
\mathbf{P}_{\text{dip}}(x,y,z) &= -\frac{\sin \chi}{A} \frac{1}{\pi} \int_{-\infty}^{+\infty} \int_{0}^{\infty} g(\frac{r}{b/2}) d(\frac{r}{b/2}) \int_{0}^{\infty} q K_1(q\frac{r}{b/2}) \cos(q\frac{c-z}{b/2}) dq \\
&= -\frac{\sin \chi}{2A} \frac{1}{\pi} \int_{-\infty}^{+\infty} \int_{0}^{\infty} g(\frac{r}{b/2}) d(\frac{r}{b/2}) \int_{0}^{\infty} q^2 \cos(q\frac{c-z}{b/2}) dq \\
&= \frac{K_1(q\frac{r}{b/2}) - \frac{K_1(q\frac{r}{b/2})}{q\frac{r}{b/2}}}{q\frac{r}{b/2}}
\end{align*}
\]

(5-29)

so that

\[
\begin{align*}
\frac{\partial \mathbf{P}_{\text{dip}}(c/2)}{\partial (\frac{x}{b/2})} &= \cos \chi \frac{\partial \mathbf{P}_{\text{dip}}(c/2)}{\partial (\frac{r}{b/2})} - \sin \chi \frac{\partial \mathbf{P}_{\text{dip}}(c/2)}{\partial \chi} = \\
&= -\frac{\sin 2\chi}{2A} \frac{1}{\pi} \int_{-\infty}^{+\infty} \int_{0}^{\infty} g(\frac{r}{b/2}) d(\frac{r}{b/2}) \int_{0}^{\infty} q^2 \cos(q\frac{c-z}{b/2}) dq \\
&= \left\{ K_1(q\frac{r}{b/2}) - \frac{K_1(q\frac{r}{b/2})}{q\frac{r}{b/2}} \right\} dq
\end{align*}
\]

(5-30)

Using the following recurrence relations (ref. 9, no. 804.3 and 804.4):

\[
\begin{align*}
2n K_n(x) &= x K_{n+1}(x) - x K_{n-1}(x) \quad (5-31) \\
2 K_n'(x) &= -K_{n-1}(x) - K_{n+1}(x) \quad (5-32)
\end{align*}
\]
it follows

$$p_{\text{dip}_{c/4}} = -\frac{\sin\chi}{A} \frac{1}{\pi} \int_{-\infty}^{+\infty} g\left(\frac{c}{b/2}\right) d\left(\frac{c}{b/2}\right) \int q K_1\left(q_{b/2}r\right) \cos\left(q_{b/2}z\right) dq +$$

$$+ \frac{\sin 2\chi}{4A^2} \frac{1}{\pi} \int_{-\infty}^{+\infty} g\left(\frac{c}{b/2}\right) d\left(\frac{c}{b/2}\right) \int q^2 K_2\left(q_{b/2}r\right) \cos\left(q_{b/2}z\right) dq +$$

$$+ 0(A^{-3})$$

(5-33)

which shows, together with (4-29) and (5-20):

$$p_{\text{dip}_{c/4}} = p_{\text{dip}_{c/2}} + p_{\text{quad}_{c/2}} + 0(A^{-3})$$

(5-34)

Likewise, the "common part" given by (5-14) may be shown to be equal to the function \(p_{\text{dip}_{c/4}}\) up to \(0(A^{-2})\). The composite field (5-26) may thus be written like:

$$p = -g\left(\frac{z}{b/2}\right) \frac{\sin\varphi}{\cosh\eta + \cos\varphi} + g\left(\frac{z}{b/2}\right) \frac{\sin\chi'}{2\pi r} + p_{\text{dip}}\left(r',\chi',z\right) +$$

$$+ 2(\cosh\eta \sin\varphi - \frac{r'}{c/2} \sin\chi') \{p_{\text{dip}}\left(c/4,\pi/2,\frac{z}{b/2}\right) + 2g\left(\frac{z}{b/2}\right)\} +$$

$$+ \frac{1}{2A^2} g''\left(\frac{z}{b/2}\right) \{\eta \sinh\eta \sin\varphi + \frac{1}{4} \sin 2\varphi - \frac{r'}{c/2} \ln\left(\frac{r'}{c/4}\right) \sin\chi'\}$$

(5-35)

where the cylindrical coordinates \(r',\chi',z\) are centered around the quarter-chord line of the wing \(x = -c/4\), thus conforming to the transformation formulae

$$x = r' \cos\chi' + c/4$$

(5-36)
y = r' \sin \chi' \quad (5-37)

z = z \quad (5-38)

5.5 The integral equation determining $g(\frac{z}{b/2})$

The velocity component $v$ on the wing surface is found by performing the integration

$$v/U(-c/4,0,z) = -\frac{1}{\rho U^2} \int_{-\infty}^{-c/4} \frac{\partial p}{\partial y} (x,0,z) \, dx \quad (5-39)$$

so that the first part of the composite field (5-35) contributes the classical lifting line downwash

$$v/U(-c/4,0,z) = \frac{g(\frac{z}{b/2})}{\rho U^2} - \frac{v_i}{U} (-c/4,0,z) \quad (5-40)$$

as derived in chapter IV.

For the contribution of the second part of the composite field to $v/U(-c/4,0,z)$, the integral $I_2$ must be evaluated, defined by:

$$I_2 = \int_{-\infty}^{-c/4} \frac{\partial}{\partial y} (\cosh \eta \sin \varphi - \frac{r'}{c/2} \sin \chi') \, dx \quad (5-41)$$

The integration limit $x = -\infty$ is for a moment replaced by a finite limit. Choosing a point $x_o = -r_o$ on the negative $x$-axis, such that the associated value of $\eta$ may be approximated by

$$\eta_o = \ln(\frac{r_o}{c/4}) \quad (5-42)$$

whereas

$$\sinh \eta_o \gtrsim \cosh \eta_o \gtrsim \frac{r_o}{c/2} \quad (5-43)$$
the integral becomes

\[ I_2 = \lim_{\eta_o \to \infty} \int_{0}^{\eta_o} \cosh \eta \, d\eta - \lim_{r_o \to \infty} \frac{1}{c/2} \int_{0}^{r_o-c/4} \, dr' = \]

\[ = \lim_{\eta_o \to \infty} \sinh \eta_o - \lim_{r_o \to \infty} \frac{r_o-c/4}{c/2} = \frac{1}{2} \quad (5-44) \]

Finally, the integral \( I_3 \) whose evaluation is needed for determining the contribution of the third part of the composite field to \( v/U(-c/4,0,z) \) is considered:

\[ I_3 = \int_{-\infty}^{-c/4} \frac{\partial}{\partial y} \left\{ \eta \sinh \eta \sin \varphi + \frac{1}{4} \sin 2\varphi - \frac{r'}{c/2} \sin \chi' \ln \left( \frac{r'}{c/4} \right) \right\} dx = \]

\[ = \lim_{\eta_o \to \infty} \int_{0}^{\eta_o} \left( \eta \sinh \eta - \frac{1}{2} \right) d\eta - \lim_{r_o \to \infty} \frac{1}{c/2} \int_{0}^{r_o-c/4} \ln \left( \frac{r'}{c/4} \right) dr' = \]

\[ = \lim_{\eta_o \to \infty} \left( \eta \cosh \eta_o - \sinh \eta_o \frac{1}{2} \eta_o \right) - \lim_{r_o \to \infty} \frac{r_o-c/4}{c/2}. \]

\[ \cdot \left\{ \ln \left( \frac{r_o}{c/4} \right) - 1 \right\} = \lim_{r_o \to \infty} \left\{ \frac{r_o}{c/2} \ln \left( \frac{r_o}{c/4} \right) - \frac{r_o}{c/2} - \frac{1}{2} \ln \left( \frac{r_o}{c/4} \right) \right\} + \]

\[ - \lim_{r_o \to \infty} \frac{r_o-c/4}{c/2} \left\{ \ln \left( \frac{r_o}{c/4} \right) - \frac{c/4}{r_o} - \frac{1}{2} \left( \frac{c/4}{r_o} \right)^2 - \ldots - 1 \right\} = 0 \quad (5-45) \]

Using the results (5-40), (5-44) and (5-45) one finds:
\[
 \frac{v}{U(-c/4,0,z)} = -\frac{g\left(\frac{z}{b/\sqrt{2}}\right)}{\rho U^2} - \frac{1}{\rho U^2} \int_{-\infty}^{-c/4} \frac{\partial}{\partial y} \left\{ p_{\text{dip}}(r',x',z) \right\} dx + 
 + g\left(\frac{z}{b/\sqrt{2}}\right) \pi c \frac{\sin \chi'}{2\pi r'} \right\} \right\} dx + 
 - \left\{ p_{\text{dip}}\left(c/4,\pi/2,\frac{z}{b/\sqrt{2}}\right) + 2g\left(\frac{z}{b/\sqrt{2}}\right) \right\} 
\]

which may be written, equating \( v/U(-c/4,0,z) \) to \( -\alpha_0(z) \):

\[
 g\left(\frac{z}{b/\sqrt{2}}\right) = \alpha_0\left(\frac{z}{b/\sqrt{2}}\right) \rho U^2 \int_{-\infty}^{-c/4} \frac{\partial}{\partial y} \left\{ p_{\text{dip}}(r',x',z) + g\left(\frac{z}{b/\sqrt{2}}\right) \frac{\sin \chi'}{2\pi r'} \right\} dx + 
 - \left\{ p_{\text{dip}}\left(c/4,\pi/2,\frac{z}{b/\sqrt{2}}\right) + 2g\left(\frac{z}{b/\sqrt{2}}\right) \right\} 
\]

The form in which (5-47) has been written suggests an iterative procedure for solving the integral equation. Referring to the composite field (5-26) it may be seen that to the order \( O(A^0) \) the function \( g(z) \) is given by \( \alpha_0(z)\rho U^2 \). Refining to the order \( O(A^{-1}) \), the dipole far field is taken into account. During this step the strength of the dipoles along the lifting line may however be taken as \( g(z) = \alpha_0(z)\rho U^2 \), since a further refinement of the lifting line far field would add irrelevant accuracy. The next refinement to \( O(A^{-2}) \) involves all the terms of the composite field (5-26). For taking this last step, the strength of the dipoles along the lifting line need not be known to a higher accuracy than \( O(A^{-1}) \), and the strength of the quadrupoles along the lifting line need not be known to a higher accuracy than \( O(A^0) \).

All this means that an iterative solution of eq. (5-47) needs no more than two iterations. The first approximation is obtained by neglecting all terms in the right hand side of (5-47) except the first one. Inserting this first approximation into the right hand side yields a second approximation while going through one more cycle concludes the iteration. Strictly speaking, it is not even necessary to take into account the third term until the last iteration.
The pressure on the surface of the wing section is according to the foregoing given by:

\[ p(0,\varphi,z) = \frac{g(z)}{\sqrt{1+\cos\varphi}} + \frac{1}{8A^2} g'' \left( \frac{z}{b/2} \right) \sin 2\varphi \]  

(5-48)

showing

\[ \lambda \left( \frac{z}{b/2} \right) = -\frac{1}{\sqrt{1+\cos\varphi}} \]  

(5-49)

\[ m \left( \frac{z}{b/2} \right) = \frac{g(z)}{\sqrt{1+\cos\varphi}} \]  

(5-50)

5.7 Comparison with the 3/4-chord method

Weissinger's 3/4-chord method, or the so-called "extended" lifting line theory, takes in pressure formulation the following form:

\[ -\alpha_o(z) = -\frac{1}{\rho U^2} \int_{-\infty}^{+c/4} \frac{\partial}{\partial y} p_{\text{dip}}(r',\chi',z) \, dx \]  

(5-51)

where the coordinates \( r', \chi' \) denote cylindrical coordinates centered around the quarter-chord line of the wing (\( x = -c/4 \)). The pressure field \( p_{\text{dip}} \) denotes the field of a dipole line which behaves close to the line like

\[ p_{\text{dip}}(r',\chi',z) \sim -\frac{\sin\chi'}{2\pi r'} \lambda \left( \frac{z}{b/2} \right) \text{ for } r' \to 0 \]  

(5-52)

Equation (5-51) is an integral equation for the function \( \lambda \left( \frac{z}{b/2} \right) \), obtained by the assumption that the downwash along the 3/4-chord line as
caused by a lifting line in the quarter-chord position, should equal the angle of incidence \( \alpha \) (\( z \)). The relation between the dipole strength distribution along the line and the lift distribution is taken to be the same as in classical lifting line theory.

Since it is known that the extended lifting line theory yields in most cases rather good results - anyway much better than the results of classical lifting line theory - it is interesting to investigate the theory from the point of view of the higher order lifting line theory developed here.

Use will be made again of the composite pressure field (5-35). Integrating \( \partial p/\partial y \) up to the point \( x = c/4 \) (i.e. the 3/4-chord point), one obtains:

\[
v/U(c/4,0,z) = -\frac{g(\frac{z}{b/2})}{\rho U^2} - \frac{1}{\rho U^2} \int_{-\infty}^{+c/4} \frac{\partial}{\partial y} \{g(\frac{z}{b/2})\pi c}{2\pi x} \sin x' + \\
+ p_{\text{dip}}(r',\chi',z) \, dx + \frac{1}{\rho U^2} \{ p_{\text{dip}}(c/4,\pi/2,z) + 2g(\frac{z}{b/2}) \} + \\
+ \frac{1}{\rho U^2} \frac{g''(\frac{z}{b/2})}{2 \pi^2} (\ln 2-1) \tag{5-53}
\]

which is, although of a different form, to \( O(A^{-2}) \) numerically equal to \( v/U(-c/4,0,z) \) as given by (5-46). Substituting eq. (5-49) and retaining only the terms of order \( O(A^{-2}) \) and lower, eq. (5-53) is transformed into:

\[
v/U(c/4,0,z) = -\frac{g(\frac{z}{b/2})}{\rho U^2 \pi c} - \frac{1}{\rho U^2} \int_{-\infty}^{+c/4} \frac{\partial}{\partial y} \{g(\frac{z}{b/2})\pi c}{2\pi x} \sin x' + \\
+ p_{\text{dip}}(r',\chi',z) \, dx + \frac{1}{\rho U^2} \frac{g''(\frac{z}{b/2})}{2\pi A^2} (\ln 2-1) \tag{5-54}
\]
It may be shown (Pistolesi's theorem) that
\[
\frac{-1}{\rho U^2} \int_{-\infty}^{+c/4} \frac{\partial}{\partial y} \left( \frac{\ell(z)}{b/2} \sin \chi' \right) dx = \frac{\ell(z)}{\rho U^2 \pi c} (5-55)
\]
yielding finally
\[
v/U(c/4,0,z) = \frac{-1}{\rho U^2} \int_{-\infty}^{c/4} \frac{\partial}{\partial y} P_{dip}(r',\chi',z) \, dx + \frac{1}{\rho U^2} \frac{\ell''(z)}{2\pi A^2/c} (\ln 2-1) (5-56)
\]
which equals (5-51) apart from the last term. The last term, of order $A^{-2}$, is the error in the 3/4-chord method. As discussed earlier, when solving eq. (5-56) the function $\ell(z)$ in the terms of order $A^{-2}$ need not be known to a higher accuracy than $O(A^0)$. This shows that for untwisted wings or wings with linear twist the error vanishes, and the 3/4-chord method is exact up to $O(A^{-2})$. In more general cases the error may be expected to be very small.
VI The far pressure field of a wing, expressed in series of Legendre-functions

6.1 The boundary value problem for the far pressure field of a wing in prolate spheroidal coordinates

In chapters IV and V, containing the derivation of the classical and higher order lifting line methods, the boundary value problem for the far pressure field was in both cases:

\[
\frac{\partial^2 p}{\partial (\frac{x}{b/2})^2} + \frac{\partial^2 p}{\partial (\frac{y}{b/2})^2} + \frac{\partial^2 p}{\partial (\frac{z}{b/2})^2} = 0 \quad (6-1)
\]

\[p \to 0 \quad \text{for} \quad x^2 + y^2 + z^2 \to \infty \quad (6-2)\]

\[p(x,y,z) \text{ singular along the line } x = y = 0, \quad \left|\frac{z}{b/2}\right| \leq 1\]

and antisymmetric with respect to the plane \(y = 0\) \( (6-3) \)

From the matching process and subsequent developments it appeared furthermore that the one interesting solution was the field behaving near the \(Z\)-axis like a two-dimensional dipole field:

\[p_{\text{far field}} \sim -\frac{g\left(\frac{z}{b/2}\right)}{A} \cdot \frac{\sin \chi}{r} (\frac{r}{b/2} \to 0) \quad (6-4)\]

In both lifting line chapters the solution of this problem was expressed in circular-cylinder coordinates, requiring for a finite lifting line a Fourier integral expression. Such an expression was found to be useful for analytical purposes, but apparently for numerical purposes a more convenient expression would be desirable. A convenient series solution of the far field boundary value problem can be found using prolate spheroidal coordinates.
Prolate spheroidal coordinates are obtained by taking an orthogonal family of confocal ellipses and hyperbolas in a plane and rotating them around the Z-axis (fig. 7). Prolate spheroidal coordinates \((\psi, \theta, \chi)\) are related to rectangular coordinates by the equations

\[
\begin{align*}
x &= \frac{b}{2} \sinh \psi \sin \theta \cos \chi \\
y &= \frac{b}{2} \sinh \psi \sin \theta \sin \chi \\
z &= \frac{b}{2} \cosh \psi \cos \theta
\end{align*}
\] (6-5, 6-6, 6-7)

Surfaces of constant \(\psi\) are prolate spheroids:

\[
\frac{x^2 + y^2}{(b/2 \sinh \psi)^2} + \frac{z^2}{(b/2 \cosh \psi)^2} = 1
\] (6-8)

The variable \(\psi\) runs from 0 to \(+\infty\). As \(\psi \to 0\) the spheroid approaches a straight line segment of length \(b\) on the Z-axis. Along this "base" of the coordinate system the transformation reads

\[
\begin{align*}
x &= y = 0 \\
z &= \frac{b}{2} \cos \theta
\end{align*}
\] (6-9, 6-10)

As \(\psi\) increases, the spheroids become more and more nearly spherical until as \(\psi \to \infty\) \(\sinh \psi \sim \cosh \psi\) and a sphere is obtained with radius \(\rho = \frac{b}{2} \sinh \psi \sim \frac{b}{2} \cosh \psi \sim \frac{1}{4} b e^\psi\).

The surfaces of constant \(\theta\) are hyperboloids of two sheets:

\[
-\frac{x^2 + y^2}{(b/2 \sin \theta)^2} + \frac{z^2}{(b/2 \cos \theta)^2} = 1
\] (6-11)

For \(\theta = 0\), the surfaces degenerate into the portion of the Z-axis from \(+\infty\) to \(b/2\). For \(\theta = \pi\) we have the corresponding portion of the Z-axis from \(-b/2\) to \(-\infty\).

When \(\theta = \pi/2\) the hyperboloid becomes the XOY-plane. Values of \(\theta\) between
0 and \( \pi/2 \) correspond to the upper sheet of the hyperboloid, while values between \( \pi/2 \) and \( \pi \) denote the lower sheet. As the system at large distances degenerates into a spherical coordinate system, the variable \( \theta \) corresponds with the angle \( \theta \) used in spherical systems to measure angles from the positive Z-axis (fig. 6).

As with spherical and cylindrical coordinates, \( \chi \) is the angle measured about the Z-axis. Surfaces of constant \( \chi \) are half-planes containing the Z-axis. The range of \( \chi \) is taken as 0 to 2\( \pi \).

The inverse transformation reads:

\[
\cosh \psi = \frac{\rho_1 + \rho_2}{b} \tag{6-12}
\]
\[
\cos \theta = \frac{\rho_1 - \rho_2}{b} \tag{6-13}
\]
\[
tg \chi = \frac{y}{x} \tag{6-14}
\]

where \( \rho_1 \) is the distance from a field point \((x,y,z)\) to the point \((0,0,-b/2)\) and \( \rho_2 \) the distance to the point \((0,0,+b/2)\):

\[
\rho_1 = \sqrt{x^2 + y^2 + (z+b/2)^2} \tag{6-15}
\]
\[
\rho_2 = \sqrt{x^2 + y^2 + (z-b/2)^2} \tag{6-16}
\]

The relation between prolate spheroidal coordinates and cylindrical coordinates will also be used frequently. It is given by:

\[
r = b/2 \sinh \psi \sin \theta \tag{6-17}
\]
\[
z = b/2 \cosh \psi \cos \theta \tag{6-18}
\]
\[
\chi = \chi \tag{6-19}
\]
Laplace's equation is in prolate spheroidal coordinates transformed into (ref. 6):

\[
\nabla^2 p = \frac{1}{(b/2)^2 \sinh^2 \psi + \sin^2 \theta} \left( \frac{\partial^2 p}{\partial \psi^2} + \coth \psi \frac{\partial p}{\partial \psi} + \frac{\partial^2 p}{\partial \theta^2} + \cot \theta \frac{\partial p}{\partial \theta} \right) +
\]

\[
\frac{1}{(b/2)^2 \sinh^2 \psi \sin^2 \theta} \frac{\partial^2 p}{\partial \chi^2} = 0 \quad (6-20)
\]

The far pressure field \( p(\psi, \theta, \chi) \) of a wing must thus satisfy eq. (6-20), such that \( p \to 0 \) for \( \psi \to \infty \). Whether a given solution of (6-20) satisfies the boundary condition (6-4) may be investigated by letting \( \psi \to 0 \) and substituting the relations (6-17) to (6-19).

6.2 General solution for the pressure field in prolate spheroidal coordinates

A general solution of (6-20) can again be found by the separation method:

\[
p = \Psi(\psi) \cdot \Theta(\theta) \cdot X(\chi) \quad (6-21)
\]

Substitution into eq. (6-20) leads to the separated equations:

\[
\frac{d^2 \Psi}{d \psi^2} + \coth \psi \frac{d \Psi}{d \psi} - \left\{ n(n + 1) + \frac{m^2}{\sinh^2 \psi} \right\} \Psi = 0 \quad (6-22)
\]

\[
\frac{d^2 \Theta}{d \theta^2} + \cot \theta \frac{d \Theta}{d \theta} + \left\{ n(n + 1) - \frac{m^2}{\sin^2 \theta} \right\} \Theta = 0 \quad (6-23)
\]

\[
\frac{d^2 X}{d \chi^2} + m^2 X = 0 \quad (6-24)
\]

or, substituting \( \xi = \cosh \psi \) and \( \mu = \cos \theta \):

\[
(\xi^2 - 1) \frac{d^2 \Psi}{d \xi^2} + 2 \xi \frac{d \Psi}{d \xi} - \left\{ n(n+1) + \frac{m^2}{\xi^2 - 1} \right\} \Psi = 0 \quad (6-25)
\]

\[
(\mu^2 - 1) \frac{d^2 \Theta}{d \mu^2} + 2 \mu \frac{d \Theta}{d \mu} - \left\{ n(n+1) + \frac{m^2}{\mu^2 - 1} \right\} \Theta = 0 \quad (6-26)
\]
The solution of (6-27) is:

\[ X = \sin(mx) \text{ or } \cos(mx) \] (6-28)

Since in our case the solution must be periodic in \( \chi \), the separation constant \( m \) is restricted to integervalues \( m = 0,1,2,... \). The other constant \( n \) can in principle take any value, but again we will have need only of integervalues \( n = 0,1,2,... \).

The equations (6-25) and (6-26) have the form of Legendre's equation. The two independent solutions of this equation are tabulated functions of the argument \( \xi \) or \( \psi \), with \( m \) and \( n \) as parameters. Usually the two independent solutions of Legendre's equation are denoted as \( P^m_n \) (Legendre function of the first kind) and \( Q^m_n \) (Legendre function of the second kind), so that the sought functions \( \Psi \) and \( \Theta \) are given as:

\[ \Psi(\psi) = P^m_n(\cosh\psi) \text{ or } Q^m_n(\cosh\psi) \] (6-29)

\[ \Theta(\theta) = P^m_n(\cos\theta) \text{ or } Q^m_n(\cos\theta) \] (6-30)

The final expression for the harmonic field \( p(\psi,\theta,\chi) \) can thus be built up by summing elementary functions:

\[ p = \sum_m \sum_n \left\{ A_{mn} P^m_n(\cos\theta) P^m_n(\cosh\psi) \sin(mx) + B_{mn} Q^m_n(\cos\theta) Q^m_n(\cosh\psi) \cos(mx) + \ldots \right\} \] (6-31)

Before going on to determine the constants in the general expressions such that the required field of a lifting line is obtained, it is useful to consider in somewhat more detail the structure and characteristics of the Legendre-functions occurring in these expressions.
6.3 Explicit expressions for the Legendre-functions; recurrence relations; orthogonality

The case \( m = 0 \)

The notation \( P^0_n \) and \( Q^0_n \) is usually simplified to \( P_n \) and \( Q_n \). The first few functions \( P_n \) are given by the expressions:

\[
\begin{align*}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1) \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x)
\end{align*}
\]

Because these functions are polynomials, usually the functions \( P_n \) are called the Legendre-polynomials. Important characteristics of the polynomials are:

\[
\begin{align*}
P_n(1) &= 1 \\
P_n(\infty) &= \infty
\end{align*}
\]

The first few Legendre-functions of the second kind are given by:

\[
\begin{align*}
Q_0(x) &= \frac{1}{2} \ln \frac{1+x}{1-x} \quad |x| < 1 \\
Q_1(x) &= P_1(x)Q_0(x) - 1 \\
Q_2(x) &= P_2(x)Q_0(x) - \frac{3}{2} x \\
Q_3(x) &= P_3(x)Q_0(x) - \frac{5}{2} x^2 + \frac{2}{3}
\end{align*}
\]
\[ Q_4(x) = P_4(x) \cdot Q_0(x) - \frac{35}{8} \frac{x^3}{3} + \frac{55}{24} x \] (6-45)

\[ Q_5(x) = P_5(x) \cdot Q_0(x) - \frac{63}{8} \frac{x^4}{4} + \frac{49}{8} \frac{x^2}{2} - \frac{8}{15} \] (6-46)

The functions \( Q_n \) are singular in the point \( x = 1 \), and approach zero for \( x \to \infty \):

\[ Q_n(1) = \infty \] (6-47)

\[ Q_n(\infty) = 0 \] (6-48)

The case \( m \neq 0 \)

In order to make a distinction between the Legendre functions for \( m \neq 0 \) and for \( m = 0 \), the functions \( P_n^m \) and \( Q_n^m \) are generally called the associated Legendre functions of the first kind and second kind respectively. They are given by:

\[
\begin{align*}
P_n^m(x) &= (1-x^2)^{m/2} \frac{d^m}{dx^m} p_n^m(x) & |x| < 1 \\
Q_n^m(x) &= (1-x^2)^{m/2} \frac{d^m}{dx^m} q_n^m(x) \end{align*}
\] (6-49)

\[
\begin{align*}
P_n^m(x) &= (x^2-1)^{m/2} \frac{d^m}{dx^m} p_n^m(x) & |x| > 1 \\
Q_n^m(x) &= (x^2-1)^{m/2} \frac{d^m}{dx^m} q_n^m(x) \end{align*}
\] (6-50)

The \( P_n^m \)-functions become zero in \( x = 1 \), and grow infinite as \( x \to \infty \).
All the \( Q_n^m \)-functions are singular in \( x = 1 \), and approach zero for \( x \to \infty \).

Recurrence relations

For the numerical evaluation of the functions \( P_n^m \) and \( Q_n^m \) the following recurrence relation is useful:

\[ (n-m) P_n^m(x) = (2n-1) x P_{n-1}^m(x) - (n+m-1) P_{n-2}^m(x) \] (6-51)
which is also valid for $Q_{n}^{m}$.

Concerning the application of (6-51) to $Q_{n}^{m}$, a warning should be given however: repeated application may eventually lead to an unacceptable loss of significant figures.

Orthogonality

Arbitrary functions on the interval $-1 < x < +1$ can be expanded into a series of Legendre functions:

$$f(x) = \sum_{n=m}^{\infty} C_n P_n^m(x)$$  \hspace{1cm} (6-52)

For determining the coefficients $C_n$ use may be made of the orthogonality of the functions $P_n^m(x)$:

$$\int_{-1}^{+1} P_n^m(x).P_r^m(x) \, dx = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{n,r}$$  \hspace{1cm} (6-53)

where $\delta_{n,r}$ is Kronecker's delta.

Just as in the case of Fourier-series, the coefficients $C_n$ may be found by multiplication of expression (6-52) with $P_n^m(x)$ and integrating:

$$C_n = \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \int_{-1}^{+1} f(x) P_n^m(x) \, dx$$  \hspace{1cm} (6-54)

6.4 The field of a line distribution of dipoles

From the boundary condition

$$\mathbf{p} \wedge - \frac{g(z/b/2)}{A} \frac{\sin \chi}{r} \left( \frac{r}{b/2} \to 0 \right)$$  \hspace{1cm} (6-4)

it appears that $m = 1$ in the general solution (6-31). Since the field
must vanish at large distances ($\psi \to \infty$) and must be singular along the lifting line ($\psi = 0$), the $\psi$-dependence of the field can only be given by the functions $Q_n^1(\cosh\psi)$. In order to avoid any singularities in the rest of the field, the $\theta$-dependence is given by the functions $P_n^1(\cos\theta)$, so that:

$$p = \sum_{n=1}^{\infty} A_n P_n^1(\cos\theta) Q_n^1(\cosh\psi)$$

(6-55)

In order to determine the coefficients $A_n$, we must investigate the behaviour of (6-55) near the lifting line ($\psi \to 0$). The functions $Q_n^1(\cosh\psi)$ are given by eq. (6-50):

$$Q_n^1(\cosh\psi) = \sinh\psi \frac{dQ_n(\cosh\psi)}{d \cosh\psi}$$

(6-56)

Now $Q_n(\cosh\psi)$ may according to section 6.3 be expressed as:

$$Q_n(\cosh\psi) = P_n(\cosh\psi) Q_n(\cosh\psi) + W_n(\cosh\psi)$$

(6-57)

where $W_n(\cosh\psi)$ is a polynomial in $\cosh\psi$. Substituting this into (6-56) shows:

$$Q_n^1(\cosh\psi) = \sinh\psi \left\{ P_n(\cosh\psi) \frac{dQ_n(\cosh\psi)}{d \cosh\psi} + \frac{dP_n(\cosh\psi)}{d \cosh\psi} Q_n(\cosh\psi) + \right.$$  

$$+ \frac{dW_n(\cosh\psi)}{d \cosh\psi} \right\} =$$  

$$= -P_n(\cosh\psi) \frac{1}{\sinh\psi} + \frac{dP_n(\cosh\psi)}{d \cosh\psi} \sinh\psi \ln(\frac{\cosh\psi+1}{\sinh\psi}) +$$  

$$+ \sinh\psi \frac{dW_n(\cosh\psi)}{d \cosh\psi}$$

(6-58)

which behaves for $\psi \to 0$ (i.e. $\cosh\psi \to 1$ and $\sinh\psi \to 0$) like:

$$Q_n^1(\cosh\psi) \sim -\frac{1}{\sinh\psi} \quad (\psi \to 0)$$

(6-59)
This result is substituted into (6-55) and the eqs. (6-17) to (6-19) are applied, from which:

\[ p \sim - \frac{\sin \chi}{2\pi r} \frac{b}{2} \left\{ 1 - \left( \frac{z}{b/2} \right)^2 \right\}^{1/2} \sum_{n=1}^{\infty} A_n \frac{p_n^1(z)}{n(b/2)} \left( \frac{r}{b/2} \to 0 \right) \] (6-60)

The coefficients \( A_n \) must thus be chosen such that

\[ g\left( \frac{z}{b/2} \right) = \frac{A}{2\pi} \left\{ 1 - \left( \frac{z}{b/2} \right)^2 \right\}^{1/2} \sum_{n=1}^{\infty} A_n \frac{p_n^1(z)}{n(b/2)} \] (6-61)

or, according to (6-54):

\[ A_n = \frac{2n+1}{2} \frac{1}{n(n+1)} \frac{2\pi}{A} \int_{-1}^{1} g\left( \frac{z}{b/2} \right) \frac{p_n^1(z)}{n(b/2)} \, d\left( \frac{z}{b/2} \right) \] (6-62)

The gradient-components \( \partial p/\partial x \), \( \partial p/\partial y \) and \( \partial p/\partial z \) are given by the expressions:

\[ \frac{\partial p}{\partial x} = \frac{\cos \chi}{b/2(\sinh \frac{2\gamma}{2} + \sin 2\theta)} \left\{ \frac{\partial p}{\partial \psi} \cosh \psi \sin \theta + \frac{\partial p}{\partial \theta} \sinh \psi \cos \theta \right\} + \right. \]

\[ - \frac{\sin \chi}{b/2 \sinh \psi \sin \theta} \frac{\partial p}{\partial x} = \]

\[ = \frac{\sin 2\chi}{2\pi b(\sinh \frac{2\gamma}{2} + \sin 2\theta)} \left\{ \sin \theta \cosh \psi \sum_{n=1}^{\infty} A_n \frac{p_n^1(\cos \theta)}{n} Q_n^2(\cosh \psi) + \right. \]

\[ \left. \cos \theta \sinh \psi \sum_{n=1}^{\infty} A_n \frac{p_n^2(\cos \theta)}{n} Q_n^1(\cosh \psi) \right\} \] (6-63)

\[ \frac{\partial p}{\partial y} = \frac{\sin \chi}{b/2(\sinh \frac{2\gamma}{2} + \sin 2\theta)} \left\{ \frac{\partial p}{\partial \psi} \cosh \psi \sin \theta + \frac{\partial p}{\partial \theta} \sinh \psi \cos \theta \right\} + \right. \]

\[ + \frac{\cos \chi}{b/2 \sinh \psi \sin \theta} \frac{\partial p}{\partial x} = \]

\[ = \frac{1}{\pi b \sin \theta \sinh \psi} \sum_{n=1}^{\infty} A_n \frac{p_n^1(\cos \theta)}{n} Q_n^1(\cosh \psi) + \tan \chi \frac{\partial p}{\partial x} \] (6-64)
\[
\frac{\partial p}{\partial z} = \frac{1}{b/2(\sinh^2 \psi + \sin^2 \theta)} \left( \frac{\partial p}{\partial \psi} \sinh \psi \cos \theta - \frac{\partial p}{\partial \theta} \cosh \psi \sin \theta \right) = \frac{\sin \chi}{\pi b (\sinh^2 \psi + \sin^2 \theta)} \left( \cos \theta \sinh \psi \sum_{n=1}^{\infty} A_n P_n^1(\cos \theta) Q_n^2(\cosh \psi) + \sin \theta \cosh \psi \sum_{n=1}^{\infty} A_n P_n^2(\cos \theta) Q_n^1(\cosh \psi) \right) \quad (6-65)
\]

simplifying in the plane y = 0 to:

\[
\frac{\partial p}{\partial x} = 0 \quad (6-66)
\]

\[
\frac{\partial p}{\partial y} = \frac{1}{\pi b \sin \theta \sinh \psi} \sum_{n=1}^{\infty} A_n P_n^1(\cos \theta) Q_n^1(\cosh \psi) \quad (6-67)
\]

\[
\frac{\partial p}{\partial z} = 0 \quad (6-68)
\]

Although these forms seem at first sight rather unwieldy, actually they appear to be very handy in numerical computations. The reason is, that in all the expressions above the same building blocks reappear, viz. the functions \( P_n^1,2 \) and \( Q_n^1,2 \) which functions can be computed very efficiently thanks to the earlier mentioned recurrence relations existing for the Legendre functions.

### 6.5 Special types of dipole-distributions

Important types of dipole distributions, needed later in chapter VII and IX, are determined by:

\[
p_1(r, \chi, z) \propto p_o \frac{\sin \chi}{r/b} (r > 0, \ |z| < b/2) \quad (6-69)
\]

\[
p_2(r, \chi, z) \propto p_o \frac{\sin \chi}{r/b} (1 + \frac{z}{b/2}) (r > 0, \ |z| < b/2) \quad (6-70)
\]
The series of the preceding paragraph is not very convenient for building up these types of distributions. In these cases it is more convenient to take a constant and a "saw-tooth" distribution of sources respectively, and to differentiate the field so obtained in order to arrive at the required dipole-distributions. Thus starting with the harmonic function

\[ \phi_1 = - p_o b P_0(\cos \theta) Q_o(\cosh \psi) = \]

\[ = - p_o b Q_o(\cosh \psi) \quad (6-71) \]

we find:

\[ \frac{\partial \phi_1}{\partial y} = \frac{\sin \chi}{b/2} \frac{1}{\sinh^2 \psi + \sin^2 \theta} \left( \frac{\partial \phi_1}{\partial \psi} \cosh \psi \sin \theta + \frac{\partial \phi_1}{\partial \theta} \sinh \psi \cos \theta \right) + \]

\[ + \frac{\cos \chi}{b/2 \sinh \psi \sin \theta} \frac{\partial \phi_1}{\partial x} = \]

\[ = 2 p_o \sin \chi \frac{\cosh \psi \sin \theta}{(\sinh^2 \psi + \sin^2 \theta) \sinh \psi} \quad (6-72) \]

which displays for \( r \to 0 \) indeed the behaviour (6-69).

Also:

\[ \frac{\partial P_1}{\partial (y/b)} = 4 p_o \sin^2 \chi \left\{ \frac{\cosh \psi}{\cosh^2 \psi - \cos^2 \theta} + \frac{\cosh^3 \psi \sin^2 \theta}{(\cosh^2 \psi - \cos^2 \theta) \sinh \psi} - \frac{2 \cosh \psi \sin^2 \theta (\cosh^2 \psi + \cos^2 \theta)}{(\cosh^2 \psi - \cos^2 \theta)^2} \right\} + \]

\[ + 4 p_o \frac{\cos^2 \chi \cosh \psi}{(\cosh^2 \psi - \cos^2 \theta) \sinh^2 \psi} \quad (6-73) \]

Similarly, starting with the harmonic function

\[ \phi_2 = - p_o b \{ P_0(\cos \theta) Q_o(\cosh \psi) + P_1(\cos \theta) Q_1(\cosh \psi) \} = \]

\[ = - p_o b \{ (1 + \cosh \psi \cos \theta) Q_o(\cosh \psi) - \cos \theta \} \quad (6-74) \]
one finds:

\[ p_2 = 2 \ p_o \ \frac{siny \ \sin \theta}{\sinh \psi (\cosh \psi - \cos \theta)} \]  \hspace{1cm} (6-75)

and

\[ \frac{\partial p_2}{\partial (y/b)} = 4 \ p_o \ \frac{\sin^2 \chi}{\cosh^2 \psi - \cos^2 \theta} \ \left\{ \frac{\sinh \psi \ \cos^2 \psi - \cosh \psi \ \sin^2 \psi}{\sinh \psi (\cosh \psi - \cos \theta)} + \\
- \sin^2 \theta \ \frac{\cosh \psi + \cos \theta}{(\cosh \psi - \cos \theta)^2} \right\} + 4 \ p_o \ \frac{\cos^2 \chi}{\sinh \psi (\cosh \psi - \cos \theta)} \]  \hspace{1cm} (6-76)
VII The swept wing executing a harmonic pitching motion

7.1 Boundary value problem
As a preliminary to the analysis of the helicopter blade, we will consider in this chapter the modifications of the previous theory necessary to account for unsteady flow. Instead of developing the equations for the most general unsteady case, a special flow problem is treated, which in its essential features is very similar to the flow around the blade of a helicopter rotor in forward flight. This is the case of the uncambered, rectangular wing subject to a harmonic pitching motion. The wing is placed in a uniform flow whose unperturbed velocity direction is not perpendicular to the mid-chord line of the wing. The pitching motion is assumed of such a frequency, that its reduced frequency \( \nu = \frac{\omega b}{U} \) is of the order unity.

The coordinate systems used are illustrated in fig. 8. The \((x,y,z)\) coordinates are chosen such that the undisturbed velocity is parallel to the X-axis, whereas the quarter-chord line of the wing lies in the plane XOZ. The "wing system" is obtained by a rotation around the Y-axis, over the sweep angle \( \Lambda \). The \(Z\)-axis coincides with the quarter-chord line of the wing. The transformations relating the two systems are given by:

\[
\begin{align*}
x_w &= x \cos \Lambda - z \sin \Lambda \\
y_w &= y \\
z_w &= z \cos \Lambda + x \sin \Lambda
\end{align*}
\]

and:

\[
\begin{align*}
x &= x_w \cos \Lambda + z_w \sin \Lambda \\
y &= y_w \\
z &= z_w \cos \Lambda - x_w \sin \Lambda
\end{align*}
\]
Assuming that the wing executes a pure pitching oscillation around the quarter-chord line with frequency \( \omega \), the wingsurface is given by the equation:

\[
y_w = - \alpha_o \cos \omega t \cdot x_w (-c/4 \leq x_w \leq 3/4 c, \ |z_w| \leq b/2) \tag{7-7}
\]

so that the coordinates of a particle of air, moving over the wing-surface are given in parameterform by:

\[
x = x_w \cos \Lambda + z_w \sin \Lambda \tag{7-8}
\]

\[
y = - \alpha_o \cos \omega t \cdot x_w \tag{7-9}
\]

\[
z = - x_w \sin \Lambda + z_w \cos \Lambda \tag{7-10}
\]

\((-c/4 \leq x_w \leq 3/4 c, \ |z_w| \leq b/2)\)

The \( y \)-component of its velocity is given by:

\[
v = \frac{Dv}{Dt} = \frac{\partial y}{\partial t} + \frac{\partial y}{\partial x_w} \dot{x}_w + \frac{\partial y}{\partial z_w} \dot{z}_w =
\]

\[
= \alpha_o \omega \sin \omega t \cdot x_w - \alpha_o \cos \omega t \cdot U \cos \Lambda \tag{7-11}
\]

where use has been made of \( \dot{x}_w = U \cos \Lambda \)

The \( y \)-component of its acceleration is:

\[
\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x_w} \dot{x}_w + \frac{\partial v}{\partial z_w} \dot{z}_w =
\]

\[
= \alpha_o \omega^2 \cos \omega t \cdot x_w + 2 \alpha_o \omega \sin \omega t \cdot U \cos \Lambda \tag{7-12}
\]
Equating $Dv/Dt$ to $-1 + \frac{\partial p}{\partial y}$, the following boundary value problem is obtained for the instantaneous pressure field:

\[
\begin{align*}
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} &= 0 \\
-\frac{1}{\rho} \frac{\partial p}{\partial y} &= \alpha_0 \omega^2 \cos \omega t \cdot x_w + 2 \alpha_0 \omega \sin \omega t \cdot U \cos \Lambda
\end{align*}
\]  

on the wingsurface \((-c/4 < x_w < 3/4 c, |z_w| \leq b/2\))  

\[ p \to 0 \quad \text{for} \quad x^2 + y^2 + z^2 \to \infty \]  

\[ p \text{ singular along the leading edge } (x_w = -c/4, |z_w| \leq b/2) \]  

such that  

\[ v_{x_w=0} = -\alpha_0 \cos \omega t \cdot U \cos \Lambda \]  

\[
7.2 \text{ The pressure field of the oscillating wing, accurate to } O(A^{-2})
\]

The singular part of the pressure field may be taken from chapter V, eq. (5-35):

\[
p_{\text{sing}} = \left\{ -g(z_w,t) \frac{\sin \varphi'}{\cosh \eta' \cos \varphi'} + g(z_w,t) \pi c \frac{\sin \chi}{2\pi r} + 
\right. 
\]

\[
+ p_{\text{dip}} \left( r, \chi, z_w, t \right) \right\} + 
\]

\[
+ 2 \left( \cosh \eta' \sin \varphi' - \frac{r}{c/2} \sin \chi \right) \left\{ p_{\text{dip}}(c/4, \pi/2, z_w, t) + 
\right. 
\]

\[
+ 2 g(z_w,t) + \frac{1}{2A} \cdot \frac{d^2 g}{dz_w^2} (z_w, t). 
\]

\[
\cdot \left\{ \eta' \sin \eta' \sin \varphi' + \frac{1}{4} \sin 2\varphi' - \frac{r}{c/2} \sin \chi \ln \left( \frac{r}{c/4} \right) \right\} 
\]  

\[ (|z_w| < b/2) \]  

(7-17)
and outside the span:

\[ P_{\text{sing}} = P_{\text{dip}} (r, \chi, z_w, t) \quad (|z_w| > b/2) \]  \hspace{1cm} (7-18)

where, expressed in spheroidal coordinates \((\psi, \theta, \chi, t)\):

\[ P_{\text{dip}} = \frac{\sin \chi}{2\pi} \sum_{n=1}^{\infty} A_n(t) \frac{1}{P_n^1} (\cos \theta) Q_n^1 (\cosh \psi) \]  \hspace{1cm} (7-19)

with \(A_n(t)\) satisfying the relation:

\[ g(z_w, t) = \frac{A}{2\pi} \left\{ 1 - \frac{z_w^2}{b/2} \right\}^{\frac{1}{2}} \sum_{n=1}^{\infty} A_n(t) \frac{1}{P_n^1} \left( \frac{z_w}{b/2} \right) \]  \hspace{1cm} (7-20)

In these equations the coordinates \(\chi\) and \(r\) denote cylindrical coordinates centered around the \(z\)-axis. The prolate spheroidal coordinates \((\psi, \theta, \chi)\) are defined likewise. Since the \(z_w\)-axis lies along the quarter-chord line of the wing, the elliptical coordinates \(\eta'\) and \(\varphi'\) must be defined by:

\[ x_w = c/2 \cosh \eta' \cos \varphi' + c/4 \]  \hspace{1cm} (7-21)

\[ y_w = c/2 \sinh \eta' \sin \varphi' \]  \hspace{1cm} (7-22)

The complete pressure field also contains a regular part, associated with boundary condition (7-14). Substituting (7-21) into (7-14) and using

\[ \text{grad } p = \frac{1}{c/2(\cosh^2 \eta' - \cos^2 \varphi')} \left\{ a_n \frac{\partial p}{\partial \eta} + a \frac{\partial p}{\partial \varphi} \right\} \]  \hspace{1cm} (3-24)

the condition (7-14) is transformed into:
where \( \nu \) is the reduced frequency \( \nu = \frac{\omega b}{U} \), and \( A \) the aspect ratio. Eq. (7-23) is satisfied by the two-dimensional harmonic field

\[
\begin{align*}
\left( -\frac{\partial}{\partial \eta} \right)_{\eta = 0} &= \frac{\nu}{A} \alpha_o \sin \omega t \cos A \sin \phi' + \\
+ \frac{1}{8} \left( \frac{\nu}{A} \right)^2 \alpha_o \cos \omega t (\sin \phi' + \sin 2\phi')
\end{align*}
\]

(7-23)

Since it has been assumed that \( \nu \) is of the order \( O(1) \), the regular pressure field is of the order \( A^{-1} \). A higher order approximation would add pressure fields of the order \( (A^{-3}) \), which shows that (7-24) is already the regular near field to the required accuracy. For the same reason, only the far field of the first term in (7-24) needs consideration in the present analysis. This far field is given by

\[
P_{\text{far}} = \frac{1}{\rho U^2} \frac{\nu}{2} \alpha_o \sin \omega t \cos A \sin \phi' \frac{\cosh \psi' \sin \theta'}{(\sinh^2 \psi' + \sin^2 \theta') \sinh \psi'}
\]

(7-25)

as derived in chapter VI. The primed coordinates \( \psi', \theta', \chi' \) indicate a coordinate system centered around the mid-chord line \( x_w = c/4 \). However, since the field (7-25) is of order \( A^{-2} \), only errors of order \( A^{-3} \) are introduced when for convenience the dashes are omitted and the coordinate system is taken to be centered around the quarter-chord line \( x_w = 0 \). The regular part of the composite field is therefore to \( O(A^{-2}) \) given by:
\[ p_{\text{reg}} = \frac{1}{\rho U^2} \left( \frac{\nu}{A} \alpha_o \sin \omega t \cos \lambda \cdot \frac{\sin \chi \cosh \psi \sin \theta}{(\sinh^2 \psi + \sin^2 \theta) \sinh \psi} + \right. \\
\left. + \frac{\nu}{A} \alpha_o \sin \omega t \cos \lambda \left( e^{-\eta} \sin \varphi - \frac{1}{2} \pi c \frac{\sin \chi}{2\pi A} \right) \right) + \\
\left. + \frac{1}{8} \left( \frac{\nu}{A} \right)^2 \alpha_o \cos \omega t \left( e^{-\eta} \sin \varphi + \frac{1}{2} e^{-2\eta} \sin 2\varphi \right) \right) \\
(|z_w| < b/2) \quad (7-26) \\

Outside the wingspan \( p_{\text{reg}} \) is composed of just the far field (7-25).

7.3 The integral equation determining \( g(z_w, t) \)
If it is wished to compute the \( y \)-component of the perturbation velocity in a field point \((x_o, y_o, z_o)\) at the instant of time \( t_o \), the use of the composite field is straightforward, at least in a numerical computation. The problem is equivalent to determining the \( v \)-velocity of a particle of air coming from infinity upstream, and passing the point \((x_o, y_o, z_o)\) at time \( t_o \). During its motion, the coordinates of the particle as a function of time are given in linearized theory by:
\[ x(t) = x_o + U(t-t_o) \] (7-27)

\[ y(t) = y_o \] (7-28)

\[ z(t) = z_o \] (7-29)

so that, using (7-1) to (7-3):

\[ x_w(t) = \{x_o + U(t-t_o)\} \cos \Lambda - z_o \sin \Lambda \] (7-30)

\[ y_w(t) = y_o \] (7-31)

\[ z_w(t) = z_o \cos \Lambda + \{x_o + U(t-t_o)\} \sin \Lambda \] (7-32)

Using furthermore the inverse coordinate transformations

\[ \eta' = \text{arccosh} \left( \frac{\rho_2 + \rho_1}{c} \right) \] (7-33)

\[ \phi' = \text{arccos} \left( \frac{\rho_2 - \rho_1}{c} \right) \] (7-34)

where

\[ \rho_1 = \{(x_w - \frac{3}{4}c)^2 + y_w^2\}^{\frac{1}{2}} \] (7-35)

\[ \rho_2 = \{(x_w + c/4)^2 + y_w^2\}^{\frac{1}{2}} \] (7-36)

\[ r = \{(x_w^2 + y_w^2\}^{\frac{1}{2}} \] (7-37)

\[ \chi = \text{arcsin} \left( \frac{y_w}{r} \right) \] (7-38)

\[ \psi = \text{arccosh} \left( \frac{\rho_3 + \rho_4}{b} \right) \] (7-39)

\[ \theta = \text{arccos} \left( \frac{\rho_3 - \rho_4}{b} \right) \] (7-40)
where
\[ \rho_3 = \left( x_w^2 + y_w^2 + (z_w + b/2)^2 \right)^{\frac{1}{2}} \] (7-41)
\[ \rho_4 = \left( x_w^2 + y_w^2 + (z_w - b/2)^2 \right)^{\frac{1}{2}} \] (7-42)

the functions \( z_w(t), \eta'(t), \varphi'(t), r(t), \chi(t), \psi(t) \) and \( \theta(t) \) pertaining to the considered particle are known, and the function \( \frac{\partial p}{\partial y}(t) = \frac{\partial p}{\partial y_w}(t) \) as experienced by the particle can be found by differentiating the composite pressure field of the oscillating wing with respect to \( y_w \). A numerical integration
\[ v(x_o, y_o, z_o, t_o) = -\frac{1}{\rho} \int_{-\infty}^{t} \frac{\partial p}{\partial y_w}(t) \, dt \] (7-43)
then yields the sought velocity perturbation.

In order to calculate \( v \) in a point of the quarter-chord line of the wing at the wingsection \( z_w = z_{w_o} \), one has to take
\[ x_o = z_{w_o} \sin \alpha \] (7-44)
\[ y_o = 0 \] (7-45)
\[ z_o = z_{w_o} \cos \alpha \] (7-46)
and, from (7-30) to (7-32):
\[ x_w(t) = U(t-t_o) \cos \alpha \] (7-47)
\[ y_w(t) = 0 \] (7-48)
\[ z_w(t) = z_{w_o} + U(t-t_o) \sin \alpha \] (7-49)
In this case the numerical integration of $\partial p / \partial y (t)$ meets several difficulties however, because of the singularities of $\partial p / \partial y (t)$ at the leading edge $x = -c/4$ and at the quarter-chord line $x = 0$. The latter singularities are simplest to deal with: taking $\partial p / \partial y (t)$ due to the dipole-lines together with $\partial p / \partial y (t)$ due to the two-dimensional dipoles of the "common parts" of the pressure field, only a logarithmic singularity remains. This presents no serious difficulties in a numerical integration, if a suitable integration procedure is chosen (see chapter X).

The singularities along the leading edge must be treated analytically however. For this purpose the integral (7-43) is broken up in such a way, that some parts of the pressure field are integrated numerically only between the limits $t = -\infty$ and $t = t_1$, whereas the remaining interval $t_1 < t < t_0$ is treated analytically:

$$v(z_0, t_0) = -\frac{1}{\rho} \int_{-\infty}^{t_0} \frac{\partial p}{\partial y} (t) \, dt =$$

$$= -\frac{1}{\rho} \int_{-\infty}^{t_0} \frac{\partial}{\partial y} \left[ p_{\text{dip}} (r, x, z_0, t) + g(z_0, t) \pi c \sin \chi \frac{\sin \chi}{2\pi r} \right] +$$

$$+ \rho U^2 \frac{1}{2} \alpha_0 \sqrt{\frac{\rho}{A}} \sin \omega t \cos \chi \left( \sin \chi \cosh \psi \sin \theta - \pi c a \frac{\sin \chi}{2\pi r} \right) +$$

$$- \frac{1}{\rho} \left( \int_{-\infty}^{t_1} + \int_{t_1}^{t_0} \right) \frac{\partial}{\partial y} \left[ -g(z_0, t) \frac{\sin \varphi'}{\cosh \eta' + \cos \varphi'} \right] +$$

$$+ 2(\cosh \eta' \sin \varphi' - \frac{r}{c/2} \sin \chi) \left( p_{\text{dip}} (c/4, \pi/2, z_0, t) + 2g(z_0, t) \right) +$$

$$+ \frac{1}{2\lambda^2} \frac{d^2 g}{dz_0^2} (z_0, t) \left( \eta' \sin \eta' \sin \varphi' + \frac{1}{4} \sin 2\varphi' +$$

$$- \frac{r}{c/2} \sin \chi \xi(\frac{r}{c/4}) + \rho U^2 \frac{1}{A} \frac{\alpha_0}{\sqrt{\rho}} \sin \omega t \cos \chi e^{-\eta' \sin \varphi'} +$$

$$+ \rho U^2 \frac{1}{8} \frac{\sqrt{\rho}}{A} \sin \omega t (e^{-\eta' \sin \varphi'} + \frac{1}{2} e^{-2\eta' \sin 2\varphi'}) \right) \, dt \quad (7-50)$$
where the time $t_1$ is chosen such, that at $t = t_1$ the distance between the considered particle and the leading edge is of the order of a chord-length, i.e. $|t_1 - t_0| = 0\left(\frac{c/2}{U \cos \Lambda}\right)$.

First of all the following integral will be considered:

$$v_1 = -\frac{1}{\rho} \int_{t_1}^{t_0} \frac{\partial}{\partial y_w} \left[-g(z_w(t),t) \frac{\sin \eta'(t)}{\cosh \eta'(t) + \cosh \eta'(t)}\right] \, dt =$$

$$= \frac{1}{\rho} \int_{t_1}^{t_0} \frac{g(z_w(t),t) \, dt}{c/2 \sinh \eta'(t) \{\cosh \eta'(t) - 1\}} \quad (7-51)$$

Between $t_1$ and $t_0$ the time function $g(z_w(t),t)$ is expanded in a Taylor-series and truncated:

$$g(z_w(t),t) = g_o + \left(\frac{Dg}{Dt}\right)_o (t-t_0) + \frac{1}{2} \left(\frac{D^2g}{Dt^2}\right)_o (t-t_0)^2 \quad (7-52)$$

Substituting into (7-51) the series expansion (7-52) and

$$t-t_0 = \frac{x_w}{U \cos \Lambda} = -\frac{c/2}{U \cos \Lambda} (\cosh \eta' - \frac{1}{2}) \quad (7-53)$$

the integral $v_1$ becomes

$$v_1 = -\frac{1}{\rho} \frac{g_o}{U \cos \Lambda} \int_{\eta'_1}^{\eta'_1} \frac{d\eta'}{\cosh \eta' - 1} +$$

$$+ \frac{1}{\rho} \frac{(Dg/Dt)_o c/2}{U^2 \cos^2 \Lambda} \int_{\eta'_1}^{\eta'_1} \frac{\cosh \eta' - 1}{\cosh \eta' - 1} \, d\eta' +$$

$$- \frac{1}{\rho} \frac{(D^2g/Dt^2)_o c^2/8}{U^3 \cos^3 \Lambda} \int_{\eta'_1}^{\eta'_1} \frac{\cosh^2 \eta' - \cosh \eta' + 1}{\cosh \eta' - 1} \, d\eta' \quad (7-54)$$
where \( \eta'_1 = \text{arccosh} \left\{ \frac{t_1 - t_o}{c/2} \left( U \cos \Lambda + \frac{1}{2} \right) \right\} \) \hspace{1cm} (7-55)

One has to keep in mind the special meaning attached to these otherwise divergent integrals, as explained in chapter III, section 3.3. Using eq. (3-38) the integrals are worked out, and the result is non-dimensionalized using the relations

\[
g = g^* \rho U^2
\]

\[
t = \tau \frac{c/2}{U}
\]

resulting in:

\[
v_1/U(z_{w_0}, t_o) = -\frac{g^*(z_{w_0}, t_o)}{\cos \Lambda \coth(\eta'_1/2)} + 
\]

\[
\left( \frac{Dg^*}{Dt} \right)_o \left\{ \eta'_1 - \frac{1}{2} \coth(\eta'_1/2) \right\} + 
\]

\[
\left( \frac{D^2g^*}{Dt^2} \right)_o \left\{ \sinh \eta'_1 - \frac{1}{4} \coth(\eta'_1/2) \right\}
\]

It is useful to consider the expressions for \( g^*_o \), \( (Dg^*/Dt)_o \) and \( (D^2g^*/Dt^2)_o \) in somewhat more detail. This has been done in appendix A. The results show that \( (Dg^*/Dt)_o \) is of order \( A^{-1} \) compared with \( g^*_o \), and \( (D^2g^*/Dt^2)_o \) of order \( A^{-2} \). Thus, one would not add any relevant refinement by taking into account more terms of the Taylor expansion (7-52), since this would add to \( v_1/U \) only terms of \( O(A^{-3}) \).

Next the integral \( v_2(z_{w_0}, t_o) \) is considered:

\[
v_2(z_{w_0}, t_o) = -\frac{1}{\rho} \int_{t_1}^{t_o} \frac{\partial}{\partial y} \left[ \frac{e U^2}{A} \alpha_o \sin \omega t \cos \Lambda e^{-\eta' \sin \phi} \right] dt
\]

\hspace{1cm} (7-59)
which is treated in the same way as \( v_1(z_w, t_o) \). Since the integrand is of order \( A^{-1} \), the Taylor series expansion of \( \sin \omega t \) may be truncated after the second term, so that (7-59) is transformed into:

\[
\frac{v_2}{U(z_w, t_o)} = \frac{v}{A} \alpha \sin \omega t_o \int_{\eta'_1}^{0} e^{-\eta'_1} d\eta' +
\]

\[
- \frac{v^2}{2 A^2} \frac{\alpha \cos \omega t_o}{\cos \Lambda} \int_{\eta'_1}^{0} \left( \cosh \eta'_1 - \frac{1}{2} \right) e^{-\eta'_1} d\eta' +
\]

\[
- \frac{v}{A} \alpha \sin \omega t_o \int_{\pi}^{\varphi'} \sin \varphi' d\varphi' +
\]

\[
- \frac{v^2}{2 A^2} \frac{\alpha \cos \omega t_o}{\cos \Lambda} \int_{\pi}^{\varphi'} \left( \cos \varphi' + \frac{1}{2} \right) \sin \varphi' d\varphi' =
\]

\[
= \frac{v}{A} \alpha \sin \left( \frac{v}{2 A} t_o \right) \left( e^{-\eta'_1} - \frac{1}{2} \right) +
\]

\[
- \frac{v^2}{2 A^2} \frac{\alpha \cos \left( \frac{v}{2 A} t_o \right)}{\cos \Lambda} \left( - \frac{1}{2} \eta'_1 - \frac{1}{2} e^{-\eta'_1} + \frac{1}{4} e^{-2\eta'_1} + \frac{3}{8} \right)
\]

(7-60)

where \( \varphi'_1 = \arccos \left( \frac{1}{2} \right) \).

Finally, the integration between \( t_1 \) and \( t_o \) of the terms of order \( A^{-2} \) in (7-50) must be performed. The Taylor-series expansions to \( O(A^{-2}) \) of these terms reduce to just the first, time-independent, term. If we accept a slight error in the terms of \( O(A^{-2}) \), an obvious approximation would then be, to replace also their total contribution to \( v(z_w, t_o) \) by the quasi-steady results:
\[ v_3/U(z_w, t) = -\frac{1}{\rho U} \int_{-\infty}^{t_o} 2(p_{\text{dip}}(c/4, \pi/2, z_w, t) + 2g(z_w, t)) \, dt \]

\[ \frac{\partial}{\partial y_w} (\cosh \eta' \sin \phi' - \frac{r}{c/2} \sin \chi) \, dt \]

\[ \eta' - \frac{1}{\cos \Lambda} \{ \frac{p_{\text{dip}}}{\rho U} (c/4, \pi/2, z_w, t_o) + 2g(z_w, t_o) \} \quad (7-61) \]

\[ v_4/U(z_w, t_o) = -\frac{1}{\rho U} \int_{-\infty}^{t_o} \frac{1}{2A} \frac{d^2 g}{dz_w^2}(z_w, t) \frac{\partial}{\partial y_w} \{ \eta' \sinh \eta' \sin \phi' + \frac{1}{4} \sin 2\phi' - \frac{r}{c/2} \sin \chi \ln \left( \frac{r_{c/4}}{r} \right) \} \, dt \]

\[ \eta' - \frac{1}{2} e^{-2\eta' \sin 2\phi'} \, dt \]

\[ \eta' = \frac{3}{32} \left( \frac{\nu}{A} \right)^2 \frac{\alpha \cos \omega t_o}{\cos \Lambda} \quad (7-63) \]

The results (7-58) through (7-63) may be substituted into (7-50), and \( v/U(z_w, t) \) equated to \( -\alpha \cos \omega t \cos \Lambda \) (see eq. (7-16)), to derive the final integro-differential equation determining the function \( g(z_w, t) \).

Since the present chapter is intended solely to expose some of the principles of the unsteady analysis and to derive several results usable in the helicopter analysis, the analysis of the oscillating wing will not be pursued any further.
VIII The boundary conditions pertaining to the blade of a helicopter-rotor in steady forward flight

8.1 Coordinate systems and other notations to be used

Fig. 9 shows the coordinate systems to be used.

Firstly, there is the system of rotor axes \( X_r, Y_r, Z_r \), as shown in the figure. The \( X_OY_r \)-plane is taken as the tip path plane of the rotor.

The system of "flow-axes" \( XYZ \) (not completely shown in the figure) is obtained by a rotation about the \( Y_r \)-axis over an angle equal to the so-called rotor angle of attack \( \alpha_r \), so that the \( X \)-axis points in the direction of flight. The unperturbed flow velocity \( U \) is directed towards the negative \( X \)-axis.

There is furthermore a "local" system attached to the blades, having its origin at half the blade span. The \( X^+_b \)-axis is parallel to the tip path plane, and points against the rotational motion of the blade. The \( Z^+_b \)-axis lies along the quarter-chord line of the blade pointing towards the blade tip. In the following often the short notation \( r_b \) will be used for \( r_b = z_b + R/2 \), where \( R \) is the bladelength.

The relative position of the "local" system of axes with respect to the rotorsystem is specified by the coning angle \( \alpha_o \) and the azimuth angle \( \psi_b = \Omega t \), where the angular velocity \( \Omega \) is assumed constant, thereby ruling out any influence of "lead and lag" motion of the blades.

The rectangular blade is assumed to be rigid and uncambered. The linear blade twist is assumed to be built in such a way that the quarter-chord line of the blade is straight. The shape of the blade is thus given by the equation:

\[
y_b = \theta(z_b) \quad x_b = (\theta_r - \psi, \frac{r_b}{R}) x_b
\]  

(8-1)
where $\theta_r$ is the angle of incidence (nose-up positive) with respect to the $X_0,b,b,b$-plane, at the blade root. The angle $\varepsilon$ denotes the total twist of the blade, positive for a section incidence decreasing towards the blade tip. Because the blade executes a periodical pitching motion when moving around the azimuth, the angle $\theta_r$ is a function of time, written as:

$$\theta_r = \theta_o + b_1 \cos \psi_b - a_1 \sin \psi_b$$  \hspace{1cm} (8-2)

The signs in expression (8-2) have been chosen such, that the coefficients $a_1$ and $b_1$ correspond to the usual notations for the unit flapping angles with respect to the control plane of the rotor. The latter is the plane relative to which the blades execute a pure flapping motion, without pitching (fig. 10). The flapping angle $\beta$ is then usually represented by:

$$\beta = a_o - a_1 \cos \psi_b - b_1 \sin \psi_b$$  \hspace{1cm} (8-3)

The transformation formulae relating the coordinates $(x,y,z)$ to the coordinates $(x_b,y_b,z_b)$ are given by:

$$x = - y_b \cos a_o \sin \alpha_r + (z_b - R/2) \sin a_o \sin \alpha_r +$$

$$- \cos \psi_b \{ y_b \sin a_o \cos \alpha_r + (z_b + R/2) \cos a_o \cos \alpha_r \} +$$

$$- \sin \psi_b \; x_b \cos \alpha_r$$  \hspace{1cm} (8-4)

$$y = - \cos \psi_b \; x_b + \sin \psi_b \{ y_b \sin a_o + (z_b + R/2) \cos a_o \}$$  \hspace{1cm} (8-5)

$$z = y_b \cos a_o \cos \alpha_r - (z_b - R/2) \sin a_o \cos \alpha_r +$$

$$- \cos \psi_b \{ y_b \sin a_o \sin \alpha_r + (z_b + R/2) \cos a_o \sin \alpha_r \} +$$
The inverse transformation reads:

\[
\begin{align*}
x_b &= - x \cos_\alpha r \sin_\psi_b - y \cos_\psi_b - z \sin_\alpha r \sin_\psi_b \\
y_b &= - x (\sin_\alpha r \cos_\alpha + \cos_\alpha r \sin_\alpha \cos_\psi_b) + \\
&\quad + y \sin_\alpha \sin_\psi_b + \\
&\quad - z (\sin_\alpha r \sin_\alpha \cos_\psi_b - \cos_\alpha r \cos_\alpha) - R/2 \sin 2\alpha r \\
z_b &= x (\sin_\alpha r \sin_\alpha - \cos_\alpha r \cos_\alpha \cos_\psi_b) + \\
&\quad + y \cos_\alpha \sin_\psi_b + \\
&\quad - z (\sin_\alpha r \cos_\alpha \cos_\psi_b + \sin_\alpha \cos_\alpha) - R/2 \cos 2\alpha r
\end{align*}
\]  

(8-7)  

(8-8)

(8-9)

8.2 The velocities and accelerations of a particle moving along the blade surface

Substituting the formulae (8-1) and (8-2) into (8-4) to (8-6), an expression in parameter form is obtained for the blade surface in terms of the flow-coordinates \((x,y,z)\). In order to remain consistent with the linearized flow equations small angle assumptions are introduced, resulting in the following linearized expressions for the coordinates of a point of the bladesurface:

\[
\begin{align*}
x &= - r_b \cos_\psi_b - x_b \sin_\psi_b \\
y &= - x_b \cos_\psi_b + r_b \sin_\psi_b
\end{align*}
\]  

(8-10)  

(8-11)
\[ z = (\theta_0 - \varepsilon r_b / R) x_b - (r_b - R) a_o + \]
\[ + x_b (b_1 \cos \psi_b - a_1 \sin \psi_b) + x \alpha_r \quad (8-12) \]

The velocities of a particle of air moving along the blade surface are thus found to be:

\[ u = \frac{Dx}{Dt} = \frac{\partial x}{\partial \psi_b} \Omega + \frac{\partial x}{\partial r_b} \dot{r}_b + \frac{\partial x}{\partial x_b} \dot{x}_b = \]
\[ = -(\Omega r_b + \dot{r}_b) \cos \psi_b + (\Omega r_b - \dot{x}_b) \sin \psi_b \quad (8-13) \]

\[ v = \frac{Dy}{Dt} = \frac{\partial y}{\partial \psi_b} \Omega + \frac{\partial y}{\partial r_b} \dot{r}_b + \frac{\partial y}{\partial x_b} \dot{x}_b = \]
\[ = (\Omega r_b - \dot{x}_b) \cos \psi_b + (\Omega r_b + \dot{r}_b) \sin \psi_b \quad (8-14) \]

\[ w = \frac{Dz}{Dt} = \frac{\partial z}{\partial \psi_b} \Omega + \frac{\partial z}{\partial r_b} \dot{r}_b + \frac{\partial z}{\partial x_b} \dot{x}_b = \]
\[ = - \Omega b_1 x_b \sin \psi_b - \Omega a_1 x_b \cos \psi_b + \]
\[ + \dot{x}_b \{ b_1 \cos \psi_b - a_1 \sin \psi_b + (\theta_0 - r_b \varepsilon / R) \} + \]
\[ + \dot{r}_b \{- \varepsilon (x_b / R) - a_o \} - U \alpha_r \quad (8-15) \]

In linearized flow theory these velocity components were written as:

\[ u = - U + u' \quad (8-16) \]
\[ v = v' \quad (8-17) \]
\[ w = w' \quad (8-18) \]

where the primed quantities indicate small perturbations, of the order...
of \( U \) times a small angle. From (8-13) and (8-14) the velocities \( \dot{x}_b \) and \( \dot{r}_b \) are solved:

\[
\dot{x}_b = \Omega r_b + U \sin \psi_b - u' \sin \psi_b - v' \cos \psi_b \tag{8-19}
\]

\[
\dot{r}_b = -\Omega x_b + U \cos \psi_b - u' \cos \psi_b + v' \sin \psi_b \tag{8-20}
\]

so that, substituting into (8-15) and linearizing, the velocity component \( w \) on the bladesurface is found:

\[
w = \Omega r_b (\theta_o - r_b \varepsilon / R) + \Omega x_b \{ \varepsilon x_b / R + a_o \} - U \alpha_r - \frac{1}{2} U a_l + \\
\cos \psi_b \left[ -\Omega a_l x_b + \Omega b_l r_b - U \{ \varepsilon x_b / R + a_o \} \right] + \\
\sin \psi_b \left[ -\Omega b_l x_b + U(\theta_o - r_b \varepsilon / R) - \Omega a_1 r_b \right] + \\
+ \sin 2\psi_b \left\{ \frac{1}{2} b_l U \right\} + \cos 2\psi_b \left\{ \frac{1}{2} a_l U \right\} \tag{8-21}
\]

Substituting \( x_b = 0 \) and introducing the advance ratio \( \mu \) of the rotor:

\[
\mu = \frac{U}{\Omega R} \tag{8-22}
\]

finally results in the quarter-chord velocity \( w_{x_b=0} \) given by:

\[
\left( \frac{w}{\Omega R} \right)_{x_b=0} = r_b / R (\theta_o - \varepsilon r_b / R) - \mu a_r - \frac{1}{2} \mu a_l + \\
\cos \psi_b \left\{ b_l r_b / R - \mu a_o \right\} + \\
\sin \psi_b \left\{ - a_1 r_b / R + \mu(\theta_o - \varepsilon r_b / R) \right\} + \\
+ \sin 2\psi_b \left\{ \frac{1}{2} \mu b_l \right\} + \cos 2\psi_b \left\{ \frac{1}{2} \mu a_l \right\} \tag{8-23}
\]
The acceleration in z-direction of a particle of air moving along the blade is obtained by differentiating (8-21) and non-dimensionalizing:

\[ \frac{1}{\Omega^2 R} \frac{Dw}{Dt} = - \theta_0 x_b/R + 4 \epsilon x_b/R \cdot r_b/R + a_o r_b/R + \]
\[ + 2 \cos \psi_b \left\{ - b_1 x_b/R + \mu \theta_0 - a_1 r_b/R - 2 \mu \epsilon r_b/R \right\} + \]
\[ + 2 \sin \psi_b \left\{ a_1 x_b/R - b_1 r_b/R + 2 \mu \epsilon x_b/R + \mu a_0 \right\} + \]
\[ + 2 \mu b_1 \cos 2\psi_b - \sin 2\psi_b (\mu^2 \epsilon + 2 \mu a_1) \quad (8-24) \]

8.3 The boundary value problem for a helicopter blade

The linearized boundary value problem for the pressure field around a helicopter blade can now be formulated as follows:

\[ \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0 \quad (8-25) \]

On the blade surface, given by:

\[ x/R = - r_b/R \cos \psi_b - x_b/R \sin \psi_b \quad (8-26) \]
\[ y/R = - x_b/R \cos \psi_b + r_b/R \sin \psi_b \quad (8-27) \]
\[ z/R = a_o (1-r_b/R) - a_r r_b/R \cos \psi_b \quad (8-28) \]

\((- c/4 \leq x_b \leq c/4, \quad 0 \leq r_b/R \leq 1)\)

the component \(\partial p/\partial z\) of the pressure gradient must attain the value

\[ - \frac{1}{\rho \Omega^2 R} \frac{\partial p}{\partial z} = x_b/R.F_1(\psi_b, r_b/R) + F_2(\psi_b) + r_b/R.F_3(\psi_b) \quad (8-29) \]
where

\[ F_1(\psi_b, r_b/R) = -\theta - 2 b_1 \cos \psi_b + 2(a_1 + 2 \mu \varepsilon) \sin \psi_b + 4 \varepsilon r_b/R \]

\[ (8-30) \]

\[ F_2(\psi_b) = 2 \mu \theta \cos \psi_b + 2 \mu a_0 \sin \psi_b + 2 \mu b_1 \cos 2\psi_b + \]

\[ - \mu(\varepsilon + 2 a_1) \sin 2\psi_b \]

\[ (8-31) \]

\[ F_3(\psi_b) = a_0 - 2(a_1 + 2 \mu \varepsilon) \cos \psi_b - 2 b_1 \sin \psi_b \]

\[ (8-32) \]

Boundary condition at infinity:

\[ p \to 0 \quad \text{for} \quad x^2 + y^2 + z^2 \to \infty \]

\[ (8-33) \]

A singularity of the pressure field \((p \to -\infty)\) is required along the leading edge \(x_b = -c/4, 0 \leq r_b/R \leq 1\), of such a value that along the quarter-chord line \(x_b = 0, 0 \leq r_b/R \leq 1\) the velocity component \(w\) attains the value

\[ \left( \frac{w}{R} \right)_{x_b=0} = r_b/R(\theta - \varepsilon r_b/R) - \mu a_1 \cos \psi_b \left( b_1 r_b/R - \mu a_0 \right) + \]

\[ + \sin \psi_b \left\{ -a_1 r_b/R + \mu(\theta - \varepsilon r_b/R) \right\} + \sin 2\psi_b \left\{ \frac{1}{2} \mu b_1 \right\} + \]

\[ + \cos 2\psi_b \left\{ \frac{1}{2} \mu a_1 \right\} \]

\[ (8-34) \]
IX Solution of the boundary value problem for a helicopter blade

9.1 Asymptotic expansion of the blade's pressure field to order $A^{-2}$

For simplicity, first the case of the single-bladed rotor will be considered. Approximating the component of the pressure gradient $\partial p/\partial y_b$ by $\partial p/\partial z$, the boundary value problem for the pressure field is not very different from that of the oscillating wing treated in chapter VII.

The singular part of the pressure field is to order $O(A^{-2})$ (compare eq. (7-17)):

$$
\frac{p_{\text{sing}}}{\rho \Omega^2 R^2} = \frac{P_{\text{dip}}}{\rho \Omega^2 R^2} (r, \chi, z_b, \psi_b) + \left\{ H\left(\frac{z_b}{R/2} + 1 \right) - H\left(\frac{z_b}{R/2} - 1 \right) \right\} .
$$

$$
- \left[ g^* (z_b, \psi_b) \frac{1}{2A} \sin \chi - \frac{g^* (z_b, \psi_b)}{\cosh \eta' + \cosh \eta'} \sin \psi' + 2 \left( \cosh \eta' \sin \psi' - \frac{r}{c/2} \sin \chi \right) \left\{ \frac{P_{\text{dip}}}{\rho \Omega^2 R^2} \left( \frac{c}{4}, \frac{\pi}{2}, z_b, \psi_b \right) + 2 g^* (z_b, \psi_b) \right\} +

\right. \\
+ \frac{d}{d\left(\frac{z_b}{R/2}\right)^2} \left( z_b, \psi_b \right) \frac{1}{2A} \left\{ \eta' \sinh \eta' \sin \psi' + \frac{1}{4} \sin 2 \phi' +

\right. \\
- \frac{r}{c/2} \sin \chi \ln\left(\frac{r}{c/4}\right) \right] \tag{9-1}
$$

with

$$
\frac{P_{\text{dip}}}{\rho \Omega^2 R^2} (\psi, \theta, \chi, \psi_b) = \frac{\sin \chi}{2\pi} \sum_{n=1}^{\infty} A_n^* (\psi_b) P_n^1 (\cos \theta) Q_n^1 (\cosh \psi) \tag{9-2}
$$

$$
g^* (z_b, \psi_b) = \frac{A}{2\pi} \left\{ 1 - \left( \frac{z_b}{R/2} \right)^2 \right\}^{1/2} \sum_{n=1}^{\infty} A_n^* (\psi_b) P_n^1 \left( \frac{z_b}{R/2} \right) \tag{9-3}
$$

and $H(x)$ Heaviside's unit function, defined as unity for $x > 0$, and zero for $x < 0$.

In order to determine the regular part of the pressure field, boundary condition (8-29) is again rewritten in terms of the coordinates $\eta'$ and $\phi'$:
\[-\frac{1}{\rho \Omega^2 R^2} \frac{\partial \rho}{\partial \eta'} \bigg|_{\eta'=0} = \frac{F_1(\psi_b, r_b/R)}{8 A^2} (\sin 2\varphi' + \sin \varphi') + \]
\[
+ \{F_2(\psi_b) + r_b/R.F_3(\psi_b)\} \frac{\sin \varphi'}{2 A} \] (9-4)

The harmonic field satisfying (9-4) is to the order 0(A^{-2}) given by (compare eq. (7-26)):

\[
\frac{P_{\text{reg}}}{\rho \Omega^2 R^2} = \left[ \{F_2(\psi_b) + r_b/R.F_3(\psi_b)\} \frac{1}{2 A} \left\{e^{-\eta'} \sin \varphi' - \frac{1}{2} \pi c \frac{\sin \chi}{2\pi R} \right\} + \frac{F_1(\psi_b, r_b/R)}{8 A^2} \{e^{-\eta'} \sin \varphi' + \frac{1}{2} e^{-2\eta'} \sin 2\varphi'\} \right] \left\{H(z_b/R^2 + 1) - H(z_b/R^2 - 1)\right\} + \]
\[
+ \frac{F_2(\psi_b)}{4 A^2} \frac{\sin \chi \cos \psi \sin \theta}{(\sinh^2 \psi + \sin^2 \theta) \sinh \psi} + \frac{F_3(\psi_b)}{8 A^2} \frac{\sin \chi \sin \theta}{(\cosh \psi - \cos \theta) \sinh \psi} \] (9-5)

9.2 The integration of the pressure gradient

In order to compute the perturbation velocity along the quarter-chord line of the blade at the particular blade section \(r_{b_o}\) when the blade is in the azimuth position \(\psi_{b_o}\), one must integrate the pressure gradient "experienced" by a particle of air reaching the point \((x_o, y_o, z_o)\) at time \(t_o\) where

\[
t_o = \psi_{b_o} / \Omega \] (9-6)
\[
x_o/R = - r_{b_o} / R \cos \psi_{b_o} \] (9-7)
\[
y_o/R = r_{b_o} / R \sin \psi_{b_o} \] (9-8)
During its motion the coordinates of the particle as a function of time are given in linearized theory by:

\[
x(t) = x_o - U(t-t_o) \tag{9-10}
\]
\[
y(t) = y_o \tag{9-11}
\]
\[
z(t) = z_o \tag{9-12}
\]

Using the coordinate transformations (8-7) to (8-9), the position of the particle relative to the blade is a known function of time. In linearized form the functions \(x_b(t)\), \(y_b(t)\) and \(z_b(t)\) are thus given by:

\[
x_b/R(t) = \left\{ r_b/R \cos \psi_b + \frac{U}{R} (t - \psi_b/\Omega) \right\} \sin \psi_b(t) + \\
- \frac{r_b}{R} \sin \psi_b \cos \psi_b(t) \tag{9-13}
\]
\[
y_b/R(t) = \left\{ r_b/R \cos \psi_b + \frac{U}{R} (t - \psi_b/\Omega) \right\} \{ \alpha_r a_0 \cos \psi_b(t) \} + \\
+ \frac{r_b}{R} \sin \psi_b \cos \psi_b(t) + a_0 (1 - r_b/R) - \alpha_r \frac{r_b}{R} \cos \psi_b \tag{9-14}
\]
\[
z_b/R(t) = \left\{ r_b/R \cos \psi_b + \frac{U}{R} (t - \psi_b/\Omega) \right\} \cos \psi_b(t) + \\
+ \frac{r_b}{R} \sin \psi_b \sin \psi_b(t) - \frac{1}{2} \tag{9-15}
\]

where \(\psi_b(t) = \Omega t\). \tag{9-16}
Together with the transformations (7-33) through (7-42) — replacing of course the index \( w \) by \( b \) — the function \( \frac{\partial p}{\partial z}(t) \), \( \frac{\partial p}{\partial y_b}(t) \) experienced by the considered particle can be found by differentiating the composite pressure field with respect to \( y_b \). A numerical integration

\[
w(\mathbf{r}_b, \psi_b) = -\frac{1}{\rho} \int_{-\infty}^{\psi_b/\Omega} \frac{\partial p}{\partial z}(t) \, dt
\]

then yields the sought velocity perturbation. In the case of the helicopter blade it is convenient to non-dimensionalize the time \( t \) as follows:

\[
t = \tau/\Omega
\]

so that

\[
\frac{w}{\Omega R} (\mathbf{r}_b, \psi_b) = -\int_{-\infty}^{\psi_b/\Omega} \frac{\partial (p/\rho \Omega^2 R^2)}{\partial (z/R)} \, d\tau
\]

This integral is broken up again in the same way as shown in chapter VII for the swept, oscillating wing:

\[
\frac{w}{\Omega R} (\mathbf{r}_b, \psi_b) = -\left\{ \int_{-\infty}^{\psi_b/\Omega - \Delta \psi_b} + \int_{\psi_b/\Omega - \Delta \psi_b}^{\psi_b/\Omega} \right\} \frac{\partial (p/\rho \Omega^2 R^2)}{\partial (z/R)} \, d\tau
\]

where \( \Delta \psi_b \) represents a relative distance between the particle and the blade of the order of a chordlength. In the time-interval \( \psi_b/\Omega - \Delta \psi_b < \tau < \psi_b/\Omega \), the particle trajectory is approximated by

\[
x_b(t) = x_b(t-t_o) = (\Omega r_{b_o} + U \sin \psi_b)(t-t_o)
\]

\[
y_b(t) = 0
\]

\[
z_b(t) = z_b(t-t_o) = z_{b_o} + U \cos \psi_b(t-t_o)
\]
One can now use the results of chapter VII (first redimensionalizing, and then replacing \( U \cos \Lambda \) by \( \Omega r_b + U \sin \psi_b \)) to write down the following expression for \( \frac{w}{\Omega R} (r_b, \psi_b) \):

\[
\frac{w}{\Omega R} (r_b, \psi_b) = -\int_{-\infty}^{0} \frac{\partial}{\partial (y_b/R)} \left[ \frac{p_{\text{dip}}}{\rho \Omega^2 R^2} (r, \chi, z_b, \psi_b) \right] + \\
+ \left[ \frac{F_2(\psi_b)}{4 A^2} \frac{\sin \chi \cosh \psi \sin \theta}{(\sinh^2 \psi + \sin^2 \theta) \sinh \psi} \right] + \frac{F_3(\psi_b)}{8 A^2} \frac{\sin \chi \sin \theta}{(\cosh \psi - \cos \theta) \sinh \psi} + \\
+ \{H(\frac{z_b}{R/2} + 1) - H(\frac{z_b}{R/2} - 1)\} \left[ \frac{g^*(z_b, \psi_b)}{2 A} \frac{\sin \chi}{r/R} \right]
\]

\[
- \left[ \frac{F_2(\psi_b) + r_b/R \cdot F_3(\psi_b)}{2 A} \frac{\sin \chi}{r/c/4} \right] \frac{\psi_b - \Delta \psi}{r_b} + \\
- \left[ \frac{F_2(\psi_b)}{2 A} \frac{r_b}{r/c/4} \right] \frac{\sin \chi}{r/R} \left[ g^*(z_b, \psi_b) \frac{\sin \phi}{\cosh \eta' + \cos \phi} \right]
\]

\[
+ \frac{F_2(\psi_b) + r_b/R \cdot F_3(\psi_b)}{2 A} \frac{e^{-\eta' \sin \phi}}{r\sin \psi_b} \right] \frac{\partial}{\partial \tau} + \\
- \frac{g^* \coth(\eta_1/2)}{r_b \sin \psi_b} + \frac{(Dg^*/D\tau)_o}{2A(r_b/R + \mu \sin \psi_b)} \left[ \frac{1}{2} \coth(\eta_1/2) - \eta_1 \right] + \\
- \frac{(D^2 g^*/D \tau^2)_o}{8A^2(r_b/R + \mu \sin \psi_b)} \left[ \frac{1}{4} \coth(\eta_1/2) - \sinh \eta_1' \right] + \\
- \frac{1}{r_b/R + \mu \sin \psi_b} \left[ \frac{p_{\text{dip}}}{\rho \Omega^2 R^2} \left( \frac{c}{4} \sin \frac{\pi}{2} z_b, \psi_b \right) + 2g^*(z_b, \psi_b) \right]
\]
\[
\begin{align*}
& F_2(\psi_b) + r_b / R \frac{F_3(\psi_b)}{2A(r_b / R + \mu \sin \psi_b)} \{e^{-\eta_1'} - \frac{1}{2}\} + \\
& \frac{F_3(\psi_b) \mu \cos \psi_b + (dF_2/d\psi_b) + r_b / R (dF_3/d\psi_b)}{4A^2(r_b / R + \mu \sin \psi_b)^2} \{e^{-\frac{1}{2} \eta_1'} + \\
& + \frac{1}{4} e^{-2\eta_1'} - \frac{1}{2} e^{-\eta_1'} + \frac{3}{8}\} \frac{F_1(r_b / R, \psi_b)}{r_b / R + \mu \sin \psi_b}
\end{align*}
\]

where \( \eta_1' = \text{arccosh} \{ 2A \Delta \psi_b (r_b / R + \mu \sin \psi_b) + \frac{1}{2} \} \)

9.3 The multibladed rotor

The case of the multibladed rotor requires only a relatively small modification of the derived expression for \( w(\Omega R) \), if it is assumed that the pressure gradient induced by any blade in the vicinity of any other blade is so small, that its influence upon the solution of the near field boundary value problem may be neglected. This assumption is evidently only permissible if the blades are not spaced too closely, i.e. if the total number of blades is not very large.

Under these assumptions \( w(\Omega R) \) as derived for the single-bladed rotor must be supplemented by:

\[
\begin{align*}
& \Delta \frac{w}{\Omega R}(r_b, \psi_b) = - B - 1 \sum_{b=1}^{\psi_b} \int_{\psi_b}^{\psi_b} \frac{\partial}{\partial (y_b / R)} \left[ \frac{Pd\rho}{\rho \mu^2 \gamma} (r_b, \chi, z_b, \psi_b) + \\
& + \frac{F_2(\psi_b)}{4A^2 (\sin \psi_b \cos \psi_b \sin \theta) + \frac{F_3(\psi_b)}{8A^2 \left( (\cos \psi_b - \cos \theta) \sinh \psi \right)} + \\
& + \{H(z_b / R/2) + H(z_b / R/2 - 1)\} - g(z_b, \psi_b) \left( \frac{\sin \psi'}{\cosh \eta' + \cos \psi'} \right) - \frac{1}{2A} \frac{\sin \chi}{\Omega / R}\right]
\end{align*}
\]
whereas in the accompanying transformation formulae (9-10) to (9-16) only (9-16) needs replacement by:

\[
\psi_b(t) = \Omega t + b \frac{2\pi}{B}
\]  

(9-27)

where \( B \) is the total number of blades.

It will be noted that in the above expression the near-field terms of order \( O(A^{-2}) \) have been neglected, since their contribution may be expected to be very small.

9.4 Some remarks on the numerical solution of the resulting integral equation

An integral equation for the unknown function \( g^*(r_b, \psi_b) \) is obtained by equating \( \frac{W}{\Omega} \) \( (r_b, \psi_b) \) as determined by the equations (9-24) and (9-26) to \( \frac{W}{\Omega} \) \( x_b = 0 \) as specified by eq. (8-34).

The most straightforward method for solving the resulting equation is, to use a collocation method. One tries to satisfy the equation only in a finite number of points \( \left( \frac{z_b}{R/2} \right) (r = 1, 2, \ldots, r_{\text{max}}) \) along the blade, at a finite number of azimuth positions \( \psi_b \) \( (s = 1, 2, \ldots, S) \).

The function \( g^*(z_b, \psi_b) \) will then be represented by a finite sum, instead of by the infinite series (9-3). Now a convenient refinement appears to be, to write \( g^*(z_b, \psi_b) \) as the sum of a linear function of \( z_b \) and a sum of Legendre-functions:
\[ g(z, \psi_b) = A \left( 1 + \frac{z_b}{R/2} \right) \left[ A_{o, o} + \sum_{k=1}^{K} \left\{ A_{o, k} \cos(k \psi_b) + B_{o, k} \sin(k \psi_b) \right\} \right] + \]

\[ + \frac{A}{2} \frac{\pi}{1 - (\frac{z_b}{R/2})^2} \sum_{n=1}^{N} \left[ A_{n, o} + \sum_{k=1}^{K} \left\{ A_{n, k} \cos(k \psi_b) + B_{n, k} \sin(k \psi_b) \right\} \right] \]

where \( 2K + 1 = S \),

and \( N + 1 = r_{\text{max}} \)

(9-28)

Using expression (9-28) for the assumed form of \( g(z, \psi_b) \), the following expressions are obtained for \( \frac{p_{\text{dip}}}{\rho \Omega^2 R^2} (r, \chi, z, \psi_b) \), \( (Dg^{\star} / Dt)_o \) and \( (D^2 g^{\star} / Dt^2)_o \) occurring in eqs. (9-24) and (9-26):

\[ \frac{p_{\text{dip}}}{\rho \Omega^2 R^2} (r, \chi, z, \psi_b) = \]

\[ = - \frac{\sin \chi}{\sinh \psi (\cosh \psi - \cos \theta)} \left[ A_{o, o} + \sum_{k=1}^{K} \left\{ A_{o, k} \cos(k \psi_b) + B_{o, k} \sin(k \psi_b) \right\} \right] + \]

\[ + \frac{\sin \chi}{2\pi} \sum_{n=1}^{N} \left[ A_{n, o} + \sum_{k=1}^{K} \left\{ A_{n, k} \cos(k \psi_b) + B_{n, k} \sin(k \psi_b) \right\} \right] \]

(9-31)

\[ (Dg^{\star} / Dt)_o = 2\mu \cos \psi_b A_s \left[ A_{o, o} + \sum_{k=1}^{K} \left\{ A_{o, k} \cos(k \psi_b) \right\} + \right. \]

\[ + B_{o, k} \sin(k \psi_b) \left\} \right] + A \left( 1 + \frac{z_b}{R/2} \right) \sum_{k=1}^{K} \left\{ A_{o, k} \sin(k \psi_b) + B_{o, k} \cos(k \psi_b) \right\} + \]

\[ - 2\mu \cos \psi_b A_s \left[ A_{o, o} + \sum_{n=1}^{N} \left\{ A_{n, n+1} \right\} \right. \]

(9-32)
\[ \left[ A_{n,o} + \sum_{k=1}^{K} \{ A_{n,k} \cos(k\psi_b) + B_{n,k} \sin(k\psi_b) \} \right] + \\
+ \frac{A}{2\pi} \left\{ 1 - \left( \frac{Z_b}{R/2} \right)^2 \right\}^{1/2} \sum_{n=1}^{N} P_{n}^1 \left( \frac{Z_b}{R/2} \right) \sum_{k=1}^{K} k \{ -A_{n,k} \sin(k\psi_b) + B_{n,k} \cos(k\psi_b) \} \]

\[ (D \psi /D\tau)^o = 4\mu \cos\psi_b \sum_{k=1}^{K} k \{ -A_{o,k} \sin(k\psi_b) + B_{o,k} \cos(k\psi_b) \} + \\
-4\mu^2 \cos^2\psi_b \sum_{n=1}^{N} n(n+1) P_n^r \left( \frac{Z_b}{R/2} \right) \]

\[ \left[ A_{n,o} + \sum_{k=1}^{K} \{ A_{n,k} \cos(k\psi_b) + B_{n,k} \sin(k\psi_b) \} \right] + \\
-4\mu \cos\psi_b \sum_{n=1}^{N} n(n+1) P_n^r \left( \frac{Z_b}{R/2} \right) \sum_{k=1}^{K} k \{ -A_{n,k} \sin(k\psi_b) + B_{n,k} \cos(k\psi_b) \} + \\
+ B_{n,k} \cos(k\psi_b) - \frac{A}{2\pi} \left\{ 1 - \left( \frac{Z_b}{R/2} \right)^2 \right\}^{1/2} \sum_{n=1}^{N} P_{n}^1 \left( \frac{Z_b}{R/2} \right) \sum_{k=1}^{K} k \{ -A_{n,k} \sin(k\psi_b) + B_{n,k} \cos(k\psi_b) \} + \\
+ B_{n,k} \sin(k\psi_b) \]

Substituting the above expressions into eqs. (9-24) and (9-26), one may compute by numerical integration the so-called "influence coefficients" \((\omega_{R,r,s}^{w,n,k})\) and \((\omega_{R,r,s}^{w,n,k})\). These coefficients denote the value of
\[ \frac{w}{\Omega R} \] in the collocation point \( \{ \frac{z_b}{R/2}, \psi_b \} \) due to a function \( g^*(z_b, \psi_b) \) in which all the coefficients (and also the functions \( F_1, F_2 \) and \( F_3 \)) have been set equal to zero, except the coefficient \( A_{n,k} \) (or \( B_{n,k} \)) which is chosen as unity.

The functions \( F_1(\psi_b, r_b/R), F_2(\psi_b) \) and \( F_3(\psi_b) \) occurring in (9-24) and (9-26) depend upon the flapping coefficients \( a, a_1 \) and \( b_1 \), as well as on the parameters defining the geometry and operating conditions of the rotor. One may now compute in a similar manner the influence coefficients, where \( (\frac{w}{\Omega R})_{r, s+1} \) denotes the influence coefficient associated with the terms occurring in \( F_1, F_2 \) and \( F_3 \) that are independent of \( a, a_1 \) or \( b_1 \) (see eqs. (8-30) to (8-32)).

The calculation of the influence coefficients can be economized considerably by taking two precautions during the numerical integrations:

1) The integrands will show a number of relatively large "peaks" (and actually a logarithmic singularity at \( \tau=\psi_b \)), due to one of the rotor blades passing closely underneath the particle of air whose perturbation velocity is calculated. The numerical integrations can now be speeded up considerably, by "synchronizing" the integration steps with the "peaks" of the integrand, such that the peaks coincide with the ends of the integration intervals. A numerical integration procedure must be chosen which is able to deal with the - infinite - peak at \( \tau=\psi_b \). A suitable procedure is the Gauss-Chebyshev integration. The peak-times of the integrand will be denoted by \( \tau_j \) \((j=0,1,2,\ldots)\) with \( \tau_{j+1} < \tau_j \), while \( \tau_0 = \psi_b \). Denoting the integrand in any of the integrations with respect to \( \tau \) by \( G(\tau) \), the Gauss-Chebyshev formula reads:

\[
\int_{\tau_{j+1}}^{\tau_j} G(\tau) \, d\tau = \frac{1}{I} \sum_{i=1}^{I} G(\tau_{ij}) \left\{ (\tau_{ij} - \tau_{j+1}) (\tau_j - \tau_{ij}) \right\}^{\frac{1}{2}}
\]

(9-34)
where \( \tau_{ij} = \frac{\tau_{i,j} + \tau_{j,i} + 1}{2} + \frac{\tau_{i,j} - \tau_{j,i} + 1}{2} \cos \left( \frac{(2i-1)\pi}{2I} \right) \) (9-35)

and the \( \tau_{j}'s \) are the solutions of the equation:

\[
tg \left[ \tau_{j} + j \cdot \frac{2\pi}{B} - \left( H(\psi_b - \pi) + \frac{1}{2} \right) \right] = \frac{(r_b/R)_s \cos \psi_b + \mu(\tau_{j} - \psi_b)_s}{(r_b/R)_s \sin \psi_b_s}
\]

\((\sin \psi_{b_s} \neq 0)\) (9-36)

or:

\[\tau_{j} = H \left( \psi_{b_s} - \frac{\pi}{2} \right) \tau_{j} - j \cdot \frac{2\pi}{B} \quad (\sin \psi_{b_s} = 0)\] (9-37)

2) The evaluation of the integrands at any time \( \tau_{ij} \) requires the evaluation of \( P_n^l(\cos \theta(\tau_{ij})) \) and \( Q_n^l(\cosh \theta(\tau_{ij})) \). Now these Legendre functions are most conveniently evaluated by means of the recurrence relations given in chapter VI. Although not strictly necessary, it is also possible to evaluate \( \cos \{k \psi_b(\tau_{ij})\} \) and \( \sin \{k \psi_b(\tau_{ij})\} \) by means of recurrence relations. This means that during the computation of the influence coefficient \( \left( \frac{\psi}{\omega} \right)^{n+k}_{r,s+A-n,k} \) at the same time all the numerical data become available to compute simply the influence coefficients \( \left( \frac{\psi}{\omega} \right)^{n+k}_{r,s+A-n,k} \) \((n=1,2,...,N, k=0,1,...,K)\)

Having evaluated the influence coefficients, the integral equation for the function \( g(z, \psi) \) reduces to a set of \( r_{\text{max}} \) linear algebraic equations:

\[
\sum_{n=0}^{N} \left( \frac{\psi}{\omega} \right)^{n+k}_{r,s+A-n,k} A_{n,o} + \sum_{k=1}^{K} \left( \frac{\psi}{\omega} \right)^{n+k}_{r,s+A-n,k} A_{n,k} + \left( \frac{\psi}{\omega} \right)^{n+k}_{r,s+a-o} a_{n,k} + \left( \frac{\psi}{\omega} \right)^{n+k}_{r,s+a_1} a_{n,k}
\]
These equations must be supplemented by three "aero-elastic" equations relating the flapping coefficients $a_0, a_1, b_1$ to the pressure distribution over the blades. According to eq. (9-1) and (9-5), the pressure distribution over the blades is given by:

\[
\frac{p}{\rho \Omega^2 R^2} = -g(z_b, \psi_b) \frac{\sin \varphi'}{1 + \cos \varphi'} + 2 \sin \varphi' \frac{p_{\text{dirp}}}{\rho \Omega^2 R^2} \left( \frac{c}{4} \frac{\pi}{2}, z_b, \psi_b \right) + \\
2g(z_b, \psi_b) + \frac{d}{d(z_b^2) R/2} \frac{1}{8 A^2} \sin 2 \varphi' + \\
\frac{F_2(\psi_b) + r_b/R F_3(\psi_b)}{2 A} \sin \varphi' + \frac{F_1(\psi_b, r_b/R)}{8 A^2} (\sin \varphi' + \frac{1}{2} \sin 2 \varphi')
\]

(9-39)

so that the lift distribution is, counting the lift positive in negative $Y_b$-direction:
The aerodynamic moment $M_a$ of the lift with respect to the bladeroot is determined by performing the integration

\[
\frac{M_a(\psi_b)}{\rho \Omega^2 R^4} = \int_0^1 \frac{\ell(z_b, \psi_b)}{\rho \Omega^2 R^2} \cdot \frac{F_2(\psi_b) + r_b/R \cdot F_3(\psi_b)}{2A} + \frac{F_1(\psi_b, r_b/R)}{8A^2} \, (r_b/R) \, d(r_b/R) =
\]

\[
= \pi c \left( \frac{2}{3} A_0 \right) \left[ A_{1,0} + \sum_{k=1}^{K} \cos(k \psi_b) + B_{1,k} \sin(k \psi_b) \right] +
\]

\[
+ \frac{1}{6} c A \left[ A_{1,0} + \sum_{k=1}^{K} \cos(k \psi_b) + B_{1,k} \sin(k \psi_b) \right] +
\]

\[
+ \frac{1}{10} c A \left[ A_{2,0} + \sum_{k=1}^{K} \cos(k \psi_b) + B_{2,k} \sin(k \psi_b) \right] +
\]

\[
+ \frac{c}{2} \Sigma_{n=1}^{N} \left[ A_{n,0} + \sum_{k=1}^{K} \cos(k \psi_b) + B_{n,k} \sin(k \psi_b) \right] +
\]

\[
+ \frac{\pi c}{8A^2} F_2(\psi_b) + \frac{\pi c}{12A^2} F_3(\psi_b) +
\]

\[
+ \frac{\pi c}{32A^2} \left\{ -a_1^2 \cos \psi_b + 2(a_1 + 2 \mu \varepsilon) \sin \psi_b \right\} + \frac{\pi c}{12A^2} \varepsilon (9-41)
\]

where

\[
I_o = \int_0^1 \frac{\sin \theta^*}{\sinh \psi^*(\cosh \psi^* - \cos \theta^*)} \, (r_b/R) \, d(r_b/R) (9-42)
\]
\[
I_n = \int_0^1 p_n^1(\cos \theta^*) Q_n^1(\cosh \psi^*) (r_b/R) d(r_b/R)
\]  
(9-43)

with \( \theta^* \) and \( \psi^* \) defined by:

\[
\cos \theta^* = (\rho_3/R)^* - (\rho_4/R)^*
\]  
(9-44)

\[
\cosh \psi^* = (\rho_3/R)^* + (\rho_4/R)^*
\]  
(9-45)

\[
(\rho_3/R)^* = \left( -\frac{1}{16A^2} + (r_b/R)^2 \right)^{1/4}
\]  
(9-46)

\[
(\rho_4/R)^* = \left( \frac{1}{2} + (r_b/R-1)^2 \right)^{1/4}
\]  
(9-47)

For centrally hinged blades it can be derived (ref. 11) that both the \( \cos \psi_b^- \) - and \( \sin \psi_b^- \) -components of \( M_{a,b} \) should be zero, while the constant component \( M_{a,c} \) should equal

\[
M_{a,c} = I \Omega^2 a_0
\]  
(9-48)

or:

\[
M_{a,c} = \frac{2\pi}{\gamma} \frac{c}{\rho} a_0
\]  
(9-49)

where \( I \) is the moment of inertia of the blade with respect to the bladeroot, and \( \gamma \), "Lock's number", is defined as

\[
\gamma = 2\pi \rho c R^4
\]  
(9-50)

The three "aero-elastic" equations completing the system (9-38) thus read:

\[
\left( \frac{2}{3} A-I_0 \right) a_0 + \frac{A}{6\pi} A_{1,o} + \frac{A}{10\pi} A_{2,o} + \frac{1}{2\pi} \sum_{n=1}^{N} I_n A_{n,o} +
\]

\[
+ \left( \frac{1}{12A} - \frac{2}{\gamma} \right) a_0 = \frac{\theta_0}{32A^2} - \frac{\varepsilon}{12A^2}
\]  
(9-51)
The set of algebraic equations (9-38) supplemented by (9-51) through (9-53) can now be solved by any of the available standard methods, to obtain the coefficients $A_{n,k}$, $B_{n,k}$, $a_0$, $a_1$ and $b_1$.

It should be recognized that the system is strictly speaking non-linear: in order to determine the influence-coefficients in the algebraic equations (9-38), one must already know the value of the flapping coefficients $a_0$, $a_1$ and $b_1$. Since the influence of the flapping-coefficients is not very large, it is sufficient to base the computation of $A_{n,k}$ and $B_{n,k}$ upon an initial estimate of $a_0$, $a_1$ and $b_1$. If necessary, the whole computation might be iterated.

9.5 Numerical results
For a typical case, some results are shown in figure 11. Fig. 11 shows some isobar contours on the surface of a blade at the advancing side of the rotordisc. From the latter figure two important conclusions can be drawn:

1) The physical assumptions underlying the lifting line approximations were, that the characteristic length scale for spanwise pressure variations is the wingspan, and the characteristic length scale for chordwise pressure variations is the chordlength. These assumptions can be seen to be justified up to blade stations very close to the blade tip.
2) The higher-order terms of the pressure field, i.e. the terms of order $O(A^{-2})$, affect rather strongly the pressure distribution in the tip region. Their effect is such, that for a given lift coefficient the leading edge pressure peak is increased, compared with the purely two-dimensional pressure distribution.

It should be emphasized that the results such as shown are just preliminary. The number of collocation points used (11 azimuth-wise and 5 spanwise) are really insufficient to avoid oscillations of the loading function $g^*(z_b, \psi_b)$ in spanwise direction, although artificially "smoothed out" curves of $g^*$ do not appear to be very sensitive to the particular choice of spanwise collocation positions, on condition that the flapping coefficients $a_0$, $a_1$ and $b_1$ are kept fixed. The pressure plot in fig. 11 has been based on such a "smoothed out" function $g^*(z_b, \psi_b)$.

In a separate computer program the function $g^*$ thus obtained was approximated by a larger number of Legendre functions, before the surface pressure distribution was computed.
10.1 Higher-order lifting line theories

Having obtained the full higher-order lifting line expressions for the case of the helicopter blade, including all relevant unsteady phenomena of inviscid theory, we can now judge the validity of existing methods based on (velocity) potential theory, derived in a partially intuitive manner.

The existing higher-order lifting line method, sometimes applied to the case of the helicopter rotor, is Weissinger's 3/4-chord point method. This method has been considered already in chapter V, and was shown to be exact up to and including terms of $O(A^{-2})$ under certain conditions. These conditions are not satisfied, however, when the method is applied to the helicopter blade, since in unsteady flows Pistolesi's theorem, which was needed for the derivation of the method, does not remain valid.

Another disadvantage of the 3/4-chord point method is, that it does not provide information about the pressure distribution over the wing surface, so that pitching moments cannot be analyzed.

10.2 Classical lifting line approaches to rotor flow analysis

From chapter IX it follows that the pressure field of a helicopter blade is, accurate to $O(A^{-1})$:

\[
\frac{p}{\rho c_R^2} = -g^*(z_b, \psi_b) \frac{\sin \varphi'}{\cosh \eta + \cos \varphi'} + \frac{\partial p}{\rho c_R^2} \partial \frac{\partial}{\partial x, z_b, \psi_b} + \frac{\partial p}{\rho c_R^2} \partial \frac{\partial}{\partial x, z_b, \psi_b} + \frac{\partial p}{\rho c_R^2} \partial \frac{\partial}{\partial x, z_b, \psi_b}
\]

which gives on the blade section $(z_b, \psi_b)$ rise to a perturbation velocity component in $Y_b$-direction:

\[
\frac{\partial (z_b, \psi_b)}{\partial (z_b, \psi_b)} = - \int_{-\infty}^{\psi_b} \frac{\partial (p/\rho c_R^2)}{\partial (z_b)} d\tau
\]
to be equated to \( \frac{w}{\Omega R} x_b =_0 \) as written out in eq. (8-34).

Now one may split off from expression (10-1) a field

\[
\frac{p_1}{\rho \Omega R^2} = -g(z_b, \psi_b) \frac{\sin \psi'}{\cosh \eta' + \cos \psi'}
\]

(10-3)

thus writing (10-1) in the form:

\[
\frac{p}{\rho \Omega R^2} = -g(z_b, \psi_b) \frac{\sin \psi'}{\cosh \eta' + \cos \psi'} - \{g(z_b, \psi_b) - g(z_b, \psi_b)\}.
\]

\[
\cdot \frac{\sin \psi'}{\cosh \eta' + \cos \psi'} + \frac{p_{\text{dip}}}{\rho \Omega R^2} (r, \chi, z_b, \psi_b) + \frac{g(z_b, \psi_b)}{2A} \frac{\sin \chi}{r/R} +
\]

\[
+ \frac{F_2(\psi_b) + r_b/R F_3(\psi_b)}{2A} e^{-\eta' \sin \psi'}
\]

(10-4)

Since the azimuth-position \( \psi_b \) of the blade is a function of time, the field (10-3) represents the field associated with a two-dimensional aerofoil whose lift is a periodic function of time. The downwash \( \frac{w_1}{\Omega R}(z_b, \psi_b) \) associated with this periodic part of the pressure field depends on the time-function \( g(z_b, \psi_b) \) via the two-dimensional functional relationship:

\[
\frac{w_1}{\Omega R}(z_b, \psi_b) = f_{\text{two-dim}} \{g(z_b, \psi_b)\} =
\]

\[
= \int_{-\infty}^{\psi_b} g(z_b, \psi_b) \frac{\partial}{\partial(z/R)} \frac{\sin \psi'}{\cosh \eta' + \cos \psi'} \, d\tau
\]

(10-5)

so that the integral equation as stated by (10-1), (10-2) and (8-34) may be written like:
\[ \frac{w}{\Omega R} x_b =_0 = \int_{\text{two-dim}} \left[ g (z_{b_0}, \psi_{b_0}) + \frac{w_i}{\Omega R} (z_{b_0}, \psi_{b_0}) \right] \psi_{b_0} \]

\[ - \int_{-\infty}^{\infty} \frac{\partial}{\partial (z/R)} \left[ -g (z_{b_0}, \psi_{b_0}) - g (z_{b_0}, \psi_{b_0}) \right] \frac{\sin \phi'}{\cosh \sigma' + \cos \phi'} + \]

\[ \frac{F_2 (\psi_{b_0}) + r_{b_0} / R F_3 (\psi_{b_0})}{2 A} e^{-\eta' \sin \phi'} \right] d\tau \quad (10-6) \]

where \( \frac{w_i}{\Omega R} (z_{b_0}, \psi_{b_0}) \) is the so-called "induced downwash", associated with the far field and common-part terms in (10-1). If the integral in the right-hand side of (10-6) is neglected, the solution of the integral equation may be written symbolically:

\[ g (z_{b_0}, \psi_{b_0}) = \int_{\text{two-dim}} \left[ \frac{w}{\Omega R} x_b =_0 - \frac{w_i}{\Omega R} (z_{b_0}, \psi_{b_0}) \right] \quad (10-7) \]

which means, that the lift-distribution function \( g (z_{b_0}, \psi_{b_0}) \) follows from the two-dimensional unsteady relations between the periodically time-varying lift and "effective" angle of attack of the considered section \( z_{b_0} \).

Expression (10-7) corresponds with a commonly known procedure in helicopter rotor analysis: one accounts for unsteady effects by calculating the periodically time fluctuating effective angle of attack of the blade section, and equating the lift of the section to the lift as experienced by a two-dimensional aerofoil whose angle of attack varies in the same way as the effective angle of attack of the considered blade section. The present analysis indicates that serious errors may be incurred by this procedure: several terms, even a term of order \( O(A^0) \), are in fact neglected in the full integral equation.
XI Conclusions

1) The common practice in the classical lifting line analysis of helicopter blades to account for unsteady flow is, to equate the unsteady lift of a blade section to the unsteady lift of a two-dimensional aerofoil which moves through a periodical gust field, where the gust distribution corresponds to the time-variations of the induced velocity. This procedure is incorrect. It introduces errors of the order $O(A^0)$ into the lifting line analysis, which itself may be accurate up to $O(A^{-1})$.

2) The 3/4-chord lifting line method can be proved to be exact up to the order $O(A^{-2})$ under certain conditions. These conditions are not satisfied when the 3/4-chord method is applied to helicopter blades.

3) Using the theory of the acceleration potential in combination with a matched asymptotic expansion technique, lifting line methods can be developed for application to the helicopter blade which avoid all the above mentioned problems.

4) The methods mentioned under 3) are efficient in numerical computations, since the two-dimensional integrations over the skewed helical vortex sheets needed in the velocity method are reduced to one-dimensional integrations using the acceleration potential.

5) The matched asymptotic expansion treatment of lifting line theory yields the complete pressure distribution over the helicopter blades, which is an advantage over existing lifting line methods.

6) Preliminary results indicate that the pressure distribution over the tip region of a blade deviates rather much from two-dimensional distributions. This observation could be of importance in relation to the development of blade sections for high critical Mach-numbers.
XII References


4. R. Dat: Représentation d'une ligne portante animée d'un mouvement arbitraire par une ligne de doublets d'accélération, La Recherche Aérospatiale, nov.-déc. 1969.


Appendix A, Series expressions for the function $g(z, t)$ and its derivatives

According to eq. (7-20), the function $g(z, t)$ may be written like:

$$g^*(z, t) = \frac{A}{2\pi} \left\{ 1 - \left(\frac{z}{b/2}\right)^2 \right\} \sum_{n=1}^{\infty} A_n^*(t) P_n^\prime \left(\frac{z}{b/2}\right)$$  \hspace{1cm} (A-1)

so that (see eq. (6-26)):

$$\frac{\partial g^*}{\partial (\frac{z}{b/2})} = \frac{A}{2\pi} \left\{ -2 \left(\frac{z}{b/2}\right) \sum_{n=1}^{\infty} A_n^*(t) P_n^\prime \left(\frac{z}{b/2}\right) + \right.$$  

$$+ \left\{ 1 - \left(\frac{z}{b/2}\right)^2 \right\} \sum_{n=1}^{\infty} A_n^*(t) P_n^{\prime\prime} \left(\frac{z}{b/2}\right) \right\} =$$

$$= - \frac{A}{2\pi} \sum_{n=1}^{\infty} A_n^*(t) n(n+1) P_n \left(\frac{z}{b/2}\right)$$  \hspace{1cm} (A-2)

and

$$\frac{\partial^2 g^*}{\partial (\frac{z}{b/2})^2} = - \frac{A}{2\pi} \sum_{n=1}^{\infty} A_n^*(t) n(n+1) P_n^\prime \left(\frac{z}{b/2}\right)$$  \hspace{1cm} (A-3)

If $A_n^*(t)$ is written in the form of a Fourier series:

$$A_n^*(t) = \sum_{k=0}^{\infty} \{ A_{n, k} \cos(k \omega t) + B_{n, k} \sin(k \omega t) \} =$$

$$= \sum_{k=0}^{\infty} \{ A_{n, k} \cos(\frac{k \omega \tau}{2A}) + B_{n, k} \sin(\frac{k \omega \tau}{2A}) \}$$  \hspace{1cm} (A-4)

then

$$g_0^* = \frac{A}{2\pi} \left\{ 1 - \left(\frac{z_0}{b/2}\right)^2 \right\} \sum_{n=1}^{\infty} A_n^* \left(\frac{z_n}{b/2}\right) \sum_{k=0}^{\infty} \left\{ A_{n, k} \cos(\frac{k \omega \tau}{2A}) + \right.$$  

$$+ B_{n, k} \sin(\frac{k \omega \tau}{2A}) \right\}$$  \hspace{1cm} (A-5)
\[
\frac{\partial g}{\partial t}_o = \frac{\partial g}{\partial \theta} \frac{\sin \Lambda}{A} + \frac{\partial g}{\partial \tau} \frac{\sin \Lambda}{A}
\]

\[
= -\frac{\sin \Lambda}{2\pi} \sum_{n=1}^{\infty} n(n+1) \frac{z_{\theta o}^2}{\pi A} \sum_{k=0}^{\infty} \frac{z_{\tau o}^2}{\pi A} + \frac{\sin \Lambda}{\pi A} \sum_{n=1}^{\infty} \frac{z_{\theta o}^2}{\pi A} \sum_{k=0}^{\infty} \frac{z_{\tau o}^2}{\pi A}
\]

\[
+ B_n k \sin \left(\frac{\tau o}{2A}\right) + \frac{\sin \Lambda}{2\pi} \sum_{n=1}^{\infty} n(n+1) \frac{z_{\theta o}^2}{\pi A} \sum_{k=0}^{\infty} \frac{z_{\tau o}^2}{\pi A} + k \tau o
\]

\[
\cdot \left\{ -A_n k \sin \left(\frac{\tau o}{2A}\right) + B_n k \cos \left(\frac{\tau o}{2A}\right) \right\} \quad (A-6)
\]

and

\[
\frac{\partial^2 g}{\partial \tau^2}_o = \frac{\partial^2 g}{\partial \theta^2} \frac{\sin \Lambda}{A} + 2 \frac{\partial^2 g}{\partial \theta^2 \partial \tau} \frac{\sin \Lambda}{A} + \frac{\partial^2 g}{\partial \tau^2} \frac{\sin \Lambda}{A}
\]

\[
= -\frac{\sin^2 \Lambda}{2\pi A} \sum_{n=1}^{\infty} n(n+1) \frac{z_{\theta o}^2}{\pi A} \sum_{k=0}^{\infty} \frac{z_{\tau o}^2}{\pi A} + \frac{\sin^2 \Lambda}{\pi A} \sum_{n=1}^{\infty} \frac{z_{\theta o}^2}{\pi A} \sum_{k=0}^{\infty} \frac{z_{\tau o}^2}{\pi A}
\]

\[
+ \frac{\sin \Lambda}{2\pi} \sum_{n=1}^{\infty} n(n+1) \frac{z_{\theta o}^2}{\pi A} \sum_{k=0}^{\infty} \frac{z_{\tau o}^2}{\pi A} + k \tau o
\]

\[
\cdot \left\{ -A_n k \sin \left(\frac{\tau o}{2A}\right) + B_n k \cos \left(\frac{\tau o}{2A}\right) \right\} +
\]

\[
\frac{\nu^2}{8\pi A} \left\{ 1 - \frac{z_{\theta o}^2}{\pi A} \right\} \sum_{n=1}^{\infty} n \frac{z_{\tau o}^2}{\pi A} \sum_{k=0}^{\infty} k^2 \left\{ -A_n k \cos \left(\frac{\tau o}{2A}\right) +\right\}
\]

\[
- \frac{\nu^2}{8\pi A} \left\{ 1 - \frac{z_{\theta o}^2}{\pi A} \right\} \sum_{n=1}^{\infty} n \frac{z_{\tau o}^2}{\pi A} \sum_{k=0}^{\infty} k^2 \left\{ -A_n k \cos \left(\frac{\tau o}{2A}\right) +\right\}
\]

\[
\cdot \left\{ -A_n k \sin \left(\frac{\tau o}{2A}\right) \right\} \quad (A-7)
\]
Fig. 1: Wake vorticity of a helicopter blade.
Fig. 2: Representation of a two-dimensional thin aerofoil by a pressure dipole distribution
Fig. 3: Representation of a three-dimensional lifting wing by a pressure dipole distribution.
Fig 4: Elliptical coordinate system
Fig. 5: Notations straight, rectangular wing.
Fig. 6: Notations in rectangular, cylindrical, and polar coordinates.
Fig. 7: Prolate spheroidal coordinates
Fig. 8: Notations swept, rectangular wing.
Fig. 9: The rotor coordinate systems.
direction of flight

blade velocities

flapping angle $\beta$
  w.r. to control plane
  $\equiv$ variation of angle of incidence w.r. to tip path plane

Fig.10: Control plane and tip path plane.
Fig. 11: Typical isobar contours on advancing blade
(note: chord and span not drawn to same scale, real aspect ratio $A=12.7$)