EXPLICIT AND SEMI-IMPLICIT CHARACTERISTIC BASED SPLIT (CBS) SCHEMES FOR VISCOELASTIC FLOW CALCULATIONS

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Key words: CBS scheme, artificial compressibility (AC), semi-implicit, Oldroyd-B, Maxwell, PTT, viscoelastic flow

Abstract. A fully explicit characteristic based split (CBS) based artificial compressibility (AC) method and a semi-implicit CBS scheme, for viscoelastic flow over a circular cylinder placed in a rectangular channel are presented in this paper. In the explicit form the pressure equation is solved explicitly by introducing an artificial compressibility parameter, while a pressure Poisson equation is implicitly solved in the semi-implicit form. Three different constitutive models have been employed to investigate flow past a circular cylinder. They are the Oldroyd-B, upper convected Maxwell (UCM) and simplified Phan-Thien/Tanner (PTT) models. The main difference from the previously published results is that in the present study the artificial dissipation is not switched on until it is absolutely essential. No loss of convergence to steady state was observed in any of the results presented in this paper. Comparison of the present numerical solution with other available numerical data shows that the CBS algorithm is in excellent agreement at lower Deborah numbers. However, at higher Deborah numbers, the present results differ substantially from other numerical results. The two presented schemes give almost identical drag force values and the solutions presented demonstrate that both schemes are suitable for viscoelastic flow calculations.

1 INTRODUCTION

It is well known that the viscoelastic flow regimes are one of the most complex fluid regimes due to macromolecular properties of such fluids. The numerical solution to viscoelastic flow problems is important in several industrial applications such as polymeric liquids synthesis, food processing and metallic/non-metallic solidification. Mathematical representation of viscoelastic fluids are often given by hyperbolic constitutive equations.
Numerical solution of such equations are often subjected to strong extra stress oscillations, especially at higher Deborah numbers. It is also obvious from the available literature that obtaining convergence to steady state is extremely difficult at higher Deborah numbers.

In order to avoid numerical instability due to high elasticity in viscoelastic flows, most of papers suggest splitting the extra stress into elastic and viscous parts of the polymeric contributions of the constitutive equation. This approach together with interpolation of the velocity-gradient/vorticity or re-structured momentum equation are used for better accuracy. Several methods have been proposed and being widely used in the finite element, finite volume or finite difference literature. These include elastic viscous split stress (EVSS)$^{1,2,3,4}$ discrete elastic viscous split stress (DEVSS)$^{4,5}$, explicitly elliptic momentum equation (EEME)$^{6,7}$, adaptive viscoelastic stress splitting (AVSS)$^{8,9}$, EVSS gradient (EVSS-G)$^{10,11}$, DEVSS gradient (DEVSS-G/DG)$^{11,12,13}$, discrete adaptive viscoelastic stress splitting gradient (DAVSS-G/DG)$^{14,15}$, DEVSS vorticity (DEVSS-$\omega$)$^{16}$, DAVSS-$\omega$.$^{16}$

Many of the reported work using the Oldroyd-B model for flow over a circular cylinder, at the higher Deborah numbers, suffer from loss of convergence to steady state on very fine meshes. Phan-Thien and Dou$^{3,16,17}$ identified that the solution breaks down due to the stress oscillations in the wake along with large values of primary (first) normal stress difference at the front stagnation point of a cylinder. Fan$^{4}$ suggested that the solutions resulted from the higher Deborah numbers may be numerical artifacts. Fan also indicated lack of convergence of the solution beyond 0.7. It was shown by Caola et al.$^{12}$ using the fine finite element mesh and DEVSS-G based discontinuous Galerkin method, that reaching a steady state beyond a Deborah number of unity is difficult. This is due to the linear increase in the extra stress at the rear stagnation point. The proposed CBS scheme is expected to reduce the stress oscillations at higher Deborah numbers due to the inherent higher order stabilization of convective terms and fractional step based stabilization for pressure variables.

Over the last ten years, a unified computational fluid dynamics solver, referred to as the CBS algorithm, for viscous compressible and incompressible flow calculations has been investigated by Zienkiewicz, Nithiarasu and co-workers$^{18-23}$. It is based on a reference particle following the characteristic direction in the time-space domain to establish the self-adjoint equation leading to temporal semi-discrete formulation. By using a Taylor series expansion, consistent, extra second-order time terms are obtained in a non-moving frame. These terms in addition to increasing the stabilization, also provides second order accuracy in time. On the other hand, in order to circumvent BB or LBB restrictions, the CBS scheme uses fractional step method. Both fully explicit$^{24,25}$ and semi-implicit forms$^{26-28}$ were presented and developed in the past for Newtonian fluids. The extension of the CBS scheme in its explicit form was previously carried out using the Oldroyd-B model$^{18}$. In the present paper, we provide details of the extension of the CBS scheme to general viscoelastic flows. Also, two variations of the CBS scheme are compared in this paper for viscoelastic flow past a circular cylinder. The extra stress equations are solved
using standard simplified, explicit characteristic Galerkin method\textsuperscript{29}.

Three different constitutive models, the Oldroyd-B, upper convected Maxwell (UCM) and Phan-Thien/Tanner (PTT)\textsuperscript{30} models, have been employed in this work to model viscoelastic flows. Both the fully explicit and semi-implicit forms are studied and presented for a wide range of Deborah numbers. Section 2 gives the governing equations for the three different viscoelastic fluid models and CBS scheme. In Section 3, qualitative and quantitative results are shown over wide range of Deborah numbers. Finally, the conclusions derived from the present research are described in Section 4.

2 MATHEMATICAL FORMULATIONS

2.1 Governing equations

The non-dimensional isothermal fluid dynamics equations for viscoelastic flows in conservative form can be written as

$$\frac{\partial W}{\partial t} + \frac{\partial F_j}{\partial x_j} + \frac{\partial G_j}{\partial x_j} = 0$$

where the increase time-rate vector

$$W = (\rho, \rho u_1, \rho u_2)^T,$$

the convective acceleration vector

$$F_j = \left( \rho u_j, \frac{\delta_{1j} p}{Re} + \rho u_1 u_j, \frac{\delta_{2j} p}{Re} + \rho u_2 u_j \right)^T,$$

and the viscous diffusion vector

$$G_j = \left( 0, \frac{1}{Re} \left( -\tau_{1j} - \tau_{2j} \right), \frac{1}{Re} \left( -\tau_{21j} - \tau_{22j} \right) \right)^T$$

and the Newtonian deviatoric stress tensor

$$\tau_{ij}^1 = (1 - \alpha) \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right)$$

with the non-Newtonian deviatoric stress tensor ($\tau_{ij}^2 = \tau_{ij}^e + \tau_{ij}^v$), which is given by the constitutive equation of extra stress, i.e.

$$\tau_{ij}^e = \frac{-\alpha De}{\alpha + \varepsilon De \ tr[\tau_{ij}^2]} \left[ \frac{\partial \tau_{ij}^2}{\partial t} + \frac{\partial}{\partial x_k} \left( u_k \tau_{ij}^2 \right) - \tau_{ik}^2 \frac{\partial u_j}{\partial x_k} - \tau_{jk}^2 \frac{\partial u_i}{\partial x_k} \right]$$

$$+ \left( \frac{\alpha}{\alpha + \varepsilon De \ tr[\tau_{ij}^2]} - 1 \right) \tau_{ij}^v$$

3
\[ \tau_{ij}^v = \alpha \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) \] (7)

In the above governing equations, \( p \) is the pressure, \( \rho \) is the density, \( u_j \) are the velocity components, \( \delta_{ij} \) is the Kronecker delta, \( \tau_{ij}^e \) is the polymer-contributed to elastic stress, \( \tau_{ij}^v \) is the polymer-contributed to viscous stress, \( \alpha = \frac{\eta_{m0}}{\eta_0} \) in which \( \eta_{m0} \) represents the polymer-contributed viscosity and \( \eta_0 \) represents the zero shear rate viscosity, \( Re \) is the Reynolds number and \( De \) is the Deborah number defined as

\[ Re = \frac{\rho u_n L}{\eta_0}; \quad De = \frac{\lambda u_n}{L} \] (8)

where subscript \( \infty \) indicates a free stream value, \( L \) is a characteristic length, \( \lambda \) is the relaxation time and \( \eta_0 = \eta_n + \eta_{m0} \) in which \( \eta_n \) represents the Newtonian dynamic viscosity. Apparently, when \( \varepsilon = 0 \) and \( 0 < \alpha < 1 \), the constitutive equation describes the Oldroyd-B model and the upper convected Maxwell (UCM) model is obtained when \( \varepsilon = 0 \) and \( \alpha = 1 \). Both models explicitly represent the elastic stress (Equation (6)) and the viscous stress (Equation (7)) and contribute to polymeric solutions as the last term disappears in Equation (6). It should also be noted that the PTT model is obtained by substituting \( \alpha = 1 \) and \( \varepsilon \neq 0 \).

### 2.2 Characteristic based split (CBS) formulation

The CBS scheme is based on the characteristic-Galerkin procedure and a fractional step method. It is widely used to carry out fluid dynamic calculations. In this paper, explicit CBS-AC and semi-implicit CBS schemes have been employed to solve viscoelastic flow. Following the intermediate momentum at the first step, the pressure expression is given from the mass conservation at step2. The velocity field is corrected at step3. Finally, the constitutive equations are solved at the fourth step. The semi-discrete form of these steps are written as

**Step1: Intermediate momentum**

\[ \Delta U^*_j = U^*_j - U^n_j \]
\[ = \Delta t \left[ \frac{\partial}{\partial x_k} (u_k U_j) + \frac{1}{Re} \frac{\partial \tau_{ij}^1}{\partial x_i} + \frac{1}{Re} \frac{\partial \tau_{ij}^2}{\partial x_i} \right]^{n} + \frac{(\Delta t)^2}{2} \left\{ u_m \frac{\partial}{\partial x_m} \left[ \frac{\partial}{\partial x_k} (u_k U_j) \right] \right\}^{n} \] (9)

where \( U^n_j = U_j(t_n) = \rho u^n_j \); \( \Delta t = t^{n+1} - t^n \) and \( * \) indicates an intermediate quantity.
Step2: Pressure

\[
\frac{1}{Re} \left( \frac{1}{c^2} \right)^n \Delta p \approx \frac{1}{Re} \left( \frac{1}{\beta^2} \right)^n (p^{n+1} - p^n) \\
= - \Delta t \left[ \partial U_j^n / \partial x_j + \theta_1 \Delta U_j^n / \partial x_j - \Delta t \theta_1 \left( \partial^2 p^n / \partial x_j \partial x_j + \theta_2 \partial^2 \Delta p / \partial x_j \partial x_j \right) \right]
\]

(10)

where \( c \) is the speed of sound which assumes density changes are related to pressure changes for small compressibility or elastic deformability and approaches infinity for incompressible flows. However, a wave speed \( c \) is reasonably replaced by an artificial compressibility parameter \( \beta \) to calculate incompressible flow solution. For the explicit CBS-AC scheme, \( \beta \) depends on convective velocity, diffusive velocity and viscoelastic velocity fields.

Step3: Momentum correction

\[
\Delta U_j = U_j^{n+1} - U_j^n = \Delta U_j^* - \frac{\Delta t}{Re} \frac{\partial p^{n+\theta_2}}{\partial x_j}
\]

(11)

where \( 0.5 \leq \theta_1 \leq 1 \) and \( \theta_2 = 0 \) is in explicit forms. \( 0.5 \leq \theta_1 \leq 1 \) and \( 0.5 \leq \theta_2 \leq 1 \) is in semi-implicit forms. In this work, \( \theta_1 = 1 \) and \( \theta_2 = 0 \) are employed with the fully explicit scheme whilst \( \theta_1 = 1 \) and \( \theta_2 = 1 \) chosen for the semi-implicit scheme.

Step4: Constitutive equation

\[
\Delta \tau_{ij}^2 = \tau_{ij}^{2n+1} - \tau_{ij}^{2n} \\
= \Delta t \left[ -\partial \left( u_k \tau_{ij}^2 \right) / \partial x_k - \alpha + \varepsilon De \right] \left[ \frac{\partial \left( \tau_{ij}^2 \right)}{\partial x_k} \right] \tau_{ij}^2 \\
+ \Delta t \left[ \tau_{ik}^2 \partial u_j / \partial x_k + \tau_{jk}^2 \partial u_i / \partial x_k + \alpha \left( \partial u_i / \partial x_j + \partial u_j / \partial x_i - 2 \partial u_k / 3 \partial u_k \delta_{ij} \right) \right] \tau_{ij}^2 \\
+ \frac{(\Delta t)^2}{2} \left\{ u_m \frac{\partial \tau_{ij}^2}{\partial x_m} \left[ \partial \left( u_k \tau_{ij}^2 \right) / \partial x_k + \alpha + \varepsilon De \right] \left[ \frac{\partial \left( \tau_{ij}^2 \right)}{\partial x_k} \right] \tau_{ij}^2 \right\} \\
+ \frac{(\Delta t)^2}{2} \left\{ u_m \frac{\partial \tau_{ij}^2}{\partial x_m} \left[- \left( \tau_{ik}^2 \partial u_j / \partial x_k + \tau_{jk}^2 \partial u_i / \partial x_k \right) \right] \right\} \\
+ \frac{(\Delta t)^2}{2} \left\{ u_m \frac{\partial \tau_{ij}^2}{\partial x_m} \left[ -\alpha \left( \partial u_i / \partial x_j + \partial u_j / \partial x_i - 2 \partial u_k / 3 \partial u_k \delta_{ij} \right) \right] \right\} 
\]

(12)

The standard Galerkin method is employed for spatial discretization. The following spatial discretization of the variables are employed.
\[ U_j = N_u \tilde{U}_j; \quad \Delta U_j = N_u \Delta \tilde{U}_j; \quad \Delta U_j^* = N_u \Delta \tilde{U}_j^*; \quad u_j = N_u \tilde{u}_j; \quad p = N_p \tilde{p}; \]
\[ \Delta p = N_p \Delta \tilde{p}; \quad \tau_{ij}^* = N_s x_i \tau_{ij}^2 \]

where \( N \) are the shape functions and \( ^\sim \) indicates a nodal quantity.

Applying the standard Galerkin approximation with the divergence theorem, we get the following weak forms, i.e.

**Step1: Weak form of intermediate momentum**

\[
\int_\Omega N_u^T \Delta U_j^* d\Omega = \Delta t \left[ - \int_\Omega N_u^T \frac{\partial}{\partial x_k} (u_k U_j) d\Omega - \frac{1}{Re} \int_\Omega \frac{\partial N_u^T}{\partial x_i} (\tau_{ij}^1 + \tau_{ij}^2) d\Omega \right]^n + \frac{\Delta t^2}{2} \left[ \int_\Omega \frac{\partial}{\partial x_m} (u_m N_u^T) \left( - \frac{\partial}{\partial x_k} (u_k U_j) \right) d\Omega \right]^n + \Delta t \left[ \int_\Gamma N_u^T t_ad\Gamma \right]^n
\]

(14)

In the above equation \( t_d = [\tau_{ij}/Re]n \) indicates the part of the traction corresponding to the Newtonian stresses only and \( n \) are the components of the outward normal to the boundaries. As the pressure term is completely removed from the first step, we have only deviatoric stresses part of the traction left in the equation.

**Step2: Weak form of pressure equation**

**explicit scheme**

\[
\int_\Omega N_p^T \frac{1}{Re} \left( \frac{1}{\beta^2} \right)^n d\Omega = -\Delta t \int_\Omega N_p^T \frac{\partial}{\partial x_j} U_j^* d\Omega \]
\[
- \Delta t \int_\Gamma N_p^T \left( \Delta U_j^* - \frac{\Delta t}{Re} \frac{\partial p^*}{\partial x_j} \right) n_j d\Gamma
\]
\[
+ \Delta t \int_\Omega \frac{\partial N_p^T}{\partial x_j} \left( \Delta U_j^* - \frac{\Delta t}{Re} \frac{\partial p^*}{\partial x_j} \right) d\Omega
\]

(15)

In the above equation, pressure and \( \Delta U_j^* \) terms are integrated by parts and \( n_j \) are the components of the outward normal to the boundaries.

**Semi-implicit scheme**

\[
\int_\Gamma N_p^T \frac{1}{Re} \frac{\partial p^*}{\partial x_j} n_j d\Gamma - \int_\Omega \frac{\partial N_p^T}{\partial x_j} \frac{1}{Re} \frac{\partial p^*}{\partial x_j} d\Omega = \frac{1}{\Delta t} \int_\Omega N_p^T \frac{\partial U_j^*}{\partial x_j} d\Omega
\]

(16)

Here, pressure term is integrated by parts.

**Step3: Weak form of momentum correction**
explicit scheme

\[
\int_{\Omega} \mathbf{N}_u^T \Delta \mathbf{U}_j d\Omega = \int_{\Omega} \mathbf{N}_u^T \Delta \mathbf{U}_j^* d\Omega + \Delta t \int_{\Omega} \frac{\partial \mathbf{N}_u^T}{\partial x_j} \frac{1}{Re} p^n d\Omega - \Delta t \int_{\Gamma} \mathbf{N}_u^T \mathbf{t}_p d\Gamma
\]  

(17)

Semi-implicit scheme

\[
\int_{\Omega} \mathbf{N}_u^T \Delta \mathbf{U}_j d\Omega = \int_{\Omega} \mathbf{N}_u^T \Delta \mathbf{U}_j^* d\Omega + \Delta t \int_{\Omega} \frac{\partial \mathbf{N}_u^T}{\partial x_j} \frac{1}{Re} p^{n+1} d\Omega - \Delta t \int_{\Gamma} \mathbf{N}_u^T \mathbf{t}_p d\Gamma
\]  

(18)

In the above two equations \( \mathbf{t}_p = [(p^n + \theta_2 \Delta p)/Re] \mathbf{n} \) only indicates the part of the traction corresponding to the pressure which was removed from step1. It is simply ignored and assumed to be zero as the full traction is prescribed and employed in step1.

**Step 4: Weak form of constitutive equation**

\[
\int_{\Omega} \mathbf{N}_e^T \Delta \sigma_{ij} d\Omega = \Delta t \left[ - \int_{\Omega} \mathbf{N}_e^T \frac{\partial}{\partial x_k} \left( u_k \sigma_{ij} \right) d\Omega - \int_{\Omega} \mathbf{N}_e^T \frac{\alpha + \varepsilon De}{\alpha De} \frac{\sigma_{ij}}{\sigma_{ij}} d\Omega \right]^{n} 
+ \Delta t \left[ \frac{\mathbf{N}_e^T}{\mathbf{N}_e^T} \left( \tau_{jk} \frac{\partial u_j}{\partial x_k} + \tau_{ik} \frac{\partial u_i}{\partial x_k} \right) d\Omega \right]^{n} 
+ \Delta t \left[ \frac{\alpha}{\alpha De} \int_{\Omega} \mathbf{N}_e^T \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) d\Omega \right]^{n} 
+ \frac{(\Delta t)^2}{2} \left[ \int_{\Omega} \frac{\partial}{\partial x_m} \left( u_m \mathbf{N}_e^T \right) \left( \frac{\partial u_j}{\partial x_k} \right) d\Omega \right]^{n} 
- \frac{(\Delta t)^2}{2} \left[ \int_{\Omega} \frac{\partial}{\partial x_m} \left( u_m \mathbf{N}_e^T \right) \frac{\alpha + \varepsilon De}{\alpha De} \frac{\sigma_{ij}}{\sigma_{ij}} d\Omega \right]^{n} 
- \frac{(\Delta t)^2}{2} \left[ \int_{\Omega} \frac{\partial}{\partial x_m} \left( u_m \mathbf{N}_e^T \right) \left( \tau_{jk} \frac{\partial u_j}{\partial x_k} + \tau_{ik} \frac{\partial u_k}{\partial x_k} \right) d\Omega \right]^{n} 
- \frac{(\Delta t)^2}{2} \left[ \int_{\Omega} \frac{\partial}{\partial x_m} \left( u_m \mathbf{N}_e^T \right) \frac{\alpha}{\alpha De} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) d\Omega \right]^{n} 
\]  

(19)

It should be noted that the convection stabilizing treatment indicates last four terms by the characteristic-Galerkin procedure.

Once the finite element approach is worked out by the above weak forms, the matrix expression is convenient to establish, i.e.

**Step 1: Intermediate momentum**

\[
\Delta \mathbf{U}^* = -\mathbf{M}_u^{-1} \Delta t \left[ \left( \mathbf{C}_u \mathbf{U} + \mathbf{K}_{r} \mathbf{U} + \mathbf{G}_{r^2} \mathbf{U}^2 - \mathbf{f}_u \right) - \Delta t (\mathbf{K}_u \mathbf{U}) \right]^{n}
\]  

(20)
Step 2: Pressure

\[
(M_p + \Delta t^2 \theta_1 \theta_2 H) \Delta \tilde{p} = \Delta t [G_u \tilde{U}^n + \theta_1 G_u \Delta \tilde{U}^* - \Delta t \theta_1 H \tilde{p} - f_p]^n
\]

(21)

Step 3: Momentum correction

\[
\Delta \tilde{U} = \Delta \tilde{U}^* - M^{-1}_{u} \Delta t \left[ G_{p}^T (\tilde{p}^n + \theta_2 \Delta \tilde{p}) \right]
\]

(22)

Step 4: Constitutive equation

\[
\Delta \tilde{\tau}^2 = -M_{\tau^2} \Delta t \left[ C_{\tau^2} \tilde{\tau}^2 + K_{\tau^2} \tilde{\tau}^2 - K_{\tau v} \bar{u} - D_{\tau} \tilde{\tau}^2 \right] + HOT^n
\]

(23)

where the velocity, pressure and extra stress variables are approximated using equal interpolation functions at all computational points in the domain and are given as

\[
M_u = \int_{\Omega} N_u^T N_u d\Omega; \quad K_{\tau} = \int_{\Omega} B^T \left( \frac{1 - \alpha}{Re} \left( I_0 - \frac{2}{3} \text{mm}^T \right) \right) B d\Omega;
\]

\[
C_u = \int_{\Omega} N_u^T (\nabla^T (u N_u)) d\Omega; \quad G_{\tau^2} = \int_{\Omega} \frac{1}{Re} (\nabla N_u)^T \nabla \tau^2 d\Omega;
\]

\[
K_{\tau} = -\frac{1}{2} \int_{\Omega} (\nabla^T (u N_u))^T (\nabla^T (u N_u)) d\Omega; \quad f_u = \int_{\Gamma} N_u^T t d\Gamma;
\]

\[
M_p = \int_{\Omega} N_p^T \left( \frac{1}{Re^2} \right) N_p d\Omega; \quad H = \int \frac{1}{Re} (\nabla N_p)^T \nabla N_p d\Omega;
\]

\[
G_u = \int_{\Omega} (\nabla N_p)^T N_u d\Omega; \quad f_p = \Delta t \int_{\Gamma} N_p^T \left[ N_u \tilde{U}^n + \theta_1 \left( \Delta \tilde{U}^* - \frac{\Delta t}{Re} \nabla \tilde{p}^{n+\theta_2} \right) \right] n^T d\Gamma;
\]

\[
G_p = \int \frac{1}{Re} (\nabla N_p)^T N_u d\Omega; \quad C_{\tau^2} = \int_{\Omega} N_{\tau^2}^T (\nabla^T (u N_{\tau^2})) d\Omega;
\]

\[
M_{\tau^2} = \int_{\Omega} N_{\tau^2}^T N_{\tau^2} d\Omega; \quad K_{\tau v} = \frac{\alpha + \varepsilon D e}{\alpha D e} \text{tr} \left[ \tau^2_{ij} \right] M_{\tau^2};
\]

\[
K_{\tau v} = \int_{\Omega} \frac{\alpha}{D e} B^T I_0 m d\Omega; \quad D_{\tau} = \int_{\Omega} N_{\tau^2}^T \tilde{\tau}^2 \nabla N_u d\Omega;
\]

\[
G_u = \int_{\Omega} (\nabla N_p)^T N_u d\Omega
\]

(24)

In the above the strain shape function matrix \( B \) is given as

\[
B = SN_u
\]

(25)

where \( S \) is an strain matrix operator derived from the deviatoric stress and strain relations.

For a two dimensional case, it can be written as

\[
S = \begin{bmatrix}
\frac{\partial}{\partial x_1} & 0 \\
0 & \frac{\partial}{\partial x_2} \\
\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1}
\end{bmatrix}
\]

(26)
\[
\mathbf{m} = [1, 1, 0]^T \tag{27}
\]
\[
\mathbf{m}_I = [1, 1, 1]^T \tag{28}
\]
and
\[
\mathbf{I}_o = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix} \tag{29}
\]

3 RESULTS AND DISCUSSION

3.1 Steady state convergence

The steady state convergence criteria is based on the $L_2$ norm of the velocity field. Here the norm of difference between time step $n + 1$ and $n$ velocities is normalized by the Euclidean norm of the velocity at time step $n + 1$. The $L_2$ norm is given as

\[
\|e\|_2 = \frac{\sum_{i=1}^{NN} (\|u\|_{n+1}^i - \|u\|_n^i)^2}{\left[\sum_{i=1}^{NN} (\|u\|_{n+1}^i)^2\right]^{1/2}} \tag{30}
\]

where $NN$ are the number of nodes. The above tolerance was reduced to a value of at least $10^{-7}$ to assume a steady state.

![Figure 1: Viscoelastic flow over a circular cylinder placed in a channel.](image)

3.2 Flow over a circular cylinder placed in a channel

The computational domain consists of a stationary circular cylinder placed in a channel at a distance $12R$ from inlet section, where $R$ is the radius of the cylinder. The distance from the centre of the cylinder to the channel wall is equal to $2R$. The total distance from
Figure 2: Viscoelastic flow over a circular cylinder. (a) Unstructured mesh1 ($\Delta d = 0.097$); (b) Unstructured mesh2 ($\Delta d = 0.041$); (c) Unstructured mesh3 ($\Delta d = 0.008$); (d) Hybrid mesh4 ($\Delta d = 0.001$); (e) Unstructured mesh3. Nodes: 27451; Elements: 53516; (f) Hybrid mesh4. Nodes: 35506; Elements: 69918.

inlet to exit section is $28R$. The geometry and boundary conditions are shown in Figure 1.

Figure 2(a)-(e) shows the four meshes in the vicinity of the cylinder. All meshes are refined close to the wall. The element size near the cylinder wall are respectively 0.097, 0.041, 0.008 and 0.001 for the four meshes. The hybrid mesh4 consists of 22 structured layers to capture high elastic stress contribution around the cylinder surface. The mesh of the full domain for the unstructured mesh3 and hybrid mesh4 are shown in Figure 2(f) and (g) respectively.

Three different viscoelastic flow models are employed with both explicit and semi-implicit CBS schemes at different Deborah numbers. The parameter $\varepsilon = 0$ gives the Oldroyd-B fluid at the viscosity rate $\alpha = 0.41$ and the UCM fluid at $\varepsilon = 0$ and $\alpha = 1$. For PTT fluid, $\varepsilon = 0.39$ and $\alpha = 1$ are used in this study. The Reynolds number is assumed
Table 1: Meshes convergence at $Re = 0$ and $De = 3$ by using explicit form with the Oldroyd-B model.

to be zero which assumes convective terms of the momentum equation are removed.

The boundary conditions in non-dimensional form are given below\textsuperscript{3,16,17}. The inlet and exit velocity profiles and extra stress are

\begin{equation}
u_1 = 1.5 \left(1 - \frac{x_2^2}{4}\right) \tag{31}\end{equation}

\begin{equation}u_2 = 0 \tag{32}\end{equation}

\begin{equation}\tau_{11}^2 = \frac{9}{8}(De)x_2^2 \tag{33}\end{equation}

\begin{equation}\tau_{12}^2 = -\frac{3}{4}x_2 \tag{34}\end{equation}

and $\tau_{22}^2 = 0$. On the channel solid walls and cylinder surface, no slip conditions are assumed. The pressure is zero at exit for semi-implicit scheme. The extra stresses on the channel walls for Oldroyd-B fluid are given as

\begin{equation}\tau_{11}^2 = 0.82De \left(\frac{\partial u_1}{\partial x_2}\right)^2 \tag{35}\end{equation}

\begin{equation}\tau_{12}^2 = 0.41 \frac{\partial u_1}{\partial x_2} \tag{36}\end{equation}

and $\tau_{22} = 0$. In cylindrical coordinates, the extra stresses on the cylinder surface are written as

\begin{equation}\tau_{\theta\theta} = 0.82De \left(\frac{\partial u_\theta}{\partial r}\right)^2 \tag{37}\end{equation}

\begin{equation}\tau_{r\theta} = 0.41 \frac{\partial u_\theta}{\partial r} \tag{38}\end{equation}
and $\tau_{rr} = 0$. It should be noted, however, that no need to explicitly prescribe the extra stresses on the channel walls and on cylinder surface for UCM and PTT models.

The detail of mesh convergence at a Deborah number of 3 using the Oldroyd-B model are given in Table 1. It shows that the drag forces converge consistently on the unstructured meshes. However, the hybrid mesh with finer elements show a higher drag force than the converged solution on the unstructured mesh.

The drag force around the circular cylinder per unit length is calculated in non-dimensional form as

$$ F_x = \int_0^{2\pi} \left[ (-p + \tau_{11}^2 + \tau_{11}^0) \cos\theta + (\tau_{12}^1 + \tau_{12}^2) \sin\theta \right] Rd\theta $$

(39)

![Figure 3](image-url)

Figure 3: Oldroyd-B fluid flow over a circular cylinder at $De = 1$ using both fully explicit scheme (left) and the semi-implicit scheme (right). (a) $\tau_{11}^2$ contours. $\tau_{11}^{2}_{\text{min}} = -4.07$, $\tau_{11}^{2}_{\text{max}} = 191.59$; (b) $\tau_{11}^1$ contours. $\tau_{11}^{1}_{\text{min}} = -0.81$, $\tau_{11}^{1}_{\text{max}} = 159.80$; (c) $\tau_{12}^1$ contours. $\tau_{12}^{1}_{\text{min}} = -38.77$, $\tau_{12}^{1}_{\text{max}} = 42.86$; (d) $\tau_{12}^2$ contours. $\tau_{12}^{2}_{\text{min}} = -45.73$, $\tau_{12}^{2}_{\text{max}} = 48.75$; (e) $\tau_{22}^2$ contours. $\tau_{22}^{2}_{\text{min}} = -0.76$, $\tau_{22}^{2}_{\text{max}} = 19.63$; (f) $\tau_{22}^1$ contours. $\tau_{22}^{1}_{\text{min}} = -0.38$, $\tau_{22}^{1}_{\text{max}} = 21.28$.

Figure 3 shows the extra stress contours at $De = 1$ from explicit and semi-implicit forms using the Oldroyd-B model. As seen, the large normal stress in the $x_1$ direction occurs on the top and bottom cylinder surfaces. Both schemes give some minor spatial oscillations of $\tau_{11}^2$ in the wake, but the semi-implicit scheme is more stable than explicit scheme. The extra oscillations generated by the explicit scheme may be attributed to inadequate dissipation. It is important to note that we have not added any extra dissipation.
Figure 4: Oldroyd-B fluid flow over a circular cylinder. (a) Drag force, $D$, distribution; (b) Pressure component of drag force, $D_p$, distribution; (c) Viscoelastic stress component of drag force, $D_s$, distribution.

Figure 5: Oldroyd-B fluid flow over a circular cylinder at $De = 3$ using both fully explicit scheme (left) and the semi-implicit scheme (right). (a) $u_1$ contours. $u_{1\text{min}} = 0.0$, $u_{1\text{max}} = 3.17$; (b) $u_1$ contours. $u_{1\text{min}} = 0.0$, $u_{1\text{max}} = 3.15$; (c) $u_2$ contours. $u_{2\text{min}} = -0.97$, $u_{2\text{max}} = 1.07$; (d) $u_2$ contours. $u_{2\text{min}} = -0.94$, $u_{2\text{max}} = 1.04$.

Figure 4(a) shows the comparison of the variation of the drag force at different Deborah numbers. The results show an excellent comparison with the literature up to a Deborah number of unity. However, beyond this limit the drag force did not increase dramatically like the ones published previously. The pressure and viscoelastic stress (including Newtonian and non-Newtonian stresses) components of drag force are also depicted in Figure 4(b) and (c).

Figure 5 shows contours of horizontal velocity component $u_1$, vertical velocity component $u_2$ and pressure at $De = 3$ using both the variations of the CBS schemes. As seen the contours are symmetric with respect to the horizontal centreline of the domain.
The extra stress contours at the Deborah number of three are shown in Figure 6. As seen, the minor spatial oscillations of $\tau_{11}^2$ still exists in the wake. Some oscillations are also noted in $\tau_{12}^2$ contours. When the Deborah number is increased, the maximum extra stress components $\tau_{11}^2$, $\tau_{22}^2$ and $\tau_{12}^2$, becomes larger due to a very thin shear boundary layer around the cylinder surface. Table 2 presents the drag force values for different Deborah numbers up to a value of three.

The drag forces from UCM and PTT models are compared with Alves et al.\textsuperscript{31}, Fan et al.\textsuperscript{4} and Phan-Thien et al.\textsuperscript{3} in Tables 3 and 4.

### Table 2: Drag force calculations by using CBS schemes with the Oldroyd-B model at higher Deborah numbers.

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4 CONCLUSIONS

This research was aimed at using fully explicit and semi-implicit CBS schemes to solve viscoelastic flow over a stationary circular cylinder placed in a rectangular channel. Three viscoelastic fluid models; the Oldroyd-B, upper convected Maxwell (UCM) and Phan-
Thien/Tanner (PTT) models were studied in this paper. One of the major problems of Oldroyd-B model is the convergence to steady state beyond a Deborah number of unity. In this paper we have proved that the CBS scheme gives steady state results up to a Deborah number of three. The major conclusions derived are:

- The CBS scheme is validated for Oldroyd-B, UCM and PTT fluids over a wide range Deborah numbers. It is clear that most of the models agree excellently at Deborah numbers below unity.
- At higher Deborah numbers, CBS scheme gives converged solution but the agreement with other numerical solution is poor. This could be due to the lack of convergence reported by other researchers.
- The CBS scheme can work for UCM and PTT fluids over a circular cylinder even though the majority part of the scheme are explicit in nature.
- At lower Deborah numbers, the explicit scheme is faster than the semi-implicit scheme. However, at higher Deborah numbers, the semi-implicit scheme is faster. This conclusion could be different if the extra artificial dissipation is switched on.
- The lack of extra dissipation results in some minor extra stress oscillations. Further investigation on using extra artificial damping for fully explicit scheme is necessary.
Table 4: Drag force calculations by using CBS schemes with the PTT model.

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5 Acknowledgements

This research is funded by the Engineering and Physical Sciences Research Council (EPSRC) grant no. EP/C515498/1.

REFERENCES


Figure 6: Oldroyd-B fluid flow over a circular cylinder at $De = 3$ using both fully explicit scheme (left) and the semi-implicit scheme (right). (a) Primary normal stress difference contours. $N_{1\text{min}} = -55.84$, $N_{1\text{max}} = 647.35$; (b) Primary normal stress difference contours. $N_{1\text{min}} = -24.11$, $N_{1\text{max}} = 593.37$; (c) $\tau_{11}^2$ contours. $\tau_{11\text{min}}^2 = -5.42$, $\tau_{11\text{max}}^2 = 646.92$; (d) $\tau_{11}^2$ contours. $\tau_{11\text{min}}^2 = -0.82$, $\tau_{11\text{max}}^2 = 594.15$; (e) $\tau_{12}^2$ contours. $\tau_{12\text{min}}^2 = -112.56$, $\tau_{12\text{max}}^2 = 124.39$; (f) $\tau_{12}^2$ contours. $\tau_{12\text{min}}^2 = -137.52$, $\tau_{12\text{max}}^2 = 147.96$; (g) $\tau_{22}^2$ contours. $\tau_{22\text{min}}^2 = -0.75$, $\tau_{22\text{max}}^2 = 62.26$; (h) $\tau_{22}^2$ contours. $\tau_{22\text{min}}^2 = -0.38$, $\tau_{22\text{max}}^2 = 56.49$.


