Wigner molecule on the top of a quantum dot

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Quantum dots in GaAs heterostructures display a complex inner structure in a large magnetic field. The distribution of electrons over different Landau levels results in the formation of alternating compressible and incompressible regions. We investigate theoretically the situation when only a few electrons are confined to the uppermost Landau level. We show that these extra electrons form a Wigner molecule. The magnetic field controls the number of electrons in the molecule, and this creates the possibility of manipulating few-electron objects. Moreover, in this way one can manipulate *fractional* charges. We discuss briefly the stability of the molecule in the presence of a fluctuating donor potential and the slow charge dynamics, which is intrinsic for the system under consideration.

It is known that for a *homogeneous* two-dimensional (2D) electron gas the energy density as a function of electron density n shows a cusp at

$$n = (p/q)(H/\Phi_0), \quad (1)$$

p, q being integer numbers, and $\Phi_0 = 2\pi\hbar c/e$. This indicates that the electron gas is *incompressible* at this point but is compressible elsewhere.² The charged excitations are separated by an energy gap E_g from the incompressible ground state. This is why the incompressible state may be thought of as an insulator, whereas the compressible one is a conventional metal. The gap drops quickly with increasing denominator q , so that only few incompressible states can actually be observed. Those with $q = 1$ correspond to completely filled Landau levels of noninteracting electrons and give rise to the integer quantum Hall effect. Those with $q \neq 1$ are Laughlin liquids and manifest themselves in the fractional quantum Hall effect. This picture seems to be correct over a wide range of electron-electron interaction strength.

In recent years, results have appeared that give an insight into the physics of the *inhomogeneous* 2D electron gas.^{1,3,4} The resulting picture can be summarized as follows. In the first approximation the density profile of the 2D gas is defined by the system of gate electrodes. The potential to be compensated by the gates is of the order of volts, which is much larger than the energy scale of the inner electron motion, $\hbar\omega_c, E_g$. Therefore the applied magnetic field cannot change the density profile much. The major effect of a magnetic field is the formation of a narrow strip of incompressible phase near the point where the density satisfies the magic relation (1). These insulating strips separate compressible, metallic regions. The strip width was shown³ to be much smaller than the geometric size of the system defined by gates but, at least for the widest strips, much larger than the magnetic length. The width decreases quickly with increasing q . The picture becomes irrelevant if the width is of the order of the magnetic length. Hence, incompressible states with large q do not occur.

To illustrate this picture further, let us consider a quantum dot with a certain density profile, $n_{\max}$ being the electron density on the top (see Fig. 1). In addition, following Ref. 1, let us assume that there are no incompressible states occurring with $q > 1$ so that we

can make use of a simple Landau-level picture. If the magnetic field exceeds $H_c = n_{\max}\Phi_0$, all the electrons are in the first Landau level. If $H_c > H > H_c/2$, the electrons in the second Landau level are confined near the top of the quantum dot, as shown in Fig. 1. The density profile remains almost the same, the potential is flat within two metallic regions and drops by a value of E_g/e inside the insulating region. It is important to note that the wave functions of the electrons in the second Landau level have almost no overlap with the wave functions of empty states in the first Landau level. That implies that the number of electrons in each Landau level is an integer. It is worthwhile to spell out what are the limits of validity of this approach. In fact, the only requirement is that the size of the dot is larger than "microscopic" lengths such as the Bohr radius a_B , magnetic length l_H , and screening length. For quantum dots in the quantum Hall regime, the electron concentration is of the order of l_H^{-2} , so that the radius $R \simeq l_H\sqrt{N}$, N being the number of confined electrons. Thus, the theory holds if N is big enough, say, $N > 100$. Starting with this number, the plain electrostatics determines electron distribution in the dot and the effects of kinetic energy and magnetic field can be considered as small perturbations.

Below we present a simple extension of the theory taking into account the discreteness of electrons. This allows us to understand what happens near H_c where only a few electrons in the second Landau level are confined near the

FIG. 1. Quantum dot in quantum Hall regime ($n_{\max} > eH/ch > n_{\max}/2$). A half of the top view (a), the density profile (b), and the potential energy profile are shown. The electrons occupy two spin-split Landau levels forming two compressible regions 1, 2 and the separating incompressible strip. The unperturbed density profile is shown as a dotted line.

top of the quantum dot. Let us first outline the results. As it was stressed above, at $H > H_c$ all electrons in the dot are in the first Landau level forming a compressible state. If H is far below H_c , there are two compressible regions corresponding to electrons in the first and in the second Landau level. The scenario of the formation of the second compressible region appears to be rather complex consisting of four stages (see Fig. 2)

First stage: When H becomes smaller than H_c , the electrons of the first Landau level form an incompressible flat region on the top of the dot, since the density of the first Landau level cannot exceed $H/\Phi_0 < n_{\max}$. In the flat region, the density is exactly H/Φ_0 . The density distortion creates a potential well. No electron can come yet to the second Landau level, since the depth of the well is smaller than E_g . The radius of the flat region and the well depth increase with decreasing the magnetic field.

Second stage: At $H = H_{c1} < H_c$ the well depth approaches E_g and the first electron appears on the second Landau level near the center of the dot. The electrostatic repulsion of the electrons prevents the second electron from entering the flat region at the same field. This will happen only at a lower field H_{c2} , when the well becomes deeper. The stable configuration of two electrons on the flat region is such that the distance between electrons is much smaller than the radius a of the flat region but much larger than the magnetic length. The configuration is then completely determined by the interplay of the Coulomb repulsion between the electrons in the second Landau level and the confining potential of the well. Thus, what we term a “Wigner molecule” is formed near the top of the dot. With further lowering of the field, more electrons join the molecule. The size of the molecule increases at the same time as the average distance between electrons gets smaller.

Third stage: For a certain number of electrons, the size of the molecule becomes comparable with the radius of the flat region. Now only a narrow strip of the incompressible phase separates the compressible phase of the electrons in the first Landau level and the Wigner crystal. Such a formation is not yet a good metal since the distance between extra electrons exceeds the magnetic length and their wave functions do not overlap much.

Fourth stage: The concentration of electrons in the second Landau level grows, and at some field they begin to overlap in the central part of the structure, thus forming a true metallic state.

These results can be easily achieved if one takes into account the discrete nature of electrons on the second Landau level. The discreteness is ensured by a large incompressible flat region. It should be much larger than

the magnetic length. As a result, the first three stages of crossover wave functions of the second-level electrons overlap neither each other nor wave functions of empty states in the first level. The extra electrons behave like classical charged particles displaying Wignerian ordering.

Let us give now quantitative arguments in support of the picture outlined. We assume that the dot radius $R \gg l_H \equiv \sqrt{\hbar c/eH}$, $a_B, l_g \equiv \epsilon(E_g/e^2)l_H^2$ which allows us to make use of the perturbative approach of Ref. 3. We assume the magnetic field decreases starting from a value high enough for all electrons in the dot to be on the first Landau level. When H approaches H_c , a flat region with constant density $n(r) = H/\Phi_0$, starts to develop around the top. The new density distribution can be represented as a sum of the density $n(r)$ without magnetic field and small perturbation $\delta n(r)$ to be evaluated. We expand the unperturbed density near the top $n(r) = n_{\max} - n''r^2/2$ making use of the axial symmetry of the problem. This holds since $n(r)$ reaches the maximal value at the center of the dot and the radius of the incompressible region is assumed to be much smaller than the characteristic scale of variation of $n(r)$. Our goal now is to find the relation between the radius a of the incompressible flat region of the dot and the magnetic field. We shall solve the three-dimensional electrostatic problem $\Delta\phi = 0$ for $\phi(r, z)$ under the following conditions:

$$\begin{aligned} \delta n(r) &= \text{const} - n(r) \\ &= (H/\Phi_0) - n_{\max} + n''r^2/2 \quad \text{at } r < a, z = 0, \end{aligned} \quad (2)$$

$$\phi(r) = \text{const} = 0 \quad \text{at } r > a, z = 0 \quad (3)$$

$\delta n(r), \phi$ are continuous at $r = a$, the 2D gas plane corresponds to $z = 0$. The normal derivative of potential at $z = \pm\delta, \delta \rightarrow 0$ in the incompressible flat region ($r < a$) is equal to $d\phi/dz = \pm(2\pi e/\epsilon)\delta n(r)$. The solution

$$\phi(r) = (8e/9\epsilon)n''(a^2 - r^2)^{3/2} \quad (4)$$

exists if

$$a = \sqrt{(3n_{\max}/n'')[(H_c - H)/H_c]}. \quad (5)$$

The field H_{c1} at which the first electron enters the dot can be found from the condition $e\phi(0) = E_g$. Whereas the approach we use is essentially phenomenological, the determination of the gap energy E_g is a subject of microscopic theory. What is this energy for the conditions we have specified? The incompressible state with $n = H/\Phi_0$ is thought to be a ferromagnetic one.⁵ In the limit $a_B \gg l_H$ corresponding to almost noninteracting electrons this is a filled lowest Landau level with all the electron spins aligned. The quasiparticle is then an electron with an *opposite* spin. The quasiparticle energy is only determined by exchange interaction:

$$E_g = \sqrt{(\pi/2)}(e^2/\epsilon l_H), \quad (6)$$

$l_g = \sqrt{\pi/2}l_H$. This is related to the exchange enhancement of the g factor.⁶ In reality, one should expect lower values of E_g, l_g . This is due to Landau-level mixing and spreading of the electron wave function in the vertical direction. Therefore we prefer not to specify the value of E_g here.

The critical field H_{c1} is given by

FIG. 2. Four stages (a, b, c, d) the dot passes with decreasing magnetic field. The top views and potential energy profiles are shown. Small circles correspond to the electrons on the second Landau level.

$$(H_c - H_{c1})/H_c = (3n''\epsilon^2 E_g^2/64e^4)^{1/3} (1/n_{\max}). \quad (7)$$

For concreteness, we can use a simple model of a quantum dot density profile,⁷ which corresponds to a parabolic confining potential. In this case, $n(r) = n_{\max} \sqrt{1 - (r/R)^2}$, $n'' = n_{\max}/R^2$ and Eq. (7) becomes

$$(H_c - H_{c1})/H_c = 1.228(l_g/R)^{2/3}, \quad (8)$$

$$a(H_{c1}) = 1.919R(l_g/R)^{1/3}. \quad (9)$$

Here we have given numerical values of all coefficients. This shows clearly that $H_c - H_{c1} \ll H_c$ and $l_H \ll a \ll R$ justifying the perturbation approach we made use of. Another important parameter is the lack of electrons in the flat N_c in comparison with the unperturbed density profile, $N_c \equiv 0.565(Rl_g^2/l_H^3)^{2/3} \gg 1$.

We note that if the number of electrons in the second Landau level is much smaller than N_c , the confining potential well is not perturbed much. The extra electrons drop to the bottom of the well which then can be treated as a parabolic one. Let us treat the electrons as classical particles. To determine the equilibrium configuration, we shall minimize the energy of the form

$$E_N = [E_g - e\phi(0)]N - \frac{e\phi''(0)}{2} \sum_i r_i^2 + \frac{1}{2} \sum_{i,j}^N \frac{e^2}{\epsilon|r_i - r_j|}. \quad (10)$$

Here N is the number of extra electrons, r_i are their coordinates. The problem closely resembles the atomic model proposed by J. J. Thompson. It has been also treated quantum mechanically in this century.⁸ We shall emphasize the interesting possibility to change the number of electrons in the molecule by changing magnetic field. Indeed, magnetic field changes a according to Eq. (5), thus changing the well depth $\phi(0)$. As a result, we obtain a series of magnetic fields at which the number of extra electrons changes from $N - 1$ to N :

$$H_{cn} = H_{c1} + \beta_n[(H_c - H_{c1})/N_c^{2/3}], \quad (11)$$

β_n being numerical factors. The corresponding magnetic fields are plotted for several n in Fig. 3. The average distance between the extra electrons r_a is of the order of $[\epsilon\phi''(0)/e]^{-1/3} \simeq (a^2 l_H^2/l_g)^{1/3} \simeq l_H(R^4/l_H^3 l_g)^{1/9}$. We see that $a \gg r_a \gg l_H$. This justifies our initial assumptions and our classical approach.

These transition magnetic fields can be measured experimentally¹ if one studies the transport properties of the dot. Due to Coulomb effects, conductance of the dot displays a sequence of peaks, each peak corresponds to addition of one electron to the dot. External charges may shift the potential of the conducting part of the dot changing the condition for such an addition and thus influencing the conductivity. Thus the dot works as a very sensitive electrometer.⁹

We should stress that in distinction from Ref. 1 the processes of adding an electron to the dot and internal charge transfer between second and first Landau levels are effectively decoupled. The point is that the capacitance of the dot is much larger than the effective capacitance of the Wigner molecule. The latter follows from the fact that $R \gg a \gg r_a$. If the ratio of capacitances is small but finite, the change of number of the electrons in the molecule will shift the potential of the conducting island with respect to the leads. This gives rise to a change of conductance at transition magnetic field.

FIG. 3. Configurations of the Wigner molecule vs magnetic field. Solid lines separate the domains where a certain configuration corresponds to the energy minimum. Dashed-dotted lines indicate where this configuration becomes unstable with respect to one-electron tunneling.

The series of the transition fields given by β_n is a signature of the Wigner molecule. One could then extract $H_c - H_{c1}$ and N_c from the measurements. This gives a possibility to determine E_g experimentally.

At further decreasing of the magnetic field, the number of extra electrons becomes of the order of N_c . This occurs at $H_{c1} - H \simeq (H_c - H_{c1})$ indicating the third stage of the crossover we mentioned. Still at this magnetic field the electron wave functions do not overlap and the resulting state resembles a Wigner lattice. The crossover to the true metallic state is expected to occur when the density of extra electrons corresponds to the filling factor 0.2 – 0.15.² This corresponds to $H \approx 0.8H_{c1}$.

We have discussed in detail the case when the incompressible state on the top of the dot corresponds to $p = q = 1$. The approach can be trivially generalized to the case of arbitrary p, q . The changes to be made are to substitute $l_H \rightarrow l_H \sqrt{q/p}$ into the equations and to use qE_g of the corresponding incompressible state. The interesting detail is that the Wigner molecule can be formed by quasiparticles with fractional charge if $q \neq 1$. The idea of fractional charges forming a Wigner lattice was previously put forward by L. P. Kowenhover.

We address below two additional questions: what might be the dynamical properties of the system and how stable is the molecule with respect to the fluctuations of the donor potential.

There are three reasons to expect very slow rates of transitions which change the number of electrons in the molecule. The transition has to consist of an electron transfer from the center of the dot to the compressible region and a simultaneous flipping of the spin. Similar spin flip processes have been considered in Ref. 10 to explain the equilibration of edge channel populations, which inevitably involve spin-orbit or spin-spin interactions. These interactions are believed to be small in GaAs, which is the first reason for the rate to be slow.

The second reason is an exponentially small overlap between the wave functions of an electron in the flat and in the compressible region. If we just estimate this overlap for experimentally relevant parameters, we could obtain astronomical relaxation times. This forces us to think of more exotic tunneling mechanisms which could include, for example, subsequent tunneling.

The third reason is intrinsic to tunneling to and from Wigner arrays¹¹ and is related to many-body effects. From general principles, the tunneling rate is proportional to the square of the matrix element between the initial and final states. If *all* the electrons in the molecule have to be shifted during the transition (compare differ-

ent configurations in Fig. 3), the matrix element is proportional to the overlap between their wave functions in the initial and final positions. This gives an important extra suppression to the tunneling rate. However, this suppression can be relaxed far from the position of equilibrium transition where the energy difference between the initial and final states becomes larger. To illustrate this, let us consider the transition between $N = 1$ and $N = 2$. Right below H_{c2} , the process is possible only if accompanied by shifting the electron in the center of the flat. Lowering the field, we can reach the point H_{c2}^* where such shift is not necessary. It is the field at which the bottom of the effective potential (which is the sum of the confining potential and the potential created by the extra electron in the center) reaches the value of E_g (see Fig. 4). Below this point, the transition rate increases tremendously. We have calculated these threshold fields for $N = 1, \dots, 7$ and plot them also in Fig. 3. They are of the same scale as H_{cn} and indicate the stability domains of possible metastable states.

For practical purposes, it is important to be sure of the stability of the molecule with respect to the fluctuations of the donor potential. The relevance of such a concern can be seen for the low density 2D gas itself. If there were no fluctuations, the electrons in this case would have formed a regular Wigner lattice. However, for most of the structures available, the fluctuations are strong enough to trap the electrons at random positions and hence destroy the lattice. Naively, one could expect this also for the object we describe. Fortunately, the situation is not so bad for two reasons. First, the molecule is formed not far from the compressible region which screens the fluctuations. Second, if the fluctuations are strong but smooth, the molecule can be trapped as a whole keeping its own structure.

To give some estimations, we adopt a simplified model of the donor potential. All donors are assumed to lie in the same plane separated by a distance d from the 2D plane, and their concentration equals the maximum electron density. The fluctuations of such a potential

FIG. 4. The potential energy profile for $N = 1$ and $N = 2$. One-electron tunneling may occur only if the potential energy minimum is lower than $-E_g$.

$\langle \phi^2 \rangle - \langle \phi \rangle^2 \simeq e^2 n_{\max} / \epsilon^2$ are correlated at distances of the order of d . The ideal situation would occur if $a < d$. In this case, the fluctuations will be screened by the metallic part of the dot. The molecule can keep its structure even in the opposite limit. Indeed, the potential harmonics that distort the structure have wave vectors of the order of the inverse distance between extra electrons $1/r_a$. The electrostatic consideration shows that these harmonics will be exponentially suppressed, $\langle \phi^2 \rangle - \langle \phi \rangle^2 \simeq e^2 n_{\max} / \epsilon^2 \exp(-d/r_a)$, if $r_a \ll d$. They become negligible if $d \gg r_a \ln(r_a/l_H)$.

Let us conclude by giving some numbers. We have shown that a Wigner molecule can be formed on the top of the quantum dot. Let us take for estimations $l_H = 10$ nm, $l_g = 13$ nm, dot radius $R = 300$ nm. This choice is not so good for the perturbation approach we use, but it corresponds to experimentally relevant parameters. Extra electrons appear on the top of the quantum dot at $H_{c1} = 0.85H_c$, when the radius of the incompressible flat is about 200 nm. If the number of extra electrons is less than $N_c = 8$, they are confined near the center of the dot. They are separated by a distance $\simeq 50$ nm in average. The number of extra electrons is controlled by the magnetic field and the field change required to add one electron is of the order of $0.02 H_c$.

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lap and quantum effects are important. Comparing the classical and quantum results, it is important to note the axial symmetry of the problem. The classical configurations can be rotated without energy change and are degenerate from the formal point of view. This degeneracy is lifted in quantum mechanics and the ground state is an eigenfunction of the angular momentum operator with all the electrons spread over a circle. In reality, the axial symmetry is approximate. We assume that either small ellipticity of the dot or small fluctuations of donor potential restore the uniqueness of the classical configuration.

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