IMAGE RESTORATION

a linear stochastic filtering approach
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SUMMARY

Image processing in general, and digital image processing in particular, has received considerable attention in recent years. As major research areas we mention image coding, image restoration and enhancement, and image (scene) analysis.

In this Ph.D. thesis attention is focused on the problem of image restoration by means of a digital computer. Observed or recorded images are not only corrupted by random observation noise, but also often degraded by the imaging medium. Defocusing, imaging with camera motion, and imaging through atmospheric turbulence are examples of such degradations. In image restoration we try to remove these kinds of degradations by developing algorithms based on an adequate mathematical description of the degrading phenomena to arrive at an improved image which is as close to the ideal image — an image we would have obtained in the absence of these degradations — as possible under an objective evaluation criterion.

In particular, we pay attention to the digital restoration of images which may be degraded by linear space-invariant blur (defocusing, camera motion) and additive white noise, uncorrelated with the image data. Inverse filtering techniques, which aim at perfect reconstruction of the image by using the convolutional inverse of the blur introduced by the imaging medium, turn into poor restoration techniques in the presence of noise. Therefore, a more successful approach is to take explicitly the presence of noise into account and to define the image-restoration problem as an optimization problem in which a suitable optimality criterion, e.g. the mean-square error between the original image and its restored version, is minimized (in a statistical sense). Both recursive and non-recursive optimal restoration filter schemes can be defined. To reduce processing time and/or storage requirements, we prefer an optimal recur-
sive filter scheme. In one dimension this is given by the recursive Kalman filter, so that the intriguing question then arises of how to extend such a filter to two dimensions with particular application to image restoration. In this thesis we have addressed ourselves to several aspects of Kalman filtering in two dimensions. Parts of the material have been published in [9], [10], [11], [13], [54].

This thesis is organized as follows. In Chapter 1 the general image-restoration problem is discussed as well as the need for using a priori knowledge, and the choice of a linear stochastic filtering approach. Further, as general background, the one-step prediction and filtering algorithms are given for the discrete one-dimensional Kalman filter.

In Chapter 2 attention is paid to modeling the a priori knowledge concerning the original undistorted image into discrete two-dimensional autoregressive types of models. At first a stochastic image representation is given, in which an image can be regarded as a sample from a two-dimensional random field. Then, this field is described by means of the first- and second-order moments. To reduce the number of degrees of freedom in this description the concepts of homogeneity, separability, isotropy and autoregression are introduced. Next, a general linear two-dimensional autoregressive type of model is defined for homogeneous images. The model coefficients are found in a linear mean-square error fitting procedure. Important aspects in defining these so-called minimum variance models are the choice of a processing order (causality in two dimensions), the fact that semicausal and noncausal model representations are not driven by white noise, and the possibility of a model output with negative spectral components. Model stability and model separability, two other important topics in developing two-dimensional image models, are also discussed. Finally, to compare several image models based on the same image statistics some quality criteria are introduced, including the Cramér-Rao bound. Experimental results of this comparison are given.

In Chapter 3 the restoration of noisy images by means of two-dimensional recursive estimators is considered. In particular, two early generalizations of the Kalman filter concept to two dimensions are dealt with: the one-step predictor of Habibi and the filter of Kak based on the Differential Pulse Code Modulation or DPCM image model. They are compared with regard to optimality, stability and performance. Two new performance measures are introduced. The first gives insight into the distortion to the original image caused by an estimator, while the second measures the noise-reducing capability of an estimator. Both estimators are compared experimentally by using these measures of performance.

In Chapter 4 attention is paid to a common problem shared by several recursive estimation procedures proposed for noise reduction of image data, namely that these linear estimators may effectively reduce the noise, but that they distort the edges in an image and reduce image contrast as well. To decrease this edge-blurring phenomenon a simple and computationally fast scan-ordered one-dimensional Kalman filter is introduced, which uses additional structural information about the edges in the noisy image. This filter behaves like the original noise-reducing Kalman filter if no edges are present, but has a greatly improved step response in the case in which an edge is detected. Further, a frequency analysis of the edge-preserving filter is given and an attempt is made to extend this filter to two dimensions. Results of several experiments are presented to demonstrate the feasibility of our approach.

In Chapter 5 a Kalman filter is derived for optimal line-by-line recursive restoration of images degraded not only in a stochastic way by additive white noise, but also in a deterministic way by linear space-invariant blur. Because of the computational and storage burden imposed by this filter attention is paid to the reduction of processing time and storage requirements. We mention a method for approximating the full gain matrix in its steady state by a quasi-Toeplitz matrix with a very small number of nonzero elements, and a diagonalization procedure using circulant matrix approximations in order to reduce the line-by-line recursive Kalman filter to a set of scalar Kalman filters suitable for parallel processing of the data in a transform domain. Experimental results on noisy defocused and noisy motion-blurred images are given.
This thesis is concerned with the digital restoration of images by means of linear stochastic filtering methods. In particular, we try to apply the recursive Kalman filter to the restoration of images which may be degraded by linear space-invariant blur and additive white noise uncorrelated with the data. Kalman filtering theory is well established in one dimension, and an intriguing question is how to extend this one-dimensional (I-D) filter concept to two dimensions with particular application to image restoration. An important aspect in this approach is the stochastic representation of images and their modeling by means of two-dimensional (2-D) autoregressive types of models.

In this first chapter we will discuss the general image-restoration problem, the need for using a priori knowledge in image restoration and the choice of a recursive linear filtering approach.

Further, as general background, we discuss briefly the one-step prediction and filtering algorithms for the discrete I-D Kalman filter.

1.1. IMAGE RESTORATION AS A LINEAR STOCHASTIC FILTERING PROBLEM

Images are produced to provide useful information about a phenomenon of interest. Unfortunately, since physical imaging systems are not perfect, a recorded image will almost certainly be a degraded version of an original image or scene. For example, in aerial photography and remote
sensing, which may be used for geographical purposes, mensuration, conservation studies and weather prediction, the images obtained are to some degree degraded by atmospheric turbulence, aberrations of the optical system, and relative motion between camera and object. More down to earth, in the medical field radiographic images are usually of low resolution and low contrast, while in electron microscopy electron micrographs are degraded by the spherical aberration of the electron lens.

The common problem confronting researchers in these fields is how to restore the image data to improve image quality. A complicating factor is the phenomenon of random noise, which is inevitably mixed with the data, and may originate from the image-formation process, the transmission medium, the recording process, or any combination of these, and which is a significant obstacle to the perfect restoration of images.

A general block diagram of an image-restoration scheme is given in Fig. 1.1. We shall concern ourselves exclusively with 2-D monochromatic images. Such an image can be characterized by a real function of two spatial variables \( i \) and \( j \), representing the intensity of the image. We assume that an ideal image – equal to the original image or scene \( x(i,j) \) – would be obtained if our imaging system were perfect. But since this is not the case, we get a degraded image \( y(i,j) \). The purpose of image restoration is to operate on the degraded image \( y(i,j) \) in order to get an improved image \( \hat{x}(i,j) \) which is as close to the original image \( x(i,j) \) as possible, subject to a suitable optimality criterion.

In particular we will be concerned with linear image restoration, in which \( \hat{x}(i,j) \) is constrained to be the best linear (optimal) estimate of \( x(i,j) \). In this thesis we use discrete spatial variables exclusively and consider the restoration problem from the point of view in which a spatially sampled degraded image \( y(i,j) \) is given and in which we try to reconstruct the spatially sampled original image \( x(i,j) \). Topics dealing with sensor, digitizer and display degradations are not considered here; see [4], [74].

We may note that the term "image enhancement" often used for improving image quality is broader in scope than the term "image restoration." While image restoration is concerned with removing some degradations in order to obtain the original image as closely as possible, in image enhancement we want to place the image in a form suitable for our purposes. We do not necessarily want the original image. It is a technique that allows certain aspects of the image to be more readily and accurately processed by the human visual system, although possibly at the cost of other aspects. For example, we may want to oversharpen edges in an image or to use pseudo colors.

**Modeling**

It is important to note that the term restoration implies the use of a priori knowledge. In the absence of such information, there is no possibility of even discovering that there is anything wrong with a given image. It is a priori information that leads us to the conclusion that, for example, graininess of a photograph is an artefact and not part of the scene [85]. Of course, the a priori information is frequently little more than a reasonable assumption.

About the original, undistorted image we would like to have a priori structural information. However, the world as captured in an image is, in general, so complex that a priori deterministic models of an image still seem to be beyond reach. Thus simple stochastic descriptions for images have been used. In this thesis the image is mainly considered as a sample of a 2-D random field. Characteristic features of the image are...
then described by means of the first- and second-order moments. This subject will be treated in Chapter 2.

In general, the degradation introduced by the imaging system may be very complex [34], [85]. However, in many cases of practical importance, such as camera motion, atmospheric turbulence and blur due to the optical transfer function of lenses, the imaging system can be modeled as a linear system [32]. Then, the degraded image can be modeled by the following 2-D superposition summation:

\[
y(i,j) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} g(i,j;k,l)x(k,l) + w(i,j),
\]

(1.1.1)

where \(y(i,j)\) is the degraded image, \(x(k,l)\) is the original image, \(w(i,j)\) is an additive noise term and \(g(i,j;k,l)\) is the impulse response or point-spread function (PSF) of the linear imaging system. If the linear system is spatially invariant, then (1.1.1) reduces to a 2-D convolution summation:

\[
y(i,j) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} g(k,l)x(i-k,j-l) + w(i,j).
\]

(1.1.2)

Concentrating next on this linear space-invariant case, we may ask how to determine the PSF \(g(k,l)\). In some cases we have the degrading system at our disposal, for example the camera-film system which we used to obtain our images. Then, we can measure the PSF of the lens and film. It must be noted, however, that some of the degrading effects of the lens and film are space-variant and nonlinear [34]. In many other cases the degrading system is not available; but can be inferred from the underlying physical process such as atmospheric turbulence and motion degradation. Then, we postulate a reasonable model for a linear space-invariant degrading system and calculate its PSF. If the degradation is of an unknown nature or if the phenomenon underlying the degradation is too complex for an analytical determination of \(g(k,l)\) the only possible alternative is to estimate it from the degraded image itself. For example, if there is some reason to believe that the original scene contains a sharp point, then the image of that point can be taken as the PSF. This would be the case in an astronomical image, where the image of a faint star could be used as an estimate of the PSF. If the original image contains sharp lines or sharp edges, then it is sometimes possible to determine \(g(k,l)\) from these sharp lines or edges [34], [77]. Another approach is to estimate the unknown PSF from the degraded image by taking averages of image segments in the log-spectral domain [18], [86].

In addition to a knowledge of the PSF, one needs to know (at least in theory) both the statistical properties of the noise itself and how it is correlated with the image. We mentioned already that the noise may originate from the image-formation process, the transmission medium, the recording process, or any combination of these. We assume that these different noise sources may be lumped together as a single additive white-noise term, uncorrelated with the image. The concept of white noise is a mathematical abstraction, but a convenient model, provided the noise bandwidth is much larger than the image bandwidth. As far as the hypothesis of image and noise being additive and uncorrelated is concerned, there are applications where this is not fully satisfactory, e.g. in the case of film grain noise or in the case of quantum limited images such as x-rays and nuclear scan images in medical applications [20], [25], [91].

Optimal nonrecursive filtering

We now turn to the problem of restoring images degraded by linear space-invariant blur in the presence of additive white noise uncorrelated with the data. One of the first methods proposed for image restoration was aimed solely at removing the effects of blur and essentially ignores the presence of additive noise. This so-called inverse filter uses the convolutional inverse of the blur introduced by the imaging system. In the presence of noise it becomes a poor restoration technique [93]. Therefore, in our approach, we explicitly take the presence of noise into account and try to find an optimum linear filter with PSF \(b(i,j)\) which, when acting upon the noisy degraded image \(y(i,j)\), will give an estimate of the original image \(x(i,j)\) under some optimality cri-
If the restored image is meant for a human observer, this optimality criterion should in some manner incorporate the properties of the human visual system. Such properties, however, are very hard to describe mathematically and therefore cannot be included in a simple optimization procedure. One optimality criterion that has been extensively used on account of its mathematical simplicity is the mean-squared error (MSE).

A linear restoration filter that minimizes (in a statistical sense) the mean-squared error between the original image $x(i,j)$ and its estimate $\hat{x}(i,j)$ is the least-squares or noncausal Wiener spatial filter [77], described in the frequency domain by:

$$H(\omega_1,\omega_2) = \frac{G^*(\omega_1,\omega_2)}{\|G(\omega_1,\omega_2)\|^2 + S_x(\omega_1,\omega_2)}$$

where $H(\omega_1,\omega_2)$ and $G(\omega_1,\omega_2)$ are the 2-D discrete Fourier transforms (DFT's) of the PSF's of the filter and the blur, respectively, and where $S_x(\omega_1,\omega_2)$ and $S_y(\omega_1,\omega_2)$ are the power spectral-density functions of the original image and the noise, respectively. The symbol $^*$ denotes a complex conjugate. In deriving (1.1.3) it is assumed that the image $y(i,j)$ is available for all $(i,j)$. In reality, we will only have a finite piece of it, which will lead to approximations [37], [72]. Note that in the absence of noise (1.1.3) reduces to the inverse filter mentioned earlier:

$$H(\omega_1,\omega_2) = \frac{1}{G(\omega_1,\omega_2)}$$

This filter is optimal in the mean-square error sense, if no noise is present. The Wiener-restored image is described in DFT form as

$$\hat{x}(\omega_1,\omega_2) = H(\omega_1,\omega_2) Y(\omega_1,\omega_2),$$

where $\hat{x}(\omega_1,\omega_2)$ and $Y(\omega_1,\omega_2)$ are the 2-D DFT's of $\hat{x}(i,j)$ and $y(i,j)$, respectively. Note that the Wiener spatial filter (1.1.3) requires a priori knowledge about the PSF $g(k,l)$ of the blur and of the image and noise autocovariance functions to evaluate the spectral-density functions $S_x(\omega_1,\omega_2)$ and $S_y(\omega_1,\omega_2)$. Other frequency-domain methods do not have all these requirements. For example, the constrained least-squares filter of Hunt [38] eliminates the requirement of covariance knowledge. In this case the resulting spatial filter can be described in the frequency domain by

$$H(\omega_1,\omega_2) = \frac{G^*(\omega_1,\omega_2)}{\|G(\omega_1,\omega_2)\|^2 + \lambda G^*(\omega_1,\omega_2) S_x(\omega_1,\omega_2)}$$

where $\lambda$ is a Lagrange multiplier found by iteration and $C(\omega_1,\omega_2)$ is the 2-D DFT of a constraint matrix $c(i,j)$. The relation (1.1.6) is similar to (1.1.3): a family of filters can be derived, of which (1.1.3) is a special case. Stockham et al. [86] and Cannon [18] describe a blind deconvolution filter

$$H(\omega_1,\omega_2) = \sqrt{\frac{1}{\|G(\omega_1,\omega_2)\|^2 + S_x(\omega_1,\omega_2)}}$$

where $G(\omega_1,\omega_2)$ is estimated from the degraded image by taking averages of image segments in the log-spectral domain. For a PSF $g(k,l)$ with zero phase, this filter is the geometric mean of the Wiener filter (1.1.3) and the inverse filter (1.1.4). Note that images corrupted by multiplicative noise may be filtered by first taking the logarithm, then filtering with a linear spatial filter as described, and finally exponentiating. This process is called homomorphic filtering [67].

At this point we want to make several observations concerning the filters described. Note first that they are nonrecursive and in principle require the block processing of the entire image or substantial sections of the image. Hence the computational and storage burden of these filters can be quite high. Further, these filters are formulated in the frequency domain by means of their transfer functions. It should be emphasized that frequency-domain analysis, although very attractive, can only be utilized for space-invariant linear degradations. For nonlinear and space-varying linear degradations, the frequency-domain analysis cannot be used to advantage. Even in the space-invariant case, where
most of our effort will be invested, the frequency-domain transfer function suffers from the major disadvantage that all initial conditions of the filter are ignored. Therefore, a more attractive approach to image restoration is to use a spatial-domain technique with emphasis on reducing the computational and storage burden.

Optimal recursive filtering

Once the PSF of a nonrecursive filter is known, many techniques are available [58] to synthesize a recursive digital filter which approximately achieves a given PSF. For example, there exist methods for designing a recursive filter whose PSF is close to the desired filter PSF in the mean square [80] or minimax [57] sense. In our approach, however, we want to develop an optimal recursive linear filter in the spatial domain, which recursively minimizes the mean-squared error between the original image \( x \) and its estimate \( \hat{x} \), without having to approximate another given PSF. In one dimension such a filter scheme is given by the recursive Kalman filter, which offers considerable computational savings over nonrecursive methods [93]. An interesting question is how to extend such a filter to two dimensions. The first attempt to extend Kalman filtering to the processing of image data was performed by Nahi and Assefi [62]. However, even though the observed image is a 2-D data array, it was treated as one dimensional by scanning the image line by line and applying a 1-D Kalman filter. The design of a 1-D Kalman filter (see Section 1.2) relies heavily on a dynamic (recursive) representation of the observed or recorded signal. Hence, to develop such a technique in two dimensions, attention must be paid to modeling the observed or recorded image into a recursive equation. Problems encountered are the lack of a 2-D spectral factorization theorem in designing recursive image models, the choice of a processing order (causality in two dimensions), the possibility of a nonwhite input and a model output with negative spectral components (see Chapter 2). Habibi [36] was the first to generalize Kalman filtering to two dimensions. He derived a 2-D causal recursive linear filter based on a 2-D discrete image model.

This filter, which is essentially a one-step predictor, will be discussed in Chapter 3 of this thesis for the case of additive noise only and compared with a true filter variant. Unfortunately, these generalizations do not preserve the optimality of the original Kalman filter. In Chapter 5 an attempt is made to derive an optimal filter in two dimensions, finally resulting in a Kalman filter for vector observations, where the filter processes the image line by line.

Performance

The reader is reminded once more that the use of the word optimal in both the nonrecursive and recursive filter descriptions refers strictly to a mathematical design concept (minimizing MSE) and not to optimum response for the human visual system. The MSE criterion weights all errors equally, regardless of their location in the image, even though it is known that the human eye demands a more faithful reproduction of the regions where the intensity changes rapidly than of those that change little. Further, it is known that the sensitivity of the eye to a given error in intensity depends strongly upon the intensity itself [85]. In fact, our present lack of knowledge about visual perception precludes a general formulation of the image-restoration problem which takes into account observer preference and capability. In Chapter 4, however, an attempt is made to include the preference of the human observer for sharp undistorted edges in the restored image, by providing a simple 1-D line-scanning filter with edge information. Some other eye-adapted filters using a simple model of the human visual system are described in [38], [53], [71].

To evaluate filter performance we may distinguish two main methods, namely: evaluating the displayed estimates visually, and using filter performance measures. The visual evaluation of the displayed images is important because the aim is to produce a restored image. However, the results will always be observer-dependent, in contrast to performance measures which make a more objective evaluation possible. A commonly used measure of filter performance, which will be introduced in Chapter
3, is the mean-square error improvement or improvement in signal-to-noise ratio (SNR) after filtering. However, this measure favors noise suppression over resolution and is therefore too optimistic a measure in the sense that an improvement in SNR is not necessarily accompanied by an equivalent improvement in visual quality*. To overcome this problem we introduce two additional performance measures in Chapter 3 for the case of restoring noisy images. These measures make it possible to evaluate separately how well a filter removes the noise and to what extent it distorts the original image. For the case of noisy blurred images we introduce in Chapter 5 a frequency-weighted performance measure to accentuate the recovery of high-frequency components in the restored image.

1.2. ONE-STEP PREDICTION AND FILTERING ALGORITHMS FOR THE DISCRETE 1-D KALMAN FILTER

In this section both the one-step prediction and the filtering algorithms are given for the discrete 1-D Kalman filter. These equations are well-known (see for example [6], [48], [65]) and are given here for notational purposes and because we will often refer to these equations when discussing extensions of the 1-D filter concept to two dimensions.

The key assumption in 1-D discrete-time Kalman filtering is that the original signal process, given a covariance description, can be modeled by a first-order linear vector dynamic system

\[
\begin{align}
S(k+1) &= A(k) S(k) + B(k) W(k), \\
X(k) &= C(k) S(k),
\end{align}
\]

where \( S(k) \) is the signal vector, \( S \) the state vector of the system and \( W \) the input noise vector, uncorrelated with the signal and with properties

\[
E[S(k)] = 0, \quad E[W(k)W^T(k)] = \Sigma \quad \delta_{k_1 k_2},
\]

where \( \delta_{k_1 k_2} \) is the Kronecker delta function. The matrices \( A, B \) and \( C \) with proper dimensions are, respectively, the system, drive, and observation matrices. All these matrices are in general functions of \( k \).

Equation (1.2.1) is called a state-space representation of the signal \( x \), where the state vector is defined as a minimal set of variables \( s(k) = [s_1(k), \ldots, s_n(k)]^T \), such that information about these variables at \( k > k_1 \), along with the input \( u(k) \) for all \( k > k_1 \), uniquely determines the output \( x(k) \) for all \( k > k_1 \). Every second-order process whose covariance function possesses a nonnegative rational spectrum has a state-space representation with a finite-dimensional state space [48].

Further, we assume that observations (or measurements) of the original signal are made in the presence of additive white observation noise, yielding the following observation equation:

\[
y(k) = x(k) + W(k), \tag{1.2.2}
\]

where \( y \) is the observation vector and \( W(k) \) the observation noise vector with the following properties:

\[
E[W(k)] = 0, \quad E[W(k)W^T(k)] = \Sigma \quad \delta_{k_1 k_2},
\]

If the observations are made through a medium introducing linear distortion and additive noise, (1.2.2) can be replaced by

\[
y(k) = H(k)x(k) + W(k), \tag{1.2.3}
\]

where \( H \) is the distortion matrix.

The initial state \( s(0) \) is assumed to be a random vector with a known a priori covariance matrix \( P(0) \).
One-step prediction

The one-step prediction problem is the following. Given the first-order linear vector dynamic model (1.2.1) and the observation equation (1.2.2), we want to find an estimate $\hat{s}(k+1)$ of the vector $s(k+1)$, which is a linear function of the observations $y(0), y(1), \ldots, y(k)$ minimizing the error

$$E[(s(k+1) - \hat{s}(k+1))^TW(s(k+1) - \hat{s}(k+1))],$$

(1.2.4)

where $W$ is any nonnegative definite matrix. For example, $W = I$ is a proper choice. It can be shown that the optimal solution is independent of the choice of $W$ [6]. The one-step predictor for the state variables of the linear dynamic system described by (1.2.1) and (1.2.2) are given in Table 1.1 for the time-invariant case. Equations (1.2.5a)-(1.2.5c) have the following simple interpretation. First we project in (1.2.5a) the last estimate $\hat{s}(k)$ ahead by using the dynamics of the signal model (1.2.1). Then we update this estimate by using the new information in the observation, the so-called innovation.

One-step predictor equation

$$\hat{s}(k+1) = A\hat{s}(k) + F(k) [y(k) - C\hat{s}(k)].$$

(1.2.5a)

prediction term innovation term

Predictor gain

$$F(k) = A P(k) C^T [C P(k) C^T + \Sigma_u]^{-1}.$$  
(1.2.5b)

Predictor mean-square error equation

$$P(k+1) = [A - F(k) C] P(k) [A - F(k) C]^T + B \Sigma_u B^T + F(k) \Sigma_u F(k)^T.$$  
(1.2.5c)

Table 1.1. One-step predictor algorithm

Equation (1.2.5b) gives the gain matrix for the updating in (1.2.5a). The remaining equation (1.2.5c) is the predictor mean-square error equation necessary to calculate the new gain matrix, where $P(k+1)$ is the covariance matrix of the prediction error vector $\{s(k+1) - \hat{s}(k+1)\}$. Note that (1.2.5a) sometimes is written in the following form

$$\hat{s}(k+1) = [A - F(k) C] \hat{s}(k) + F(k) y(k).$$

(1.2.6)

If an estimate of the state $\hat{s}(k)$ is obtained, the estimate of the original signal vector can be found by

$$\hat{x}(k) = C\hat{s}(k).$$

(1.2.7)

Discrete Linear filtering

The filtering problem is to determine the estimate $\hat{s}(k)$, given the observations $y(0), y(1), \ldots, y(k)$. This filtered (zero-step prediction) estimate $\hat{s}(k)$ can be obtained from the one-step prediction estimate $\hat{s}(k+1)$ denoted as $\hat{s}_{\text{pred}}(k+1)$ by using (see [65]):

$$\hat{s}(k) = A^{-1} \hat{s}_{\text{pred}}(k+1).$$

(1.2.8)

Leaving out the details of the algebraic manipulations (see [68]), this results in the recursive filter equations as given in Table 1.2.

Filter equation

$$\hat{s}(k) = A \hat{s}(k-1) + F(k) [y(k) - C A \hat{s}(k-1)].$$

(1.2.9a)

Filter gain

$$F(k) = Q(k-1) [Q(k-1) + \Sigma_u]^{-1}.$$  
(1.2.9b)

Error covariance equation

$$Q(k) = A [I - F(k) C] Q(k-1) A^T + B \Sigma_u B^T.$$  
(1.2.9c)

Table 1.2. Discrete filter algorithm
In (1.2.9c) $Q(k)$ is the covariance matrix of the prediction error vector $s(k+1)-s_{\text{pred}}(k+1)$. In terms of the covariance matrix $P^0(k)$ of the filter error vector $s(k)-s_{\text{filter}}(k)$ the matrix $Q(k)$ can be written as

$$Q(k) = A P^0(k) A^T + B \sum_{u} B^T.$$

(1.2.10)

**Optimal Linear Smoothing**

Suppose we are given the observations $y(0), \ldots, y(N)$ and we wish to find an optimal estimate $\hat{s}(k/N)$ of the state $s(k)$, where $0 \leq k \leq N$. This estimation procedure is known as fixed-interval smoothing [31], where the initial and final time instances 0 and N are fixed and where the smoothed estimate of $s(k)$ is based on all observations from 0 to N. An optimal smoother can be thought of as a suitable combination of two optimal estimators, one running in a forward direction over the data, yielding an estimate $\hat{s}_F(k)$ of $s(k)$, and the other running backwards, yielding an estimate $\hat{s}_B(k)$ of $s(k)$. Suppose that $\hat{s}_F(k)$ and $\hat{s}_B(k)$ are two unbiased estimators of the same state $s(k)$. If $\hat{s}_F(k)$ and $\hat{s}_B(k)$ are uncorrelated, i.e. if

$$E[(s(k)-\hat{s}_F(k))(s(k)-\hat{s}_B(k))^T] = \mathbf{0},$$

(1.2.11)

then the optimal smoothed estimate of $s(k)$ is obtained as follows [29]:

$$\hat{s}(k/N) = P(k/N) (P_F^{-1}(k) \hat{\hat{s}}_F(k) + P_B^{-1}(k) \hat{\hat{s}}_B(k)),$$

$$P(k/N) = (P_F^{-1}(k) + P_B^{-1}(k))^{-1},$$

(1.2.12)

where the covariance matrices $P_F(k)$ and $P_B(k)$ are defined by

$$P_F(k) = E[(s(k)-\hat{s}_F(k))(s(k)-\hat{s}_F(k))^T],$$

(1.2.13a)

$$P_B(k) = E[(s(k)-\hat{s}_B(k))(s(k)-\hat{s}_B(k))^T].$$

(1.2.13b)

We will refer a few times to this fixed-interval smoothing problem when recursively filtering image data, because in our application the com-
The purpose of this chapter is to discuss several aspects of modeling images into discrete 2-D autoregressive type of models based on the second-order statistics of the images to be presented.

In Section 2.1 a stochastic image representation is given. To reduce the number of degrees of freedom in the image autocovariance matrix, the concepts of homogeneity, separability, isotropy and autoregression are introduced. Next, in Section 2.2 a general linear 2-D autoregressive type of model is introduced for homogeneous images. The model coefficients are found in a linear \( \text{MSE} \) fitting procedure. Special attention is focused on the meaning of causality in two dimensions and on the covariance structure of the model error. To obtain an adequate fit between the actual image autocovariance function and the autocovariance function of the output of the model this covariance structure must be exploited, in particular, because semicausal and noncausal minimum variance models are not driven by white noise. Some examples of fitting for causal, semicausal and noncausal models are given in Section 2.4.

Separability and stability, two other important topics in developing 2-D image models, will be discussed in Sections 2.3 and 2.5, respectively.

Next, in Section 2.6, some quality criteria are introduced, including the Cramér-Rao bound, to compare several models based on the same image autocovariance function.
Finally, in Section 2.7, some experimental results of this comparison are given.

2.1. STOCHASTIC IMAGE REPRESENTATION

Let the 2-D array of real numbers

\[ X = \{ x(i,j) \}, \quad i=1, \ldots, M, \quad j=1, \ldots, N, \]  

represent a discrete monochromatic original image. The variable \( x(i,j) \) represents the intensity value of a picture element (pixel) at spatial coordinate \((i,j)\), where \( i \) and \( j \) are the vertical and horizontal position variables, respectively. When using a stochastic image characterization, the image representation (2.1.1) can be regarded as a sample (realization) from a random field \( \{ x(i,j) \} \) defined on a finite \((MxN)\) subset \( U \) of the 2-D lattice \( IxI^2 \), where \( I \) is the set of integers. This random field can be completely described in a probabilistic way by means of a joint probability density function of \( MN \) random variables. To find a suitable description, the \( MN \times MN \) matrix \( X \) is converted to an \( MN \times 1 \) column vector \( x \) through the use of an \( Mn \) operational vector \( v_n \) and an \( MN \times MN \) matrix \( N_n \) defined as:

\[
 v_n = \begin{bmatrix} 0 & 1 & \cdots & n-1 & 0 & 1 & \cdots & n & \cdots & 0 & 1 & \cdots & n \end{bmatrix}, \quad N_n = \begin{bmatrix} 0 & \cdots & 1 \end{bmatrix},
\]

where \( I \) and \( O \) are the \( MN \times MN \) identity and the \( MN \times MN \) null matrix, respectively [73]. Then, the vector representation of the image \( X \) is given by

\[ x = \sum_{n=1}^{M} N_n x^t v_n. \]  

In essence, the vector \( v_n \) extracts the \( n \)th row from \( X \) and the matrix \( N_n \) places this row into the \( n \)th segment of the \( MN \times 1 \) vector \( x \). Thus \( x \) contains the row-scanned elements of \( X \). The ordering thus obtained is called a lexicographic ordering. The reverse operation of converting the vector \( x \) into matrix form is given by

\[ X = \sum_{n=1}^{M} v_n x^t N_n. \]  

Now it becomes possible to discuss the vector \( x \) in (2.1.3) as being a sample from a vector random process and to consider probability density functions used in multivariate analysis as possible candidates for a stochastic image description [39]. Actually, the number of candidates to be considered is very small. For most practical cases, the multivariate Gaussian probability density function (p.d.f.) is the only p.d.f. that is used as a model for multivariate data. The utility of the multivariate Gaussian p.d.f. is its ability to embody the dependence between the components of the vector \( x \) (spatial correlation in the case in which \( x \) represents a lexicographic-ordered image) directly in the form of an autocovariance matrix, i.e.,

\[ p(x) = \frac{1}{(2\pi)^{\frac{MN}{2}} |R|^{\frac{1}{2}}} \exp \{-\frac{1}{2}(x-\mu)^t R^{-1}(x-\mu)\}, \]

where \( R \) is the autocovariance matrix of \( x \), \( \mu \) is the mean vector and \( |R| \) denotes the determinant of \( R \). An important restriction at this point is the nonnegativity property of the elements of \( X \) (intensity of the incident image energy), i.e.

\[ x(i,j) \geq 0, \quad \forall(i,j) \in U. \]

We assume that the components of the mean vector \( \mu \) are each nonnegative and sufficiently larger than the corresponding variance components (diagonal elements of \( R \)), so that restriction (2.1.6) is violated only very infrequently by this model. However, we have to be aware of the limitations of using the Gaussian density function. A histogram of the intensities of an arbitrary pixel over an ensemble of images is seldom
unimodal, and only satisfies the Gaussian assumption weakly [39].

**Second-order statistics**

We may conclude from the arguments above that a complete description of a random field by means of a suitable p.d.f. is seldom possible. However, in many cases it may be sufficient to use the first- and second-order moments to describe a random field, because important quantities such as mean values and autocovariances can be obtained. In particular, these first- and second-order moments may completely determine the a priori knowledge needed for linear MSE filtering of noise-corrupted image data, while in the Gaussian case both moments completely describe the random field.

The first moment (the mean value) is given by

\[ E[x(i,j)] = \mu(i,j), \quad \forall (i,j) \in U, \]  

(2.1.7)

and the second central moment (the autocovariance), which describes the dependence between two arbitrary pixels \( x(i,j) \) and \( x(s,t) \), is given by

\[ r(i,j;s,t) = E[(x(i,j)-\mu(i,j))(x(s,t)-\mu(s,t))], \quad \forall (i,j),(s,t) \in U, \]  

(2.1.8)

where \( E \) denotes averaging over the ensemble of images. Because we work with real data, it holds that \( r(i,j;s,t) = r(s,t;i,j) \). The image autocovariance matrix \( R \) can now be found as follows (We set \( \mu = 0 \)):

\[ R = E[x x^T] = E\left\{ \sum_{m=1}^{M} \sum_{n=1}^{N} x_m v_n x_m v_n^T X N \right\} \]  

\[ = \sum_{m=1}^{M} \sum_{n=1}^{N} E[ N \sum_{m=1}^{M} v_n x_m v_n^T X N ] \]  

\[ = \sum_{m=1}^{M} \sum_{n=1}^{N} R_{m,n} N_n^T, \]  

(2.1.9)

where \( R_{m,n} \) is the \( N \times N \) autocovariance matrix of the \( m \)th and \( n \)th row of \( X \) with the following structure:

\[
\begin{bmatrix}
    r(m,1;n,1) & r(m,1;n,2) & \cdots & r(m,1;n,N) \\
    r(m,2;n,1) & r(m,2;n,2) & \cdots & r(m,2;n,N) \\
    \vdots & \vdots & \ddots & \vdots \\
    r(m,N;n,1) & r(m,N;n,2) & \cdots & r(m,N;n,N)
\end{bmatrix}
\]  

(2.1.10)

Thus, it is possible to express \( R \) in partitioned form as the \( MN \times MN \) matrix

\[
\begin{bmatrix}
    R_{1,1} & R_{1,2} & \cdots & R_{1,N} \\
    R_{2,1} & R_{2,2} & \cdots & R_{2,N} \\
    \vdots & \vdots & \ddots & \vdots \\
    R_{N,1} & R_{N,2} & \cdots & R_{N,N}
\end{bmatrix}
\]  

(2.1.11)

It is obvious that the image autocovariance matrix \( R \) can be used as statistical knowledge in designing image models. The number of degrees of freedom of maximally \( \frac{1}{2}(MN^2+MN) \) in this symmetric matrix \( \text{R}=R^T \), however, makes this hardly feasible.

In the following we will pay attention to the reduction of the number of degrees of freedom by assuming certain image properties. Before doing so, it is good to note that in modeling images we try to find a balance between reducing complexity on the one hand, which benefits the simplicity of algorithms based on these models, and guarding against oversimplification on the other, which can have a disadvantageous influence on the filter results.
Homogeneity, separability and isotropy

A considerable reduction in the number of degrees of freedom can be achieved by assuming the random field to be homogeneous (wide sense stationary). Then the mean value

\[ \mathbb{E} [x(i,j)] = \mu, \quad \forall (i,j) \in \mathcal{U}, \quad (2.1.12) \]

is space-invariant and the autocovariance is translation-invariant, i.e. depends only on the difference between the coordinates

\[ r(i,j;s,t) = \mathbb{E}[(x(i,j)-\mu)(x(s,t)-\mu)] = r((i-s),(j-t)) = r(k,l), \quad \forall (i,j),(s,t) \in \mathcal{U}. \quad (2.1.13) \]

Since we work with real data, it holds that \( r(k,l) = r(-k,-l) \), i.e. the autocovariance is centro-symmetric.

The autocovariance matrix \( R_{m,n} \) of the \( m \)th row and \( n \)th row of \( X \) in (2.1.10) can now be written as follows:

\[
R_{m,n} = \left[ \begin{array}{ccc}
    r(m-n,0) & r(m-n,-1) & \ldots & r(m-n,N+1) \\
    r(m-n,1) & r(m-n,0) & \ldots & \ldots \\
    \ldots & \ldots & \ldots & \ldots \\
    r(m-n,N) & r(m-n,N-1) & \ldots & r(m-n,0) \\
\end{array} \right] \quad (2.1.14)
\]

with \( R_{m,n} = R_{n,m}^{\top} \).

As a consequence, the image autocovariance matrix \( R \) is symmetric and has a block Toeplitz form with \( N^2 \) blocks, each block also being Toeplitz with dimension \( N \times N \):

\[
R = \left[ \begin{array}{cccc}
    R_0 & R_{-1} & \ldots & R_{M-1} \\
    R_1 & R_0 & \ldots & \ldots \\
    \ldots & \ldots & \ldots & \ldots \\
    R_{M-1} & \ldots & R_{-1} & R_0 \\
\end{array} \right] \quad (2.1.15)
\]

Note that the elements of a Toeplitz matrix \( A \) are characterized by \( a(i,j) = a(i-j) \). The number of degrees of freedom in this case is maximally \( 2MN-M \).

Reduction in the number of degrees of freedom can also be achieved by introducing the concepts of separability or of isotropy. Introducing separability means that the variability of pixel intensities in the vertical direction (columns) will not be related to the variability in the horizontal direction (rows). Then the autocovariance matrix \( R \) can be factorized into a Kronecker product [17]:

\[
R = R_v \otimes R_h
\]

where \( R_v \) and \( R_h \) denote the \( M \times M \) autocovariance matrix of the columns and the \( N \times N \) matrix of the rows of \( X \), respectively, defined by

\[
R_v = \mathbb{E}_v \{ (x_{v,-1}) (x_{v,-1})^\top \},
\]

\[
R_h = \mathbb{E}_h \{ (x_{h,-1}) (x_{h,-1})^\top \}.
\]

By using the operators \( \mathbb{E}_v \) and \( \mathbb{E}_h \) we average over all possible columns and rows of the ensemble of images, where
The number of degrees of freedom in the case of separability will be maximally \( (M^2 + N^2 + M + N) \).

Combination of homogeneity and separability yields for the autocovariance in (2.1.13):

\[
\rho(i,j;s,t) = \rho((i-s),(j-t)) = \rho(k,l) \rho(k) \rho(l)
\]

Then, the autocovariance matrices \( R_v \) and \( R_h \) of the columns and the rows, respectively, have the following symmetric Toeplitz form:

\[
R_v = \begin{bmatrix}
\rho_v(0) & \rho_v(1) & \cdots & \rho_v(M-1) \\
\rho_v(1) & \rho_v(0) & \cdots & \\
& & \ddots & \\
\rho_v(M-1) & \cdots & \rho_v(1) & \rho_v(0)
\end{bmatrix}
\]

and

\[
R_h = \begin{bmatrix}
\rho_h(0) & \rho_h(1) & \cdots & \rho_h(N-1) \\
\rho_h(1) & \rho_h(0) & \cdots & \\
& & \ddots & \\
\rho_h(N-1) & \cdots & \rho_h(1) & \rho_h(0)
\end{bmatrix}
\]

The number of degrees of freedom is now reduced to maximally \( M+N \).

Let us next investigate the concept of isotropy. From the foregoing we have seen that the autocovariance is in general a function of four independent variables, while in the homogeneous case it is a centro-symmetric function of two variables. The autocovariance may still possess a higher degree of symmetry, when it is rotation-invariant. In that case, the autocovariance is a function of one variable: the Euclidean distance between two pixels. Random fields with this property are called homogeneous and isotropic random fields \([99]\) with

\[
\rho(i,j;s,t) = \rho((i-s)^2 + (j-t)^2) = \rho(d)
\]

with \( d = \sqrt{(i-s)^2 + (j-t)^2} \).

The image autocovariance matrix \( R \) has a symmetric block Toeplitz structure with \( M^2 \) blocks, each block also being a symmetric Toeplitz matrix with dimension \( M \times M \). The number of degrees of freedom is equal to the number of geometric distances between the pixels in the image and this is equal to maximally \( (M^2 + M) + (N-M)M \), with \( M \ll N \). Note that the image autocovariance matrix is not separable in this case.

**Autoregressive processes**

A further reduction in the number of degrees of freedom can be achieved by introducing the concept of autoregression, which makes the number of degrees of freedom independent of the image size \( M \times N \).

In discrete time-series analysis \([16]\), one often makes the assumption that a weakly stationary process \( \{x(t)\} \) is generated by a causal linear time-invariant system, whose input is a zero-mean white-noise process \( \{u(t)\} \) with variance \( \sigma^2 \):

\[
x(t) = x(t) + \sum_{i=1}^{\infty} \psi(i) u(t-i), \quad -\infty < t < \infty,
\]

where \( \psi(i) \) are the weight coefficients of the previous input values \( u(t-i) \). Alternatively, (2.1.23) can be written as a weighted sum of previously generated values \( x(t-1), x(t-2), \ldots \) and an input \( u(t) \)

\[
x(t) = \sum_{i=1}^{\infty} a(i) x(t-i) + u(t), \quad -\infty < t < \infty,
\]

If the coefficients \( a(i) \) are zero for \( i > m \), then the process is called an autoregressive process of order \( m \), denoted by AR\((m)\),
\[ x(t) = \sum_{i=1}^{m} a(i) x(t-i) + u(t). \] (2.1.25)

Since \( u(t) \) has zero mean and the process \( u(t) \) is stationary, it follows that \( E[x(t)] = 0 \). By using the \( z \)-transform in negative powers of \( z \) we can write (2.1.25) as

\[ [1 - \sum_{i=1}^{m} a(i) z^{-i}] X(z) = U(z), \] (2.1.26)

and by defining the transfer function \( H(z) \) as

\[ H(z) = \frac{1}{1 - \sum_{i=1}^{m} a(i) z^{-i}} \] (2.1.27)

we find the relation

\[ X(z) = H(z) U(z). \] (2.1.28)

The stability of (2.1.25) is guaranteed if all poles of \( H(z) \) lie inside the unit circle.

To find the autocovariance function \( r(k) \) of the process \( \{x(t)\} \) of (2.1.25) both sides are multiplied by \( x(t-k) \) for \( k > 0 \) after which the expectation is taken. Then we find

\[ r(k) - \sum_{i=1}^{m} a(i) r(k-i) = 0, \quad \text{for} \ k > 0. \] (2.1.29)

Note that the expectation \( E[x(t-k)u(t)] \) vanishes when \( k > 0 \), since \( x(t-k) \) can only involve \( u(.) \) up to time \( t-k \) (causality assumption). The equations (2.1.29) are called the normal equations [19]. The variance \( \sigma_u^2 \) of the input process can be expressed as

\[ \sigma_u^2 = E[u^2(t)] = r(0) - \sum_{i=1}^{m} a(i) r(i). \] (2.1.30)

In particular, for an AR(1) process it follows from (2.1.29) that

\[ r(k) = a(1) r(k-1), \quad k > 0. \] (2.1.31)

The solution of (2.1.31) is given by

\[ r(k) = \sigma_u^2 a(1)^k, \quad k=0,1,2,\ldots \] (2.1.32)

Since \( r(k) = r(-k) \) for a real process, we find

\[ r(k) = \sigma_u^2 a(1)^{|k|}, \quad k=0,\pm1,\pm2,\ldots \] (2.1.33)

where \( \sigma_u^2 \) is the variance of the output process and \( a(1) \) is the correlation coefficient between neighboring sequence elements, with \( |a(1)| < 1 \). This autocovariance function is often called the exponentially decaying autocovariance function

\[ r(k) = \sigma_u^2 \rho |k|, \quad \text{with} \ \rho = a(1). \] (2.1.34)

It should be noted that until now we have treated the AR-process in terms of past and present values. Analogously, Woods [94] defines a process which depends on the past as well as on the future values of \( x(t) \):

\[ x(t) = \sum_{i=-m}^{m} a(i) x(t-i) + u(t), \quad -\infty < t < \infty, \] (2.1.35)

where

\[ i) \ E[x(t-k)u(t)] = 0, \quad \forall k \neq 0, \] (2.1.36)

and

\[ ii) \ E[u(t-k)u(t)] = \begin{cases} \sigma_u^2 & k=0 \\ (-a(k)\sigma_u^2) & |k| \leq m, \\ 0 & \text{elsewhere} \end{cases} \] (2.1.37)

with \( \sigma_u^2 \) the variance of the input. This two-sided AR-process is often called a two-sided Markov process of order \( m \). Note from (2.1.37) that, unlike the one-sided AR-process, the two-sided Markov process is not
driven by a white-noise input process. It can easily be shown [14] that this colored noise process may cancel the poles of the model transfer function outside the unit circle. Therefore, it provides the possibility of replacing a two-sided Markov (m) process by a one-sided AR(m) process. A second difference is that we have to restrict the values of \( a(i) \) in (2.1.35) to such values that the autocovariance function of \( u(t) \) in (2.1.37) is nonnegative definite [94].

It is possible to directly extend a one-sided AR(m) process or a two-sided Markov (m) process to two dimensions by assuming the image-generating process to be a separable autoregressive process in both the vertical and horizontal directions. Then, for the image autocovariance matrix it holds according to (2.1.16) that

\[
R = R_v \otimes R_h.  
\]

By the introduction of separability the number of degrees of freedom is reduced to two plus the sum of the orders of both processes. If, for example, both processes are AR(1) processes, then it holds that

\[
R_v = \begin{bmatrix}
1 & \rho_v & \rho_v^2 & \cdots & \rho_v^{M-1} \\
\rho_v & 1 & \rho_v & \cdots & \rho_v^{M-2} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\rho_v^{N-1} & \cdots & \rho_v & 1
\end{bmatrix}; \quad R_h = \begin{bmatrix}
1 & \rho_h & \rho_h^2 & \cdots & \rho_h^{N-1} \\
\rho_h & 1 & \rho_h & \cdots & \rho_h^{N-2} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\rho_h^{M-1} & \cdots & \rho_h & 1
\end{bmatrix}.  
\]

(2.1.39)

where \( \sigma_v^2 \) and \( \sigma_h^2 \) are the variances in the vertical and the horizontal direction, respectively, and \( \rho_v \) and \( \rho_h \) are the correlation coefficients between neighboring pixels in the vertical and horizontal directions. The number of degrees of freedom is thus reduced to 4 (\( \sigma_v^2, \sigma_h^2, \rho_v, \rho_h \)) and the image autocovariance function can be written as

\[
n(k, k) = \sigma_x^2 \rho_v^{k_1} \rho_h^{k_2},  
\]

(2.1.40)

where \( \sigma_x^2 = \sigma_v^2 + \sigma_h^2 \) is the variance of the image. In the next section we will describe the general problem of modeling images, given certain image statistics, as (not necessarily separable) 2-D autoregressive types of models (2-D stochastic difference-equation models).

Summarizing this section, we have given a second-order description of a random (image) field. By assuming certain image properties a reasonable reduction of degrees of freedom can be obtained. This may result in the design of relatively simple filter structures. The effects of certain assumptions, particularly the homogeneity assumption, on the final filter results will be extensively treated in the following chapters when dealing with filter applications.

2.2. 2-D STOCHASTIC DIFFERENCE-EQUATION MODELS

Because it is our aim to restore distorted images by means of a Kalman filter, it is necessary to model the original image, given certain image statistics, as a recursive equation. We assume an image to be a sample from a discrete homogeneous random field, which can be described by means of a space-invariant mean \( \mu \) (2.1.12) and a translation-invariant autocovariance function \( r_{\text{image}}(k, k) \) (2.1.13). For different estimation procedures of these moments, we refer to [52]. For convenience and without lack of generality we set the mean at \( \mu = 0 \). In analogy with the 1-D case we could conceive of the following general scheme for finding a 2-D stochastic difference equation:

\[
\begin{align*}
(x(i,j)) &= r_{\text{image}}(k, k) = R(k_x, k_y) = R(k_x, k_y) = x(i,j) + \sum_{p=1}^{N-1} a(p) x(i-p,j-q) + u(i,j) \\
\end{align*}
\]

where

- \( a(p) \) are the coefficients of the difference equation
- \( R(k_x, k_y) \) is the autocovariance function
- \( u(i,j) \) is the noise term
- \( x(i,j) \) is the original image
- \( h(i,j) \) is the impulse response
- \( S(k_x, k_y) \) is the power spectrum

The image can be modeled as a linear system with input \( u(i,j) \) and output \( x(i,j) \). The system is described by the difference equation

\[
x(i,j) = \sum_{p=1}^{N-1} a(p) x(i-p,j-q) + u(i,j)  
\]

where \( a(p) \) are the coefficients of the difference equation and \( u(i,j) \) is the noise term. The system can be represented by its transfer function

\[
H(k_x, k_y) = \frac{1}{1 - \sum_{p=1}^{N-1} a(p) e^{j(k_x p + k_y q)}}  
\]

The transfer function describes the relationship between the output and input. The impulse response \( h(i,j) \) is the output of the system when the input is a unit impulse. The power spectrum \( S(k_x, k_y) \) is the Fourier transform of the autocovariance function of the input noise.

The stochastic difference equation can be written as

\[
x(i,j) = \sum_{p=1}^{N-1} a(p) x(i-p,j-q) + u(i,j)  
\]

where

- \( a(p) \) are the coefficients of the difference equation
- \( x(i,j) \) is the original image
- \( u(i,j) \) is the noise term
- \( h(i,j) \) is the impulse response
- \( S(k_x, k_y) \) is the power spectrum

The image can be modeled as a linear system with input \( u(i,j) \) and output \( x(i,j) \). The system is described by the difference equation

\[
x(i,j) = \sum_{p=1}^{N-1} a(p) x(i-p,j-q) + u(i,j)  
\]

where \( a(p) \) are the coefficients of the difference equation and \( u(i,j) \) is the noise term. The system can be represented by its transfer function

\[
H(k_x, k_y) = \frac{1}{1 - \sum_{p=1}^{N-1} a(p) e^{j(k_x p + k_y q)}}  
\]

The transfer function describes the relationship between the output and input. The impulse response \( h(i,j) \) is the output of the system when the input is a unit impulse. The power spectrum \( S(k_x, k_y) \) is the Fourier transform of the autocovariance function of the input noise.
For 2-D random fields, however, factorization of a 2-D spectral-density function $S(z_1,z_2)$ (rational or not) as the magnitude square of a rational function $H(z_1,z_2)$ is not possible in general. The reason for this is that it is generally not possible to factorize a 2-D polynomial as a product of lower-order polynomials. An attempt to generalize spectral factorization to two dimensions is made in [24] by application of the Wiener-Doob technique. The resulting 2-D factors do not generally have finite dimensions, but can be approximated as such.

Due to the lack of a 2-D spectral-factorization theorem, several authors have developed stochastic difference equations in modeling images based on various ideas. Nahi and Assefi [60][61] scan an image line by line. The periodic nature of the scanning procedure results in an output which will be a nonstationary random process. This introduces complexities in the design of a stationary model which in turn necessitates additional approximations. The resulting model is basically a 1-D model. An advantage of this model is the opportunity for vector processing [63], by which it is possible to process several lines simultaneously.

In contrast to Nahi's model, the Differential Pulse Code Modulation (DPCM) model of Habibi [36], the Markov Models of Woods [94] and the Partial Differential-Equation (PDE) models of Jain [41] are two-dimensional by nature.

By extending these 2-D models, it is possible to create many different stochastic difference-equation models for homogeneous images, just by expressing a pixel intensity in a finite number of other pixel intensities. Our linear model for stationary images will then have the following autoregressive form:

$$x(i,j) = \sum_{p,q \in S} a(p,q) x(i-p,j-q) + u(i,j),$$

(2.2.1)

in which $u(i,j)$ can be viewed as the process input or the error in generating $x(i,j)$ by $x(i-p,j-q)$ for all $(p,q) \in S$, where $S$ is a set of index pairs $(p,q)$, which are independent of $(i,j)$, with $(0,0) \notin S$. The set $S$ defines the type of model which is used to describe the image. We assume $E[u(i,j)] = 0$; then $E[x(i,j)] = 0$ because of the homogeneity assumption.

Let us investigate the choice of the set $S$ further. In general, the question will be which values of the 2-D field can be indicated as past and which as future signal values in generating a particular value $x(i,j)$ by means of (2.2.1). Since causality is primarily a time-domain concept and since in an image usually neither of the two independent variables involves time, the concepts of past and future within an image in fact have no meaning. However, any recursive algorithm is sequential by nature, so that an artificial time coordinate has to be introduced, i.e., a scanning path, which yields restrictions on the set $S$. In the remainder of this section the following causality conditions are used.

**Quarter-plane causality**

As a simple extension of the 1-D causality concept to two dimensions, we can define the so-called Quarter-plane (QP) causality [24], where the set $S$ is defined as:

$$S = \{p, q: p > 0, q > 0; p+q > 0\}.$$  

(2.2.2)

Then the past and the future of an arbitrary image point $x(i,j)$ are as given in Fig. 2.1. Often one is only interested in representations where the coefficients $a(p,q)$ are nonzero only over a finite window $W$, which is a subset of $S$. In Fig. 2.2 an example of a QP model is given, with $W = \{(0,1),(1,0),(1,1)\}$. The point $x(i,j)$ is indicated by "x" and the points $x(i-p,j-q)$ with $(p,q) \in W$ are indicated by "O". Because of the finite size $MxN$ of the image field initial values are needed. For QP models the pixels of two boundaries are required as initial values.
Another definition of causality in two dimensions finds its origin in the line-scanning procedure [24]. For these so-called nonsymmetric half-plane (NSHP) models the set $S$ is defined as

$$S = \{p,q: (p > 0, q > 0) \cup (p > 0, q < 0)\}. \tag{2.2.3}$$

This implies another past and future definition; see Fig. 2.3.

In Fig. 2.4 an example of a NSHP model is given, with $W = \{(0,1), (1,0), (1,1), (1,-1)\}$. Now the pixels of three boundaries are required as initial values in generating the $M \times N$ image field. Note that QP models can be considered as a subclass of NSHP models.

**Semicausality and noncausality**

A model is called semicausal [41] when it is causal in one direction and noncausal in the other. With causality in the "i" variable, the set $S$ is defined as

$$S = \{p,q: (p > 0, q > 0) \cup (p > 0, q < 0)\}. \tag{2.2.4}$$

In Fig. 2.5 an example is given of a semicausal model with $W = \{(0,1), (1,0), (1,1), (1,-1), (0,-1)\}$. Now the pixels of two opposite boundaries are required as boundary values and the pixels of the upper boundary as initial values.

In the noncausal case, a model is noncausal in both directions, so that the pixels of four boundaries are required as boundary values (see Fig. 2.6). The set $S$ is defined as

$$S = \{p,q: \forall (p,q) \neq (0,0)\}. \tag{2.2.5}$$

In Fig. 2.6. Example of a noncausal model
The semicausal and noncausal models just defined can lead to smaller prediction errors than the causal models, as will be shown in Section 2.4. However, before recursive filter structures based on these models can be derived, a transformation must be performed [40][42][50] (See also Section 5.1).

After the choice of a set $S$ is made, a particular model specified by the window $W$ is fitted to a given image autocovariance function $r_{\text{image}}(k,\ell)$ by using a linear MSE fitting procedure. This means that the weight coefficients $a(p,q)$ are chosen such that the variance of $u(i,j)$,

$$E[u^2(i,j)] = E[(x(i,j) - \sum_{p,q \in W} a(p,q)x(i-p,j-q))^2], \quad (2.2.6)$$

is minimal. Minimizing (2.2.6) by setting

$$\frac{\partial^2 E[u^2(i,j)]}{\partial a(k,\ell)} = 0, \quad (k,\ell) \in W, \quad (2.2.7)$$

yields the set of normal equations

$$E[(x(i,j) - \sum_{p,q \in W} a(p,q)x(i-p,j-q))x(i-k,j-\ell)] = 0, \quad (k,\ell) \in W. \quad (2.2.8)$$

By recalling the definition of the autocovariance function $r_{\text{image}}(k,\ell)$ (See (2.1.13)), (2.2.8) can be rewritten as

$$r_{\text{image}}(k,\ell) - \sum_{p,q \in W} a(p,q)r_{\text{image}}(k-p,\ell-q) = 0, \quad (k,\ell) \in W. \quad (2.2.9)$$

Thus, to find the best linear MSE coefficients $a(p,q)$ the set of equations in (2.2.9) must be solved by using the autocovariance function $r_{\text{image}}(k,\ell)$ of the original image. Note that (2.2.8) can be written as

$$E[u(i,j) \times (i-k,j-\ell)] = 0, \quad (k,\ell) \in W, \quad (2.2.10)$$

which states the orthogonality of $u(i,j)$ and $x(i-k,j-\ell)$ within the subset $W$ of $S$.

Once the MSE coefficients $a(p,q)$ have been obtained, the variance of $u(i,j)$, denoted as $\sigma^2$, can be derived as follows:

$$\sigma^2 = E[u^2(i,j)] = E[u(i,j)x(i,j) - \sum_{p,q \in W} a(p,q)x(i-p,j-q)]$$

$$- E[u(i,j)x(i,j)] - \sum_{p,q \in W} a(p,q)E[u(i,j)x(i-p,j-q)]. \quad (2.2.11)$$

The second term is zero due to (2.2.10); therefore,

$$\sigma^2 = E[u^2(i,j)] = E[u(i,j)x(i,j)]$$

$$= E[(x(i,j) - \sum_{p,q \in W} a(p,q)x(i-p,j-q))x(i,j)]$$

$$= r_{\text{image}}(0,0) - \sum_{p,q \in W} a(p,q)r_{\text{image}}(p,q). \quad (2.2.11)$$

It should be noted that when the homogeneous field $\{x(i,j)\}$ is Gaussian, the best linear MSE estimate is then equal to the best possible MSE estimate, obtained by calculating the conditional expectation of $x(i,j)$, given $x(i-p,j-q), (p,q) \in W$, which may in general be a nonlinear function of $x(i-p,j-q)$ [99].

Equations (2.2.9) and (2.2.11) can be combined and written in matrix-vector notation, resulting in the so-called Yule-Walker equations:

$$R_n a = b, \quad (2.2.12)$$

where $R_n$ is the autocovariance matrix of the homogeneous image, and

$$a = [1, \ldots, -a(p,q), \ldots]^T, \quad (p,q) \in W,$$

$$b = [\sigma^2, 0, \ldots, 0]^T.$$
Consider for instance the following quarter-plane causal model with spatial structure:

\[
\begin{bmatrix}
x(i,j) = \alpha(0,1)x(i-1,j) + \beta(1,0)x(i, j - 1) + \epsilon(i,j)
\end{bmatrix}
\]  

(2.2.13)

The best linear MSE coefficients \( a(p,q) \) and the variance \( \sigma^2 \) can be found by solving (2.2.12) for this particular model structure, which yields:

\[
\begin{bmatrix}
r_{\text{image}}(0,0) & r_{\text{image}}(0,1) & r_{\text{image}}(1,0) \\
r_{\text{image}}(0,1) & r_{\text{image}}(0,0) & r_{\text{image}}(-1,0) \\
r_{\text{image}}(1,0) & r_{\text{image}}(-1,0) & r_{\text{image}}(0,0)
\end{bmatrix}
\begin{bmatrix}
a(0,1) \\
a(1,0)
\end{bmatrix} = \begin{bmatrix}1
\sigma^2 \end{bmatrix},
\]

(2.2.14)

By observing (2.2.14) it follows that \( R_u \) is symmetric, but not Toeplitz. This is an important difference with the 1-D case, where the nonnegative definite Toeplitz structure of \( R_u \) and its nonnegative definite extension play an important role in developing fast algorithms in estimating the weight coefficients of an AR-process [90]. As a consequence, the nonnegative definiteness of \( R_u \) must be investigated. Besides that, we may remark that for a given nonnegative definite \( R_u \), a unique solution of (2.2.14) is obtained. However, for a given admissible set of coefficients \( a(p,q) \) there is not a unique \( R_u \) as the solution of (2.2.14). Moreover, the solution of (2.2.14) does not assure model coefficients which would yield stable models [84]. Model stability will be investigated in Section 2.5.

The stochastic difference-equation model (2.2.1) with constant MSE coefficients \( a(p,q) \) derived in (2.2.12) can be described by the rational transfer function in negative powers of \( z_1 \) and \( z_2 \):

\[
H(z_1, z_2) = \frac{1}{1 - \sum_{p,q \in \mathbb{W}} a(p,q) z_1^{-p} z_2^{-q}}.
\]

(2.2.15)

If we use \( \{u(i,j)\} \) as input to the model with transfer function (2.2.15) and \( \{x(i,j)\} \) as model output, and if we define the spectral density function of \( \{u(i,j)\} \) as

\[
S_u(z_1, z_2) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} r_u(k,l) z_1^{-k} z_2^{-l},
\]

with \( r_u(k,l) \) the autocovariance function of the model input, and the spectral density function of \( \{x(i,j)\} \) as

\[
S_x(z_1, z_2) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} r_{\text{model}}(k,l) z_1^{-k} z_2^{-l},
\]

with \( r_{\text{model}}(k,l) \) the autocovariance function of the model output, then the following relation holds between model input and model output [41]

\[
S_x(z_1, z_2) = \frac{S_u(z_1, z_2)}{(1 - \sum_{p,q \in \mathbb{W}} a(p,q) z_1^{-p} z_2^{-q})(1 - \sum_{p,q \in \mathbb{W}} a(p,q) z_1^{-p} z_2^{-q})}.
\]

(2.2.16)

Unlike causal minimum variance models, semicausal and noncausal minimum variance models are not driven by white noise. This can be shown as follows. The autocovariance function \( r_u(k,l) \) is given by

\[
r_u(k,l) = E[u(i,j)u(i-k,j-l)]
\]

\[
= E[u(i,j)x(i-k,j-l)] - \sum_{p,q \in \mathbb{W}} a(p,q) E[u(i,j)] x(i-k-p,j-l-q)].
\]

(2.2.17)

The best linear MSE coefficients \( a(p,q) \) of the general model (2.2.1) are calculated by using the orthogonality of \( u(i,j) \) and \( x(i-k,j-l) \) within the subset \( \mathbb{W} \) of \( S \), i.e.

\[
E[u(i,j)x(i-k,j-l)] = 0, \quad \forall (k,l) \in \mathbb{W}.
\]

(2.2.18)
If we assume that the field \( \{x(i,j)\} \) generated by the model is a wide-sense Markov field [70], [89], [94], then the best linear MSE estimate of \( x(i,j) \) in terms of \( \{x(i,j): (i,j) \in W\} \) equals its estimate in terms of the entire past \( \{x(i,j): (i,j) \in S\} \). This means that

\[
E[u(i,j) | (i-k, j-l)] = 0, \quad \forall (k,l) \in S, \quad (2.2.19)
\]

where the set \( S \) specifies the entire past of a model according to the underlying causality condition. By using orthogonality condition (2.2.19) instead of (2.2.18) and recalling the centro-symmetric structure of the autocovariance function, we can find the following explicit expressions for \( r_u(k,l) \) in (2.2.17).

In the causal case, where \( S \) is defined by (2.2.3), only the first term of (2.2.17) contributes to the autocovariance function for \( (k,l) = (0,0) \) (\( \beta^2 \) according to (2.2.11)), i.e.,

\[
r_u(k,l) = \begin{cases} 
\beta^2 & (k,l) = (0,0) \\
0 & \text{elsewhere}
\end{cases}, \quad (2.2.20)
\]

a function which describes a white-noise process.

In the semicausal case, where \( S \) is defined by (2.2.4), the first term of (2.2.17) again contributes to the autocovariance function for \( (k,l) = (0,0) \) and the second term for \( k+p=0 \), \( q+q=0 \), with \( (k,l), (p,q) \in W \), i.e.,

\[
r_u(k,l) = \begin{cases} 
\beta^2 & (k,l) = (0,0) \\
- \beta^2 a(0,l) & (0,l) \in W \\
0 & \text{elsewhere}
\end{cases}, \quad (2.2.21)
\]

a function which describes a white-noise process in the "i" variable and a colored-noise process with nearest-neighbor support in the "j" variable.

Finally, in the noncausal case, where \( S \) is defined by (2.2.5), the first term of (2.2.17) again contributes to the autocovariance function for \( (k,l) = (0,0) \), while the second term contributes for \( k+p=0, \ell+q=0 \), with \( (k,l), (p,q) \in W \), i.e.,

\[
r_u(k,l) = \begin{cases} 
\beta^2 & (k,l) = (0,0) \\
- \beta^2 a(k,l) & (k,l) \in W \\
0 & \text{elsewhere}
\end{cases}, \quad (2.2.22)
\]

a function which describes a colored-noise process with bounded (nearest neighbor) support.

2.3. SEPARABILITY OF IMAGE MODELS

An often-used assumption for the image autocovariance function \( r_{\text{image}}(k,l) \) is separability, which means that the variability of the pixel intensities in the vertical direction (columns) is not related to the variability in the horizontal direction (rows). This assumption can lead to autoregressive types of representations in both directions of the image (see Section 2.1).

In the previous section we have seen that for a given image autocovariance function \( r_{\text{image}}(k,l) \) it is possible to create different stochastic difference-equation models. However, independently of the possible separability of the underlying image autocovariance function, the model output autocovariance function \( r_{\text{model}}(k,l) \) can be either separable or not. Separability of the model output autocovariance function is an important issue in developing recursive filters, because it can lead to sequential 1-D column and row operations on the data [73].

Consider the general linear model for homogeneous images

\[
x(i,j) = \sum_{p,q \in W} a(p,q) x(i-p,j-q) + u(i,j), \quad (2.3.1)
\]

where \( u(i,j) \) is chosen to represent the model input. If \( x(i,j) \) and \( u(i,j) \) are seen as elements of the \( M \times N \) matrices \( X \) and \( U \), respectively, then (2.3.1) can be written in matrix-vector notation by using the conversion formula given in (2.1.3) as

\[
A x = u \quad (2.3.2)
\]
or
\[ x = A^{-1}u, \]  

(2.3.3)

provided A is nonsingular. The MNxMN matrix A is called the model operator. The inverse matrix \( A^{-1} \) transforms the MNx1 input vector \( u \) into the MNx1 output vector \( x \). The autocovariance matrix of the output vector is given by
\[ E[x x^T] = A^{-1} E[u u^T]A^{-T}. \]  

(2.3.4)

If the output autocovariance matrix can be separated into a Kronecker product of an MXM column autocovariance matrix and an NxN row autocovariance matrix, then (2.3.4) can be written as
\[ E[x x^T] = (A_v^{-1} E[u_v u_v^T]A_v^{-T}) \otimes (A_h^{-1} E[u_h u_h^T]A_h^{-T}). \]  

(2.3.5)

By using the properties of a Kronecker product that [17]
\[ (C \otimes D)(G \otimes H) = CG \otimes DH \]  

(2.3.6)

and
\[ (C \otimes D)^{-1} = C^{-1} \otimes D^{-1}, \]  

(2.3.7a)
\[ (C \otimes D)^T = C^T \otimes D^T \]  

(2.3.7b)

it is possible to write (2.3.5) as
\[ E[x x^T] = (A_v^{-1} E[u_v u_v^T]A_v^{-T}) \otimes (A_h^{-1} E[u_h u_h^T]A_h^{-T}). \]  

(2.3.8)

Thus, for separability of the output autocovariance matrix \( E[x x^T] \) it is only necessary for both the model operator \( A \) and the input autocovariance matrix \( E[u u^T] \) to be separable into a column model operator \( A_v \) and a row model operator \( A_h \), a column autocovariance matrix \( E[u_v u_v^T] \) and a row autocovariance matrix \( E[u_h u_h^T] \), respectively.

If for the input process \( \{u(i,j)\} \) uncorrelated variables are assumed with zero mean and unit variance, then the MNxMN diagonal autocovariance matrix \( E[u u^T] \) is separable into
\[ E[u u^T] = I_v \otimes I_h, \]  

(2.3.9)

where \( I_v \) and \( I_h \) denote the MXM identity matrix of the columns and the NxN identity matrix of the rows of \( U \), respectively. Thus, in this case the model output autocovariance matrix \( E[x x^T] \) is separable, if the model operator \( A \) is separable. Jain and Angel [40] prove the nonseparability of the model operator of a specific model (nearest-neighbor model) by determining the eigenvalues of the model operator by means of diagonalization techniques and by recognizing the nonseparability of these eigenvalues. In general, for an arbitrary model, this can be a very tedious procedure. To avoid this situation we propose the following method to investigate the separability of the model operator \( A \) based on its structure [54].

Because of the lexicographic ordering of the elements of \( x \) and \( u \) the matrix \( A \) has a block Toeplitz structure with \( M^2 \) blocks, each block being also Toeplitz with dimension MxN. Thus we obtain
\[ A = \begin{bmatrix} A_1 & A_0 & A_{-1} & \ldots \; & \ldots & A_1 \end{bmatrix}, \; A_k = \begin{bmatrix} a(k,1) & a(k,0) & a(k,-1) & \ldots \; & \ldots \end{bmatrix}, \]  

(2.3.10)

where \( {A_k} = \ldots A_1 A_0 A_{-1} \ldots \) is the so-called defining sequence [79] for the block-Toeplitz structure of \( A \), and where \( \{a(k,\xi)\} = \ldots a(k,1), a(k,0), a(k,-1) \ldots \) is the defining sequence for the block \( A_k \). Note that the elements \( a(k,\xi) \) are just equal to minus the weight coefficients \( a(p,q) \) in (2.3.1) for \( p-k \) and \( q=\xi \).

For separability of \( A \) (2.3.10) into a Kronecker product
where $A_v$ and $A_h$ are $M \times M$ and $N \times N$ matrices, respectively, it must hold that

$$A = A_v \otimes A_h,$$  \hspace{1cm} (2.3.11)

where $A_v$ and $A_h$ are $M \times M$ and $N \times N$ matrices, respectively. Thus, the defining sequences $\{a(k,\ell)\}$ must be pairwise linearly dependent over all $k$.

When applying this result to the model matrix $S$, which describes in rectangular notation the elements $a(p,q)$ of a model with $(p,q) \in \mathbb{W}U (0,0)$, i.e.

$$S = \begin{bmatrix} a_v(0,0) & a_v(1,0) & a_v(0,1) & a_v(-1,0) & a_v(-1,1) \\ a_h(0,0) & a_h(1,0) & a_h(0,1) & a_h(-1,0) & a_h(-1,1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$  \hspace{1cm} (2.3.12)

with $A_v = a_v(1)A_v$, and where the constants $a_v(k)$ are the elements of $A_v$. Thus, all rows of $S$ must be pairwise linearly dependent. Note that the element $a(0,0)$ is equal to 1. Conversely, it can easily be seen that, if all rows of $S$ are pairwise linearly dependent, then the model operator $A$ is separable.

Instead of testing the separability of the model operator $A$ by using the theory of Kronecker products, it is also possible to test the separability of the 2-D transfer function $H(z_1,z_2)$ of the model (2.3.1) in the $z$-transform domain. In that case the model matrix $S$ can be seen as the coefficient matrix of the denominator polynomial of the transfer function. For separability of the transfer function $H(z_1,z_2)$ into a product $H_v(z_1)H_h(z_2)$, the same test as described above must be performed on the rows of the model matrix $S$.

If for the input process $\{u(i,j)\}$ a correlated noise process is chosen, then for separability of the output autocovariance matrix of the model both the separability of the model operator $A$ and that of the model input autocovariance matrix must be tested. Because of the lexicographic ordering of the elements of $u$ and the homogeneity assumption of the sequence $\{u(i,j)\}$, the input autocovariance matrix $E[u \cdot u^T]$ has a block Toeplitz structure with $N^2$ blocks, each block being also Toeplitz with dimension $N \times N$. It can be shown in the same way as for the model operator $A$ that the input is separable if all the rows of the input autocovariance function $r_u(k,\ell) = E[u(i,j)u(i-k,j-\ell)]$, $V(k,\ell)$, given in matrix notation with $(k,\ell) = (0,0)$ situated in the center of the matrix, are pairwise linearly dependent.

As model fitting and image coding are the inverted processes of the foregoing, the reverse question is also very interesting. What is the structure of the error process $\{u(i,j)\}$, given the structure of the model operator $A$? It can be shown in the same way as above that the error autocovariance function $r_u(k,\ell)$ is separable if both the input and the model are separable.

## 2.4. Some Model-Fitting Examples

In this section some causal, semicausal and noncausal models will be fitted to the often-used [28], [31], [36], [41], [51], [61], [69] separable exponentially decaying image autocovariance function

$$r_{\text{image}}(k,\ell) = \sigma_x^2 \rho_v^{|k|} \rho_h^{|\ell|}, \quad 0 < \rho_v, \rho_h < 1,$$  \hspace{1cm} (2.4.1)

where $\sigma_x^2$ is the image variance and $\rho_v$ and $\rho_h$ are the vertical and horizontal correlation coefficients, respectively. For real-world images the correlation coefficients are restricted to nonnegative values. We note once more the limited number of degrees of freedom in this description.
(See also (2.1.40)). For convenience we set the image variance at \( \sigma^2 = 1 \).

Causal model

Consider the quarter-plane causal DPCM model with spatial structure:
\[
\begin{align*}
  & j-1 & j \\
  \cdots & \cdots & \cdots \\
  i-1 & \circ & \circ \\
  i & \circ & x \\
  \cdots & \cdots & \cdots
\end{align*}
\]

\[
x(i,j) = a(1,0)x(i-1,j) + a(0,1)x(i,j-1) + a(1,1)x(i-1,j-1) + u(i,j),
\]

(2.4.2)

The best linear MSE coefficients \( a(p,q) \) and the variance \( \beta^2 \) can be calculated with (2.2.12). This results in
\[
x(i,j) = \rho_v x(i-1,j) + \rho_h x(i,j-1) + u(i,j),
\]

with
\[
\rho_v = a(1,0), \quad \rho_h = a(0,1), \quad \rho_v \rho_h = -a(1,1).
\]

The variance \( \beta^2 \) is given by
\[
\beta^2 = (1 - \rho_v^2)(1 - \rho_h^2).
\]

(2.4.4)

The model matrix
\[
S = \begin{bmatrix} \rho_v \rho_h & -\rho_v \\ -\rho_h & 1 \end{bmatrix} = \begin{bmatrix} -\rho_v \\ -\rho_h \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

(2.4.5)

has linearly dependent rows (and columns) and can be written as a vector product. Thus, the model operator of (2.4.3) is separable. Calculation of the autocovariance function of the error process \( \{u(i,j)\} \) yields

\[
r_u(k,l) = \begin{cases} \beta^2 & (k,l) = (0,0) \\ 0 & \text{elsewhere} \end{cases}
\]

(2.4.6)

which is a separable function, describing a white-noise process. Conversely, if the white-noise sequence \( \{u(i,j)\} \) is used as input to the model, then, in accordance with (2.2.19) and (2.2.20), the output of the model will describe a wide-sense Markov field with separable autocovariance function (2.4.1). This can also be seen in the following way. Because of the separability, this 2-D model can be written as two coupled 1-D models: a horizontal and a vertical equation, respectively,

\[
\begin{align*}
  x(i,j) &= \rho_h x(i,j-1) + \tilde{x}(i,j), \\
  \tilde{x}(i,j) &= \rho_v \tilde{x}(i-1,j) + u(i,j),
\end{align*}
\]

(2.4.7a, 2.4.7b)

where \( u(i,j) \) is uncorrelated in the "i" variable and \( \tilde{x}(i,j) \) in the "j" variable. Note that (2.4.7a) describes an AR(1) process in the "j" variable and (2.4.7b) an AR(1) process in the "i" variable. The autocovariance functions of both processes are exponentially decaying as was indicated in (2.1.34). Thus (2.4.3) exactly fits the separable exponentially decaying image autocovariance function \( r_{\text{image}}(k,l) \) in (2.4.1) for all \( (k,l) \).

Semicausal model

Consider a semicausal model with the following spatial structure:
\[
\begin{align*}
  & j-1 & j+1 \\
  \cdots & \cdots & \cdots \\
  i-1 & \circ & \circ \\
  i & \circ & x \\
  \cdots & \cdots & \cdots
\end{align*}
\]

\[
x(i,j) = a(0,1)x(i,j-1) + a(1,0)x(i,j+1) + a(1,1)x(i-1,j) + u(i,j),
\]

(2.4.8)

Again the best linear MSE coefficients \( a(p,q) \) and the variance \( \beta^2 \) can
be derived from (2.2.12). This results

\[ x(i,j) = \alpha(x(i,j-1) + x(i,j+1)) - \rho_v \alpha(x(i-1,j-1) + x(i-1,j+1)) + \rho_v x(i-1,j) + u(i,j), \]  

(2.4.9)

with \( \alpha = \frac{\rho_h}{1 + \rho_h} \).

The variance \( \beta_{SC}^2 \) is given by

\[ \beta_{SC}^2 = \frac{(1-\rho_v^2)(1-\rho_h^2)}{1 + \rho_h^2}. \]  

(2.4.10)

Because the image autocovariance function (2.4.1) is real and circularly symmetric with \( \gamma_{image}(k,l) = \gamma_{image}(|k|,|l|) \), it holds for the coefficients \( a(p,q) \) that \( a(p,q) = a(p,|q|) \). The model matrix \( S \) is given by

\[ S = \begin{bmatrix} \rho_v & -\rho_v \\ -\rho_v & 1 & -\rho_v \\ -\rho_v & 1 & -\rho_v \end{bmatrix} = \begin{bmatrix} -\alpha & 1 & -\alpha \\ 1 & 1 & 1 \end{bmatrix}, \]  

(2.4.11)

which is separable. Calculation of the error autocovariance function \( \gamma_u(k,l) \), written in matrix notation with \( (k,l) = (0,0) \) situated in the center of a 5x5 matrix, yields

\[ \gamma_u(k,l) = \beta_{SC}^2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 1 & -\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \]  

(2.4.12)

Investigating (2.4.12) reveals that the rows (and columns) are pairwise linearly dependent and thus, the error autocovariance function \( \gamma_u(k,l) \) is a separable function describing a white-noise process in the "i" variable and a colored-noise process with nearest-neighbor support in the "j" variable. Conversely, if \{u(i,j)\} with autocovariance function (2.4.12) is used as input to the model, then in accordance with (2.2.19) and (2.2.21), the output of the model will describe a wide-sense Markov field with separable autocovariance function (2.4.1). This can also be seen in the following way. By using the separability of both the model and the input, the 2-D model (2.4.9) can be written as two coupled 1-D models:

\[ x(i,j) = \rho_v x(i-1,j) + \gamma_u(i,j), \]  

(2.4.13a)

\[ v(i,j) = a(x(i-1,j) + x(i,j+1)) + u(i,j), \]  

(2.4.13b)

where \( x(i,j) \) is uncorrelated in the "i" variable and where \( u(i,j) \) has a one-step (nearest-neighbor) correlation in the "j" variable. It can easily be shown that the two-sided Markov(1) process (2.4.13b) with coefficient \( \alpha = \frac{\rho_h}{1 + \rho_h} \) can be replaced by an AR(1) process with coefficient \( \rho_h \). Thus, the semicausal model (2.4.9) with nonwhite input (2.4.12) exactly fits the separable exponentially decaying autocovariance function \( \gamma_{image}(k,l) \) in (2.4.1) for all \( (k,l) \).

**Noncausal model**

Consider the noncausal nearest-neighbor model with the following spatial structure:

\[
\begin{align*}
& j-1 & j & j+1 \\
& . & . & . & . & . \\
& i-1 & . & \cdot & \cdot & x(i,j) = a(1,0)x(i-1,j) + a(-1,0)x(i+1,j) + a(0,1)x(i,j-1) \\
& i & . & \cdot & \cdot & + a(0,-1)x(i,j+1) + u(i,j). \\
& i+1 & . & \cdot & \cdot & . & . & . \\
\end{align*}
\]  

(2.4.14)

The best linear MSE coefficients \( a(p,q) \) and the variance \( \beta_{NC}^2 \) can again be derived from (2.2.12). This results in
\[ x(i,j) = a_1 x(i-1,j) + a_2 x(i+1,j) + a_3 x(i-1,j-1) + a_4 x(i+1,j+1) + u(i,j), \] (2.4.15)

with
\[ a_1 = \frac{\rho_v (1-\rho_h^2)}{1+\rho_v^2 \rho_h + 3 \rho_v^2 \rho_h^2}, \quad a_2 = \frac{\rho_h (1-\rho_v^2)}{1+\rho_v^2 \rho_h + 3 \rho_v^2 \rho_h^2}. \]

The variance \( \beta_{NC}^2 \) is given by
\[ \beta_{NC}^2 = \frac{(1-\rho_v^2)(1-\rho_h^2)}{1+\rho_v^2 \rho_h + 3 \rho_v^2 \rho_h^2}. \] (2.4.16)

The model matrix \( S \) has the following nonseparable structure:
\[
S = \begin{bmatrix}
0 & -\alpha_1 & 0 \\
-\alpha_2 & 1 & -\alpha_2 \\
0 & -\alpha_1 & 0
\end{bmatrix}. \] (2.4.17)

Calculation of the autocovariance function \( r_u(k,\ell) \) yields the nonseparable function
\[
r_u(k,\ell) = \beta_{NC}^2 \begin{bmatrix}
0 & -\alpha_1 & 0 \\
-\alpha_2 & 1 & -\alpha_2 \\
0 & -\alpha_1 & 0
\end{bmatrix}. \] (2.4.18)

\[ \gamma_1 = \frac{\rho_v (3 \rho_v^2 - 2 \rho_h^2)}{1+\rho_v^2 \rho_h + 3 \rho_v^2 \rho_h^2}, \quad \gamma_2 = \frac{\rho_h (3 \rho_h^2 - 2 \rho_v^2)}{1+\rho_v^2 \rho_h + 3 \rho_v^2 \rho_h^2}, \quad \gamma_3 = \frac{\rho_h (3 \rho_h^2 - 2 \rho_v^2)}{1+\rho_v^2 \rho_h + 3 \rho_v^2 \rho_h^2}, \quad \gamma_4 = \frac{\rho_v (3 \rho_v^2 - 2 \rho_h^2)}{1+\rho_v^2 \rho_h + 3 \rho_v^2 \rho_h^2}, \] which is not of bounded support, but for convenience is given here in matrix form of size 7x7. Note that in this case the model error \( u(i,j) \) does not satisfy orthogonality condition (2.2.19).

Conversely, if \( \{u(i,j)\} \) is used as input to the model and if we choose the nonseparable, but bounded Markov input (2.2.22)
\[
r_u(k,\ell) = \rho_v (3 \rho_v^2 - 2 \rho_h^2), \quad r_{image}(k,\ell) = \rho_h (3 \rho_h^2 - 2 \rho_v^2), \] given here in 3x3 matrix notation, then the nonseparable autocovariance function \( r_{model}(k,\ell) \) of the wide-sense Markov field generated by the model fits the separable autocovariance function \( r_{image}(k,\ell) \) to a good approximation, provided \( r_u(k,\ell) \) is nonnegative definite [14], [94].

An exact fit between \( r_{image}(k,\ell) \) in (2.4.1) and \( r_{model}(k,\ell) \) can be obtained by using the following noncausal model with spatial structure:
\[
x(i,j) = a_1 x(i-1,j) + a_2 x(i+1,j) + a_3 x(i-1,j-1) + a_4 x(i+1,j+1) + u(i,j). \] (2.4.21)
The variance $\sigma^2_{NC}$ is given by

$$
\sigma^2_{NC} = \frac{(1-p_v^2)(1-p_h^2)}{(1+p_v^2)(1+p_h^2)}.
$$

(2.4.22)

Because the image autocovariance function (2.4.1) is circularly symmetric, it holds for the coefficients $a(p,q)$ that $a(p,q)=a(|p|,|q|)$. It can easily be verified that (2.4.21) is separable, and that the model error $u(i,j)$ exactly satisfies (2.2.19). Therefore, this model with Markov input (2.2.22) for this particular model exactly fits the separable autocovariance function (2.4.1).

We may observe that the causal model (2.4.3), the semicausal model (2.4.9) and the noncausal model (2.4.21) all exactly fit the separable autocovariance function $r_{image}(k,l)$ for all $(k,l)$. Thus from a fitting point of view these models are equivalent. Comparison of the variances of the model errors for these models yields the following inequality:

$$
\sigma^2_{C} > \sigma^2_{SC} > \sigma^2_{NC}.
$$

(2.4.23)

In Section 2.6 some more quality criteria are introduced to compare different models representing the same image statistics.

### 2.5. Model Stability

In this section some stability criteria will be given for causal, semicausal and noncausal models. In general, a model is considered to be stable if the output is bounded when a bounded input is used, usually referred to as Bounded-Input, Bounded-Output (BIBO) stability. When the general 2-D model in (2.2.1) is characterized by its 2-D transfer function

$$
B(z_1,z_2) = \frac{1}{B(z_1,z_2)} = \frac{1}{1-\sum_{p,q \in W} a(p,q)z_1^p z_2^q},
$$

(2.5.1)

the BIBO stability condition can be related to the position of the zeros of the denominator bivariate polynomial $B(z_1,z_2)$.

For quarter-plane causal models BIBO stability is guaranteed if and only if (iff) [80]

$$
B(z_1,z_2) \neq 0, \quad \text{when } |z_1| > 1 \text{ and } |z_2| > 1.
$$

(2.5.2)

A stability criterion, equivalent to (2.5.2) but computationally much simpler, was developed by Huang in [35] and by Strintzis [88] for the case (most common in practice) in which $B(z_1,z_2)$ is a finite-order polynomial:

$$
B(z_1,z_2) \neq 0, \quad \text{when } |z_1| = 1 \text{ and } |z_2| > 1;
$$

$$
B(z_1,a) \neq 0, \quad \text{for some } a, |a| > 1 \text{ and } |z_1| > 1.
$$

(2.5.3)

The stability condition in (2.5.3) not only holds for QP causal models but also for NSHP causal models [24]. We noted already that there is no need to restrict our models to spatially causal models. Justice and Shanks [47] derived stability criteria for the $n$-dimensional case and dropped the requirement that the model should be (though it may be) causal in any variable. For the two-dimensional case, in which $B(z_1,z_2)$ is of finite or infinite order these stability conditions are:

- **Causal models**

  $$
  B(z_1,z_2) \neq 0, \quad \text{when } |z_1| > 1 \text{ and } |z_2| > 1.
  $$

  (2.5.4)

- **Semicausal models**

  $$
  B(z_1,z_2) \neq 0, \quad \text{when } |z_1| > 1 \text{ and } |z_2| = 1.
  $$

  (2.5.5)

- **Noncausal models**

  $$
  B(z_1,z_2) \neq 0, \quad \text{when } |z_1| = |z_2| = 1.
  $$

  (2.5.6)
Note that (2.5.4) is equal to the result obtained in (2.5.2). In a recent comment on [47], Shaw points out [81] that when using semi-causal models, condition (2.5.5) is necessary and sufficient for the existence of a stable inverse, but to insure that \( B(z_1,z_2) \) and its stable inverse have the same unit-impulse response support extending over the symmetric half-plane region instead of over a quarter-plane region, the additional condition (2.5.7) is necessary and sufficient:

\[ B(1,z_2) \text{ has no linear phase component when } |z_2| = 1. \]  
(2.5.7)

The fact that a polynomial has no linear phase component can be interpreted with the aid of the Nyquist theory. If the unit circle is defined to be the Nyquist contour, the above phase condition means that the Nyquist plot (in this case, \( B(1,\exp(j\omega_2)) \), \( \omega_2 = 0 \) to \( \omega_2 = 2\pi \)) does not encircle or pass through the origin. In other words, if \( \text{Ind}(B(1,z_2)) \) indicates the number of encirclements of zero of the Nyquist plot of \( B \) along the Nyquist contour \( |z_2| = 1 \), then \( B(1,z_2) \) has no linear phase component if \( \text{Ind}(B(1,z_2)) = 0 \). Note that \( B(1,z_2) \) should not have any roots on the unit circle. Shaw's proof of (2.5.7) is based on the fact that if the support of the stable inverse of \( B(z_1,z_2) \) only extends over a quarter-plane region, then \( \text{Ind}(B(1,z_2)) \neq 0 \), and therefore, in order to not have quarter-plane support it is necessary and sufficient that \( B(1,z_2) \) have no linear phase component. Let \( M_1 \) be the degree of \( B(1,z_2) \) in \( z_2^{-1} \) and \( M_2 \) be the degree of \( B(1,z_2) \) in \( z_2 \), where \( M_1,M_2 > 0 \); then investigating condition (2.5.7) is equivalent to testing if \( B(1,z_2) \) has exactly \( M_1 \) zeros inside the unit circle and \( M_2 \) zeros outside it.

**Example 2.5.1**

Consider the semi-causal model as given in (2.4.9) with separable transfer function

\[ B(z_1,z_2) = \frac{1}{(1-\rho_2 z_1^{-1})(1-\alpha(z_2^{-1}+z_2))}, \]  
(2.5.8)

where

\[ \alpha = \frac{\rho_1}{1+\rho_2 \rho_1^2}. \]

Condition (2.5.5) yields

\[ (1-\rho_2 z_1^{-1})(1-2\alpha \cos \theta) \neq 0, \quad |z_1| > 1. \]  
(2.5.9)

This condition is satisfied iff

\[ |\rho_1| < 1 \text{ and } |\rho_2| < 1, \]  
(2.5.10)

or equivalently

\[ |\rho_1| < 1 \text{ and } |\rho_2| < 1. \]  
(2.5.11)

Thus condition (2.5.5) is always satisfied. From condition (2.5.7) it follows that \( B(1,z_2) \) has one zero inside the unit circle and one zero outside it for \( |\rho_1| < 1 \), which means that \( B(1,z_2) \) has no linear phase components when \( |z_2| = 1 \). Therefore, \( B(z_1,z_2) \) and its stable inverse have the same support extending over the symmetric half-plane.

In the noncausal case as well some remarks can be made: condition (2.5.6) is necessary and sufficient for the existence of a stable inverse, but to insure that \( B(z_1,z_2) \) and its stable inverse have the same full-plane support the additional conditions (2.5.12a) and (2.5.12b) are necessary and sufficient (private communication with Shaw [82]):

\[ B(z_1,1) \text{ has no linear phase component when } |z_1| = 1; \]  
(2.5.12a)

\[ B(1,z_2) \text{ has no linear phase component when } |z_2| = 1. \]  
(2.5.12b)
2.6. MODEL QUALITY

In Section 2.2 we have introduced a general 2-D autoregressive type of model based on the autocovariance function of the original image. For a given image autocovariance function it is possible to consider various models. The choice of a model (and its input) may influence the restoration result, because it is the image model which summarizes the a priori information about the original image.

To compare various models, representing the same image, three quality criteria will be used; each measuring a different model aspect [10].

The first one calculates the variance of the model error for each chosen MSE image model (see (2.2.11)); i.e.,

$$\beta^2 = \text{E}[u^2(i,j)] = r_{\text{image}}(0,0) - \sum_{p,q \in W} a(p,q) r_{\text{image}}(p,q). \quad (2.6.1)$$

A small value of $\beta^2$ is preferred. In image coding (DPCM) a model with a low value of $\beta^2$ yields a large data compression. In image restoration by means of a Kalman filter, a small value of $\beta^2$ directly influences the Kalman gain. The smaller this value, the smaller the Kalman gain is and the more important the prediction part of the Kalman filter will be.

The variance comparison alone is not sufficient for choosing image models. If $\{u(i,j)\}$ is used as input to the model, it is often the covariance structure of $\{u(i,j)\}$ which provides an adequate fit between $r_{\text{image}}(k,l)$ and $r_{\text{model}}(k,l)$ (semicausal and noncausal cases). Besides that, the order of a model may be lower than that of the image autocovariance function. In that case only a few points of the image autocovariance function may be fitted. Therefore, we have introduced a second criterion which measures the fit between $r_{\text{image}}(k,l)$ and $r_{\text{model}}(k,l)$ for all $(k,l)$ by using a normalized squared-error fit criterion

$$\epsilon^2 = \frac{\sum_{k,l} [r_{\text{image}}(k,l) - r_{\text{model}}(k,l)]^2}{\sum_{k,l} [r_{\text{image}}(k,l)]^2}. \quad (2.6.2)$$

If an attempt is made to match the autocovariance functions generated by several models and the autocovariance function of one class of images, it is often more convenient to calculate the squared error $\epsilon^2$ in the frequency domain. With the help of Parseval's theorem it is possible to rewrite $\epsilon^2$ as follows [78]:

$$\epsilon^2 = \frac{\sum_{k,l} [S_{\text{image}}(k,l) - S_{\text{model}}(k,l)]^2}{\sum_{k,l} [S_{\text{image}}(k,l)]^2}, \quad (2.6.3)$$

where $S_{\text{image}}(k,l)$ and $S_{\text{model}}(k,l)$ are the spectral-density functions of the image and the model, respectively. The influence the choice of input has on how well $r_{\text{model}}(k,l)$ fits $r_{\text{image}}(k,l)$ will be illustrated in the following example. A more comprehensive comparison for several classes of images and for a number of different models can be found in [14].

Example 2.6.1

Fitting the noncausal nearest-neighbor model as given in (2.4.14) onto the 2-D separable exponentially decaying autocovariance function $r_{\text{image}}(k,l)$ (2.4.1) will be investigated for $p_w$ and $p_h$ equal to 0.3 for a white input and for a Markov input (2.2.22). In Fig. 2.7(a), (b), and (c) the plots are given of $r_{\text{image}}(k,l)$, $r_{\text{model}}(k,l)$ (white input), and $r_{\text{model}}(k,l)$ (Markov input).

The values of the fitting error $\epsilon^2$ (2.6.2) are:

- White input: $\epsilon^2 = 127.72$
- Markov input: $\epsilon^2 = 0.59$

This example illustrates the necessity of choosing a correct or approximately correct covariance structure of the input.
We would like to introduce a third test for model quality, namely the Cramer-Rao bound for unbiased estimators, which is a lower bound for the mean-square error in estimating a parameter (signal value) from a disturbed observation [65].

In the following, this lower bound will be derived for estimators based on the following model:

\[ x(i,j) = \sum_{p,q} a(p,q) x(i-p,j-q) + bu(i,j), \]  

where the variance of \( u(i,j) \) is assumed to be equal to the variance of \( x(i,j) \), i.e. \( \sigma_u^2 = \sigma_x^2 r_{image}(0,0) \), and where the observations are of the form

\[ y(i,j) = x(i,j) + w(i,j). \]

The observation noise \( w(i,j) \) is assumed to be zero-mean white Gaussian noise with variance \( \sigma_w^2 \), uncorrelated with \( x(i,j) \). Then the Cramer-Rao bound in estimating \( x(i,j) \) from observation \( y(i,j) \), given \( x(i,j) \) and the probability density function \( p(w(i,j)) \), will be for unbiased estimators

\[ \mathbb{E}[(x(i,j) - \hat{x}(i,j))^2] \geq \frac{1}{\sigma_x^2 I^{-1}} \]

where the right-hand term is the inverse of the Fisher information \( I \).

From (2.6.5), the conditional density function \( p(y(i,j) | x(i,j)) \) is

\[ p(y(i,j) | x(i,j)) = \frac{1}{\sigma_w \sqrt{2\pi}} \exp \left( -\frac{1}{2\sigma_w^2} (y(i,j) - x(i,j))^2 \right) \]

and

\[ \log p(y(i,j) | x(i,j)) = - \log \sigma_w \sqrt{2\pi} - \frac{1}{2\sigma_w^2} (y(i,j) - x(i,j))^2. \]

It follows from (2.6.4) that

\[ x(i,j) = f(x(i-p,j-q), (p,q) \in W; u(i,j)) \]

Thus (2.6.8) can be written as

\[ \log p(y(i,j) | x(i,j)) = - \log \sigma_w \sqrt{2\pi} - \sum_{p,q} a(p,q) x(i-p,j-q) + u(i,j))^2. \]

Note that if the field \( \{x(i,j)\} \) is known, \( u(i,j) \) will be known for a given set \( a(p,q) \). By differentiating (2.6.10) with respect to its conditions \( x(i-p,j-q), (p,q) \in W \) and \( u(i,j) \) we obtain the vector

\[ \frac{\partial}{\partial a(p,q)} \log p(y(i,j) | x(i,j)) = \sum_{p,q} a(p,q) x(i-p,j-q) - b u(i,j). \]
\[ f = \frac{1}{\sigma_w^2} \sum_{p,q \in \mathbb{W}} [a(p,q)x(i-p,j-q) - bu(i,j)] \]

By using (2.6.11), we obtain for the Fisher information \( I \):

\[ E\left[ \left( \frac{\partial \log p(y(i,j) | x(i,j))}{\partial x(i,j)} \right)^2 \right] = \frac{1}{\sigma_w^2} \sum_{p,q \in \mathbb{W}} a^2(p,q) + b^2. \]  

This lower bound contains all coefficients necessary to define a model of form (2.6.4), so that this bound can be used as a criterion of quality. The model with the smallest lower bound will be preferred in the sense of the mean-square error between the estimate \( \hat{x}(i,j) \) from the observation \( y(i,j) \) and the original \( x(i,j) \).

2.7. MODEL-QUALITY EXPERIMENT

In this section we will test the quality of some causal models by the three criteria given in the previous section. The test will be performed for two classes of images. As the first class of images we consider images with a separable exponentially decaying autocovariance function (see also 2.4.1)

\[ r_{\text{image}}(k,t) = \sigma_x^2 \rho_t^{\left| k \right|} \rho_h^{\left| t \right|}, \quad 0 < \rho_t, \rho_h < 1, \quad (2.7.1) \]

where \( \sigma_x^2 \) is the image variance and \( \rho_t \) and \( \rho_h \) are the vertical and horizontal correlation coefficients, respectively. For the second class of images we consider an isotropic exponentially decaying autocovariance function

\[ r_{\text{image}}(k,t) = \sigma_x^2 \rho^{|k| + |t|}, \quad 0 < \rho < 1. \]  

The shape parameter \( s(\text{real}) \) in (2.7.2) is chosen to obtain a class of images which is more general than the often-used class for \( s=1 \). The models we use in our tests and their spatial structure are listed in Table 2.1.

<table>
<thead>
<tr>
<th>( \text{causal (QP)} )</th>
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TABLE 2.1. Models used in experiment

Some typical results of testing the quality are given in Fig. 2.8(a), (b) and (c) for the class of images with separable autocovariance function (2.7.1). Note that in this case model \( C_2 \) becomes equivalent to \( C_1 \) because the MSE coefficient \( a(1,-1) \) vanishes. In these figures the horizontal correlation coefficient \( \rho_h \) of the image is equal to 0.9 and the vertical correlation coefficient \( \rho_v \) of the image ranges from 0.10 to 0.95. As can be seen from these figures, model \( C_2 \) is preferable since both the variance of the model error \( \sigma^2 \) and the Cramér-Rao bound \( I^{-1} \) are the smallest of all investigated models, while the squared error \( e^2 \) is zero for all values of \( \rho_v \) and \( \rho_h \). This was to be expected because
this separable model is a direct extension of an AR(1) process in both
the horizontal and vertical directions (See Section 2.4). Although not
depicted here, experiments for $\rho_h=0.8, 0.7$ and $0.6$ show the same order
of preference.

In Fig. 2.9(a), (b) and (c) some typical results are given for the
class of images with an isotropic character. The shape parameter $s$
is taken as 1 and the correlation coefficient $\rho$ ranges from 0.10 to
0.95. From these figures we observe that contrary to the separable case
model $C_2$ is now not preferable to the other models. Although the dif-
f erences in $S^2$ and $I^{-1}$ are small for the models $C_1$ through $C_5$, there is
a remarkable difference in the squared error $\epsilon^2$ for the models $C_1$ and
$C_2$ in comparison with the other models. An advantage of the models $C_1$
and $C_2$, however, is their relative insensitivity to variations in the
shape parameter $s$ due to their spatial structure [10],[14]. It is noted
that the models $C_1$ through $C_5$ are compared for a white-noise model input.
Similar experiments for semicausal and noncausal models with different
types of model input are described in [14].

We may conclude that there is a noticeable relation between the
image autocovariance function given and the choice of a model. The three
quality criteria, each measuring a different model aspect, facilitate
this choice.

Fig. 2.8. Model errors for three criteria: separable autocovariance
function $\rho_h=0.9$, (a) Variance of model error, (b) Squared
error in model autocovariance function, (c) Cramér-Rao bound.

Fig. 2.9. Model errors for three criteria: isotropic autocovariance
function $s=1$, (a) Variance of model error, (b) Squared error
in model autocovariance function, (c) Cramér-Rao bound.
This chapter deals with the restoration of noisy images by means of 2-D recursive estimators. In particular, we discuss ways to evaluate the performance of two recursive point-to-point estimators of the Kalman type, based on the DPCM image model.

The first is the one-step predictor proposed by Habibi (1972), the second the filter proposed by Kak (1976).

Although these 2-D generalizations of the 1-D Kalman filter concept are known to be suboptimal (Strintzis (1976)), they are intuitively appealing, easy to implement and relatively fast, especially if the rapid convergence of the filter parameters to a steady state is exploited. This explains the continuing interest in these types of filters in the field of digital image restoration, but above all it motivates one to search for methods to evaluate their performance.

This chapter is organized as follows. In Section 3.1 both estimators will be introduced, and convergence as well as stability will be discussed.

As stability does not guarantee good filter performance, in Section 3.2 two performance measures are introduced. The first gives insight into the distortion to the original image caused by an estimator, while the second measures the noise-smoothing capability of an estimator.

In Section 3.3 both estimators will be compared experimentally by
applying these measures of performance.

3.1. TWO-DIMENSIONAL ESTIMATORS FOR THE DPCM IMAGE MODEL

One approach in recursive estimation of noisy images is the conversion of the 2-D information into a form with only one variable. In Nahi [61] and Nahi and Assefi [62] a 1-D recursive Kalman filter has been developed, which is applied to the scan-ordered image. This technique does not directly use a 2-D recursive model for the image. However, in [36] Habibi does make use of such a model to derive a 2-D recursive estimator. He assumes that an image can be represented by a zero-mean homogeneous discrete MxN random field with a separable exponentially decaying autocovariance function. Under these assumptions the image can be modeled by the 2-D stochastic difference equation

\[ x(i,j) = \rho_v x(i-1,j) + \rho_h x(i,j-1) + u(i,j), \quad (3.1.1) \]

where \( x(i,j) \) represents the intensity value at spatial coordinate \( (i,j) \) and \( \rho_v \) and \( \rho_h \) are the vertical and horizontal correlation coefficients, respectively, and where \( u(i,j) \) is a zero-mean homogeneous white-noise input process with variance \( \sigma_u^2 \). This model, referred to as the Differential Pulse Code Modulation or DPCM model because of its early application in the field of image coding, has already been discussed in Chapter 2. Assuming that the observations \( y(i,j) \) can be expressed as

\[ y(i,j) = x(i,j) + w(i,j), \quad (3.1.2) \]

where \( w(i,j) \) is a zero-mean homogeneous white-noise field with variance \( \sigma_w^2 \) and uncorrelated with the data, Habibi develops a point-to-point estimator to estimate \( x(i,j) \) based on the observations \( y(m,n) : 1 \leq m \leq i-1, 1 \leq n \leq j-1 \); that is, the estimator predicts one step ahead in diagonal direction. Similarly, Kak [77] derives a point-to-point estimator for the DPCM model based on the observations \( y(m,n) : 1 \leq m \leq i-1, 1 \leq n \leq j-1 \); now the estimator is a true filter. Both quarter-plane causal estimators will successively be discussed in this section (see also [11]).

One-step predictor of Habibi

Motivated by the structure both of the 1-D Kalman filter (see Section 1.2) and of the DPCM image model, Habibi proposes a 2-D recursive estimator of the following form:

\[ \hat{x}(i,j) = \gamma_1(i,j) \hat{x}(i-1,j) + \gamma_2(i,j) \hat{x}(i,j-1) + \gamma_3(i,j) \hat{x}(i-1,j-1) + F(i,j)y(i-1,j-1), \quad (3.1.3) \]

to obtain the best linear MSE estimate \( \hat{x}(i,j) \) of the state \( x(i,j) \) from the noisy observations \( y(i-1,j-1) \) given by (3.1.2). Note that Habibi uses (3.1.1) as dynamic system (state-space equation) of the original image generating process, with scalar state \( x(i,j) \). It is then necessary to find \( \gamma \)'s and a gain \( F \) such that the error

\[ e(i,j) = x(i,j) - \hat{x}(i,j) \quad (3.1.4) \]

is orthogonal to all observations involved in the estimation process, i.e. for all \( y(m,n) \) in the rectangular region \( 1 \leq m \leq i-1, 1 \leq n \leq j-1 \). By expanding \( x(i,j) \) and \( \hat{x}(i,j) \) using (3.1.1)-(3.1.3) we find that the error is given by

\[ e(i,j) = \left( \rho_v \gamma_1(i,j) + \rho_h \gamma_2(i,j) \right) x(i-1,j) + \left( \rho_v \gamma_2(i,j) + \rho_h \gamma_3(i,j) \right) x(i,j-1) + \left( \rho_v \rho_h \gamma_3(i,j) + F(i,j) \right) y(i-1,j-1), \quad (3.1.5) \]

and if we set

\[ \gamma_1(i,j) = \rho_v, \quad (3.1.6a) \]
\[ \gamma_2(i,j) = \rho_h, \quad (3.1.6b) \]
\[ \gamma_3(i,j) = -\rho_v \rho_h - F(i,j), \quad (3.1.6c) \]
then we obtain for the estimation error a recursive expression with only one unknown, i.e. the gain $F(i,j)$:

$$
e(i,j) = \rho_v e(i-1,j) + \rho_h e(i,j-1) - (\rho_v \rho_h + F(i,j)) e(i-1,j-1) + u(i,j) - F(i,j) w(i-1,j-1).$$  \hspace{1cm} (3.1.7)

By constructing $e(i-1,j)$ orthogonal to all $y(m,n)$ in the rectangle $1 \leq m \leq i-2$, $1 \leq n \leq j-1$; $e(i,j-1)$ to all $y(m,n)$ in the rectangle $1 \leq m \leq i-1, 1 \leq n \leq j-2$; and $e(i-1,j-1)$ in the rectangle $1 \leq m \leq i-2, 1 \leq n \leq j-2$, this requirement reduces - due to the recursivity of (3.1.7) - to the orthogonality of $e(i,j)$ to all $y(m,n)$ in the intersection of the above three rectangular regions. Thus $F(i,j)$ must be determined by making $e(i,j)$ orthogonal to $y(m,n)$ at the remaining points in the rectangle $1 \leq m \leq i-1, 1 \leq n \leq j-1$, i.e.

$$E[e(i,j)y(m,n)] = 0, \quad \{m,n: m=1, n=1, \ldots, j-1, i=1, n=1, \ldots, i-1, n=j-1, \ldots \}.$$  \hspace{1cm} (3.1.8)

which leads to an overdetermined set of $(i+j-3)$ equations in the single unknown coefficient $F(i,j)$ (See Fig. 3.1, shaded area). One may settle for a suboptimal solution where only one single orthogonality condition is met, i.e.

$$E[e(i,j)y(i-1,j-1)] = 0.$$  \hspace{1cm} (3.1.9)

In [36], $F(i,j)$ is calculated from (3.1.9) under the erroneous assumption of optimality. However, straightforward calculation of $F(i,j)$ from (3.1.9), by using (3.1.1)-(3.1.4) and (3.1.6), yields

$$F(i,j) = \frac{E[x(i,j) - \rho_v \hat{x}(i,j) - \rho_h \hat{x}(i,j-1) + \rho_v \rho_h \hat{x}(i-1,j-1) x(i-1,j-1)]}{E[x(i-1,j-1) - \hat{x}(i-1,j-1) x(i-1,j-1)] + \sigma_w^2}.$$  \hspace{1cm} (3.1.10)

where $\sigma_w^2$ is the variance of the observation noise. The covariances between $x$ and $\hat{x}$ in (3.1.10) can be obtained recursively from the following relation:

$$E[x(i,j) \hat{x}(m,n)] = \rho_v E[x(i,j) \hat{x}(m-1,n)] + \rho_h E[x(i,j) \hat{x}(m,n-1)]$$

$$- (\rho_v \rho_h + F(m,n)) E[x(i,j) \hat{x}(m-1,n-1)]$$

$$+ F(m,n) E[x(i,j) x(m-1,n-1)],$$  \hspace{1cm} (3.1.11)

by using some initial conditions. Note that for the computation of a specific gain $F(i,j)$ all covariances starting with $E[x(i,j) \hat{x}(1,1)]$ and recursively proceeding towards the point $(i,j)$ have to be evaluated. This leads to an excessive number of computations even for moderately sized images. However, $F(i,j)$ converges rapidly to a steady-state value. As an example, Fig. 3.2 shows a plot of the gain $F(i,j)$ for the first 32x32 picture elements of an image with $\rho_v \rho_h = 0.9$ and a signal-to-noise ratio $\text{SNR}=1$, where $\text{SNR}$ is defined as the ratio of the signal variance $\sigma_x^2$ to the variance of the noise $\sigma_w^2$. The plot shows that $F(i,j)$ has a transient fluctuation near the $i=0$ and $j=0$ boundaries, which dies down at a relatively small distance from these boundaries, where $F(i,j)$ attains practically a constant value. Hence, one needs to evaluate $F(i,j)$ for a few rows and columns only and to apply the steady-state value for the remaining part of the image.

The suboptimality of the one-step predictor described above has been discussed by various authors [71, 68, 87]. Willsky [93] points out that the suboptimality of the Habibi estimator arises essentially because $x(i,j)$ is merely a local state whereas one should estimate the

![Fig. 3.1. Orthogonality region for determining $F(i,j)$](image)
global state to obtain optimality. This latter approach has been inves-
tigated by Woods and Radewan [95] and appears to result in an excessive
computational load. In Chapter 5 of this thesis the subject of local
and global states and the derivation of an optimal 2-D Kalman filter
will be discussed in more detail.

Now we turn our attention to the stability of the one-step predictor
as defined in (3.1.3) with coefficients (3.1.6). If we assume the gain
\( F(i,j) \) to be an unknown constant, denoted by \( F \), then stability regions
for the gain can be derived by interpreting (3.1.3) as a linear shift-
invariant system with input \( y(i-1,j-1) \) and output \( \hat{x}(i,j) \). The 2-D trans-
fer function of the system in the \( z \)-domain is then given by

\[
H(z_1, z_2) = \frac{Fz_1^{-1}z_2^{-1}}{1 - \rho_h z_1^{-1} - \rho_v z_2^{-1} + (\rho_h \rho_v + F)z_1^{-1}z_2^{-1}}. \tag{3.1.12}
\]

By applying the BIBO stability theorem for quarter-plane causal systems
as formulated by Huang [35] (See also (2.5.3)), this system is BIBO
stable iff the following conditions simultaneously hold:

\[
\left| \frac{\rho_h \rho_v + F}{1 - \rho_v} \right| < 1, \tag{3.1.13a}
\]

\[
\left| \frac{\rho_h \rho_v + F}{1 - \rho_v} \right| < 1, \tag{3.1.13b}
\]

\[
|\rho_v| < 1. \tag{3.1.13c}
\]

Since \( \rho_v \) is the vertical correlation coefficient of the random field,
it is reasonable to assume that condition (3.1.13c) is satisfied. Con-
ditions (3.1.13a) and (3.1.13b) yield the following stability regions
for the gain:

\[
-1 < \rho_h (1 + \rho_h) < F < (1 + \rho_v)(1 - \rho_h), \tag{3.1.14a}
\]

\[
-1 < \rho_h (1 - \rho_v) < F < (1 + \rho_v)(1 + \rho_h). \tag{3.1.14b}
\]

The one-step predictor is stable if \( F \) meets both criteria simultaneously,
i.e. if \( F \) lies in the intersection of the stability regions given in
(3.1.14a) and (3.1.14b). When \( \rho_v = \rho_h = 0 \) with \( 0 < \rho < 1 \) this results in
the following stability region:

\[
-(1 - \rho)^2 < F < 1 - \rho^2. \tag{3.1.15}
\]

In Section 3.3 the stability region will be calculated for a number
of values of the filter parameters and compared with bounds on the gain
that can be derived from the measures of the filter performance to be
discussed in Section 3.2.
The filter of Kak

The second recursive point-to-point estimator to be considered here is the filter proposed by Kak [77] (See also [49],[60]). It is again based on the 1-D Kalman-filter structure and on the DPCM model given in (3.1.1) and the observations given in (3.1.2) and has the following form:

\[
\hat{x}(i,j)=\gamma_4(i,j)\hat{x}(i-1,j)+\gamma_5(i,j)\hat{x}(i,j-1)+\gamma_6(i,j)\hat{x}(i-1,j-1)+F(i,j)y(i,j).
\] (3.1.16)

If we choose the filter parameters as

\[
\gamma_4(i,j) = \rho_v[1-F(i,j)],
\] (3.1.17a)

\[
\gamma_5(i,j) = \rho_h[1-F(i,j)],
\] (3.1.17b)

\[
\gamma_6(i,j) = -\rho_v\rho_h[1-F(i,j)],
\] (3.1.17c)

we obtain the following recursive expression for the estimation error:

\[
e(i,j)=(1-F(i,j))[\rho_v e(i-1,j-1)+\rho_h e(i,j-1)+\rho_v\rho_h e(i-1,j)+u(i,j)]+
\]

\[
-F(i,j)w(i,j).
\] (3.1.18)

The orthogonality conditions that are to be met if we wish \(\hat{x}(i,j)\) to be the best linear MSE estimate of \(x(i,j)\) again lead to an overdetermined set of equations. In this case we determine a suboptimal solution for \(F(i,j)\) from the single orthogonality condition

\[
E[e(i,j)y(i,j)] = 0,
\] (3.1.19)

which yields

\[
F(i,j) = \frac{E[x(i,j)-\rho_v\hat{x}(i-1,j)-\rho_h\hat{x}(i,j-1)+\rho_v\rho_h\hat{x}(i-1,j-1)]x(i,j)}{E[x(i,j)-\rho_v\hat{x}(i-1,j)-\rho_h\hat{x}(i,j-1)+\rho_v\rho_h\hat{x}(i-1,j-1)]x(i,j)^2}.
\] (3.1.20)

The gain \(F(i,j)\) of the filter of Kak also rapidly converges to a steady state. In Fig. 3.3 the gain \(F(i,j)\) is plotted for the first 32x32 picture elements of an image with \(\rho_v=\rho_h=0.9\) and SNR=1. So we conclude also for this estimator that it is only necessary to evaluate \(F(i,j)\) for a few rows and columns and to apply the steady-state value for the remaining part of the image. Equation (3.1.20) can be rewritten as

\[
F(i,j) = \frac{E[\hat{x}(i-1,j)\hat{x}(i,j-1)\hat{x}(i-1,j-1)]x(i,j)^2}{E[\hat{x}(i-1,j)\hat{x}(i,j-1)\hat{x}(i-1,j-1)]x(i,j)^2}.
\] (3.1.22)

where \(\sigma_x^2\) is the variance of the image. From this expression it is clear that when the signal-to-noise ratio is large, i.e. \(\sigma_x^2 \gg \sigma_w^2\), the gain \(F(i,j)\) approaches unity, which causes \(\gamma_4\), \(\gamma_5\) and \(\gamma_6\) to approach zero. This implies that in this case the estimate \(\hat{x}(i,j)\) in (3.1.16) is almost entirely based on the current observation \(y(i,j)\). Conversely, when the signal-to-noise ratio is extremely small, the estimate is almost entirely based on the previously obtained estimates, which implies heavy smoothing. Applying the same analysis to the one-step predictor of Habibi in (3.1.3) reveals that its behavior is similar to that of the filter of Kak when the signal-to-noise ratio approaches zero, but that the smoothing is hardly reduced when the signal-to-noise ratio is large. This is an important difference between both estimators.

The filter of Kak is BIBO stable iff the following conditions simultaneously hold:
Fig. 3.3. Convergence of \( F(i,j) \) for the filter of Kak.

If the following constraints on \( \rho_v \) and \( \rho_h \) are met:

\[
0 < \rho_v < 1, \\
0 < \frac{\rho_v}{1 + \rho_v} < \rho_h < \frac{\rho_v}{1 - \rho_v} < 1,
\]

then conditions (3.1.23a)-(3.1.23c) lead to the following stability regions for \( F \), which must be simultaneously fulfilled:

\[
- \frac{(1+\rho_v)(1-\rho_h)}{1+\rho_v + \rho_h + \rho_v \rho_h} < F < \frac{(1+\rho_v)(1+\rho_h)}{1+\rho_v + \rho_h + \rho_v \rho_h}, \tag{3.1.25a}
\]

\[
F < - \frac{(1-\rho_v)(1+\rho_h)}{\rho_v + \rho_h + \rho_v \rho_h} \quad \text{or} \quad F > - \frac{(1-\rho_v)(1-\rho_h)}{\rho_v + \rho_h + \rho_v \rho_h}, \tag{3.1.25b}
\]

\[
- \frac{(1-\rho_v)}{\rho_v} < F < \frac{(1+\rho_v)}{\rho_v}. \tag{3.1.25c}
\]

The restrictions (3.1.24a) and (3.1.24b) vanish if \( \rho_v = \rho_h = \rho \) with \( 0 < \rho < 1 \). In that case the stability regions (3.1.25a)-(3.1.25c) result in

\[
- \frac{(1-\rho)^2}{2\rho^2} < F < \frac{(1+\rho)^2}{2\rho}, \tag{3.1.26}
\]

In Section 3.3 it will become apparent that under the restrictions \( \rho_v = \rho_h = \rho \) with \( 0 < \rho < 1 \), the steady-state value \( F_{ss} \) of the gain \( F(i,j) \) will attain a value between zero and one. Through (3.1.26) it implies that the filter of Kak is then always BIBO stable, because in (3.1.26) the lower bound is negative and the upper bound is always greater than one.

Note that it is possible to combine both the one-step predictor and the filter described above to obtain a smoothed estimate (See Section 1.2). For example, one may start the one-step predictor at the upper left of the image and proceed to the right and downward, while one may start the filter at the lower-right corner and proceed to the left and upward. Then we have two uncorrelated estimates available for each picture element. The smoothing, however, is not optimal, because not all observations are used in the estimation procedure due to the quarter-plane structure of the estimators (See Fig. 3.4).
3.2. MEASURES OF FILTER PERFORMANCE

In the previous section the stability of the two recursive point-to-point estimators was discussed in terms of the stability theorem as formulated by Huang. However, in spite of the fact that stability is a necessary condition it does not guarantee good filter performance. In this section we will discuss filter performance under the assumption that we are in a simulation environment where we have at our disposal the original MxN discrete image \( X \) with picture elements \( x(i,j) \), the noise field \( W \) with elements \( w(i,j) \) which is added to \( X \) on a pixel-by-pixel basis yielding the corrupted image \( Y \) with elements \( y(i,j) \), and the filter output \( \tilde{X} \) with elements \( \tilde{x}(i,j) \).

A commonly used measure of filter performance is the improvement in signal-to-noise ratio which may be defined in decibels (dB) as

\[
\eta_1 = 10 \log_{10} \frac{\sum_{i=1}^{M} \sum_{j=1}^{N} (y(i,j)-x(i,j))^2}{\sum_{i=1}^{M} \sum_{j=1}^{N} (x(i,j)-x'(i,j))^2}. \tag{3.2.1}
\]

Note that the numerator in (3.2.1) is proportional to the noise variance and hence inversely proportional to the signal-to-noise ratio before filtering. The denominator is proportional to the mean-square reconstruction error and hence inversely proportional to the signal-to-noise ratio after filtering. Estimation procedures may be compared by applying the same input to each of them and comparing the resulting values of \( \eta_1 \).

Obviously, a difference in \( \eta_1 \) is the result of different values of the denominator in (3.2.1). By exploiting the additivity of the observation noise and the linearity of the estimators, two more performance measures \( \eta_2 \) and \( \eta_3 \) can be derived from the denominator in (3.2.1).

Let the linear estimator be represented by the linear operator \( L \), operating on the sum of \( X \) and \( W \) and yielding the output \( \tilde{X} \). The filter procedure may then be denoted as

\[
\tilde{X} = L(X+W) = L(X)+L(W) = X' + W'. \tag{3.2.2}
\]

So the filter output may be viewed as the sum of a filtered version of the original image and a filtered version of the noise field. Now because \( x(i,j) \) and \( w(i,j) \) are assumed to be uncorrelated random variables the sample covariances between \( x(i,j) \) and \( w(i,j) \), between \( x(i,j) \) and \( w'(i,j) \), and between \( x'(i,j) \) and \( w'(i,j) \) will be zero. The mean-square reconstruction error MSE may then be rewritten as

\[
\text{MSE} = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} (\tilde{x}(i,j)-x(i,j))^2 \tag{3.2.3}
\]

\[
= \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} (x'(i,j)-x(i,j))^2 + \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} (w'(i,j))^2,
\]

where \( x'(i,j) \) and \( w'(i,j) \) are the elements of \( X' \) and \( W' \), respectively.

Here

\[
\sigma_d^2 = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} (x'(i,j)-x(i,j))^2 \tag{3.2.4}
\]

is the variance of the distortion caused by the filter to the original image and
\[
\sigma_n^2 = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} (w(i,j))^2
\]

is the variance of the remaining noise, that is, the noise still present after filtering. The performance measures we propose here are normalized versions of (3.2.4) and (3.2.5), i.e.

\[
\eta_2 = \frac{\sigma_n^2}{\sigma_x^2}
\]

and

\[
\eta_3 = \frac{\sigma_n^2}{\sigma_v^2}
\]

The use of the performance measures \( \eta_2 \) and \( \eta_3 \) makes it possible to evaluate separately how well a filter removes the noise and how badly it distorts the original image. Performance measure \( \eta_2 \) evaluates the distortion to the original image caused by the filtering procedure. In the case of perfect reconstruction \( \eta_2 = 0 \). Measure \( \eta_3 \) evaluates the normalized variance of the remaining noise. Again, \( \eta_3 \) yields a value of zero in the case of perfect reconstruction. When \( \eta_3 \) exceeds a value of one, the noise variance is enlarged by the filtering procedure, which implies noise amplification instead of reduction. An example of the use of \( \eta_2 \) and \( \eta_3 \) is given in Section 3.3.

3.3. EXPERIMENTAL RESULTS

The performance of the estimators has been investigated by using test images of size 256x256 picture elements, each quantized in eight bits. Here we restrict the discussion to the test image shown in Fig. 3.5, which is a typical example. The original image is corrupted by additive signal-independent zero-mean white Gaussian noise, with a signal-to-noise ratio SNR between 0.01 and 10.00. Figure 3.6 shows the test image with SNR=1.00. The corrupted image is filtered from left to right and from top to bottom, after which the performance measures \( \eta_2 \) and \( \eta_3 \) are evaluated.

In the experiments described here we use the steady-state value \( F_{ss} \) of the gain for the entire image. This steady-state value is chosen equal to \( F(10,10) \). The effect of this approximation on the final filter result is negligible. When the vertical and horizontal correlation coefficients are estimated separately from the test image of Fig. 3.5, we find a value of 0.94 for both coefficients. It is our experience, however, that in many cases the filter performance may be improved by using a somewhat lower value of the correlation coefficients in the computation of the filter parameters. This is supported by the fact that joint estimation of the vertical, horizontal and diagonal correlation coefficients for the image of Fig. 3.5, under the constraints of equality of the vertical and horizontal coefficients and separability of the two-dimensional autocorrelation function, yields a value of 0.79. In our experiments we have chosen \( \rho_v = \rho_h = \rho \), while filter performance has been calculated for a number of values of \( \rho \), i.e. \( \rho = 0.6, 0.7, 0.8 \) and 0.9.

We first discuss the performance of the one-step predictor defined in (3.1.10). The values of the performance measures \( \eta_2 \) and \( \eta_3 \) are plotted in Fig. 3.7 and Fig. 3.8, respectively, as a function of the steady-state gain \( F_{ss} \). Notice that \( F_{ss} \) itself depends on both SNR and \( \rho \). It can be shown that for \( \rho_v = \rho_h = \rho \) and some suitable initial conditions, (3.1.10) can for \( F_{ss} < 1 \) be very well approximated by \( F_{ss} = (1 - \rho)^2 \) SNR.
For future reference this relationship is expressed in numerical form in Table 3.1.

![Fig. 3.7. Performance \( \eta_2 \) versus \( F_{ss} \) for the one-step predictor of Habibi for different values of \( \rho \).](image)

![Fig. 3.8. Performance \( \eta_3 \) versus \( F_{ss} \) for the one-step predictor of Habibi for different values of \( \rho \).](image)

<table>
<thead>
<tr>
<th>( \text{SNR} )</th>
<th>( \rho = 0.6 )</th>
<th>( \rho = 0.7 )</th>
<th>( \rho = 0.8 )</th>
<th>( \rho = 0.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>4.1 .10^{-3}</td>
<td>2.6 .10^{-3}</td>
<td>1.3 .10^{-3}</td>
<td>3.6 .10^{-4}</td>
</tr>
<tr>
<td>0.1</td>
<td>4.1 .10^{-2}</td>
<td>2.6 .10^{-2}</td>
<td>1.3 .10^{-2}</td>
<td>3.6 .10^{-3}</td>
</tr>
<tr>
<td>1.00</td>
<td>4.1 .10^{-1}</td>
<td>2.6 .10^{-1}</td>
<td>1.3 .10^{-1}</td>
<td>3.6 .10^{-2}</td>
</tr>
</tbody>
</table>

Table 3.1: Values of \( F_{ss} \) for the one-step predictor of Habibi.

From Fig. 3.7 it is clear that for a fixed value of the parameter \( \rho \), the distortion is relatively large for very small values of \( F_{ss} \), that is, for very small values of the signal-to-noise ratio. From (3.1.3) and (3.1.6) it is understood that in this situation the new estimate is almost entirely based on the previous estimates. For increasing values of \( F_{ss} \) the distortion decreases when we move away from pure prediction in a direction which is automatically corrected through the increasing weight of the observation. When \( F_{ss} \) further increases, the weight of the observation increases, but so does the coefficient of the diagonally situated data point. This causes the variance of the distortion to increase as well. The shaded region in Fig. 3.7 indicates that the variance of the distortion exceeds the variance of the original image.

For the interpretation of Fig. 3.8 it should be noted that we now consider that part of the input of the one-step predictor which consists of the white noise field \( W \). Obviously, the filtered noise is no longer white. For very small values of \( F_{ss} \), i.e. for large values of the noise variance \( \sigma^2 \), \( \eta_3 \) is forced to be small. When \( F_{ss} \) increases, the absolute values of the weighting coefficients in the linear combination either remain constant or increase, which results in a decreasing performance in terms of the performance measure \( \eta_3 \). The shaded region in Fig. 3.8 indicates that the noise variance has been amplified instead of reduced by the one-step predictor.

Further we may observe from Figs. 3.7 and 3.8 that for decreasing values of \( \rho \), i.e. a weaker coupling between the estimates, the influence of the observation increases through a greater value of \( F_{ss} \).

Naturally, one is interested in both \( \eta_2 \) and \( \eta_3 \). Figure 3.9 shows \( \eta_2 \) and \( \eta_3 \) for some values of \( \rho \), where now \( F_{ss} \) varies along each curve. The shaded region in Fig. 3.9 represents the union of the shaded regions in Figs. 3.7 and 3.8. Filtering with values of \( F_{ss} \) which fall in this shaded region would serve no useful purpose. Table 3.2 presents the corresponding performance bounds for \( F_{ss} \), which may be compared with the stability regions computed from (3.1.15). From these results it is concluded that the filter performance measures lead to tighter bounds on the gain coefficient than the stability constraints. This is in agreement with the intuitively correct statement that filter performance decreases before instability occurs.

The experiments were repeated for the filter of Kak defined in (3.1.16) and (3.1.17). The steady-state values \( F_{ss} \) of the gain coefficient are given in Table 3.3. The performance measures \( \eta_2 \) and \( \eta_3 \) versus \( F_{ss} \) are plotted in Figs. 3.10 and 3.11, respectively. It has already
been pointed out in Section 3.1 that the coefficients of the previous estimates, i.e. $\gamma_4$, $\gamma_5$, and $\gamma_6$ in (3.1.17), tend to zero when $F_{ss}$ approaches unity. In the limit the filter output is equal to the filter input. This implies no distortion of the original image, i.e. $n_2$ tends to zero, and no change in the variance of the noise, i.e. $n_3$ tends to a value of one. This explains the behavior of $n_2$ and $n_3$ depicted in Figs. 3.10 and 3.11.

Because neither of the two performance measures exceeds a value of 1, it is concluded that filter performance does not yield a bound on the gain coefficient for the filter of Kak. It has already been shown that this estimator is always BIBO stable. So, it is concluded that the filter of Kak may be applied in any situation, i.e. for all values of the signal-to-noise ratio and of $p$.

The measures $n_2$ and $n_3$ provide detailed insight into the performance of the linear estimators. A more global measure is the improvement in signal-to-noise ratio, i.e. the performance measure $n_1$ as defined in (3.2.1). Under the assumptions given in Section 3.2 it holds that

\begin{tabular}{|c|c|c|c|c|}
\hline
$p=0.6$ & $p=0.7$ & $p=0.8$ & $p=0.9$ \\
\hline
SNR=0.01 & 9.8 .10^{-3} & 9.6 .10^{-3} & 9.3 .10^{-3} & 8.2 .10^{-3} \\
SNR=0.10 & 8.2 .10^{-2} & 7.6 .10^{-2} & 6.5 .10^{-2} & 4.4 .10^{-2} \\
SNR=1.00 & 3.8 .10^{-1} & 3.4 .10^{-1} & 2.6 .10^{-1} & 1.6 .10^{-1} \\
SNR=10.00 & 8.2 .10^{-1} & 7.6 .10^{-1} & 6.5 .10^{-1} & 4.4 .10^{-1} \\
\hline
\end{tabular}
\[ \eta_1 = 10 \log_{10} \frac{1}{\eta_3 + \text{SNR} \cdot \eta_2} \text{ dB.} \] (3.3.1)

For both estimators discussed here, the values of \( \eta_1 \) are plotted in Fig. 3.12 as a function of SNR for the case \( \rho = 0.9 \). From this typical example it can be seen that the filter of Kak has better overall performance than the one-step predictor for all values of the signal-to-noise ratio. Furthermore, it becomes clear that for \( \rho = 0.9 \) it does not serve any useful purpose to apply the one-step predictor if the signal-to-noise ratio prior to filtering exceeds a value of 3.3.

Various aspects of the above discussion are illustrated by the following examples. Both estimators are applied to the noise-corrupted image of Fig. 3.6 (SNR=1) with \( \rho = 0.9 \). The resulting image from the one-step predictor is shown in Fig. 3.13, where \( F_{\text{ss}} = 0.036, \eta_1 = 5.2 \text{ dB,} \eta_2 = 0.26 \text{ and } \eta_3 = 0.04 \). The result from the filter of Kak is given in Fig. 3.14, where \( F_{\text{ss}} = 0.16, \eta_1 = 7.3 \text{ dB,} \eta_2 = 0.08 \text{ and } \eta_3 = 0.10 \). The values of \( \eta_2 \) and \( \eta_3 \) for both estimators indicate a better noise-smoothing behavior for the one-step predictor, but at the cost of more distortion to the original image. As an example, Fig. 3.15 shows the distortion to the original image caused by the one-step predictor. Notice the way the edges are smoothed in this image. Ways to suppress this phenomenon are described in Chapter 4.

Figure 3.16 gives an example of the diminishing performance of the one-step predictor. Here SNR=5, \( \rho = 0.9 \) and \( F_{\text{ss}} = 0.18 \). True instability of the same filter occurs, for example, when SNR=6, \( \rho = 0.9 \) and \( F_{\text{ss}} = 0.22 \). The result is shown in Fig. 3.17.

We may conclude from these experiments that for the one-step predictor the performance measures lead to tighter bounds on the gain coefficient than follow from the stability constraints. This does not occur in the case of the Kak filter. Therefore, this filter may be applied in all situations, i.e. for all values of the signal-to-noise ratio and \( \rho \).
In the preceding chapter the restoration of noisy images has been performed by 2-D recursive estimators based on the homogeneous DPCM image model. The noise is effectively reduced by these linear estimation techniques, but the edges in the image are blurred and image contrast is reduced as well. These effects decrease the subjective quality of the restored image.

In this chapter a simple and computationally fast scan-ordered 1-D Kalman filter (predictor) is derived, which uses additional structural information about the edges in the noisy image. This filter behaves like the original noise-reducing Kalman filter if no edges are present, but has a greatly improved step response if an edge is detected. In this way the edge-blurring phenomenon is effectively reduced.

In Section 4.1 we discuss several ways to control the distortion caused by the estimation process by making use of additional structural information concerning the edges in the noisy image. Next, we introduce an edge-preserving recursive noise-reducing algorithm based on the improvement of the step response of the steady-state version of the scan-ordered 1-D scalar Kalman predictor. To gain insight into the performance of this algorithm, in Section 4.2 a frequency analysis is given.

In Section 4.3 the 1-D edge-preserving algorithm is extended to two dimensions by using a separable estimator structure.
In Section 4.4 we treat the problem of detecting edges in a noisy image. Experimental results are given in Section 4.5.

4.1. THE EDGE-PRESERVING RECURSIVE NOISE-REDUCING ALGORITHM

A number of Kalman filters, based on different dynamic models, has been introduced in the literature for noise reduction in image data. We have already mentioned the 1-D Kalman filters of Nahi [61] and Nahi and Assefi [62] based on a scan-ordered 1-D dynamical model, the 2-D estimators of Habibi [36] and Kak [77], based on the quarter-plane causal DPCM model, and the estimators proposed by Jain [40], [42] based on semicausal and noncausal models.

All these models have in common the assumption that the spatial correlation within an image can be approximated by a 2-D exponentially decaying autocovariance function. This leads to estimators which are insensitive to abrupt changes. Experiments show that the noise is effectively reduced, but that the edges are blurred by the linear (space-invariant) filtering operation and that image contrast is reduced as well. Depending on the intended use of the restored image the edge distortion may be undesirable. If the image is meant for processing by the human visual system, then it is desirable to have sharp and undistorted edges, especially because sharp edges are among the most effective stimulus configurations in visual perception [53], [66].

Several researchers have attempted to control the distortion caused by the estimation process by making use of additional structural information concerning the edges in the noisy image. We distinguish the following three levels of operation.

On level 1 structural information about the image is incorporated into the model. An estimator based on such a model should result in an improvement of the visual restoration result. Nahi and Habibi [64] interpret edges as locations where the statistical properties of an image change. They describe a nonlinear statistical model that uses edge-detection techniques to segment an image into two regions, such as an object and its background. Then both regions are filtered with conventional linear MSE filters, where each filter is based on the statistics of the corresponding region. This approach appears to be rather complicated if more than two regions are present in an image.

On level 2 the estimator is adapted to the presence of an edge. Anderson and Netravali [3] work on this level by varying the finite impulse response of a 2-D nonrecursive linear filter.

On level 3, Panda and Kak [68] propose a spatial-domain postprocessor, which follows the linear MSE estimator.

The edge-preserving recursive noise-reducing algorithm to be described here operates on level 2 (see also [9]). A scan-ordered 1-D scalar Kalman predictor is derived based on the assumption that no abrupt system changes are present, and a second system is designed to extract information concerning the size and location of the edges from the noisy image in order to determine whether a change has occurred and to adjust the predictor accordingly. The reasoning behind this structure is [92], that since changes occur rather infrequently, we do not wish to degrade the performance of the predictor under normal conditions requiring the state estimate to be directly sensitive to system changes.

To apply a 1-D Kalman predictor to an image the 2-D image data has to be converted into a 1-D signal via a scanning procedure. Nahi and Assefi [62] showed that although the image is assumed to be spatially homogeneous, the output of the scanner is nonstationary due to the periodic nature of the scanning procedure. This introduces additional complexities in the design of the dynamic model, finally resulting in a 1-D first-order linear vector-difference equation.

Here we follow a simpler method using a concatenation of scan lines. The nonstationarity at the end of the \( n \)th scan line and the beginning of the \((n+1)^{th}\) scan line is modeled as an edge. The vertical correlation in the image is not taken into consideration in defining the image model. As we shall see, however, (vertical) structural information is used to control the predictor. It is assumed that the autocovariance function of the concatenation of scan lines is exponentially decaying, i.e.
\[ \tau_{\text{image}}(l) = \sigma_x^2 \rho^{|l|}, \quad (4.1.1) \]

where \( \sigma_x^2 \) is the image variance, \( \rho \) the (horizontal) correlation coefficient, and \( l \) the increment or the shift in position. Under these assumptions the image can be modeled by the 1-D scalar stochastic difference equation (See Section 2.1)

\[ x(j+1) = \rho x(j) + u(j), \quad (4.1.2) \]

where \( x(j) \) is the scalar state representing the original pixel intensity at point \( j \), and where \( u(j) \) is a zero-mean white-noise input process with variance \( \sigma_u^2 \). Assuming that the observations \( y(j) \) can be expressed as

\[ y(j) = x(j) + w(j), \quad (4.1.3) \]

where \( w(j) \) is a zero-mean white-noise process with variance \( \sigma_w^2 \) and uncorrelated with the data, a scalar Kalman predictor can be derived (see Table 1.1) to obtain a best linear MSE estimate \( \hat{x} \) of the state \( x \)

\[ \hat{x}(j+1) = \rho \hat{x}(j) + F(j) \{ y(j) - \hat{x}(j) \}, \quad (4.1.4a) \]

or

\[ \hat{x}(j+1) = [\rho - F(j)] \hat{x}(j) + F(j) y(j). \quad (4.1.4b) \]

The predictor gain is given by

\[ F(j) = \frac{\sigma_u^2}{\sigma_x^2 + \sigma_u^2 \rho^2}. \quad (4.1.5) \]

For the predictor mean-square error \( P(j) \) it holds that

\[ P(j+1) = [\rho - F(j)]^2 P(j) + (1 - \rho^2) \sigma_x^2 + F^2(j) \sigma_u^2. \quad (4.1.6) \]

The initial conditions are

\[ \hat{x}(0) = 0, \quad P(0) = \tau_{\text{image}}(0) = \sigma_x^2. \quad (4.1.7) \]

From our experiments, to be described in Section 4.5, we have observed that after a small number of iterations the Kalman predictor equation (4.1.4) converges to a steady-state. Thus, (4.1.4b) can to a good approximation be written with constant coefficients \( a \) and \( b \) as

\[ \hat{x}(j+1) = a \hat{x}(j) + b y(j). \quad (4.1.8) \]

The step response of this steady-state predictor on a unit step function can now be derived as follows. The \( z \)-transform of (4.1.8), after a time shift, is

\[ \hat{x}(z) = a z^{-1} x(z) + b z^{-1} y(z). \quad (4.1.9) \]

Then, for the transfer function of the Kalman predictor it holds that

\[ H(z) = \frac{x(z)}{y(z)} = \frac{b z^{-1}}{1 - a z^{-1}} = \frac{b}{z-a}. \quad (4.1.10) \]

The predictor is stable if it has no pole outside the unit circle, i.e. if \( |a| < 1 \). The \( z \)-transform of a unit step function at \( j=0 \) is

\[ Y(z) = \sum_{j=0}^{\infty} y(j) z^{-j} = \frac{z}{z-1}. \quad (4.1.11) \]

The response of the predictor in the \( z \)-domain is

\[ \hat{x}(z) = Y(z) H(z) \]

\[ = \frac{z}{z-1} \cdot \frac{b}{z-a}. \quad (4.1.12) \]

The inverse \( z \)-transform of (4.1.12) is determined by using Cauchy's residue theorem [46]. Thus,
This step response \( \hat{x}(j) \) is given in Fig. 4.1.

![Fig. 4.1. Response \( \hat{x}(j) \) on unit step function.](image)

The steady-state value for \( j \to \infty \) directly follows from (4.1.13b) by recalling that \( |a| < 1 \); thus

\[
\hat{x}_{ss} = \frac{b}{1-a} .
\]

(4.1.14)

Now we discuss the feasibility of improving the step response by using edge information. The edge information is obtained from the noisy image through an edge-detection operator. In Section 4.4 attention will be paid to the specific choice of an edge-detection scheme for noisy images. Here we restrict the discussion to the following separable edge detector:

\[
H_{\text{edge}} = \frac{1}{3} \begin{bmatrix}
-1 & 0 & 1 \\
-1 & 0 & 1 \\
-1 & 0 & 1
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
1 & 1 & 1
\end{bmatrix} \begin{bmatrix}
-1 & 0 & 1
\end{bmatrix},
\]

(4.1.15)

which is the most sensitive to vertical edges. Convolution of the image with this operator yields the so-called edge weights.

To derive the edge-controlled predictor we first consider a noise-free image with one ideal vertical edge. Note that for an ideal vertical edge the edge weights correspond to the height of the edge. Along a horizontal scan line the observations \( y(j) \) represent a step function. If this image is convolved with \( H_{\text{edge}} \) the output values or edge weights \( v(j) \) along a horizontal scan line form a block function, as shown in Fig. 4.2. For the sake of convenience we have set \( j = 0 \) for the first value of \( v(j) \neq 0 \).

![Fig. 4.2. Edge weights \( v(j) \) from 3x3 edge detector on observations \( y(j) \).](image)

The edge weights can now be used to improve the step response. This is achieved by adding the edge weights \( v(j) \) with a scaling coefficient \( c \) to the observations \( y(j) \). So the predictor input is

\[
y_1(j) = y(j) + c \cdot v(j).
\]

(4.1.16)

In this way the predictor is adjusted according to whether or not an edge is detected, that is, the input is temporarily increased at an edge with a positive derivative and temporarily reduced at an edge with a negative derivative. The factor \( c \) is determined such that the predictor output \( \hat{x}(j) \) reaches its steady-state \( \hat{x}_{ss} \) as quickly as possible.

The \( z \)-transform of the input signal \( y_1(j) \) is
\[ Y_1(z) = \sum_{j=0}^{\infty} y_1(j)z^{-j} = \sum_{j=0}^{\infty} y(j)z^{-j} + c \sum_{j=0}^{\infty} v(j)z^{-j} \]
\[ = \frac{1}{z-1} + c \cdot \frac{z+1}{z}. \quad (4.1.17) \]

The response of the predictor in the z-domain is
\[ \hat{x}(z) = Y_1(z) H(z) \]
\[ = \left( \frac{1}{z-1} + c \cdot \frac{z+1}{z} \right) \frac{b}{z-a} \]
\[ = b \frac{z^2 + z - c}{z(z-1)(z-a)}. \quad (4.1.18) \]

Applying the power-expansion method [46] yields the first value of \( \hat{x}(j) \):
\[ \hat{x}(1) = bc. \quad (4.1.19) \]

The remaining values of \( \hat{x}(j) \) are determined by using the residue theorem (4.1.13a) for \( j=2,3, ... \), at the poles \( p_1=1 \) and \( p_2=a \), from which we obtain
\[ \hat{x}(j) = \left( \frac{b}{z-a} \right) \left[ 1 - \left( \frac{c}{2} \cdot \frac{a}{z-a} \right)^{j-1} \right], \quad j=2,3, ... \quad (4.1.20) \]

From (4.1.20) it is clear that the prediction output reaches its steady-state \( \hat{x}(j) \) at \( j=2,3, ... \), if \( ca^2 - a = 0 \), or if
\[ c = \frac{a}{1-a}. \quad (4.1.21) \]

Note that \( \hat{x}(j) \) is the predictor output which results when the observations consist of a unit step at \( j=1 \). This improved step response is shown in Fig. 4.3.

The predictor equation now becomes
\[ \hat{x}(j+1) = a \hat{x}(j) + b (y(j) + c v(j)), \quad (4.1.22) \]
with \( c = \frac{a}{1-a^2} \).

Fig. 4.3. Improved step response

If (4.1.22) is used to filter a noisy image, then in relatively flat areas the value of \( v(j) \) is small and the filter behaves like the original one-step predictor, while at an edge the value of \( v(j) \) is such that a step is immediately followed. The total computational effort per restored pixel lies in the order of 11 scalar multiplications or additions, if the separability of the edge detector (4.1.15) is exploited.

In Section 4.5 (experimental results) both the 3x3 edge detector (4.1.15) as well as the following separable 3x4 edge detector
\[ H_{\text{edge}} = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \quad (4.1.23) \]

will be applied for detecting edges in noisy images. For this detector the following optimal value of \( c \) can be completely analogously derived:
\[ c = \frac{2a}{14a^2 - a^3}. \quad (4.1.24) \]

Note that if the total image is available in advance, it is possible to apply a one-step predictor in one direction and a true filter in the reverse direction to obtain an optimal smoothed estimate, which exactly uses all available data [79] (See Fig. 4.4).

A filter equation with edge information can be derived in the same way as for the predictor. In the steady-state the filter equation is of
the following form:
\[ \hat{x}(j) = a' \hat{x}(j-1) + b' \{ y(j) + c' v(j) \} , \]
with
\[ a' = \rho (1 - \gamma_8) , \quad b' = \gamma_8 , \quad c' = \frac{-a'}{1-a'^2} . \]  
(4.1.25)

Fig. 4.4. Optimal smoothing with one-step predictor and filter.

4.2. FREQUENCY ANALYSIS OF THE EDGE-PRESERVING PREDICTOR

In this section we investigate the frequency response of the edge-preserving predictor as defined in (4.1.22) for the edge detector (4.1.15). We are particularly interested in the influence of variations in the value of the optimization factor $c$ on the frequency response of the predictor. This is due to the fact that the optimal value of $c$ in (4.1.22) has been found by optimizing the step response for a noise-free image with one vertical edge. In practical situations, however, there are two opposite effects which may necessitate adjustment of the value of $c$.

First, not all edges will be vertically oriented. In the line-scanning approach the jump in the scan signal due to a not exactly vertically oriented edge is the same as for a vertical edge. The edge detector (4.1.15), however, measures a lower value of the edge weight $v(j)$. This effect can be compensated by increasing the value of $c$.

Second, because of the presence of noise in the image artefacts may be introduced by the edge detector. Therefore, the influence of the edge detector must be less than in the case of noise-free images. This can be obtained by decreasing the value of $c$.

Now in order to investigate the influence of variations in $c$ on the frequency response of the predictor, (4.1.22) can be interpreted as a linear system with input $y(j)$ and output $\hat{x}(j+1)$ as given in Fig. 4.5.

\[ H_t(z) = (1 + c \ H_\text{edge}(z)) \ H(z) , \]  
(4.2.1)

where $H_\text{edge}(z)$ is the transfer function of the edge detector in (4.1.15) and $H(z)$ that of the one-step predictor in (4.1.10).

In the case $c=0$ no edge information is used and the system in Fig. 4.5 reduces to the orginally proposed one-step predictor with its poor step response. The frequency response can then be found by substituting $z=\text{e}^{j\omega \Delta t}$ with sampling interval $\Delta t=1$ in (4.1.10), yielding

\[ H(\omega) = \frac{b\ e^{-j\omega}}{1-a\ e^{-j\omega}} . \]  
(4.2.2)

The magnitude and the phase of the frequency response are given by

\[ |H(\omega)| = \left| \frac{b\ e^{-j\omega}}{1-a\ e^{-j\omega}} \right| = \frac{b}{\sqrt{1+a^2-2a\cos \omega}} ; \]  
(4.2.3)

and

\[ \arg H(\omega) = \arg \left( b\ e^{-j\omega} \right) - \arg \left( 1-a\ e^{-j\omega} \right) = -\omega - \arctg \left( \frac{a\sin \omega}{\sqrt{1-a^2}} \right) . \]  
(4.2.4)
They are shown in Fig. 4.6 for $0 \leq \omega \leq \pi$:

![Figure 4.6](image1.png)

Fig. 4.6. Magnitude and phase of the frequency response of the one-step predictor ($a=0.69$).

In Fig. 4.6 we observe that the one-step predictor is basically a low-pass filter. The noise is smoothed out by this low-pass filter to a great extent, but so are the edges which contain high-frequency information of the image.

In the case $c > 0$ the high frequency information extracted from the noisy image by means of the edge detector is used in the estimation procedure. For convenience, the frequency response of the edge detector will be examined first, followed by the investigation of the frequency response of the total system (4.2.1).

Because of separability the transfer function of the edge detector (4.1.15) can be written as

$$H_{\text{edge}}(z) = \frac{1}{3} H_h(z) H_v(z)$$

$$= \frac{1}{3} (z^{-1} - z^{-M}) (z^{-N} + 1 + z^{-N}),$$  \hspace{1cm} (4.2.5)

where the vertical operator extends over three concatenated scan lines and where $N$ is the number of pixels per scan line. The frequency response of the horizontal differentiating operator $[-1  0  1]$ is then given by

$$H_h(\omega) = e^{j\omega} - e^{-j\omega} = 2 j \sin \omega,$$  \hspace{1cm} (4.2.6)

with magnitude

$$|H_h(\omega)| = 2 \sin \omega, \hspace{1cm} 0 \leq \omega \leq \pi,$$  \hspace{1cm} (4.2.7)

and phase (see Fig. 4.7)

$$\arg (H_h(\omega)) = \frac{\pi}{2}, \hspace{1cm} 0 \leq \omega \leq \pi.$$  \hspace{1cm} (4.2.8)

![Figure 4.7](image2.png)

Fig. 4.7. Magnitude and phase of the frequency response of the horizontal differentiating operator.

Note in Fig. 4.7 the differentiating character of the operator with attenuation of the highest frequencies. This prevents amplification of the highest (noisy) frequencies. The frequency response of the vertical summing operator $[1  1  1]^t$ is given by

$$H_v(\omega) = e^{-j\omega N} + 1 + e^{j\omega N} = 1 + 2 \cos \omega N,$$  \hspace{1cm} (4.2.9)
with magnitude
\[ |H_y(\omega)| = 1 + 2\cos \omega N, \quad 0 \leq \omega \leq \pi, \quad (4.2.10) \]
and phase (see Fig. 4.8)
\[ \arg H_y(\omega) = 0 \quad [\pm \pi], \quad 0 \leq \omega \leq \pi. \quad (4.2.11) \]

A filter with such a response is called a comb filter. Elementary harmonics which repeat each scan line and cause no vertical intensity variations may pass this filter, while elementary harmonics which give rise to vertical intensity variations are attenuated by it.

![Magnitude and phase of the vertical summing operator, N=8.](image)

The magnitude of the frequency response of the combined horizontal and vertical edge operator has the comb structure of Fig. 4.8, but as envelope the band-pass structure of Fig. 4.7.

After investigating the frequency response of both the one-step predictor (low-pass character) and the edge detector (band-pass character), their combined behavior for different values of \( c \) will be analyzed. The total response is given by
\[
H_T^c(z) = (1 + c H_{\text{edge}}(z)) H(z)
= \left(1 + \frac{1}{2} c \frac{(z^{-1} - z^{-1})}{(z^{-N} + z^{-1})} \right) \frac{b}{z-a}.
\quad (4.2.12)
\]
To facilitate the analysis we first assume that the image has identical rows. In that case the edge detector (4.1.15) can be written as a horizontal differentiating operator \([-1 \quad 0 \quad 1] \) only and (4.2.12) can be simplified to
\[
H_T^c(z) = (1 + c(z^{+1} - z^{-1})) \frac{b}{z-a}.
\quad (4.2.13)
\]
The transfer function \( H_T^c(z) \) has two zeros:
\[
n_1 = -1 - \sqrt{1 + 4c^2}, \quad n_2 = -1 + \sqrt{1 + 4c^2},
\quad (4.2.14a)
\]
and two poles: \( p_1 = a, \quad p_2 = 0. \quad (4.2.14b)\)

For increasing \( c \), starting from 0 the zero \( n_1 \) moves over the real axis from \(-\infty\) to \(-1\), and similarly the zero \( n_2 \) moves over the real axis from 0 to +1. For different values of \( c \), in Fig. 4.9 a pole/zero plot and the magnitude of the frequency response is shown, where the constants \( a \) and \( b \) are chosen (in accordance with the first experiment in Section 4.5) as \( a=0.69 \) and \( b=0.25 \). The closer \( a \) comes to the unit circle, the higher the attenuation is for these specific frequencies, i.e. \( n_1 \) attenuates the highest frequencies and \( n_2 \) the lowest frequencies. Thus, by increasing the value of \( c \) the frequency response changes from a low-pass to a band-pass characteristic.

An exact cancellation of the pole \( p_1 = a \) (recursive part of the system) by a zero \( n_2 = a \) (nonrecursive part) takes place for the optimal value of
\[
c = \frac{a}{1-a^2} \quad \text{as given in (4.1.21).} \quad \text{In that case (4.2.13) simplifies to}
\]
\[
H_T^c(z) = \frac{b}{1-a^2} (a + \frac{1}{z}).
\quad (4.2.15)
\]
Inverse z transformation of \((4.2.15)\) yields the impulse response

\[
h_1^T(j) = \frac{b}{1-\alpha} \{a \delta(j) + \delta(j-1)\}, \tag{4.2.16}
\]

where \(\delta(j)\) is the Kronecker delta. Thus, for the optimal value of \(c\) the recursive predictor equation \((4.1.22)\) has a finite impulse response.

Next, the frequency analysis will be performed without constraints on the image data. In that case the influence of \(c\) on the frequency response must be obtained directly from \((4.2.12)\). In Fig. 4.10 the magnitude of the frequency response of the total system is given for \(c=3.00\) and \(N=256\) pixels per scan line.

Observe that the envelope of the magnitude function is equal to the characteristic in Fig. 4.9e, but that due to the comb filter action of the vertical summing operator vertical intensity variations in the image are now attenuated.

Note that a similar frequency analysis can be performed for the total system, if the edge detector \((4.1.23)\) is used.

Summarizing this section, we have shown that for increasing values of \(c\) the frequency response of the total 1-D line scanning system varies from a low-pass character with much noise smoothing and a lot of edge smearing to a band-pass character with less noise reduction, but better...
edge-preserving properties. Besides that, due to the comb filter property vertically oriented structures are favored over structures with other orientations.

4.3. SEPARABLE 2-D ESTIMATOR WITH EDGE INFORMATION

In this section we investigate the possibility of extending the 1-D edge-preserving algorithm to two dimensions for the case in which the original image can be represented by the separable exponentially decaying autocovariance function (2.4.1). In that case the image can be recursively described by means of the quarter-plane causal DPCM model

\[ x(i,j) = p_v x(i-1,j) + p_h x(i,j-1) - p_v p_h x(i-1,j-1) + u(i,j) \]  

(4.3.1)

Because of the separability of this 2-D model (see Section 2.4), it can be rewritten as two coupled 1-D models: a horizontal and a vertical equation, respectively

\[ x(i,j) = \rho_h x(i,j-1) + \hat{x}(i,j), \]  

(4.3.2a)

\[ \hat{x}(i,j) = \rho_v \hat{x}(i-1,j) + u(i,j). \]  

(4.3.2b)

Thus, given the separable image autocovariance function we obtain a separable white-noise driven image model. The next step is to investigate the possible separability of an estimator based on this separable model. If the observations are of the form

\[ y(i,j) = x(i,j) + w(i,j), \]  

(4.3.3)

where \( w(i,j) \) is a homogeneous zero-mean white input process uncorrelated with the data, then we know from Chapter 3 that a one-step predictor (Habibi) may be defined with steady-state form

\[ \hat{x}(i,j) = (1-F) \{ \rho_v \hat{x}(i-1,j) + \rho_h \hat{x}(i,j-1) - p_v p_h \hat{x}(i-1,j-1) \} \]  

(4.3.4)

or a true filter (Kak) with steady-state form

\[ \hat{x}(i,j) = (1-F) \{ \rho_v \hat{x}(i-1,j) + \rho_h \hat{x}(i,j-1) - p_v p_h \hat{x}(i-1,j-1) \} \]  

(4.3.5)

\[ + F y(i,j). \]

The coefficient matrices for both estimators are respectively

\[ S_{\text{Habibi}} = \begin{bmatrix} \rho_v + F & -\rho_v \\ -\rho_h & 1 \end{bmatrix}, \quad S_{\text{Kak}} = \begin{bmatrix} \rho_v p_h (1-F) & -\rho_v (1-F) \\ -\rho_h (1-F) & 1 \end{bmatrix} \]

(4.3.6)

In both cases the rows of these matrices are pairwise linearly independent, i.e. the transfer functions of these estimators are not separable. Thus, starting from a separable 2-D model neither estimator structures are separable. For the filter of Kak, however, only a slight modification is needed to obtain a separable filter. If the diagonal term \( \rho_v p_h (1-F) \) is replaced by \( \rho_v p_h (1-F)^2 \), then the coefficient matrix \( S_{\text{Kak}} \) has pairwise linearly dependent row, and the resulting separable filter is given by

\[ \hat{x}(i,j) = (1-F) \{ \rho_v \hat{x}(i-1,j) + \rho_h \hat{x}(i,j-1) - p_v p_h \hat{x}(i-1,j-1) \} \]  

(4.3.7)

\[ + F y(i,j). \]

From (3.1.26) we know that under the restrictions \( p_v = p_h = p \) with \( 0 < p < 1 \), it holds that \( F \) attains a value between zero and one. This results in the following inequality for the weighting coefficients of the diagonal term:

\[ 0 < \rho_v p_h (1-F)^2 < \rho_v p_h (1-F) < 1, \]  

(4.3.8)

yielding a somewhat lower weighting of the diagonal term in the case of the separable filter. Given a practical situation where \( p_v = p_h = 0.9 \) and
SNR=1, the steady-state value of the gain $F_{ss}$ is 0.16. Substitution of these values in (4.3.8) yields $0 < 0.57 < 0.68 < 1$. In Section 4.5 the performance of the 2-D separable filter (4.3.7) with these parameter values will be compared with the results of the Kak filter (4.3.5). The transfer function of the separable filter (4.3.7) is given by

$$F_{ss}(z_1, z_2) = H_v(z_1)H_h(z_2) = \frac{1}{[1 - \rho_v z_1^{-1}(1 - F_{ss})][1 - \rho_h z_2^{-1}(1 - F_{ss})]}.$$  

(4.3.9)

from which the frequency response may be obtained by substituting $z_1 = e^{j\omega_1}$ and $z_2 = e^{j\omega_2}$. The magnitude of the frequency response is shown in Fig. 4.11, indicating a low-pass character in both horizontal and vertical directions.

Fig. 4.11. Magnitude of the frequency response of the 2-D separable filter

Because of the separability (4.3.7) can be written as two coupled 1-D estimators:

$$\hat{x}(i,j) = \rho_h(1-F_{ss}) \hat{x}(i,j-1) + F_{ss} \hat{x}(i,j),$$  

(4.3.10a)

$$\hat{\hat{x}}(i,j) = \rho_v(1-F_{ss}) \hat{x}(i-1,j) + F_{ss} y(i,j).$$  

(4.3.10b)

The step response of both equations may be improved by using edge information from two different edge detectors which are the most sensitive in the vertical and the horizontal direction, respectively (see Fig. 4.12).

Fig. 4.12. 2-D separable filter with edge information

Experimental results of this 2-D separable filter with edge information are given in Section 4.5.

4.4. EDGE DETECTION IN NOISY IMAGES

In this section we discuss in more detail the problem of detecting edges in noisy images. There are several ways in which the edges in an image can be defined. Each of the many edge detection schemes developed by various researchers is based implicitly or explicitly on a specific definition of an edge. In an image consisting of homogeneous regions without texture or noise we define an edge as the boundary between neighboring regions that differ in constant gray level. The profile of such an edge corresponds to the step function used in Section 4.1. In the presence of noise (and possibly texture) we define an edge as the boundary between neighboring regions that differ in average gray level. A number
of edge-detection schemes is based on this definition of an edge [77].

We base our choice of a specific scheme on two considerations, i.e.
a strong preference for a relatively simple edge detector and the results
of the comparative studies of Fram and Deutsch [26], [27] who investig­
gated three types of edge-detecting and edge-fitting operators in a
noisy environment. Therefore, we restrict ourselves to the class of
noise-smoothing difference operators, i.e. operators in which some kind
of integration precedes the differentiation process. Practical implemen­
tation of these operators is carried out by means of a 2-D discrete con­
volution of the noisy image y(i,j) with a convolution operator h(i,j),
and results in the edge weights v(i,j).

Three aspects of this class of edge detectors deserve our attention:
the shape of the convolution function, its size, and its directional
sensitivity. Argyle [5] and MacLeod [55], [56] propose convolution func­
tions with exponentially decreasing weights, which are intuitively
appealing because pixels possibly lying on neighboring edges will con­
tribute little to the resulting edge weights. Rosenfeld [75], [76] uses
convolution functions with equal weights and adjustable
size. This size is critical: when large operators are used the noise dependence decrea­
ses, but the edges of small objects cannot be detected correctly. As for
the directional sensitivity of the edge detector, it is noted that the
1-D predictor processes the image line by line. Therefore, our edge de­
tector should be the most sensitive to vertical edges.

In order to find an edge detector suitable for our application, some
computer experiments were carried out in which the size and the shape
of the convolution function were varied [21]. The smallest operator we
tested was of size 3x3, the largest of size 9x9. We investigated three
operator shapes with the common property of a discontinuity in the cen­
ter, but with different weighting coefficients: equal weights, linearly
decreasing, and exponentially decreasing. For the test picture and sig­
mal-to-noise ratio discussed in Section 4.5 we found that both a simple
equal weight operator of size 3x3 (4.1.15) and an equal weight operator
of size 3x4 (4.1.23) form an attractive compromise between size and per­
formance, where performance has been judged by visual inspection of the
image displaying the resulting edge weights. Note that the noise-smoothing
behavior increases with its size, while the improvement of the step
response decreases.

4.5. EXPERIMENTAL RESULTS

The performance of the edge-preserving estimators has been investig­
gated by using test images of size 256x256 picture elements, each quan­
tized in eight bits. Here we restrict the discussion to the test image
shown in Fig. 3.5 and to its noise-corrupted version with SNR=1 as shown
in Fig. 3.6. The performance is evaluated by means of the measures \( \eta_1, \eta_2 \) and \( \eta_3 \) as defined in Section 3.2.

This section is organized as follows. First we discuss the performance
of the edge-preserving predictor (4.1.22), and next that of the edge­
preserving filter (4.1.25). Then their combined behavior as a smoother
will be investigated. Next, the performance of the 2-D separable filter
and its edge-controlled version will be compared. Finally, at the end
of this section we describe an experiment with the edge-preserving pre­
dictor to combine noise smoothing and edge preservation in an even better
way.

Experiment 1: Edge-preserving predictor

We compare the performance of the edge-preserving predictor for the
case in which the edge information is obtained from a 3x3 edge operator
(4.1.15) as well as from a 3x4 edge operator (4.1.23). The steady-state
value \( F = 0.25 \) of the gain is attained after 14 iterations in the sense
that the absolute difference between two subsequent values of \( F \) as com­
puted through (4.1.5) and (4.1.6) is less than \( 10^{-5} \). The value of \( p \) was
found to be 0.94 for this image. Thus, for the coefficients of the one­
step predictor it holds that \( b_F = 0.25 \) and \( a_F = 0.69 \). The optimal
values of \( c \) for the 3x3 and the 3x4 edge operator are \( c = 1.32 \) and \( c = 1.56 \),
respectively. In Table 4.1 the values of the performance measures \( \eta_1, \eta_2 \) and \( \eta_3 \) are given separately for the edge-preserving predictor for the
3x3 and the 3x4 edge operator. In order to evaluate the effect of varia­
tions in the factor \( c \) on the performance, some other values besides the optimal values are used.

<table>
<thead>
<tr>
<th>((3x3))</th>
<th>((3x4))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c )</td>
<td>( \eta_1 ) (dB)</td>
</tr>
<tr>
<td>0.75</td>
<td>6.18</td>
</tr>
<tr>
<td>0.90</td>
<td>6.25</td>
</tr>
<tr>
<td>1.30</td>
<td>6.25</td>
</tr>
<tr>
<td>1.50</td>
<td>6.16</td>
</tr>
<tr>
<td>2.10</td>
<td>5.17</td>
</tr>
</tbody>
</table>

Table 4.1. Values of performance measures for the edge-preserving predictor for 3x3 and 3x4 edge operators

From this table we observe that for both the 3x3 and the 3x4 edge operator the largest improvement in signal-to-noise ratio (\( \eta_1 \)) is obtained for a somewhat lower value of \( c \) than the optimal value \( c=1.32 \) and correspondingly \( c=1.56 \). The influence of artefacts introduced by the edge detectors due to the presence of noise in the image demand somewhat lower values of \( c \). Furthermore, we may observe that for increasing values of \( c \) the value \( \eta_2 \) decreases (less distortion to original image), while \( \eta_3 \) increases (more remaining noise). This is caused by the fact that for increasing values of \( c \) the frequency response of the edge-preserving predictor moves away from a low-pass character with much noise reduction and a lot of edge smearing to a band-pass character with less noise-reduction but better edge-preserving properties (See also Fig. 4.9). This may be preferable, because in high-contrast regions, the human visual system readily accepts more noise to obtain a greater resolution [93]. Note that for large values of \( c \) the value of \( \eta_2 \) is going to increase. An overall comparison of the estimation results in Table 4.1 reveals slightly better performance for the 3x4 edge operator.

In Fig. 4.13 the estimation result is shown without using edge information (\( c=0 \)), while in Fig. 4.14 the estimation result is shown for the optimal value \( c=1.56 \) of the 3x4 edge operator. The performance \( \eta_1 \) increases from 4.89 to 6.30 dB. Note that the edges in Fig. 4.14 are less distorted. The edge weights resulting from the 3x4 edge operator are shown in Fig. 4.15, where all edge weights are made positive for display purposes.

Fig. 4.13. Prediction estimate without edge information, \( c=0 \)

Fig. 4.14. Prediction estimate with edge information from 3x4 operator, \( c=1.56 \)

Fig. 4.15. Edge-weight picture of 3x4 operator

**Experiment 2: Edge-preserving filter**

Similar experiments were performed for the edge-preserving filter as defined in (4.1.25). For the filter coefficients it holds that
The optimal values of $c'$ for the 3x3 and 3x4 edge operators are $c' = 1.43$ and $c' = 1.67$, respectively. In Table 4.2 the values of the performance measures $\eta_1$, $\eta_2$, and $\eta_3$ are given for both edge operators and for different values of $c'$.

<table>
<thead>
<tr>
<th></th>
<th>(3x3)</th>
<th></th>
<th>(3x4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c'$</td>
<td>$\eta_1$ (dB)</td>
<td>$\eta_2$</td>
<td>$\eta_3$</td>
</tr>
<tr>
<td>0</td>
<td>6.02</td>
<td>.15</td>
<td>.10</td>
</tr>
<tr>
<td>0.45</td>
<td>6.63</td>
<td>.11</td>
<td>.11</td>
</tr>
<tr>
<td>0.60</td>
<td>6.67</td>
<td>.11</td>
<td>.11</td>
</tr>
<tr>
<td>0.75</td>
<td>6.65</td>
<td>.11</td>
<td>.11</td>
</tr>
<tr>
<td>1.43</td>
<td>5.93</td>
<td>.12</td>
<td>.14</td>
</tr>
</tbody>
</table>

Table 4.2. Values of performance measures for the edge-preserving filter for 3x3 and 3x4 edge operators

From this table we observe that for both the 3x3 and the 3x4 edge operator the largest improvement in signal-to-noise ratio ($\eta_1$) is obtained for significantly lower values of $c'$ than the optimal values. This is caused by the fact that the filter responds quicker to the presence of artefacts than the predictor in the previous experiment. Furthermore, it can be observed that the edge-preserving filter with edge information from the 3x4 operator leads to slightly better results than that with edge information from the 3x3 operator.

Comparison of both Tables 4.1 and 4.2 reveals that the improvement in performance brought about by using edge information is largest for the predictors. However, the filter has already a better performance than the predictor without edge information. For the predictors the distortion caused by the estimation procedure to the original image ($\eta_2$) is most significantly reduced by the edge information.

Experiment 3: Edge-preserving smoothing

If the total image is available in advance an optimal smoothed estimate can be obtained by applying a one-step predictor from left to right and from top to bottom and a true filter in reverse direction from right to left and from bottom to top and then by combining these estimates. If no edge information is used we obtain an improvement $\eta_1 = 6.62$ dB. If, for example, edge information is used through the 3x3 operator with $c = c' = 0.75$, the improvement is $\eta_1 = 7.23$ dB.

Experiment 4: 2-D separable edge-preserving filter

In this experiment the performance of the 2-D separable edge-preserving filter as described in Section 4.3 (Fig. 4.12) will be investigated. First, the noisy image in Fig. 3.6 is filtered in the horizontal and vertical directions without edge information with filter parameters $\rho_v = \rho_h = 0.9$ and $\rho_{ss} = 0.16$. The estimation result is shown in Fig. 4.16, where the improvement $\eta_1 = 6.20$ dB. This result may be compared with the Kak filter result shown in Fig. 3.14 of Section 3.3 ($\eta_1 = 7.30$ dB). It appears that the separable filter has a more pronounced noise-reducing character (low-pass filter), but at the cost of blurred edges.
edges and the other to horizontally oriented edges \((c_v \approx c_h = 0.75)\). The improvement \(\eta_1\) increases to 7.43 dB. From the estimation result in Fig. 4.17 it can be seen that this filter also has a good noise-reducing behavior, but that now the edges are obviously less distorted than in Fig. 4.16. Note that the line structure which is present in the 1-D estimation results has almost disappeared in these 2-D filter results.

Experiment 5: Edge-preserving prediction with emphasis on edge weights

Finally, we describe an experiment with the edge-preserving predictor (4.1.22) in order to combine noise reduction and edge preservation in an even better way. From (4.1.14) we know that the steady-state value of the response on a unit step for the predictor is

\[
x_{ss} = \frac{b}{1-a}.
\]  

If we choose \(b=1-a\), then \(x_{ss}\) of the edge-preserving predictor will be equal to the height of the step in the observations. The resulting estimator, which is not optimal in the MSE sense, yields encouraging results for large values of \(a\). Besides an intensified noise reduction, the edges are weighted more severely. If the noisy image in Fig. 3.6 is filtered with the following equation:

\[
\hat{x}(j) = 0.95 \hat{x}(j-1) + 0.05 \{y(j-1) + 9.99 v(j-1)\},
\]  

\((a=0.95, b=0.05\) and \(c=9.99)\), then the improvement \(\eta_1\) of the image, as shown in Fig. 4.18, is 7.93 dB. Next, the noisy image is filtered with (4.5.2), but now with edge information obtained from the original image (Fig. 3.5) with the 3x4 edge operator. The filter result is shown in Fig. 4.19; the improvement \(\eta_1\) is 12.43 dB. Note that the edge information from the noisy image introduces artefacts in the estimation result in Fig. 4.18. These artefacts are not present in Fig. 4.19, where the edges are obtained from the noise-free image. The estimation result in Fig. 4.19 can be seen as an upper bound in performance for the estimator (4.5.2).

In summary, these experiments lead to the conclusion that use of edge information may significantly increase the visual quality of the filtered images.
RESTORATION OF IMAGES DEGRADED BY BLUR AND NOISE

In this chapter a Kalman filter is derived for optimal line-by-line recursive restoration of images degraded in a deterministic way by blur and in a stochastic way by additive white noise.

In Section 5.1 the problem of optimal Kalman filtering in two dimensions is discussed. This results in the choice of a Kalman filter for vector observations, i.e. where the processor observes a line at a time instead of a point at a time. Then, in order to derive such a line-by-line recursive filter a state-space model is formed for the sequence of image vectors representing the original image. In Section 5.2 the imaging system is modeled as a linear space-invariant system with a known point-spread function, resulting in the so-called observation equation. Both state-space model and observation equation are coupled in Section 5.3 to produce a first-order linear dynamic model of the total image-recording system suitable for the derivation of a line-by-line recursive Kalman filter.

Because of the computational and storage burden imposed by this filter attention must be paid to the reduction of processing time and storage requirements. Two techniques will be discussed. First, in Section 5.4 a method is described for approximating the full gain matrix in its steady state by a quasi-Toeplitz matrix with a very small number of nonzero elements. Second, in Section 5.5 a diagonalization procedure is described which uses circulant matrix approximations in order to reduce the dynamic
model of the total image-recording system to a set of decoupled equations in the Fourier domain. Then, the line-by-line recursive Kalman filter reduces to a set of scalar Kalman filters suitable for parallel processing of the data in the Fourier domain.

Experimental results are given in Section 5.6.

5.1. THE CONVERSION OF A 2-D IMAGE MODEL INTO A 1-D STATE-SPACE MODEL

In the preceding chapters we have dealt with the problem of recursive restoration of noisy images. In this chapter we will extend this problem to the recursive restoration of images degraded not only in a stochastic way by additive white noise, but also in a deterministic way by blur. The first work along this line was that of Aboutalib and Silverman [1], for the case of linear motion blur in one direction. Their approach is based on modeling the motion (1-D PSF) by a linear dynamic model, while the original image is modeled as the output of a line scanner as described by Nahi and Aseffi [62]. Then, by joining both models into one state-space model a 1-D Kalman filter can be derived. Aboutalib, Murphy and Silverman [2] extend this approach to the case of general motion blurs.

In our approach [13] we use explicitly a 2-D model description of the original image and of the blur. We assume, in accordance with Chapter 2, that an undistorted image can be modeled as the 2-D scalar output of a linear system, say \( L_1 \), driven by a (non) white-noise process. Further, we assume that the blur (causal or noncausal) can be adequately modeled by a 2-D linear system, say \( L_2 \), with the undistorted image as input to the system. The noise, although it may have its origin in any part of the imaging system, is modeled as an additive process at the output of \( L_2 \).

\[
x(i,j) = \rho_v x(i-1,j) + \rho_h x(i,j-1) - \rho_v \rho_h x(i-1,j-1) + u(i,j).
\]

Then, given the degraded image as the scalar output \( y(i,j) \) of the linear system \( L \) (See Fig. 5.1), we try to find a linear MSE estimator which recursively restores the image one pixel at a time. In accordance with Chapter 3, we could conceive of extending the point-to-point estimators of Habibi and Kak to the reduction of both blur and noise (see also [68]). However, these estimators do not preserve the MSE optimality, due to an invalid 2-D state definition [94]. Therefore, in order to find an optimal 2-D Kalman filter for scalar observations, we first try to define a valid 2-D state-space description of the original image.

We define, in accordance with Woods [95], [98], a 2-D state (vector) as a set of original pixel intensity values*. The dimension of the state vector and the way it evolves in time (processing order) has to be such that it includes all past and present information necessary to determine uniquely all future responses (pixel intensity values). This requires a specific ordering of the pixels in the plane, which can be obtained from a suitable QP causal or NSHP causal model description of the original homogeneous image.

As an example, a 2-D state-space representation will be derived for the DPCM image model, in order to compare this state-space representation with that used by Habibi and Kak to define their point-to-point estimators. The DPCM image model is given by (see also 3.1.1)

\[
x(i,j) = \rho_v x(i-1,j) + \rho_h x(i,j-1) - \rho_v \rho_h x(i-1,j-1) + u(i,j).
\]

Then, the following valid state vector can be defined at the point \((i,j)\) (See Fig. 5.2):

\[
s(i,j) = [x(i,j), x(i,j-1), \ldots, x(i,1); x(i-1,1), \ldots, x(i-1,j)]^T.
\]

* A more complicated state description for modeling images is given by Murphy and Silverman [59].
This state vector contains that part of the pixels of the line \( i-1 \) which is needed to determine future responses on the line \( i \) when the QP causal DPCM model moves to the right along the line \( i \). Further, it contains that part of the pixels of the line \( i \) which is needed to determine future responses on the line \( i+1 \). Note that the dimension of the state vector is proportional to the horizontal size of the image and to the number of lines considered by the 2-D image model. It is convenient to call this full state the \textit{global} state and to define a \textit{local} state as that part of the global state needed to compute only the next output value \( x(i,j+1) \) given the present input. For the DPCM model (5.1.1) the local state is defined by

\[
\phi_{\text{local}}(i,j) = [x(i,j), x(i-1,j), x(i-1,j+1)]^T. \tag{5.1.3}
\]

Note that in the ordinary 1-D case the local state is equal to the global state. By using the global state definition (5.1.2) the following state-space representation can be obtained:

\[
s(i,j+1) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} s(i,j) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(i,j), \tag{5.1.4a}
\]

\[
x(i,j) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} s(i,j), \tag{5.1.4b}
\]

where \( u \) is a white-noise vector process uncorrelated with the state vector \( s \), and where the matrices \( A \) and \( B \) are determined by the model (5.1.1) and where \( \epsilon = [1,0,...,0]^T \). When this model is combined with a suitable observation equation, an optimal 2-D Kalman filter for scalar observations can be defined. In this formulation the suboptimality of the estimators of Habibi and Kak is due to the fact that instead of the global state \( s \) in (5.1.4a) only one element \( x(i,j) \) is estimated. A closer observation of (5.1.4) reveals, however, that the optimal Kalman filter for scalar observations based on this state-space model involves an enormous computational and storage burden. For the DPCM image model, where the global state is of dimension \( N \), the number of computations per output point lies in the order of \( O(N^3) \). For image models where the global state vector extends over several image lines this number is even larger. Therefore, Woods and Radewan [95] develop for an NSHP model description of the original image a reduced updated Kalman filter, where only the local part of the global state is estimated, resulting in a suboptimal filter [96]. One can say in general that optimal point-to-point estimation of an image is a rather inefficient procedure. A more efficient scheme can be obtained by processing a line per recursion step instead of a point. The global state will then have the same high dimension (extending over one or more image lines), but now each iteration of this Kalman filter for vector observations will yield \( N \) estimates. In this way the number of computations can be reduced by a factor \( N \), while the storage requirements remain the same. Another advantage of this vector filter is that semicausal and noncausal model representations can also be used in defining a state-space model. Therefore, in the remainder of this chapter we restrict ourselves to this Kalman filter for vector observations.

We seek a state-space representation of the sequence of vectors \( \{x(i), i=1,...,M\} \), where \( x(i) = [x(i,1),...,x(i,N)]^T \) denotes the vector of original picture elements on line \( i \). Such a representation is given by the first-order vector dynamical system

\[
s(i+1) = As(i) + Bu(i), \tag{5.1.5a}
\]

\[
x(i) = Cs(i), \tag{5.1.5b}
\]
where \( s \) is the state vector and \( u \) is a white-noise vector process uncorrelated with \( s \). The identification of the dimension and parameters of this system will be performed for the following 2-D model of a homogeneous image (see 2.2.2):

\[
x(i,j) = \sum_{p,q \in \mathbb{Z}} a(p,q)x(i-p,j-q) + u(i,j),
\]

where \( \mathbb{Z} = \{p,q : 0 \leq p \leq p_2, -q_1 \leq q \leq q_2\} \).

The spatial support of this model and the boundary conditions necessary to evaluate the image inside an \( M \times N \) viewing area are shown in Fig. 5.3. Note that this semicausal support includes the support of QP causal and NSHP causal models. By taking advantage of a shift in position noncausal models may also be included.

If we define \( a(0,0) = -1 \), then the model (5.1.6) can be rewritten as:

\[
-a(p_2,q_2)x(i-p_2,j-q_2) \ldots -a(p_2,0)x(i-p_2,j) \ldots -a(p_2,-q_1)x(i-p_2,j+q_1) + \\
\vdots \\
-a(0,q_2)x(i,j-q_2) \ldots -a(0,0)x(i,j) \ldots -a(0,-q_1)x(i,j+q_1) = u(i,j)
\]

(5.1.7)

A state-space representation can now be constructed as follows. For fixed line index \( i \), we apply (5.1.7) for all \( (i,j) \) in the vector \( x(i) \) and combine these \( N \) equations into a 1-D matrix-vector equation:

\[
\sum_{p=0}^{p_2} \left[ A_p x(i-p) + E^L_p x^L(i-p) + E^R_p x^R(i-p) \right] = B_0 u(i).
\]

(5.1.8)

Here \( u(i) = [u(i,1), \ldots, u(i,N)]^T \), \( x^L(i-p) = [x(i-p,1), \ldots, x(i-p,N+q_1)]^T \), \( x^R(i-p) = [x(i-p,-1), \ldots, x(i-p,N+q_1)]^T \) and \( A_p, p=0,1, \ldots, p_2 \) are \( N \times N \) band Toeplitz matrices of the form:

![Diagram](image-url)
The vectors $x^L(i-p)$ and $x^R(i-p)$ for $i=1, \ldots, M$, $p=0, \ldots, p_2$ are the left- and right-side boundary conditions. Both conditions, combined with a top-side boundary condition $x^T(i-p) = x(i-p)$ for $p=1, \ldots, p_2$ are necessary to evaluate the image inside the $M \times N$ viewing area according to (5.1.7). These boundary conditions, which can be seen as initial values, must be specified prior to evolving $x(i)$ via (5.1.8). However, they must be consistent with the statistical model of the image field. The simplest choice is to set $x^L(i-p)$, $x^R(i-p)$ and $x^T(i-p)$ equal to the mean value of the image field, in our case zero. A more precise approach is suggested by Murphy and Silverman [59]. They consider random boundary conditions so as to properly characterize a rectangular region obtained from a homogeneous random field and model them as a state-space model which can be generated independently of the image field. Woods and Ingle [97] refine this approach to admit image-dependent boundaries. For simplicity, and in accordance with our experiments as described in Section 5.6, we have set the boundary conditions equal to zero. Then, (5.1.8) simplifies to

$$\sum_{p=0}^{p_2} A_p x(i-p) = B_0 w(i). \tag{5.1.9}$$

In order to obtain a recursive equation, (5.1.9) can be written as

$$A_0 x(i) = - \sum_{p=1}^{p_2} A_p x(i-p) + B_0 w(i). \tag{5.1.10}$$

If we define a 2-D state again as a set of picture elements and if we combine the picture elements of $p_2$ image lines into the following state vector summarizing past and present,

$$s(i) = [x(i,1), \ldots, x(i,N); x(i-1,1), \ldots, x(i-1,N); \ldots \ldots [x(i-p_2^2+1,1), \ldots, x(i-p_2^2+1,N)]^T$$

$$= [x(i)]^T, \ldots, [x(i-p_2^2+1)]^T]^T, \tag{5.1.11}$$

then the $p_2$th-order matrix-vector equation (5.1.10) can be replaced by a first-order state-space representation of the form (5.1.5). To put (5.1.10) into the form (5.1.5) it is necessary to premultiply both sides of (5.1.10) by the matrix $A_0^{-1}$. Because the rows in $A_0$ are linearly independent, the inverse of the matrix always exists. Then, the $p_2 \times N$ matrix $A$, the $p_2 \times N$ matrix $B$, and the $N \times p_2$ matrix $C$ have the following structure:

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_{p_2} \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ \end{bmatrix}, \quad B = \begin{bmatrix} B_0 \\ \vdots \\ 0 \\ \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \end{bmatrix}, \tag{5.1.12}$$

where $\hat{A}_p = -A_0^{-1}A_p$, $p=1, \ldots, p_2$ and $\hat{B}_0 = A_0^{-1}B_0$. Note that although $A_0$ has a banded structure its inverse has not. In the following example a state-space representation will be derived for a first-order ($p_2=1$) semicausal model, which is used in our experiments.

**Example 5.1.1.**

Consider a semicausal model with spatial support (see also 2.4.8)

$$x(i,j) = a(0,1)x(i,j-1) + a(0,-1)x(i,j+1) + a(1,0)x(i-1,j) + a(1,1)x(i-1,j+1) + a(1,-1)x(i-1,j-1) + u(i,j). \tag{5.1.13}$$

This model can be written in matrix-vector notation according to (5.1.10) as

$$A_0 x(i) = - A_1 x(i-1) + B_0 u(i), \tag{5.1.14}$$

where $A_p$, $p=0,1$ are $N \times N$ tri-diagonal Toeplitz matrices.
and \( B_0 = I \). The state-space representation can now be found as follows. Premultiplying both sides of (5.1.14) by \( A_0^{-1} \) yields

\[
x(i) = -A_0^{-1} A_1 x(i-1) + A_0^{-1} u(i) \quad (5.1.15)
\]

or

\[
s(i) = A s(i-1) + B u(i),
\]

\[
x(i) = C s(i), \quad (5.1.16)
\]

where \( A = -A_0^{-1} A_1 \), \( B = A_0^{-1} \) and \( C = I \).

If, for instance, the semicausal model (5.1.13) is used to describe a class of images with a separable exponentially decaying autocovariance function, it holds for the MSE coefficients of the semicausal model that

\[
a(0,1) = a(0,-1) = \frac{\rho_h}{1 + \rho_h} = a_x,
\]

\[
a(1,0) = \rho_v,
\]

\[
a(1,1) = a(1,-1) = -a \rho_v,
\]

and (5.1.15) reduces to

\[
x(i) = \rho_v x(i-1) + A_0^{-1} u(i), \quad (5.1.17)
\]

while the input process \( u(i) \) has the following autocovariance matrix:

\[
\Sigma_u(k) = E[u(i)u^T(i+k)] = \beta^2 A_0 \delta(k). \quad (5.1.18)
\]

The input variance \( \beta^2 \) of the model (5.1.13) is defined by

\[
\beta^2 = \frac{(1-\rho_h^2)(1-\rho_v^2)}{1+\rho_h^2} \sigma_x^2. \quad (5.1.19)
\]

The stability of this model has already been investigated in Example 2.5.1.

### 5.2. The Observation Equation

In the preceding section we have found a state-space representation of the sequence of original image vectors. Now, in this section we pay attention to modeling the imaging medium, the subsystem \( L_2 \) in Fig. 5.1, and the observation noise process into an observation equation.

We assume the linear system \( L_2 \) to be spatially invariant; then the observed or recorded image can be modeled by the following 2-D convolution summation (see Chapter 1):

\[
y(i,j) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} g(k,l) x(i-k,j-l) + w(i,j), \quad (5.2.1)
\]

where \( w(i,j) \) is a zero-mean white-noise process uncorrelated with the data and where \( g(k,l) \) is the impulse response or point-spread function (PSF) of the system \( L_2 \). Equation (5.2.1) is the general expression for 2-D discrete convolution in which the imaging system has an infinite-duration impulse response \( g(k,l) \). In real life the observation at point \((i,j)\) is affected by the signal (original image intensities) only in a small neighborhood around the point \((i,j)\), and the contribution of signal samples far away from \((i,j)\) is insignificant. In other words, the imaging system can be approximated quite well by a system with finite-duration impulse response (FIR) and, therefore, the summation in (5.2.1) may be performed over a finite set of points only. Then, (5.2.1) can be replaced by

\[
y(i,j) = \sum_{k=-k_1}^{k_2} \sum_{l=-l_1}^{l_2} g(k,l) x(i-k,j-l) + w(i,j). \quad (5.2.2)
\]
To combine this observation equation with the state-space representation of the sequence of vectors \{x(i)\}, it is necessary to transfer (5.2.2) into vector-matrix form. This will be done under the assumption that the degraded image is observed in the rectangular viewing area \(1 \leq i \leq M\) and \(1 \leq j \leq N\). To describe the degraded image correctly within this area, we have to introduce a region outside where the original pixel intensity values are assumed to be known. This region is denoted by Slepian in his work on motion degraded images [83] as an admissible a priori set and is shown in Fig. 5.4 for a general noncausal blur.

In obtaining a matrix-vector formulation of (5.2.2), we first assume the pixel intensity values to be equal to zero within this set. Eliminating the inner summation of (5.2.2) yields

\[
y(i) = \sum_{k=-k_1}^{k_2} G(k) x(i-k) + w(i). \tag{5.2.3}
\]

Here \(x(i)\) is the image vector \([x(i,1), \ldots, x(i,N)]^T\), \(y(i)\) the observation vector \([y(i,1), \ldots, y(i,N)]^T\), \(w(i)\) the observation noise vector \([w(i,1), \ldots, w(i,N)]^T\), and \(G(k)\) is an \(N \times N\) matrix with band-Toeplitz structure.

The outer summation can be eliminated as follows. Define a sequence of column vectors \(z(i) = [x(i+k_1), \ldots, x(i-k_2)]^T\) for all \(i, i=1,2, \ldots, M\) and a matrix \(H = [G(-k_1), \ldots, G(k_2)]\). Then (5.2.3) can be written as

\[
y(i) = H z(i) + w(i), \tag{5.2.4}
\]

where \(y(i)\) is the \(N \times 1\) observation vector, \(w(i)\) the \(N \times 1\) observation noise vector and \(z(i)\) the \(K \times 1\) vector obtained by stacking \(K=k_1+k_2+1\) image vectors. The distortion matrix \(H\) has dimension \(N \times K\).

If \(x(i,j)\) is nonzero outside the \(M \times N\) viewing area, i.e., nonzero within the admissible a priori set, then a correct degraded image-description can be obtained by modeling the admissible a priori set as part of the image field. Then the \(N \times 1\) image vector \(x(i)\) is augmented to an \((N+1) \times 1\) image vector

\[
x'(i) = [x(i-k_2), \ldots, x(i,1), \ldots, x(i,N+k_1)]^T. \tag{5.2.5}
\]

For the sequence of augmented image vectors \(x'(i), i=-k_2, \ldots, M+k_1\) a new state-space representation can be defined according to Section 5.1. Besides that, the observation equation (5.2.4) must be adapted to this new situation. This will be illustrated in the following examples, where an observation equation will be derived for a causal distortion (linear
camera motion in one direction) and a noncausal distortion (out-of-focus lens system).

**Example 5.2.1.**

Suppose that there is a relative motion between an object and the imaging system during exposure. When this motion is merely a translation with constant velocity \( V \) along the horizontal axis during the exposure-time interval \([0,T]\), the extent of the motion blur is equal to \( VT \) and the PSF is spatially invariant with the following form (with continuous variables \((n,v)\)) \([1]\):

\[
g(n,v) = \frac{1}{VT}, \quad 0 \leq v \leq VT, \quad n = 0, \quad 0 \text{ elsewhere.} \tag{5.2.6}
\]

In our experiments where the motion extends over \( \ell_2 \) (discrete) picture elements, we make use of the following discrete PSF:

\[
g(k,\ell) = \frac{1}{\ell_2 + 1}, \quad \ell=0,1,\ldots,\ell_2, \quad k=0, \quad 0 \text{ elsewhere.} \tag{5.2.7}
\]

The degraded image can be seen as a family of identically degraded line images, because the dependency in the vertical direction can be dropped. If \( x(i,j) \) is zero outside the \( M \times N \) viewing area, then (5.2.2) with PSF (5.2.7) can be written as

\[
y(i,j) = \frac{1}{\ell_2 + 1} \sum_{\ell=0}^{\ell_2} x(i,j-\ell) + w(i,j), \tag{5.2.8}
\]

or in vector-matrix notation as

\[
y(i) = G x(i) + w(i), \tag{5.2.9}
\]

where \( G \) is an \( N \times N \) lower triangular matrix.

If \( x(i,j) \) is nonzero outside the \( M \times N \) viewing area, a strip of width \( \ell_2 \) pixels is needed at the left border of the image where the original pixel intensity values are assumed to be known (admissible a priori set). Then the \( N \times 1 \) image vector \( x(i) \) is augmented to an \((N+\ell_2) \times 1\) image vector

\[
x'(i) = [x(i,1-\ell_2),\ldots,x(i,1),\ldots,x(i,N)]^T, \tag{5.2.10}
\]

and a new observation equation can be defined as

\[
y(i) = G' x'(i) + w(i), \tag{5.2.11}
\]

where the \( N \times (N+\ell_2) \) distortion matrix \( G' \) has the following form:

\[
G' = \frac{1}{\ell_2 + 1} \begin{bmatrix}
1 & \ldots & \ldots & 0 \\
0 & \ddots & \ldots & \vdots \\
0 & \ldots & \ddots & \vdots \\
0 & \ldots & \ldots & 1
\end{bmatrix}.
\]

Note that the matrix \( G' \), unlike \( G \), is not truncated at the upper-left side.
Example 5.2.2.

The PSF of a defocused lens system with a circular aperture can be approximated by a cylinder whose radius \( R \) depends on the extent of the focus defect \([18]\), thus

\[
\begin{align*}
g(n, \psi) &= \frac{1}{\sqrt{1 + \psi^2}}, & \quad \sqrt{1 + \psi^2} \ll R, \\
&= 0, & \quad \sqrt{1 + \psi^2} > R.
\end{align*}
\]

(5.2.12)

In the discrete case this circular PSF must be approximated by means of a number of pixels in a discrete grid structure.

In our experiments we only make use of the following simple discrete approximation of (5.2.12):

\[
\begin{align*}
g(k, \ell) &= \frac{1}{9}, & \quad \sqrt{k^2 + \ell^2} \ll \sqrt{2}, \\
&= 0, & \quad \text{elsewhere}.
\end{align*}
\]

(5.2.13)

The observation equation (5.2.2) then has the following form:

\[
y(i, j) = \frac{1}{9} \sum_{k=-1}^{1} \sum_{\ell=-1}^{1} x(i-k, j-\ell) + w(i, j).
\]

(5.2.14)

If \( x(i, j) \) is zero outside the \( M \times N \) viewing area, and if we define a vector \( z(i) = [x(i+1), x(i), x(i-1)]^T \), then (5.2.14) can be written as

\[
y(i) = Hz(i) + w(i).
\]

(5.2.15)

Here \( H = [G(-1), G(0), G(1)] \) and the \( NxN \) submatrices \( G(k) \), \( k=-1,0,1 \) are given by

\[
G(k) = \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

If \( x(i,j) \) is nonzero outside the \( M \times N \) viewing area, then an admissible a priori set must be defined, as given in Fig. 5.4 for \( k_1=k_2=1 \) and \( \ell_1=\ell_2=1 \). Then a new state-space representation can be formulated for the sequence of augmented image vectors \( x'(i) = [x(i,0), \ldots, x(i,M+1)]^T \). The observation equation then has the following form:

\[
y(i) = H' z'(i) + w(i),
\]

(5.2.16)

where \( z'(i) = [x'(i+1), x'(i), x'(i-1)]^T \) and where the \( N(N+2) \) submatrices \( G'(k) \), \( k=-1,0,1 \) of \( H' \) are given by

\[
G'(k) = \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Note that even for this simple example already three image lines must be combined. For a defocusing with large radius \( R \) this can lead to very high dimensions of the vector \( z(i) \).

5.3. THE OPTIMAL LINE-BY-LINE RECURSIVE KALMAN FILTER

In this section the state-space representation of the sequence of original image vectors (5.1.5) will be combined with the observation equation (5.2.4) to produce a first-order linear dynamical model suitable for the derivation of an optimal line-by-line recursive Kalman filter.

For convenience, both equations with dimensionality indications will be repeated here for the case \( x(i,j) \) is zero outside the \( M \times N \) viewing area. The state-space representation of the original image is given by
where \( x(i) \) is the \( N \times 1 \) image vector, \( u(i) \) the \( N \times 1 \) noise input vector, \( s(i) \) the \( P \times 2N \) state vector, and where the dimensions of the matrices \( A, B \) and \( C \) are \( P \times N, P \times 2N \) and \( N \times 2N \), respectively. The observation equation is given by

\[
y(i) = H z(i) + w(i),
\]

where \( y(i) \) is the \( N \times 1 \) observation vector, \( w(i) \) the \( N \times 1 \) observation noise vector and \( z(i) \) the \( K \times N \) vector of stacked image vectors \( (K = k_1 + k_2 + 1) \), and where \( H \) is an \( N \times K \) distortion matrix.

In order to find a dynamical model of the total image-recording system, both (5.3.1) and (5.3.2) must be coupled. Strictly speaking, (5.3.1b) must be replaced by (5.3.2). However, this is only possible if the dimensions of both equations are adapted to each other. Because we work with low-order image models, we assume that the dimension of the state-vector \( s(i) \) in (5.3.1) is less than or equal to the dimension of \( z(i) \), i.e. \( p_2 \leq K \). In that case the following dynamical model is used:

\[
x(i+1) = A' z(i) + B' u(i),
\]

\[
y(i) = H z(i) + w(i).
\]

Here the observation equation (5.3.3b) is equal to (5.3.2) and the state vector \( s(i) \) has been replaced by the higher-dimensional vector \( z(i) \). The matrices \( A \) and \( B \) of (5.3.1a) are replaced by the higher-dimensional sparse matrices \( A' \) and \( B' \) of dimension \( K \times N \) and \( N \times K \), respectively, containing \( A \) and \( B \) as submatrices (see (5.1.12)).

Here each null matrix \( O \) and each identity matrix \( I \) has dimension \( N \times N \).

Given the dynamical model (5.3.3), a Kalman filter for obtaining the best linear MSE estimate \( \hat{x}(i) \) in terms of \( y(i) \) can be derived. This results in the following equations (see also table 1.2)

\[
\hat{x}(i) = A' \hat{x}(i-1) + F(i) [y(i) - H A' \hat{x}(i-1)],
\]

\[
F(i) = Q(i-1) H^T [H Q(i-1) H^T + \Sigma_u^{-1}]^{-1},
\]

\[
Q(i) = A' [I - F(i)H] Q(i-1) A'^T + B' \Sigma_u B'^T,
\]

where \( F(i) \) is the \( K \times N \) gain matrix and where \( \Sigma_u \) and \( \Sigma_v \) are the \( N \times N \) covariance matrices of \( u(i) \) and \( w(i) \), respectively. In terms of the matrix \( F^0(i) \) of the filter error vector \( [z(i) - \hat{x}(i)] \) the matrix \( Q(i) \) in (5.3.4b) and (5.3.4c) is defined by

\[
Q(i) = A' F^0(i) A'^T + B' \Sigma_u B'^T.
\]

The vector \( \hat{x}(i) \) represents estimates of \( k_1 + k_2 + 1 \) image lines about line \( i \); and since the filter is successively applied to all lines of the degraded image, the aggregate result is multiple estimates for each image line. For example, the estimates \( \hat{x}(i-k_1) \) through \( \hat{x}(i+k_2) \) all contain \( \hat{x}(i) \), since
that the gain matrix in (5.3.4) converges to a steady-state value $F_{ss}$. Because no observations are needed to calculate this gain matrix, its computation can be performed in advance and $F_{ss}$ can be used in (5.3.4a) for all $i, \ i=1,2,...,M$. A disadvantage is the still large storage requirement and the fact that the number of iterations required for obtaining $F_{ss}$ from (5.4.3b) and (5.4.3c) consumes, in our experiments, up to 80% of the total processing time needed to filter an image. Therefore, we have to find a restoration scheme which reduces both the processing time and storage requirements needed to calculate the gain matrix.

One such scheme is to partition the image into vertical strips with strip width $W$ (expressed in number of pixels), where $W \ll N$, and to restore the strips independently. The gain matrix calculated for the first strip, which is of a reasonable dimension, can be used for all other strips. Some problems encountered involve the choice of the strip width and the amount of overlap of the strips necessary to reduce artefacts at the strip boundaries, see [98], [59]. However, since artefacts at the strip boundaries cannot be totally avoided, we prefer a suboptimal restoration scheme which filters the entire $M \times N$ image without breaking it into strips, but with a reduced number of computations and a smaller amount of computer memory. For the case of noisy images (no blur present) Panda and Kak [69] describe a method of expanding a small-sized (16x16) gain matrix calculated for one strip, obtained by using statistical knowledge of the entire image, into a large gain matrix, say 256x256. Here we follow a similar approach for images with a horizontal motion degradation in the presence of additive white noise. Examination of the steady-state matrices $F_{ss}$ for different 32x32 images reveals the following relationships for the elements $f_{ss}(i,j)$ of $F_{ss}$:

i) For elements in the middle rows (around the 16th) it holds that

$$f_{ss}(i,j) \approx f_{ss}(i-j). \quad (5.4.1)$$

ii) For $|i-j| > k$, the value of $f_{ss}(i-j)$ drops off rapidly, where $k$ depends on SNR and on the extent of the motion blur.
We may expect the previous two observations made on 32x32 gain matrices to be even more valid for, say, 256x256 images. In other words, if one were to actually derive the 256x256 gain matrix $F_{SS}$, a large inner portion of it (away from the boundary effects) is expected to conform to the two observations just made, i.e. the gain matrix has a quasi-Toeplitz structure and has many zero elements due to relation ii) (sparseness).

Bearing both conditions in mind, it is possible to construct an approximated version of an $N \times N$ gain matrix $F_{SS}$ from a $W \times W$ gain matrix $F_{SS}'$, where $W \ll N$. Suppose for the sake of simplicity that $W$ and $N$ are even. Then the expanded $N \times N$ gain matrix $F_{SS}$ has the following form:

$$
F_{SS} = \begin{bmatrix}
 f_{SS}(1,1) & \ldots & f_{SS}(1,W) \\
 \vdots & & \vdots \\
 f_{SS}(W,1) & \ldots & f_{SS}(W,W) \\
 f_{SS}(W/2,1) & \ldots & f_{SS}(W/2,W) \\
 0 & \ldots & 0 \\
 f_{SS}(W,1) & \ldots & f_{SS}(W,W)
\end{bmatrix}
$$

(5.4.2)

Computationally, this is a very attractive solution, because expanding a small matrix to a large matrix according to (5.4.2) is less time consuming and has fewer storage requirements than finding the steady-state value of the large matrix $F(i,j)$ according to (5.3.4b) and (5.3.4c). Besides that, due to the sparseness of (5.4.2) storage schemes such as the row-pointer/column index scheme [22] can be used to store the data efficiently and to reduce the number of computations in the filter equation (5.3.4a) by avoiding computations which involve zeros. In Section 5.6 an example of this method is given for a 128x128 noisy motion blurred test image by expanding a 32x32 gain matrix. Note that for every new filter situation a structure analysis of the small gain matrix must be performed. Therefore, in the next section a mathematically more tractable solution will be described.

5.5. USE OF CIRCULANT MATRIX APPROXIMATION

In this section the computational and storage burden imposed by the line-by-line recursive Kalman filter will be reduced by using a diagonalization procedure. One first looks for a transformation which diagonalizes both the model matrices in the state-space representation of the original image and the distortion matrix in the observation equation. Then the dynamical model reduces to a set of decoupled equations and the line-by-line recursive Kalman filter based on this model reduces to a set of scalar Kalman filters suitable for parallel processing of the data in the transform domain. Via an inverse transformation the filtered data is presented in the data domain.

The state-space representation of the semicausal model in Example 5.1.1 contains model matrices which are tri-diagonal, symmetric Toeplitz matrices. These matrices can be effectively diagonalized by means of the sine transformation [44]. For the case where the original image is observed in the presence of additive white noise, uncorrelated with the data, Jain proposes an efficient filter scheme [42] based on this transformation for the semicausal model.

If the observed image is, besides being corrupted by observation noise, also degraded by blur, then the distortion matrix in the observation equation has a more or less complex structure and the sine transform technique fails. For the case of linear motion blur in a horizontal direction, where the distortion matrix has a simple band-Toeplitz struc-
tation (Example 5.2.1), we can make use of the property that a Toeplitz matrix can be approximated by a circulant matrix, which is known to be asymptotically equivalent to the Toeplitz matrix [43]. A circulant matrix has the attractive property that it can easily be diagonalized by means of the discrete Fourier transform (DFT) [4] as can be seen from the following.

Let $Q$ be an $N\times N$ band-Toeplitz matrix of the following form:

$$
Q = \begin{bmatrix}
q(0) & q(-1) & \cdots & q(-l) \\
q(1) & & & \\
& \ddots & & \\
& & \ddots & \\
& & & q(k) \\
\end{bmatrix}
$$

which is approximated by an $N\times N$ circulant matrix $Q^c$:

$$
Q^c = \begin{bmatrix}
q(0) & q(-1) & \cdots & q(-l) & q(k) & \cdots & q(1) \\
q(-1) & & & & & \ddots & \\
\vdots & & & & & \ddots & q(k) \\
& & & & & & \ddots \\
& & & & & & q(1) \\
\end{bmatrix}
$$

where each row is a circular right shift of the row above, and where the first row is a circular right shift of the last row. It is known [4] that the $N\times N$ circulant matrix $Q^c$ is diagonalized by

$$
\Lambda = \Phi^{-1} Q^c \Phi.
$$

Here $\Lambda$ is an $N\times N$ diagonal matrix whose elements $\lambda(j,j)$ are the eigenvalues of $Q^c$ and where $\Phi$ is an $N\times N$ unitary matrix of eigenvectors of $Q^c$, for which it holds that

$$
(\Phi^*)^T \Phi = I,
$$

where $I$ is the $N\times N$ identity matrix. For the elements $\varphi(i,j)$ of $\Phi$ it holds further that

$$
\varphi(i,j) = \exp\left\{j \frac{2\pi}{N} i j\right\}, \quad \text{for } i,j=0,1,\ldots,N-1,
$$

and for the eigenvalues $\lambda(j,j)$ of $Q^c$, from now on denoted by $\lambda(j)$, it holds that

$$
\lambda(j) = q(0) + \sum_{i=-\ell}^{k} q(i) \exp(- j \frac{2\pi}{N} i j) + \sum_{i=-\ell}^{l-1} q(i) \exp(- j \frac{2\pi}{N} i j),
$$

for $j=0,1,\ldots,N-1$.

Now it follows from (5.5.7) that the circulant matrix $Q^c$ can be simply diagonalized by computing the DFT of the cyclic sequence $q(0),q(1),\ldots,q(N-1)$.

This diagonalization procedure and the construction of a set of scalar filters in the transform domain will be performed for the case that the original image can be modeled by the semicausal model (5.1.17) and in which the observed image has a linear motion blur in a horizontal direction over $\ell_2$ picture elements in the presence of additive white noise, uncorrelated with the data (Example 5.2.1). We assume that $x(i,j)$ is
zero outside the viewing area. By combining (5.1.17) and (5.2.9) the following dynamic model can be constructed:

\[ A_0 x(i+1) = \rho v A_0 x(i) + u(i), \quad (5.5.8a) \]
\[ y(i) = G x(i) + w(i), \quad (5.5.8b) \]

where

\[
A_0^c = \begin{bmatrix}
1 & 0 & 0 \\
-\rho & 1 & 0 \\
-\rho & 0 & 1 \\
\end{bmatrix}, \quad G = \frac{1}{s^2+1} \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
\end{bmatrix},
\]

and \( \Sigma_u(k) = \beta^2 A_0^c \delta(k), \Sigma_y(k) = \sigma_w^2 I_{n \times n} \delta(k). \) (5.5.8d)

First the tri-diagonal symmetric Toeplitz matrix \( A_0 \) and the lower-triangular band-Toeplitz matrix \( G \) are approximated by circulant matrices by inserting some element according to (5.5.2), resulting in the circulant matrices \( A_0^c \) and \( G^c \):

\[
A_0^c = \begin{bmatrix}
1 & -\rho & 0 \\
-\rho & 1 & 0 \\
0 & -\rho & 1 \\
\end{bmatrix}, \quad G^c = \frac{1}{s^2+1} \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
\end{bmatrix}.
\]

We then obtain

\[ A_0^c x(i+1) = \rho v A_0^c x(i) + u(i), \quad (5.5.10a) \]
\[ y(i) = G^c x(i) + w(i), \quad (5.5.10b) \]

Next we define the vectors

\[ \tilde{x}(i) = \Phi^{-1} x(i), \quad \tilde{u}(i) = \Phi^{-1} u(i), \quad \tilde{y}(i) = \Phi^{-1} y(i), \quad \tilde{w}(i) = \Phi^{-1} w(i), \]

which are the DFTs of the vectors \( x(i), u(i), y(i) \) and \( w(i) \), respectively, and premultiply (5.5.10a) by \( \Phi^{-1} \) to obtain with (5.5.3)

\[ \Lambda \tilde{x}(i+1) = \rho v \Lambda \tilde{x}(i) + \tilde{u}(i), \quad (5.5.12) \]

where \( \Lambda \) is the matrix of eigenvalues of \( A_0^c \). Since \( \Lambda \) is a diagonal matrix (5.5.12) reduces to the set of equations

\[ \lambda_A(j) \tilde{x}(i+1,j) = \rho v \lambda_A(j) \tilde{x}(i,j) + \tilde{u}(i,j) \]

or

\[ \tilde{x}(i+1,j) = \rho v \tilde{x}(i,j,j) + \frac{1}{\Lambda_A(j)} \tilde{u}(i,j,j), \]

where \( i=0,\ldots,M-1, j=0,\ldots,N-1 \). Note that this diagonalization procedure could also be performed for the general semicausal model (5.1.10), due to the Toeplitz structure of the matrices \( A_p, p=0,1,2,\ldots,p_2 \). Then, we obtain a set of \( p_2 \) order equations, which can easily be combined to a set of first-order vector equations. From (5.5.8d), (5.5.9) and (5.5.11) it holds that

\[ \Sigma_u(k) = E[\tilde{u}(i)\tilde{u}(i+k)\Phi^T] = \Phi^{-1} E[u(i)u(i+k)\Phi^T] \Phi = \beta^2 \Lambda \delta(k). \]

Hence, \( \tilde{u}(i) \) is a white input process and (5.5.14) represents a set of decoupled first-order autoregressive processes.

In a similar way, by using (5.5.3) and (5.5.11) the observation equation (5.5.10b) can be written as

\[ \tilde{y}(i) = \Lambda \tilde{x}(i) + \tilde{w}(i) \]
or
\[ \hat{y}(i,j) = \lambda_G(j) \hat{x}(i,j) + \hat{w}(i,j), \]
(5.5.17)

where \( \lambda_G \) is the matrix of eigenvalues \( \lambda(j) \) of \( G^C \). Equations (5.5.14) and (5.5.17) form a set of \( N \) decoupled equations suitable for the derivation of \( N \) scalar Kalman filters. A linear MSE estimate \( \hat{x}(i,j) \) of \( x(i,j) \) is then given by
\[ \hat{x}(i,j) = \hat{x}(i,j) - \rho \hat{x}(i-1,j) + F(i,j) \{ \hat{y}(i,j) - \rho \lambda_G(j) \hat{x}(i-1,j) \}, \]
(5.5.18)

where \( F(i,j) \) is the scalar Kalman gain. Once \( \hat{x}(i,j) \) is known for all \( i,j \), the spatial domain estimates \( \hat{x}(i,j) \) are given by
\[ \hat{x}(i) = \hat{x}(i), \]
(5.5.19)

representing the inverse DFT of the vector \( \hat{x}(i) \). The resulting efficient filter scheme using the fast fourier transform (FFT) is illustrated in Fig. 5.5.

Fig. 5.5. Parallel filter scheme.

The number of computations (multiplications and additions) for an \( N \times N \) image required for the line-by-line recursive Kalman filter (without using the sparseness of the matrix structures) would be maximally of the order of \( O(N^4) \). By using the diagonalization of the circulant matrices, all matrix multiplications are replaced by scalar multiplications.

With the FFT for the transformations the total number of computations for the \( N \times N \) image is of the order \( O(N^2 \log_2 N) \), which yields a considerable reduction. Some examples of this approach and a comparison with the results obtained with the optimal line-by-line recursive Kalman filter are given in the next section.

5.6. EXPERIMENTAL RESULTS

In this section the performance of the line-by-line recursive Kalman filter will be investigated in several image-restoration experiments by using test images of different sizes, each quantized in eight bits. The a priori knowledge necessary in the restoration procedure is obtained in the following way. Because the original undistorted image is available, it is possible to estimate its autocovariance function. Then, in order to obtain a parameter reduction, a parametrical autocovariance function (separable exponentially decaying (2.7.1), isotropic exponentially decaying (2.7.2)) is fitted to the estimated autocovariance function. Then a number of different 2-D image models is fitted to the given parametrical autocovariance function by means of a linear MSE fitting procedure. Which model ultimately serves as a model to describe the image depends on the results obtained with the three model quality criteria given in Section 2.6. Here we restrict the discussion to the DPCM image model (2.4.2) and the semicausal model given in (2.4.8).

Both models may exactly describe a class of images with a separable exponentially decaying autocovariance function, although other autocovariance functions can approximately be described as well.

The distortion introduced by the imaging system is simulated by a discrete convolution of the original image with a distortion operator. We restrict ourselves to linear motion blur and defocusing. Note that in this simulation environment the point-spread function of the distortion is known exactly. Therefore, the restoration results can be seen as upper bounds in performance.

Next, computer-generated zero-mean white Gaussian noise, uncorrelated with the image data, is added to the blurred image.

The following filter experiments will be performed. First, to gain
insight into the noise-smoothing behavior of the optimal line-by-line recursive Kalman filter, a noise corrupted test image without blur will be filtered. Second, some experiments with noisy motion-blurred images will be described, followed by a restoration experiment with a noisy defocused image. Finally, some experiments will be described with motion-blurred images by using the suboptimal restoration schemes described in Section 5.4 and 5.5.

The performance of the filter will be evaluated in the following way. For the case in which only observation noise is present in the observed image, the filter performance will be measured with (3.2.1):

$$\eta_1 = 10 \log_{10} \left( \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} [y(i,j)-x(i,j)]^2}{\sum_{i=1}^{M} \sum_{j=1}^{N} [\hat{x}(i,j)-x(i,j)]^2} \right) \text{dB}, \quad (5.6.1)$$

the improvement in signal-to-noise ratio after filtering. If in addition to the noise a deterministic degradation is present (motion blur, defocusing), image sharpness is affected due to the loss of high-frequency information of the image. Now, to emphasize the deblurring property of the filter, i.e. the recovery of high-frequency content, a new performance measure is introduced, which is derived from (5.6.1) in the following way. By using Parseval's theorem, it is possible to rewrite (5.6.1) in the frequency domain. Then, it is possible to introduce a frequency weighting, which leads to the following frequency-weighted performance measure:

$$\eta_4 = 10 \log_{10} \left( \frac{\sum_{\omega_1=1}^{M} \sum_{\omega_2=1}^{N} W(\omega_1,\omega_2) \left| Y(\omega_1,\omega_2)-X(\omega_1,\omega_2) \right|^2}{\sum_{\omega_1=1}^{M} \sum_{\omega_2=1}^{N} W(\omega_1,\omega_2) \left| \hat{X}(\omega_1,\omega_2)-X(\omega_1,\omega_2) \right|^2} \right) \text{dB}, \quad (5.6.2)$$

where $X(\omega_1,\omega_2)$, $Y(\omega_1,\omega_2)$ and $\hat{X}(\omega_1,\omega_2)$ are the DFT's of $x(i,j)$, $y(i,j)$ and $\hat{x}(i,j)$, respectively, and where $W(\omega_1,\omega_2)$ is a frequency-weighted function. In our experiments, to accentuate the recovery of high-frequency components a paraboloid is chosen, which favors the high frequencies over the lower frequencies in such a way that a better correspondence with the human visual interpretation of the results is obtained [23].

**Experiment I: Optimal restoration of noisy images**

In experimentally testing the noise-smoothing behavior of the optimal line-by-line recursive Kalman filter, we restrict ourselves to a 64x64 test image of a human face, shown in the upper-left corner of Fig. 5.6. The autocovariance function is approximated by an isotropic exponentially decaying autocovariance function with $\rho=0.86$ and $\sigma=0.6$. The noise-corrupted version with SNR=2 is shown in the upper-right corner. In the lower-left corner the filtered image is shown using the semicausal model with nonwhite noise input ($\eta_4=5.59$ dB), and in the lower-right corner the filter result is shown using the DPCM model with white-noise input ($\eta_4=5.63$ dB).

![Fig. 5.6. Filter results for the human face, SNR=2](image)
Experiment 2: Optimal restoration of noisy motion-blurred images

A 32x32 test image representing the letter "G", as shown in the upper-left corner of Fig. 5.7, is artificially degraded with a linear space-invariant motion blur in horizontal direction over five pixels. Then noise is added with SNR=100 (upper-right corner). The autocovariance function of the original image is approximated as a separable exponentially decaying autocovariance function with \( \rho_v = \rho_h = 0.8 \). The image is modeled by the semicausal model (2.4.8) (See also (5.1.17)). The pixel intensity values within the a priori admissible set at the left border of the image are modeled as part of the image field (See Example 5.2.1). In the lower part of Fig. 5.7 the optimally filtered image is shown. The improvement, measured with the frequency-weighted performance measure \( \eta_4 \), is 4.1 dB.

This filter experiment was repeated for another value of the signal-to-noise ratio, i.e. SNR=10. The estimation result of this filter is shown in the lower part of Fig. 5.8 with an improvement \( \eta_4 \) of 3.9 dB.

When comparing both filter results in Figs. 5.7 and 5.8, we observe a somewhat lower deconvolution action of the filter in Fig. 5.8. This is caused by the fact that the presence of more observation noise in the image to be filtered lowers the weighting of the innovation term in the Kalman filter equation.

When comparing both filter results in Figs. 5.7 and 5.8, we observe a somewhat lower deconvolution action of the filter in Fig. 5.8. This is caused by the fact that the presence of more observation noise in the image to be filtered lowers the weighting of the innovation term in the Kalman filter equation.

Fig. 5.7. Filter result for the letter "G"; linear motion blur over 5 pixels, SNR=100.

Fig. 5.8. Filter result for the letter "G", linear motion blur over 5 pixels, SNR=10.

Experiment 3: Optimal restoration of noisy defocused images

In this experiment the 32x32 test image representing the letter "G" is artificially defocused by means of the discrete approximation (5.2.13) of the circular PSF (5.2.12) as discussed in Example 5.2.2, after which noise is added. To gain insight into the deconvolution action of the filter in the presence of noise, once again two values of the signal-to-noise ratio are chosen: SNR=100 and SNR=10. The pixel intensity values within the a priori admissible set are modeled as part of the image field,
while the image is modeled again by the semicausal model with \( p_v = p_h = 0.8 \). The filter result for SNR=100 is shown in Fig. 5.9 (lower part). The improvement in performance is \( \eta_4 = 3.2 \) dB. The filter result for SNR=10, not depicted here, shows again a somewhat lower deconvolution action with respect to Fig. 5.9. The improvement \( \eta_4 \) is 2.9 dB.

Now that we have described several experiments with the optimal line-by-line recursive Kalman filter, attention will be paid to the suboptimal restoration schemes as discussed in Sections 5.4 and 5.5, respectively, for the case of linear motion blur in the presence of additive white noise.

**Fig. 5.9.** Filter result for the letter "G", defocused image, SNR=100.

Experiment 4: Suboptimal restoration of noisy motion-blurred images

First processing time and storage requirements are reduced by means of the technique described in Section 5.4, in which the full-gain matrix is approximated by means of a quasi-Toeplitz matrix with a small number of nonzero elements. A 128x128 test image of an aerial sight, shown in the upper-left corner of Fig. 5.10, is artificially degraded with linear motion blur in the horizontal direction extending over 9 pixels and with additive noise with SNR=100 (upper-right corner). The autocovariance function of the original image is approximated as an isotropic exponentially decaying autocovariance function with \( p=0.71 \) and \( s=0.9 \). The image is modeled by the semicausal model (2.4.8). Initially, a 32x32 gain matrix with \( k=16 \) (See equation (5.4.1)), calculated for one strip of the image and obtained by using statistical knowledge of the entire image, is expanded to a 128x128 gain matrix, after which the entire image is filtered with the steady-state gain matrix. However, to reduce boundary effects imposed by this filter, the a priori admissible set at the left border of the image is modeled as part of the image field by augmenting the Nx1 image vector to an (N+2x2)xl image vector, where \( x \) describes the extent of the horizontal motion blur.

**Fig. 5.10.** Filter result for the aerial sight; linear motion blur over 9 pixels, SNR=100.
The dimension of the full-gain matrix then becomes \((N+2\times)N\). Therefore, in our experiment a 41x32 gain matrix is expanded to a 137x128 gain matrix. The filter result with this expanded steady-state gain matrix is shown in the lower part of Fig. 5.10. Observe that the filtered image regains its original sharpness, although some details are lost. The improvement \(\eta_4\) is 2.6 dB.

Second, the transform-domain filter method as described in Section 5.5, is used to filter the image in the Fourier domain by means of a parallel bank of scalar Kalman filters.

As an initial experiment, the noisy motion-blurred letters "G" as shown in Figs. 5.7 and 5.8 are filtered with the transform-domain filter (5.5.18) so that these results can be compared with the filter results obtained by the optimal line-by-line recursive Kalman filter in experiment 2. The original image is again modeled with the semicausal model with \(\rho_v=0.8\). Because of the circulant-matrix approximations, we can think of the image as a cylinder instead of a flat array. Due to the uniform background intensity of this test image, the left-side boundary condition for this image is therefore automatically fulfilled. The filter results for both images are \(\eta_4=4.0\) dB and 3.8 dB, respectively, and yield only a slight decrease in filter performance of less than 3% when compared to the optimal filter results (4.1 dB and 3.9 dB).

Comparison of the filter results of both suboptimal restoration schemes as defined in Sections 5.4 and 5.5 for the 128x128 noisy motion-blurred image of the aerial sight (Fig. 5.10) and for several other noisy motion-blurred test images yield about the same filter performance. Therefore, the transform technique of Section 5.5 is to be preferred, due to its mathematical tractability, low demand on computer memory and speed.

To gain insight into the speed of the transform-domain filter, a 256x256 test image of an aerial sight, shown in the upper-left part of Fig. 5.11, is artificially degraded with linear motion blur in the horizontal direction extending over 9 pixels and with additive noise with SNR=100 (upper-right corner of Fig. 5.11). The autocovariance function of the original image is approximated as a separable exponentially decaying autocovariance function with \(\rho_v=0.9\). The image is modeled with the semicausal model (2.4.8). The filtered image is shown in the lower part of Fig. 5.11. The improvement is 3.1 dB. We know from Section 5.5 that the number of computations lies in the order of \(O(N^2 \log_2 N)\). Actual processing time on the Amdahl V/7-8 computer of the Computation Centre of the Delft University of Technology is less than 6 sec (Fortran, extended compiler; optimization level 2).

In summary, we may conclude from the experiments of this section that both optimal and suboptimal filter schemes yield acceptable filter results, although the filter performance of the latter is decreased a few percent more than the former for the case of noisy motion-blurred images. The suboptimal transform-domain filter technique is preferred because of its mathematical tractability, low demand on computer memory and speed.

Fig. 5.11. Filter result for the aerial sight; linear motion blur over 9 pixels, SNR=100.
5.7. CONCLUSIONS AND RECOMMENDATIONS

An optimal line-by-line recursive Kalman filter has been developed for restoring images degraded not only in a stochastic way by additive white noise, but also in a deterministic way by linear space-invariant blur. To reduce processing time and storage requirements, two practical suboptimal restoration schemes were presented and compared experimentally with the optimal filter scheme.

Extension of the results to the situation where the additive noise is colored is straightforward if the noise can be modeled by a partial difference-equation model of the form (5.1.6). In that case, one must simply augment the state-space representation of the image by the appropriate state-space representation of the noise. Further, if smoothed rather than filtered estimates of the image are desired we can easily combine two estimators (optimal or suboptimal), one running in a forward direction over the data and the other running backwards. Also edge information can be used in the filtering procedure to improve the visual quality of the restored images. Of particular interest and the subject of current investigation is the extension of the suboptimal restoration scheme in Section 5.5 to the more general case of noncausal (space-invariant) blurring (defocusing). In that case the resulting filter in the transform domain reduces to a parallel bank of low-order Kalman filters, in which the order of each filter is determined by the underlying image model and the extent of the blur [15]. Finally, we mention the ability of the line-by-line recursive Kalman filter to handle the more general situation of a nonstationary image covariance function and space-variant blur.

MAJOR NOTATIONS

FUNCTIONS

- \(e(\cdot)\)
- \(g(\cdot)\)
- \(h(\cdot)\)
- \(p(\cdot)\)
- \(r(\cdot), r(\cdot)\)
- \(u(\cdot), u(\cdot)\)
- \(v(\cdot)\)
- \(w(\cdot), w(\cdot)\)
- \(x(\cdot), x(\cdot)\)
- \(\hat{r}(\cdot), \hat{r}(\cdot)\)
- \(y(\cdot), y(\cdot)\)

- \(B(\cdot)\)
- \(C(\cdot)\)
- \(F(\cdot), F(\cdot)\)
- \(G(\cdot)\)
- \(H(\cdot), H(\cdot)\)
- \(S(\cdot)\)
- \(W(\cdot)\)
- \(X(\cdot), X(\cdot)\)
- \(\hat{r}(\cdot), \hat{r}(\cdot)\)
- \(Y(\cdot), Y(\cdot)\)

- \(s(\cdot)\)
- \(\hat{s}(\cdot)\)
- \(\hat{s}(\cdot, \cdot)\)
- \(u(\cdot)\)

error function
point-spread function of imaging system
filter point-spread function
probability-density function
autocovariance function
input-noise function
edge-weighted function
observation-noise function
original signal (image)
estimated signal (image)
observed signal (image)
denominator polynomial
constraint function
filter (predictor) gain
transfer function of imaging system
filter transfer function
power spectral-density function
frequency-weighting function
transform-domain signal (image)
transform-domain estimated signal (image)
transform-domain observed signal (image)
state vector
estimated state vector
optimal smoothed estimate of state vector
input-noise vector
### VECTORS
- \( a, b, f \): coefficient vectors
- \( x \): stacked vector form of discrete image
- \( \mu \): mean vector

### SCALARS
- \( a, b, c, \alpha, \beta, \gamma, \delta, \omega, \tau \): real constants
- \( i, j, k, l, m, n, M, N, p, q, s, t \): integer constants

### SETS
- \( I \): set of integers
- \( U \): subset of \( I \)
- \( S \): set of index pairs \((p,q)\)
- \( W \): subset of \( S \)

### MATRICES
- \( A, B, C \): model matrices
- \( I \): identity matrix
- \( O \): null matrix
- \( H \): distortion matrix
- \( Q \): circulant matrix
- \( R \): image autocovariance matrix
- \( S \): model coefficient matrix
- \( W \): noise field
- \( X, \hat{X}, Y \): original, estimated, observed image
- \( \Sigma \): matrix of eigenvectors
- \( \Lambda \): matrix of eigenvalues

### GENERAL SYMBOLS
- \( E \): expectation operator
- \( I \): Fisher information
- \( L \): linear system
- \( L \): linear operator
- \( \delta \): Kronecker delta
- \( \varepsilon^2 \): squared error
- \( \eta \): performance measure
- \( \otimes \): Kronecker product
REFERENCES


De beeldverwerking, in het bijzonder de digitale beeldverwerking, heeft de laatste jaren veel aandacht gekregen. Belangrijke onderzoekgebieden zijn beeldcodering, beeldreconstructie, beeldverbetering en beeldanalyse.

In dit proefschrift wordt aandacht besteed aan het probleem van de beeldreconstructie met behulp van een digitale rekenmachine. De beelden die men tot zijn beschikking heeft zijn in vrijwel alle gevallen opnamen van een scène of object, zoals televisiebeelden, foto's, met röntgen- of gammastraling verkregen opnamen, etc. Bij het tot stand komen van deze opnamen kan op velerlei wijze vervorming en ruis geïntroduceerd worden. De ruis kan het gevolg zijn van de korrelstructuur van fotografische emulsies (korreleruis) of afkomstig zijn van elektronische beeldversterkers en opnemapparatuur. Bij vervorming kan gedacht worden aan sferische aberraties van lenzen, bewegingsscherpte, onjuiste fociering, niet-lineaire eigenschappen van fotografische materialen, geometrische vervorming of vervorming ten gevolge van atmosferische turbulenties. De beeldreconstructie stelt zich ten doel deze vervormingen te reduceren door gebruik te maken van algoritmen die gebaseerd zijn op mathematische modellen van deze vervormingsverschijnselen. Hierbij wordt getracht het gereconstrueerde beeld zo goed mogelijk - in de zin van een objectief evaluatiecriterium - bij het originele onvervormde beeld te laten aansluiten.

In het bijzonder wordt in dit proefschrift aandacht besteed aan de digitale reconstructie van beelden die behept zijn met lineaire verschuivingsinvariante vervorming (defocussering, bewegingsscherpte) en additieve witte ruis, ongecorreleerd met het beeld. Inverse filtermethoden die een perfecte reconstructie nastreven door invertering van het vervormingsproces blijken bij aanwezigheid van ruis povere resultaten.

De indeling van het proefschrift is als volgt. In hoofdstuk 1 wordt het algemene beeldreconstructieprobleem besproken. Toegelicht wordt de noodzaak om over a priori informatie te beschikken. Tevens wordt ingegaan op de keuze van een lineaire stochastische filteraanpak. Als algemene achtergrondkennis worden tenslotte in dit hoofdstuk de een-stapspredictie- en filter-algoritmen gegeven voor het discrete één-dimensionale Kalmanfilter.

In hoofdstuk 2 wordt aandacht besteed aan het modelleren van de a priori kennis omtrent het oorspronkelijke onvervormde beeld in de vorm van discrete twee-dimensionale autoregressieve modellen. Allereerst wordt ingegaan op de stochastische representatie van beelden, waarbij een beeld gezien kan worden als een realisatie van een twee-dimensionaal stochastisch veld. Vervolgens wordt dit veld beschreven met behulp van de momenten van de eerste en tweede orde. Teneinde het aantal vrijheidsgraden in deze beschrijvingswijze te reduceren worden de volgende beeldeigenschappen geïntroduceerd: homogeniteit, scheidbaarheid, isotropie en autoregressie. Op basis van een homogene beeldbeschrijving wordt dan een algemeen twee-dimensionaal autoregressief model gedefinieerd. De modelcoëfficiënten worden gevonden m.b.v. een kleinste-kwadratenmethode voor aanpassing. Belangrijke aspecten bij het definieren van deze zogenaamde minimum-variantiemodellen zijn die van een verwer-
een scheibare filterstructuur. Aan de hand van een aantal experimenten wordt de bruikbaarheid van deze methode gedemonstreerd.

Tenslotte wordt in hoofdstuk 5 een optimaal Kalmanfilter afgeleid dat zowel vervorming als ruis verwijdert uit een beeld. Dit filter reconstrueert een beeld lijnsgewijs in plaats van puntsgewijs. Vanwege de aanzienlijke rekeninspanning en geheugenruimte die voor dit filter nodig is wordt gezocht naar methoden om deze computerlast te verminderen. Allereerst wordt ingegaan op een methode waarbij de versterkingsmatrix in zijn eindwaarde wordt benaderd door een quasi-Toeplitz matrix met een gering aantal van nul verschillende elementen. Vervolgens wordt een diagonaliseringsmethode beschreven die gebruik maakt van circulantmatrix-benaderingen. Het lijn voor lijn werkende Kalmanfilter wordt daarmee gereduceerd tot een aantal scalaire parallelwerkende Kalmanfilters in een transformatiedomein. Experimentele resultaten op verruiste gedefocuseerde en verruiste bewegingsvormde beelden worden besproken.

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STELLINGEN

behoorende bij het proefschrift van J. Biemond.

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1. Het grote accent dat bij het ontwerpen van filters in de literatuur wordt gelegd op het vinden van een optimale oplossing is niet in overeenstemming met de geringe aandacht die wordt besteed aan de keuze van het ontwerpcriteria ten opzichte waarvan optimaliteit gezocht wordt.

2. Het verdient aanbeveling het onderzoek naar schattingmethoden voor de puntspreidingsfunctie van de vervorming in een beeld, in het bijzonder vanuit het vervormde beeld zelf, te intensiveren.

3. Een functie in twee variabelen die positief definitief is op een rechthoek behoeft geen positief-definiete uitbreiding te bezitten op het gehele vlak. Een gevolg hiervan is dat twee-dimensionale parametrische spectrumschattingmethoden die gebaseerd zijn op gecorrigeerde autocorrelatiewaarden aanleiding kunnen geven tot negatieve spectrale waarden.

4. De vertaling van 2-D stabiliteitstheorema's in numerieke testmethoden voor stabiliteit heeft zich tot dusverre beperkt tot causale modellen. Gezien het belang van semicausale en nietcausale modellen in de beeldreconstructie dient dit onderzoek zich ook tot deze modellen uit te strekken.

5. Door drempeling van de contourinformatie in een beeld kan de gevoeligheid van het hiermee gestuurde filter voor artefacten worden verminderd.

6. Niets doet meer afbreuk aan de kwaliteit van en de goede sfeer op een congres dan dat sprekers zonder kennisgeving wegblijven.


8. Elke auto zou voorzien moeten zijn van een signalering die defecten aan de verlichting meldt.

9. Door een technisch mankement aan computers of andere gecompliceerde machines toe te schrijven aan oorzaken buiten de menselijke invloedsbereik, gaat men gemakshalve voorbij aan het feit dat het mensen zijn die deze machines ontwerpen, bouwen, bedienen en onderhouden.


11. Een opvallend kenmerk van de Nederlandse orgelwereld is, dat het aantal bezoekers van een orgelconcert slechts in geringe mate wordt bepaald door de vakbekwaamheid van de organist, maar in veel sterke mate door persoonsverheerlijking en religieuze sentiment.

12. De gemeente Alkemade mag slechts dan de exploitatie van het pontvaart De Kaag in particuliere handen doen overgaan, indien zij als lokale overheid ervoor zorgdraagt dat de huidige dienstverlening, ook op langere termijn, gevaarloos blijft.

13. Nog steeds blijven er watersportliefhebbers te zijn die niet beseffen dat voor het veilig navigeren in de Nederlandse kustwateren of op het IJsselmeer meer nodig is dan een autokaart van Nederland.