Modelling of Separation using Euler Methods

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<td>E.M. Houtman, P.G. Bakker, E. de Vries and J.C. van den Berg</td>
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<tr>
<td><strong>Abstract</strong></td>
<td>The modelling of separation from smooth surfaces within a method based on a finite volume discretization of the Euler equations has been investigated. A theoretical analysis is given of the flow field near a separation in a conical flow. Three numerical boundary condition techniques to enforce separation are presented. Numerical tests on a cone with elliptic cross-section show that smooth surface separation can be obtained, which is at least qualitatively in agreement with experiments.</td>
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<td><strong>Keyword(s)</strong></td>
<td>Inviscid separation, Euler method, Kutta condition, vortex flow</td>
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<td>P.G. Bakker</td>
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SUMMARY

The modelling of separation from smooth surfaces within a method based on the Euler equations has been investigated. Euler methods have shown to be able to simulate separation from geometries with sharp edges without explicit prescription of a Kutta condition. Separation from smooth surfaces, however, so far can not be simulated with Euler methods. It is clear that such solutions are allowed within the Euler flow model, but a correct implementation of a Kutta condition is not available at the moment. The separation from a smooth surface has been investigated using the conical flow concept, in which all essential three-dimensional properties are kept.

The flow in the neighbourhood of the separation line has been investigated by some local solutions of the equations for inviscid conical flow. These local solutions give the structure of the flow and relations between pressure distribution, streamline curvature and the shape of the shear-layer. The first model is a review of the model developed by J.H.B. Smith, which describes separation from a smooth surface in a conical homentropic flow. Main characteristic of this flow model is the requirement that the separation line leaves the solid wall tangentially. The second model is a model based on the matching of two conical stagnation flow solutions at the separation surface. In this model the separation line leaves the solid wall at a finite angle, and the separation is embedded in a flow with continuous distributed vorticity.

The formation of separation and vortical flow in numerical solutions of the discretized Euler equations has been studied on a cone with an elliptic cross-section. Three different procedures to enforce separation from a smooth surface within a conical Euler method have been developed and analyzed. The main issue concerns the prescription of the flow direction along the specified separation line on the surface of the body.

The first model prescribes the flow direction at a certain specified point on the surface. However, this model can be used only in combination with the flux-vector splitting scheme of van Leer. With this model separation from smooth surfaces can be obtained. The success of this model is based on an incorrect treatment of contact discontinuities of van Leer's scheme. The effect of the forced secondary separation seems to be in agreement with experimental observations. The second model is based on the prescription of the flow direction in a cell near the solid surface. It is based on a model successfully used by Kwong and Myring. This model, however, delivered unsatisfactory results. No converged solutions could be obtained, and the separation was delayed to a significant distance downstream of the prescribed separation position. The third model uses a closed cell boundary normal to the surface. This introduces a sharp edge in the geometry, resulting in separation. Computational results show that this model enforces separation at the prescribed position under conditions of a favourable pressure gradient (positive in cross-flow direction). This model can be used for all types of discretization schemes. At the separation position, however, an unphysical pressure jump is calculated. For all test cases investigated, no significant influence on the convergence behaviour could be observed.

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<th>Description</th>
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<tr>
<td>$A$</td>
<td>length of major axis of ellipse; Flux Jacobian</td>
</tr>
<tr>
<td>$A_1$</td>
<td>attachment of primary separation</td>
</tr>
<tr>
<td>$a$</td>
<td>real constant in Joukowsky transformation</td>
</tr>
<tr>
<td>$a_1, a_2$</td>
<td>coefficients in vortex sheet equation</td>
</tr>
<tr>
<td>$B$</td>
<td>length of minor axis of ellipse</td>
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<tr>
<td>$b_1, b_2$</td>
<td>coefficients in pressure equation</td>
</tr>
<tr>
<td>$C_L$</td>
<td>lift coefficient</td>
</tr>
<tr>
<td>$C, C_1, C_2$</td>
<td>constants determining trajectories</td>
</tr>
<tr>
<td>$c$</td>
<td>speed of sound</td>
</tr>
<tr>
<td>$c_p$</td>
<td>nondimensional static pressure coefficient</td>
</tr>
<tr>
<td>$c_0$</td>
<td>speed of sound in conical stagnation point</td>
</tr>
<tr>
<td>$e$</td>
<td>internal energy</td>
</tr>
<tr>
<td>$e_t$</td>
<td>total energy per unit of mass</td>
</tr>
<tr>
<td>$F, G, H$</td>
<td>flux vectors in $x, y, z$-direction respectively</td>
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<tr>
<td>$F_{NFF}$</td>
<td>numerical flux function</td>
</tr>
<tr>
<td>$F_{S_{\pm 1}}$</td>
<td>states at upstream and downstream side of separation</td>
</tr>
<tr>
<td>$F^+, F^-$</td>
<td>split flux vectors</td>
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<td>$f$</td>
<td>spatial discretization of governing equations</td>
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<td>$f_1, f_2$</td>
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<tr>
<td>$\mathcal{H}$</td>
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<td>$h_t$</td>
<td>total enthalpy</td>
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<tr>
<td>$i, j, k$</td>
<td>indices in computational space</td>
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<td>$k_1, k_2, k_3, k_4$</td>
<td>intermediate vectors in Runge Kutta system</td>
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<tr>
<td>$M$</td>
<td>Mach number</td>
</tr>
<tr>
<td>$m$</td>
<td>index of cell face</td>
</tr>
<tr>
<td>$N_f$</td>
<td>number of faces of control volume</td>
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<tr>
<td>$\tilde{n}$</td>
<td>unit normal vector</td>
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<tr>
<td>$\tilde{r}, \tilde{s}$</td>
<td>base vectors in rotated coordinate system</td>
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<tr>
<td>$p$</td>
<td>static pressure</td>
</tr>
<tr>
<td>$p_t$</td>
<td>total pressure</td>
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<tr>
<td>$Q$</td>
<td>state vector of conserved variables</td>
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<td>$\bar{q}$</td>
<td>velocity vector</td>
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<tr>
<td>$R$</td>
<td>gas constant</td>
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<tr>
<td>$R_{ij}^\top$</td>
<td>limiter function applied to $i$-direction</td>
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<tr>
<td>$r$</td>
<td>radius</td>
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<tr>
<td>$S$</td>
<td>boundary of control volume; ratio of minor- and major axis of ellipsis</td>
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<td>$S_1, S_2$</td>
<td>primary- and secondary separation</td>
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<tr>
<td>$s$</td>
<td>entropy; coordinate along surface</td>
</tr>
<tr>
<td>$T$</td>
<td>static temperature; rotation matrix</td>
</tr>
<tr>
<td>$t$</td>
<td>time; scaled circumferential parameter</td>
</tr>
<tr>
<td>$\tilde{r}$</td>
<td>tangential vector</td>
</tr>
<tr>
<td>$u, v, w$</td>
<td>velocity components in orthogonal coordinate system</td>
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LIST OF SYMBOLS

*u*, *u₂*, *uₙ*  orthogonal velocity components on surface along conical ray, tangential and normal to surface respectively
*u₀*  velocity in conical stagnation point
*u₁*, *u₂*, *w₁*, *p₁*  coefficients
*V*  control volume
*w₁*, *w₂*  distances of concentric circles
*x, y, z*  Cartesian coordinate system
*z*  scaled entropy
*ẑ₁, ẑ₂, ẑ₃, ẑ₄*  vectors in complex planes
*ẑ*, ẑ*, ẑ*  rotated Cartesian coordinate system
*α*  angle of incidence; ratio of speeds of sound in Osher scheme
*βₐ*, *βₜ*  angles in Kwong and Myring method
*Γ*  circulation
*Γₖ*  subpath in state space for Osher scheme
*γ*  ratio of specific heats (cₚ/cᵥ = 1.4); surface streamline angle
*Δᵢ*, ⁰⁻¹  forward and backward differences in i-direction
*δ*  angle of rotation along rotated z-axis
*ε*  small constant
*ζ*, *η*  non-dimensional conical coordinates
*η₁*, *η₂*  closure ratio of cell face in separation model
*θ*  polar angle
*θᵅ*  half top angle of elliptic cone in z, y-plane
*κ*  constant in higher order interpolation function
*κₕ*  streamline curvature
*Λ*  sweep angle between y-axis and leading edge
*λ*  eigenvalue
*μ₀₀*  Mach angle based on free-stream Mach number
*ρ*  density; radius in complex plane
*σ*  arc length; translation parameter in conformal mapping
*φ*  circumferential angle; function of potential
*ϕₐ*, *ϕₜ*  angles in Kwong and Myring method
*Ψ₀*, ¹₀  intermediate quantities in Osher scheme
*Ωᵢᵣₖ*  control volume
*Ω*  vorticity vector
*ω*  angle between separation line and surface
*δΩ*  cell face of control volume

Subscripts

*ARS*  Approximate Riemann Solver
*av*  averaged quantity
*ax*  component along conical ray
*B*  boundary condition
*extrap*  extrapolated quantity
*le*  leading edge
LIST OF SYMBOLS

OSH Osher
r radial direction
sub subsonic
t tangential direction
VL van Leer
W solid wall
Σ cross-plane normal to z-axis
∞ free-stream condition

Superscripts

L, R left and right to a certain cell face
S sonic conditions
Chapter 1

INTRODUCTION

Flow separation and the formation of vortices in three-dimensional flow is an important aspect in advanced airplane design. The understanding of the physics of these phenomena and the capability to simulate these types of flows is necessary for the improvement of the aerodynamic characteristics and the controllability of these airplanes. Several types of separation and formation of vortices can be distinguished. Well-known is the separation and the formation of a tip vortex system downstream of an airplane wing. This type of flow can be simulated with inviscid flow models based on the linearized potential equations, in which the wake is represented by a planar vortex sheet whose strength is determined by the Kutta Joukowsky condition of finite velocity at the trailing edge. More complicated is separation and the formation of a coherent vortex system at (sharp) leading edges of swept wings. This type of flow is often utilized in order to increase the lift capabilities. At high angles of incidence also separation from smooth surfaces, for example cones or bodies, can occur. Although in most cases the rotation can still be represented by a vortex sheet, the location and form of the vortex sheet can no longer be assumed a priori. Especially for separated flows from smooth surfaces, the separation position is determined by an interaction between the inviscid flow and the boundary layer on the surface. In order to simulate this viscous phenomenon, without using information of the flow topology in advance, a method based on the Navier-Stokes equations would be appropriate. However, Navier Stokes codes require large computer memory and time, which makes their usability in a design environment very limited up to now. Codes based on an inviscid flow model can deliver an alternative approach to the problem.

In the past some successful results were obtained with methods based on linearized potential models (Smith 1984; Hoeffmackers 1989). Characteristic for these methods is, that the vortical flow is modelled by fitting vortex sheets and vortex filaments in the flow field. These methods require the global structure (topology) of the flow field to be known in advance; the determination of the position and strength of the different vortex elements is part of the solution process. For complex flow fields, where the global topology of the flow is not known in advance, or for flow fields which are dominated by compressibility effects, application of these methods may lead to incorrect results.

More recently, modelling of inviscid vortical flow is pursued with methods based on the Euler equations. The major advantages over potential flow methods are abilities to handle shock waves and convection of vorticity. Numerical simulations with Euler methods have shown that separation can be generated by discontinuities in the geometry slope (sharp edges) or by a shock wave (Newsome 1985; Houtman and Bannink 1991; Marconi 1987). It has been shown, but not fully understood, that the amount of vorticity produced at sharp edges is in agreement with experimental results for flows at high Reynolds numbers. The global structure of vortices in the numerical simulations and in the experiments compare very well. Separation from a smooth surface, however, is not modelled by Euler methods. This will lead to disagreement between simulation and experiment if separation effects from a smooth surface are significant, for example the primary separation of swept wings with rounded leading edges, or the secondary separation at the leeward side of a delta wing. In order to extend the area of application for methods based on inviscid flow models, modelling of separation from smooth surfaces would be necessary.

One of the first methods of prescribing separation from smooth surfaces, was provided by Smith (1977).
This method essentially relies on vortex sheets within potential flow models. Some favourable results for slender cone flows were shown by Fiddes (1980). More recently, progress has been made in modelling of smooth surface separation within Euler methods. Besides the advantage that no a priori specification of the flow topology is required, Euler methods are also able to handle strong shock waves and vorticity convection. Klopfenstine and Nielsen (1981) have reported three-dimensional separated flow over tangent ogive bodies at incidence by specifying the direction of the flow along a separation streamline. They were using a space-marching Euler code, based on a finite-difference discretization. Their results showed improvements in surface pressure distribution and flow field structures, which agreed well with experimental data if the separation position was known a priori. Impressively, results for conical flow around a circular cone were obtained by Marconi (1987), who claims to specify separation according to the analysis of Smith (1977). In this method, based on the $\lambda$-scheme from Moretti (1979), separation was forced by the introduction of a double-point on the surface, in which a jump in tangential velocity was specified. However, problems with the pressure behaviour directly downstream of the separation were observed. A viscous/inviscid interaction scheme was developed by Kwong and Myring (1989), who showed results of the coupling of a time-marching Euler code with a three-dimensional integral boundary layer method. The boundary layer calculations were used to predict the separation location. Within the Euler solver the separation was forced by the specification of the flow direction at each side of the separation stream surface.

The primary goal of this project is the development of a general boundary condition procedure, in order to prescribe separations within numerical methods based on a finite volume discretization of the Euler equations. This work is done in conjunction with a theoretical analysis of the flow near separation from a smooth surface in inviscid conical flow. Provisional results of this combined approach are reported by de Vries (1992). This theoretical analysis can provide a guidance for the development of the boundary condition technique. Furthermore it can be used to validate the numerical results of separated flows from smooth surfaces. This report starts with an introduction about the physics of separation and the conical flow concept. In the following chapter 3 the local analytical descriptions of inviscid separated flow are outlined. The numerical method, with which the simulations of conical flows around cones with an elliptic cross-section are performed, is described in chapter 4. Some characteristics of the conical flow around elliptic cones are presented in chapter 5. The different models to prescribe separation in the numerical simulation method are introduced in chapter 6, together with some results of separated flow calculated with these models. This report ends with an analysis of the boundary condition procedure and some recommendations for future work.
Chapter 2

PHYSICS OF SEPARATION AND VORTEX FLOW

2.1 Basic flow separation structures

Flow separation is an important phenomenon in fluid flow, which may appear in different forms on bodies in fluid flow. In many cases separation of flow is a viscous flow phenomenon, and the position of separation is determined by the viscous shear stresses, which are dependent on the Reynolds number. Only in the cases of discontinuities in the geometry, for example sharp edges, the position of separation is fixed.

From a theoretical point of view separation in two-dimensional flow can be described by the classical boundary layer theory. When a two-dimensional boundary layer undergoes an increasing pressure, it follows that the skin friction decreases and that the displacement thickness increases. The point where the skin friction itself changes sign is called the separation point. At this point a streamline leaves the surface. As long as this streamline does not proceed into the outer flow field, but remains within the boundary layer, it does not alter the basic behaviour of the main stream. An example of this small-scale separation is the local circulating flow inside a boundary layer. On the other hand, large-scale separations, characterized by a layer containing vorticity entering into the main stream can have a large influence on the flow field.

In three-dimensional flow the skin friction is a vector with two components. These will only vanish simultaneously in isolated points. The structure of the flow in the neighbourhood of these points is found by a local linearization technique yielding a linear system for the description of the flow topology. The isolated points where the skin friction vanishes, appear to be singular points of this linear system. Determining the eigenvalues of this linear system, several types of structures can be distinguished. In those cases where streamlines, forming a stream-surface, leave the surface from a singular point, separation is clearly defined. Besides this type of separation, in the literature also the so-called "open-type" separation is distinguished. This type of separation is not connected to a singular point, but can be characterized by a strong convergence of skin friction lines. The line of convergence can be identified as the separation line. Close to this line, the streamlines are moving away from the body surface.

In general large-scale separation is characterized by boundary layers, which separate from the surface and move into the flow field. These layers form regions of concentrated vorticity, which roll up into a vortex system (Fig. 2.1).

For the limit case \( Re \to \infty \) the inviscid flow model is recovered. This type of flow can be described by the Euler- or, in the case of irrotational flow, by potential equations. No-slip boundary conditions and boundary layers disappear, and shear layers reduce to contact discontinuities (separation surface). Contact discontinuities are streamsurfaces with a continuous pressure and a jump in the tangential velocity. Other quantities (depending in the appropriate flow model) may show also a discontinuous behaviour. An analysis of the inviscid irrotational flow in the vicinity of a separation line on a smooth surface, carried out by Smith (1977), delivered an important result of the behaviour of the separation surface. It was concluded that vorticity can only be shed from the surface and convected into the flow field (along a vortex sheet) if the separation surface leaves the body tangentially. Developing a method to describe separation from a smooth surface, it is of particular importance to incorporate the
local behaviour into the global model of the flow.

2.2 Conical flow

The analysis of three-dimensional separation and the numerical simulations will be done for conical flow situations. In some cases, conically shaped geometries in a supersonic flow, generate a flow field with a conical similarity. This means that the velocity and the flow quantities defining the state of the gas, e.g. pressure and density, are constant along straight lines emanating from a common point. This point is usually called the conical centre, which coincides with the apex (top) of the conical geometry. Well-known examples of configurations of practical interest, which can be described with conical flow theory are cones and delta wings with arbitrary cross-section, inlet configurations and wing-body junctions in supersonic flow with attached (bow)-shock waves. For subsonic flow and supersonic flows in which the bow shock is not attached at the apex, conical flow is not possible, because the far-field boundary conditions are not conical. Within the conical flow concept all essential three-dimensional flow properties are kept. Studying conical flows, the analytical/computational effort is substantially reduced, since the dependence on the axial coordinate is removed from the equations of motion.

The structure of a conical flow may be represented on a unit sphere centred at the conical centre. The velocity vector $\vec{q}_t$ with the Cartesian velocity components $u$, $v$ and $w$, may be orthogonally decomposed into a radial component $\vec{q}_r$, normal to the sphere and a component $\vec{q}_t$ tangential to the sphere (Fig. 2.2). Integration of the vector field $\vec{q}_t$ gives lines on the sphere, which are called conical
streamlines. For a conical flow, streamlines passing a common ray from the conical centre form a conical shaped streamsurface. The intersection of this streamsurface with the unit sphere, gives the conical streamline again. Similarly, the intersection of the conical streamsurfaces with a plane surface \( \Sigma \) can also be taken. This method is better suited for graphical presentation. Such a plane is usually called a cross-flow plane, and is placed normal to the main flow direction or the principal axis of the conical configuration. Consider a flow in a right-handed Cartesian \( x, y, z \)-coordinate system with the origin in the conical centre (see Fig. 2.3), and the main flow directed along the \( x \)-axis. Due to the

![Figure 2.3: Velocity components in conical flow representation](image)

conical similarity, the primitive variables depend only on the non-dimensional coordinates \( \eta = y/z \) and \( \zeta = z/z \). The velocity vector \( \vec{q} \) can be decomposed non-orthogonally in a vector along the conical ray \( \vec{q}_r \) and a vector in the cross-plane \( \vec{q}_\Sigma \) (Bakker 1991). The raywise velocity vector \( \vec{q}_r \) and the cross flow velocity vector \( \vec{q}_\Sigma \) are given by:

\[
\vec{q}_r = (u, u\eta, u\zeta)^T
\]

\[
\vec{q}_\Sigma = (0, v - u\eta, w - u\zeta)^T
\]

The vector field \( \vec{q}_\Sigma \) gives the direction of the conical streamlines. These conical streamlines may be found from an integration of the vector field \( \vec{q}_\Sigma \) written in the form of a differential equation:

\[
\frac{d\eta}{d\zeta} = \frac{v - u\eta}{w - u\zeta}
\]

Points where the velocity components \( v - u\eta \) and \( w - u\zeta \) both vanish, are singular points in the cross-flow plane. They are called conical stagnation points. The conical stagnation points have a special meaning; in the three dimensional flow field they represent special rays along which the velocity vector is directed. Knowledge about the number of and the position of these points gives a valuable insight in the topology of the conical flow.

### 2.3 Survey of the flow field over a delta wing

The flow field over a delta wing appears to be an interesting example where separation from a smooth surface occurs. Therefore in this report some numerical computations will be done on delta wing configurations. The flow field at the leeward side of a delta wing at various angles of incidence, may show a variety of flow patterns. At small angles of incidence attached flow appears, while at
larger angles of incidence separated flow with the formation of vortices will occur. At higher Mach numbers the flow field may be complicated by the appearance of embedded shock waves. Assuming conical flow, which is a valid assumption for inviscid supersonic flow, the parameters determining the flow structure are the free stream Mach number, the angle of incidence and the geometry of the delta wing. The most important parameter in the wing geometry seems to be its leading edge sweep. As long as the the leading edge of the wing is "aerodynamically" sharp, the exact geometry does not have a large influence on the flow pattern. For supersonic delta wing flows, the possible flow patterns are investigated and summarized by Stanbrook and Squire (1964). They proposed a diagram in which the relationship between the above mentioned parameters and the flow pattern at the leeward side of a delta wing is expressed. An extension of this diagram, made by Miller and Wood (1984), is presented in Fig. 2.4. The variables are the Mach number \(M_N\) based on the

![Figure 2.4: Classification of delta wing leeward side flow patterns](image)

velocity component normal to the leading edge and the angle of incidence \(\alpha_N\) normal to the leading edge. Seven regions are distinguished, each representing a particular flow type, depending on the set \(M_N, \alpha_N\). In Fig. 2.4 the main flow characteristics are given in a cross-sectional view: separation bubbles, primary and secondary vortices, embedded shocks. The internal boundaries in the diagram are affected by parameters of secondary importance such as the Reynolds number, nose radius, taper ratio, wing profile shape. The classical delta wing vortex formation is normally found in region 1. Two strong and stable vortices emanate from the leading edges affecting the leeward side flow field to a large extent. The flow reattaches the wing surface along a rather straight ray passing through the apex of the wing. A three-dimensional sketch of the flow field is shown in Fig. 2.5, and the conical representation in a cross-flow plane normal to the wing in Fig. 2.6. Outboard of the reattachment line (A1) the surface flow turns outward and generally separates due to a strong adverse pressure gradient. Then, a secondary vortex is formed counter-rotating to the primary one. The surface flow exhibits a separation line and, as a consequence, also a secondary reattachment line. In a similar way tertiary vortices may be induced further outboard. At high subsonic speeds the overall features of the flow
structure show a noticeable effect of compressibility. With increasing Mach number the vortex core moves closer to the wing surface and closer to the plane of symmetry; in addition, the vortex flattens out. Moreover, due to the acceleration of the flow in the rapidly swirling vortex areas, local supersonic regions and even local embedded shocks may be present at free stream Mach numbers well below unity. Those regions can be found both on top of the vortex as well between vortex and wing surface (regions 3 to 6 in Fig. 2.4).
Chapter 3

THEORETICAL MODELLING OF INVISCID SEPARATION

Even for inviscid flow, separation is a rather complex phenomenon. Because the structure of the flow near separation is characterized by several types of discontinuities and high gradients, it is very difficult to obtain an accurate solution of the flow field with methods based on an approximate flow model. Analytical studies of the flow in the vicinity of separation lines are undertaken in order to give some insight in the structure of flow separation, and to provide a guidance for choosing appropriate local representations and boundary conditions for the subsequent numerical flow model. Two analytical flow models will be described here.

3.1 Modelling separation from a smooth surface in potential flow

A well-known analytical description of a separation in inviscid three-dimensional flow is the model developed by Smith (1977). The analysis of the inviscid homentropic irrotational flow in the vicinity of a separation line on a smooth surface was performed within the framework of slender body theory. The assumption of homentropic flow excludes the study of vortex sheets in flows with strong shock waves, and the study of vortex sheets near closed bubbles of fluid, which in steady conditions are not penetrated by fluid from the main stream. The model is an extension of the classical coiled vortex sheet model for separation from a salient edge to separation from a smooth surface. This research was motivated by the observation that slender bodies with smooth cross sections (such as cones) placed in a flow at large angles of incidence, showed separations similar to those at bodies with sharp edges, like delta wings. Due to the absence of sharp edges, the separation position is not fixed, but is the result of an interaction of the boundary layer on the smooth body and the external flow. It was shown (Smith 1977; Fiddes 1980), however, that this type of flow could also be modelled by an inviscid flow model, if the position of the separation is prescribed. The determination of the position in the flow field and the strength of the vortex sheet is a part of the solution of the mathematical flow model.

The analysis of Smith has demonstrated that vorticity can be shed from a smooth surface into an inviscid homentropic irrotational flow and concentrates in a vortex sheet. Two important results were obtained:

- vorticity can be shed from the surface only if the vortex sheet leaves the body surface tangentially;
- the curvature of the sheet, measured in a plane normal to the separation line, is infinite at its base ("singular" separation) or is locally equal to the surface curvature ("smooth" separation).

As a consequence of the assumption of irrotational flow and the elimination of closed bubbles, the circulation of the vortex sheet can be defined. The circulation along a path which begins at a point on one side of the sheet and ends at the adjacent point on the opposite side of the sheet, depends only on the choice of the point on the sheet. If the point lies on the separation line at the base of the sheet, the circulation along such a path is defined as the circulation of the sheet at that position on the separation line. It is taken that vorticity is being shed from a point on the separation line if and only if the circulation is varying with the position on the separation line at this point and the point is not a stagnation point of the mean flow. An example in which vorticity is not shed is provided by the
inviscid model of the flow past a combination of a body and a lifting wing. The plane vortex sheet representing the wake from the wing lies along the body without being fed with vorticity from the body.

The first result was obtained from the following reasoning. Assume that the sheet is not tangential to the body, then it will appear that vorticity is not being shed. Since the body and the vortex sheet are continuously curved, there are unique normals to both the body and the vortex sheet at the point considered. On the separation line both normals are distinct because it was assumed that the sheet is not tangential to the body. The component of the flow velocity along each of the normals must vanish because the sheet and the body are streamsurfaces of a steady flow. Hence the flow velocity on both sides of the sheets is perpendicular to both normals, which implies that it is parallel to the tangent of the separation line. The second boundary condition on the sheet is that the pressure is continuous across it. For steady, homentropic and irrotational flow, equality of pressure is identical to equality of flow velocity. This means that the flow velocity on the two sides of the separation line is either: (1) zero, (2) equal, non-zero, and in opposite direction, (3) non-zero and identical. In case of possibility (1) the mean velocity will be equal to zero, which corresponds to a stagnation point of the mean flow. It is obvious that in that case no vorticity can be shed. Possibility (2) implies a flow with the same velocity as the mean flow, but in a reversed direction, which is hardly conceivable. This leaves possibility (3), according to which the velocity is the same on both sides of the sheet. Also in this case no vorticity can be shed. However, the circulation along such a path does not change with time and consequently, the circulation of the vortex sheet is not varying with position along the separation line. So in this case also vorticity is not being shed.

However, if the vortex sheet is tangential to the body, then vorticity can be shed. This configuration is sketched in Fig. 3.1. First, distinguish between an upstream (denoted by a 1 in Fig. 3.1) and a downstream side (denoted by a 2 in Fig. 3.1), the downstream side being the side where the sheet lies close to the surface. On that side the velocity vector must be parallel to the separation line. On the upstream side, however, the normals to the surface and to the separation line are identical, so the only restriction to be imposed on the flow direction is that it must be parallel to the common tangent plane of the sheet and the body. Again, the magnitude of both velocities are equal but this still allows there to be a non-zero component of the mean velocity normal to the separation line. Thus, there is no stagnation point in the mean flow, and the circulation of the sheet varies along the separation line. As a result, vorticity can be shed into the flow sheet. An important condition is the requirement of
the flow direction at the downstream side of the separation to be parallel to the separation line. This condition serves as a Kutta condition for smooth separation, in order to fix the separation line. Smith (1982) derived an expression for the circulation in terms of upstream quantities. On the separation line the velocity components \( u, v \) and \( w \) are defined as follows; \( u \) is directed along the separation line, \( v \) is normal to the separation line and tangential to the surface, \( w \) is normal to the separation line and normal to the surface. The equality of pressure at the separation line in a steady homentropic flow can be written as:

\[
\begin{align*}
    u_1^2 + u_1^2 + w_1^2 &= u_2^2 + v_2^2 + w_2^2
\end{align*}
\] (3.1)

where the subscripts 1 and 2 denote the situations at the separation line just upstream and just downstream respectively. Due to the surface boundary condition we have:

\[
    w_1 = w_2 = 0
\] (3.2)

and the Kutta condition is given by:

\[
    v_2 = 0
\] (3.3)

Defining \( \sigma \) as the arc length along the separation line and \( \Gamma \) as the circulation about a contour surrounding the the vortex sheet, the following expression for the jump in the velocity component along the separation line can be derived:

\[
    u_2 - u_1 = \frac{\partial}{\partial \sigma} (\phi_2 - \phi_1) = \frac{d \Gamma}{d \sigma}
\] (3.4)

where \( \phi \) is the function of the potential. Combining expressions (3.1–3.4) it is possible to rewrite the Kutta condition in terms of total circulation and upstream quantities:

\[
    v_1^2 = \frac{d \Gamma}{d \sigma} \left( \frac{d \Gamma}{d \sigma} + 2 u_1 \right)
\] (3.5)

The second result about the curvature of the vortex sheet and the corresponding pressure distribution was obtained from a more complex local analytical analysis (Smith 1977; F.T. Smith 1978), which will not be repeated here. Looking at a section through the surface and the sheet normal to the separation line, it appeared that the distance of the sheet from the wall (\( z \)) is given by:

\[
    z = a_1 (y - y_s)^{1/2} + a_2 (y - y_s)^{3/2} + \ldots
\] (3.6)

where \( y - y_s \) is the distance along the wall downstream of separation, the dots denote higher order terms. To this shape of the sheet corresponds an expression for the pressure upstream of the separation line:

\[
    p = p_s - b_1 (y - y_s)^{1/2} + b_2 (y - y_s)^{3/2} + \ldots
\] (3.7)

\( a_1 \) and \( b_1 \) are proportional to one-another and \( p_s \) is the pressure at the separation line. For \( a_1 \neq 0 \) the vortex sheet has an infinite curvature at the base of the sheet and the pressure gradient on the surface just upstream of the separation is also infinite. This case is referred to as "singular" separation. For the case in which the leading terms vanish \( (a_1 = 0, b_1 = 0) \), both the curvature and the pressure gradient remain finite, which is called "smooth" separation. This case, however, will probably occur only in isolated points along the separation line.

Numerical simulations with vortex sheet methods of separation from smooth surfaces were performed by Fiddes (1980) and Hoeijmakers (1989). Their results showed a good agreement with the above
mentioned local analytical description of inviscid separation from a smooth surface on elliptic cones and delta wings. These inviscid pressure distributions were used as an input to conventional boundary layer methods (Fiddes 1980; de Bruin and Hoeijmakers 1987) in an attempt to predict the separation position in an iterative manner. It was shown that this approach converged to the position of "smooth" separation (finite curvature of the vortex sheet), which was in general far upstream of the experimentally observed position of separation. This behaviour may be explained by the infinite adverse pressure gradient just upstream of the separation, which a boundary layer can not withstand. The converged viscous-inviscid procedure gives a solution for which the curvature and the pressure gradient at separation are finite, the "smooth" separation. A triple-deck method was necessary to prescribe separations which were in agreement with experiments (Fiddes 1980).

3.2 Modelling of separation in conical flow with distributed vorticity

At the High Speed Laboratory an analytical model for the description of separation and attachment in conical flow has been developed (van Horssen 1984; Ottochian 1985; Abdulgani 1990). The flow is assumed to be inviscid, conical, non-heat-conducting and steady, but in contrast to the previous Smith-model, the flow may be afflicted with continuous distributed vorticity originating for example from curved shock waves. This model is based on a matching of two analytical solutions for the conical flow in the vicinity of a corner.

![Figure 3.2: Definition of velocity in spherical coordinate system](image)

For the analysis, the conical Euler equations in a spherical coordinate system are used (see Fig. 3.2), where the transformation form Cartesian to spherical coordinates is given by:

\[
x = r \cos \theta ; \quad y = r \sin \theta \cos \phi ; \quad z = r \sin \theta \sin \phi
\]  \hspace{1cm} (3.8)

and the velocity components \( u \), \( v \) and \( w \) in the spherical coordinate system are:

\[
u = \frac{dr}{dt} ; \quad v = r \frac{d\theta}{dt} ; \quad w = r \sin \theta \frac{d\phi}{dt}
\]  \hspace{1cm} (3.9)

The conical Euler equations are derived from the three-dimensional Euler equations, assuming that all variables are independent of \( r \). For an isenthalpic flow these equations may be written as:
• conservation of mass
\[
\left(1 - \frac{v^2}{c^2} \right) \frac{\partial v}{\partial \theta} - \frac{vw}{c^2} \frac{\partial w}{\partial \theta} - \frac{1}{\sin \theta} \frac{v w}{c^2} \frac{\partial v}{\partial \phi} + \frac{1}{\sin \theta} \left(1 + \frac{w^2}{c^2} \right) \frac{\partial w}{\partial \phi} = \frac{u}{c^2} (w^2 - v^2) + v \cot \theta + 2u = 0
\]  
(3.10a)

• r-momentum
\[
v \frac{\partial u}{\partial \theta} + \frac{w}{\sin \theta} \frac{\partial u}{\partial \phi} = v^2 - w^2 = 0
\]  
(3.10b)

• θ-momentum
\[
v \frac{\partial v}{\partial \theta} + \frac{w}{\sin \theta} \frac{\partial v}{\partial \phi} - w^2 \cot \theta + uv + \frac{c^2}{\gamma \rho} \frac{\partial \rho}{\partial \theta} = 0
\]  
(3.10c)

• φ-momentum
\[
v \frac{\partial w}{\partial \theta} + \frac{w}{\sin \theta} \frac{\partial w}{\partial \phi} + uv \cot \theta + uw + \frac{c^2}{\gamma \rho \sin \theta} \frac{\partial \rho}{\partial \phi} = 0
\]  
(3.10d)

where \(u, v\) and \(w\) are the velocity components in \(r, \theta\) and \(\phi\)-direction, \(c\) the speed of sound and \(p\) the static pressure. This system of equations is closed by the energy equation in algebraic form:
\[
h = \frac{c^2}{\gamma - 1} + \frac{1}{2} (u^2 + v^2 + w^2) = \frac{\gamma p}{\rho (\gamma - 1)} + \frac{1}{2} (u^2 + v^2 + w^2) = \text{Constant}
\]  
(3.11)

From the general energy equation and the other equations, some equations for the entropy can be derived (Bakker 1973):
\[
v \frac{\partial s}{\partial \theta} + \frac{w}{\sin \theta} \frac{\partial s}{\partial \phi} = 0
\]  
(3.12a)
\[
T \frac{\partial s}{\partial \phi} = v \frac{\partial w}{\partial \theta} \sin \theta - u \frac{\partial u}{\partial \phi} - v \frac{\partial v}{\partial \phi} + w \sin \theta (u + v \cot \theta)
\]  
(3.12b)

where \(s\) is the entropy, \(T = \frac{c^2}{\gamma R}\) the static temperature and \(R\) the gas-constant. From Eq. (3.12a) it can be seen the entropy remains constant in the direction \(\frac{d \phi}{d \theta} = \frac{w}{v \sin \theta}\), which is the direction of a conical streamline.

The conical flow in the vicinity of a corner will be analyzed and it is assumed that the corner point is a conical stagnation point when the flow velocity is directed to the corner line. In order to develop an analytical description of the flow in the neighbourhood of a conical stagnation point, some series expansions for the velocity components and the pressure are developed and substituted into the governing equations (3.10a–3.10d). These equations will be made dimensionless by scaling the velocities with a speed of sound \(c_0\) and the pressures with the pressure \(p_0\). For these quantities the values in the conical stagnation point are used. The stagnation point will be placed at \(\theta = 0\). For the stagnation point yields that \(u = u_0 \neq 0, v = w = 0\) and \(p = p_0 \neq 0\), so we can derive the following
expansions from the stagnation point:

\[
\begin{align*}
\bar{u} & = \frac{u}{c_0} = u_0 + u_1(\phi)\theta^a + \ldots \\
\bar{v} & = \frac{v}{c_0} = v_1(\phi)\theta^b + \ldots \\
\bar{w} & = \frac{w}{c_0} = w_1(\phi)\theta^c + \ldots \\
\bar{p} & = \frac{p}{p_0} = 1 + p_1(\phi)\theta^d + \ldots
\end{align*}
\]

(3.13)

The exponents \(a, b, c\) and \(d\) are positive constants, and the functions of \(\phi\) are smooth. Various combinations of \(a, b, c\) and \(d\) are investigated by van Horssen (1984), which will not be treated here. One of these combinations, which gave physical relevant solutions (separation and reattachment), is:

\(a = 2, b = c = 1\) and \(d = 2\). Substitution of these expansions into Eqs. (3.10a)–(3.11) and skipping the higher order terms gives:

\[
\begin{align*}
2u_0 + 2v_1 + \frac{dw_1}{d\phi} & = 0 \quad (3.14a) \\
2u_1v_1 - v_1^2 - w_1^2 + w_1 \frac{du_1}{d\phi} & = 0 \quad (3.14b) \\
u_0v_1 + v_1^2 - w_1^2 + w_1 \frac{dv_1}{d\phi} + \frac{2}{\gamma} p_1 & = 0 \quad (3.14c) \\
u_0w_1 - \frac{1}{\gamma} \frac{dp_1}{d\phi} & = 0 \quad (3.14d)
\end{align*}
\]

This is a system of ordinary nonlinear differential equations with respect to \(\phi\). This system can be rescaled by choosing a new parameter \(t\), which is defined by \(dt = \frac{1}{w_1} d\phi\). In this way the derivatives with respect to \(\phi\) can be replaced by:

\[
\frac{dg}{d\phi} = \frac{dg}{dt} \frac{dt}{d\phi} = \frac{\dot{g}}{w_1}
\]

(3.15)

so we obtain the following set of ordinary differential equations (with a reordering of the equations):

\[
\begin{align*}
\dot{u}_1 & = -2w_1(u_0 + v_1) \quad (3.16a) \\
\dot{v}_1 & = -u_0v_1 - v_1^2 + w_1^2 - \frac{2}{\gamma} p_1 \quad (3.16b) \\
\dot{p}_1 & = \gamma u_0 w_1^2 \quad (3.16c) \\
\dot{w}_1 & = -2u_1v_1 + v_1^2 + w_1^2 \quad (3.16d) \\
\dot{\phi} & = v_1 \quad (3.16e) \\
\dot{\theta} & = v_1 \theta \quad (3.16f)
\end{align*}
\]

The last equation (3.16f) is derived from Eq. (3.15) and the definition for a streamline \(\frac{d\phi}{d\theta} = \frac{w}{\nu \sin \theta}\).

The equations (3.16a–3.16c) do not depend on \(u_1, \theta\) or \(\phi\), so they can be decoupled from the equations (3.16d–3.16f). This means that only the first three equations have to be solved simultaneously for \(u_1, v_1\) and \(p_1\), after which \(u_1, \theta\) and \(\phi\) can be determined.
For the construction of a flow field with a separation, two areas are defined at each side of the separation streamline. For both areas separately a solution will be determined, which will then be matched. The situation is shown in Fig. 3.3, where the two flow fields are bounded by the lines 1 and 2 and the lines 3 and 2 respectively; the lines 1 and 3 denote the surface, while line 2 denotes the separation streamline. The intersection of these three lines is the conical stagnation point, where the cross-flow velocity components vanish. It is obvious that for a separation, the flow on the surface must be directed towards the stagnation point, while the flow must leave the surface on the separation streamline. This is shown in Fig. 3.3 by the boundary conditions on the different lines:

- upstream side:
  - line 1: $v_1 < 0 \land w_1 = 0$
  - line 2: $v_1 > 0 \land w_1 = 0$

- downstream side:
  - line 3: $v_1 < 0 \land w_1 = 0$
  - line 2: $v_1 > 0 \land w_1 = 0$

The matching conditions for a flat surface are:

- $|\phi_1 - \phi_2| + |\phi_2 - \phi_3| = \omega_{12} + \omega_{23} = \pi$

- Static pressure ($p_1$) on line 2 must be equal for both areas.

This means that solutions have to be found in a ($p_1, v_1, w_1$) space, which start at the plane $w_1 = 0$, have a trajectory where $w_1 \neq 0$ and end at the plane $w_1 = 0$. From Eq. (3.16a) it is clear that in general solution trajectories starting in the plane $w_1 = 0$ will never leave this plane, because $w_1 = 0$ everywhere in this plane. The only possibility for trajectories leaving the plane $w_1 = 0$ are singular points, where:

$$\dot{w}_1 = 0 \land \dot{v}_1 = 0 \land \dot{p}_1 = 0$$

(3.17)

The singular points of system Eqs. (3.16a)–(3.16c) are the intersection of the surface: $u_0 v_1 + v_1^2 - w_1^2 - \frac{2}{\gamma} p_1 = 0$ and the plane $w_1 = 0$. The singular points are not isolated, but form a singular line, which is the parabola $u_0 v_1 + v_1^2 + \frac{2}{\gamma} p_1 = 0$ in the plane $w_1 = 0$ with top $(w_1, v_1, p_1) = (0, -\frac{1}{2} u_0, \frac{1}{2} \gamma u_0^2)$ (see Fig. 3.4). The character of the solutions in the neighbourhood of the singular points is determined by an analysis of the linearized system around the singular point $(\ddot{w}_1, \ddot{v}_1 = 0, \ddot{p}_1)$. Substitution of $v_1 = \ddot{v}_1 + v^*$, $w_1 = w^*$ and $p_1 = \ddot{p}_1 + p^*$ into Eqs. (3.16a)–(3.16c) gives:

$$\dot{w}^* = -2 w^*(u_0 + \ddot{w}_1) - 2 w^* v^*$$

$$\dot{v}^* = -2 p^* - (u_0 + 2 \ddot{v}_1) v^* + (w^*)^2 - (v^*)^2$$

$$\dot{p}^* = \gamma u_0 (w^*)^2$$

(3.18)
and skipping the higher order terms, the following linear system is obtained:

\[
\begin{align*}
\dot{w}^* &= -2w^* (u_0 + \hat{v}_1) \\
\dot{v}^* &= -(u_0 + 2\hat{v}_1)v^* \\
\dot{p}^* &= 0
\end{align*}
\]  

(3.19)

The character of the singular points in the \(w_1, v_1\) plane is determined by the eigenvalues of system Eq. (3.19). These eigenvalues, \(\lambda_1 = -2(u_0 + \hat{v}_1)\) and \(\lambda_2 = -(u_0 + 2\hat{v}_1)\) are real, so we have the following structures (see Fig. 3.4):

- if \(\hat{v}_1 < -u_0\): unstable node, where trajectories depart
- if \(\hat{v}_1 = -u_0\): higher order singularity
- if \(-u_0 < \hat{v}_1 < -\frac{1}{2} u_0\): saddle point
- if \(\hat{v}_1 = -\frac{1}{2} u_0\): higher order singularity
- if \(-\frac{1}{2} u_0 < \hat{v}_1\): stable node, where trajectories arrive

The system Eq. (3.19) can be solved analytically, which gives the following expressions for the trajectories in the \(w_1, v_1\)-plane:

\[
\begin{align*}
w^* &= C_1e^{-2(u_0 + \hat{v}_1)t} \\
v^* &= C_2e^{-(u_0 + 2\hat{v}_1)t}
\end{align*}
\]  

(3.20)

or eliminating the \(t\) variable:

\[
w^* = C \cdot |v^*| \left(\frac{2u_0 + 2\hat{v}_1}{u_0 + 2\hat{v}_1}\right)
\]  

(3.21)

where \(C_1, C_2\) and \(C\) are arbitrary constants. Using the nonlinearized \(p_1\)-equation from Eq. (3.18) the following expression for the pressure can be obtained:

\[
p^* = -\frac{7u_0}{4(u_0 + \hat{v}_1)}(w^*)^2
\]  

(3.22)
Several remarks about the solutions of the system Eqs. (3.16a)–(3.16c) can be made. From the analysis of the structure of the singular points on the parabola in the \( w_1 = 0 \)–plane (Fig. 3.4), it is clear that trajectories depart from the parabola in the area \( v_1 < -u_0 \) and arrive at the parabola in the area \( v_1 > -\frac{1}{2} u_0 \). Furthermore, from Eq. (3.16a) it is clear that the trajectories do not cross the plane \( w_1 = 0 \). This means that the trajectories lie in either the half space \( w_1 < 0 \) or \( w_1 > 0 \). Two different types of flow are possible. First we have a node-type structure, given by the trajectories which start at the parabola at \( v_1 < -u_0 \) and arrive at the parabola at \(-\frac{1}{2} u_0 < v_1 < 0\). It is clear that, since the azimuthal velocity \((v)\) does not change sign, these solutions do not fulfill the boundary conditions for a separation flow as given in Fig. 3.3. The second type of flow is a saddle point structure. In this case the trajectories start at \( v_1 < -u_0 \) and arrive at the parabola at \( v_1 > 0 \). From the boundary conditions given in Fig. 3.3 it is clear that the points corresponding to the separation line (2) must lie on the parabola in the area \( v_1 > 0 \) and \( p_1 < 0 \), while the points corresponding to the surface streamlines must lie on the parabola in the area \( v_1 < -u_0 \) and \( p_1 < 0 \). From Eq. (3.16c) it can be seen that \( p_1 \) is a monotonic function along a trajectory, from which it may be concluded that trajectories arriving at the parabola at a certain \((p_1)_2\) in the \( p_1 < 0 \) area, come from a point with \( p_1 \leq (p_1)_2 \). Similarly, the angle \( \phi \) is a monotonic function, which can be seen from Eq. (3.16e). The difference in \( \phi \) between beginning and end point gives the angle \( \omega \) between surface and separation streamline.

### 3.2.1 Vorticity distribution and potential flow

The vorticity vector in a spherical coordinate system is given by:

\[
\vec{\Omega} = \text{rot} \vec{q} = \frac{1}{r} \left( \begin{array}{c}
\frac{1}{\sin \theta} \left( \frac{\partial w}{\partial \phi} \sin \theta + w \cos \theta - \frac{\partial v}{\partial \theta} \right) \\
\frac{1}{\sin \theta} \left( \frac{\partial u}{\partial \phi} - r \sin \theta \frac{\partial w}{\partial r} - w \sin \theta \right) \\
r \frac{\partial v}{\partial r} + v - \frac{\partial u}{\partial \theta}
\end{array} \right)
\]  

(3.23)

with \( \vec{q} = (u, v, w)^T \). This vector becomes for conical flow and small \( \theta \):

\[
r \vec{\Omega} = \left( \begin{array}{c}
\frac{1}{\theta} \left( \frac{\partial w}{\partial \theta} \theta + w - \frac{\partial v}{\partial \phi} \right) \\
\frac{1}{\theta} \left( \frac{\partial u}{\partial \phi} - w \theta \right) \\
v - \frac{\partial u}{\partial \theta}
\end{array} \right)
\]  

(3.24)

Substitution of the system Eq. (3.13) and using the system Eqs. (3.16a)–(3.16d) gives the vorticity vector:

\[
r \vec{\Omega} = \left( \begin{array}{c}
\frac{1}{w_1} \left( w_1^2 + u_0 w_1 + v_1^2 + \frac{2}{\gamma} p_1 \right) \\
\theta \frac{v_1}{w_1} (v_1 - 2u_1) \\
\theta (v_1 - 2u_1)
\end{array} \right)
\]  

(3.25)
3.2 Modelling of separation in conical flow with distributed vorticity

The system of differential equations has a subset of potential flow solutions. Using conical stagnation flow solutions of the full potential equations, reported by Bakker (1991), the following analytical expressions for coefficients of Eq. (3.13) can be derived:

\[
\begin{align*}
  u_1 &= a_2 \cos 2\phi - \frac{1}{2} u_0 \\
  v_1 &= 2a_2 \cos 2\phi - u_0 \\
  w_1 &= -2a_2 \sin 2\phi \\
  p_1 &= -\gamma (2a_2^2 - a_2 u_0 \cos 2\phi)
\end{align*}
\]

(3.26)

It is easily verified that these expressions are a solution of (3.14a–3.14d), and since \( w = w_1 = 0 \)
for \( \phi = \cdots, -\pi, -\frac{1}{2} \pi, 0, \frac{1}{2} \pi, \pi \cdots \) these solutions have orthogonal separatrices. The constant \( a_2 \) is a parameter, which can be related to \( v_1 \) at the separation line \( (\phi = \frac{1}{2} \pi): \ a_2 = -\frac{1}{2} (u_0 + (v_1)_2). \) These trajectories \( (0 \leq \phi \leq \frac{1}{2} \pi) \) are formed by the intersection of a paraboloid given by:

\[
f_1 (w_1, v_1, p_1) = u_0 v_1 + v_1^2 + w_1^2 + \frac{2}{\gamma} p_1 = 0
\]

(3.27)

and the plane surface given by:

\[
f_2 (w_1, v_1, p_1) = 2 v_1 + \gamma u_0 p_1 = 2(v_1)_2 - \gamma^2 \frac{u_0}{2} (v_1)_2 (u_0 + (v_1)_2)
\]

(3.28)

Substitution of the expressions Eq. (3.26) into the vorticity vector Eq. (3.25), shows that these trajectories are a set of potential solutions, since the vorticity is zero.

Using relation Eq. (3.25), the vorticity in the flow field can be obtained. Special attention, however, has to be paid to the vorticity vector in the singular points at the paraboloid in the \( w_1 = 0 \) plane, where the vorticity becomes undetermined. The behaviour of the vorticity vector near the singular points can be analysed by substitution of the solutions of the linearized system. Substitution of \( v_1 = \hat{v}_1 + v^* \), \( w_1 = w^* \) and \( p_1 = \hat{p}_1 + p^* \) and Eq. (3.22), with \( u_0 \hat{v}_1 + (\hat{v}_1)^2 + \frac{2}{\gamma} \hat{p}_1 = 0 \) into the first component of the vorticity vector gives:

\[
r \Omega_1 = \frac{1}{w_1} \left( w_1^2 + u_0 v_1 + v_1^2 + \frac{2}{\gamma} p_1 \right) = \left( 1 - \frac{u_0}{2 (u_0 + \hat{v}_1)} \right) w^* + \frac{v^*}{w^*} (u_0 + v_1 + v^*)
\]

(3.29)

We have to determine the limit solution for \( w^* \to 0 \land v^* \to 0 \). Using the solution Eq. (3.20) gives the following possibilities:

- **Curves not on the paraboloid \( (v^* \neq 0, C_2 \neq 0) \):**
  - for \( v_1 < -u_0 \): \( \Omega_1 = \lim_{t \to -\infty} \frac{C_2}{C_1} (u_0 + \hat{v}_1) e^{u_0 t} = 0 \)
  - for \( v_1 > -\frac{1}{2} u_0 \): \( \Omega_1 = \lim_{t \to +\infty} \frac{C_2}{C_1} (u_0 + \hat{v}_1) e^{u_0 t} = \infty \)

- **Curves on the paraboloid \( (v^* = 0, C_2 = 0) \):**
  - for all \( v_1 \): \( \Omega_1 = 0 \)

From this analysis it can be concluded that all trajectories departing from the parabola at \( v_1 < -u_0 \) have a zero vorticity if the requirement \( v_1 - 2u_1 = 0 \) at the starting point is fulfilled; these points
correspond to the surface streamlines (1) and (3) in Fig. 3.3. On the other hand, the trajectories arriving at the $v_1 > -\frac{1}{2} u_0$ side of the parabola have an infinite vorticity, except the trajectories on the paraboloid itself; these points correspond to the separation streamline (2) in Fig. 3.3.

From Eq. (3.26) it is clear that the trajectories on the paraboloid create a separation in an irrotational flow with an angle $\omega_s = 90^\circ$. Trajectories inside of the paraboloid create separation flows with an opening angle $\omega_s < 90^\circ$, while trajectories outside of the paraboloid create separation flows with $\omega_s > 90^\circ$. Making appropriate combinations of inner and an outer solutions, local descriptions of separating flows with a certain required angle (for flat surfaces $180^\circ$) can be found.

### 3.2.2 Physical interpretation of coefficients

The different terms in the expansions Eq. (3.13) can be related to some physical quantities. The angle between a streamline on the surface or on the separation surface and the ray from the conical centre is given by:

$$\tan \gamma = \frac{v}{u} = \frac{v_1 \theta}{u_0 + u_1 \theta}$$

(3.30)

from which the curvature $\kappa_s$ of the streamlines at the stagnation point ($\theta = 0 \land r = 1$) can be derived:

$$\kappa_s (\theta = 0) = \frac{d (\tan \gamma)}{d \theta} (\theta = 0) = \frac{v_1}{u_0}$$

(3.31)

From these equations it is clear that $v_1$ is related to the curvature of the streamlines. Furthermore it follows that $p_1 = \frac{d^2 p}{d \theta^2}$, thus giving the second derivative of the static pressure in radial direction. The first derivative of the static pressure in the stagnation point is equal to zero, so the stagnation point is a local extremum. From the fact that $p_1$ is negative, one can conclude that this extremum is a maximum. The curvature of streamlines ($v_1$) and the second derivative of the pressure ($p_1$) at the stagnation point are coupled via the parabolic relation $u_0 v_1 + v_1^2 + \frac{2}{3} p_1 = 0$.

### 3.2.3 Numerical implementation

In general, no analytical solutions of the equations (3.16a–3.16f) have been found. Therefore a numerical method will be used to integrate the system of ordinary differential equations. For this purpose a simple 4-step Runge-Kutta method is used. Using $\vec{x} = (w_1, v_1, p_1)^T$ and $\vec{f}(\vec{x}) = (w_1(\vec{x}), v_1(\vec{x}), \dot{p}_1(\vec{x}))^T$ this process can be written as:

$$\begin{align*}
\vec{k}_1 &= \Delta t \vec{f}(\vec{x}_0) \\
\vec{k}_2 &= \Delta t \vec{f}(\vec{x}_0 + \frac{1}{2} \vec{k}_1) \\
\vec{k}_3 &= \Delta t \vec{f}(\vec{x}_0 + \frac{1}{2} \vec{k}_2) \\
\vec{k}_4 &= \Delta t \vec{f}(\vec{x}_0 + \vec{k}_3) \\
\vec{z}_1 &= \vec{x}_0 + \frac{1}{6} (\vec{k}_1 + 2 \vec{k}_2 + 2 \vec{k}_3 + \vec{k}_4) \\
\vec{z}_0 &= \vec{z}_1
\end{align*}$$

(3.32)
It is obvious that the numerical procedure can not start at the singular curve. Therefore the solution procedure is started in a point \( (\theta_1 + \delta \theta, \omega^*, p_1 + \delta p^*) \) close to the singular curve, where \( \delta \theta, \omega^* \) and \( p^* \) are the small quantities from the solution of the linearized system Eqs. (3.20),(3.22). From this point the interpolation is proceeded in both positive and negative \( \tau \)-direction. The process is stopped if \( |w_1| < \epsilon \), where \( \epsilon \) is a small quantity \( (\epsilon \approx 10^{-10}) \). A large positive value of the constant \( C \) from Eq. (3.21) will give a trajectory outside, but close to the paraboloid with a separation angle close to 90°. On the other hand a small positive value of \( C \) will give a trajectory outside of the paraboloid with a separation angle approaching 180°. Starting with negative values of \( C \) will lead to trajectories inside of the paraboloid, with separation angles less than 90°. The relation between the coefficients \( C \) and the separation angle \( \omega \) is not known a priori, although it is close to the relation: \( \tan C \approx \tan \omega \). The desired value of \( C \) is found by a simple iteration procedure, which uses the results of previous calculations in order to make a new estimate.

3.2.4 Results

A three-dimensional view of the paraboloid with some trajectories on it, denoted by the black lines, is given in Fig. 3.5. Some trajectories arriving at the same point at the positive \( v_1 \)-side, but making different separation angles are shown in a top-view (along the \( w_1 \)-axis) and in a front-view (along the \( p_1 \)-axis) in Fig. 3.6. From this we can see that if the separation angle increases, also the difference between the curvatures \( (v_1) \) and the second derivatives of the pressure \( (p_1) \) between the surface streamline and the separation streamline increases. In Fig. 3.7, Fig. 3.8 and Fig. 3.9 the distribution of \( w_1, v_1 \) and \( p_1 \) is shown as a function of the angle \( \phi \) for the separation angles 30°, 60° and 90° respectively. The total scaled vorticity \( \| r \Omega \| \) for these situations is plotted as a function of the roll angle \( \phi \) in Fig. 3.10. It can be observed that the vorticity level decreases if the separation angle goes to an angle of 90°, the irrotational flow solution. Some streamlines and isobars for the cases \( \omega_s = 30° \) and \( \omega_s = 60° \) are plotted in Fig. 3.11 and Fig. 3.12 respectively. Remarkable are the highly curved streamlines at the downstream side of the separation and the low pressure gradients in that area.

Figure 3.5: Three-dimensional view of paraboloid with some trajectories
A large number of calculations have been made for different separation angles and curvatures of streamlines on the surface. The whole separation configuration can be fixed by a combination of two independent parameters. For example, choosing a curvature $(v_1)_3$ on the windward side on the surface and the separation angle $\omega_{23}$, the other parameters $((v_1)_2,(v_1)_1,(p_1)_1,(p_1)_2)$ are fixed. The results of the calculations are shown in a diagram with the parameters $(v_1)_1$ and $(v_1)_2$ on the axes (Fig. 3.13). In this figure lines of constant $(v_1)_2$ and $\omega$ are plotted. The relation between the coefficient $C$, which determines the starting curve, and the separation angle $\omega$ is given in Fig. 3.14 for various values of the streamline curvature on the separation surface $(v_1)_2$. This figure shows that for a large curvature of the streamlines on the separation surface $((v_1)_2)$ indeed the relation $\tan C \approx \tan \omega$ is obtained.
Figure 3.7: Distribution of velocity and pressure for $\omega_s = 30^\circ$

Figure 3.8: Distribution of velocity and pressure for $\omega_s = 60^\circ$
Figure 3.9: Distribution of velocity and pressure for $\omega = 90^\circ$

Figure 3.10: Distribution of scaled total vorticity
Figure 3.11: Streamlines and isobars for $\omega_z = 30^\circ$

Figure 3.12: Streamlines and isobars for $\omega_z = 60^\circ$
Figure 3.13: Diagram of possible separations

Figure 3.14: Relation between coefficient $C$ and separation angle $\omega$
Chapter 4

DISCRETIZATION OF THE EULER EQUATIONS

4.1 Governing equations

The governing equations of motion for a non-viscous and non-heatconducting gas are the Euler equations. The Euler equations, expressing conservation of mass, momentum and energy for a compressible perfect gas, will be formulated in conservative form. In a Cartesian coordinate system the Euler equations in a conservative differential form are given by:

\[
\frac{\partial Q}{\partial t} + \frac{\partial F(Q)}{\partial x} + \frac{\partial G(Q)}{\partial y} + \frac{\partial H(Q)}{\partial z} = 0
\]

(4.1)

where \( Q \) is the vector of the conserved variables:

\[
Q = (\rho, \rho u, \rho v, \rho w, \rho e)\]

(4.2)

and \( F(Q) \), \( G(Q) \), and \( H(Q) \) are the flux vectors, given by:

\[
F(Q) = (\rho u, \rho u^2 + p, \rho u v, \rho u w, \rho u e)\]
\[
G(Q) = (\rho v, \rho u v, \rho v^2 + p, \rho v w, \rho v e)\]
\[
H(Q) = (\rho w, \rho u w, \rho v w, \rho w^2 + p, \rho w e)\]

(4.3)

Here \( \rho \) is the density; \( u, v, w \) are the Cartesian velocity components in the \( x, y, z \) directions respectively; \( p \) is the static pressure; \( e \) is the total energy per unit of mass given by \( e = e + \frac{1}{2}(u^2 + v^2 + w^2) \), in which \( e \) is the internal energy per unit of mass; \( h \) is the total enthalpy given by \( h = e + p/\rho \). For a calorically perfect gas the equation of state may be expressed as:

\[
p = (\gamma - 1)\rho e
\]

(4.4)

in which the ratio of specific heats \( \gamma = c_p/c_v \) may be considered as constant \( (\gamma = 1.4) \). These equations fully describe the three-dimensional inviscid perfect gas flow.

Solutions of the Euler equations in general may contain discontinuities (shock waves, shear layers). Since the differential form expressed by Eq. (4.1) is not valid at these discontinuities, the equations are written in an integral form, in which discontinuities are captured as "weak" solutions:

\[
\int \int \int_V \frac{\partial Q}{\partial t} dV + \int \int_S (F(Q) \cdot n_x + G(Q) \cdot n_y + H(Q) \cdot n_z) dS = 0
\]

(4.5)

where \( \vec{n} = (n_x, n_y, n_z) \) (with \( |\vec{n}| = 1 \)) is the outward unit normal vector on the boundary \( S \) of the control volume \( V \). Making use of the invariance property of the Euler equations under rotation of the coordinate system, equation Eq. (4.5) can be simplified with:

\[
F(Q) \cdot n_x + G(Q) \cdot n_y + H(Q) \cdot n_z = T^{-1}(\theta, \phi) F(T(\theta, \phi) Q)
\]

(4.6)

where \( T(\theta, \phi) \) is the rotation matrix, which transforms the momentum components of the state vector \( Q \) to a new Cartesian \( \vec{x}, \vec{y}, \vec{z} \) coordinate system in which the \( \vec{z} \)-axis is aligned with the unit normal on
the control volume boundary. This new Cartesian coordinate system has the base vectors \( \vec{n} \), \( \vec{s} \) and \( \vec{t} \) in \( \vec{z}, \vec{y} \) and \( \vec{z} \)-direction respectively (Fig. 4.1), which are:

\[
\begin{align*}
\vec{n} &= (n_x, n_y, n_z)^T = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi)^T \\
\vec{s} &= (s_x, s_y, s_z)^T = (-\sin \theta, \cos \theta \cos \phi, \cos \theta \sin \phi)^T \\
\vec{t} &= (t_x, t_y, t_z)^T = (0, -\sin \phi, \cos \phi)^T
\end{align*}
\]

The rotation matrix \( T \) is defined as:

\[
T = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & n_x & n_y & n_z & 0 \\
0 & s_x & s_y & s_z & 0 \\
0 & t_x & t_y & t_z & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \cos \theta & \sin \theta \cos \phi & \sin \theta \sin \phi & 0 \\
0 & -\sin \theta & \cos \theta \cos \phi & \cos \theta \sin \phi & 0 \\
0 & 0 & -\sin \phi & \cos \phi & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]  \hspace{1cm} (4.8)

4.2 Finite volume discretization

A straightforward and simple discretization of Eq. (4.5) with the substitution of Eq. (4.6) for a subdivision of the control volume \( V \) into disjunct cells \( V_{ijk} \) (finite volumes) is:

\[
\Delta V_{ijk} \frac{dQ_{ijk}}{dt} + \sum_{m=1}^{N_f} T^{-1}(\theta_{ijk,m}, \phi_{ijk,m}) F(T(\theta_{ijk,m}, \phi_{ijk,m}) Q_{ijk,m}) \Delta S_{ijk,m} = 0 \hspace{1cm} (4.9)
\]

where \( \Delta V_{ijk} \) is the volume of cell \( V_{ijk} \), \( Q_{ijk} \) is the mean value of \( Q \) over \( V_{ijk} \) and is collocated at the centre of the finite volume. The second part of the equation is the summation of the total fluxes normal to the surface \( \Delta S_{ijk,m} \) of the \( N_f \) cell faces of \( V_{ijk} \). This total flux is assumed to be constant over the cell face. For practical reasons (simple implementation) a structured grid with hexahedral cells is used, where \( V_{ijk}, V_{ijk+1} \) and \( V_{ijk+1} \) are the neighbouring cells of \( V_{ijk} \) (Fig. 4.2). Thus the second (steady) part of Eq. (4.9) may be written as (with each unit normal in positive \( i, j \) or \( k \)
4.2 Finite volume discretization

\[ F_{i,j,k} = \sum_{m=1}^{N_f} T^{-1}(\theta_{i,j,k,m}, \phi_{i,j,k,m}) F(T(\theta_{i,j,k,m}, \phi_{i,j,k,m}) Q_{i,j,k,m}) \Delta S_{i,j,k,m} = \]

\[ \bar{F}_{i+\frac{1}{2},j,k} \Delta S_{i+\frac{1}{2},j,k} - \bar{F}_{i-\frac{1}{2},j,k} \Delta S_{i-\frac{1}{2},j,k} + \bar{F}_{i+\frac{1}{2},j,k} \Delta S_{i+\frac{1}{2},j,k} - \bar{F}_{i+\frac{1}{2},j,k} \Delta S_{i-\frac{1}{2},j,k} \]  

(4.10)

where \( \bar{F}_{i+\frac{1}{2},j,k} \) = \( T^{-1}(T_{i+\frac{1}{2},j,k} Q_{i+\frac{1}{2},j,k}) \) and similar expressions for \( \bar{F}_{i-\frac{1}{2},j,k}, \bar{F}_{i+\frac{1}{2},j,k}, \bar{F}_{i+\frac{1}{2},j,k}, \) and \( \bar{F}_{i-\frac{1}{2},j,k} \). The surface \( \Delta S_{i+\frac{1}{2},j,k} \) is the interface of the cells \( V_{i,j,k} \) and \( V_{i+1,j,k} \). The matrix \( T_{i+\frac{1}{2},j,k} = T(\theta_{i+\frac{1}{2},j,k}, \phi_{i+\frac{1}{2},j,k}) \), where the angles \( \theta_{i+\frac{1}{2},j,k} \) and \( \phi_{i+\frac{1}{2},j,k} \) define the unit-normal \( n_{i+\frac{1}{2},j,k} \) on \( \Delta S_{i+\frac{1}{2},j,k} \) in positive i-direction.

The total flux vectors \( \bar{F}_{i+\frac{1}{2},j,k} \) etc. in Eq. (4.10) have to be calculated by some numerical flux function. For the calculation of the numerical flux some functions belonging to the family of upwind schemes are used. These upwind schemes are based on the Godunov principle. This means that the intercell flux is computed by means of an (approximate) solution of the one-dimensional Riemann problem applied to a direction normal to the control volume interfaces, and hence introducing some physical properties of the flow into the discretized formulation. With respect to a Cartesian frame \((x, y, z)\) the 1D-Riemann problem is formulated as:

\[ \frac{\partial Q}{\partial t} + \frac{\partial F(Q)}{\partial x} = 0 \]  

(4.11)

where \( Q \) and \( F \) are given by Eq. (4.2) and Eq. (4.4). The exact solution of this problem is a similarity solution \((Q(x, t) = Q_{ex}(x/t))\), and as a consequence the state at \( x = 0 \) \((Q_{ex}(0))\) is constant for all \( t > 0 \). The exact solution of the Riemann problem requires the solution of a nonlinear algebraic equation by a numerical procedure, which is too expensive for practical computations. Therefore several approximate solutions have been developed. Three different types of schemes are to be
considered here; the flux-vector-splitting scheme of van Leer (1982) and flux-difference-splitting schemes of Osher (1982) and Roe (1981). For the interface $S_{i+\frac{1}{2},jk}$ the numerical flux function $F_{NFF}$ may be written in the form:

$$\hat{F}_{i+\frac{1}{2},jk} = T_{i+\frac{1}{2},jk}^{-1} F_{NFF}(T_{i+\frac{1}{2},jk} Q_{i+\frac{1}{2},jk}^L, T_{i+\frac{1}{2},jk} Q_{i+\frac{1}{2},jk}^R)$$  (4.12)

where $Q_{i+\frac{1}{2},jk}^L$ and $Q_{i+\frac{1}{2},jk}^R$ are the states at either side of the cell interface, obtained from an interpolation between some states $Q_{ijk}$ in the centres of the finite volumes. For example, in a spatially first order accurate system, the states are assumed constant within each volume, so we get $Q_{i+\frac{1}{2},jk}^L = Q_{ijk}$ and $Q_{i+\frac{1}{2},jk}^R = Q_{i+1,jk}$.

### 4.2.1 The van Leer scheme

In the flux vector splitting method the flux terms are split and discretized directionally according to the sign of the associated propagation speeds. The original van Leer flux vector splitting (van Leer 1982) is given by (with the introduction of the Mach number $M_\varepsilon = u/c$):

- if $M_\varepsilon \leq -1$: $F^+ = 0 \land F^- = F$
- if $M_\varepsilon \geq 1$: $F^+ = F \land F^- = 0$
- if $|M_\varepsilon| < 1$: $F^\pm = F_{\text{sub}}^{\pm} = \left(\begin{array}{c} 1 \\ \frac{2c}{\gamma} \left(\frac{\gamma - 1}{2} M_\varepsilon \mp 1\right) \\ v \\ w \\ \frac{2c^2}{\gamma^2 - 1} \left(1 \mp \frac{\gamma - 1}{2} M_\varepsilon \right)^2 + \frac{1}{2}(v^2 + w^2) \end{array}\right)$  (4.13)

where $F_{\text{sub}}^{\pm} = \pm \frac{\rho c}{4} (M_\varepsilon \pm 1)^2$. For the computations described in this report a modification to the energy component of the split flux is used, as proposed by Hanel (1987). He introduced a valid splitting of the energy component, which preserves a constant total enthalpy of the flow:

$$F_{\text{sub}}^S = \pm \frac{\rho c}{4} (M_\varepsilon \pm 1)^2 h_4$$  (4.14)

The numerical flux function for the flux-vector-splitting scheme is taken to be:

$$F_{VL}(Q^L, Q^R) = F^+(Q^L) + F^-(Q^R)$$  (4.15)

### 4.2.2 The Osher scheme

A more refined approach is delivered by schemes of the flux-difference-splitting type (Osher (1982), Roe (1981)). The major difference between these schemes and the van Leer scheme is the fact that contact discontinuities are captured with a higher resolution, especially when they are aligned with grid cell faces. The Osher scheme has the advantage over the Roe scheme that the flux function is
continuously differentiable, so that it can be used within a Newton method. Following Osher we can write the approximate solution of the Riemann problem as:

\[ F_{ARS}(Q^L, Q^R) = F^+(Q^L) + F^-(Q^R) \]

\[ = F(Q^R) - F^+(Q^R) + F^+(Q^L) = F(Q^R) - \int_{Q^L}^{Q^R} A^+(Q) \, dQ \]  \hfill (4.16a)

\[ = F(Q^L) - F^-(Q^L) + F^-(Q^R) = F(Q^L) + \int_{Q^L}^{Q^R} A^-(Q) \, dQ \]  \hfill (4.16b)

\[ = \frac{1}{2} \left( F(Q^L) + F(Q^R) - \int_{Q^L}^{Q^R} |A(Q)| \, dQ \right) \]  \hfill (4.16c)

where \( A(Q) = \frac{\partial F}{\partial Q} \), \( A^+(Q) = \frac{\delta F^+}{\delta Q} \), \( A^-(Q) = \frac{\delta F^-}{\delta Q} \), and \( |A(Q)| = A^+(Q) - A^-(Q) \). The essence of the method lies in the way the integrals are evaluated. In Osher's approach the integrals are computed along a path in the state space, \( Q = Q(s); \ 0 \leq s \leq 1 \), where \( Q(0) = Q_0 \) and \( Q(1) = Q_1 \) (see Fig. 4.3). This path is divided into a sequence of subpaths \( \Gamma_k \), \( k = 1, 2, 3 \), each one

\[ Q_{1/3} \quad \Gamma_2 \quad Q_{2/3} \]

\[ Q^L = Q_0 \quad Q^R = Q_1 \]

<table>
<thead>
<tr>
<th>path:</th>
<th>( \Gamma_1 )</th>
<th>( \Gamma_2 )</th>
<th>( \Gamma_3 )</th>
</tr>
</thead>
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<tr>
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<td>( u )</td>
<td>( u + c )</td>
</tr>
<tr>
<td>Riemann invariants:</td>
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<td>( u )</td>
<td>( \frac{u - 2}{\gamma - 1} )</td>
</tr>
<tr>
<td></td>
<td>( z = \ln \left( \frac{p}{p_1^r} \right) )</td>
<td>( p )</td>
<td>( z )</td>
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Figure 4.3: Definition of Osher path

connecting the states \( Q_{(k-1)/3} \) and \( Q_{k/3} \). Each subpath is supposed to be tangential to an eigenvector \( R_m \) of \( A(Q) \), which makes the evaluation of the integrals very easy (Spekreijse 1987; Osher and Solomon 1982). Formally there are five subpaths, since there are five independent eigenvectors, but the subpaths tangential to the eigenvectors with the corresponding constant eigenvalue \( u \) may be replaced by a single path (\( \Gamma_2 \)). Following the (physical)-variant, introduced by Spekreijse (1987), the paths are visited in order of increasing eigenvalue. The unknown intermediate states \( Q_{1/3} \) and \( Q_{2/3} \) are computed by means of the Riemann invariants, which are constant along a certain subpath. The different eigenvalues and Riemann invariants along the integration paths are defined in Fig. 4.3. Using these invariants, we obtain 10 linear equations for the 10 unknowns of \( Q_{1/3} \) and \( Q_{2/3} \), which can be
easily solved. The state vectors are expressed in the variables \(c, u, v, w\) and \(z\), where \(c = \sqrt{\gamma pp^{-1}}\) is the speed of sound, and \(z = \ln(p \rho^{-\gamma})\) is an scaled entropy. This selection of dependent variables makes the evaluation of Osher’s numerical flux function efficient. The intermediate state vectors \(Q = (c, u, v, w, z)^T\) are:

\[
Q_{1/3} = \left( \frac{\gamma - 1}{2} \Psi_0 - \Psi_1, \frac{\Psi_1 + \alpha \Psi_0}{1 + \alpha}, v_0, w_0, z_0 \right)^T
\]

\[
Q_{2/3} = \left( \frac{\alpha}{2} \Psi_0 - \Psi_1, \frac{\Psi_1 + \alpha \Psi_0}{1 + \alpha}, v_1, w_1, z_1 \right)^T
\]

(4.17a)

(4.17b)

where

\[
\Psi_0 = u_0 + \frac{2}{\gamma - 1} c_0 ; \quad \Psi_1 = u_1 - \frac{2}{\gamma - 1} c_1 ; \quad \alpha = \exp \left( \frac{z_1 - z_0}{2\gamma} \right)
\]

(4.18)

Along the subpaths \(\Gamma_1\) and \(\Gamma_3\) the eigenvalues may change sign (only once because the eigenvalue \(\lambda\) is a monotonic function), resulting in sonic points \(Q^S_1\) and \(Q^S_3\) along \(\Gamma_1\) and \(\Gamma_3\) respectively. The states \(Q^S_1\) and \(Q^S_3\) are also computed from the Riemann invariants along the subpaths \(\Gamma_1\) and \(\Gamma_3\) with the additional relations \(u_1^S - c_1^S = 0\) and \(u_3^S + c_3^S = 0\) respectively. This gives:

\[
Q^S_1 = \left( \frac{\gamma - 1}{\gamma + 1} u_0 + \frac{2}{\gamma + 1} c_0, \frac{\gamma - 1}{\gamma + 1} u_0 + \frac{2}{\gamma + 1} c_0, v_0, w_0, z_0 \right)^T
\]

\[
Q^S_3 = \left( \frac{\gamma - 1}{\gamma + 1} u_1 + \frac{2}{\gamma + 1} c_1, \frac{\gamma - 1}{\gamma + 1} u_1 + \frac{2}{\gamma + 1} c_1, v_1, w_1, z_1 \right)^T
\]

(4.19a)

(4.19b)

If we define the following eigenvalues in the different endpoints of the subpaths (with \(u_{1/3} = u_{2/3} = u_{1/2}\) : \(\lambda_0 = u_0 - c_0\), \(\lambda_{1/3} = u_{1/2} - c_{1/3}\), \(\lambda_{1/2} = u_{1/2} + c_{2/3}\) and \(\lambda_1 = u_1 + c_1\), Osher’s approximate Riemann solver can be written as (Spekreijse 1987):

\[
F_{OSH}(Q_0, Q_1) = \begin{cases} F(Q_0) & \text{(if } \lambda_0 \geq 0 \text{ then)} \\ \text{sign}(\lambda_{1/3}) F(Q_1) & \text{(if } \lambda_{1/3} \leq 0 \text{ then)} \\ \text{sign}(\lambda_{1/2}) F(Q_1) & \text{(if } \lambda_{1/2} \leq 0 \text{ then)} \\ \text{sign}(\lambda_1) F(Q_1) & \text{(if } \lambda_1 \leq 0 \text{ then)} \end{cases}
\]

(4.20)

4.3 Higher order spatial discretizations

The order of accuracy of the spatial discretization is determined by the way the states \(Q^L_{i+\frac{1}{2}jk}\) and \(Q^R_{i+\frac{1}{2}jk}\) for the numerical flux evaluation Eq. (4.12) are calculated. As mentioned before a scheme with first order accuracy is achieved by assuming constant states in each volume:

\[
Q^L_{i+\frac{1}{2}jk} = Q_{ijk} ; \quad Q^R_{i+\frac{1}{2}jk} = Q_{i+1jk}
\]

(4.21)

First order accuracy, however, is too low for practical applications and discontinuities not aligned with the grid are smeared out disastrously. As has been noted by van Leer (1977) the order of accuracy
can be improved by using a more accurate interpolation to calculate the different components $q$ of the state vectors $Q$ at both sides of a cell face, which can be written in a general form as:

\[
q^{L}_{i+\frac{1}{2},jk} = q_{ijk} + \frac{1}{4} \left\{ (1 + \kappa) \Delta_i + (1 - \kappa) \nabla_i \right\}
q^{R}_{i-\frac{1}{2},jk} = q_{ijk} - \frac{1}{4} \left\{ (1 - \kappa) \Delta_i + (1 + \kappa) \nabla_i \right\}
\]  \hspace{1cm} (4.22)

where $\Delta_i = q_{i+1,jk} - q_{ijk}$ and $\nabla_i = q_{ijk} - q_{i-1,jk}$ and $\kappa \in (-1, 1)$. Similar formulae yield in the other coordinate directions. This expression contains for instance the following difference schemes: ($\kappa = -1$) the one-sided second-order scheme; ($\kappa = 0$) the Fromm scheme; ($\kappa = 1/3$) the "third-order" accurate upwind biased scheme; ($\kappa = 1$) the central difference scheme. These schemes, however, suffer from spurious non-monotonicity (wiggles or over- and undershoots). To overcome this a limiter is introduced, which has the properties of second order accuracy in the smooth part of the flow field and steepening of discontinuities without introducing non-monotonicity. In the present method the van Albada limiter (van Albada et al. 1982) is used. For the van Albada limiter the interpolation functions are:

\[
q^{L}_{i+\frac{1}{2},jk} = q_{ijk} + \frac{R^{L}_{i,jk}}{4} \left\{ (1 + \kappa R^{L}_{i,jk}) \Delta_i + (1 - \kappa R^{L}_{i,jk}) \nabla_i \right\}
q^{R}_{i-\frac{1}{2},jk} = q_{ijk} - \frac{R^{R}_{i,jk}}{4} \left\{ (1 - \kappa R^{R}_{i,jk}) \Delta_i + (1 + \kappa R^{R}_{i,jk}) \nabla_i \right\}
\]  \hspace{1cm} (4.23)

where the limiter function $R^{\ell}_{i,jk}$ in $i$-direction is defined as:

\[
R^{\ell}_{i,jk} = \frac{2 \Delta_i \nabla_i + 2 \epsilon^2}{\Delta_i^2 + \nabla_i^2 + 2 \epsilon^2}
\]  \hspace{1cm} (4.24)

The constant $\epsilon$ is a small number ($\epsilon \approx 10^{-7}$), which is made proportional to the grid size ($\epsilon \sim (\text{Constant} \Delta x)^{3/2}$), where $\Delta x$ is a characteristic mesh width (Venkatakrishnan 1993). This constant has a twofold objective: first to avoid dividing by zero, and second to switch off the limiting in near-constant regions of the flow. In regions of near-constant flow where $\epsilon^2$ is dominant over $(\Delta_i)^2$ and $(\nabla_i)^2$ the unlimited $\kappa = 0$ scheme is recovered. The computations in this report with the van Albada limiter are performed with $\kappa = 0$, which corresponds to the Fromm scheme.

### 4.4 Boundary treatment

The treatment of the boundary conditions is very consistent to the numerical flux calculations at the internal cell walls following the Osher scheme (Spekreijse 1987; Osher and Chakravarthy 1981). This is a consequence of the fact that the Osher scheme is based on the Riemann invariants, which is also the case with a proper treatment of boundary conditions.

The complete procedure is as follows. The flux at the boundary of the domain will be determined partially by the state vector ($Q_0$ or $Q_1$) near the boundary and partially by the boundary conditions. Let us consider the case where the state $Q_1$ is the state at the boundary in a rotated Cartesian frame with the $x$-axis in a direction normal to the boundary surface (Fig. 4.4). The first step is to determine the state vector $Q_B$ at the boundary, depending on $Q_0$ and the prescribed boundary conditions. Then the numerical flux function $F_{NFF}(Q_0, Q_B)$ gives the boundary flux vector. The state at the boundary $Q_B$ is determined such that it satisfies the boundary conditions and the equality:

\[
F(Q_B) = F_{NFF}(Q_0, Q_B)
\]  \hspace{1cm} (4.25)
With Eq. (4.16b) it is clear that the last requirement implies:

$$\int_{Q_0}^{Q_B} A^+(Q) \, dQ = 0 \quad (4.26)$$

This means that $Q_B$ should be connected with $Q_0$ by a path such that the eigenvalues $\lambda_k$ satisfy:

$\lambda_k \leq 0$ along $\Gamma_k$. The number of subpaths depends on the type of the boundary conditions, which is in fact equivalent to the number of outgoing characteristics. The endpoints of $\Gamma_k$ are again computed by means of the Riemann invariants. The following types of boundary conditions are distinguished:

- **supersonic inflow** ($u_0 < -c_0$): all boundary conditions ($u_B, v_B, w_B, c_B, z_B$) are prescribed.

- **subsonic inflow** ($-c_0 < u_0 < 0$): 4 boundary conditions ($u_B, v_B, w_B, c_B$) are prescribed. There is one interior point $Q_i$; $Q_0$ and $Q_i$ are connected by $\Gamma_1$ and $Q_i$ and $Q_B$ are connected by $\Gamma_2$, so we have the relations:

$$u_0 + \frac{2}{\gamma - 1} c_0 = u_i + \frac{2}{\gamma - 1} c_i \; ; \; v_0 = v_i \; ; \; w_0 = w_i \; ; \; z_0 = z_i \quad (4.27)$$

$$u_i = u_B \; ; \; p_i = p_B$$

Together with the 4 boundary conditions, the states $Q_i$ and $Q_B$ can be calculated as follows:

$$c_i = \frac{\gamma - 1}{2} (u_0 - u_B) + c_0$$

$$\rho_i = \exp \left\{ \left( \ln \left( \frac{c_i^2}{c_1^2} \right) - z_0 \right) \frac{1}{\gamma - 1} \right\}$$

$$\rho_B = \frac{\gamma p_B}{c_B^2} = \frac{\gamma p_i}{c_i^2} = \rho_i \frac{c_i^2}{c_B^2} \quad (4.28)$$

$$p_B = \frac{\rho_B c_B^2}{\gamma}$$

$$z_B = \ln p_B - \gamma \ln \rho_B$$

- **subsonic outflow** ($0 < u_0 < c_0$): 1 boundary condition ($p_B$) prescribed. The states $Q_0$ and $Q_B$ are connected by the integral path $\Gamma_1$, so we have the relations:

$$u_B + \frac{2}{\gamma - 1} c_B = u_0 + \frac{2}{\gamma - 1} c_0 \; ; \; v_B = v_0 \; ; \; w_B = w_0 \; ; \; z_B = z_0 \; ; \; p_B \quad (4.29)$$
from which the resulting components can be obtained:

\[
\rho_B = \exp\left\{ \left( \ln(p_B) - x_0 \right) \frac{1}{\gamma} \right\}
\]

\[
c_B = \sqrt{\frac{\gamma p_B}{\rho_B}}
\]

\[
u_B = u_0 + \frac{2}{\gamma - 1} (c_0 - c_B)
\]

- **supersonic outflow**: no boundary conditions prescribed, \( Q_B = Q_0 \).

- **solid wall**: this can be treated in two ways:

  1. Prescribe \( u_B = 0 \); this is equivalent to the treatment of subsonic outflow. The states \( Q_0 \) and \( Q_B \) are connected by the integral path \( \Gamma_1 \), so we have:

\[
u_B + \frac{2}{\gamma - 1} c_B = u_0 + \frac{2}{\gamma - 1} c_0 \; ; \; v_B = v_0 \; ; \; w_B = w_0 \; ; \; z_B = z_0
\]

from which \( c_B \) can be obtained:

\[
c_B = \frac{\gamma - 1}{2} u_0 + c_0
\]

2. Treat the solid wall as a symmetry plane, so we have:

\[
u_B = -u_0 \; ; \; v_B = v_0 \; ; \; w_B = w_0 \; ; \; c_B = c_0 \; ; \; z_B = z_0
\]

With a correct treatment of the flux at a boundary wall, the boundary flux vector should only contain a pressure term, \( F_{NFF} (Q_0, Q_B) = (0, p_W, 0, 0, 0)^T \). Using method (1), this is not the case for the van Leer and the Roe schemes (for all \( Q_0 \) except the case \( u_0 = 0 \)), and for the Osher scheme if \( u_0/c_0 > 1 \) (which will only occur in the initial phase of the computation of a supersonic flow around a blunt body). Using method (2), this will give physically correct results for the van Leer and the Roe scheme, and also for the Osher scheme for \( u_0/c_0 > 1 \), although the requirement of Eq. (4.25) is not fulfilled. Furthermore, in the case of the Osher scheme for \( u_0/c_0 \leq 1 \) method (1) and (2) will give the same pressure \( p_W \) on the wall. Therefore method (2) is preferred.

In the previous part the state vector \( Q_0 \) is assumed to be known. For the first-order accurate discretized equations it is obvious that the state vector \( Q_0 \) is equal to the state vector in the centre of the boundary cell. For higher order accurate discretized equations, however, \( Q_0 \) should be obtained from the interpolation procedure described in section 4.3. Suppose that the cell face \( \partial Q_{i+\frac{1}{2}jk} \) is a part of the boundary of the computational domain. In that case the states \( Q_{i+\frac{1}{2}jk}^{L} \) and \( Q_{i-\frac{1}{2}jk}^{R} \) are determined by the states \( Q_{i-1jk}, Q_{ijk} \) and \( Q_{i+1jk} \). The higher order interpolation functions can not be applied in this case, because the state vector \( Q_{i+1jk} \) corresponds to a cell outside of the computational domain, which is not a part of the flow field. As an alternative, a simple extrapolation and interpolation of the two internal states could be used to obtain the states \( Q_{i+\frac{1}{2}jk}^{L} \) and \( Q_{i-\frac{1}{2}jk}^{R} \) respectively. A more sophisticated approach, which is as far as possible consistent with the interpolation procedure in the internal field, will be followed in this report. Several different situations with corresponding treatment can be distinguished.
• Internal boundary
  In cases where a boundary volume is connected to an other boundary volume (e.g. cells at the cutting plane in the wake of a grid around an airfoil), the limiter interpolations can be performed with the data from those neighbouring volumes.

• Symmetry plane
  Many flow problems have a symmetry plane. This means that it is sufficient to consider only a half-space, and thus reducing the computational time. In the case of a symmetry plane the scalar quantities as the pressure and the density or the speed of sound and the entropy are copied to the state vector outside of the domain and the velocity vector is copied according to the reflection principle (see Fig. 4.5):

  \[
  (u_n)_0 = -(u_n)_1 ; (v_n)_0 = (v_n)_1 ; (w_n)_0 = (w_n)_1 ; c_0 = c_1 ; z_0 = z_1 \quad (4.34)
  \]

  Figure 4.5: Determination of state at symmetry plane

  With the state vector outside of the domain defined, the limiter functions can be applied to calculate the state vector at the boundary.

• Solid wall
  For a finite volume discretization of the Euler equations only the pressure at the cell face on a solid wall is required. Several methods use a pressure extrapolation from internal cells towards the surface, which may be based on a discretization of the normal momentum equation (Rizzi 1978). Here a method is used, which is consistent with the internal flux evaluation. For solid boundaries without curvature, the calculation of the external state \(Q_0\) (see Fig. 4.5) is equivalent to the treatment of a symmetry plane. For curved boundaries, however, the normal momentum equation may be taken into account; but this will not be done in the present report.

• Inflow and outflow
  In any other case simple linear interpolations are used. If \(\partial \Omega_{i+\frac{1}{2}jk}\) is a part of the boundary domain, simple inter- and extrapolation formulae are used:

  \[
  q_{i-\frac{1}{2}jk}^R = q_{ijk} - \frac{1}{2}(q_{ijk} - q_{i-1,jk}) ; \quad q_{i+\frac{1}{2}jk}^R = q_{ijk} + \frac{1}{2}(q_{ijk} - q_{i-1,jk}) \quad (4.35)
  \]

  In most situations these boundaries are located in smooth (undisturbed) regions of the flow, where the van Albada limiter functions Eq. (4.23) reduce to similar linear interpolations.

4.5 Solution procedure

In the original code, the system of nonlinear discretized equations is solved by means of a multigrid technique. For the computations in this report about the conical vortical flow around elliptic cones
this procedure appeared to be inefficient, so this procedure will not be described here. Consider the first- or second order accurate discretization of the Euler equations given by equation Eq. (4.9) to be written as:

$$\Delta V_m \frac{\partial Q_m}{\partial t} + F_m(Q_m) = 0$$
(4.36)

The solution procedure used here is based on an implicit time integration method, which can also be used as a smoothing method within the multigrid method. For the system of equations Eq. (4.36) a backward time-integration method can be written as:

$$\Delta V_j \frac{\Delta Q_j^{n+1}}{\Delta t} = -F(Q_j^{n+1})$$
(4.37)

where \( F(Q_j^{n+1}) \) denotes the spatial discretization evaluated at time level \( n + 1 \), and \( \Delta Q_j^{n+1} = Q_j^{n+1} - Q_j^n \). Because Eq. (4.37) is a system of non-linear equations, this can not be solved directly. Therefore a Newton linearization is used, which can be written as:

$$F(Q_j^{n+1}) = F(Q_j^n) + \left[ \frac{\partial F}{\partial Q} \right]_j \Delta Q_j^{n+1}$$
(4.38)

Substitution of Eq. (4.38) into Eq. (4.37) with \( \mathcal{H} = \left[ \frac{\partial F}{\partial Q} \right] \) gives:

$$\left[ \frac{\Delta V}{\Delta t} I - \mathcal{H} \right]_j^n \Delta Q_j^{n+1} = -F(Q_j^n)$$
(4.39)

For the limit \( \Delta t \to \infty \) Newton's root finding method is obtained, which should theoretically lead to quadratic convergence if the Jacobian matrix \( \mathcal{H} \) is evaluated correctly. The system Eq. (4.39) represents a large banded block matrix, whose bandwidth is dependent on the order of accuracy of the spatial discretization and on the dimensions of the grid. Especially for the three-dimensional second-order discretized equations the bandwidth is very large. The construction of this matrix and the solution of the system requires an enormous amount of memory and CPU-time, which goes far beyond the capacities of most computers. Rather than solve Eq. (4.39) directly, a number of strategies have been developed in order to reduce the computational work, but maintaining as far as possible a high convergence rate. Requiring second order accurate steady solutions, it is common practice to replace the true Jacobian matrix \( \mathcal{H} \) in the left hand side of Eq. (4.39) by a much simpler matrix \( \mathcal{H}^1 \) based on the first-order accurate equations. For steady flows this has no effect on the accuracy of the right hand side discretization. The matrix for a three-dimensional first-order system is a septadiagonal block matrix, where the blocks itself are \( 5 \times 5 \)-matrices. This system, however, is certainly for three-dimensional problems still too large to solve directly, so most implicit methods use iterative methods. One of the oldest methods to solve the system Eq. (4.39) iteratively is an approximate factorization such that each time step involves alternating direction sweeps across the domain, and only requires the solution of uncoupled block tridiagonal systems (Anderson et al. 1988). As an alternative several variants of standard relaxation can be used (Edwards and McRae 1992; Schröder and Hartmann 1992). In contrast with approximate factorization methods, relaxation methods in general are not subjected to a stability condition on the time-step. Schröder and Hartmann (1992) have reported that the overall performance of the Gauss-Seidel relaxation method is much better than the approximate factorization method. In this report some versions of the Gauss-Seidel relaxation method are used.
4.6 Conical flow mode

The Euler code is developed for the simulation of three-dimensional flows, but can also be used to calculate flows with conical similarity. For that purpose the code is equipped with a mode to calculate conical flows at reduced costs. In this mode grids are used with only one volume in the $k$-index direction, while the corner points of the volumes at constant $i$, $j$-indices lie on a straight line from the conical centre (see Fig. 4.6). Due to the conical similarity, the state vectors at the cell boundaries $S_{ijk-\frac{1}{2}}$ and $S_{ijk+\frac{1}{2}}$ are identical to the state vector at the cell centre:

$$Q_{ijk-\frac{1}{2}} = Q_{ijk+\frac{1}{2}} = Q_{ijk} \quad (4.40)$$

This makes the flux evaluation at these cell boundaries straightforward:

$$\bar{F}_{ijk-\frac{1}{2}} \Delta S_{ijk-\frac{1}{2}} = T_{ij,k-\frac{1}{2}}^{-1} F(T_{ij,k-\frac{1}{2}} Q_{ijk}) \Delta S_{ijk-\frac{1}{2}} \quad (4.41a)$$

$$\bar{F}_{ijk+\frac{1}{2}} \Delta S_{ijk+\frac{1}{2}} = T_{ij,k+\frac{1}{2}}^{-1} F(T_{ij,k+\frac{1}{2}} Q_{ijk}) \Delta S_{ijk+\frac{1}{2}} \quad (4.41b)$$

Figure 4.6: Finite volume in conical mode
Chapter 5

NUMERICAL SIMULATION OF CONICAL FLOWS

5.1 Computational grid

The test case chosen for this investigation is a conical flow around a cone with an elliptic cross-section. The computations will be performed with a zero slip angle, so only a half-space has to be considered. Due to the conical similarity only one grid plane in a certain cross-plane has to be constructed; the other plane follows from a linear scaling. The elliptic cone configuration with a semi apex angle $\theta_c$, measured in the $x-y$-plane and the minor-to-major axis ratio $S$ ($S = B/A$), is given in Fig. 5.1. The technique used to generate grids, is an algebraic technique employing a series of conformal mappings. This process is showed in Fig. 5.2. The basic grid in the complex $z_1$-plane consists of circles, where the inner boundary being the unit circle, is given by:

$$z_1 = \rho e^{i\phi} ; \quad 1 \leq \rho \leq \rho_c \quad \wedge \quad 0 \leq \phi \leq \pi$$

(5.1)

The $\rho$-spacing is substituted to a distribution function based on hyperbolic goniometric functions (Thompson 1987), with which the mesh width ratios near the surface and the outer boundary can be adjusted. This grid is mapped to the $z_2$-plane by the conformal transformation:

$$z_2 = \frac{z_1 - \sigma}{1 - \sigma z_1}$$

(5.2)

This mapping translates the centres of the circles along the imaginary axis, except the unit circle. The next mapping is a mirroring along the line $z_2 = r e^{i\pi}$:

$$z_3 = \frac{\|z_2\|^2 i}{z_2}$$

(5.3)
Figure 5.2: Conformal mappings for grid generation

The final mapping is given by the Joukowski transformation and a scaling, in order to transform the circles into ellipses:

\[ z_4 = \left( z_3 + \frac{a^2}{z_3} \right) \frac{A}{1 + a^2} \]  \hspace{1cm} (5.4)

For a grid in the \( z = \text{Constant} \) plane, the coordinates \( y \) and \( z \) are:

\[ y = \mathcal{R}(z_4) \hspace{1cm} z = \mathfrak{I}(z_4) \]  \hspace{1cm} (5.5)

In order to obtain a grid for the specified cone geometry and the outer grid boundaries at the \( z \)-axis (\( H_1 \) and \( H_2 \) in Fig. 5.1) the following parameters are used:

\[ \rho_e = \frac{\sigma + w_2}{1 + \sigma w_2} \]  \hspace{1cm} (5.6a)

\[ \sigma = \frac{-(1 + w_1 w_2) - \sqrt{(1 - w_1^2)(1 - w_2^2)}}{w_1 + w_2} \]  \hspace{1cm} (5.6b)

\[ w_1 = \frac{1}{2} \left( h_1 - \sqrt{h_1^2 + 4a^2} \right) \]  \hspace{1cm} (5.6c)

\[ w_2 = \frac{1}{2} \left( h_2 + \sqrt{h_2^2 + 4a^2} \right) \]  \hspace{1cm} (5.6d)

\[ a = \sqrt{\frac{1 - S}{1 + S}} \]  \hspace{1cm} (5.6e)

\[ h_1 = \left( 1 + a^2 \right) \frac{H_1}{A} \]  \hspace{1cm} (5.6f)

\[ h_2 = \left( 1 + a^2 \right) \frac{H_2}{A} \]  \hspace{1cm} (5.6g)
The outer grid boundary parameters $H_1$ and $H_2$ can be adjusted to the bow shock wave for supersonic flow cases. A first order estimate is given by:

\begin{align}
H_1 &= -\tan (\mu_\infty - \alpha_\infty + S\theta_c) \\
H_2 &= \tan (\mu_\infty + \alpha_\infty + S\theta_c)
\end{align}

(5.7a)  (5.7b)

where $\mu_\infty = \arcsin \left( \frac{1}{M_\infty} \right)$ is the Mach angle and $\alpha_\infty$ is the angle of incidence.

5.2 Characteristics of leading edge separation and vortical flow

A number of test cases is selected for which the computations are performed. The basic cone geometry has a sweep angle $\Lambda$ of 65° (or $\theta_c = 25^\circ$), which corresponds to the delta wing used in the International Vortex Flow Experiment (Elsenaar and Eriksson 1987). The test cases selected are located in the regions (0), (1) or (2) of the Stanbrook-Squire diagram (Fig. 2.4). In these cases the leeward flow is not complicated by the existence of embedded shocks, which cause vorticity in the flow field and eventually shock-induced separation.

A number of simulations has been performed for the subsonic free-stream Mach number $M_\infty = 0.5$. A cross-sectional view of the grid used for these cases is shown in Fig. 5.3. Formally there are no conical solutions for pure subsonic flow, but from experimental observations it is known that high-subsonic flow around slender delta wings and cones may be regarded as conical in a large part of the flow field. Only near the apex and the trailing edge, the conical flow assumption is not satisfied. The effect of inconsistent boundary conditions at subsonic flow conditions can be seen in Fig. 5.4, where the total pressure in the flow field along the $y$-axis is plotted for different cones and outer
Figure 5.4: Distribution of total pressure along y-axis for different cones

grid boundary positions. These results are obtained from numerical simulations for a flow at zero angle of incidence around two different elliptic cones with a minor-to-major axis ratio $S = B/A$ of 0.1 and 1 respectively. The total pressure distribution shows a jump from the free stream level ($p_1 / p_{\infty}$ = 1) to a higher (unphysical) level in the first cell at the outer boundary. The results show that this pressure jump becomes smaller when the radius of the outer grid boundary is decreased and when the minor-to-major axis ratio of the ellipse is decreased. Besides the total pressure all other quantities show a jump in the first cell at the outer boundary of the grid. This has also a consequence for the pressure level on the surface, which is not calculated correctly. This means that calculations of subsonic conical flows are of no practical interest. The global behaviour of the flow, however, is identical to the flow in a certain cross-section of a simulation of the three-dimensional flow around a delta wing, and these simulations can be used to investigate aspects of vortical flow and the behaviour of separation models at low computational costs. The distribution of the total pressure ($p_1 / p_{\infty}$) and some conical streamlines is given in Fig. 5.5 for the flow around an flat plate delta wing at an angle of incidence $\alpha = 10^\circ$. Due to the sharp leading edge the flow separates and rolls up into a vortex. Within the shear-layer originating at the leading edge and the vortex a significant loss of total pressure ($\pm 25\%$) is calculated, which is characteristic for simulations of vortex flows with discrete Euler methods.

The existence of a sharp edge is not a prerequisite for the occurrence of separation. Simulations of flows around elliptic cones with a finite thickness show that above a certain angle of incidence leading edge separation occurs. An example for the flow around a cone with a thickness of 4\% at $10^\circ$ angle of incidence is shown in Fig. 5.6. In this figure some conical streamlines and lines of constant magnitude of vorticity $| \tau \Omega |$ are plotted. The maximal vorticity occurs at the leading edge near the position of separation. Furthermore a local maximum of vorticity appears in the vortex core. The separation,
Figure 5.5: Total pressure distribution and conical streamline pattern, $M_\infty = 0.5$, $\alpha = 10^\circ$, $B/A = 0.0$

Figure 5.6: Conical streamlines and lines of constant $|r\Omega|$, $M_\infty = 0.5$, $\alpha = 10^\circ$, $B/A = 0.04$
however, does not take place at the leading edge where the vorticity reaches a maximum, but at a small distance downstream. This can be observed in a detail plot of the leading edge region given in Fig. 5.7. The vorticity is transported along the shear layer, but the magnitude decreases away from the leading edge. The separation seems to leave the body surface with an angle close to $90^\circ$. This is confirmed by an analysis of the topology of the flow in the proximity of the singular point. This analysis uses local bilinear interpolations of the cross-flow velocities from Eq. (2.2) between 4 cell-centres. The eigenvalues and eigenvectors of the linearised system of the cross-flow give insight in the character of the stagnation points and the direction of the separatrices. It appears that the separation angle at the leading edge is $81^\circ$. For this subsonic Euler flow, without origins of vorticity like shock waves, the separation should leave the body tangentially (section 3.1). However, due to discretization errors, vorticity is created, which will make a non-tangential separation possible (section 3.2). In the case of an infinitely thin wing, the leading edge separation takes place at the leading edge, which is also the place where the maximal vorticity occurs (see Fig. 5.8). The separation also leaves the body tangential to the surface.

The leading edge region is very interesting for the study of the behaviour of the flow near the separation point. In order to make a clear distinction between the lower and the upper side of the cone, the behaviour of some velocity components at the surface will not be shown as a function of the spanwise coordinate $y/y_{le}$, but as a function of the scaled distance $s$ along the surface in a cross-section. This coordinate is defined in Fig. 5.9, where the symmetry plane at the lower (windward) surface corresponds to $s = -\frac{1}{2}$, the leading edge to $s = 0$ and the symmetry plane at the upper (leeward) surface to $s = \frac{1}{2}$. The distribution of the axial velocity $u_{ax}/c_{\infty}$, the component along the conical ray, and the cross velocity $u_{\phi}/c_{\infty}$, the component normal to conical ray and tangential to surface (see Fig. 5.9), for a case without separation ($\alpha = 2.5^\circ$) and a case with separation ($\alpha = 10^\circ$)
Figure 5.8: Detail of leading edge region, $M_{\infty} = 0.5$, $\alpha = 10^\circ$, $B/A = 0.0$

Figure 5.9: Definition of velocities in surface coordinate system
is shown in Fig. 5.10. In this figure the positions of the attachment at the windward side ($A_0$), the
primary separation ($S_1$) and the corresponding primary attachment ($A_1$) are indicated at the points
where the cross-velocity $u_x$ changes sign. A detail of the leading edge region is shown in Fig. 5.11.
At the leading edge large gradients in the cross-velocity occur, which will together with the high
streamline curvature lead to losses of entropy and cause separation. For the separated flow case the
distribution of the axial velocity $u_a$ shows a jump at the position of the primary separation.

The effect of the grid size is shown in Fig. 5.12, where the surface pressure distribution for the same
geometry and flow conditions with grid sizes of $48 \times 32, 96 \times 64$ and $192 \times 128$ cells is given. Due to
the grid refinement the pressure induced by the vortex decreases, while area of the $c_p$ peak becomes
smaller. At the leading edge the pressure becomes lower and the gradients are higher on the finer
grids. A similar behaviour can be observed in Fig. 5.13, where the magnitude of the vorticity $|r\Omega|$ is
plotted along a grid line through the vortex core, as indicated in the small sub-figure. The magnitude
of the vorticity in the vortex core and in the shear layer emanating from the leading edge increases,
while the width of the areas with vorticity decreases on the finer grids. Also an increase of loss of
total pressure (Fig. 5.14) on the finer grids can be observed. This may be explained by the fact that the
decrease of discretization errors on the finer grids are counterbalanced by the increase of the errors
due to the higher gradients.

A parametric study has been performed, which includes the effects of the leading edge curvature, the
grid dimensions and the angle of incidence. These studies have been performed for the subsonic flow
case ($M_{\infty} = 0.5$). The object was to study influence of the different parameters on the occurrence of
separation and vortices in a flow around an elliptic cone. In order to study the effect of the leading
edge curvature, calculations are performed for various minor-to-major axis ratios of the elliptic cross-
section. These calculations are done for three different grid sizes, $48 \times 32, 96 \times 64$ and $192 \times 128$ cells
Figure 5.11: Distribution of $u_{zz}/c_\infty$ and $u_z/c_\infty$ on surface in leading edge region, medium and fine grid.

Figure 5.12: Pressure distribution on elliptic cones with different grid sizes.
Figure 5.13: Distribution of total vorticity along curve through vortex core

Figure 5.14: Distribution of total pressure along curve through vortex core
at angles of incidence up to 10°. The results of these calculations are summarized in Fig. 5.15, where

![Diagram illustrating primary separation on elliptic cone, \( M_\infty = 0.5 \)]

Figure 5.15: Onset of primary separation on elliptic cone, \( M_\infty = 0.5 \)

the boundaries between attached and separated flows are given as a function of the minor-to-major axis ratio \( (B/A) \) and the angle of incidence for the three different grid sizes. From this figure it can be observed that for a decreasing leading edge curvature (increasing \( B/A \)), separation occurs at an increasing angle of incidence. For the thin cones \( (B/A < 0.025) \) separation takes place at lower angles of incidence for the finer grids. At the higher \( B/A \)-ratios there is not much difference for the three grids investigated. A possible explanation for this behaviour may be that the level of numerical dissipation remains constant due to the fact that the possible decrease of the truncation errors by grid refinement is cancelled by the higher gradients in the flow variables.

Finally some examples of shock-induced separation will be shown. This type of separation will be demonstrated by simulations of the supersonic flow \( (M_\infty = 1.79) \) around a circular cone \( (\theta_c = 5^\circ) \) at high angle of attack \( (\alpha = 15^\circ \text{ and } 25^\circ) \). The topology of the flow pattern for these two cases may be observed in Fig. 5.16, where some conical streamlines are shown. For the lower angle of attack case \( (\alpha = 15^\circ) \) the nodal singularity at the leeward generator, which exists at low angles of attack, has lift off from the surface. The streamline pattern shows three conical stagnation points in the leeward symmetry plane; a nodal point on the surface, and a saddle point and a nodal point above the surface. The conical Mach number (Mach number based on the velocity component normal to a conical ray) remains subsonic in the whole flow field between the surface and the bow-shock for this angle of attack, which restrains the formation of embedded cross-flow shocks. If the angle of attack is increased to \( \alpha = 25^\circ \), the conical Mach number reaches supersonic values near the surface \( (((u_d/c)_{max} = 2.096) \) and a strong embedded shock (indicated in Fig. 5.16) is formed. The flow pattern at the leeward side changes due to the appearance of a saddle point on the surface (separation) and a spiral singularity (Fig. 5.16). This separation, at a short distance downstream of the shock, is
a result of the production of a large amount of vorticity by the embedded shock. The total vorticity distribution for the two cases is given in Fig. 5.17. Due to numerical truncation errors a small amount of vorticity is produced near the surface at $\alpha = 15^\circ$. For $\alpha = 25^\circ$ however, an area with distributed vorticity with a much higher level appears downstream of the embedded shock and in the vortex. The axial ($u_{ax}$) and cross-flow velocity ($u_y$) distribution and the pressure distribution ($-c_p$) at the surface are given in Fig. 5.18. This shows the supersonic cross-flow for the high angle of attack and the cross-flow shock with the separation downstream. The separation is also characterized by a sudden increase of the axial velocity. In the pressure distribution the position of the separation obtained from the cross-flow velocity distribution, is indicated. The separation coincides with a local maximum of the pressure, which is in agreement with the theoretical analysis given in section 3.2. The separation angle, obtained from a bilinear representation of the cross-flow velocity components, is $76^\circ$. No attempts have been made to make a comparison between the numerical solution and the theoretical analysis, because it appeared to be impossible to extract accurate values for the second derivative of the pressure or the first derivative of the cross-flow velocities (analogous to $p_1$ and $v_1$ from Eq. (3.13)) with respect to the surface coordinate.
5.2 Characteristics of leading edge separation and vortical flow

Figure 5.17: Vorticity distribution around circular cone without and with shock-induced separation; $\theta_c = 5^\circ$, $M_{\infty} = 1.79$, $\alpha = 15^\circ, 25^\circ$

Figure 5.18: Distribution of $u_{az}/c_{\infty}$, $u_s/c_{\infty}$ and $-c_p$ at the surface of a circular cone; $\theta_c = 5^\circ$, $M_{\infty} = 1.79$, $\alpha = 15^\circ, 25^\circ$
Chapter 6

FORCED SEPARATION FROM A SMOOTH SURFACE

6.1 Introduction

The specification of a separation position shows an analogy to the Kutta condition in potential flow theory. This condition is used to circumvent the ambiguity of potential solutions, so that infinite velocities at the sharp trailing edge of an airfoil will not appear. The removal of infinite velocities will be achieved by the specification of a certain amount of circulation in the flow around an airfoil. The magnitude of the circulation can be adjusted to let the flow separate at the sharp trailing edge. It is reasonable to assume that the extra condition determines the relevant inviscid flow solution. The Kutta condition usually must be prescribed explicitly in finite-difference methods based on the inviscid flow equations. On the other hand, in finite volume methods based on the Euler equations in general no extra boundary conditions have to be specified at the sharp trailing edge of an airfoil. It appeared that the explicit implementation of a condition similar to the Kutta condition is not necessary, since separation occurs automatically at the sharp trailing edge. Although the mechanism responsible for the implicit Kutta condition is not yet understood, it is in general contributed to a dissipation due to numerical truncation errors. Due to this property finite volume Euler methods deliver the relevant solutions for practical geometries with sharp trailing edges. The implicit existence of a Kutta condition in the numerical discretization, however, has prevented the research to a correct implementation of a Kutta condition in finite volume Euler methods. A basic example of a flow where an explicit Kutta condition is required, is the two-dimensional inviscid flow around an ellipse. Numerical experiments (Pulliam 1990) have shown that various Euler methods produce strong mesh- and parameter-dependent solutions having different values of lift. The mechanism which produces the lifting results is not understood. Also the way how to influence the automatic circulation production remains unknown. This may be regarded as a major drawback of Euler methods, since an important part of the inviscid flow theory has been lost.

Another aspect of flow models based on the Euler equations that has to be mentioned, is that, in contrast with flow models based on the linearized potential equations, the continuous Euler equations are able to capture vortex sheets (contact discontinuities) as an integral part of the solution. However, discretized Euler methods in general will smear out vortex layers over a number of cells, due to the numerical viscosity. The vortex sheet will be captured as a discontinuity in the very special case, when the vortex sheet is perfectly aligned with one of the grid lines and when a discretization scheme is used which can recognize contact discontinuities.

Several methods of prescribing separation from smooth surfaces within Euler methods have been reported (Marconi 1987; Klopper and Nielsen 1981; Kwong and Myring 1989). The outstanding question concerns the implementation of appropriate boundary conditions which enforce separation and which are also in agreement with an analytical framework. Two analytical descriptions are given in chapter 3. Although both descriptions are different, and also give a different behaviour of the flow near the separation line, they have in common that the flow at the separation line at the downstream side is directed along the separation line. The development of a general boundary condition technique for the prescription of separation should be focussed on the specification of the flow direction on the surface along the required separation line. No other quantities such as the direction of the velocity

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6.1 Introduction

Vectors near separation or the amount of vorticity shed off should be specified; the determination of these quantities has to be left to be specified by the solution of the appropriate flow model.

The concept of shedding vorticity from a smooth surface according to the work of Smith (1977) has been used by Marconi (1987) in a conical space-marching Euler method. Separation was forced on various elliptic cones in supersonic flow at specified separation points. The method used was based on a the so called $\lambda$-scheme (Moretti 1979). According to the assumptions of Smith, a double point at the intersection of the vortex sheet and the surface was introduced. At the point at the downstream side of the separation a zero cross-flow velocity (conical stagnation) was prescribed, while the cross-flow velocity at the windward side was determined by the global solution. The cross-flow velocity at the windward side in general remained finite. The pressure across the sheet was continuous. Difficulties were encountered in computing the static pressure in a small region downstream of the separation. Results are shown for slender cones in supersonic flow with prescribed first- and secondary separation for which the separation positions were obtained from experimental results. An improved agreement with experimental pressure distributions was observed for the cases with prescribed separation.

Klopfser and Nielsen (1980) have computed three-dimensional separated flow over tangent ogive bodies at incidence in supersonic flow by specifying the direction of the separation streamline. This procedure was implemented in a space-marching finite-difference Euler method. In contrast with the procedure of Marconi, no double-point was introduced. At the desired separation point on the surface the pressure, density and total enthalpy are obtained from an arithmetic average of the points on either side of the separation surface, while the velocity vector is obtained from the average quantities and two specified flow angles. Remarkable was that the flow direction was chosen such that the velocity vector was directed along the separation surface, and not along the separation line on the body surface. Hence this method of boundary conditions is not in agreement with a theoretical analysis. Parametric studies of these flow angles showed that variation of these angles had little effect on the amount of vortex shedding from the surface. The position of separation, however, has a large effect on the vortex shedding and total circulation of the induced vortex.

Based on the method of Klopfser and Nielsen, a separation model was implemented in a finite-volume Euler code by Kwong and Myring (1989). For this finite volume method separation was forced by the specification of flow quantities in two cells at either side of the separation. In their approach, the pressure and the density in two cells at either side of the desired separation line was calculated by an arithmetic average or an extrapolation from other field points, the magnitude of the velocity was obtained from the constant total enthalpy relationship, while the flow direction was specified by two angles in each point, which are shown in Fig. 6.1. The row of cells in which the flow properties are averaged and extrapolated are located in a direction normal to the separation line. The separation surface in this model is placed between the cells $S$ and $+1$. The pressure and the density in these cells are calculated by some averaging, for which two models are used: the mean-value model and the upwind biased model. The mean value model equations are:

$$F_S = 0.5 \left( F_{S-1} + F_{S+1} \right) ; \quad F_{S+1} = 0.5 \left( F_S + F_{S+2} \right) \quad (6.1)$$

where the quantities $F$ denote the pressure or the density. The main (cross)-flow is directed into the increasing index direction. Using the constant total enthalpy, all quantities are known by the specification of the flow angles at both sides of the separation surface, $\phi_a$, $\phi_c$, $\beta_a$ and $\beta_c$. In order to avoid the linear relationships for pressure and density across four cells, some upwind biasing has been introduced. For the upstream point this is found by taking the mean of the average value and the extrapolated value in the upstream direction. For the downstream point the gradient is assumed to be
zero, which is based on experimental observations. The equations are:

$$F_S = 0.5 \left( F_{av} + F_{extrap} \right) ; \quad F_{S+1} = F_{S+2} \quad (6.2)$$

where the averaged and extrapolated values are:

$$F_{av} = 0.5 \left( F_{S-1} + F_{S+1} \right) ; \quad F_{extrap} = 2 F_{S-1} - F_{S-2} \quad (6.3)$$

These models have been applied to the flow over a circular cone and the flow over a three calibre tangent ogive cylinder. The numerical results showed an improvement in both the surface pressure distribution and the leeside flow field structure. Disadvantage of this model is number of parameters (four flow angles) to be prescribed, for which no physical basis is derived. Also no examples of the effect on the convergence behaviour has been reported. It may be expected that due to this model some inconsistency is introduced in the discretized system.

### 6.2 Model I; Prescription of conical stagnation point

At first a boundary condition procedure was investigated, which is in agreement with physical and analytical observations. Within this model a conical stagnation point on the surface will be prescribed by forcing the flow direction along the conical ray through this point. The separation model is applied to a certain cell boundary on the surface of the model. The flux calculation procedure for this cell boundary is similar to the treatment of a solid wall boundary as described in section 4.4, with one extra condition. Assume that we have a state vector $Q_0$ near the boundary. The flux at this boundary will be determined by this state vector and the state vector $Q_B$. In order to perform the flux calculation, the state $Q_0$ will be calculated in a rotated Cartesian coordinate system, with the rotation matrix Eq. (4.8). In this system the $z$-axis is directed along the normal of the surface. An extra rotation along the $z$-axis is applied, in order to define the state vector in a coordinate system in which the $y$-axis is directed along the conical ray, and thus along the separation line. The modified rotation matrix (for the velocity vector) is given by:

$$T_s = \begin{pmatrix}
\cos \theta & \sin \theta \cos \phi & \sin \theta \sin \phi \\
- \sin \theta \cos \delta & \cos \theta \cos \phi \cos \delta - \sin \phi \sin \delta & \cos \theta \sin \phi \cos \delta + \cos \phi \sin \delta \\
\sin \theta \sin \delta & - \cos \theta \cos \phi \sin \delta - \sin \phi \cos \delta & - \cos \theta \sin \phi \sin \delta + \cos \phi \cos \delta
\end{pmatrix} \quad (6.4)$$
The state vector $\bar{Q}_B$ in the rotated coordinate system is calculated according to the solid wall boundary treatment, given in Eq. (4.33), except the cross-flow velocity $w_B$, which is kept zero. Thus the boundary state is given by:

$$\bar{u}_B = -\bar{u}_0 \ ; \ \bar{v}_B = \bar{v}_0 \ \text{or} \ \sqrt{\bar{v}_0^2 + \bar{w}_0^2} \ ; \ \bar{w}_B = 0 \ ; \ c_B = c_0 \ ; \ \bar{z}_B = z_0 \quad (6.5)$$

This procedure is consistent with the numerical flux calculation at the other boundary cells. For the case of the solid wall the Riemann problem has one in-going characteristic, one outgoing characteristic and three characteristics remain in the surface plane, because the corresponding eigenvalue $u = u_B = 0$. Due to this property we have the freedom to specify a number of boundary conditions which varies from one to four.

The combination of the state vectors $\bar{Q}_0$ and $\bar{Q}_B$ can also be interpreted as a contact discontinuity, because both states have an equal pressure and velocity towards the contact surface. It depends on the type of the approximate Riemann solver if this contact discontinuity is treated correctly. For a correct treatment of the contact discontinuity, a variation in the velocity components tangential to the contact surface ($\bar{v}$ and $\bar{w}$) will not influence the flux at the contact surface (the pressure).

It can easily be shown that the van Leer flux-vector splitting scheme does not recognize contact discontinuities. Assume that $|\bar{u}_0/c_0|<1$, then the van Leer scheme is given by:

$$\vec{F}_{VL} (\bar{Q}_0, \bar{Q}_B) = F_{sub}^+ (\bar{Q}_0) + F_{sub}^- (\bar{Q}_B) \quad (6.6)$$

Substituting Eq. (6.5) into Eq. (4.13) gives the following flux vector at a solid wall:

$$\vec{F}_{VL} (\bar{Q}_0, \bar{Q}_B) = \frac{\rho_0 c_0}{4} \left( \frac{\bar{u}_0}{c_0} + 1 \right)^2 \begin{pmatrix} 0 \\ \frac{4 \bar{v}_0}{c_0} \left( \left( \frac{\gamma - 1}{2} \right) \frac{\bar{u}_0}{c_0} + 1 \right) \\ \bar{v}_0 - \bar{v}_B \\ \bar{w}_0 \\ \frac{1}{2} (\bar{v}_0^2 + \bar{w}_0^2 - \bar{v}_B^2) \end{pmatrix} \quad (6.7)$$

in which splitting errors in the tangential momentum terms and the energy term appear. It is clear that the fourth component in general will not vanish, while either the third or the fifth component can be kept zero by prescribing $\bar{v}_B = \bar{v}_0$ or $\bar{w}_B = \sqrt{\bar{v}_0^2 + \bar{w}_0^2}$ respectively. These non-zero components may be regarded as numerical dissipation terms, which makes the van Leer scheme less accurate in calculating shear-layers. The success of the separation model, however, is a consequence of these splitting errors.

For a more refined approximate Riemann solver, such as the flux-difference-splitting scheme of Osher, the proposed separation model will not work. This can be shown easily as follows. Substitution of Eq. (6.5) into the equations for the Osher scheme and using the assumption that the velocity component normal to the surface is subsonic ($|\bar{u}_0/c_0|<1$), gives the eigenvalues:

$$\lambda_0 = \bar{u}_0 - c_0 < 0 \ ; \ \lambda_{1/3} = c_{1/3} < 0 \ ; \ \lambda_{1/2} = 0 \quad (6.8)$$

$$\lambda_{2/3} = c_{2/3} > 0 \ ; \ \lambda_1 = -\bar{u}_0 + c_0 > 0$$

where the intermediate speeds of sound are given by: $c_{1/3} = c_{2/3} = \frac{\gamma - 1}{2} \bar{u}_0 + c_0$. The flux vector at the solid boundary according to the Osher scheme can then be written as:

$$\vec{F}_{OSH} (\bar{Q}_0, \bar{Q}_B) = \begin{pmatrix} 0 \\ \frac{c_{1/3}^2}{\gamma} \exp \left\{ \frac{1}{\gamma - 1} \left( \ln \left( \frac{c_{1/3}^2}{\gamma} \right) - x_0 \right) \right\} \\ 0 \\ 0 \\ 0 \end{pmatrix}^T \quad (6.9)$$
For this scheme the velocities in the surface (\(\tilde{v}_B\) and \(\tilde{w}_B\)) have no influence on the flux vector, and can be chosen arbitrarily. It is obvious that in this case the separation model will not work.

### 6.2.1 Results

Although the separation model can only be used with a numerical flux calculation based on the van Leer scheme, a number of test cases have been calculated in order to investigate the applicability of the model. Some early results were shown by de Vries (1992). He has shown that this model can be used to force separation at the leading edge of elliptic cones for cases in which no "natural" separation is present. Attempts to prescribe secondary separation at the leeward side of an elliptic cone were not very successful due to problems with the convergence of the multigrid solution procedure. These problems could be resolved by using a single-grid solution procedure. An example of the flow around a cone with a thickness \(B/A = 0.04\) is shown in Fig. 6.2, where the conical streamline pattern is given. Secondary separation is prescribed at a spanwise position of \(y/y_{le} = 0.84\), and the occurrence

![Figure 6.2: Conical streamlines for flow with prescribed secondary separation (Model I), \(M_{\infty} = 0.5, \alpha = 10^\circ, B/A = 0.04\)](image)

of a secondary vortex can be observed in the streamline pattern.

The behaviour of the axial- and the cross-velocity for the case without and with secondary separation is given in Fig. 6.3. The position of the prescribed secondary separation is indicated; it can be seen that the cross velocity becomes zero at that point. Between the secondary separation and the leading edge (\(S = 0\), an area with a positive cross-flow (towards the symmetry plane) occurs. The secondary separation has a remarkable effect on the pressure distribution (Fig. 6.4). The pressure peak induced by the primary vortex is shifted towards the symmetry plane, while the peak height is decreased. The pressure distribution between the secondary separation position and the leading edge shows no
Figure 6.3: Distribution axial- \((u_x/c_\infty)\) and cross-velocity \((u_y/c_\infty)\) on surface without and with prescribed secondary separation (Model I), \(M_\infty = 0.5, \alpha = 10^\circ, B/A = 0.04\)

Figure 6.4: Surface pressure distribution without and with prescribed secondary separation (Model I), \(M_\infty = 0.5, \alpha = 10^\circ, B/A = 0.04\)
suction peak due to the secondary vortex as in the case of the shock-induced secondary vortex on the circular cone (see Fig. 5.18). The number of grid cells in the secondary vortex on the elliptic cone is probably too low for a good representation of the secondary vortex. Although no direct comparisons with experimental data can be made, the global behaviour is in agreement with observations in various experiments of the flow around delta wings. A detail of the region near the secondary separation is shown in Fig. 6.5, where conical streamlines with lines of constant raywise vorticity \( \Omega_{\alpha} \) (component of the vorticity vector along the conical ray) are plotted. An area with a negative raywise vorticity occurs at the secondary separation. In contrast with the primary separation (also visible), no shear-layer with a high vorticity level can be detected. This is a consequence of the larger sizes of the grid cells and the smaller dimensions of the vortical region and hence a lower resolution as compared to the leading edge region.

The effect on the convergence of the solution procedure is shown in Fig. 6.6, where the norm of the mean residual and the lift force coefficient \( C_L \) are plotted for three cases. These are: a case without prescribed separation; a case with separation prescribed at the leading edge and a case with separation prescribed at \( y/y_{le} = 0.84 \). It is clear that the separation model has no influence on the convergence behaviour. The prescription of a separation at the "natural" separation position does not increase the convergence, while the prescription of a secondary separation does not decrease the convergence. The occurrence of a secondary vortex, however, decreases the lift.

A number of calculations have been performed with different positions of the prescribed secondary separation. This position was moved between the primary (leading edge) separation and the primary attachment. The results are summarized in Fig. 6.7, where the positions \( (y/y_{le}) \) of the conical stagnation points at the leeward side of the ellipse are given as a function of the prescribed secondary
Figure 6.6: Convergence history of calculations with and without prescribed secondary separation (Model I), \( M_\infty = 0.5, \alpha = 10^\circ, B/A = 0.04 \)

Figure 6.7: Effect of position of prescribed separation (Model I), \( M_\infty = 0.5, \alpha = 10^\circ, B/A = 0.04 \)
separation. It appeared that a secondary vortex could only be obtained if the secondary separation was prescribed at a position with a positive pressure gradient in cross-flow direction. This region extends from \( y/y_{le} \approx 0.70 \) to \( y/y_{le} \approx 0.91 \). In most cases the real separation position is delayed in downstream direction by one or two grid cells. The appearance of a secondary vortex shifts the primary vortex towards the symmetry plane, which can be observed in the movement of the primary attachment \( A_1 \). The prescription of a separation between the primary attachment and the position near the location of the minimum pressure due to the primary vortex (\( y/y_{le} \approx 0.70 \)) did not give a separation and a vortex, but only a point with vanishing cross-flow velocity. The same yields for a small region near the location of the primary separation. The positions of the secondary separation and attachment move towards each other for increasing span. When these positions coincide, no secondary vortex appears. The position of the secondary attachment \( A_2 \) is located in the span region \( 0.92 < y/y_{le} < 0.96 \). Prescribing a conical stagnation point at such a position does not lead to the appearance of a secondary vortex. The introduction of a disturbance is evidently not strong enough to cause a separation at a downstream location. The position of the primary attachment can be shifted a little by prescribing a conical stagnation point in the region \( 0.46 < y/y_{le} < 0.54 \). The position of the primary separation on the other hand can not be influenced by the prescription of a conical stagnation point.

### 6.3 Model II; Improved Kwong and Myring model

A second model like the method of Kwong and Myring (1989) has been investigated. The original procedure of Kwong and Myring requires the specification of four angles defining the flow direction in two points at each side of the required separation. No mathematical/physical base for the proper choice of these flow directions is given. In the present approach the velocity vector in only one point at the centre of a boundary cell is forced into a plane normal to the surface and parallel to the separation line. Assuming a saddle-type structure of the cross-flow, there is always a curve where the velocity vector is tangential to the normal plane through the separation line. Using the rotation matrix Eq. (6.4), the velocity components in the rotated coordinate system are calculated. For a velocity vector lying in the \( \tilde{x}, \tilde{y} \)-plane, the component \( \tilde{w} \) must be zero. The other flow quantities, like the pressure and the total enthalpy, must remain constant, so the new velocity components are calculated by:

\[
\tilde{u}^{\text{new}} = \tilde{u} ; \quad \tilde{v}^{\text{new}} = \sqrt{\tilde{u}^2 + \tilde{w}^2} ; \quad \tilde{w}^{\text{new}} = 0
\]

(6.10)

and the velocity components in the Cartesian frame are calculated by the inverse of the matrix Eq. (6.4). This procedure is applied before each residual calculation. In contrast to the original procedure of Kwong and Myring, no averaging of pressure and density is used. It must be noted however, that this approach is not a boundary condition technique.

#### 6.3.1 Results

This separation model is applied to a test case which was also calculated by Kwong and Myring; a circular cone with a semi apex angle of 5° at \( M_{\infty} = 1.79 \) and \( \alpha = 12.65^\circ \). Separation has been prescribed at 135° from the windward generator. The conical streamline pattern for the case without and with separation is shown in Fig. 6.8. It appeared that the separation was delayed to a position more downstream. The distribution of the pressure and the cross-velocity \( u_\parallel/c_\infty \) at the surface is shown in Fig. 6.9, where it can be seen that there is a significant distance between the position of the prescription of the separation and the real separation (the cross-flow velocity becomes negative). Also
Figure 6.8: Conical streamlines around circular cone; without and with prescribed separation (Model II), $M_\infty = 1.79$, $\alpha = 12.65^\circ$, $\theta_c = 5^\circ$

Figure 6.9: Distribution of pressure and cross-velocity at surface of circular cone without and with prescribed separation (Model II), $M_\infty = 1.79$, $\alpha = 12.65^\circ$, $\theta_c = 5^\circ$
an large jump in the static pressure at the prescribed separation occurs, which is unphysical. Kwong and Myring have reported such a pressure jump for their two dimensional calculations. It was not present in their three-dimensional calculations, probably due to the averaging of the pressure and the density over the separation. Finally the convergence behaviour is shown in Fig. 6.10. The prescription

![Graph showing convergence history of calculations with and without prescribed separation (Model II).](image)

Figure 6.10: Convergence history of calculations without and with prescribed separation (Model II) of separation has a disastrous effect on the convergence. After a limited number of iterations, the residual remains at a constant level. This may be an indication that the updating of the flow variables due to the solution procedure and the updating of the velocity vector due to the separation model at the specified separation point counteract each other. No further attempts have been made to improve the convergence and the results of this model, due to the absence of a proper mathematical and physical background.

### 6.4 Model III; Artificial edge

The last separation model investigated is inspired by the observation that a sharp edge in the contour of a body surface will normally induce separation. The subsequent separation model uses an artificial edge on a smooth surface, which is introduced by an impermeable cell boundary normal to the surface (see Fig. 6.11). This is achieved by the replacement of the usual internal flux calculation by a flux calculation according to a solid wall for the two volumes \( V_{ij+k} \) and \( V_{i+j+k} \) on each side of the boundary surface \( S_{i+j+k} \) separately. The fluxes at the intermediate cell face for the two volumes become:

\[
V_{ij+k} : \quad \tilde{F}_{i+j+k} = (1 - \eta_l) \tilde{F}_{NFF} \left( \tilde{Q}_{i+j+k}^L, \tilde{Q}_{i+j+k}^R \right) + \eta_l \tilde{F}_{NFF} \left( \tilde{Q}_{i+j+k}^L, \tilde{Q}_S^R \right)
\]

\[
V_{i+j+k} : \quad \tilde{F}_{i+j+k} = (1 - \eta_l) \tilde{F}_{NFF} \left( \tilde{Q}_{i+j+k}^L, \tilde{Q}_{i+j+k}^R \right) + \eta_l \tilde{F}_{NFF} \left( \tilde{Q}_S^L, \tilde{Q}_{i+j+k}^R \right)
\]  

(6.11)
where the state vectors $\tilde{Q}_s^R$ and $\tilde{Q}_s^L$ (in the Cartesian coordinate system belonging to the cell face $S_{i+\frac{1}{2}jk}$) are calculated according to the boundary condition procedure for solid walls:

$$\tilde{Q}_s^R = \left( e_{i+\frac{1}{2}jk}^L, -\tilde{u}_{i+\frac{1}{2}jk}^L, \tilde{v}_{i+\frac{1}{2}jk}^L, \tilde{w}_{i+\frac{1}{2}jk}^L, z_{i+\frac{1}{2}jk}^L \right)^T$$

$$\tilde{Q}_s^L = \left( e_{i+\frac{1}{2}jk}^R, -\tilde{u}_{i+\frac{1}{2}jk}^R, \tilde{v}_{i+\frac{1}{2}jk}^R, \tilde{w}_{i+\frac{1}{2}jk}^R, z_{i+\frac{1}{2}jk}^R \right)^T \quad (6.12)$$

and the quantities $\eta_1$ and $\eta_2$ specify the amount of closure of the cell face. These quantities have a value from 0 (no solid cell face) to 1 (fully closed cell face) and can be specified for the two volumes separately. The latter is inspired by the Smith model, in which a conical stagnation occurs only at the downstream side of the separation. This should correspond to a closure value of 0 for the upstream cell and a closure value of 1 for the downstream cell.

It may be expected that this model will give a separation whose direction is dictated by the angle between the cell boundary and the surface, which is usually about 90°. It is clear that this is not in agreement with the results from the analytical models. On the other hand, separation within Euler methods caused by discretization errors at sharp edges are generally accepted. This may be an practical argument for the acceptance of the subsequent separation model.

### 6.4.1 Results

A number of simulations with this model have been performed on the same configuration as in the case of model I, an elliptic cone with a sweep angle $\Lambda = 65^\circ$, $B/A = 0.04$, $M_\infty = 0.5$ and $\alpha = 10^\circ$. The streamline pattern of a simulation with a prescribed secondary separation at $y/y_{le} = 0.84$ is given in Fig. 6.12, which shows that this model also produces a separation at the prescribed position. The flow field shows a good agreement with the flow field for the separation model I given in Fig. 6.2. A detail of the streamlines near the secondary vortex, together with lines of constant raywise vorticity are shown in Fig. 6.13. Besides the secondary vortex near the surface, also a small sub-vortex near the shear-layer of the primary separation is observed. The distribution of the axial- and the cross-velocity ($u_{ax}/c_\infty$ and $u_c/c_\infty$) at the surface is given in Fig. 6.14 for the case without and with prescribed secondary separation. The region of the secondary vortex is indicated by a positive cross-velocity.
Figure 6.12: Conical streamlines with prescribed secondary separation (Model III), 
$M_\infty = 0.5, \alpha = 10^\circ, B/A = 0.04$

Figure 6.13: Detail of secondary vortex with lines of constant raywise vorticity $\omega_{\alpha\zeta}$ (Model III)
Figure 6.14: Distribution of axial- and cross-velocity at surface; without and with secondary separation (Model III)

As in the case of model I the primary vortex is shifted towards the symmetry plane at the root chord. In contrast with model I, the separation is not delayed downstream (compare Fig. 6.3). Also the axial velocity distribution is not affected by the appearance of a secondary separation. In Fig. 6.15 the pressure distributions for the separation models I and III are compared with each other and with the case without secondary separation. The pressure distributions for the cases with secondary separation show a good agreement, except the region near the separation. Separation model III shows a jump from a relatively high pressure at the upstream side to a relatively low pressure at the downstream side of the separation. This pressure jump is the result of the implementation of a closed cell face in the separation model. This will lead to an increase or decrease of pressure at the cell face for the cases of a cross-flow directed towards or from the cell face respectively. The isobars in the region near the secondary separation for the two separation models are given in Fig. 6.16. In the neighbourhood of the separation, the isobar patterns are different. Model I shows a local maximum in the pressure at the separation, which is in agreement with the expected isobar pattern of a conical stagnation point, while model III shows a local maximum at the upstream side of the separation and a local minimum at the downstream side of the separation. Further away from the separation the isobar patterns agree very well.

The unphysical pressure jump at the separation predicted by model III may be influenced by the closure parameters \( \eta_1 \) and \( \eta_2 \). Therefore the effect of the parameters \( \eta_1 \) and \( \eta_2 \) is investigated. The results are shown in Fig. 6.17 and 6.18, where the cross-velocity and the pressure distribution at the surface for different combinations is plotted. First the closure ratios for both volumes at each side of the required separation are decreased equally to values of \( 2/3 \) and \( 1/3 \) (see Fig. 6.17). The effect on the cross-velocity and the pressure distribution however is minimal, although a small delay of the
Figure 6.15: Pressure distribution without and with secondary separation (Model I and III)

Figure 6.16: Isobars ($c_p$) near secondary separation (Model I and III)
Figure 6.17: Distribution of cross-velocity $u_{x}/c_{\infty}$ and pressure $-c_{p}$ at surface for different closure values (Model III)

Figure 6.18: Distribution of cross-velocity $u_{x}/c_{\infty}$ and pressure $-c_{p}$ at surface for different closure values (Model III)
secondary separation can be observed for the case \( \eta_1 = \eta_2 = 1/3 \). Furthermore different values of closure ratio for the two volumes were tried out. The results for the combinations \( \eta_1 = \eta_2 = 1 \), \( \eta_1 = 0 \land \eta_2 = 1 \) and \( \eta_1 = 1 \land \eta_2 = 0 \) are shown in Fig. 6.18. It is remarkable that the combination \( \eta_1 = 0 \land \eta_2 = 1 \) gives the smallest pressure jump. The parameter \( \eta_1 \) is connected to the cell at the downstream side of the secondary separation.

The convergence behaviour for a case without and a case with prescribed secondary separation is shown in Fig. 6.19. The prescription of separation has no significant influence on the convergence.

![Figure 6.19: Convergence history (Model III)](image)

The result of a large number of calculations with different positions of secondary separation is shown in Fig. 6.20. The results with this model show a similar behaviour as with model I (Fig. 6.7). In Fig. 6.20 different results of the prescription of a conical stagnation point on a smooth surface can be observed.

- Prescription of a stagnation point near an existing separation on a strongly curved geometry will not influence the separation position. Only minor shifts of the existing separation occur, which are less than one cellwidth. At the prescribed stagnation point a singular point with vanishing cross-flow velocity occurs.

- Prescription of a stagnation point between the primary attachment and the primary separation with a positive pressure gradient in cross-flow direction will lead to a secondary separation with a corresponding attachment on the surface. The region where a secondary separation can be obtained, is more extended than in the case of model I. There is almost no difference between the prescribed position and the actual position of the separation. The positions of the secondary separation and the primary attachment seem to be shifted towards the symmetry plane, but this shift is smaller
than the cellsize at the corresponding positions. Furthermore it is even more clear than in the case of model I, that a prescription of a stagnation point at a certain position will not give a flow field with an attachment at the prescribed position and a separation somewhere else. It appeared that the prescription of a stagnation point in the region $0.91 < \frac{y}{y_{le}} < 0.975$ led to a separation at that point and an attachment between the separation and the leading edge. Examining Fig. 6.20 it is clear that solutions are possible with an attachment at the prescribed position and a separation at a position towards the symmetry plane. These solutions, however, can only be obtained by the prescription of a stagnation point at the separation.

- Prescription of a stagnation point near the existing primary attachment will shift the existing attachment towards the prescribed position. This shift of the primary attachment position, however, has a minimal influence on the global flow field.

- Prescription of a stagnation point in a region with a negative pressure gradient, between primary attachment and symmetry plane or between primary attachment and position of the suction peak of the primary vortex, gives an isolated point on the surface with vanishing cross-velocity.
Chapter 7

CONCLUSIONS

The modelling of separation from a smooth surface within a conical inviscid flow based on the Euler equations has been investigated. This investigation consists of a theoretical part in which analytical descriptions of the flow in the neighbourhood of the separation line have been considered, and of a numerical part in which several models for the enforcing of separation from a smooth surface in a discretized Euler flow model have been analyzed.

Two different analytical descriptions of the flow near separation on a smooth surface are presented. The choice for a certain model depends on the type of the flow field in which the local description has to be embedded.

The first model is a review of the model developed by Smith (1977), which describes separation from a smooth surface in a conical homentropic flow. Some expressions for the shape of the separation surface (vortex sheet) and the surface pressure distribution are given. Main characteristic of this flow model is the requirement that the separation line leaves the solid wall tangentially. The flow has an adverse pressure gradient upstream of the separation line, which becomes infinite at the separation line itself.

The second model is based on the matching of two conical stagnation flow solutions at the separation surface. In this model the separation line leaves the solid wall at a finite angle. This local description is embedded in a flow with continuous distributed vorticity. This model contains a subset of potential flow solutions, but these solutions give a separation with an angle of $90^\circ$ with respect to the solid wall, in which no vorticity is shed from the surface. In contrast to the Smith model no singularities in the pressure distribution occur. The surface pressure distribution is continuous up to the first derivative; the second derivative with respect to the coordinate along the wall shows a jump over the separation line. The flow field is determined by two independent parameters, for example a streamline curvature on the wall or the separation surface and the separation angle. The other parameters are then found by the model.

Three different procedures to enforce separation from a smooth surface within a conical Euler method have been developed and analyzed. The main objective for the development of these models was the prescription of the flow direction along the specified separation line on the surface of the body. These separation models have been tested in subsonic and supersonic conical flow simulations around cones with an elliptic cross-section.

The first model prescribes the flow direction at a certain specified point on the surface. However, this model can only be used in combination with the flux-vector splitting scheme of van Leer. With this model separation from smooth surfaces can be obtained. The success of this model is based on an incorrect treatment of contact discontinuities of van Leer's scheme. Some results of flows containing secondary separation are shown. The effect of the forced secondary separation seems at least qualitatively to be in agreement with experimental observations.

The second model is based on the prescription of the flow direction in a cell near the solid surface. It is based on a model successfully used by Kwong and Myring. This model, however, delivered unsatisfactory results. No converged solutions could be obtained, and the separation was delayed to a significant distance downstream of the prescribed separation.

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The third model uses a closed cell boundary normal to the surface. This introduces a sharp edge in the geometry, which will induce separation. Computational results show that this model enforces separation at the prescribed position under conditions of a favourable pressure gradient (positive in cross-flow direction). This model can be used for all types of discretization schemes. At the separation, however, an unphysical pressure jump is calculated. For the test cases investigated, the use of this model had no significant influence on the convergence behaviour.

Further research could be concentrated on the following items:

- The relation between the analytical descriptions of the flow in the neighbourhood of a separation. For the embedding of the model with distributed vorticity into a flow without sources of vorticity (shockwaves), this model should be extended with irrotational solutions in which vorticity is shed into the flow field.

- The derivation of a proper mathematical and physical implementation of Kutta-type conditions in finite volume Euler codes. It is likely that there is a strong connection to the treatment of the Kutta condition in two-dimensional flow problems. The implementation of an explicit Kutta condition for smooth bodies is still unknown, while the mechanism of the implicit Kutta condition at sharp edges is far from understood.

- The extension of the boundary condition technique to three-dimensional problems, for which a comparison with experimental data could be made. This procedure should also be incorporated in a viscous/inviscid iteration scheme, in which the separation position is predicted by a boundary layer calculation.
REFERENCES


REFERENCES


