DELTFT UNIVERSITY OF TECHNOLOGY

DEPARTMENT OF AERONAUTICAL ENGINEERING

Memorandum M-240

SIMULATION OF PATCHY ATMOSPHERIC TURBULENCE, BASED ON MEASUREMENTS OF ACTUAL TURBULENCE

by

G.A.J. van de Moesdijk

DELTFT - THE NETHERLANDS

October 1975
DELFt UNIVERSITY OF TECHNOLOGY

Department of Aeronautical Engineering

Memorandum M-240

Simulation of patchy atmospheric turbulence,
based on measurements of actual turbulence

by

G.A.J. van de Moesdijk

Paper to be presented at the AGARD/FMT-GCP Joint
Symposium on: "Flight Simulation/Guidance Systems
Netherlands.

DELFt - THE NETHERLANDS
<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Summary</td>
<td>1</td>
</tr>
<tr>
<td>1. Introduction</td>
<td>2</td>
</tr>
<tr>
<td>2. Patchiness of atmospheric turbulence, description and model development</td>
<td>3</td>
</tr>
<tr>
<td>3. The model of patchiness</td>
<td>6</td>
</tr>
<tr>
<td>4. Measured patchy characteristics and model prediction</td>
<td>10</td>
</tr>
<tr>
<td>5. Digital simulation of patchy turbulence velocities</td>
<td>13</td>
</tr>
<tr>
<td>6. Concluding remarks</td>
<td>16</td>
</tr>
<tr>
<td>7. Acknowledgements</td>
<td>17</td>
</tr>
<tr>
<td>8. References</td>
<td>18</td>
</tr>
<tr>
<td>Figures</td>
<td></td>
</tr>
</tbody>
</table>
SUMMARY

Pilot dissatisfaction with the characteristics of Gaussian simulated turbulence in flight simulation stimulated a research program to determine the relevant non-Gaussian aspects of actual atmospheric turbulence needed in a realistic turbulence simulation. A model describing the so-called patchy characteristics of atmospheric turbulence as sensed by the pilot is developed in which the degree of patchiness is defined in mathematical terms. Results of actual measurements of patchy characteristics analyzed in a method indicated by the model are compared to the model characteristics. Finally a digital simulation of real-time patchy turbulence velocities is presented.
1. INTRODUCTION

In flight simulation to study aircraft handling qualities, the validity of a piloted simulation experiment highly depends not only on the quality of simulator hardware but also on the degree of sophistication of the mathematical models used to represent the aircraft and its environment. Consequently realistic representation of atmospheric turbulence velocities is considered to be of importance to the outcome of a piloted simulation experiment. The serious deficiencies in pilot acceptance often encountered, when a simulation of turbulence velocities is performed based on the Gaussian assumption, urged the search for a more refined turbulence model, in which relevant non-Gaussian characteristics were to be included.

This paper describes the formulation and validation of such a more realistic turbulence model intended to be used in piloted flight simulation. Non-Gaussian aspects of atmospheric turbulence, called patchiness or patchy characteristics have been defined in mathematical terms. An existing technique of simulating patchy turbulence, Refs. 3 and 4, is further elaborated upon, allowing the patchiness to vary more widely. In an attempt to verify the usefulness of the newly proposed model, the patchy characteristics of actually measured turbulence velocities have been analysed and compared to the model characteristics.

In the next chapter some basic considerations of patchiness leading to a model of patchiness are presented.
2. PATCHINESS OF ATMOSPHERIC TURBULENCE, DESCRIPTION AND MODEL DEVELOPMENT

The structure of atmospheric turbulence possesses a certain element of surprise to a pilot. During flight in turbulence often a variability in intensity is found to exist, indicating local regions of relatively higher energy concentrations. Large deviations in aircraft response may therefore suddenly occur without prior warning to the pilot. In this context the term patchiness or patchy characteristics is often used. If atmospheric turbulence is represented in simulation studies by a Gaussian random process, these elements are lacking. In terms of probability concepts, actual atmospheric turbulence produces more frequently large gusts than a realisation of a Gaussian process would have and there are also more relatively quiet periods. In view of the low activity regions, a higher probability of small turbulence velocities may be expected as compared to the probability of occurrence, when turbulence activity is supposed to be homogeneously distributed, see Fig. 1. Furthermore, a higher probability of large turbulence velocities may be expected, resulting from the regions of relatively high energy concentrations. The measured probability density distributions of actual turbulence velocities have indeed been found to consistently deviate from the well-known Gaussian distribution function see e.g. Refs. 1 and 2. Since the above described features have been recognized as being rather essential to many simulator studies, a number of attempts have recently been made to simulate patchy characteristics.

Instead of proceeding directly towards a description of the model of patchiness and the simulation of patchy turbulence velocities, it seems more desirable to review some basic considerations about the nature of patchiness, which will eventually result into the development of a model of patchiness.

Consider more closely a time-history of patchy atmospheric turbulence velocities as shown in Fig. 2. Note the distinct bursts of activity occurring in random order. Such a time-history can easily be viewed as created by a random process, modulated in amplitude by another random process. This may be illustrated by a sinusoidal amplitude-modulated by another sine wave having a different frequency, leading to a very regular form of "patchiness", see Fig. 3. Applying this concept to atmospheric turbulence, turbulence may be assumed to consist of a small scale Gaussian random process, representing the basic irregular motion of the air, amplitude modulated by an independent
Gaussian process of larger scale, representing for instance the influence of convective currents in the air. The concept of amplitude-modulating one Gaussian process by another independent Gaussian process to create patchy characteristics has been originally indicated by Reeves, see Ref. 3 in his design of simulation of non-Gaussian turbulence velocities. A third independent Gaussian process is added to the process just mentioned to obtain probability density distributions more comparable to those of actual turbulence measurements. The system is illustrated in the block-diagram of Fig. 4. In a previous paper, published in AGARD CP-140, Ref. 5, and more thoroughly in Refs. 4 and 6, it is shown that the system indicated by the block-diagram of Fig. 4 is capable of producing non-Gaussian turbulence velocities having a class of probability density distributions between the Gaussian and a so-called "Bessel distribution", see Fig. 5. The expression for this class of non-Gaussian probability density distributions is, according to Ref. 6:

$$p_w(x) = \frac{(1+Q^2)^{\frac{1}{2}}}{\pi \sigma_w} \int_{0}^{\infty} \left[ 2 \left( \frac{2}{1+2\zeta^2Q^2} \right)^{\frac{1}{2}} \exp \left[-\zeta^2 - \frac{1}{\sigma_w} \frac{(x-\zeta)^2}{1+2\zeta^2Q^2} \right] d\zeta \right]$$

(2.1)

The parameter Q, appearing in eq. (2.1) expresses the symmetrical deviations from the Gaussian distribution function. It has a fixed relationship to the fourth order central moment of the probability density distribution function, which is frequently used in statistics to express symmetrical deviations from the Gaussian curve:

$$m_4 = \frac{9q^4+6q^2+3}{(1+q^2)^2} \sigma_w^4$$

(2.2)

where in both eqs. (2.1) and (2.2) the index w is either u, $\nu$ or $\nu_w$.

In Ref. 6, it is further shown that the well-known power spectral densities due to von-Karman or Dryden, see Fig. 6, can be maintained in the simulation, independently from a particular choice of the probability density distribution. This implies that Gaussian as well as non-Gaussian distributed turbulence velocities can be generated having exactly the same power spectral densities. The effect of amplitude modulation need therefore not to be seen in the power spectral density distribution due to von-Karman or Dryden.

To this end solely a particular non-Gaussian distribution function characterized by its central moments and in particular the fourth order central moment indicates the degree of patchiness. It is shown, however, in the aforementioned Refs. 5 and 6 that time-histories can be created by the system,
shown in Fig. 4, having exactly the same power spectral density and probability density distribution, yet differing in patchy characteristics. Hence, knowledge of the non-Gaussian distribution function and the power spectral density function, pertaining to a particular time-history of turbulence velocities, does not suffice to completely characterize patchiness in this time-history. Therefore additional analysis is required to express the patchiness in mathematical terms.

The additional analysis required may be achieved by considering the properties of the square of the turbulence velocities $w^2(t)$, and in particular the power spectral density of $w^2(t)$. The model of patchiness to be treated in the next Section is based on the properties of this power spectral density.
3. THE MODEL OF PATCHINESS

As stated in the previous Section, patchiness can be considered as caused by local regions of relatively higher energy concentrations, appearing in a measured turbulence record. Therefore it is assumed in this paper that a pilot assesses patchiness on his memory of the fluctuations of the instantaneous energy during the immediately preceding interval of time. Since the mean value of the square of the turbulence velocities \( \bar{w^2(t)} \) equals the average power of the turbulence velocities, a little reflection will show that the stochastic process \( w^2(t) \) may be considered as a representation of the momentary fluctuations of the energy about its mean value.

The pilot's memory of the variations of the instantaneous energy \( w^2(t) \) during an arbitrary period of time may be represented mathematically as a "window" or weighting function, see Fig. 7. The representation of the instantaneous energy \( w^2(t) \) is multiplied by the weighting function and the product is integrated with respect to time. The resulting signal is \( z(t) : \)

\[
z(t) = \int_{0}^{\infty} w^2(t-\nu)h(\nu)d\nu \tag{3.1}
\]

which in fact is a convolution in the time domain.

Let \( h(\nu) \) have the form shown in Fig. 7

\[
h(\nu) = \begin{cases} 
\frac{1}{\tau} \exp \left[ -\frac{\nu}{\tau} \right] & \nu > 0 \\
0 & \nu < 0 
\end{cases} \tag{3.2}
\]

In the frequency domain the exponential weighting function is represented by a simple linear filter, characterized by the transfer function, see Fig. 7:

\[
H(j\omega) = \frac{1}{1+j\omega\tau} \tag{3.3}
\]

The output of the memory filter \( z(t) \) can be expected to react on the individual patches of turbulence. Consequently the variance \( \sigma^2_z \) of the fluctuations \( z(t) \) about its mean value will be larger than in the case of non-patchy turbulence, in which the energy is homogeneously distributed. Therefore the variance of the output \( z(t) \) of the memory-filter \( \sigma^2_z(\tau) \) as a function of the time-constant \( \tau \), the "length" of the pilot's memory is considered to be a descriptive function of patchiness as sensed by the pilot. Since a Gaussian representation of the
turbulence velocities has minimal patchy characteristics (no modulation) the variance of the output \( z(t) \) of the memory-filter, in case the turbulence velocities were purely Gaussian, is taken as a reference line. In this way a dimensionless patchiness parameter is defined, see Refs. 5 and 6:

\[
P = \frac{\sigma_z^2(\tau)}{\sigma_z^2(\tau)_{\text{Gaussian}}} \quad (3.4)
\]

Based on the simulation of patchy turbulence velocities as indicated by the block-diagram of Fig. 4, closed analytical expressions have been derived for the patchiness parameter \( P \), assuming a Dryden spectral density for the turbulence velocities, see Ref. 6.

In deriving the expressions for the patchiness parameter \( P \), use is made of the power spectral density of the square of the turbulence velocities. Consider again the block-diagram of Fig. 4. A random Gaussian process \( a(t) \) is amplitude-modulating a different independent Gaussian process \( b(t) \). The resulting process is \( u(t) \) having patchy characteristics. It can be shown mathematically that the power spectral density of the square of \( u(t) \), \( u^2(t) \) consists of three additive parts. There is also a dirac delta function at \( \omega = 0 \) due to the non-zero mean value of the resulting process. The first of the three contributions is due to the interaction of the modulating frequencies with themselves. The second contribution results from the interaction of the carrier-frequencies with themselves and the third contribution is caused by the interaction of the modulating frequencies and the carrier-frequencies. The elements of the second order power spectral density are shown in Fig. 8.

For the third independent process \( c(t) \) to be added to \( u(t) \), a cut-off frequency has been selected and the variances of \( a(t) \) and \( b(t) \) are decreased, such that the shape of the second order power spectral density of \( u(t) \) is conserved. It can be shown mathematically see Ref. 6, that the total area under the second order power spectral density curve, i.e. the power spectral density of the square of the variable, equals the value of the fourth order central moment of the probability density distribution of the turbulence velocities. Consequently the second order power spectral density shows how the contributions to the fourth order moment are arranged in the frequency domain. As mentioned earlier in Section 2, the fourth order central moment expresses the symmetrical deviations of a particular non-Gaussian distribution from the Gaussian curve. As such it indicates the presence of patchiness. Different shapes of the second order power spectral density may exist, while the total area under the curve, i.e. the fourth order central moment remains the same.
Returning to the memory filter concept, the value of the variance $\sigma_z^2$ of the low-pass filter at a certain value of $\tau$ expresses the contribution to the fourth order central moment in a frequency band determined by the cut-off frequency of the filter $\omega = \frac{1}{\tau}$. In Fig. 9 the contribution to the fourth order central moment in case the turbulence velocities were purely Gaussian, minimal patchy characteristics, is also shown. From this Figure it may be concluded that the relative importance of the patchy characteristics in terms of frequencies can be determined by comparing the area $\sigma_z^2$ non-Gaussian to the area $\sigma_z^2$ Gaussian, thus constructing the patchiness parameter as defined in eq. (3.4)

$$p_\tau = \frac{\sigma_z^2(\tau)}{\sigma_z^2(\tau)}_{\text{non-Gaussian}}$$

In Ref. 6 closed analytical expressions have been derived for the patchiness parameter $P_\tau$, assuming a Dryden spectral density for the turbulence velocities. $P_\tau$ as a function of the time-constant of the memory filter appears to be dependent on two parameters $Q$ and $R$. The parameter $Q$ has been introduced earlier in eq. (2.1). It determines the fourth order central moment of the probability density distribution. The parameter $R$ is a characteristic "modulating factor" of the turbulence patches and equals in Fig. 4 the ratio of the cut-off frequencies of filter A, the output of which is the modulating random process and filter B, the output of which is the carrier-process.

A different value of $R$ alters the shape of the second order power spectral density, but has no effect on the first order power spectral density due to Dryden.

The results of the calculations for $P_\tau$, derived in Ref. 6 are:

Longitudinal turbulence velocity

$$P_{\text{ug}} = \frac{2Q^4 + 2Q^2 + 1}{(1 + Q^2)^2} + \frac{Q^4(R+1)(-\frac{L_{ug}}{V} + 2\tau)}{1 + Q^2 \left[ (R+1) \frac{L_{ug}}{V} + 2R\tau \right]} + \frac{Q^4(R+1)(-\frac{L_{ug}}{V} + 2\tau)}{(1 + Q^2)^2 \left[ (R+1) \frac{L_{ug}}{V} + 2R\tau \right]}$$

(3.5)
Vertical and Lateral turbulence velocity

\[ P_{\tau w_g} = \frac{2Q^4 + 2Q^2 + 1}{(1+Q^2)^2} + \frac{Q^4(R+1)(2\frac{V}{L_{w_g}})^3}{(1+Q^2)^2(2\frac{V}{L_{w_g}} - \tau + R+1)(5\frac{V^2}{L_{w_g}}R^2 + 6\frac{V}{L_{w_g}} - \tau + 2)} \]

\[ 2 = \frac{2\frac{V}{L_{w_g}} - \tau(R+1)}{2\frac{V}{L_{w_g}} - \tau + R+1} + \frac{\frac{V^2}{L_{w_g}}R^2(R+1)^2}{(2\frac{V}{L_{w_g}} - \tau + R+1)^2} \]

\[ 2Q^4(R+1)(2\frac{V}{L_{w_g}})^3 \]

\[ + \frac{(1+Q^2)^2(2\frac{V}{L_{w_g}} - \tau + R+1)(5\frac{V^2}{L_{w_g}}R^2 + 6\frac{V}{L_{w_g}} - \tau + 2)}{(1+Q^2)^2(2\frac{V}{L_{w_g}} - \tau + R+1)(5\frac{V^2}{L_{w_g}}R^2 + 6\frac{V}{L_{w_g}} - \tau + 2)} \]

By replacing the index \( w_g \) by \( v_g \), the expression for \( P_{\tau v_g} \) is obtained from eq.(3.6). The model expressions, eqs.(3.5) and (3.6) are shown in Fig. 10 and Fig. 11, respectively at different values of \( Q \) or the fourth order central moment and of the parameter \( R \). From these Figures it can be seen that a pilot may be expected to sense the same patchiness, as expressed by \( P_{\tau} \) at a certain value of \( \tau \) for different combinations of the parameter \( Q \) and \( R \).

In the next Section a comparison is made between measured patchy characteristics of actual atmospheric turbulence and the model characteristics.
4. MEASURED PATCHY CHARACTERISTICS AND MODEL PREDICTION

A primary test to verify a model of non-Gaussian, patchy turbulence is to compare the model characteristics to those of actually measured turbulence. Therefore, actual measurements of turbulence in terms of true gust velocities along the aircraft-fixed body axes have been analyzed to obtain the patchy characteristics as defined by the model.

The turbulence measurement program was conducted by the National Aeronautical Establishment of Canada in 1968.

As a first indication of possible patchy characteristics the probability density distribution of each turbulence velocity record has been derived and the first four central moments have been estimated. A typical example of the results of these measured probability density distributions is shown in Fig. 12. A non-Gaussian probability density distribution, according to eq. (2.1) has been fitted to the measured distributions. For comparison a Gaussian probability density distribution has been shown as well. From this Figure it can be seen that the non-Gaussian probability density distribution, defined in Ref. 6 provides a much better description of the actual probability density distribution of the turbulence velocities than the Gaussian distribution.

To derive the patchy characteristics from the data-record, the turbulence velocity record is squared, yielding \( w^2(t) \).

The squared actual turbulence velocity record is used as the input signal to a first order linear filter having an output \( z(t) \), according to Fig. 7. The variance of the output \( z(t) \) is calculated at various values of the time-constant \( \tau \).

In this way the behavior of \( \sigma_z^2(\tau) \) of the actual turbulence velocities is measured, according to the model of patchiness. To obtain the patchiness parameter \( P \), the measured values of \( \sigma_z^2(\tau) \) measured have to be divided by \( \sigma_z^2(\tau) \) Gaussian. Therefore a hypothetical turbulence record has been thought of in a statistical sense. This hypothetical turbulence has the same statistical properties, i.e. mean value, variance and scale length as the measured turbulence record, but possesses a purely Gaussian distribution function. All statistical properties of a purely Gaussian process can be determined from the knowledge of the power spectral density distribution. From a mathematical elaboration, given in Ref. 6, it follows that the variance \( \sigma_z^2(\tau) \) Gaussian can be calculated from:

\[
\sigma_z^2(\tau) \text{ Gaussian} = \int_0^\infty \int_0^\infty \phi_{\omega \omega}(\lambda) \phi_{\omega \omega}(\omega - \lambda) |\mathcal{H}(j\omega, \tau)|^2 d\lambda d\omega  \tag{4.1}
\]

- 10 -
where $\phi_{u u} (\omega)$ is the auto-power spectral density of the turbulence velocities. The von-Karman spectral functions are assumed to accurately represent the power spectral densities of the actual turbulence velocities. These functions are, see Ref. 6 and Fig. 6:

\[ \phi_{u u} (\omega) = \frac{2}{\pi} \sigma_u^2 \frac{\bar{u}^2}{V} \frac{1}{1 + \left( 1.339 \frac{\bar{u}}{V} \omega \right)^2}^{5/6} \]  

**Longitudinal turbulence velocity, $u_k$**

\[ \phi_{w w} (\omega) = \frac{1}{\pi} \sigma_w^2 \frac{\bar{w}^2}{V} \frac{1 + 8/3 \left( 1.339 \frac{\bar{w}}{V} \omega \right)^2}{1 + \left( 1.339 \frac{\bar{w}}{V} \omega \right)^2}^{11/6} \]  

**Vertical and Lateral turbulence velocity, $w_g; v_g$**

Replacing the index $w_g$ by $v_g$ yields the expression for the lateral turbulence velocity.

As can be seen by inserting eqs. (4.2) or (4.3) into eq. (4.1), $\sigma_z^2 (\tau)$ depends on the variance $\sigma_w^2$ and the scale length $L_w$ of the turbulence velocity. By means of standard analysing techniques the values of the variance $\sigma_w^2$ and the scale length $L_w$ have been estimated from the available turbulence records. The values of $\tau$ finally have been obtained by taking the ratio, according to eq. (3.4):

\[ P^w_{\text{measured}} = \frac{\sigma_z^2 (\tau)_{\text{measured}}}{\sigma_z^2 (\tau)_{\text{Gaussian calculated}}} \]

A complete description of the method of analysing the available NAE, 1968 turbulence data is given in Ref. 7, as well as the least-squares fitting technique used to identify the parameters $Q$ and $R$ from the model. It may be sufficient here to present some results of the analysis.
In the analysis of the majority of the turbulence flights, the patchiness parameter $P_T$ showed a behaviour which agreed well with the model prediction. Some typical results are presented in Figs. 13 and 14.

In Fig. 13 the analyses of two different data records for the longitudinal turbulence velocity have been presented. The values of the fourth order central moments, 3.29 and 3.28 respectively, indicate very nearly the same rather small deviations from the Gaussian distribution function. However, the patchiness as indicated by $P_T$ at a certain value of $\tau$, e.g. 30 seconds exhibits a significant difference in patchiness as defined by the model. Note that the measured values and model prediction correspond very well. A next example is given in Fig. 14, for the vertical turbulence velocity, at a somewhat larger value of the fourth order central moment. Although the two values of the fourth order central moment do not differ significantly, viz $m_4 = 4.52$ and 3.94 respectively, the difference in patchiness as defined by the model is very large. At an arbitrary value of $\tau$ of 30 seconds, the difference amounts to about 6 units. The complete analysis of the NAE 1968 turbulence flights is documented in Ref. 7.

Finally a digital simulation of patchy atmospheric turbulence velocities is presented in the next Section.
5. DIGITAL SIMULATION OF PATCHY TURBULENCE VELOCITIES

To simulate atmospheric turbulence velocities, having patchy characteristics as defined by the model a general purpose digital computer may be used. The system is again indicated by the block-diagram of Fig. 4. The reason for the choice of the equivalent discrete time system over analog equipment is the relative simplicity obtained, as will become clear from the following.

Consider again the block-diagram of Fig. 4. The heart of the turbulence simulation consists of three filters, the output signals of which are mixed as shown in the Figure. Each filter is fed by an independent Gaussian white noise process, thus requiring in principle three white noise generators to construct one turbulence velocity component. To facilitate the generation of a complete three dimensional turbulence velocity field, three orthogonal turbulence velocity components have to be simulated simultaneously, requiring a total of nine independent Gaussian white noise generators and nine analog filters in case of an isotopic turbulence velocity field. Therefore a substantial amount of analog equipment would be needed for a real-time simulation.

In a digital simulation the generation of the turbulence velocities is performed by a rather simple computer program. The nine filters required can each be represented by first or second order differential equations, where the second order differential equation can be transformed into two coupled first order differential equations. Each filter can therefore be represented by the vector valued linear state differential equation:

\[ \dot{x}(t) = Ax(t) + Bn(t) \]  \hspace{1cm} (5.1)

where \( n(t) \) is the Gaussian white noise input.

A is either a scalar or a \((2x2)\) matrix.

B is either a scalar or a two component column vector.

Solving this vector valued differential equation for the time-interval \([t_{k-1}, t_k]\), for \( k = 0, 1, 2, \ldots, N \), results in the following expression, see Ref. 8:

\[ x(t_k) = \phi(t_k, t_{k-1}) x(t_{k-1}) + \int_{t_{k-1}}^{t_k} \phi(t_k, \tau) Bn(\tau) d\tau \]  \hspace{1cm} (5.2)
where \( \phi(t_k, t_{k-1}) \) is the transition matrix, which is found solving the so-called fundamental equation:

\[
\phi = A\phi
\]  

(5.3)

over the time-interval \( [t_{k-1}, t_k] \).

Solution of this equation for constant A, characterizing a time-invariant system yields:

\[
\phi(t_k, t_{k-1}) = e^{A(t_k-t_{k-1})}
\]

where A is the scalar or (2x2) matrix of eq. (5.1).

According to Ref. 8:

\[
e^{A(t_k-t_{k-1})} = I + A(t_k-t_{k-1}) + \frac{A^2(t_k-t_{k-1})^2}{2!} + \ldots.
\]  

(5.4)

To enable digital computation of eq.(5.2) the Gaussian zero-mean white noise process \( n(\tau) \) is replaced by a sequentially uncorrelated Gaussian zero-mean white noise sequence \( n(t_{k-1}) \), which can be thought of as generated by sampling the white noise process \( n(t) \). The sample \( n(t_{k-1}) \) is obviously considered constant over the time interval of integration \( [t_{k-1}, t_k] \). Bearing in mind that the vector B is constant for a time-invariant system the integral of eq.(5.2) is constant in the time-interval \( [t_{k-1}, t_k] \).

Therefore eq.(5.2) can be rewritten as:

\[
x(t_k) = Cx(t_{k-1}) + Dn(t_{k-1})
\]  

(5.6)

Calculation of the constants C and D can be performed off-line for a given turbulence situation, i.e. for certain values of the parameter \( Q, R, \sigma_u, L_u \).

Therefore the simulation of patchy atmospheric turbulence reduces to a simple computer program solving the vector valued linear equations, expressed by eq.(5.6).

Using the assumption of ergodicity and stationarity the required nine white noise sequences \( n_i(t_{k-1}), i = 1, 2, 3, \ldots, 9 \), of sufficient length can be stored on magnetic tape, using sample sequences of one Gaussian white noise generator.

The digital program can easily be expanded to a non-stationary turbulence situation, as will occur for instance during an approach to land. The constant matrices A and C as well as the vectors B and D then become time-dependent and the calculation is more complicated, but does not affect the basic solution in terms of eq.(5.6) in the interval \( [t_{k}, t_{k-1}] \).
In Fig. 15 the analysis of patchy characteristics of digitally simulated turbulence velocities has been presented. The result shows that basically the same patchy characteristics in artificially generated turbulence velocities can be obtained, as found from measurements of patchy characteristics of actual turbulence velocities.
6. CONCLUDING REMARKS

This paper has discussed the development and application of a newly proposed turbulence model defining patchiness from a pilot's point of view. It has been shown that the patchiness of atmospheric turbulence can be simulated with respect to all statistical properties, considered relevant to simulation studies. Good correspondence has been found between the model and measurements of actual atmospheric turbulence. Simulated time histories of turbulence velocities possessing prescribed patchy characteristics can be generated in real-time with simple programs on any general purpose digital computer.
7. ACKNOWLEDGEMENTS

The author wishes to express his gratitude to the National Aeronautical Establishment of Canada for making the 1968 turbulence data available for analysis and to H.L. Jonkers of Delft University of Technology for his very useful remarks and carefully reading of the manuscript.
6. REFERENCES

1. Dutton, J.A.
   Some observed properties of atmospheric turbulence.


3. Reeves, P.M.

4. Reeves, P.M.,
   Campbell, G.S., et al.

5. Gerlach, O.H.,
   Van de Moesdijk, G.A.J.,
   Van der Vaart, J.C.

6. Van de Moesdijk, G.A.J.

7. Van de Moesdijk, G.A.J.
   Patchiness of actual atmospheric turbulence at altitudes below 1000 ft, as derived from NAE Measurements with a T-33. Delft University of Technology, Report VTH-204 (to be published).

8. Sage, A.P.,
   Melsa, J.L.
Fig. 1. The Gaussian - and a possible non-Gaussian distribution function
Fig. 2. Time history of patchy turbulence velocities
$x(t) = a \sin \mu t \cdot b \sin \lambda t$

Fig. 3. Time history of a sinusoidally amplitude-modulated sine wave.
Fig. 4. Block diagram representing the generation of a non-Gaussian process having an arbitrary fourth order moment: \(3\sigma^4 < m_4 < 9\sigma^4\)
Fig. 5. Normalized probability density functions of $w(t)$ for various values of $Q$. 
Fig. 6a. Comparison between the power spectral densities due to Von Karman and due to Dryden; longitudinal turbulence velocities.

\[ \omega_n = \omega \frac{Lg}{V} (\text{rad}) \]
Fig. 6b. Comparison between the power spectral densities due to Von Karman and Dryden; vertical and lateral turbulence velocities.
a. Time series of the signals $w(t)$ and $w^2(t)$:

\[
\frac{1}{\tau} \exp \left( -\frac{\gamma^2}{\tau} \right) \quad (\gamma > 0)
\]

window, representing pilot's memory

10 sec

time (sec)

b. Linear filter, representing pilot's memory

Fig. 7. System to sense patchiness
Contribution of the carrier process

Contribution of the modulating process

Contribution due to interaction

Power spectral density of a squared patchy turbulence velocities

Fig. 8. The elements of the second order power spectral density of a possible patchy turbulence velocity time-history.
Fig. 9. Construction of the patchiness parameter $P_t$.
Fig. 10: The patchiness $P_T$, as sensed by the pilot; $\tau$=time constant of the memory filter.
Fig. 11: The patchiness $P_T$, as sensed by the pilot, is time constant of the memory filter.

Vertical or lateral turbulence velocities

- $Q = \infty, R = 0.05$
- $Q = 10, R = 0.05$
- $Q = 1.0, R = 0.1$
- $Q = 10, R = 0.1$

Gaussian turbulence
Fig. 12. Measured probability distribution of turbulence velocities and fitted non-Gaussian distribution. The Gaussian distribution is indicated by the dashed line.
Fig. 13. Comparison between the patchiness of two data records, Ref. 5.
Fig. 14. Comparison between the patchiness of two data records, Ref. 5.
Fig. 15. Patchiness analysis of digitally generated vertical turbulence velocities.