ON INTERNAL WAVES IN A DENSITY-STRATIFIED ESTUARY

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Report No. 2 - 91

Prepared for the J.M. Burgers Centre for Fluid Mechanics

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1991
## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Introduction</strong></td>
<td></td>
</tr>
<tr>
<td>Salt intrusion in estuaries</td>
<td>1</td>
</tr>
<tr>
<td>Internal waves</td>
<td>2</td>
</tr>
<tr>
<td><strong>Elements of internal-wave theory</strong></td>
<td>3</td>
</tr>
<tr>
<td><strong>Special levels</strong></td>
<td></td>
</tr>
<tr>
<td>(1) Turning points</td>
<td>5</td>
</tr>
<tr>
<td>(2) Critical layers</td>
<td>5</td>
</tr>
<tr>
<td>An estuarine critical layer</td>
<td>7</td>
</tr>
<tr>
<td><strong>Internal waves generated by a periodically corrugated bed</strong></td>
<td>8</td>
</tr>
<tr>
<td>Linearized two-layer model</td>
<td>9</td>
</tr>
<tr>
<td>Finite-amplitude model for continuous stratification</td>
<td>11</td>
</tr>
<tr>
<td>Observations in the Rotterdam Waterway</td>
<td>11</td>
</tr>
<tr>
<td><strong>Concluding remarks</strong></td>
<td></td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>13</td>
</tr>
<tr>
<td>References</td>
<td>14</td>
</tr>
</tbody>
</table>
On internal waves in a density-stratified estuary

Introduction

In this article some field observations, made in recent years, of internal wave motions in a density-stratified estuary are presented. In order to facilitate the appreciation of the results, and to make some quantitative comparisons, the relevant theory is also summarized. However, before doing so it seems in place to discuss briefly the origins of stratification in estuaries.

Salt intrusion in estuaries

The difference in density between saline seawater and fresh water discharged by rivers, though small, may have a marked influence on the flow and turbulence in estuaries. A physical or mathematical description of the water motion and the transport of solutes or sediments requires knowledge of the processes involved.

In an estuary in which the tidal currents are weak or even absent, the denser seawater intrudes into the estuary along its bed while on top of it a layer of less dense, fresh water flows towards the sea. The arrested salt wedge thus formed may extend over long reaches of the estuary. Gravity imparts stability to the density stratification at the interface between salt and fresh water so as to almost completely suppress the exchange of mass between the layers.

If significant tidal currents do occur, the salt wedge will move up the estuary during flood and retreat in the seaward direction during ebb. The tidal velocities usually are much larger than the fresh-water velocity. As a result, the tide enhances the turbulence production and mixing between bottom and surface layers. The transition in salinity and density between these layers - the halocline - is more diffuse than in the case of the arrested salt wedge. A schematic diagram is shown in Fig. 3.1.
Figure 3.1: Diagram of the distribution of the salinity \( S \) in a partially mixed estuary during ebb.

In the example presented in this figure the density stratification still causes a substantial reduction in turbulence levels. However, in many estuaries the tidal currents can become so strong, or the fresh-water discharge so small, that the density gradients in the vertical direction become quite small. In such a well-mixed situation mainly a longitudinal density gradient remains, and the turbulence is only weakly affected by stratification.

The density structure of an estuary may often be characterized by an estuarine Richardson number, \( R_{iE} \), defined as

\[
R_{iE} = \frac{\Delta \rho g Q_F h}{\rho^o A_M U_T^2}
\]  

(3.1)

where \( \Delta \rho \) is the density difference between seawater and fresh water, \( g \) the acceleration of gravity, \( Q_F \) the fresh-water discharge, \( h \) a typical water depth, \( \rho^o \) a reference density of water, \( A_M \) the area of cross-section of the estuary near the mouth and \( U_T \) the rms tidal water velocity at that location. The estuarine Richardson number represents the ratio of the work needed to mix the discharged fresh water across the vertical and the available tidal kinetic energy. As a consequence, strong stratification is to be expected for large \( R_{iE} \) and well-mixed conditions prevail when \( R_{iE} \) is small.

The density structure of a particular estuary may change in course of time. For example, the estuary can be strongly stratified during high fresh-water discharges or neap tides, and well mixed when the discharge is low or during spring tides. Transient phenomena may play a part when the density structure adapts to a new situation, especially during droughts.

The isohalines as shown in Fig. 3.1 slope upwards in the direction of the sea. Gravity therefore implies horizontal pressure gradients which tend to bring the isohalines to the horizontal. The related currents are a flow landward along the bed and seaward at the surface. This flow, which is often referred to as the gravitational circulation, distorts the vertical distributions of the horizontal velocities owing to tidal currents and fresh-water discharge.

Internal waves

A further consequence of stable stratification is the possibility of sustained internal-wave motion when the water body is disturbed. Internal waves sometimes manifest themselves as large-amplitude oscillations in the interior of the fluid, which, however, are hardly noticeable at the
free surface because the density differences are small. These waves are found on a wide range of scales, from the scale of the estuary itself (the motion of the salt wedge can be conceived as an internal tidal wave) down to the scales of the turbulent motions.

In this article attention is devoted to estuarine internal waves, the wavelengths of which are of an intermediate scale. In particular we consider two types of waves in some detail, that is, waves at a critical layer in shear flow and trapped internal waves over a corrugated bed.

**Elements of internal-wave theory**

To introduce the subject we give a concise summary of the linearized theory of internal waves in a continuously stratified fluid, also see the reviews of Turner (1973) and Lighthill (1978). Various simplifications are needed to arrive at a tractable mathematical model. The fluid is assumed to be inviscid and non-rotating. The density differences are small with respect to the reference density, the undisturbed flow is steady and uniform in the horizontal x-direction and the wave motion is two-dimensional in vertical planes. The pressure $p$, density $\rho$ and velocity $(u, w)$ are decomposed in basic-state variables (denoted by capitals) and perturbation quantities denoted by a tilde,

$$
\begin{align*}
    p &= P(x) + \tilde{p}(x, z, t) \\
    \rho &= \rho_0 + \tilde{\rho}(x, z, t) \\
    (u, w) &= [U(z), 0] + [\tilde{u}(x, z, t), \tilde{w}(x, z, t)] 
\end{align*}
$$

(3.2)

where $t$ is time and $z$ the vertical coordinate (positive in the upward direction). The undisturbed pressure distribution is hydrostatic that $P' = -gR'$, where the prime denotes differentiation with respect to $z$. Restricting, for the moment, the analysis to small-amplitude perturbations of the basic state, the linearized equations of motion, continuity and conservation of mass become

$$
\begin{align*}
    \rho_0 (\tilde{u}_t + U \tilde{u}_x + U' \tilde{w}) + \tilde{p}_x &\approx 0 \\
    \rho_0 (\tilde{w}_t + U \tilde{w}_x) + \tilde{\rho}_z + g \tilde{\rho} &\approx 0 \\
    \tilde{u}_x + \tilde{w}_z &= 0 \\
    \tilde{\rho}_t + U \tilde{\rho}_x + R' \tilde{w} &\approx 0
\end{align*}
$$

(3.3)

Substituting a harmonic, progressive-wave solution of the form

$$
\{\tilde{u}, \tilde{w}, \tilde{p}, \tilde{\rho}\} = \{\tilde{u}(z), \tilde{w}(z), \tilde{p}(z), \tilde{\rho}(z)\} \exp[i(\omega t - kx - \int m(z) dz)]
$$

(3.4)

gives after some rearrangement

$$
mtw^2 = \text{constant}
$$

(3.5)

and

$$
\tilde{w}'' + \frac{k^2 N^2}{(\omega - kU)^2} + \frac{kU''}{\omega - kU} - k^2 - m^2 \tilde{w} = 0
$$

(3.6)

Here a caret denotes an amplitude, $\omega$ is the wave frequency, $[k, m(z)]$ is the wavenumber vector and $N$ the buoyancy frequency given by

$$
N^2 = -\frac{g}{\rho_0} R'
$$

(3.7)
Figure 3.2: Schematic representation of the wave pattern generated by an oscillating cylinder in a constant-$N$ fluid at rest ($\omega/N = 0.508$). The wavenumber vector is in the direction of the phase velocity.

The phase velocity $\vec{c}_p$ of the wave is given by $\vec{c}_p = (k, m)\omega/(k^2 + m^2)$.

Let us assume a moment that the fluid is unrestricted and that the properties of the undisturbed flow vary only little within one wavelength. Eq. 3.6 then yields as an approximate dispersion relation for a simple wave as given by Eq. 3.4.

$$(\omega - kU)^2 \approx N^2 \frac{k^2}{k^2 + m^2} \equiv N^2 \cos^2 \theta$$

Here $\omega - kU$ is the intrinsic frequency relative to the moving fluid and $\theta$ the angle between the wavenumber vector and the $x$-axis. Apparently, wave motion is possible only if the absolute value of the intrinsic frequency is less than the buoyancy frequency.

The continuity equation implies that $k\bar{u} + m\bar{w} \approx 0$. Hence the motions of fluid particles are all in lines perpendicular to the wavenumber vector, that is, these motions lie in the planes of constant phase. As a result the group velocity $\vec{c}_G = (\partial\omega/\partial k, \partial\omega/\partial m)$, which is the velocity of wave-energy propagation, also is normal to the wavenumber vector and hence to the phase velocity. Wave energy is transmitted away from a source along rays which coincide with the directions of the particle motions. These rays are represented mathematically as the characteristics of Eqs. 3.3 and 3.4, and lie at angles $\pm \theta$ to the vertical, see Fig. 3.2.

An alternative method to describe linear internal waves that is equivalent with ray theory, is based on the notion of vertical standing modes. This way of analysis is particularly suited, if the presence of a bottom and a free surface is to be taken into account. The approach in terms of normal modes can be conceived to result from the superposition of two wave systems as given by Eq. 3.4; however, one with $+m(z)$ and the other with $-m(z)$. The vertical velocity $\bar{w}$ then is given by an expression of the form

$$\bar{w} = \bar{w}_s(z) \exp[i(\omega t - k z)]$$

and the wave propagates in the horizontal direction only. The amplitude is found to satisfy Eq. 3.6 with $m$ set equal to zero. This equation is then called the Taylor-Goldstein equation.
Figure 3.3: Example of a wave pattern in the halocline with $Ri > 1/4$ for $\omega/N(0) = 0.5$, a: turning point $[\omega - kU(-z_T) = N(-z_T)]$, b: turning point $[\omega + kU(z_T) = N(z_T)]$, c: critical layer $[\omega - kU(z_C) = 0]$, d: critical layer $[\omega + kU(-z_C)] = 0$.

In the case of a flat bed and negligible free-surface disturbances (rigid-lid approximation) the boundary conditions at the bed and the surface are $\bar{w}_l = 0$, and an eigenvalue problem results that yields the phase velocities $\omega/k$ of the wave modes.

**Special levels**

Returning to the description in terms of rays, two conditions can be identified under which a group of waves can no longer propagate vertically.

(1) **Turning points**

It follows from Eq. 3.8 that if both $U$ and $N$ decrease with $z$, for example, $\cos^2 \theta$ will become equal to one at a certain level for a wave group propagating upwards. The rays then turn to the vertical and the wave energy is totally reflected at that level. Fig. 3.3 (case b) gives an example of such a turning point. A similar result holds for a wave the energy of which propagates downwards with $U$ and $N$ decreasing in that direction, see Fig. 3.3 (case a). Thus a layer with high $N$ values (the halocline is an example) can become a waveguide for high-frequency waves. The interactions between these waves and turbulence affect the vertical transfer of mass and momentum through the halocline (Uittenbogaard, 1991).

(2) **Critical layers**

The second condition arises in shear flow when the distribution of the undisturbed velocity $U(z)$ is such that a certain (critical) level, $z = z_C$, the intrinsic frequency becomes equal to zero. Eq. 3.8 shows that then $\theta \to \pm \pi/2$ and the rays become nearly horizontal, see Fig. 3.3
Figure 3.4: Periodic nonlinear waves at a critical layer, a: stationary waves for $Ri = 0$, b: stationary waves for $Ri = O(1)$, c: example of developing waves for $Ri = O(1)$ (a and b after Maslowe, 1986).

(cases c and d). The vertical wave number $m(z) \to \infty$ as $z \to z_C$ and, according to Eq. 3.5, the vertical velocity component $\hat{w} \to 0$. This means that a wave group does not succeed in passing through the critical layer. The continuity equation indicates that $\hat{u} \to \infty$ as $z \to z_C$. Apparently the wave energy in a wave group is not conserved. What is conserved is the wave action, defined as wave energy/intrinsic frequency.

The behaviour of a wave near a critical layer according to linear theory depends on the local value of the gradient Richardson number $Ri$,

$$Ri = \frac{N^2}{U^2}$$

(3.10)

If $Ri$ is much greater than $1/4$ at the critical level, the mean flow will absorb most of the wave energy (Booker and Bretherton, 1967). If $Ri < 1/4$ wave energy can get through the critical layer and in bounded shear flows the wave can become unstable through the mechanism of over-reflection (Lindzen and Barker, 1985). In the halocline an internal wave group eventually tends to travel to its appropriate critical level when $Ri > 1/4$, see Fig. 3.3.

The results summarized above indicate that linear inviscid theory must break down at a critical layer. In order to continue the solution across the critical-layer singularity inherent
Figure 3.5: Acoustic image showing developing waves at a critical layer in the Rotterdam Waterway. The grey-tones represent differences in density. The water depth was 15.2 – 15.5m. The wavelength of the highest waves (occurring between 10.12 and 10.13 hours) was about 25 m.

to this theory, either nonlinear terms or molecular (diffusive) effects must be restored to the basic equations. In estuaries and other geophysical flows, where the Reynolds and Péclet numbers are typically very large, nonlinear effects will be more important in a critical layer than molecular effects (Maslowe, 1986).

A striking result of the nonlinear analysis is that the absorption of wave energy by the mean flow in the critical-layer region when \( Ri > 1/4 \) does not necessarily occur. This demonstrates that finite-amplitude waves then can exist at a critical layer. Such modes cannot exist according to linear theory. These waves are interesting for the application under consideration, since \( Ri > 1/4 \) is the rule rather than the exception in the halocline.

The structure of the nonlinear waves also depends on the Richardson number. The basic streamline pattern for steady periodic waves in the limit of zero \( Ri \) is the Kelvin cats-eye configuration, see Fig. 3.4a. When \( Ri = O(1) \) buoyancy forces the flow near the corners of the cats-eyes to be more nearly horizontal than in the unstratified case (Fig. 3.4b). Within the cats-eyes mixing renders the fluid homogeneous. Thin shear layers develop along the edges of the cats-eyes, which break down in small-scale turbulence. Waves of this kind have been observed in the atmosphere, and have been considered to be the origin of clear-air turbulence. An important limitation of these results is the assumption of steady flow. Fig. 3.4c, which is based on observations in the atmosphere (Atlas et al., 1970) and numerical calculations (Collins, 1982) shows in schematic form the evolution of a nonlinear wave at a critical layer.

An estuarine critical layer

We now turn to a discussion of a critical layer observed during one of the field surveys mentioned in the Introduction. The measurements were made in the Rotterdam Waterway, a man-made navigation channel that discharges fresh water from the River Rhine into the North Sea.
Figure 3.6: Distributions of mean density, mean velocity and estimated Richardson number when the acoustic image of Fig. 3.5 was taken. The distribution of the Richardson number varied little in the period 9.31 – 10.41 h.

The density structure of this estuary varies from stratified during neap tides and high river discharges to partially mixed during spring tides and low discharges. The measuring site selected had a flat bed and was located near the axis of the navigation channel west of Maassluis at km 1023. The measurements were made during a spring tide on 20 April 1988 when the ebb current was maximal. Vertical distributions of mean density and horizontal velocity were measured from an anchored survey vessel. At the same time a second vessel recorded acoustic images of the density structure as well as the bed using a 210 kHz transducer while drifting freely.

Fig. 3.5 shows an acoustic image with large-amplitude internal waves at the halocline, and Fig. 3.6 distributions of density, velocity and estimated Richardson number (Eq. 3.10) at the time the image was taken. Numerical analysis using the Taylor-Goldstein equation suggests that the highest internal waves observed had wavelengths of about 25 m. Under these conditions a critical layer existed at a depth between 7 and 9 m below the surface.

According to linear theory all of the internal-wave energy at the critical layer would be absorbed by the mean flow at the large Richardson numbers shown in Fig. 3.6. Nevertheless waves were observed, and this must be attributed to the nonlinear effects described. The acoustic image suggests that the highest waves are still developing and breaking is likely to ensue. Waves of this kind have also been observed by others (e.g., Geyer and Smith, 1987) and there seems to be some resemblance with the waves sketched in Fig. 3.4c.

The measurements are not detailed enough to make further comparisons with theory. Also, it is not easy to say anything about the mechanism that generated these waves. One could think of disturbances owing to turbulence near the bed (where $R_i$ was small) or upstream topographical features. Anyway, the initial perturbation must have been sufficiently strong so as to overcome the linear absorption mechanism.

Internal waves generated by a periodically corrugated bed

A major source of internal wave activity in stratified and partially mixed estuaries is formed by the ubiquitous bottom topographical features, such as bed ripples, dredged trenches and channels, sills (e.g., in fjords), banks, groynes and variations in width of the channel. Since the background velocity and density distributions depend on the phase of the tide, the internal-wave response shows great spatial and temporal variability. In particular internal waves generated by bottom topography can attain large amplitudes and even break under certain conditions.
Occasionally there seems to be a relation between internal waves and sediment transport.

**Linearized two-layer model**

As a relatively simple case we consider the steady flow of two layers of different densities over sinusoidal bottom topography given by

\[ \eta(x) = \hat{\eta} \sin kx \]  

(3.11)

where \( \hat{\eta} \) is the amplitude of the corrugations and \( k \) the wavenumber. The unperturbed interface is at mid-depth. The solution of the Taylor-Goldstein equation for small amplitudes satisfying the linearized boundary conditions (\( \hat{\omega} = 0 \) at \( z = h \) and \( \hat{\omega} = kU \hat{\eta} \cos kx \) at \( z = 0 \)) then gives

\[ \delta(x) = \frac{F_2^2}{F^2 \cosh \frac{k h}{2} - \frac{2}{k h} \sinh \frac{k h}{2}} \hat{\eta} \sin kx \]  

(3.12)

where \( \delta \) is the vertical displacement of the interface; the internal Froude numbers \( F \) and \( F_2 \) are given by

\[ F^2 = \frac{U_1^2 + U_2^2}{\Delta \rho g h}, \quad F_2^2 = \frac{U_2^2}{\Delta \rho g h} \]  

(3.13)

Here \( \Delta \rho \) is the density difference between the layers, and \( U_1 \) and \( U_2 \) are the water velocities in upper and lower layers. The Froude number \( F \) plays a similar role as the Mach number in gas dynamics.

(for caption see next page)
Figure 3.7: Acoustic images taken along the axis of the navigation channel, a: from 15.31 - 15.38h (part); the flow is from right to left; b: from 15.46 - 15.52h; the flow is from right to left; c: from 16.00 - 16.12h (part); the flow is from left to right. The dark band running across the image was caused by a towed conductivity meter used to verify that grey- tones on the images corresponded with the density structure.
The response to topography of arbitrary shape can be analyzed using Fourier transforms. In the case of internally subcritical flow \( F < 1 \) over an isolated obstacle, a system of stationary lee waves downstream of the obstacle is formed.

If \( F < 1 \) the denominator in Eq. 3.12 can become equal to zero for a certain Froude number, and the fluid then is resonantly excited. The linearized theory turns out to predict the Froude number at which resonance occurs quite well. However, finite-amplitude effects must be taken into account to calculate wave heights.

**Finite-amplitude model for continuous stratification**

A relatively simple way to accomplish this is to make use of the well-known fact that the equation for the stream function becomes linear for a certain class of velocity and density distributions (the boundary condition at the bed remains nonlinear). For the present purpose this equation can be written for steady flow as (Yih, 1965)

\[
\nabla^2 \psi + \frac{b^2}{h^2} \psi = \frac{\Delta \rho}{\rho_0} \frac{g}{U h} (z - A h \tan \frac{b}{2})
\]

where \( \Delta \rho \) is now the top to bottom density difference, \( \bar{U} \) the mean (horizontally and depth averaged) water velocity, \( A \) a constant related to the mean-velocity distribution, and \( b \) a constant given by

\[
b^2 = (k h)^2 + \frac{\pi^2}{(1 + a)^2}
\]

The solution given by Yih was extended to allow for a corrugated bed by introducing the constant \( a \) \( (0 < a < 1) \).

The stream function is of the form \( \psi = \psi_1 + \psi_2 \), where \( \psi_1 \) represents the horizontally averaged velocity distribution, and \( \psi_2 \) the trapped wave owing to the bottom topography. A simple expression for periodic mod-one waves as observed is

\[
\psi_2 = \frac{\Delta \rho}{\rho_0} \frac{g h^2}{\bar{U} b^2} B \cos k x \sin \pi \frac{k}{1 + a}
\]

where \( B \) is a constant related to the wave height.

The imposed linearity requires that the density distribution satisfies

\[
\rho = \rho_b - \frac{\Delta \rho}{U h} \psi
\]

where \( \rho_b \) is the density at the bed.

Assuming the wave height and the wavelength of the bottom corrugations to be known, a relationship between the constants \( B \) and \( a \) is found from the condition that \( \psi = 0 \) at the bed. Since the Froude number \( F = \pi/b \) also depends on \( a \), the wave height can be calculated as a function of the Froude number. Some results of this calculation will be presented together with field observations.

**Observations in the Rotterdam Waterway**

Echo-sounding and side-scan sonar soundings made in the Rotterdam Waterway near km 1016 showed the presence of a series of thirteen ridges at the bed with their crests at approximately
Figure 3.8: Assumed velocity and density distributions and calculated heights of internal waves versus Froude number for various wavelengths \( L \) (\( h = 16.6\,\text{m}, 2\tilde{h} = 1.38\,\text{m} \)), \( \alpha \): assumed distributions, \( \ldots \ldots \ldots \ldots \): uniform velocity distribution, \( \ldots \ldots \ldots \ldots \): velocity distribution approximately as observed, \( b \): wave heights, \( \ldots \ldots \ldots \ldots \): resonance, \( \chi \) onset of gravitational instability. The error bars indicate the internal Froude numbers at the times the images of Figs. 3.7a and b were taken.

right angles to the axis of the navigation channel. The wavelengths \( L \) varied from 28 to 74m and the heights of the ridges from 1.15 to 1.80m. These ridges extended from at least 50m south to 25m north of the axis. Depending on position, the water depth near the axis was 16.3 - 18.0m at the time of the measurements.

The resonance phenomenon to the described was observed on the decreasing flood during a normal tide on 21 October 1987. Velocity and density distributions were measured from a vessel anchored at the axis of the navigation channel at km 1016. Internal Froude numbers were computed from these data. A second vessel recording accurate images of density distribution and bed, was sailing up and down along the axis between km 1015.5 and km 1016.5 at the same time.

The flow became internally subcritical to first-mode waves at about 15.20h, where upon the possibility of resonance arose. A wave pattern clearly related to the bottom topography was observed in the period 15.26 - 16.12h. The heights of the topography generated waves gradually increased and became maximal in the period 15.46 - 15.52h, see Figs. 3.7a and b. The greatest wave height observed in this period was 6.6m (over the fifth ridge from right in Fig. 3.7b), which is about 40% of the water depth. The maximal angle of inclination of the halocline was about 30°. The crests of the waves in Fig. 3.7b coincide with those of the ridges in most cases, which is typical of a flow that is nearing resonance while slackening.

After resonance the waves damped rapidly, which may be attributed to the effective viscosity of the background turbulence. In addition internal waves developed that showed inverted
Figure 3.9: Observed wave heights and calculated maximal wave heights versus wavelength, observed from 15.31 - 15.38h (Fig. 3.7a), 15.39 - 15.44h, 15.46 - 15.52h (Fig. 3.7b) Solid and dashed lines represent the two theoretical cases shown in Fig. 3.8.

profiles with the troughs of the waves over the crests of the ridges, which is typical of a sub-resonance response, see Fig. 3.7c (between 430 and 580m).

Wave heights computed using the finite-amplitude model outlined, are shown in Fig. 3.8b for the assumed mean velocity and density distributions of Fig. 3.8a. Only the solutions for Froude numbers greater than the resonance value are shown, since the subresonance response is inherently unsteady. Time-dependence caused by the tide may also be nonnegligible for the larger wavelengths shown \((L > \text{about} 50\text{m})\). Of the two velocity distributions considered the nonuniform one (solid lines) mimics the observed distribution. The low velocities near the bed are caused by turbulent bed friction, while those near the free surface are due to the gravitational circulation. The halocline shown is more diffuse than the observed one, but this has to be accepted in order to retain the linearity of the model (Eq. 3.17).

All theoretical curves in Fig. 3.8 terminate, before resonance conditions are met, when gravitational instability \((\partial p/\partial z > 0)\) arises. Regions of mixed fluid or rotors would then be expected to develop near the bed, which is likely to inhibit further wave growth since this would reduce the topographic forcing.

According to the theory resonance would occur for \(L \approx 30\text{m}\) when the acoustic image with the highest waves (Fig. 3.7b) was taken. However, this image shows large wave heights for other wavelengths as well. A similar result was obtained for the preceding images. Presumably the variability of the flow, which results in substantial variations in the Froude number \(F\), causes this effect. Fig. 3.9 shows that the model is reasonably successful in producing large wave heights like those revealed by the acoustic images.

Concluding remarks

In this article only a small number of internal-wave phenomena has been considered. Little or nothing has been said about stability problems, the formation of solitary waves and related nonlinear effects, interactions between waves, frontogenesis, the influence of the earth's rotation, upstream influence owing to isolated bottom topography, and so on. The choice made was merely prompted by the field observations.
The increase in shear caused by internal waves of the scale considered increases the turbulence production so that internal-wave energy is transferred to the kinetic energy of the turbulence. Furthermore the stability of the flow as characterized by the gradient Richardson number is reduced. As a result of these processes the vertical transfer of mass and momentum will increase. As the increase in shear is concentrated in the halocline, the vertical exchange between bottom and surface layers may be substantially enhanced. However, although mesoscale internal waves have been observed in many estuaries, little systematic work seems to have been done on the prediction of their influence on turbulence and mixing.

Acknowledgements

The field work and related analysis was done in cooperation with Dr. G. Abraham of Delft Hydraulics and Dr. J.D. Pietrzak who was supported from a Royal Society fellowship. References are Pietrzak et al. (1990, 1991) and Kranenburg et al. (1991). The field measurements were made by Rijkswaterstaat, Ministry of Transport and Public Works, which institution also supplied financial support.

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