TECHNISCHE HOGESCHOOL DELFT VLIEGTUIGBOUWKUNDE

Report VTH-96

Compressible inviscid flow near the
end of pointed afterbodies

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1. Introduction.

When the supersonic inviscid flow around a pointed afterbody is analyzed by means of potential flow theory a singularity is found to exist in the rear point. The question as to whether a shock wave is attached to this point or a stagnation point is formed therefore still remains to be answered. Of course, the real flow will be determined largely by the boundary layer. It is of interest however to consider the inviscid flow in order to know whether, for instance by means of boundary layer control, the drag of afterbodies could be decreased. For a shock wave to originate at the point a conical flow solution should exist satisfying the boundary condition given by the tangent cone in the point considered. It can be shown that the axisymmetric conical flow around a cone extending upstream to infinity does not exist physically because limit lines appear in the solution. Axial symmetry is not thought to be essential to this result. Then it can be concluded that throughout the compressible flow region a stagnation point exists at the end of a pointed afterbody. After a brief review of the properties of limit lines a discussion of the flow with axial symmetry around a cone extending upstream will be given.

2. Limit lines in compressible flows.

A limit line or surface in a compressible flow can be defined as a boundary of the flow field, which the streamlines do not cross as they are turned back. Such a physically absurd result indicates that the assumptions underlying the theory are not justified. Often in this case solutions with shock waves can be

found satisfying the given boundary conditions. The properties of limit lines have been investigated extensively. A treatise on limit lines in axisymmetric flow was given by Tsien¹. He finds that for an axisymmetric isentropic flow for which a Legendre transformation is possible that this transformation becomes singular at the limit line. By this transformation the characteristics in the physical plane are mapped on the characteristics in the hodograph plane except at the limit line where the characteristics of one family in the physical plane form an envelope, which transforms in the hodograph plane into the limiting hodograph. At the limiting hodograph the streamlines are not curved and are tangent to the characteristics in the hodograph plane. At the limit line the pressure gradient and acceleration become infinite whereas the streamline has a cusp. The solution cannot be continued beyond the limit line, neither with a continuous nor with a discontinuous solution.

3. Conical flow with axial symmetry around a cone extending upstream.

A spherical-polar and a cylindrical coordinate system as shown in fig. 1 will be used. The spherical-polar coordinate system has the origin in the apex of the cone and the angle θ is measured between the downstream part of the axis of revolution and the radius R (θ is positive in counterclockwise sense). The cylindrical coordinate system has the same origin and the x axis is along the downstream part of the axis of revolution. The y axis is perpendicular to the x axis. In the spherical-polar coordinate system the radial velocity is denoted by u_0 and the tangential velocity by v_0 . The velocity along the x axis is

called u and along the y axis v . The relation between the velocity components in the two systems can be written as

$$u = u_0 \cos \theta - v_0 \sin \theta \quad (1)$$

$$v = u_0 \sin \theta - v_0 \cos \theta \quad (2)$$

Conical flow with axial symmetry involves one independent variable and the following ordinary differential equations can be derived using spherical-polar coordinates and θ as independent variable.

The condition that the flow is free of rotation can be expressed as

$$v_0 = \frac{du_0}{d\theta} \quad (3)$$

while Bernoulli's equation can be written as

$$u_0^2 + v_0^2 + \frac{2}{\gamma-1} a^2 = \frac{\gamma+1}{\gamma-1} a_{*}^2 \quad (4)$$

where a_{*} is the critical speed of sound.

Combining Euler's equations with the equation of continuity, and the introduction of the velocity of sound gives:

$$\left(1 - \frac{v_0^2}{a^2}\right) \left(\frac{d^2 u_0}{d\theta^2} + u_0\right) + u_0 + v_0 \cot \theta = 0 \quad (5)$$

Making use of the cylindrical coordinate system and taking u as independent variable the flow is governed by the equation

$$v v_{uu} = 1 + v_u^2 - \frac{(u+v v_u)^2}{a^2} \quad (6)$$

The properties of conical flow with axial symmetry were investigated by Busemann². Especially the supersonic flow over the axially symmetric cone at zero angle of attack has been given much attention and has become known as the Taylor and Maccoll solution, which has been tabulated by Kopal³. The conical flow over a downstream pointing circular cone exhibits the same features.

Let q_c be the velocity on the surface of the cone in the direction of the apex and A in fig. 1b the corresponding point in the hodograph plane. From Eqs. (1), (2) and (3) can be deduced that

$$v_u = \frac{\frac{dv}{d\theta}}{\frac{du}{d\theta}} = -\cot \theta. \quad (7)$$

The tangent to the streamline in the hodograph plane is therefore perpendicular to the radius in the physical plane. The normal to the streamline in the hodograph plane makes the angle θ with the u axis. In A the streamline in the hodograph plane is perpendicular to the surface of the cone OA. The acceleration in A is therefore equal to zero as also can be concluded from the following expression, derived with the aid of Eqs. (2) and (5):

$$\frac{dq}{dt} = \frac{v_o}{R} \left(\frac{d^2 u_o}{d\theta^2} + u_o \right) = - \frac{v_o (u_o + v_o \cot \theta)}{R} = - \frac{v_o v}{R \sin \theta (1 - \frac{v_o^2}{a^2})} \quad (8)$$

The tangent to the streamline in the physical plane is tangent to the cone, perpendicular to the radius in the physical plane. The acceleration is zero. It follows from Eqs. (1), (2), (6) and (7) that the normal to the streamline in the hodograph plane makes the angle θ with the u axis. In A the streamline is perpendicular to the surface of the cone OA. The acceleration in A is therefore equal to zero as also can be concluded from the following expression, derived with the aid of Eqs. (2) and (5):

$$v_{uu} = \frac{1 - \frac{v_o^2}{a^2}}{v \sin^2 \theta} \quad (9)$$

In A $v_{uu} < 0$ since $v < 0$ and $v_o = 0$; the streamline in the hodograph plane is in A curved to the right, which can also be concluded from the expression for the radius of curvature, given by Busemann²:

$$\rho = \frac{N}{1 - \frac{v_o^2}{a^2}}, \quad (10)$$

where N = length of the normal between the point considered and the point where the u axis is intersected by the normal. If the denominator of the right hand side is positive, then the streamline lies in between the normal and the centre of the circle of curvature.

The flow from A to A' is accelerated. The velocity component v_o is zero on the surface of the cone and approaches $-a$ in A', the point in the hodograph plane corresponding to a limit cone in the physical plane. The flow at the limit cone exhibits properties similar to that at the limit line in axisymmetric isentropic flows and of which the results found by Tsien are mentioned before. A difference however is that Tsien considered flow for which a Legendre transformation is regular, except of course at the limit line, whereas the transformation of the conical flow is singular throughout the flow field. Velocities and pressures on a radius through the origin are constant and correspond to one point in the hodograph plane. The limit cone

is the envelope of the Mach cones since on the limit cone $|v_0| = a$. It also follows that there is a second limit cone for which $v_0 = +a$. The second limit cone is situated inside the cone to which the flow is to be tangent. Numerical values for the angles of the limit cones are given in ref. 3, where they are called second-order discontinuities of the equations of flow. As can be seen from Eq.(8) the acceleration becomes infinite at the limit cones and so does the pressure gradient. The radius of curvature of the streamline in the hodograph plane becomes infinite as can be seen from Eq.(10) and also from Eq.(9). Further can be shown that the streamline in the physical plane has a cusp at the limit cone. By differentiating Eq.(7) with respect to u follows:

$$v_{uu} = \frac{1}{\sin^2 \theta} \frac{d\theta}{du} \quad (11)$$

The distance along a streamline in the physical plane can be expressed as

$$ds = R \cot (\pi - \theta + \tan^{-1} \frac{v}{u}) d\theta \quad (12)$$

and with Eq. (11) as

$$ds = v_{uu} R \sin^2 \theta \cot (\pi - \theta + \tan^{-1} \frac{v}{u}) du \quad (13)$$

so that it follows that on the limit cones, where $v_{uu} du$ goes through zero the streamlines return. The continuation of the solution beyond the limit cone is not possible.

The behaviour of the streamlines is not investigated in further detail since it is clear by now that the solution is not of physical interest. Also it is not possible to use part of the solution as can be done for the flow around an upstream pointing cone. Such a flow can be obtained by rotating fig. 1a around the

y axis and fig.1b around the u axis. The result is shown in fig.2. The flow is now towards the cone. A part of the conical flow solution excluding the limit cone can be matched to a parallel flow with a shock wave, since just in front of the shock wave $|v_0| > a$. It is impossible to get a shock wave when the direction of flow is reversed since in the region of flow in between the cone surface and the first limit cone $|v_0| < a$. It is therefore concluded that conical flow with axial symmetry around a cone extending upstream does not exist.

Throughout the compressible flow region a stagnation point is formed at the end of a pointed afterbody ending as a circular cone with a vertex angle different from zero. A qualitative sketch of the supersonic flow around a body of revolution pointed at both ends is given in fig. 3. The Mach number is chosen so large that the bow shockwave is attached to the foremost point. At the tail a shock wave stands somewhat upstream of the rear point around which a subsonic region exists bounded downstream by a sonic line.

4. References.

- ¹ Tsien, H.S., The "limiting line" in mixed subsonic and supersonic flow of compressible fluids, NACA TN 961, November, 1944.
- ² Busemann, A., Die achsensymmetrische kegelige Ueberschallströmung, Luftfahrtforschung, Band 19, No.4, pp. 137-145, May, 1942.
- ³ Staff of the computing section, Center of Analysis (under direction of Zdenek Kopal), Tables of supersonic flow around cones. Massachusetts Institute of Technology, Technical Report 1, 1947.

FIRST LIMIT CONE

a. PHYSICAL PLANE

b. HODOGRAPH PLANE

FIG. 1. CONICAL FLOW WITH AXIAL SYMMETRY AROUND A CONE EXTENDING UPSTREAM

a. PHYSICAL PLANE

b. HODOGRAPH PLANE

FIG. 2. CONICAL FLOW WITH AXIAL SYMMETRY AROUND A CONE EXTENDING DOWNSTREAM

FIG. 3 . SUPERSONIC INVISCID FLOW AROUND A BODY OF REVOLUTION POINTED AT BOTH ENDS.