Optimum Wing Area, Aspect Ratio and Cruise Altitude for Long Range Transport Aircraft

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ABSTRACT

An analytical method has been developed for calculating a Figure Of Merit (FOM) for long-range jet aircraft with engines sized for the cruise condition. This FOM represents the added weight fractions of the fuel load, the powerplant installation and the wing structure. Its minimum value results in a maximum of the payload fraction. Closed-form relationships were derived for optimum wing loading, wing aspect ratio, design lift coefficient and initial cruise altitude, for a specified cruise Mach number. Various combinations of partial and constrained optima have been derived both for given sweepback angle and (mean) thickness/chord ratio of the wing and for the case of constant drag rise Mach number parameters have to be matched to the lift coefficient. All results are illustrated for a hypothetical long-range passenger transport.
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\( W_D \) - Design weight (N)
\( W_{to} \) - Maximum Take-off Weight (N)
\( w \) - Specific weight of wing material (N/m³)
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**INDICES**

\( cr \) - Cruising at initial altitude
\( comp \) - Compressibility
\( des \) - Design condition
\( f \) - Fuel
\( H \) - Horizontal tailplane, except in \( R_H \)
\( is \) - Isolated, uninstalled
\( md \) - Minimum drag condition
\( P \) - Powerplant
\( p \) - Payload
\( par \) - Parasitic
\( ref \) - Reference
\( sl \) - Sea Level
\( to \) - Take-Off
\( W \) - Wing
Chapter 1

INTRODUCTION

One of the principal issues in conceptual aircraft design is the choice of wing area, wing aspect ratio and the cruise altitude at the initial cruise weight. For specified combinations of payload, range and cruise speed, these design variables have a large impact on the design lift coefficient and the aerodynamic design of the wing, on the weight distribution and on the choice of the engine thrust at take-off. Although the freedom of choice for wing geometry and altitude is usually limited due to many practical constraints, it is useful to consider the unconstrained optimization problem, since its results indicate in which direction technical improvements will pay off most. However, in the present study some attention will also be paid to constrained optimization.

The Figure Of Merit (FOM) to be used for transport aircraft is the economic performance, e.g. Direct Operating Costs (DOC). Since DOC are rather specific to an airline and moreover contain many unknown factors, a useful alternative criterion is the Maximum Take-Off Weight (MTOW) for specified payload, or the payload for specified MTOW. The latter case will be considered in the present study, which means in effect that the economic value of the payload is dominating over cost factors such as fuel cost, engine price and development and production cost of the airframe.

It will be obvious that this point of departure results in a simplification of the problem, which can only be accepted when the results have a rather general validity. Indeed, the aim of the present study is to derive generalized results — preferably in analytical form — which may serve to interpret and augment numerical optimization studies. Several analytical criteria have been derived in [5, 6] for the independent variables wing loading, aspect ratio and cruise altitude, based on linearized equations. Since these have only validity in a very small region of these variables, they will be improved in the present report on the basis of a more fundamental sensitivity relationship between the weight of a wing and its geometry.

Chapter 2 treats the sensitivity of wing weight with respect to variation of the relevant parameters. The result is used in Chapter 3 to derive a Figure Of Merit in a closed-form analytical equation. Partial optima for the independent design variables are derived in Chapter 4 and are illustrated by means of an example of a hypothetical long-range aircraft design. A discussion of the results follows in Chapter 5, together with the derivation of unconstrained combined optima and the effects are shown of imposing several realistic design constraints. Chapter 6 gives an overview of the most significant effects of compressibility drag on the various optima. Concluding remarks in Chapter 7 form a summary of the most pertinent results and procedures for optimization.
Chapter 2

WING WEIGHT SENSITIVITY

Empty mass prediction methods for application in the conceptual design stage lean heavily on statistics and empiricism. For the purpose of optimization, however, it is imperative to pay attention to 'functional' sensitivity of the wing weight to the relevant design parameters. This means that ideally for the various wing structural component loading cases should be considered, which determine the material thicknesses. Such an approach entails a number of steps and iterations. For example, the computation of loads on the various structural elements requires considerations of many load distributions — e.g. fuel/payload combinations, design speeds and altitudes — whereas the wing structural mass distribution itself, which is unknown a priori, affects the inertia loads. Moreover, a station-by-station analysis will be required to take into account that for each structural element the critical loading varies in spanwise direction.

Another complication is caused by the fact that certain components — such as the upper and lower panels — are subject to optimization: selections must be made of skin/stringer combinations, rib distances, etc. This may easily result in a laborious structural design process for each wing geometry, which is very objectionable to the efficiency of the overall optimization process. In the present study we avoid this complication by using a general relationship which complies with most published weight prediction methods and is probably representative of industrial practice. Weight contributions of the various parts of a wing, as shown schematically in Figure 2.1, are summarized in the following generalized expression,

$$W_W = \sum^{} \Delta W_W = \sum^{} f_i (n_{ult} W_D)^{p_i} S^q A^x = \frac{(t/c)^y (cos\Lambda)^z}{i}$$ (2.1)

For each component the factor of proportionality will be a constant or a function of wing taper, material specific density, a mean stress level, structural configuration, etc. The design ultimate load factor $n_{ult}$ and All-Up Weight $W_D$ refer to a critical loading condition. The exponents $p_i$, $q_i$, $x_i$, $y_i$, and $z_i$ in Equation 2.1 are generally different for the various structural components. Their significance is discussed most clearly in early RAE reports [2] and in Shanley’s book on wing structural weight [8]. A recent publication by the present author [12] derives a calculation procedure which can be used in a typical CAD-environment.

These references show that — for given design conditions, wing taper, type of material and stress level, etc., — the design sensitivity of the material required to withstand the bending load is characterized by: $p = 1$, $q = 0.5$, $z = 1.5$, $y = -1$, and $z = -2$. These values have to be modified if the mean compressive stress is allowed to increase with the loading intensity. For shear web material the exponents theoretically amount to: $p = 1$, $q = 0.5$, $x = 0.5$, $y = 1$, and $z = -2$. With regard to wing ribs the situation is much more complicated, since these
Figure 2.1. Subdivision of wing weight

have to be sized for several different loading cases with no clearly critical case for each wing configuration. There is in addition a choice of rib distance involved.

For leading and trailing edge structures, high lift devices, control surfaces and spoilers, the critical loading cases are not determined directly by $n_{ull}W_D$, nor by the wing aspect ratio. Important factors here are the various skin panel areas, the structural configuration and retraction mechanism, the chord extension due to flaps, etc. It turns out that for a given size of aircraft and configuration of high-lift devices the single most significant parameter is the wing area. In this case all exponents of $\Delta W_W$ in Equation 2.1 would have a value zero, except $q_i = 1$. Other weight components are even more difficult to analyse: non-optimum weight penalties for non-tapered skins, joints, cut-outs, mountings and connections, etc. If a fair approximation of these is available, e.g. on the basis of existing design practice, their value may be assumed constant or equal to a constant fraction of the wing weight. In that cases all exponents in Equation 2.1 are equal to zero, except $p_i$ which can be either zero or one.

A typical practical wing weight formula having the following appearance can be derived [12]:

$$W_W = 0.06 \frac{w}{\sigma_r} n_{ull}W_D \frac{b^3 f(\lambda)}{S (t/c)\cos^2 \Lambda} + wt_{ss}S$$  \hspace{1cm} (2.2)$$

In this expression the first term represents the weight of the upper and lower wing panels, required to withstand the bending load due to lift, $n_{ull}W_D$. The meaning of the various factors is as follows:
Chapter 2. WING WEIGHT SENSITIVITY

- The factor 0.06 is a non-dimensional constant of proportionality, accounting for average values of bending relief due to wing and engine masses, structural efficiency, spar locations, etc. It also includes 15% allowance for shear web material, although it is realized that this is proportional to the wing span instead of $b^2/S$.

- The ratio of specific wing material weight $w$ of structural material to a mean stress level at the wing root $\sigma_r$ is sensitive to the loading intensity, and hence to the All Up Weight of the aircraft. Reference [12] suggests as a statistical average for Al-alloy primary structures:

$$\frac{w}{\sigma_r} = 40 \left(1 + 1.10 \left(\frac{W_{lo}/10^6}{1} \right)^{1/4} \right) \times 10^{-6} \tag{2.3}$$

- The ultimate design load factor $n_{ult}$ and design All-Up Weight $W_D$ refer to critical combinations of manoeuvre or gust loads, fuel and payload distributions and operating speed versus altitude limits. Although this factor has a major influence on wing weight, most authors recommend to use $n_{ult} = 1.50 \times 2.50 = 3.75$, while for $W_D$ a mean value between the MTOW and Maximum Zero Fuel Weight (MZFW) is used, for example [9]:

$$W_D = \sqrt{MZFW \times MTOW} \tag{2.4}$$

This takes into account some bending moment relief due to wing fuel.

- The taper factor $f(\lambda)$ is usually of the form:

$$f(\lambda) = \frac{1 + 2\lambda}{1 + \lambda} \tag{2.5}$$

which accounts for taper ratio effects on the center of pressure location and on wing box and sheet tapering.

- The mean thickness-to-chord ratio $t/c$ is found by suitably weighing the wing thickness in spanwise direction, for example by the local chord.

- The sweep angle $\Lambda$ is theoretically related to the elastic axis, but since its location is usually not known, it is acceptable to approximate it by the mean quarter-chord line.

It is significant to note that the first term of Equation 2.2 is proportional to $(\text{span})^3$ divided by wing area. This ratio is not dimensionless and in fact increases with the size of the aircraft. This unfavourable square-cube-law effect is partly offset by the increase in mean stress level — as indicated by Equation 2.3 — and by the decreasing relative weight contribution of the secondary weight components.

The second term of Equation 2.2 represents mostly these secondary structural components, namely the weight of (Figure 2.1):

- wing ribs;
- non-optimum weight due to joints, non-tapered skins, etc.;
- mountings of undercarriages and engine nacelles;
- fixed leading and trailing edges;
- high-lift devices at the leading and trailing edges;
- ailerons, spoilers and other controls;
- miscellaneous weight components, e.g. wing tips, wing-to-fuselage fairings, etc.
The 'smeared skin thickness' $t_{ss}$ in Equation 2.2 lumps together all these contributions, smearing them out over the gross wing area. It amounts to several millimeters on small transports, up to about one centimeter for large transports with complex high-lift devices. A typical figure could be:

$$t_{ss} = 0.004 \left(1 + \sqrt{\frac{W_{to}}{10^6}}\right)$$  \hspace{1cm} (2.6)

This means that $w_t t_{ss}$ varies typically between 160 and 220 N/m². It is emphasized, however, that technology factors — e.g. the choice of composite materials for flaps and slats — can have a large effect on this.

Several remarks should be made concerning the ultimate load factor. For MTOW in excess of 50 000 lbs (222.5 kN) it is either equal to 3.75 for manoeuvre critical aircraft, or — for gust critical cases — equal to:

$$n_{ult} = 1.50 \left(1 + 0.50 K_g \rho_{st} U_{de} V_C S \frac{dC_L}{d\alpha} W_D \right)$$  \hspace{1cm} (2.7)

Using the relationship between the gust load alleviation factor $K_g$ and the mass parameter $\mu_g$ as defined in the airworthiness requirements, it can be shown that gust is critical if the wing loading is below the following value:

$$\frac{W_D}{S} = \frac{dC_L}{d\alpha} \left(\frac{0.44}{n_{man} - 1} \rho_{st} U_{de} V_C - 2.65 \rho g \bar{c}\right)$$  \hspace{1cm} (2.8)

where $n_{man}$ denotes the limit manoeuvre load factor — usually 2.50 for civil transports — and $\rho$ denotes the outside air density at a critical altitude, usually $\rho \approx 0.65$ kg/m³ @ 20 000 ft (6 150 m). Since the second term between brackets is a small fraction of the first, it will be acceptable to use a guessed constant value for the mean chord $\bar{c}$, which is defined as $S/b$. All factors in Equation 2.8 are determined by the aircraft design cruising speed (EAS) and Mach number and are therefore constant, except the lift gradient $dC_L/d\alpha$, which is also a function of the aspect ratio and the sweepback angle. If the wing loading is below the value given by Equation 2.8, we can write:

$$n_{ult} W_D = 1.50 (W_D + 0.50 K_g \rho_{st} U_{de} V_C S dC_L/d\alpha)$$  \hspace{1cm} (2.9)

and substitution into Equation 2.2 shows that for gust critical wings the bending term in fact consists of two contributions:

- one term proportional to $W_D b^3/S$;
- one term proportional to $b^3$.

In view of the complications explained in this chapter, it is preferable to use instead of Equation 2.1 the more practical Equation 2.2, with a constant load factor $n_{ult} = 3.75$. In some of the examples shown later the boundary value according to Equation 2.8 will be indicated. As long as the optimum wing loading exceeds this value, the generality of the resulting expressions is not impaired.
Chapter 3

DERIVATION OF THE FIGURE OF MERIT

The All-Up Weight of an aircraft consists of the empty weight, the fuel load and the payload:

\[ W_{to} = W_e + W_f + W_p \]  \hspace{1cm} (3.1)

In the present study the empty weight will be considered to consist of fixed and variable components:

\[ W_e = \sum W_{e,fixed} + \sum W_{e,variable} \]  \hspace{1cm} (3.2)

Since the effect of wing geometry variation and powerplant thrust are both subject of the optimization study, we write

\[ \sum W_{e,variable} = W_W + W_P \]  \hspace{1cm} (3.3)

It can be argued that due to variations in wing size there will also be secondary effects, e.g. variations in tailplane size. This can readily be taken into account in the present analysis, e.g. by increasing the wing weight by a certain percentage (e.g. 15%) allowing for the necessary growth of the horizontal tailplane when the wing area increases.

The fixed empty weight in Equation 3.2 is related to components that are associated directly with the payload, e.g. fuselage structure, services and equipment, and to other invariable weight contributions. The vertical tailplane weight is related primarily to fuselage geometry and should be considered as fixed. The precise numerical value of the fixed empty weight is irrelevant for the present study, since combination of Equations 3.1 through 3.3 into:

\[ \frac{W_p}{W_{to}} = \frac{1 - \left( W_f/W_{to} + W_p/W_{to} + W_W/W_{to} \right)}{1 + \sum \left( W_{e,fixed}/W_p \right)} \]  \hspace{1cm} (3.4)

shows that the payload fraction \( W_p/W_{to} \) is maximal for a minimum extremal value of the Figure Of Merit:

\[ FOM \overset{\text{def}}{=} \frac{W_f + W_P + W_W}{W_{to}} \]  \hspace{1cm} (3.5)

which is not affected by the term \( W_{e,fixed}/W_p \). In the following sections the three terms of this equation will be further considered.
3.1 Fuel weight fraction

The fuel load of long-range transport aircraft consists mainly of cruise fuel and reserve fuel; minor components are climb and descent fuel. In [10] the derivation has been given of a useful approximation of the total fuel load of an aircraft, which can be represented as follows:

\[
\frac{W_f}{W_{to}} = \frac{R}{R_H} \frac{C_D}{\eta C_L} \quad \text{with} \quad R_H \overset{\text{def}}{=} H/g \approx 4400 \text{km}
\]

(3.6)

The equivalent all-out range \( R \) in Equation 3.6 is the total distance that an aircraft can cover in a hypothetical cruising flight, using all available fuel. It also takes into account that Equation 3.6 is in fact the first term of a series expansion of the Bréguet range equation in the fuel weight fraction. The equivalent range \( R \) is typically equal to the design range plus about 1000 km.

The drag-to-lift ratio in Equation 3.6 is considered to be composed of three terms:

- the profile drag of the wing (plus horizontal tail), including compressibility drag;
- the parasite drag of the fuselage (plus vertical tail);
- the induced drag in the design cruise condition, including trim drag.

Hence:

\[
C_D = C_{D,\phi} + \beta C_L^2 = C_{D,p} + \frac{F_{par}}{S} + \beta C_L^2 \quad \text{with} \quad \beta \overset{\text{def}}{=} \frac{dC_D}{dC_L^2} = (\pi A\varphi)^{-1}
\]

(3.7)

The primary effects of wing area and aspect ratio variation are apparent from this expression. Although the profile drag coefficient will be affected by wing geometry and altitude through Reynolds number effects, we will ignore this complication. This is considered justified in view of the analytical character of the present study. However, for high-subsonic aircraft this is not always justified and corrections will be presented in Chapter 6.

With respect to the drag breakdown according to Equation 3.7 the following further remarks can be made:

1. The profile drag coefficient \( C_{D,p} \) can be written as follows:

\[
C_{D,p} = (C_{D,p}^H + (C_{D,p})_H \frac{S_H}{S_W})
\]

(3.8)

where \( ^H \) and \( ^W \) denote the horizontal tail and wing, respectively. It is fair to assume both profile drag coefficients equal and than \( C_{D,p} \) is simply found from the wing profile drag by means of the multiplication factor \( 1 + S_H/S_W \). The parasite drag area in Equation 3.7 is defined as:

\[
F_{par} = \sum (C_{D,\text{par}} S)
\]

(3.9)

These contributions refer to fuselage and vertical tail but should not contain the powerplant installation drag of nacelles and pylons, since this is not a constant term due to variation of the engine size. Powerplant drag will be accounted for by reducing the thrust and the overall efficiency.

2. The induced drag factor \( \varphi \) can be interpreted as the ratio of aerodynamic aspect ratio to geometric ratio. It is not identical to the Oswald factor, since it refers to the design condition. As a result, the wing aerodynamic design (e.g. camber and angle of incidence) is assumed to be matched to the design lift coefficient, resulting in a value of \( \varphi \) rather close to one, e.g. 0.90–0.95.
The installed overall powerplant efficiency in Equation 3.6 is defined as follows:

\[ \eta = \frac{(T_{is} - D_P)V}{m_P H} = \eta_{is} \left(1 - \frac{D_P}{T_{is}}\right) = \eta_{is} \left\{1 - q_o \frac{(C_D S)_P}{T_{is}/\delta}\right\} \] (3.10)

The isolated engine overall efficiency can be obtained from engine data provided by engine manufacturers. Contrary to the Thrust Specific Fuel Consumption (TSFC) \( \eta \) appears to be insensitive to altitude variation above \( \approx 9000 \) m (30 000 ft) For given Mach number the correction term \( D_P/T_{is} \) can be considered as constant, since:

- the engine nacelle and pylon sizes grow proportional to the installed thrust, hence \( (C_D S)_P \propto T_{is}/\delta \), where \( \delta \) denotes the relative ambient pressure, \( \delta \equiv p/p_{st} \).
- for given engine rating both \( T \) and \( D_P \) are approximately proportional to the ambient pressure, hence \( T_{is}/\delta \) is constant.

For modern high-bypass engines the overall efficiency is typically 0.30–0.35 for high-subsonic Mach numbers.

The flight conditions at the initial cruise altitude and for a weight practically equal to MTOW will be defined as 'design condition'. From vertical equilibrium we have:

\[ C_L S = W_{to}/q = W_{to}/(q_o \delta) \] (3.11)

where \( q_o \equiv 0.5 \gamma p_{st} M^2 \). Since Mach number variations are not considered and we maximize the payload for given take-off weight, the following applies:

\[ \delta C_L S = W_{to}/q_o = \text{constant} \] (3.12)

The drag-to-lift ratio is now obtained from Equations 3.7 and 3.11:

\[ \frac{C_D}{C_L} = \frac{C_{D,P}}{C_L} + \omega \delta + \frac{C_L}{\pi A_P} \] (3.13)

where the (constant) parasite drag parameter \( \omega \) is defined as follows:

\[ \omega \equiv q_o F_{par}/W_{to} \] (3.14)

In Equation 3.13 we consider the relative pressure \( \delta \) and the design lift coefficient \( C_L \) as independent variables, instead of the altitude and wing area or wing loading. This is logical since both factors are non-dimensional and between zero and one. Once optimum values for both factors have been established the optimum wing loading follows from Equation 3.11.

### 3.2 Powerplant weight fraction

Engines may be sized either to an initial cruise condition, with some reserve for climb potential, or by low-speed performance requirements. In the present study the first of these options will be used, since for long-range aircraft this leads to the most general results and the lowest value of the \( FOM \). Total engine thrust at take-off is then found from the thrust-to-weight ratio:

\[ (T/W)_{to} \approx T_{to}/W_{cr} = (T/W)_{cr} \times (T_{to}/T_{cr}) = \frac{C_D/C_L}{T_{cr}/T_{to}} \] (3.15)

The engine thrust lapse rate \( T_{cr}/T_{to} \) is defined by means of a generalized thrust parameter:

\[ \tau \equiv \frac{T_{cr}}{\delta T_{to}} = \frac{T_{is}}{\delta T_{to}} \left(1 - \frac{D_P}{T_{is}}\right) \] (3.16)
3.3. THE FIGURE OF MERIT

In accordance with the drag breakdown in Section 3.1 the correction for powerplant installation drag in Equation 3.16 is necessary since \( C_D \) represents airframe drag only. At high altitudes — near the tropopause — \( \tau \) has an approximately constant value, e.g. 0.8–0.9 typically. It can be obtained from engine data relating to the recommended engine rating for long-range cruising flight, and from the geometry of the nacelles.

The powerplant installation weight is considered proportional to the total take-off thrust by a factor \( \mu_T \):

\[
\mu_T \triangleq \frac{W_P}{T_{to}}
\]

(3.17)

The order of magnitude of \( \mu_T \) — which is considered as a constant factor in this study — is about 0.15–0.20 for the uninstalled engines and 0.30–0.35 for the complete powerplant installation, including nacelles, pylons, thrust reversers, etc. From Equations 3.15 through 3.17 we derive the powerplant weight fraction:

\[
\frac{W_P}{W_{to}} = \frac{\mu_T/\tau}{\delta} \frac{C_D}{C_L}
\]

(3.18)

This equation clearly shows that this weight contribution is sensitive to the initial cruise altitude. For example, increasing the cruise altitude from 10 to 11 km increases the powerplant weight by 16.5%. It is clear that in the optimization of the altitude the term \( \mu_T/\tau \) is very significant.

3.3 The Figure Of Merit

The wing weight fraction is obtained from Equation 2.2. However, instead of the span the aspect ratio will be used:

\[
A \triangleq b^2/S
\]

(3.19)

The wing area is replaced by the lift coefficient, using Equation 3.11. Furthermore we introduce the aspect ratio parameter:

\[
\phi_A \triangleq 0.06 \frac{C}{\delta^{1/2}} \frac{W_D}{\sqrt{\delta}} \frac{f(\lambda)}{(t/c) \cos^2 \Lambda}
\]

(3.20)

and the area parameter:

\[
\phi_S \triangleq \frac{W_{ass}}{g_0}
\]

(3.21)

Both non-dimensional parameters \( \phi_A \) and \( \phi_S \) will be considered as constants for the present moment, but it is emphasized that they are by no means independent of the aircraft’s design characteristics. For example, in order to allow for compressibility effects variation of the wing thickness-to-chord ratio and the sweep angle will be taken into consideration in Chapter 6, resulting in variations of \( \phi_A \). A first estimate of \( \phi_A \) and \( \phi_S \) can be made by means of the data and relationships presented in Chapter 2, Equations 2.2 through 2.6.

Substitution of Equations 3.11 and 3.19 through 3.21 into Equation 2.2 yields:

\[
\mu_W \triangleq \frac{W_W}{W_{to}} = \phi_A A \sqrt{\frac{A}{\delta C_L}} + \frac{\phi_S}{\delta C_L}
\]

(3.22)
Chapter 3. DERIVATION OF THE FIGURE OF MERIT

The $FOM$ according to Equation 3.5 is now obtained by adding the various terms given by Equations 3.6, 3.18 and 3.22:

$$FOM = \left( \frac{R/R_H}{\eta} + \frac{\mu_T/\tau}{\delta} \right) \left( \frac{C_{D,F}}{C_L} + \omega \delta + \frac{C_L}{\pi A \varphi} \right) + \phi_A A \sqrt{\frac{A}{\delta C_L}} + \frac{\phi_S}{\delta C_L} \tag{3.23}$$

It is significant that this $FOM$ is composed of three groups of contributions, which are mutually independent:

1. The powerplant contribution:

$$F_P = \frac{R/R_H}{\eta} + \frac{\mu_T/\tau}{\delta} \tag{3.24}$$

This quantity is — apart from the powerplant characteristics $\eta$, $\tau$ and $\mu_T$ — affected by the range, speed and cruise altitude. In previous publications the author has baptized Equation 3.24 as a 'Powerplant Merit Function'. Since its minimum value corresponds to a minimum MTOW this Figure Of Merit can be used as a means of selecting a suitable powerplant cycle, independent of the aircraft characteristics.

2. The airframe aerodynamic fineness ratio $C_D/C_L$. Although this quantity is affected by the aerodynamic design quality, it is also subject to optimization, through proper choice of the wing loading, the aspect ratio, the cruise altitude and other shape parameters.

3. The wing weight fraction, given by the last two terms of Equation 3.23, which are essential in the optimization: since increasing the aspect ratio only reduces the first term, there is no practical aerodynamic optimum aspect ratio.

All independent variables to be optimized are explicit in Equation 3.23, and it will appear in the next chapter that this equation is very suitable for deriving analytic criteria for the partial optima.
Chapter 4

PARTIAL AND UNCONSTRAINED OPTIMA

The $FOM$ according to Equation 3.23 is illustrated by an example for a long-range (8000 km) transport aircraft with a MTOW of 120 tons (Figure 4.1). The results are shown on Figures 4.2 and 4.3. The partial optima are defined as follows:

\[
\frac{\partial (FOM)}{\partial (C_L)} = 0 \quad \text{curve I}
\]

\[
\frac{\partial (FOM)}{\partial \delta} = 0 \quad \text{curve II}
\]

\[
\frac{\partial (FOM)}{\partial A} = 0 \quad \text{curve III}
\]

Intersection of these curves defines the unconstrained combined optima, for which the $FOM$ — and hence the payload fraction — is extremal.

4.1 Partial optimum lift coefficient

For given altitude and aspect ratio, the aircraft $L/D$ ratio achieves a maximum value for:

\[
C_L = C_{L,\text{ref}} \overset{\text{def}}{=} \sqrt{C_{D,p} \pi \varphi}
\]

(4.1)

The minimum $FOM$ as derived from Equation 3.23 by differentiation with respect to the lift coefficient, is directly related to this reference value of the lift coefficient:

\[
C_L = C_{L,\text{ref}} \left\{ 1 + \frac{1}{2} \Phi_A \sqrt{\delta_0 C_L + \Phi_S} \right\}^{1/2}
\]

(4.2)

Although this is a transcendent equation, a sufficiently accurate closed-form approximation can be obtained by substituting $C_L = C_{L,\text{ref}}$ in the r.h.s. If required, a second iteration can than be made.

The graphical representation of Equation 4.2 in Figures 4.2 and 4.3 (Curve I) shows the following properties of the partial optimum:

1. Although the altitude is present in $\delta$ and $F_P$, its effect on this partial optimum with altitude is relatively small.
2. The optimum $C_L$ is quite sensitive to aspect ratio variation, since $C_{L,\text{ref}} \propto \sqrt{A}$. However, if instead of $C_L$ we use $C_L/C_{L,\text{ref}}$ (cf. Figure 4.4) as the independent variable, the sensitivity is much smaller. This figure also shows clearly the difference between the present optimum and the 'aerodynamic' optimum ($C_L/C_{L,\text{ref}} = 1$) which results in maximum lift-to-drag ratio for given flight conditions.

4.2 Partial optimum altitude

When Equation 3.23 is minimized with respect to the altitude, the altitude variation is matched by variation of the wing loading, according to Equation 3.11. Hence for given $C_L$ the wing loading decreases with altitude, leading to an increase in wing area and hence weight. By partial differentiation of Equation 3.23 a result is found which can be written in simplified form by using the glide ratio of the wing:

$$\varepsilon_W \overset{\text{def}}{=} (C_D/C_L)_W = C_{D,\text{ref}}/C_L + C_L/(\pi A \varphi)$$

The optimum altitude is then obtained from:

$$\delta = \left\{ \frac{\eta}{\omega R/R_H} \left( \varepsilon_W \mu T/\tau + 0.5 \phi A A \sqrt{\frac{A \delta}{C_L} + \frac{\phi S}{C_L}} \right) \right\}^{1/2}$$
Figure 4.2. The Figure Of Merit affected by the lift coefficient and the altitude
Figure 4.3. The Figure Of Merit affected by the lift coefficient and the aspect ratio.
This equation can be solved readily by means of an iterative process, for example by substituting the relative pressure at the tropopause ($\delta = 0.225$) as a first approximation. In Figure 4.2 the result has been depicted as curve II, which shows that the optimum altitude is relatively insensitive to the lift coefficient.

It is interesting to compare the result with the optimum altitude for minimum weight of fuel plus powerplant, which is found by taking $C_L = C_{L,\text{ref}}$ and substituting $\phi_A = \phi_S = 0$ in Equation 4.4. The result thus found is referred to as a 'reference altitude', defined by:

$$\delta_{\text{ref}} \overset{\text{def}}{=} \left( \frac{2C_{D,p} \eta \mu T/\tau}{C_{L,\text{ref}} \omega R/R_H} \right)^{1/2}$$  \hspace{1cm} (4.5)

It can be shown that the optimum altitude including the wing weight contribution may then be approximated very closely by the following closed-form equation:

$$\delta = \delta_{\text{ref}} \left( 1 + \frac{2\phi_A A_s \delta_{\text{ref}} C_{L,\text{ref}} + \phi_S}{2C_{D,p} \mu T/\tau} \right)^{1/2} \left( \frac{C_L}{C_{L,\text{ref}}} \right)^{1/6}$$  \hspace{1cm} (4.6)

This result shows that the wing weight effect is quite substantial: the first bracketed term has a typical value of 2.50, which pushes the optimum $\delta$ up by about 60% relative to $\delta_{\text{ref}}$. Since moreover the optimum lift coefficient is increased by a factor $\approx 5/4$, the optimum wing loading increases by a factor 2. It therefore remains to be investigated how realistic the unconstrained optimum in Figure 4.2 is. This will be considered in Chapter 5.

### 4.3 Partial optimum aspect ratio

Differentiation of Equation 3.23 with respect to the aspect ratio yields the partial optimum:

$$A = C_L^{3/5} \left( \frac{2F_p \sqrt{\delta}}{3\pi \varphi \phi_A} \right)^{2/5}$$  \hspace{1cm} (4.7)

where $F_p$ is defined by Equation 3.24. This closed-form solution is depicted in Figures 4.3 and 4.4 as curve III. Since for constant $C_L$ and $\delta$ the wing loading is also fixed, Equation 4.7 defines the optimum aspect ratio for given wing loading. The present result shows that an optimum aspect ratio is found only for $\phi_A > 0$, even though there is a wing weight term which does not contain the aspect ratio. It is also clear that the optimum aspect ratio is quite sensitive to $F_p$, in particular to the range. The cruise altitude appears to have a minor influence.

Instead of optimizing the aspect ratio it can be useful to consider the span loading $W_{to}/b^2$ as the independent variable. If we divide this by the dynamic pressure, the span loading coefficient is obtained, to be interpreted as a lift coefficient referred to $(\text{span})^2$:

$$\frac{L}{\varphi b^2} = \frac{C_L}{} \overset{\text{def}}{=} C_{L,b}$$  \hspace{1cm} (4.8)

The partial optimum for the span loading is:

$$\frac{W_{to}}{b^2} = \frac{L}{\varphi b^2} = q_\delta \left( \frac{3\pi \varphi \phi_A}{2F_p \sqrt{\delta}} C_L \right)^{2/5}$$  \hspace{1cm} (4.9)

In Figure 4.3 and in Figure 4.5 — which is equivalent to Figure 4.3 — this result is indicated as curve V.
4.4 Unconstrained optima

Having derived partial optima for $C_L$, $\delta$, and $A$ we may now combine the various solutions into unconstrained optima, which is recognised in Figures 4.3 through 4.5 as the intersection of curves I, III, and IV.

a. Optimum wing geometry for given altitude. By simultaneous solution of Equations 4.2 and 4.7 we find the following result for $\partial(FOM)/\partial C_L = \partial(FOM)/\partial A = 0$:

$$C_L = \left(\frac{3}{2}\pi\varphi\right)^{3/7} \left(\frac{F_P \sqrt{\delta}}{\phi_A}\right)^{2/7} \left(C_{D,p} + \frac{\phi_S}{\delta F_P}\right)^{5/7}$$

$$A = \left(\frac{3}{2}\pi\varphi\right)^{-1/7} \left(\frac{F_P \sqrt{\delta}}{\phi_A}\right)^{4/7} \left(C_{D,p} + \frac{\phi_S}{\delta F_P}\right)^{3/7}$$

(4.10) (4.11)

As opposed to Equation 4.2 — which is a transcendental equation — the optimum lift coefficient according to Equation 4.10 is a closed-form analytical solution. For this unconstrained optimum we also find the general condition:

$$\frac{C_L}{C_{L,\text{ref}}} = \sqrt{1.50} \left(1 + \frac{\phi_S}{\delta C_{D,p} F_P}\right)^{1/2}$$

(4.12)

For long-range aircraft usually $\phi_S \ll \delta C_{D,p} F_P$, and $C_L/C_{L,\text{ref}}$ approaches the value $\sqrt{1.50}$. It is significant that the lift coefficient and aspect ratio according to Equations 4.10 and 4.11 show little variation with the altitude and the fuselage parasite drag, which makes the optimization of wing geometry practically independent of these parameters.
b. **Optimum wing loading and altitude for given aspect ratio.** A combined solution of $C_L$ and $\delta$ from Equations 4.2 and 4.4 does not result in a manageable equation. However, a very good approximation is found from Equations 4.3 and 4.4 by taking $C_L \approx 1.30 C_{L,\text{ref}}$. The value of $\delta$ thus found is then substituted into Equation 4.2 and finally the wing loading is found from Equation 3.11.

\[ A^* = C_L^{3/5} \left( \frac{2 \sqrt{\delta^*}}{3 \pi \phi_A \eta} \left( \frac{R/R_H}{\tau} + \frac{\mu_T/\tau}{\delta^*} \right) \right)^{2/5} \]

\( A^* = 0.225 \) with $\delta^*$ = 0.225. Substituting this aspect ratio into Equation 4.4 yields the optimum altitude. A second iteration may be started by substituting this value of $\delta$ into Equation 4.7.

d. **Unconstrained global optimum.** For given altitude the minimum value of $FOM$ is obtained by substituting Equations 4.10 and 4.11 into Equation 3.23. This results in the following
interesting relationship:

$$FOM = P \omega \delta + 3.50 F_P^{5/7} F_W^{2/7} \delta^{-1/7}$$  \hspace{1cm} (4.14)

where the 'Wing Merit Function' $F_W$ is defined as follows:

$$F_W \overset{\text{def}}{=} \phi_A \left( C_{D,P} + \frac{\phi_S}{\delta F_P} \right) \left( \frac{3}{2} \pi \varphi \right)^{-3/2} \left( \frac{R/R_H}{\eta} \right)^{-1/2} \omega^{-5/9}$$  \hspace{1cm} (4.15)

This Figure Of Merit — which should obviously have the lowest possible value — weighs the relative importance of the weight sensitivity parameters $\phi_A$ and $\phi_S$ against the induced drag factor $\varphi$ and the profile drag coefficient $C_{D,P}$ . Effectively this FOM can be used for trading off structural weight against aerodynamic effects due to variation in shape parameters such as thickness-to-chord ratio, sweep angle, and taper ratio.

In order to derive the global unconstrained optimum, Equation 4.14 has to be minimized with respect to the altitude, which can be done either numerically or by accepting the following approximation:

$$\delta \approx 7/4 \left\{ \left( \frac{\mu_T/\tau}{5C_{D,P}} \right) \sqrt{\phi_A C_{D,P}} \right\}^{1/3} \left( \frac{R/R_H}{\eta} \right)^{-1/2} \omega^{-5/9}$$  \hspace{1cm} (4.16)

Since this result is totally independent of the aspect ratio, wing area and lift coefficient, it can be used as a first result for the global optimum altitude. Subsequently it is substituted into Equations 4.10 and 4.11 to find the lift coefficient and aspect ratio and finally the wing loading is obtained from Equation 3.11. It will then usually appear that this global optimum is not feasible in practice and the constrained optimization treated in the following Chapter will most probably lead to a more realistic result.
Chapter 5

UNCONSTRAINED VERSUS CONstrained OPTIMA

Although the various partial optima derived in Chapter 4 can lead to results that are reasonable in isolation of each other, their combination is usually of questionable practical value. For example, for an aspect ratio of 10 Figure 4.2 defines an optimum combination of initial altitude ($\delta = 0.29$, altitude 9300 m.) and lift coefficient $C_L = 0.62$, resulting in a wing loading of 8500 N/m², approximately. Such a value is likely to be too high for acceptable low-speed performances. Also, as indicated by Figure 4.3 the unconstrained optimum lift coefficient and aspect ratio are both quite high from an aerodynamical and structural viewpoint, although the wing span in this point appears to be quite normal for the aircraft under consideration (see Figure 4.1). Having accepted that the unconstrained optimum will not always be favoured, the designer should decide in which direction to move away from the unconstrained optimum. Some considerations are given below.

5.1 Optimum wing loading and altitude

Referring to Figure 4.2 it will be clear that — once a minor deterioration of the $FOM$ is considered to be acceptable — the choice of the lift coefficient and the altitude can go in many directions away from the unconstrained optimum (point O).

Figure 5.1 depicts the partial optima from Figure 4.2 in the generalized co-ordinates $C_L/C_{L,ref}$ and $\delta/\delta_{ref}$. The reference lift coefficient — defined by Equation 4.1 — is indicated by curve Ib, which defines the partial optimum for maximum $L/D$ of the wing, and minimum fuel plus powerplant weight fraction; the reference density is defined by Equation 4.5. Three curves of type I (partial optimum $C_L$) and type II (partial optimum altitude) have been indicated in Figure 5.1, defining optimisation according to the following criteria:

- type a: minimum weight of (fuel + powerplant + wing structure);
- type b: minimum weight of (fuel + powerplant);
- type c: minimum weight of (fuel + wing structure).

The intersection of curves I and II define the unconstrained optima for $C_L$ and $\delta$ for the three cases mentioned above, $O_a$, $O_b$, and $O_c$, resp. Furthermore curves have been drawn which define constant wing loading and constant thrust loading (hence constant $L/D$). Also indicated is the
Figure 5.1. Partial optima for the lift coefficient and altitude in generalized co-ordinates

condition for minimum drag: \( C_L = \sqrt{C_{D,\infty} \pi A \varphi} \), which limits the altitude according to:

\[
\delta_{md} \geq \frac{1}{\omega} \left( \frac{C_L}{\pi A \varphi} - \frac{C_{D,E}}{C_L} \right) \quad \text{for } C_L, C_{L,ref}
\]

(5.1)

Only a very limited segment of the design space in Figure 5.1 is interesting:

- To the left of curve Ib and to the right of curve Ia all Figures Of Merit are deteriorating.
- Between curves Ia and Ib there is a trade-off between wing weight on the one hand and fuel plus powerplant weight on the other hand.
- Below curve IIa all Figures Of Merit are deteriorating.
- Above curve IIb engine size is progressively increasing, fuel weight is decreasing, but all Figures Of Merit are deteriorating.
- Between curves IIa and IIb there is a trade-off between fuel weight on the one hand and engine plus wing weight on the other hand.

Furthermore some other general observations can be made:

- The right-hand side of the diagram with its high values of the lift coefficient will probably give rise to aerodynamic problems, such as compressibility effects in the case of high subsonic flight.
The upper left-hand side features low wing loadings, resulting in a gust-sensitive aircraft.

The lower part of the diagram forces the design to a low cruise altitude. This would only be justified if there is an unescapable engine thrust limit.

The only interesting sector of the diagram is therefore the area enclosed by curves Ia, Ib, IIa and IIb. The logical choice of the design point is thus to select a combination of $C_L$ and $\delta$ that is reasonably close to point $O_a$ and at the same time satisfies the design constraints.

### 5.2 Constrained optima for the lift coefficient and altitude

The curves in Figure 5.1 for constant thrust and constant wing loading can be used to demonstrate the effects of practical constraints.

1. The unconstrained optimum for given engine(s) is either point $O_c$ or the intersection of the constant-thrust line with curve Ic. In the latter case the thrust is insufficient to attain the optimum altitude. Moving along the constant-thrust line to the left improves take-off performance, since the wing loading is decreasing. If in addition there is a constraint on the wing loading, the optimum is determined by the intersection of both constraints (point $C_1$) since to the left of this point the $\text{FOM}$ is deteriorating.

2. An inequality constraint on the wing loading can be imposed by specified low-speed performance boundaries or by a wing fuel volume constraint which is affected by the aspect ratio (see Section 5.3).

The optimum lift coefficient and altitude for given wing loading are obtained by minimizing the fuel plus powerplant weight fraction, since the wing weight is constant. Differentiating Equation 3.23 with a constraint on the wing loading yields the following relationship between $C_L$ and $\delta$:

$$C_L^2 \left(1 + 2 \frac{\eta \mu T}{\tau} \frac{R}{R_H}\right) = \pi A \varphi(C_D, \tau + \omega \delta C_L)$$

The solution of this quadratic equation has been denoted in Figures 4.2 and 5.1 as curve IV. Intersecting this curve with the specified value of the wing loading yields the constrained optimum altitude and lift coefficient (point C in Figure 5.1). It is noted here that curve IV can be approximated by a straight line between point $O_a$ and $O_b$, or by:

$$\delta \approx \delta_{md} + 0.7 \frac{\eta \mu T / \tau}{R / R_H}$$

where $\delta_{md}$ applies to the given wing loading, which follows directly from Equation 3.11 by substitution of: $C_{L,md} = \sqrt{C_D, \pi A \varphi}$.

### 5.3 Constrained optima for aspect ratio and wing loading

Practical engineering considerations may limit the aspect ratio or the lift coefficient to values below the optimum values obtained in Section 4.4. Such constraints can be readily incorporated in the partial optima according to Equations 4.2 or 4.7. In the following paragraphs several additional constraints will be considered.
a. A constraint on the wing span — for example associated with airfield size limitations — can be translated into:

\[
\frac{C_L}{A} \geq \frac{W_{to}}{b_{max}^2} \tag{5.4}
\]

If \( C_L \) and \( A \) according to Equations 4.10 and 4.11 do not comply with this condition, the optimum is found from

\[
\frac{\partial (\text{FOM})}{\partial A} = 0 \quad \text{for} \quad C_L/A = L/(qb^2) \xrightarrow{\text{def}} C_{L,b} \tag{5.5}
\]

As a result we find:

\[
C_L C_{L,b}^{-3/4} = A C_{L,b}^{1/4} = \left( \frac{F_P C_{D,p} + \phi S/\delta}{\phi_A/\sqrt{\delta}} \right)^{1/2} \tag{5.6}
\]

This relation is depicted in Figure 4.3 as curve V. Although for the present example a constraint on the span to a maximum of 40 meters will lay an active constraint on the aspect ratio, it leads to extremely high values of the lift coefficient.

b. Constraint on wing volume. For long-range aircraft a condition frequently imposed is the requirement that all fuel shall be carried inside the wing. Using a typical relationship between the available and the required tank volume,

\[
\nu S^2/b \geq W_{f}/(\rho_{fg}) \tag{5.7}
\]

the following approximation can be derived for the minimum required wing area:

\[
\rho_{fg} \nu S^{3/2} \geq W_{to} \frac{R_f/R_H}{\eta} \left\{ 2 \sqrt{\frac{C_{D,p}}{\pi \varphi}} + \omega \delta \sqrt{A} \right\} \tag{5.8}
\]

Since the wing loading obtained from this result increases proportional to \( W_{to}^{1/3} \) this constraint can be quite problematic for relatively small long-range aircraft, where it may require a large wing area — far from the optimum value.

Using a typical \( \nu = 0.05 \) — this number depends on the thickness/chord ratio, taper ratio and structural configuration — the constraint according to Equation 5.8 is depicted on Figure 4.3. The limit on \( C_L \) appears to be practically independent of the aspect ratio. Therefore the optimum lift coefficient and aspect ratio are determined by the intersection of Equations 4.7 and 5.8. If the resulting increment of the \( \text{FOM} \) is considered objectionable, the effect of cruise altitude variation should be investigated, since Equation 5.8 clearly indicates that the fuel required decreases with higher altitudes.

c. Constraint on wing weight. The solution for the unconstrained optimum wing loading and aspect ratio according to Equations 4.10 and 4.11 clearly indicates that both increase with increasing range. This is due to the fact that the optimum wing weight for this condition appears to be a certain fraction of the fuel weight. One may argue that it is undesirable to have a high wing weight, since development and production cost are directly related to the structural weight. If therefore a limit is imposed on the wing weight — resulting in an approximately constant empty weight of the aircraft — a limit is imposed on the aspect ratio, derived from Equation 3.23:

\[
A^{3/2} \leq \frac{1}{\phi_A} \left( \mu_W C_L \sqrt{\delta} - \phi_s \right) \tag{5.9}
\]
5.3. **CONSTRAINED OPTIMA FOR ASPECT RATIO AND WING LOADING**

This condition can be substituted into Equation 3.23 and the subsequent optimization in fact means that a maximum $L/D$-ratio is obtained for constant wing weight. The resulting expression appears to be rather unwieldy, but the following result is a good approximation:

$$C_L \approx \left( \frac{3}{2} C_{D,\text{ref}} \pi \varphi \right)^{3/5} \left( \mu_W \sqrt{\delta/\phi_A} \right)^{2/5} \left( 1 + \frac{\phi_S}{\delta \mu_W} \right)$$

(5.10)

Substitution into Equation 5.9 yields the constrained optimum aspect ratio, depicted in Figure 4.2 for $\mu_W = 0.12$. In the present case the constrained optimum is very close to the unconstrained optimum, but since Equation 5.10 does not contain the range — as opposed to the unconstrained optimum — this observation must be considered as a coincidence.

A final result from this paragraph is that for the wing weight-constrained optimum we find approximately:

$$C_L/C_{L,\text{ref}} = \sqrt{\frac{3}{2} \left( 1 + \frac{\phi_S}{\delta \mu_W} \right)}$$

(5.11)

Since $\phi_S \ll \delta \mu_W$ this result is practically independent of the precise value of $\phi_S$, $\delta$ and $\mu_W$. For typical values $\phi_S/\delta = 0.02$ and $\mu_W = 0.125$ we find:

$$C_L \approx 1.32 C_{L,\text{ref}}$$

(5.12)

Obviously it should be checked that Equation 5.9 is not satisfied by Equations 4.10 and 4.11, otherwise the unconstrained optimum remains valid.
Chapter 6

VARIATION OF THICKNESS RATIO AND SWEEP ANGLE

Up to this point we have ignored any variation in compressibility drag. Although for a constant Mach number this may seem reasonable\(^1\), it should be realized that variation in the design lift coefficient will affect the drag rise due to compressibility, unless compensating measures are taken, e.g. reducing the wing thickness and/or increasing the sweep angle. In the latter case the drag, fuel, and wing weight will be affected. The following terms contributing to the FOM as described by Equation 3.23 are involved.

1. The wing profile drag coefficient \(C_{D,p}\) which can be approximated by means of the ‘flat plate analogy’:

\[
C_{D,p} = 2C_F \left\{ 1 + \Phi(t/c) \cos^2 \Lambda \right\} + \Delta C_{D,\text{comp}}
\]  

(6.1)

for thickness/chord ratios less than 20%. The shape parameter \(\Phi\) — which varies typically between 2.50 and 3.50 — determines the form drag. It depends on the location of the transition point and on the location where maximum suction occurs on the upper wing surface. The compressibility drag \(\Delta C_{D,\text{comp}}\) (cf. Figure 6.1) in the design condition depends on the shape of the drag rise curve. For long-range flights at 99% of the maximum specific air range it varies usually between 5 and 10 counts, i.e. \(\Delta C_{D,\text{comp}} = 0.0005 - 0.0010\). In the present report the drag-divergence Mach number is defined as the Mach number for \(\Delta C_{D,\text{comp}} = 0.0020\).

2. The wing bending material weight — represented by the first term of Equation 3.23 — is inversely proportional to the factor \((t/c) \cos^2 \Lambda\), as is clear from Equation 3.20.

3. The induced drag can be sensitive to the wing thickness and sweep in the case that flow separation occurs in the design condition, for instance caused by a sharp nose or an unfavourable lift distribution. However, it is assumed that by proper aerodynamic design of the wing sections and twist distribution these flow conditions can avoided. The induced drag is than practically independent of \(t/c\) and \(\Lambda\).

\(^1\)In spite of compressibility effects the drag polar for a given Mach number can be considered as parabolic
The two variable terms in Equation 3.23 which are directly affected by the sweep angle and thickness/chord ratio are added:

$$\Delta\mathcal{FOM} = 2F_p C_F \Phi \frac{(t/c) \cos^2 \Lambda}{C_L} + \phi_A \frac{A^{3/2}}{\sqrt{\delta C_L}} \left\{ (t/c) \cos^2 \Lambda \right\}^{-1}$$  \hspace{1cm} (6.2)

where $\phi_A \overset{\text{def}}{=} \phi_A(t/c) \cos^2 \Lambda$ (see Equation 3.20).

With respect to variation in $t/c$ and $\cos \Lambda$ this function has a minimum value for:

$$(t/c) \cos^2 \Lambda = A^{3/4} (C_L/\delta)^{1/4} \left( \frac{\phi_A}{2F_p C_F \Phi} \right)^{1/2}$$  \hspace{1cm} (6.3)

For this theoretical optimum the first and second term of Equation 6.2 are equal.

In practice it will generally be found that this thickness/chord ratio will be too large to have any practical significance, since the condition $(t/c) \cos^2 \Lambda \leq 0.20$ will probably not be satisfied. For high-subsonic aircraft, however, the wing thickness will have an upper limit, which is determined by the ability to design a wing which has only a very limited amount of compressibility drag. This limit can be quantified approximately by means of Korn's Equation [1], which provides a simple means of estimating the limits of airfoil performance at high subsonic speeds:

$$M_D + \frac{t}{c} + 0.1 c_l = \kappa$$  \hspace{1cm} (6.4)

In this equation $M_D$ denotes the two-dimensional drag rise Mach number (Figure 6.1) — for which $\Delta c_d,\text{comp} = 0.002$ — and $c_l$ denotes the wing section design lift coefficient. The technology factor $\kappa$ in this equation amounts typically to 0.87 for $\sim 1965$ - technology wing sections, and 0.95 for supercritical sections, designed according to $\sim 1990$ state-of-the-art. In order to match Korn's Equation better with empirical data, we modify it as follows:

$$M_D + \frac{t}{c} + 0.1 c_l^{3/2} = M^*$$  \hspace{1cm} (6.5)
Figure 6.2. Upper limit of the thickness/chord ratio related to the sweep angle and drag rise Mach number

where \( M^* \approx 0.935 \) for 1990-technology. If we adapt Equation 6.5 to three-dimensional wings by means of the simple sweep theory we find:

\[
M_D \cos \Lambda + \frac{t/c}{\cos \Lambda} + 0.1 \left( \frac{C_L}{\cos^2 \Lambda} \right)^{3/2} = M^* 
\]

It should be noted here that \( M_D \) is a higher Mach number than the design cruise Mach number (see Figure 6.1):

\[
M_D = M_{des} + \Delta M \tag{6.7}
\]

If we now wish to put a constraint on the compressibility drag, an upper value for \( t/c \) is obtained from:

\[
(t/c) \cos^2 \Lambda = \cos^3 \Lambda (M^* - M_D \cos \Lambda) - 0.1 C_L^{3/2} \tag{6.8}
\]

This relationship is plotted on Figure 6.2 for a typical value of \( C_L \). It shows that for \( M_D \leq 0.75M^* \) the sweep angle can be zero. Above this value Equation 6.8 has a maximum value for:

\[
\Lambda = \arccos(0.75M^*/M_D) \tag{6.9}
\]

For this optimum sweep angle the first term of Equation 6.2 has a maximum value, while the second term has a minimum value. Since the wing weight term nearly always is dominant — except for ultra-long-range transports — Equation 6.9 indeed defines the condition of minimum \( SOM \).

The optimum sweep angle according to Equation 6.9 can be interpreted as a requirement that the Mach number component perpendicular to the mid-chord line should be equal \( 0.75M^* \). This extremely simple result indicates that the optimum sweep angle is primarily dependent on
Figure 6.3. Effect of lift coefficient, sweepback angle and lift coefficient on the FOM for $M_D = 0.825$

the design Mach number and on the aerodynamic technology. A more advanced state-of-the-art corresponds to a smaller optimum wing sweep angle. The thickness constraint defined by Equation 6.8 is, however, definitely affected by the lift coefficient. Hence it is logical to select the sweep angle first and then solve for the optimum thickness/chord ratio.

The effect of lift coefficient variation on the FOM may now be obtained by adapting the $t/c$ value for each combination of $C_L$ and $\Lambda$. The result is illustrated for the present aircraft on Figure 6.3, which also shows that for each $t/c$ a maximum value of the lift coefficient is obtained, resulting in the minimum wing area. This condition is defined by solving $C_L$ from Equation 6.8 and setting the derivative with respect to the sweep angle equal to zero, resulting in:

$$\Lambda = \arccos \left[ \frac{3M^*}{8M_D} + \left( \frac{3M^*}{8M_D} \right)^2 - \frac{t/c}{2M_D} \right]^{1/2}$$

(6.10)

Figure 6.3 shows clearly the difference between Equations 6.9 (minimum FOM ) and 6.10 (minimum S). This figure also shows that the $L/D$ curves describe a saddle surface, with a maximum w.r.t. $C_L$ and a minimum w.r.t. $\Lambda$. Since the optimum lift coefficient according to Figure 6.3 is approximately 0.55 for $A = 10$, whereas in Figure 4.3 the partial optimum $C_L = 0.63$ for the same aspect ratio, it is appropriate to revise Figure 4.3 for the present case, where the
thickness/chord ratio is varying so that for each \( C_L \) the compressibility drag remains constant. Using Equation 6.8, the profile drag and wing weight can be calculated and new contours of constant \( FOM \) are obtained. The result — depicted on Figures 6.4 and 6.5 — clearly shows that the unconstrained optimum lift coefficient and aspect ratio are rather different from the unconstrained optimum for constant \( t/c \) and \( \Lambda \) (Figure 4.3).

A similarly modified calculation of the \( FOM \) for varying \( C_L \) and altitude is depicted on Figure 6.6, while Figure 6.7 shows the partial and unconstrained maxima. Compared to Figure 4.2 these figures indicate that altitude variation effects on the \( FOM \) (curve II) are hardly affected by wing thickness variation, except at high lift coefficients where the partial optimum altitude shifts to slightly lower values of \( \delta \). However, for \( C_L > 0.5 \) there is an appreciable effect on the isomerit lines and the partial optimum for \( C_L \) shifts to a lower value, which is almost independent of the altitude.

Revised equations for the partial optima of \( A \), \( C_L \) and \( \delta \) can be obtained by setting the partial derivatives of the \( FOM \) equal to zero. The result can be expressed in terms of the value for \( (t/c) \cos^2 \Lambda \) for \( C_L = 0 \), according to Equation 6.8:

\[
f(\Lambda) \overset{\text{def}}{=} \cos^3 \Lambda (M^* - M_D \cos \Lambda) \tag{6.11}
\]

with the corresponding profile drag coefficient:

\[
C_{D,p}' \overset{\text{def}}{=} 2C_F \{1 + \Phi f(\Lambda)\} + \Delta C_{D,\text{comp}} \tag{6.12}
\]

a. The partial optimum for the aspect ratio as given by Equation 4.7 remains basically valid, but now the wing weight factor \( \phi_A \) must be adapted for variation of the thickness/chord
ratio and $\Lambda$ into:

$$\phi_A' \equiv \phi_A(t/c) \cos^2 \Lambda \quad (= \text{constant})$$  \hspace{1cm} (6.13)

resulting in, for $\partial(FOM)/\partial A = 0$:

$$A = C_L^{3/5} \left[ \frac{2F_P \sqrt{\delta}}{3\pi \varphi \phi_A'} \left\{ f(\Lambda) - 0.1 C_L^{3/2} \right\} \right]^{2/5}$$  \hspace{1cm} (6.14)

In this equation the sweep angle can be either assumed or selected according to Equation 6.9.

b. The partial optimum lift coefficient cannot easily be derived exactly but a satisfactory approximation is, for $\partial(FOM)/\partial C_L = 0$:

$$C_L = \left\{ \pi A \varphi \left( C_{D,p} + \frac{\phi_S}{\delta F_P} \right) \right\}^{1/2}$$  \hspace{1cm} (6.15)

Contrary to Equation 4.2 this expression does not contain the $\phi_A$-term since the variation in the bending moment weight term due to $C_L$ and $t/c$ variation appears to be small.

c. The partial optimum altitude as defined by Equation 4.4 remains valid, but the glide ratio of the wing according to Equation 4.3 has to be modified since $C_{D,p}$ now depends on the lift coefficient:

$$C_{D,p} = C_{D,p}' - 0.2 C_F \Phi C_L^{3/2}$$  \hspace{1cm} (6.16)

The unconstrained optimum can be obtained by simultaneous solution of Equations 6.14, 6.15 and 4.4, in combination with Equation 6.16. This must be done numerically. Since experience has shown, however, that the optimum aspect ratio and lift coefficient are insensitive to the altitude, the following procedure will lead most easily to the result.
1. Make a reasonable guess for the altitude, e.g. the tropopause, with $\delta = 0.225$. and solve Equations 6.14 and 6.15 simultaneously for $A$ and $C_L$.

2. Calculate $\delta_{ref}$ according to Equation 4.5 and solve Equation 4.6 with $C_L = C_{L, ref}$:

$$\delta = \delta_{ref} \left(1 + \frac{2 \phi_A A \sqrt{A \delta_{ref} C_{L, ref}} + \phi_S}{2 C_{D, p} \mu T / \tau} \right)^{1/2}$$  \hspace{1cm} (6.17)

where

$$\phi_A = \frac{\phi'_A}{f(A) - 0.1 C_{L, ref}^{3/2}}$$  \hspace{1cm} (6.18)

3. This new value of $\delta$ can be used improve the accuracy of $A$ and $C_L$. These are then substituted into Equation 4.3 to obtain the optimum altitude according to Equation 4.4.

Although the solution presented in this paragraph is slightly more complicated than the one derived for constant wing profile drag and $(t/c) \cos^2 \Lambda$, it is a much more realistic one. Other constraints — treated in Section 5.3 — should be adapted accordingly. For example, the wing volume limit appears to be quite sensitive to the thickness/chord ratio and should be modified accordingly. No attempt has been made to present a generalised result. Instead, explicit results — such as those of Figures 6.4 and 6.6 — with the various constraints superimposed are considered more powerful than the analytical solutions.
Figure 6.6. Effect of the lift coefficient and altitude on the FOM for $M_D = 0.825$

Figure 6.7. Partial and unconstrained optima for the case of Figure 6.6
Chapter 7

SUMMARY AND CONCLUDING REMARKS

A method has been developed for calculating a Figure Of Merit for long-range jet aircraft with engines sized for the cruise condition. This FOM represents the added weight fractions of the fuel load, the powerplant installation and the wing structure. Its minimum value results in a maximum value of the payload fraction.

The effects of variation in the initial cruise altitude and the wing geometry — aspect ratio and wing loading — are apparent from Equation 3.23:

$$FOM = F_P \left( \frac{C_{D_P}}{C_L} + \omega \delta + \frac{C_L}{\pi A \varphi} \right) + \phi_A A \sqrt{\frac{A}{\delta C_L}} + \frac{\phi S}{\delta C_L}$$

Instead of the wing loading — which may show a large variation between different aircraft — the design-lift coefficient has been used, which is related to the wing loading and the altitude:

$$C_L = \frac{W_{to}}{\delta q_0}$$

The FOM also contains the Powerplant Merit Function:

$$F_P = \frac{R}{R_H} + \frac{\mu_T/\tau}{\delta}$$

with $R_H \overset{\text{def}}{=} H/g$

which is affected only by the operational conditions and the powerplant characteristics: overall efficiency, specific weight, thrust lapse rate and installed drag. Although $F_P$ is thus a constant factor when wing geometric variations are considered — its scope can be wider when studies are made of different engine configurations and cycles for long-range jet aircraft, with engines sized for attaining a specified initial cruise altitude.

The FOM has been used primarily for deriving generalized results for the optimum aspect ratio, design lift coefficient — and hence the wing loading — and initial cruise altitude. In Chapters 4 and 5 equations have been derived for partial and constrained optima, assuming that the mean thickness/chord ratio and mean sweep angle of the wing remain constant. For high-subsonic aircraft, however, there is an upper limit to the profile thickness ratio, which is related to the sweep angle, the design lift coefficient and the aerodynamic design technology. Using a modification of Korn's Equation 6.4 the following expression was derived which approximates the relation between the wing geometry, the lift coefficient and the drag-rise Mach number.

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(Equation 6.6):

\[ M_D \cos \Lambda + \frac{t/c}{\cos \Lambda} + 0.1 \left( \frac{C_L}{\cos^2 \Lambda} \right)^{3/2} = M^* \]

The parameter \( M^* \) reflects the aerodynamic design technology; it may be assumed equal to 0.935 for \( \sim 1990 \) supercritical wings. Using this result is was shown in Chapter 6 that the \( \mathcal{FOM} \) is minimum for an optimum sweep angle (Figure 6.2) defined by Equation 6.9:

\[ \Lambda = \arccos(0.75 M^* / M_D) \]

This defines effectively a quarter-chord sweep angle resulting in a normal Mach number equal to approximately 0.70 for \( M^* = 0.935 \). It is to be noted that \( M_D \) is higher than the design Mach number for long-range cruising flight (Figure 6.1). Usually a smaller sweepback angle will be selected than the one given by Equation 6.9, in the interest of better low-speed properties. For given sweep angle the highest (and optimum) wing thickness ratio follows from Equation 6.8:

\[ t/c = \cos \Lambda (M^* - M_D \cos \Lambda) - 0.1 C_{L,I}^{3/2} / \cos^2 \Lambda \]

Several results will now be summarized for partial optima, derived analytically by setting the partial derivative of the \( \mathcal{FOM} \) equal to zero.

a. The optimum lift coefficient is closely related to:

\[ C_{L,ref} \overset{\text{def}}{=} \sqrt{C_{D,p} \pi \Lambda \varphi} \]

which defines — for given flight conditions and wing shape — the wing loading for minimum drag, and hence minimum fuel and powerplant weight. For the case of constant \( t/c \) and \( \Lambda \) Equation 5.2 defines the partial optimum, but a more refined result was derived in Chapter 6 for a constant drag rise Mach number (Equation 6.15):

\[ \frac{\partial(\mathcal{FOM})}{\partial C_L} = 0 \Rightarrow C_L = \left\{ \pi \Lambda \varphi \left( C'_{D,p} + \frac{\phi_A}{\delta F_P} \right) \right\}^{1/2} \]

where \( C'_{D,p} \) is defined by Equation 6.12. This equation shows that the optimum lift coefficient — and hence the optimum wing loading — is quite sensitive to the aspect ratio and much less sensitive to altitude variation.

b. The partial optimum aspect ratio is defined by Equation 4.7:

\[ \frac{\partial(\mathcal{FOM})}{\partial A} = 0 \Rightarrow A = C_{L,I}^{3/2} \left( \frac{2 F_P \sqrt{\delta}}{3 \pi \varphi \phi_A} \right)^{2/5} \]

In the case of constant thickness/chord and sweep angle the factor \( \phi_A \) — defined by Equation 3.20 — is a constant. In that case the optimum \( A \) is affected mainly by the lift coefficient and the range (through \( F_P \)); it is practically independent of the altitude. In the case of constant drag-rise Mach number a modified result is given by Equation 6.14. It is then found that for high \( C_L \) - values the optimum aspect ratio will be reduced (see Figure 6.5).

c. The partial optimum initial cruise altitude is related to the reference altitude, for which the fuel plus powerplant weight is minimal (Equation 4.5):

\[ \delta_{ref} \overset{\text{def}}{=} \left( \frac{2 C_{D,p} \eta_H \tau}{\sqrt{C_{L,ref} \omega R / R_H}} \right)^{1/2} \]
It can be shown that for this — rather high — altitude the following relationship applies:

\[
\frac{\text{powerplant weight}}{\text{fuel weight}} = \frac{\text{parasite drag}}{\text{minimum wing drag}}
\]

The partial optimum altitude is not sensitive to the lift coefficient and can be approximated by Equation 4.6:

\[
\delta = \delta_{ref} \left( 1 + \frac{\frac{2}{3} \phi \Delta A \sqrt{A \delta_{ref} C_{L,ref} + \phi S}}{2 C_{D,p} \mu T / \tau} \right)^{1/2} \left( \frac{C_{L}}{C_{L,ref}} \right)^{1/6}
\]

If a more refined solution is desired Equation 4.4 may be used which is, however, transcendent.

d. **Several combined optima** were derived in Section 4.4. For example, Equations 4.10 and 4.11 define the case of \( \partial(\text{FOM})/\partial C_L = \partial(\text{FOM})/\partial A = 0 \). In Chapter 6 it was found that this solution has to be modified considerably when compressibility has to be taken into account. Moreover, practical constraints on \( C_L \) — e.g. due to buffet limits — and on the aspect ratio will force the designer to select an off- optimum combination. Since the \( \text{FOM} \) is not very sensitive to small excursions away from the optimum, the following unconstrained minimum for given altitude remains useful (Equation 4.14):

\[
\min(\text{FOM}) = F_p \omega \delta + 3.50 \frac{F_p}{F_w} \frac{C_{L,ref}}{C_{L}} \delta^{-1.7}
\]

where the 'Wing Merit Function' \( F_w \) is defined by Equation 4.15. Prior to the selection of wing loading and aspect ratio this result defines the minimum weight condition achievable for given thickness/chord ratio and sweep angle. It does not apply when the design aims at obtaining a given drag rise Mach number, as explained in Chapter 6.

e. **Design constraints** have been treated in Paragraphs 5.2 and 5.3. An important case is a wing loading constraint, usually associated with desirable low-speed properties or a wing fuel volume limit. In this case the zero-lift drag coefficient is fixed and a minimum-drag condition defined by:

\[
C_{L,md} = \sqrt{C_{D,0}} \pi A \rho
\]

The corresponding relationship between \( C_L \) and \( \delta \), defining a minimum (constrained) \( \text{FOM} \) is Equation 5.2, which may be written alternatively as follows:

\[
C_L = C_{L,md} \left( 1 + 2 \frac{\text{powerplant weight}}{\text{fuel weight}} \right)^{-1/2}
\]

This is, however, not a closed-form solution since the r.h.s. contains the relative pressure. A simple approximation for the optimum altitude is:

\[
\delta \approx \delta_{md} + 0.70 \frac{\eta \mu_T / \tau}{R / R_H}
\]

and the lift coefficient is obtained from:

\[
C_L = C_{L,md} \delta_{md} / \delta
\]

All optima derived in the present report are intended as guidelines for a proper selection of initial cruise altitude and wing geometry. However, the designer has to consider more requirements than just the weight sensitivity. Instead of calculating the generalized optima, one may prefer to prepare complete diagrams with isomerit curves, such as Figures 6.3, 6.4 and 6.6.
Inclusion of pertinent design constraints, such as those mentioned in paragraph 5.3, allows the designer to show the sensitivity of the optimum to specific requirements that make it impossible to select the unconstrained optimum.

Another useful application of the present theory is that it makes clear in a generalised form the sensitivity of the $FOM$ to operational requirements — such as the range and fuselage size — and to technological factors, such as:

**aerodynamic technology:** $C_{D, p}$ and $M^*$

**structural technology:** $\phi_S$ and $\phi_A$

**powerplant technology:** $\eta$, $\mu_T$ and $\tau$.

The present theory can be extended to incorporate cost considerations through proper modification of the $FOM$. Subsequent analyses will be developed to show that very similar results are obtained to those of the present report [11].
Bibliography
