Some Kinematic Relations of Thin Elastic Shells

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### Table of contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acknowledgement</td>
<td>2</td>
</tr>
<tr>
<td>List of symbols</td>
<td>2</td>
</tr>
<tr>
<td>Introduction</td>
<td>3</td>
</tr>
<tr>
<td>1. General kinematic relations of thin shells</td>
<td>3</td>
</tr>
<tr>
<td>2. Some linear thin shell theories</td>
<td>7</td>
</tr>
<tr>
<td>2.1 Love’s first approximation</td>
<td>7</td>
</tr>
<tr>
<td>2.2 General linear kinematic relations – the relations of Byrne, Flügge, Goldenveizer, Lur’ye and Novozhilov</td>
<td>8</td>
</tr>
<tr>
<td>2.3 Relations of Love and Timoshenko</td>
<td>10</td>
</tr>
<tr>
<td>2.4 Relations of Reissner, Naghdi and Berry</td>
<td>11</td>
</tr>
<tr>
<td>2.5 Relations of Donnell and Mushtari</td>
<td>11</td>
</tr>
<tr>
<td>2.6 Relations of Vlasov</td>
<td>12</td>
</tr>
<tr>
<td>2.7 Relations of Sanders</td>
<td>13</td>
</tr>
<tr>
<td>2.8 Modified Donnell theory</td>
<td>13</td>
</tr>
<tr>
<td>2.9 Discussion of the results</td>
<td>13</td>
</tr>
<tr>
<td>3. Some nonlinear thin shell theories</td>
<td>19</td>
</tr>
<tr>
<td>3.1 Kirchhoff assumptions</td>
<td>19</td>
</tr>
<tr>
<td>3.2 Novozhilov theory</td>
<td>23</td>
</tr>
<tr>
<td>3.3 Modified Novozhilov theory 1</td>
<td>25</td>
</tr>
<tr>
<td>3.4 Modified Novozhilov theory 2</td>
<td>26</td>
</tr>
<tr>
<td>3.5 Modified Novozhilov theory 3</td>
<td>27</td>
</tr>
<tr>
<td>3.6 Discussion of the results</td>
<td>28</td>
</tr>
<tr>
<td>4. The Reduce-based Package GNKRS</td>
<td>29</td>
</tr>
<tr>
<td>5. Conclusions</td>
<td>32</td>
</tr>
</tbody>
</table>
References

Appendix A  On the linear or nonlinear 'Kirchhoff assumption results'

Appendix B  Computer results of $\varepsilon_{ij}$'s and $x_{ij}$'s of the modified Novozhilov theory

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List of symbols

$A_1$, $A_2$  Lame coefficients
$e_{ij}$  Deformation parameters
$H_1$, $H_2$, $H_3$  Lame coefficients
$R_1$, $R_2$  Principal radii of curvature
$u$, $v$, $w$  Displacements
$\theta$, $\phi$, $\omega$  Middle surface displacements
$z$  Coordinate
$\alpha_1$, $\alpha_2$  Coordinates
$\omega_1$, $\omega_2$, $\omega_3$  Deformation parameters
$\varepsilon_{ij}$  Strain components
Introduction

Since shell configurations of various wall constructions have been widely used as structural elements, the various geometrically linear shell theories, which can be used to solve many shell problems, have been extensively investigated and widely employed [1,2,3].

But there still exist many other shell problems, such as the problems involving large displacements and rotations, loss of stability, dependence of external forces on deformation etc., in which it's currently recognized that the nonlinear effects play an important role and must be taken into account by the geometrically nonlinear shell theories instead of the linear theories. Therefore more and more attention has been devoted to setting up various geometrically nonlinear shell theories [4,5]. On the other hand, the rapid development of numerical techniques for three dimensional problems, in particular the finite element technique, has made it possible to introduce the complicated refined nonlinear shell theories into the practical shell problem solving.

Therefore one of the recent trends in shell analysis is to employ the more rigorous shell theories in conjunction with the numerical techniques, which can lead to improved accuracy and permit the study of various refinements in shell buckling theories.

In this report, both the linear shell theories and nonlinear Novozhilov-like theories with the Kirchhoff-Love assumptions are reviewed and discussed in the classical notation. They will be used later to derive the various kinds of governing equations of shell buckling problems.

1. General kinematic relations

First of all, three-dimensional elasticity theory [5] is presented as following. The strain components referred to an arbitrary orthogonal coordinate system can be written as

\[
\varepsilon_{11} = e_{11} + \frac{1}{2} \left[ e_{11}^2 + \left( e_{12} + \omega_3 \right)^2 + \left( e_{13} - \omega_2 \right)^2 \right], \tag{1.1a}
\]

\[
\varepsilon_{22} = e_{22} + \frac{1}{2} \left[ e_{22}^2 + \left( e_{23} - \omega_3 \right)^2 + \left( e_{21} + \omega_2 \right)^2 \right], \tag{1.1b}
\]
\[ \epsilon_{33} = \epsilon_{33}^{e} + \frac{1}{2} \left[ \epsilon_{33}^{e} \left( \frac{1}{2} \epsilon_{13}^{e} + \omega_2 \right)^2 + \epsilon_{12}^{e} \left( \frac{1}{2} \epsilon_{12}^{e} + \omega_1 \right)^2 \right] \]

\[ \epsilon_{12} = \epsilon_{12}^{e} + \frac{1}{2} \epsilon_{12}^{e} \left( \frac{1}{2} \epsilon_{12}^{e} - \omega_3 \right) + \epsilon_{22} \left( \frac{1}{2} \epsilon_{12}^{e} + \omega_3 \right) \]
\[ + \left( \frac{1}{2} \epsilon_{13}^{e} - \omega_2 \right) \left( \frac{1}{2} \epsilon_{23}^{e} + \omega_1 \right) \]

\[ \epsilon_{13} = \epsilon_{13}^{e} + \frac{1}{2} \epsilon_{13}^{e} \left( \frac{1}{2} \epsilon_{13}^{e} + \omega_2 \right) + \epsilon_{33} \left( \frac{1}{2} \epsilon_{13}^{e} - \omega_2 \right) \]
\[ + \left( \frac{1}{2} \epsilon_{13}^{e} + \omega_1 \right) \left( \frac{1}{2} \epsilon_{23}^{e} + \omega_1 \right) \]

\[ \epsilon_{23} = \epsilon_{23}^{e} + \frac{1}{2} \epsilon_{23}^{e} \left( \frac{1}{2} \epsilon_{23}^{e} - \omega_1 \right) + \epsilon_{33} \left( \frac{1}{2} \epsilon_{23}^{e} + \omega_1 \right) \]
\[ + \left( \frac{1}{2} \epsilon_{23}^{e} + \omega_2 \right) \left( \frac{1}{2} \epsilon_{13}^{e} + \omega_2 \right) \]

Here the parameters \( \epsilon_{ij} \) and \( \omega_i \) are given by

\[ \epsilon_{11} = \frac{1}{H_1} \frac{\partial u}{\partial \alpha_1} + \frac{1}{H_1 H_2} \frac{\partial H_1}{\partial \alpha_2} \frac{\partial v}{\partial \alpha_2} + \frac{1}{H_1 H_2} \frac{\partial H_1}{\partial \alpha_3} \frac{\partial w}{\partial \alpha_2}, \]

\[ \epsilon_{22} = \frac{1}{H_2} \frac{\partial v}{\partial \alpha_2} + \frac{1}{H_2 H_3} \frac{\partial H_2}{\partial \alpha_3} \frac{\partial w}{\partial \alpha_2} + \frac{1}{H_1 H_2} \frac{\partial H_2}{\partial \alpha_1} \frac{\partial u}{\partial \alpha_2}, \]

\[ \epsilon_{33} = \frac{1}{H_3} \frac{\partial w}{\partial \alpha_3} + \frac{1}{H_1 H_3} \frac{\partial H_3}{\partial \alpha_1} \frac{\partial u}{\partial \alpha_2} + \frac{1}{H_2 H_3} \frac{\partial H_3}{\partial \alpha_2} \frac{\partial v}{\partial \alpha_2}, \]

\[ \epsilon_{12} = \frac{H_1}{H_2} \frac{\partial (v)}{\partial \alpha_1} + \frac{H_1}{H_2} \frac{\partial (u)}{\partial \alpha_2}, \]

\[ \epsilon_{13} = \frac{H_2}{H_3} \frac{\partial (w)}{\partial \alpha_1} + \frac{H_2}{H_3} \frac{\partial (u)}{\partial \alpha_3}, \]

\[ \epsilon_{23} = \frac{H_3}{H_2} \frac{\partial (w)}{\partial \alpha_2} + \frac{H_3}{H_2} \frac{\partial (v)}{\partial \alpha_3}, \]

\[ 2\omega_1 = \frac{1}{H_2 H_3} \left[ \frac{\partial}{\partial \alpha_2} \left( H_3 w \right) - \frac{\partial}{\partial \alpha_3} \left( H_2 v \right) \right], \]

\[ 2\omega_2 = \frac{1}{H_1 H_3} \left[ \frac{\partial}{\partial \alpha_3} \left( H_1 u \right) - \frac{\partial}{\partial \alpha_1} \left( H_3 w \right) \right], \]
\[ 2\omega_3 = \frac{1}{H_1H_2} \left[ \frac{\partial}{\partial a_1} [H_2v] - \frac{\partial}{\partial a_2} [H_1u] \right]. \]  

(1.2i)

Where \( H_1 \), \( H_2 \) and \( H_3 \) are the Lame coefficients

\[ H_1 = \sqrt{\left( \frac{\partial x}{\partial a_1} \right)^2 + \left( \frac{\partial y}{\partial a_1} \right)^2 + \left( \frac{\partial z}{\partial a_1} \right)^2} \]  

(1.3a)

\[ H_2 = \sqrt{\left( \frac{\partial x}{\partial a_2} \right)^2 + \left( \frac{\partial y}{\partial a_2} \right)^2 + \left( \frac{\partial z}{\partial a_2} \right)^2} \]  

(1.3b)

\[ H_3 = \sqrt{\left( \frac{\partial x}{\partial a_3} \right)^2 + \left( \frac{\partial y}{\partial a_3} \right)^2 + \left( \frac{\partial z}{\partial a_3} \right)^2} \]  

(1.3c)

Now, to get the theory of deformation of thin shells, we use the coordinate system shown in Fig. 1. The coordinates \( a_1, \ a_2, \ z \) form a triply orthogonal system.

![Fig. 1](image)

The positions of points on the middle surface of the shell are determined by the Gaussian curvilinear coordinates \( a_1 \) and \( a_2 \) of the surface. The Lame coefficients corresponding to this curvilinear system are denoted by \( A_1 \) and \( A_2 \), and the principal radii of curvature by \( R_1 \) and \( R_2 \).

Thus the Lame coefficients of above triply orthogonal system are

\[ H_1 = A_1 \left( 1 + \frac{z}{R_1} \right), \ H_2 = A_2 \left( 1 + \frac{z}{R_2} \right), \ H_3 = 1. \]  

(1.4a,b,c)
where \( A_1, A_2, R_1, R_2 \) (being functions of \( \alpha_1 \) and \( \alpha_2 \) only) must satisfy the Gauss-Codazzi relations of surface theory:

\[
\frac{\partial}{\partial \alpha_1} \left( \frac{A_2}{R_2} \right) = \frac{1}{R_1} \frac{\partial A_2}{\partial \alpha_1}, \quad \frac{\partial}{\partial \alpha_2} \left( \frac{A_1}{R_1} \right) = \frac{1}{R_2} \frac{\partial A_1}{\partial \alpha_2}, \quad \frac{\partial}{\partial \alpha_1} \left( \frac{1}{A_1} \frac{\partial A_2}{\partial \alpha_1} \right) + \frac{\partial}{\partial \alpha_2} \left( \frac{1}{A_2} \frac{\partial A_1}{\partial \alpha_2} \right) = - \frac{A_1 A_2}{R_1 R_2}. \tag{1.5a,b,c}
\]

Substituting equations (1.4) into equations (1.2), one obtains, with the aid of equations (1.5), the relations:

\[
e_{11} = \frac{1}{1 + \frac{z}{R_1}} \left( \frac{1}{A_1} \frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} + \frac{w}{R_1} \right), \tag{1.6a}
\]

\[
e_{22} = \frac{1}{1 + \frac{z}{R_2}} \left( \frac{1}{A_2} \frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} + \frac{w}{R_2} \right), \tag{1.6b}
\]

\[
e_{12} = \frac{1}{1 + \frac{z}{R_1}} \left( \frac{1}{A_1} \frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} + \frac{1}{1 + \frac{z}{R_2}} \left( \frac{1}{A_2} \frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} \right) \right), \tag{1.6c}
\]

\[
e_{1z} = \frac{\partial u}{\partial z} + \frac{1}{1 + \frac{z}{R_1}} \left( \frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} - \frac{u}{R_1} \right), \tag{1.6d}
\]

\[
e_{2z} = \frac{\partial v}{\partial z} + \frac{1}{1 + \frac{z}{R_2}} \left( \frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2} \right), \tag{1.6e}
\]

\[
e_{zz} = \frac{\partial w}{\partial z}, \tag{1.6f}
\]

\[
2\omega_1 = - \frac{\partial v}{\partial z} + \frac{1}{1 + \frac{z}{R_2}} \left( \frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2} \right), \tag{1.6g}
\]

\[
2\omega_2 = \frac{\partial u}{\partial z} - \frac{1}{1 + \frac{z}{R_1}} \left( \frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} - \frac{u}{R_1} \right), \tag{1.6h}
\]

\[
2\omega_z = \frac{1}{1 + \frac{z}{R_1}} \left( \frac{1}{A_1} \frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} \right) - \frac{1}{1 + \frac{z}{R_2}} \left( \frac{1}{A_2} \frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} \right), \tag{1.6i}
\]
Here $u,v,w$ are displacements of an arbitrary point in the direction of $\alpha_1$, $\alpha_2$ and $z$, respectively.

Therefore equations (1.1) and equations (1.6) form the most general kinematic relations of thin shells which, however, may be too complicated to be used directly for the solution of shell problems. But anyway, this general kinematic relations establish the foundation for all the following simplifications which lead to many practical linear and nonlinear shell theories.

2. Various linear thin shell theories

By summarizing certain aspects of the more widely used linear shell theories involving Love's first approximation, and analyzing where the differences originate, we expect to get better understanding of the various nonlinear thin shell theories. Much of the material here is abstracted from Leissa's [1] and Bushnell's surveys [2].

2.1 Love's first approximation

There are four assumptions in the so-called Love's first approximation. They are

1. The thickness of the shell is small compared with the other dimensions, for example, the smallest radius of curvature of the middle surface of the shell. This defines what is meant by 'thin shells' and sets the stage for the entire theory.

2. Strains and displacements are sufficiently small so that the quantities of second - and higher order magnitude in the kinematic relations may be neglected in comparison with the first - order terms. This permits one to refer all calculations to the original configuration of the shell, and ensures that the differential equations will be linear.
3. The transverse normal stress is small compared with the other normal stress components, and may be neglected.

4. Normals to the undeformed middle surface remain straight and normal to the deformed middle surface, and suffer no extension. This is known as Kirchhoff's hypothesis and categorizes the shell theories that will be discussed in this report.

2.2 General linear kinematic relations - the relations of Byrne, Flügge, Goldenweizer, Lur'ye and Novozhilov

According to the second assumption of Love's first approximation, the linear kinematic relations can be obtained by neglecting all the nonlinear terms in equations (1.1) under the additional restriction that the rotations of material fibers be also small everywhere, one obtains

\[ \epsilon_{11} = e_{11} \]  \hspace{1cm} (2.1a)
\[ \epsilon_{22} = e_{22} \]  \hspace{1cm} (2.1b)
\[ \epsilon_{12} = e_{12} \]  \hspace{1cm} (2.1c)
\[ \epsilon_{1z} = e_{1z} \]  \hspace{1cm} (2.1d)
\[ \epsilon_{2z} = e_{2z} \]  \hspace{1cm} (2.1e)
\[ \epsilon_{zz} = e_{zz} \]  \hspace{1cm} (2.1f)

Where the \( e_{ij} \)'s can be found in equations (1.6). Now in order to satisfy the Kirchhoff hypothesis, the class of displacements is restricted to the following linear relationships

\[ u(\alpha_1, \alpha_2, z) = \hat{u}(\alpha_1, \alpha_2) + z\theta(\alpha_1, \alpha_2) \]  \hspace{1cm} (2.2a)
\[ v(\alpha_1, \alpha_2, z) = \hat{v}(\alpha_1, \alpha_2) + z\psi(\alpha_1, \alpha_2) \]  \hspace{1cm} (2.2b)
\[ w(\alpha_1, \alpha_2, z) = \hat{w}(\alpha_1, \alpha_2) \]  \hspace{1cm} (2.2c)
Where $u$, $v$ and $w$ are the components of displacement at the middle surface in the $a_1$, $a_2$ and normal directions, respectively; and $\theta$ and $\psi$ are the rotations of the normal to the middle surface during deformation about the $a_1$ and $a_2$ axes, respectively; i.e.,

\begin{align}
\theta &= \frac{\partial u(a_1,a_2,z)}{\partial z} \\
\psi &= \frac{\partial v(a_1,a_2,z)}{\partial z}
\end{align}

(2.3a) (2.3b)

Substituting equations (2.2) into $\varepsilon_{1z}$, $\varepsilon_{2z}$ and $\varepsilon_{zz}$ and letting $\varepsilon_{1z} = 0$, $\varepsilon_{2z} = 0$, $\varepsilon_{zz} = 0$, one obtains

\begin{align}
\theta &= \frac{\partial \varphi}{R_1} - \frac{1}{A_1} \frac{\partial \varphi}{\partial a_1} \\
\psi &= \frac{\partial \varphi}{R_2} - \frac{1}{A_2} \frac{\partial \varphi}{\partial a_2}
\end{align}

(2.4a) (2.4b)

Thus the behaviour of a thin shell can be described with sufficient accuracy by the behaviour of the shell middle surface.

Substituting equations (2.2) into equations (2.1a, b, c) yields

\begin{align}
\varepsilon_{11} &= \frac{1}{(1 + \frac{z}{R_1})} (\varepsilon_{11} + zX_{11}) \\
\varepsilon_{22} &= \frac{1}{(1 + \frac{z}{R_2})} (\varepsilon_{22} + zX_{22}) \\
\varepsilon_{12} &= \frac{1}{(1 + \frac{z}{R_1})(1 + \frac{z}{R_2})} \left[ (1 - \frac{z^2}{R_1R_2})\varepsilon_{12} + z(\frac{z}{2R_1} + \frac{z}{2R_2})X_{12} \right]
\end{align}

(2.5a) (2.5b) (2.5c)

where $\varepsilon_{11}$, $\varepsilon_{22}$ and $\varepsilon_{12}$ are the normal and shear strains in the middle surface ($z = 0$) given by
\[\varepsilon_{11} = \frac{1}{A_1} \frac{\partial a}{\partial a_1} + \frac{\phi}{A_1 A_2} \frac{\partial A_1}{\partial a_2} + \frac{\omega}{R_1}\]  
(2.6a)

\[\varepsilon_{22} = \frac{\bar{u}}{A_1 A_2} \frac{\partial A_2}{\partial a_1} + \frac{1}{A_2} \frac{\partial \psi}{\partial a_2} + \frac{\omega}{R_2}\]  
(2.6b)

\[\varepsilon_{12} = \frac{A_1}{A_2} \frac{1}{\partial a_2} \left( \frac{\bar{u}}{A_1} \right) + \frac{A_2}{A_1} \frac{1}{\partial a_1} \left( \frac{\psi}{A_2} \right)\]  
(2.6c)

and \(\chi_{11}\) and \(\chi_{22}\) are the mid-surface changes in curvature and \(\chi_{12}\) the mid-surface twist, given by

\[\chi_{11} = \frac{1}{A_1} \frac{\partial \theta}{\partial a_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial a_2} \psi\]  
(2.7a)

\[\chi_{22} = \frac{1}{A_2} \frac{\partial \psi}{\partial a_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial a_1} \theta\]  
(2.7b)

\[\chi_{12} = \frac{A_1}{A_2} \frac{1}{\partial a_2} \left( \frac{\theta}{A_1} \right) + \frac{A_2}{A_1} \frac{1}{\partial a_1} \left( \frac{\psi}{A_2} \right) + \frac{1}{R_1} \left( \frac{1}{A_2} \frac{\partial \theta}{\partial a_2} - \frac{\theta}{A_1 A_2} \frac{\partial A_2}{\partial a_1} \right)\]  
(2.7c)

Equations (2.5 a, b, c) are the general linear kinematic relations used by Byrne, Flügge, Goldenveizer, Lur'y e and Novozhilov.

2.3 Relations of Love and Timoshenko

If in the equations (2.5) the terms \(\frac{Z}{R_1}, \frac{Z}{R_2}\) and their products are neglected as compared with unity, a new set of kinematic relations can be obtained

\[\varepsilon_{11} = \varepsilon_{11}' + z\chi_{11}\]  
(2.8a)

\[\varepsilon_{22} = \varepsilon_{22}' + z\chi_{22}\]  
(2.8b)

\[\varepsilon_{12} = \varepsilon_{12}' + z\chi_{12}\]  
(2.8c)
Here $\varepsilon$ and $\chi$ are still given by equations (2.6) and (2.7).
The above kinematic relations are the ones used by Love and Timoshenko.

2.4 Relations of Reissner, Naghdi and Berry

In equations (2.1 a,b,c) terms like $\frac{Z}{R_1}$ and $\frac{Z}{R_2}$ are also neglected, as compared to unity, which leads to

$$\varepsilon_{11} = \frac{1}{A_1} \frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} v + \frac{w}{R_1}$$

(2.9a)

$$\varepsilon_{22} = \frac{1}{A_2} \frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} u + \frac{w}{R_2}$$

(2.9b)

$$\varepsilon_{12} = \left( \frac{1}{A_1} \frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} \right) + \left( \frac{1}{A_2} \frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} \right) v$$

(2.9c)

Then, substituting equations (2.2) into the above, we can get the total strains presented by the middle surface parameters, which differ from equations (2.8) only by that $\chi_{12}$ becomes

$$\chi_{12} = \frac{A_1}{A_2} \frac{\partial}{\partial \alpha_2} \left( \frac{\theta}{A_1} \right) + \frac{A_2}{A_1} \frac{\partial}{\partial \alpha_1} \left( \frac{\theta}{A_2} \right)$$

(2.10)

These are the kinematic relations used by Reissner, Naghdi and Berry.

2.5 Relations of Donnell and Mushtari

If in equations (2.8), $\varepsilon$ are still given by equations (2.6), while in $\chi$ expressed by equations (2.7) the tangential displacements and their derivatives are neglected, then the Donnell and Mushtari kinematic relations are obtained. Thus,

$$\chi_{11} = -\frac{1}{A_1} \frac{\partial}{\partial \alpha_1} \left( \frac{1}{A_1} \frac{\partial \omega}{\partial \alpha_1} \right) - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} \frac{\partial \omega}{\partial \alpha_2}$$

(2.11a)
\[ \chi_{22} = -\frac{1}{A_2} \frac{\partial}{\partial \alpha_2} \left( \frac{1}{A_2} \frac{\partial \tilde{W}}{\partial \alpha_2} \right) - \frac{1}{A_2 A_1} \frac{\partial^2 \tilde{W}}{\partial \alpha_1 \partial \alpha_2} \]  
(2.11b)

\[ \chi_{12} = -\frac{A_2}{A_1} \frac{\partial}{\partial \alpha_1} \left( \frac{1}{2} \frac{\partial \tilde{W}}{\partial \alpha_2} \right) - \frac{A_1}{A_2} \frac{\partial}{\partial \alpha_2} \left( \frac{1}{2} \frac{\partial \tilde{W}}{\partial \alpha_1} \right) \]  
(2.11c)

2.6 Relations of Vlasov

Noticing that for a thin shell \( \frac{z}{R_1} \) (i=1, 2) is always less than unity, \( \frac{1}{1 + \frac{z}{R_1}} \), can be expanded into a geometric series; i.e.,

\[ \frac{1}{1 + \frac{z}{R_1}} = \sum_{n=0}^{\infty} (-\frac{z}{R_1})^n, \quad i=1, 2 \]  
(2.12)

Substituting equations (2.2) and (2.12) into equations (2.1 a,b,c), and neglecting the second and higher-order terms of \( z \), then the kinematic relations can be obtained as equations (2.8), however the \( \chi_{ij} \) become

\[ \chi_{11} = \frac{1}{A_1} \frac{\partial \theta}{\partial \alpha_1} + \frac{\psi}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} - \frac{1}{R_1} \left( \frac{1}{A_1} \frac{\partial \theta}{\partial \alpha_2} + \frac{\psi}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} + \frac{\tilde{W}}{R_1} \right) \]  
(2.13a)

\[ \chi_{22} = \frac{1}{A_2} \frac{\partial \psi}{\partial \alpha_2} + \frac{\theta}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_2} - \frac{1}{R_2} \left( \frac{1}{A_2} \frac{\partial \psi}{\partial \alpha_2} + \frac{\theta}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_2} + \frac{\tilde{W}}{R_2} \right) \]  
(2.13b)

\[ \chi_{12} = \left[ \frac{1}{R_1} - \frac{1}{R_2} \right] \left[ \frac{A_1}{A_2} \frac{\partial}{\partial \alpha_2} \left( \frac{\partial}{\partial \alpha_1} \right) - \frac{A_2}{A_1} \frac{\partial}{\partial \alpha_1} \left( \frac{\partial}{\partial \alpha_2} \right) \right] \]

\[ - \frac{A_2}{A_1} \frac{\partial}{\partial \alpha_1} \left( \frac{1}{2} \frac{\partial \tilde{W}}{\partial \alpha_2} \right) - \frac{A_1}{A_2} \frac{\partial}{\partial \alpha_2} \left( \frac{1}{2} \frac{\partial \tilde{W}}{\partial \alpha_1} \right) \]  
(2.13c)

Whereas the \( \xi_{ij} \) are the same as those given by equations (2.6).
2.7 Relations of Sanders

Sanders’ kinematic relations can be obtained by adding the correction factor
\[
\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \frac{1}{2A_1A_2} \left(\frac{\partial A_1}{\partial \alpha_1} - \frac{\partial A_2}{\partial \alpha_2}\right)
\]
to the \(\chi_{12}\) expression of Reissner et al. This addition is needed in order to eliminate the non-zero \(\chi_{12}\) which arises from rigid body rotation.

2.8 Modified Donnell theory

The modified Donnell theory can be obtained by neglecting the tangential displacements and their derivatives in the midsurface changes in curvature and twist of Vlasov’s theory. Thus

\[
\chi_{11} = -\frac{1}{A_1} \frac{\partial}{\partial \alpha_1} \left(\frac{1}{A_1} \frac{\partial \hat{\omega}}{\partial \alpha_1}\right) - \frac{1}{A_1A_2} \frac{\partial A_1}{\partial \alpha_1} \frac{\partial \hat{\omega}}{\partial \alpha_2} - \frac{1}{R_1^2} \hat{\omega}
\]  
(2.14a)

\[
\chi_{22} = -\frac{1}{A_2} \frac{\partial}{\partial \alpha_2} \left(\frac{1}{A_2} \frac{\partial \hat{\omega}}{\partial \alpha_2}\right) - \frac{1}{A_1A_2} \frac{\partial A_2}{\partial \alpha_2} \frac{\partial \hat{\omega}}{\partial \alpha_1} - \frac{1}{R_2^2} \hat{\omega}
\]  
(2.14b)

\(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}\) and \(\chi_{12}\) are the same as those of Donnell’s theory.

2.9 Discussion of the results.

1. In all the theories considered above, two types of expressions were found to represent the total strains. These are summarized in table 2.1.
Table 2.1: Total Strains at Any Point in a Shell

<table>
<thead>
<tr>
<th>Theory</th>
<th>$\varepsilon_{11}$, $\varepsilon_{22}$</th>
<th>$\varepsilon_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Byrne, Flügge</td>
<td>$\frac{1}{(1+\frac{z}{R_1})} (\varepsilon_{11}^* + z \chi_{11})$</td>
<td>$\frac{1}{(1+\frac{z}{R_1})(1+\frac{z}{R_2})} \left[ \left(1 - \frac{z^2}{R_1R_2}\right) \varepsilon_{12}^* + z \left(1 + \frac{z}{2R_1} + \frac{z}{2R_2}\right) \chi_{12}^* \right]$</td>
</tr>
<tr>
<td>Goldenveizer</td>
<td>$\frac{1}{(1+\frac{z}{R_2})} (\varepsilon_{22}^* + z \chi_{22})$</td>
<td></td>
</tr>
<tr>
<td>Lur'ye, Novozhilov</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Love, Timoshenko, Vlasov, Reissner, Naghdi, Berry, Sanders, Donnell, Mushtari, Modified Donnell</td>
<td>$\varepsilon_{11}^* + z \chi_{11}$</td>
<td>$\varepsilon_{12}^* + z \chi_{12}$</td>
</tr>
</tbody>
</table>

In above table, the expressions for the middle surface strains $\varepsilon_{11}$, $\varepsilon_{22}$ and $\varepsilon_{12}$ are the same according to all the theories discussed above. They are given by equations (2.6).

2. The changes in curvature and twist of the middle surface are summarized in table 2.2 and 2.3 respectively.
<table>
<thead>
<tr>
<th>Theory</th>
<th>$\chi_{11}$</th>
<th>$\chi_{22}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Byrne, Flügge, Goldenveizer, Lur'ye, Novozhilov, Love, Timoshenko, Reissner, Naghdi, Berry, Sanders,</td>
<td>$\frac{1}{A_1} \frac{\partial \theta}{\partial a_1} + \frac{\psi}{A_1 A_2} \frac{\partial A_1}{\partial a_2}$</td>
<td>$\frac{1}{A_2} \frac{\partial \psi}{\partial a_2} + \frac{\theta}{A_1 A_2} \frac{\partial A_2}{\partial a_1}$</td>
</tr>
<tr>
<td>Vlasov</td>
<td>$\frac{1}{A_1} \frac{\partial \theta}{\partial a_1} + \frac{\psi}{A_1 A_2} \frac{\partial A_1}{\partial a_2}$</td>
<td>$\frac{1}{A_2} \frac{\partial \psi}{\partial a_2} + \frac{\theta}{A_1 A_2} \frac{\partial A_2}{\partial a_1}$</td>
</tr>
<tr>
<td></td>
<td>$- \frac{1}{R_1} \left( \frac{1}{A_1} \frac{\partial \theta}{\partial a_1} + \frac{\varphi}{A_1 A_2} \frac{\partial A_1}{\partial a_2} + \frac{\omega}{R_1} \right)$</td>
<td>$- \frac{1}{R_2} \left( \frac{1}{A_2} \frac{\partial \psi}{\partial a_2} + \frac{\theta}{A_1 A_2} \frac{\partial A_2}{\partial a_1} + \frac{\omega}{R_2} \right)$</td>
</tr>
<tr>
<td>Donnell, Mushtari</td>
<td>$- \frac{1}{A_1} \frac{\partial \varphi}{\partial a_1} \left( \frac{1}{A_1} \frac{\partial \omega}{\partial a_1} \right)$</td>
<td>$- \frac{1}{A_2} \frac{\partial \varphi}{\partial a_2} \left( \frac{1}{A_2} \frac{\partial \omega}{\partial a_2} \right)$</td>
</tr>
<tr>
<td></td>
<td>$- \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial a_2} \frac{\partial \omega}{\partial a_1}$</td>
<td>$- \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial a_1} \frac{\partial \omega}{\partial a_1}$</td>
</tr>
<tr>
<td>Modified Donnell</td>
<td>$- \frac{1}{A_1} \frac{\partial \varphi}{\partial a_1} \left( \frac{1}{A_1} \frac{\partial \omega}{\partial a_1} \right)$</td>
<td>$- \frac{1}{A_2} \frac{\partial \varphi}{\partial a_2} \left( \frac{1}{A_2} \frac{\partial \omega}{\partial a_2} \right)$</td>
</tr>
<tr>
<td></td>
<td>$- \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial a_2} \frac{\partial \omega}{\partial a_1} - \frac{1}{R_1} \omega$</td>
<td>$- \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial a_1} \frac{\partial \omega}{\partial a_1} - \frac{1}{R_2} \omega$</td>
</tr>
</tbody>
</table>
Table 2.3. - Change in Twist $\chi_{12}$ of the Middle Surface.

| Byrne, Flügge, Lur'ye, Goldenveizer, Novozhilov, Timoshenko, Love | $\frac{A_1}{A_2} \frac{a}{\delta a_2} \left( \theta \right) - \frac{A_2}{A_1} \frac{a}{\delta a_1} \left( \frac{\omega}{A_2} \right) + \frac{1}{R_1} \left( \frac{1}{A_2} \frac{a}{\delta a_2} - \frac{a}{A_1 A_2} \frac{\delta A_2}{\delta a_1} \right)$
|---|---|
| Reissner, Berry, Naghdi | $\frac{A_1}{A_2} \frac{a}{\delta a_2} \left( \theta \right) - \frac{A_2}{A_1} \frac{a}{\delta a_1} \left( \frac{\omega}{A_2} \right)$
| Vlasov | $\left( \frac{1}{R_1} - \frac{1}{R_2} \right) \left[ \frac{A_1}{A_2} \frac{a}{\delta a_2} \left( \frac{\omega}{A_1} \right) - \frac{A_2}{A_1} \frac{a}{\delta a_1} \left( \frac{\omega}{A_2} \right) \right] - \frac{A_2}{A_1} \frac{a}{\delta a_1} \left( \frac{1}{A_2} \frac{\delta \omega}{\delta a_2} \right)$
| Sanders | $\frac{A_1}{A_2} \frac{a}{\delta a_2} \left( \frac{\omega}{A_1} \right) + \frac{A_2}{A_1} \frac{a}{\delta a_1} \left( \frac{\omega}{A_2} \right) + \frac{1}{2 A_1 A_2} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \left( \frac{\delta A_2 \omega}{\delta a_1} - \frac{\delta A_1 \omega}{\delta a_2} \right)$
| Mushtari-Donnell, Modified Donnell | $- \frac{A_2}{A_1} \frac{a}{\delta a_1} \left( \frac{1}{A_2} \frac{\delta \omega}{\delta a_2} \right) - \frac{A_1}{A_2} \frac{a}{\delta a_2} \left( \frac{1}{A_1} \frac{\delta \omega}{\delta a_1} \right)$
It is seen that there is a partial agreement among the theories for the expressions of the middle surface curvature changes $\chi_{11}$, $\chi_{22}$. The exceptions are the Donnell-Mushtari expressions which differ from others because of neglecting terms containing the tangential displacements $u$ and $v$, while Vlasov expressions differ from others because of replacing $1/(1+Z/R_1)$ by its series expansion in the derivation.

But there is widespread disagreement concerning the proper form for the middle surface change in twist $\chi_{12}$. The Donnell-Mushtari expression differs from the others for the same reasons given in the discussion of $\chi_{11}$ and $\chi_{22}$. The Reissner et al. expression differs because of neglecting of $Z/R_1$, compared to unity at an earlier stage in the derivation than in the Byrne, Flügge et al. formulas, Sanders' expression is derived through modifying Reissner et al.'s formula by addition of a term needed to eliminate the non-zero $\chi_{12}$ arising from rigid body rotation. Kraus [6] demonstrated that the kinematic relations of Byrne, Flügge et al. are also consistent with regard to rigid body motion. Kadi [7] found the same for the theories of love, Timoshenko, and Vlasov. Finally the Donnell-Mushtari theory gives non-zero curvature changes and twist due to rigid body translations.

3. Because of the linearization of equations (1.1) it becomes possible to express the Kirchhoff assumptions by equations (2.2).

4. The difference between the Donnell theory and the modified Donnell theory is that two additional terms $\frac{-\dot{\omega}}{R_1}$ and $\frac{-\dot{\omega}}{R_2}$ are added to the equations (2.11a) and (2.11b) of Donnell's theory respectively. These two terms are negligible as compared with the other terms in Donnell theory only in the case that the circumferential wave number $n \geq 4$. Otherwise they must be taken into account.

5. All the above theories are based on Love's first approximation. But in many cases, these assumptions will not yield satisfactory results. In certain case some of them must be discarded, such as the assumptions 3 and 4, thus broadening the scope of the theory by including the effects of transverse normal and shear deformations. These thin shell theories without Love's first approximation are not discussed in this report despite their importance for certain class of problems.
After all, all the above theories are linear, and they can only be applied to a particular class of shell problems, while the nonlinear theories which embrace all elastic deformation problems, will be discussed below.
3. Some Novozhilov-like nonlinear thin shell theories

Up to now, the search for appropriate geometrically nonlinear small strain Kirchhoff–Love type theories, which are able to describe the large deflection and stability behaviour of thin elastic shells, has already led to many successful formulations. A few of them are from references [4] to [11]. In this section, emphasis is put on the geometrically nonlinear theory given by Novozhilov [5]. Based on that theory, several other geometrically nonlinear theories in classical notation are derived with the help of the formula manipulation program Reduce. A similar approach as with the derivations of various linear theories will be followed.

3.1 Kirchhoff assumptions

According to equations (1.1) and (1.6), the Kirchhoff hypothesis \( \epsilon_{1z} = \epsilon_{2z} = \epsilon_{zz} = 0 \) can be formulated analytically as follows:

\[
\epsilon_{1z} = \frac{\partial u}{\partial z} + \frac{1}{1 + \frac{z}{R_1}} \left( \frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} - \frac{u}{R_1} \right) + \frac{1}{1 + \frac{z}{R_1}} \left( \frac{1}{A_1} \frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} v + \frac{w}{R_1} \right) \frac{\partial u}{\partial z}
\]

\[+ \frac{1}{1 + \frac{z}{R_1}} \left( \frac{1}{A_1} \frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} u \right) \frac{\partial v}{\partial z} + \frac{1}{1 + \frac{z}{R_1}} \left( \frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} - \frac{u}{R_1} \right) \frac{\partial w}{\partial z} = 0
\]

\( (3.1a) \)

\[
\epsilon_{2z} = \frac{\partial v}{\partial z} + \frac{1}{1 + \frac{z}{R_2}} \left( \frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2} \right) + \frac{1}{1 + \frac{z}{R_2}} \left( \frac{1}{A_2} \frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} u + \frac{w}{R_2} \right) \frac{\partial v}{\partial z}
\]

\[+ \frac{1}{1 + \frac{z}{R_2}} \left( \frac{1}{A_2} \frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} v \right) \frac{\partial u}{\partial z} + \frac{1}{1 + \frac{z}{R_2}} \left( \frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2} \right) \frac{\partial w}{\partial z} = 0
\]

\( (3.1b) \)

\[
\epsilon_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right] = 0
\]

\( (3.1c) \)

Meanwhile one can expand the displacement components into power series:
\[ u = u(a_1, a_2, 0) + \left( \frac{\partial u}{\partial z} \right)_0 z^2 + \frac{1}{2} \left( \frac{\partial^2 u}{\partial z^2} \right)_0 z^2 + \ldots \]  
(3.2a)

\[ v = v(a_1, a_2, 0) + \left( \frac{\partial v}{\partial z} \right)_0 z^2 + \frac{1}{2} \left( \frac{\partial^2 v}{\partial z^2} \right)_0 z^2 + \ldots \]  
(3.2b)

\[ w = w(a_1, a_2, 0) + \left( \frac{\partial w}{\partial z} \right)_0 z^2 + \frac{1}{2} \left( \frac{\partial^2 w}{\partial z^2} \right)_0 z^2 + \ldots \]  
(3.2c)

Retaining in these series only the first two terms and introducing the notations

\[ u(a_1, a_2, 0) = \hat{u}, \quad v(a_1, a_2, 0) = \hat{v}, \quad w(a_1, a_2, 0) = \hat{w}, \]

\[ \left( \frac{\partial u}{\partial z} \right)_0 = \theta, \quad \left( \frac{\partial v}{\partial z} \right)_0 = \psi, \quad \left( \frac{\partial w}{\partial z} \right)_0 = \chi \]

one obtains the following expressions for the displacements

\[ u = \hat{u}(a_1, a_2) + z\theta(a_1, a_2), \]  
(3.3a)

\[ v = \hat{v}(a_1, a_2) + z\psi(a_1, a_2), \]  
(3.3b)

\[ w = \hat{w}(a_1, a_2) + z\chi(a_1, a_2). \]  
(3.3c)

Here, \( \hat{u}, \hat{v}, \hat{w} \) are the displacements of the middle surface of the shell (which follows from equations (3.3) by setting \( z = 0 \)).

Now, if equations (3.3) are substituted into equations (3.1), we get five equations, two of which, however, are implied by one of the other three. Hence, just three of the equations are independent, namely

\[ \theta^2 + \psi^2 + (1 + x)^2 = 1 \]  
(3.4a)

\[ (1 + x) \left( \frac{1}{A_1} \frac{\partial \hat{v}}{\partial a_1} - \frac{\hat{v}}{R_1} \right) + \left( \frac{1}{A_1} \frac{\partial \hat{\psi}}{\partial a_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial a_2} \hat{u} \right) \psi + \left( 1 + \frac{1}{A_1} \frac{\partial \hat{u}}{\partial a_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial a_2} \psi + \hat{w} \right) \theta = 0 \]  
(3.4b)

\[ (1 + x) \left( \frac{1}{A_2} \frac{\partial \hat{u}}{\partial a_2} - \frac{\hat{u}}{R_2} \right) + \left( \frac{1}{A_2} \frac{\partial \hat{\psi}}{\partial a_2} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial a_1} \psi \right) \theta \]
\[ + \left( 1 + \frac{1}{A_2} \frac{\partial \psi}{\partial a_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial a_1} \dot{u} + \frac{\dot{\omega}}{R_2} \right) \psi = 0 \]  

(3.4c)

Simultaneous solution of the last two equations yields

\[ \theta = \frac{\dot{\alpha}_{31}}{\dot{a}_{33}} (1 + x) \]  

(3.5a)

\[ \psi = \frac{\dot{\alpha}_{32}}{\dot{a}_{33}} (1 + x) \]  

(3.5b)

where

\[ \dot{\alpha}_{31} = - \dot{\varepsilon}_{13} (1 + \varepsilon_{22}) + \dot{\varepsilon}_{23} \dot{\varepsilon}_{21} \]  

(3.6a)

\[ \dot{\alpha}_{32} = - \dot{\varepsilon}_{23} (1 + \varepsilon_{11}) + \dot{\varepsilon}_{13} \varepsilon_{12} \]  

(3.6b)

\[ \dot{\alpha}_{33} = (1 + \varepsilon_{11})(1 + \varepsilon_{22}) - \varepsilon_{12} \varepsilon_{21} \]  

(3.6c)

with

\[ \dot{\varepsilon}_{11} = \frac{1}{A_1} \frac{\partial \ddot{u}}{\partial a_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial a_2} \ddot{\psi} + \frac{\dot{\omega}}{R_1} \]  

(3.7a)

\[ \dot{\varepsilon}_{22} = \frac{1}{A_2} \frac{\partial \ddot{u}}{\partial a_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial a_1} \ddot{u} + \dot{\omega} \frac{\dot{\psi}}{R_2} \]  

(3.7b)

\[ \dot{\varepsilon}_{21} = \frac{1}{A_1} \frac{\partial \ddot{u}}{\partial a_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial a_2} \ddot{\psi} \]  

(3.7c)

\[ \dot{\varepsilon}_{13} = \frac{1}{A_1} \frac{\partial \ddot{u}}{\partial a_1} - \frac{\ddot{\omega}}{R_1} \]  

(3.7d)

\[ \dot{\varepsilon}_{12} = \frac{1}{A_2} \frac{\partial \ddot{u}}{\partial a_2} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial a_1} \ddot{\psi} \]  

(3.7e)

\[ \dot{\varepsilon}_{23} = \frac{1}{A_2} \frac{\partial \ddot{u}}{\partial a_2} - \frac{\dot{\omega}}{R_2} \]  

(3.7f)

Substituting equations (3.5) into the first of equations (3.4), one can obtain
\[ x = \frac{\hat{a}_{33}}{\sqrt{\hat{a}_{31}^2 + \hat{a}_{32}^2 + \hat{a}_{33}^2}} - 1 \]  

(3.8)

Recalling that

\[ \sqrt{\hat{a}_{31}^2 + \hat{a}_{32}^2 + \hat{a}_{33}^2} = \frac{1 + \Delta}{1 + E_\zeta} \]

Where \( \Delta \) is the relative change in volume and \( E_\zeta \) is the relative elongation of a line element of the shells after the deformation.

Substitution in the above equation of \( \hat{a}_{ij} \) for \( a_{ij} \), which corresponds to the replacement of \( u, v, w \) by \( \hat{u}, \hat{v}, \hat{w} \), is equivalent to considering \( \Delta \) and \( E_\zeta \) on the middle surface of the shells.

Hence, it is seen that if the shears and elongations are neglected in comparison with unity, one obtains

\[ \sqrt{\hat{a}_{31}^2 + \hat{a}_{32}^2 + \hat{a}_{33}^2} = 1 \]  

(3.9)

Thus

\[ x = \hat{a}_{33} - 1 = \hat{e}_{11} \hat{e}_{22} + \hat{e}_{11} \hat{e}_{22} - \hat{e}_{12} \hat{e}_{21} \]  

(3.10)

Substituting equation (3.10) into equations (3.5), one obtains

\[ \theta = -\hat{e}_{13}(1 + \hat{e}_{22}) + \hat{e}_{23}\hat{e}_{21} \]  

(3.11a)

\[ \psi = -\hat{e}_{23}(1 + \hat{e}_{11}) + \hat{e}_{13}\hat{e}_{12} \]  

(3.11b)

Now equations (3.10), (3.11) and (3.3) express the displacements of an arbitrary point of the shell in terms of the displacements of the corresponding point of the middle surface, which will serve as the basis for the following discussions.
3.2 Novozhilov theory

Substituting equations (1.4) and (1.5) into equations (1.1a,b,d), one obtains

\[
\varepsilon_{11} = \frac{1}{1 + \frac{z}{R_1}} \left[ \left( \frac{1}{A_1} \frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} v + \frac{w}{R_1} \right) \right]
\]

\[
+ \frac{1}{2} \left( \frac{1}{1 + \frac{z}{R_1}} \right)^2 \left[ \left( \frac{1}{A_1} \frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_1} v + \frac{w}{R_1} \right)^2 \right]
\]

\[
+ \left[ \left( \frac{1}{A_1} \frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} u \right)^2 + \left( \frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} - \frac{1}{R_1} \right)^2 \right], \tag{3.12a}
\]

\[
\varepsilon_{22} = \frac{1}{1 + \frac{z}{R_2}} \left[ \left( \frac{1}{A_2} \frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} u + \frac{w}{R_2} \right) \right]
\]

\[
+ \frac{1}{2} \left( \frac{1}{1 + \frac{z}{R_2}} \right)^2 \left[ \left( \frac{1}{A_2} \frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} v \right)^2 \right]
\]

\[
+ \left[ \left( \frac{1}{A_2} \frac{\partial u}{\partial \alpha_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} u + \frac{w}{R_2} \right)^2 + \left( \frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2} \right)^2 \right], \tag{3.12b}
\]

\[
\varepsilon_{12} = \frac{1}{1 + \frac{z}{R_1}} \left[ \left( \frac{1}{A_1} \frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} u \right) + \frac{1}{1 + \frac{z}{R_2}} \left( \frac{1}{A_2} \frac{\partial u}{\partial \alpha_2} - \frac{1}{A_2 A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} v \right) \right]
\]

\[
+ \frac{1}{\left[ 1 + \frac{z}{R_1} \right] \left[ 1 + \frac{z}{R_2} \right]} \left[ \left( \frac{1}{A_1} \frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} v + \frac{w}{R_1} \right) \left( \frac{1}{A_2} \frac{\partial u}{\partial \alpha_2} - \frac{1}{A_2 A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} v \right) \right]
\]

\[
+ \left[ \left( \frac{1}{A_1} \frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} u \right) \left( \frac{1}{A_2} \frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} u + \frac{w}{R_2} \right) \right]
\]

\[
+ \left( \frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} - \frac{1}{R_1} \right) \left( \frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2} \right) \right]. \tag{3.12c}
\]

If we set \( 1 + \frac{z}{R_1} = 1 + \frac{z}{R_2} = 1 \), and substitute equations (3.3) into (3.12), the Novozhilov geometrically nonlinear kinematic relations can be obtained as
\[ \varepsilon_{11} = \varepsilon_{11} + z\chi_{11} + z^2 \nu_{11}, \quad (3.13a) \]
\[ \varepsilon_{22} = \varepsilon_{22} + z\chi_{22} + z^2 \nu_{22}, \quad (3.13b) \]
\[ \varepsilon_{12} = \varepsilon_{12} + z\chi_{12} + z^2 \nu_{12}, \quad (3.13c) \]

Here \( \varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12} \) are the elongations and shear of the middle surface of the shell. These are given by
\[ \varepsilon_{11} = \dot{\varepsilon}_{11} + \frac{1}{2} [\dot{\varepsilon}_{11}^2 + \dot{\varepsilon}_{12}^2 + \dot{\varepsilon}_{13}^2], \quad (3.14a) \]
\[ \varepsilon_{22} = \dot{\varepsilon}_{22} + \frac{1}{2} [\dot{\varepsilon}_{21}^2 + \dot{\varepsilon}_{22}^2 + \dot{\varepsilon}_{23}^2], \quad (3.14b) \]
\[ \varepsilon_{12} = \dot{\varepsilon}_{12} + \dot{\varepsilon}_{21} + \dot{\varepsilon}_{11} \dot{\varepsilon}_{12} + \dot{\varepsilon}_{22} \dot{\varepsilon}_{21} + \dot{\varepsilon}_{13} \dot{\varepsilon}_{23}. \quad (3.14c) \]

The parameters \( \chi_{11}, \chi_{22}, \chi_{12} \), which characterize the variations of the curvature of the middle surface induced by the deformation, are given by
\[ \chi_{11} = (1 + \dot{\varepsilon}_{11}) k_{11} + \dot{\varepsilon}_{12} k_{12} + \dot{\varepsilon}_{13} k_{13}, \quad (3.15a) \]
\[ \chi_{22} = (1 + \dot{\varepsilon}_{22}) k_{22} + \dot{\varepsilon}_{21} k_{21} + \dot{\varepsilon}_{23} k_{23}, \quad (3.15b) \]
\[ \chi_{12} = k_{21} (1 + \dot{\varepsilon}_{11}) + k_{12} (1 + \dot{\varepsilon}_{22}) + k_{11} \dot{\varepsilon}_{21} + k_{22} \dot{\varepsilon}_{12} + k_{13} \dot{\varepsilon}_{13} + k_{23} \dot{\varepsilon}_{23} \quad (3.15c) \]

where
\[ k_{11} = \frac{1}{A_1} \frac{\partial \theta}{\partial \alpha_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} \psi + \frac{\chi}{R_1}, \]
\[ k_{22} = \frac{1}{A_2} \frac{\partial \phi}{\partial \alpha_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} \theta + \frac{\chi}{R_2}, \]
\[ k_{21} = \frac{1}{A_2} \frac{\partial \theta}{\partial \alpha_2} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} \psi, \]
\[ k_{12} = \frac{1}{A_1} \frac{\partial \phi}{\partial \alpha_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} \theta. \quad (3.16) \]
\[
\begin{align*}
k_{13} &= \frac{1}{A_1} \frac{\partial x}{\partial a_1} - \frac{\theta}{R_1}, \\
k_{23} &= \frac{1}{A_2} \frac{\partial x}{\partial a_2} - \frac{\psi}{R_2}.
\end{align*}
\]

The presence of the parameters \(v_{11}, v_{22}, v_{12}\) in equations (3.13) indicates that the linear law of variation of displacements through the thickness of the shell generally corresponds to a nonlinear variation in the strain components. However, for small elongations and shear \(v_{11}, v_{22}, v_{12}\) are always negligible. Therefore, the expressions for \(v_{11}, v_{22}, v_{12}\) are not written out here.

### 3.3 Modified Novozhilov theory I

If in equations (3.12) we set

\[
(1 + z/R_i)^{-1} = 1 - z/R_i,
\]
\[
(1 + z/R_i)^{-2} = 1 - 2z/R_i, \text{ where } i = 1, 2.
\]

and introduce equations (3.3), then the modified Novozhilov theory I can be obtained as

\[
\begin{align*}
\varepsilon_{11} &= \xi_{11} + z\chi_{11} \\
\varepsilon_{22} &= \xi_{22} + z\chi_{22} \\
\varepsilon_{12} &= \xi_{12} + z\chi_{12}
\end{align*}
\]

(3.17a) (3.17b) (3.17c)

where the \(\xi_{ij}\)'s and \(\chi_{ij}\)'s are listed in Appendix B.

The equivalence between the notation in the text and in the program GNKRSML is given in the following list.
<table>
<thead>
<tr>
<th>Text</th>
<th>Program GNKRSMI</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{\varepsilon}_{11}$</td>
<td>EP11</td>
</tr>
<tr>
<td>$\dot{\varepsilon}_{22}$</td>
<td>EP22</td>
</tr>
<tr>
<td>$\dot{\varepsilon}_{12}$</td>
<td>EP12</td>
</tr>
<tr>
<td>$\chi_{11}$</td>
<td>K11</td>
</tr>
<tr>
<td>$\chi_{22}$</td>
<td>K22</td>
</tr>
<tr>
<td>$\chi_{12}$</td>
<td>K12</td>
</tr>
<tr>
<td>$\ddot{u}$, $\ddot{v}$, $\ddot{w}$</td>
<td>u1, v1, w1</td>
</tr>
<tr>
<td>$\ddot{a}$, $\ddot{a}_{2}$</td>
<td>a1, a2</td>
</tr>
<tr>
<td>$\dot{b}<em>{11}$, $\dot{b}</em>{12}$, $\dot{b}_{13}$</td>
<td>EB11, EB12, EB13</td>
</tr>
<tr>
<td>$\dot{b}<em>{21}$, $\dot{b}</em>{22}$, $\dot{b}_{23}$</td>
<td>EB21, EB22, EB23</td>
</tr>
</tbody>
</table>

### 3.4 Modified Novozhilov theory 2

In equations (3.12), we first put the sum over a common denominator, and then let $z/R_i$ ($i = 1, 2$) be equal to zero only in the denominator. Afterwards, introducing equations (3.3), the modified Novozhilov theory 2 can be obtained as

\[
\varepsilon_{11} = \dot{\varepsilon}_{11} + z\chi_{11} \quad \quad \quad (3.18a)
\]

\[
\varepsilon_{22} = \dot{\varepsilon}_{22} + z\chi_{22} \quad \quad \quad (3.18b)
\]

\[
\varepsilon_{12} = \dot{\varepsilon}_{12} + z\chi_{12} \quad \quad \quad (3.18c)
\]
where the $\varepsilon_{ij}$'s and $\chi_{ij}$'s can be obtained by GNKRSM2.

3.5 Modified Novozhilov theory 3

In equations (3.12), without putting the sum over a common denominator, and introducing equations (3.3), we obtain the modified Novozhilov theory 3 as

\begin{align}
\varepsilon_{11} &= \varepsilon_{11}^{L} + \varepsilon_{11}^{N} \\
\varepsilon_{22} &= \varepsilon_{22}^{L} + \varepsilon_{22}^{N} \\
\varepsilon_{12} &= \varepsilon_{12}^{L} + \varepsilon_{12}^{N}
\end{align}

(3.19a, 3.19b, 3.19c)

with

\begin{align}
\varepsilon_{11}^{L} &= \frac{1}{1 + \frac{z}{R_1}} (\varepsilon_{11}^{L} + z\chi_{11}) \\
\varepsilon_{22}^{L} &= \frac{1}{1 + \frac{z}{R_2}} (\varepsilon_{22}^{L} + z\chi_{22}) \\
\varepsilon_{12}^{L} &= \frac{1}{(1 + \frac{z}{R_1})(1 + \frac{z}{R_2})} (\varepsilon_{12}^{L} + z\chi_{12}) \\
\varepsilon_{11}^{N} &= \frac{1}{(1 + \frac{z}{R_1})^2} (\varepsilon_{11}^{N} + z\chi_{11}^{N}) \\
\varepsilon_{22}^{N} &= \frac{1}{(1 + \frac{z}{R_2})^2} (\varepsilon_{22}^{N} + z\chi_{22}^{N}) \\
\varepsilon_{12}^{N} &= \frac{1}{(1 + \frac{z}{R_1})(1 + \frac{z}{R_2})} (\varepsilon_{12}^{N} + z\chi_{12}^{N})
\end{align}

(3.20)

where the $\varepsilon_{ij}^{L}$'s, $\varepsilon_{ij}^{N}$'s, $\chi_{ij}$'s and $\chi_{ij}^{N}$'s can be obtained by GNKRSM3.
3.6 Discussion of the results

1. In all of the above theories, the terms which involve the square of coordinate z (or higher than that) are neglected.

2. For the expressions of total strains, Novozhilov theory and modified Novozhilov theories 1 and 2 have the same form, while the modified Novozhilov theory 3 has its own form of equations (3.19).

3. It can be seen that the modified Novozhilov theory 3 is the most general form of the four types of theories discussed, while the Novozhilov theory is the simplest one. The modified Novozhilov theory 1 and 2 are improvements for the Novozhilov theory by retaining z/R₁ at different places.

4. Substitution α₁ = s, α₂ = θ, letting R₁ = s, R₂ = s tan θ, A₁ = 1 and A₂ = s sin θ into the above derived four Novozhilov-like nonlinear theories yields the corresponding four Novozhilov-like nonlinear theories of conical thin shells. Similarly, other results can be easily obtained by introducing other kinds of geometrical properties.
4. The Reduce-based package GNKRS [17]

The GNKRS (Generating Nonlinear Kinematic Relations of Shells) is written for the derivations of various linear and nonlinear kinematic relations of shells, based on the three-dimensional elasticity theory. The whole package consists of four blocks.

Block 1, expressed by file GNKRS1, is for the cases where \( \frac{z}{R_i} \) \((i=1,2)\) can be always assumed to be zero.

Block 2, expressed by file GNKRS2, is for the cases where \( \frac{z}{R_i} \) \((i=1,2)\) is retained in the kinematic relations.

Block 3, expressed by file GNKRS3, is for the cases where \( \frac{z}{R_i} \) \((i=1,2)\) assumed to be zero in the common denominator of equations (3.12).

Block 4, expressed by file GNKRS4, is for the cases
where \( \frac{1}{1 + \frac{z}{R_i}} = 1 - \frac{z}{R_i} \), \( \frac{1}{(1 + \frac{z}{R_i})^2} = 1 - \frac{2z}{R_i} \) \((i=1,2)\).

The GNKRS written in Reduce can be used interactively in the Reduce environment. For the different retentions of terms in equations (1.1), one can get all the results corresponding to the four different ways of dealing with \( \frac{z}{R_i} \).
The flow chart of the execution procedure is

```
REduce

IN GNKRsi $

PAUSE $  Y  STOP

N

INPUT deli $

CONT $

PAUSE $  Y  STOP

N

INPUT θ, ψ, χ $

CONT $

PRINT OUT THE RESULT
```
Notes:

1. In order to meet the need to make different approximations of equations (1.1), the input data deli are used. One can decide which terms in equations (1.1) are to be retained by setting deli = 1, and to be neglected by setting deli = 0. Thus various nonlinear theories can be obtained.

2. The input data θ, ψ, x represent the Kirchhoff assumptions. They can be the "linear Kirchhoff assumption results" or the "nonlinear Kirchhoff assumption results", and correspond to different accuracies of approximations. See Appendix A.
5. Conclusions

1. The motivations for investigating the nonlinear shell theories, besides the reasons that have been mentioned earlier, lie also in the fact that there are some special shell configurations for which presumably small terms may be of significance, and for which different results are obtained from different theories. Therefore it is necessary to choose the "best" or most suitable shell theory.

In ref. [18], another modified Novozhilov theory was suggested, which differs from the Novozhilov theory by using the linear Kirchhoff assumption results instead of the nonlinear ones. This theory was used in the in-plane buckling analysis of an annular plate ring, and the result shows that the nonlinear strains arising from products of in-plane strain terms are important in this buckling problem.

2. In the linear theories, Bushnell [2] and Naghdi [3] suggested that the differences attributable to retentions of \( \frac{Z}{R_i} \) (\( i = 1, 2 \)) are of little importance for most engineering problems, and it is best to choose the simplest theories. However, since up to now no results have been published about comparison studies involving the above different nonlinear Novozhilov theories obtained by different retentions of \( \frac{Z}{R_i} \) (\( i = 1, 2 \)), therefore no statements can be made about the most suitable nonlinear theory for a certain application.

3. The differences in approach of linear or nonlinear theories in the determination of strains consist in that the linear theories neglect the influence of rotations on elongations and shears, while the nonlinear theories take it into account. Besides, in linear theories, based on the assumption that the strain and rotations are small and of comparable magnitude, the parameters \( e_{11}, e_{22}, e_{33} \) are the elongations of the line elements having directions \( \hat{k}_1, \hat{k}_2, \hat{k}_3 \), the parameters \( e_{12}, e_{23}, e_{13} \) are shears between these elements, and \( \omega_1, \omega_2, \omega_3 \) are the angles of rotation of a volume element about \( \hat{k}_1, \hat{k}_2, \hat{k}_3 \). But in the general case of an arbitrary deformation, these parameters do not have so simple a geometrical meaning and cannot be identified with the strain components and the angles of rotation.

4. In all the above results of nonlinear theories, the expressions of middle surface strains and curvature changes are nonlinear. But a physical observation shows that there is a need to retain nonlinear terms in the strain-displacement but not in the curvature-displacement relations as long
as the largest reference surface rotations are less than about 20°, which is usually the case [2]. However, there are still some cases where the nonlinear curvature changes should not be neglected [18].

5. All the simplifications and assumptions used in deriving the kinematic relations should be consistent with those which will be used in the derivation of constitutive relations. For example, whether \( \frac{Z}{R_i} \) (i = 1, 2) in the force and moment expressions are negligible or not, depends on which kind of kinematic relations are employed. That is to say, if Vlasov kinematic relation is used (assuming \( \frac{1}{1 + \frac{Z}{R_i}} = 1 - \frac{Z}{R_i} \)), then \( \frac{Z}{R_i} \) in the force and moment expressions should not be neglected.

6. All the above nonlinear theories are Novozhilov-like theories, which are obtained by the different retentions of \( \frac{Z}{R_i} \) (i = 1, 2) in equations (1.1), and are valid for small strains and arbitrary rotations. On the other hand, by different retentions of terms in equations (1.1), other kinds of nonlinear theories can also be obtained. Each of them can also generate a group of nonlinear theories by the different retention of \( \frac{Z}{R_i} \) (i = 1, 2) terms.

Svalbonas [12] assumes that \( \omega_3 \) is considerably smaller than \( \omega_1 \) and \( \omega_2 \), and \( \omega_i^2 \) (i = 1, 2) is the order of magnitude of \( \varepsilon_{ij} \), thus equations (1.1) become

\[
\varepsilon_{11} = e_{11} + \frac{1}{2} \omega_2^2
\]

\[
\varepsilon_{22} = e_{22} + \frac{1}{2} \omega_1^2
\]

\[
\varepsilon_{33} = e_{33} + \frac{1}{2} (\omega_1^2 + \omega_2^2)
\]

\[
\varepsilon_{12} = e_{12} - \omega_1 \omega_2
\]

\[
\varepsilon_{13} = e_{13}
\]

\[
\varepsilon_{23} = e_{23}
\]

Bushnell [2] assumes that

\[
\varepsilon_{11} = e_{11} + \frac{1}{2} (\omega_2^2 + \omega_3^2)
\]

\[
\varepsilon_{22} = e_{22} + \frac{1}{2} (\omega_1^2 + \omega_3^2)
\]

(4.1)
\[ \varepsilon_{12} = \varepsilon_{12} - \omega_1 \omega_2 \]
\[ \varepsilon_{33} = \varepsilon_{33} \]
\[ \varepsilon_{13} = \varepsilon_{13} \]
\[ \varepsilon_{23} = \varepsilon_{23} \]

The nonlinear results corresponding to different retentions of \( \frac{Z}{R_i} \) (i = 1,2) in equations (4.1) and (4.2) can easily be obtained by our Reduce-based package.

7. For most of the presently available nonlinear theories [12][2][16] the displacements are assumed to be represented by

\[ u = \hat{u}(\alpha_1, \alpha_2) + z\theta(\alpha_1, \alpha_2) \]
\[ v = \hat{v}(\alpha_1, \alpha_2) + z\psi(\alpha_1, \alpha_2) \]
\[ w = \hat{w}(\alpha_1, \alpha_2) \]

(4.3)

Since the second term in equations (3.2c) is assumed small as compared to the first term. It is easy to see that, actually, equations (4.3) imply that the \( \varepsilon_{33}, \varepsilon_{13}, \varepsilon_{23} \) expressions in equations (1.1) must be linear. Therefore we can say that these nonlinear theories using equations (4.3) are not fully nonlinear. In fact, they are some kinds of mixed theories between the pure linear and nonlinear theories. This kind of mixing can lead to considerable simplification in the derivations of the nonlinear theories, but may also cause the loss of accuracy sometimes.

8. The linear Novozhilov relations for \( \hat{e}_{ij} \) and \( \chi_{ij} \) can be obtained from the nonlinear Novozhilov theory by A) neglecting all the nonlinear terms in the nonlinear expressions of \( \hat{e}_{ij} \) and \( \chi_{ij} \) B) letting \( \chi \) equal to zero, i.e. \( w(\alpha_1, \alpha_2, z) = w(\alpha_1, \alpha_2) \). Without step B, the Novozhilov nonlinear theory cannot degenerate to its linear theory.

9. The derivation procedures discussed in this report can be summarized as follows:
THREE DIMENSIONAL ELASTICITY THEORY

DIFFERENT RETENTIONS OF TERMS IN EQUATIONS (1.1)

DIFFERENT KIRCHHOFF ASSUMPTION RESULTS

DIFFERENT RETENTIONS OF $\frac{1}{(1+\frac{Z}{R_i})}$

DIFFERENT KINDS OF NONLINEAR THEORIES
References.


5. V. Novozhilov, Foundations of the Non-linear Theory of Elasticity, Graylock Press, 1953

6. H. Kraus, Thin Elastic Shells, John Wiley and Sons, Inc. 1967


APPENDIX A

On the linear or nonlinear "Kirchhoff assumption results"

1. In the linear theories, the Kirchhoff assumptions can be expressed as follows:

\[ \epsilon_{1z} = \frac{\delta u}{\delta z} + \frac{1}{1 + \frac{z}{R_1}} \left( \frac{1}{A_1} \frac{\delta w}{\delta a_1} - \frac{u}{R_1} \right) = 0 \]  \hspace{1cm} (1)

\[ \epsilon_{2z} = \frac{\delta v}{\delta z} + \frac{1}{1 + \frac{z}{R_2}} \left( \frac{1}{A_2} \frac{\delta w}{\delta a_2} - \frac{v}{R_2} \right) = 0 \]  \hspace{1cm} (2)

\[ \epsilon_{zz} = \frac{\delta w}{\delta z} = 0 \]  \hspace{1cm} (3)

In order to solve the above equations, noticing equation (3) one can assume

\[ u = \bar{u} + z\theta \]
\[ v = \bar{v} + z\psi \]
\[ w = \bar{w} \]  \hspace{1cm} (4)

Substituting equations (4) into equations (1-3), \( \theta \) and \( \psi \), which identically satisfy equations (1-3) can be obtained as

\[ \theta = -\left( \frac{1}{A_1} \frac{\delta \bar{w}}{\delta a_1} - \frac{\bar{u}}{R_1} \right) \]
\[ \psi = -\left( \frac{1}{A_2} \frac{\delta \bar{w}}{\delta a_2} - \frac{\bar{v}}{R_2} \right) \]  \hspace{1cm} (5)

2. In the nonlinear theories, the Kirchhoff assumptions become

\[ \epsilon_{1z} = \frac{\delta u}{\delta z} + \frac{1}{1 + \frac{z}{R_1}} \left( \frac{1}{A_1} \frac{\delta w}{\delta a_1} - \frac{u}{R_1} \right) + \frac{1}{1 + \frac{z}{R_1}} \left( \frac{1}{A_1} \frac{\delta u}{\delta a_1} + \frac{1}{A_1A_2} \frac{\delta A_1}{\delta a_2} v + w \right) \frac{\delta u}{\delta z} \]

\[ + \frac{1}{1 + \frac{z}{R_1}} \left( \frac{1}{A_1} \frac{\delta v}{\delta a_1} - \frac{1}{A_1A_2} \frac{\delta A_1}{\delta a_2} u \right) \frac{\delta v}{\delta z} \]

\[ + \frac{1}{1 + \frac{z}{R_1}} \left( \frac{1}{A_1} \frac{\delta w}{\delta a_1} - \frac{u}{R_1} \right) \frac{\delta w}{\delta z} = 0 \]  \hspace{1cm} (6)
\[ \epsilon_{zz} = \frac{\partial v}{\partial z} + \frac{1}{(1 + \frac{z}{R_2})} \left( \frac{1}{A_2} \frac{\partial w}{\partial z} - \frac{v}{R_2} \right) + \frac{1}{(1 + \frac{z}{R_2})} \left( \frac{1}{A_2} \frac{\partial v}{\partial z} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial z} u + \frac{w}{R_2} \right) \frac{\partial v}{\partial z} \]

\[ + \frac{1}{(1 + \frac{z}{R_2})} \left( \frac{1}{A_2} \frac{\partial u}{\partial z} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial z} v \right) \frac{\partial u}{\partial z} \]

\[ + \frac{1}{(1 + \frac{z}{R_2})} \left( \frac{1}{A_2} \frac{\partial w}{\partial z} - \frac{v}{R_2} \right) \frac{\partial w}{\partial z} = 0, \tag{7} \]

\[ \epsilon_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right] = 0 \tag{8} \]

To solve the above equations, if one still use the displacement assumptions presented by equations (4), one obtains

\[ \theta^2 + \psi^2 = 0 \tag{9} \]

\[ \theta = \frac{\hat{a}_{31}}{\hat{a}_{33}}, \quad \psi = \frac{\hat{a}_{32}}{\hat{a}_{33}} \]

The above results have no meaning. That is to say, equations (4) are not the solutions of equations (6-8). Therefore another kind of displacement assumptions must be introduced. Those are

\[ u = \hat{u} + z\theta \]
\[ v = \hat{v} + z\psi \]
\[ w = \hat{w} + z\chi \tag{10} \]

Substituting equations (10) into equations (6-8), one obtains

\[ \theta = -\hat{e}_{13} (1 + \hat{e}_{22}) + \hat{e}_{23}\hat{e}_{21} \]
\[ \psi = -\hat{e}_{23} (1 + \hat{e}_{11}) + \hat{e}_{13}\hat{e}_{12} \tag{11} \]
\[ \chi = \hat{e}_{11} + \hat{e}_{22} + \hat{e}_{11}\hat{e}_{22} - \hat{e}_{12}\hat{e}_{21} \]

Where \( \hat{e}_{ij} \) are given in the main report.
3. Conclusion:
1. When using Kirchhoff assumptions, different expressions of $\varepsilon_{1z}$, $\varepsilon_{2z}$, $\varepsilon_{zz}$ lead (or need) different displacement relations.

2. The displacement relations, equations (4), are called the linear Kirchhoff assumption results, since they correspond to the linear $\varepsilon_{1z}$, $\varepsilon_{2z}$, $\varepsilon_{zz}$ expressions on which Kirchhoff assumptions are based.

While equations (10) are called the nonlinear Kirchhoff assumption results, since they correspond to the nonlinear $\varepsilon_{1z}$, $\varepsilon_{2z}$, $\varepsilon_{zz}$ expressions.
APPENDIX B

Computer results of $\xi_{ij}$'s and $\chi_{ij}$'s of the modified Novozhilov theory

\[ EP_{11} = (DF(A1, AL2) * U1 * R1 \]

\[ \begin{align*}
2 & \quad 2 \quad 2 \\
+DF(A1, AL2) * V1 & \quad R1 \\
+2*DF(A1, AL2) * DF(U1, AL1) * A2 * V1 * R1 & \quad 2 \\
-2*DF(A1, AL2) * DF(V1, AL1) * A2 * U1 * R1 & \quad 2 \\
+2*DF(A1, AL2) * A1 * A2 * V1 * W1 * R1 & \quad 2 \\
+2*DF(A1, AL2) * A1 * A2 * V1 * R1 & \quad 2 \\
+DF(U1, AL1) * A2 & \quad R1 \\
+2*DF(U1, AL1) * A1 * A2 * W1 * R1 & \quad 2 \\
+2*DF(U1, AL1) * A1 * A2 & \quad R1 \\
+DF(V1, AL1) * A2 & \quad R1 \\
+DF(W1, AL1) * A2 & \quad R1 \\
-2*DF(W1, AL1) * A1 * A2 * U1 * R1 & \quad 2 \\
+A1 & \quad A2 * U1 \\
+2*DF(A1, AL1) * A2 & \quad W1 \\
\end{align*} \]

\[ \frac{2*DF(A1, AL1) * A2 * W1 * R1/(2*DF(A1, AL1) * A2 * R1)}{2*222} \]

\[ EP_{22} = (DF(A2, AL1) * U1 * R2 \]

\[ \begin{align*}
2 & \quad 2 \quad 2 \\
+DF(A2, AL1) * V1 & \quad R2 \\
-2*DF(A2, AL1) * DF(U1, AL2) * A1 * V1 * R2 & \quad 2 \\
+2*DF(A2, AL1) * DF(V1, AL2) * A1 * U1 * R2 & \quad 2 \\
+2*DF(A2, AL1) * A1 * A2 * U1 * W1 * R2 & \quad 2 \\
\end{align*} \]
\[ \begin{align*} 
+2\cdot DF(A_2, AL_1) & \cdot (A_1 \cdot A_2 \cdot U_1 \cdot R_2) \\
+ DF(U_1, AL_2) & \cdot A_1 \cdot R_2 \\
+ DF(V_1, AL_2) & \cdot A_1 \cdot R_2 \\
+2\cdot DF(V_1, AL_2) & \cdot A_1 \cdot A_2 \cdot W_1 \cdot R_2 \\
+2\cdot DF(V_1, AL_2) & \cdot A_1 \cdot A_2 \cdot R_2 \\
+ DF(W_1, AL_2) & \cdot A_1 \cdot R_2 \\
-2\cdot DF(W_1, AL_2) & \cdot A_1 \cdot A_2 \cdot V_1 \cdot R_2 \\
+A_1 \cdot A_2 \cdot V_1 \\
+A_1 \cdot A_2 \cdot W_1 \\
+2\cdot A_1 \cdot A_2 \cdot W_1 \cdot R_2 \cdot R_2 \\
\end{align*} \]

\[ EP_{12} = \]

\[ \begin{align*} 
- (DF(A_1, AL_2) \cdot DF(A_2, AL_1) \cdot U_1 \cdot R_1 \cdot R_2) \\
+ DF(A_1, AL_2) \cdot DF(A_2, AL_1) \cdot V_1 \cdot R_1 \cdot R_2 \\
- DF(A_1, AL_2) \cdot DF(U_1, AL_2) \cdot A_1 \cdot V_1 \cdot R_1 \cdot R_2 \\
+ DF(A_1, AL_2) \cdot DF(V_1, AL_2) \cdot A_1 \cdot U_1 \cdot R_1 \cdot R_2 \\
+ DF(A_1, AL_2) \cdot A_1 \cdot A_2 \cdot U_1 \cdot W_1 \cdot R_1 \\
+ DF(A_1, AL_2) \cdot A_1 \cdot A_2 \cdot U_1 \cdot R_1 \cdot R_2 \\
+ DF(A_2, AL_1) \cdot DF(U_1, AL_1) \cdot A_2 \cdot V_1 \cdot R_1 \cdot R_2 \\
- DF(A_2, AL_1) \cdot DF(V_1, AL_1) \cdot A_2 \cdot U_1 \cdot R_1 \cdot R_2 \\
+ DF(A_2, AL_1) \cdot A_1 \cdot A_2 \cdot V_1 \cdot W_1 \cdot R_1 \\
+ DF(A_2, AL_1) \cdot A_1 \cdot A_2 \cdot V_1 \cdot R_1 \cdot R_2 \\
- DF(U_1, AL_1) \cdot DF(U_1, AL_2) \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2 \\
\end{align*} \]
\[-DF(U_1, AL_2) \cdot A_1 \cdot A_2 \cdot W_1 \cdot R_2\]
\[-DF(U_1, AL_2) \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2\]
\[-DF(V_1, AL_1) \cdot DF(V_1, AL_2) \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2\]
\[-DF(V_1, AL_1) \cdot A_1 \cdot A_2 \cdot W_1 \cdot R_1\]
\[-DF(V_1, AL_1) \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2\]
\[-DF(W_1, AL_1) \cdot DF(W_1, AL_2) \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2\]
\[+DF(W_1, AL_1) \cdot A_1 \cdot A_2 \cdot V_1 \cdot R_1\]
\[+DF(W_1, AL_2) \cdot A_1 \cdot A_2 \cdot U_1 \cdot R_2\]
\[-A_1 \cdot A_2 \cdot (U_1 \cdot V_1) / (A_1 \cdot A_2 \cdot R_1 \cdot R_2)\]

\[K_{11} =\]
\[2\]
\[-(DF(A_1, AL_2) \cdot U_1 \cdot R_1)\]
\[2\]
\[+DF(A_1, AL_2) \cdot U_1 \cdot R_1 \cdot EB_{22} \cdot EB_{13}\]
\[2\]
\[-DF(A_1, AL_2) \cdot U_1 \cdot R_1 \cdot EB_{21} \cdot EB_{23}\]
\[2\]
\[+DF(A_1, AL_2) \cdot U_1 \cdot R_1 \cdot EB_{13}\]
\[2\]
\[+DF(A_1, AL_2) \cdot V_1 \cdot R_1\]
\[2\]
\[+DF(A_1, AL_2) \cdot V_1 \cdot R_1 \cdot EB_{11} \cdot EB_{23}\]
\[2\]
\[-DF(A_1, AL_2) \cdot V_1 \cdot R_1 \cdot EB_{12} \cdot EB_{13}\]
\[2\]
\[+DF(A_1, AL_2) \cdot V_1 \cdot R_1 \cdot EB_{23}\]
\[2\]
\[+2 \cdot DF(A_1, AL_2) \cdot DF(U_1, AL_1) \cdot A_2 \cdot V_1 \cdot R_1\]
\[3\]
\[+DF(A_1, AL_2) \cdot DF(U_1, AL_1) \cdot A_2 \cdot R_1 \cdot EB_{11} \cdot EB_{23}\]
\[3\]
\[-DF(A_1, AL_2) \cdot DF(U_1, AL_1) \cdot A_2 \cdot R_1 \cdot EB_{12} \cdot EB_{13}\]
-DF(A1, AL2) *DF(U1, AL1) *A2 *R1 *EB23
-2 *DF(A1, AL2) *DF(V1, AL1) *A2 *U1 *R1
-DF(A1, AL2) *DF(V1, AL1) *A2 *R1 *EB22 *EB13
+DF(A1, AL2) *DF(V1, AL1) *A2 *R1 *EB21 *EB23
-DF(A1, AL2) *DF(V1, AL1) *A2 *R1 *EB13
-DF(A1, AL2) *DF(EB11, AL1) *A2 *U1 *R1 *EB23
+DF(A1, AL2) *DF(EB22, AL1) *A2 *V1 *R1 *EB13
+DF(A1, AL2) *DF(EB12, AL1) *A2 *U1 *R1 *EB13
-DF(A1, AL2) *DF(EB21, AL1) *A2 *V1 *R1 *EB23
+DF(A1, AL2) *DF(EB13, AL1) *A2 *U1 *R1 *EB12
+DF(A1, AL2) *DF(EB13, AL1) *A2 *V1 *R1 *EB22
+DF(A1, AL2) *DF(EB13, AL1) *A2 *V1 *R1
-DF(A1, AL2) *DF(EB23, AL1) *A2 *U1 *R1 *EB11
-DF(A1, AL2) *DF(EB23, AL1) *A2 *U1 *R1
-DF(A1, AL2) *DF(EB23, AL1) *A2 *V1 *R1 *EB21
+2 *DF(A1, AL2) *A1 *A2 *V1 *W1 *R1
-DF(A1, AL2) *A1 *A2 *V1 *R1 *EB11
-DF(A1, AL2) *A1 *A2 *V1 *R1 *EB22
+DF(A1, AL2) *A1 *A2 *V1 *R1 *EB12 *EB21
\[ +DF(A1,AL2)*A1*A2*V1*R1 \]
\[ +DF(A1,AL2)*A1*A2*W1*R1 *EB11*EB23 \]
\[ -DF(A1,AL2)*A1*A2*W1*R1 *EB12*EB13 \]
\[ +DF(A1,AL2)*A1*A2*W1*R1 *EB23 \]
\[ +DF(A1,AL2)*A1*A2*R1 *EB11*EB23 \]
\[ -DF(A1,AL2)*A1*A2*R1 *EB12*EB13 \]
\[ +DF(A1,AL2)*A1*A2*R1 *EB23 \]
\[ +DF(U1,AL1)*A2*R1 \]
\[ +DF(U1,AL1)*DF(EB22,AL1)*A2*R1 *EB13 \]
\[ -DF(U1,AL1)*DF(EB21,AL1)*A2*R1 *EB23 \]
\[ +DF(U1,AL1)*DF(EB13,AL1)*A2*R1 *EB22 \]
\[ +DF(U1,AL1)*DF(EB13,AL1)*A2*R1 \]
\[ -DF(U1,AL1)*DF(EB23,AL1)*A2*R1 *EB21 \]
\[ +2*DF(U1,AL1)*A1*A2*W1*R1 \]
\[ -DF(U1,AL1)*A1*A2*R1 *EB11*EB22 \]
\[ -DF(U1,AL1)*A1*A2*R1 *EB11 \]
\[ -DF(U1,AL1)*A1*A2*R1 *EB22 \]
\[ +DF(U1,AL1)*A1*A2*R1 *EB12*EB21 \]
\[ +DF(U1,AL1)*A1*A2*R1 \]
\[ +DF(V1,AL1)*A2*R1 \]
\[
+DF(V_1, A_1) * DF(E_{B11}, A_1) * A_2 * R_1 * E_{B23} \\
+DF(V_1, A_1) * DF(E_{B12}, A_1) * A_2 * R_1 * E_{B13} \\
+DF(V_1, A_1) * DF(E_{B13}, A_1) * A_2 * R_1 * E_{B12} \\
+DF(V_1, A_1) * DF(E_{B23}, A_1) * A_2 * R_1 * E_{B11} \\
+DF(V_1, A_1) * DF(E_{B23}, A_1) * A_2 * R_1 \\
+DF(W_1, A_1) * A_2 * R_1 \\
+DF(W_1, A_1) * DF(E_{B11}, A_1) * A_2 * R_1 * E_{B22} \\
+DF(W_1, A_1) * DF(E_{B11}, A_1) * A_2 * R_1 \\
+DF(W_1, A_1) * DF(E_{B22}, A_1) * A_2 * R_1 * E_{B11} \\
+DF(W_1, A_1) * DF(E_{B22}, A_1) * A_2 * R_1 \\
+DF(W_1, A_1) * DF(E_{B21}, A_1) * A_2 * R_1 * E_{B12} \\
+DF(W_1, A_1) * A_1 * A_2 * U_1 * R_1 \\
+DF(W_1, A_1) * A_1 * A_2 * R_1 * E_{B22} * E_{B13} \\
+DF(W_1, A_1) * A_1 * A_2 * R_1 * E_{B21} * E_{B23} \\
+DF(W_1, A_1) * A_1 * A_2 * R_1 * E_{B13} \\
+DF(E_{B11}, A_1) * A_1 * A_2 * U_1 * R_1 * E_{B22} \\
+DF(E_{B11}, A_1) * A_1 * A_2 * U_1 * R_1 \\
+DF(E_{B22}, A_1) * A_1 * A_2 * U_1 * R_1 * E_{B11} \\
+DF(E_{B22}, A_1) * A_1 * A_2 * U_1 * R_1 \\
\]
\[ +DF(EB22, A11) \cdot A1 \cdot A2 \cdot W1 \cdot R1 \cdot EB13 +DF(EB22, A11) \cdot A1 \cdot A2 \cdot R1 \cdot EB13 -DF(EB12, A11) \cdot A1 \cdot A2 \cdot U1 \cdot R1 \cdot EB21 -DF(EB21, A11) \cdot A1 \cdot A2 \cdot U1 \cdot R1 \cdot EB12 -DF(EB21, A11) \cdot A1 \cdot A2 \cdot W1 \cdot R1 \cdot EB23 -DF(EB21, A11) \cdot A1 \cdot A2 \cdot R1 \cdot EB23 +DF(EB13, A11) \cdot A1 \cdot A2 \cdot W1 \cdot R1 \cdot EB22 +DF(EB13, A11) \cdot A1 \cdot A2 \cdot W1 \cdot R1 \cdot EB13 +DF(EB13, A11) \cdot A1 \cdot A2 \cdot R1 \cdot EB22 +DF(EB13, A11) \cdot A1 \cdot A2 \cdot R1 \cdot EB21 -DF(EB23, A11) \cdot A1 \cdot A2 \cdot W1 \cdot R1 \cdot EB21 -DF(EB23, A11) \cdot A1 \cdot A2 \cdot R1 \cdot EB21 +A1 \cdot A2 \cdot U1 +A1 \cdot A2 \cdot U1 \cdot R1 \cdot EB22 \cdot EB13 -A1 \cdot A2 \cdot U1 \cdot R1 \cdot EB21 \cdot EB23 +A1 \cdot A2 \cdot U1 \cdot R1 \cdot EB13 +A1 \cdot A2 \cdot W1 -A1 \cdot A2 \cdot W1 \cdot R1 \cdot EB11 \cdot EB22 -A1 \cdot A2 \cdot W1 \cdot R1 \cdot EB11 -A1 \cdot A2 \cdot W1 \cdot R1 \cdot EB22 \]
\[+A_1 \ast A_2 \ast W_1 \ast R_1 \ast E_{B12} \ast E_{B21}\]
\[+A_1 \ast A_2 \ast W_1 \ast R_1\]
\[-A_1 \ast A_2 \ast R_1 \ast E_{B11} \ast E_{B22}\]
\[-A_1 \ast A_2 \ast R_1 \ast E_{B11}\]
\[-A_1 \ast A_2 \ast R_1 \ast E_{B22}\]
\[+A_1 \ast A_2 \ast R_1 \ast E_{B12} \ast E_{B21} \big/ (A_1 \ast A_2 \ast R_1 )\]

\[K_{22} :=\]
\[-(DF(A_2, A_1)) \ast U_1 \ast R_2\]
\[+DF(A_2, A_1) \ast U_1 \ast R_2 \ast E_{B22} \ast E_{B13}\]
\[-DF(A_2, A_1) \ast U_1 \ast R_2 \ast E_{B21} \ast E_{B23}\]
\[+DF(A_2, A_1) \ast U_1 \ast R_2 \ast E_{B13}\]
\[+DF(A_2, A_1) \ast V_1 \ast R_2\]
\[+DF(A_2, A_1) \ast V_1 \ast R_2 \ast E_{B11} \ast E_{B23}\]
\[-DF(A_2, A_1) \ast V_1 \ast R_2 \ast E_{B12} \ast E_{B13}\]
\[+DF(A_2, A_1) \ast V_1 \ast R_2 \ast E_{B23}\]
\[-2 \ast DF(A_2, A_1) \ast DF(U_1, A_2) \ast A_1 \ast V_1 \ast R_2\]
\[-DF(A_2, A_1) \ast DF(U_1, A_2) \ast A_1 \ast R_2 \ast E_{B11} \ast E_{B23}\]
\[+DF(A_2, A_1) \ast DF(U_1, A_2) \ast A_1 \ast R_2 \ast E_{B12} \ast E_{B13}\]
\[-DF(A_2, A_1) \ast DF(U_1, A_2) \ast A_1 \ast R_2 \ast E_{B23}\]
\[+2 \ast DF(A_2, A_1) \ast DF(V_1, A_2) \ast A_1 \ast U_1 \ast R_2\]
\[ \begin{align*}
&\frac{2}{\text{DF}(A_2, A_{11}) \ast A_1 \ast A_2 \ast W_1 \ast R_2 \ast \text{EB}_{21} \ast \text{EB}_{23}} \\
&\frac{2}{\text{DF}(A_2, A_{11}) \ast A_1 \ast A_2 \ast W_1 \ast R_2 \ast \text{EB}_{13}} \\
&\frac{3}{\text{DF}(A_2, A_{11}) \ast A_1 \ast A_2 \ast R_2 \ast \text{EB}_{22} \ast \text{EB}_{13}} \\
&\frac{3}{\text{DF}(A_2, A_{11}) \ast A_1 \ast A_2 \ast R_2 \ast \text{EB}_{21} \ast \text{EB}_{23}} \\
&\frac{3}{\text{DF}(A_2, A_{11}) \ast A_1 \ast A_2 \ast R_2 \ast \text{EB}_{13}} \\
&\frac{2 \ast 2 \ast 2}{\text{DF}(U_1, A_{12}) \ast A_1 \ast R_2} \\
&\frac{2 \ast 3}{\text{DF}(U_1, A_{12}) \ast \text{DF}(\text{EB}_{22}, A_{12}) \ast A_1 \ast R_2 \ast \text{EB}_{13}} \\
&\frac{2 \ast 3}{\text{DF}(U_1, A_{12}) \ast \text{DF}(\text{EB}_{21}, A_{12}) \ast A_1 \ast R_2 \ast \text{EB}_{23}} \\
&\frac{2 \ast 3}{\text{DF}(U_1, A_{12}) \ast \text{DF}(\text{EB}_{13}, A_{12}) \ast A_1 \ast R_2 \ast \text{EB}_{22}} \\
&\frac{2 \ast 3}{\text{DF}(U_1, A_{12}) \ast \text{DF}(\text{EB}_{13}, A_{12}) \ast A_1 \ast R_2} \\
&\frac{2 \ast 3}{\text{DF}(U_1, A_{12}) \ast \text{DF}(\text{EB}_{23}, A_{12}) \ast A_1 \ast R_2 \ast \text{EB}_{21}} \\
&\frac{2 \ast 2 \ast 2}{\text{DF}(V_1, A_{12}) \ast A_1 \ast R_2} \\
&\frac{2 \ast 3}{\text{DF}(V_1, A_{12}) \ast \text{DF}(\text{EB}_{11}, A_{12}) \ast A_1 \ast R_2 \ast \text{EB}_{23}} \\
&\frac{2 \ast 3}{\text{DF}(V_1, A_{12}) \ast \text{DF}(\text{EB}_{12}, A_{12}) \ast A_1 \ast R_2 \ast \text{EB}_{13}} \\
&\frac{2 \ast 3}{\text{DF}(V_1, A_{12}) \ast \text{DF}(\text{EB}_{13}, A_{12}) \ast A_1 \ast R_2 \ast \text{EB}_{12}} \\
&\frac{2 \ast 3}{\text{DF}(V_1, A_{12}) \ast \text{DF}(\text{EB}_{23}, A_{12}) \ast A_1 \ast R_2 \ast \text{EB}_{11}} \\
&\frac{2 \ast 3}{\text{DF}(V_1, A_{12}) \ast \text{DF}(\text{EB}_{23}, A_{12}) \ast A_1 \ast R_2} \\
&\frac{2}{2 \ast \text{DF}(V_1, A_{12}) \ast A_1 \ast A_2 \ast W_1 \ast R_2} \\
&\frac{2 \ast 2}{\text{DF}(V_1, A_{12}) \ast A_1 \ast A_2 \ast R_2 \ast \text{EB}_{11} \ast \text{EB}_{22}} \\
&\frac{2 \ast 2}{\text{DF}(V_1, A_{12}) \ast A_1 \ast A_2 \ast R_2 \ast \text{EB}_{11}}
\end{align*} \]
-DF(V1, AL2) * A1 * A2 * R2 * EB22

+DF(V1, AL2) * A1 * A2 * R2 * EB12 * EB21

+DF(V1, AL2) * A1 * A2 * R2

+DF(W1, AL2) * A1 * R2

-DF(W1, AL2) * DF(EB11, AL2) * A1 * R2 * EB22

-DF(W1, AL2) * DF(EB11, AL2) * A1 * R2

-DF(W1, AL2) * DF(EB22, AL2) * A1 * R2 * EB11

-DF(W1, AL2) * DF(EB22, AL2) * A1 * R2

+DF(W1, AL2) * DF(EB12, AL2) * A1 * R2 * EB21

+DF(W1, AL2) * DF(EB21, AL2) * A1 * R2 * EB12

-2 * DF(W1, AL2) * A1 * A2 * V1 * R2

-DF(W1, AL2) * A1 * A2 * R2 * EB11 * EB23

+DF(W1, AL2) * A1 * A2 * R2 * EB12 * EB13

-DF(W1, AL2) * A1 * A2 * R2 * EB23

+DF(EB11, AL2) * A1 * A2 * V1 * R2 * EB22

+DF(EB11, AL2) * A1 * A2 * V1 * R2

+DF(EB11, AL2) * A1 * A2 * W1 * R2 * EB23

+DF(EB11, AL2) * A1 * A2 * R2 * EB23

+DF(EB22, AL2) * A1 * A2 * V1 * R2 * EB11

+DF(EB22, AL2) * A1 * A2 * V1 * R2
2 2
-DF(EB12, AL2) * A1 * A2 * V1 * R2 * EB21
2 2
-DF(EB12, AL2) * A1 * A2 * W1 * R2 * EB13
2 3
-DF(EB12, AL2) * A1 * A2 * R2 * EB13
2 2
-DF(EB21, AL2) * A1 * A2 * V1 * R2 * EB12
2 2
-DF(EB13, AL2) * A1 * A2 * W1 * R2 * EB12
2 3
-DF(EB13, AL2) * A1 * A2 * R2 * EB12
2 2
+DF(EB23, AL2) * A1 * A2 * W1 * R2 * EB11
2 2
+DF(EB23, AL2) * A1 * A2 * W1 * R2
2 3
+DF(EB23, AL2) * A1 * A2 * R2 * EB11
2 3
+DF(EB23, AL2) * A1 * A2 * R2
2 2
+A1 * A2 * V1
2 2
+A1 * A2 * V1 * R2 * EB11 * EB23
2 2
2 2
+A1 * A2 * V1 * R2 * EB23
2 2
+A1 * A2 * W1
2 2
2 2
2 2
2 2
+A1 * A2 * W1 * R2 * EB12 * EB21
2 2
+A1 * A2 * W1 * R2
\[ 2 \begin{align*}
-A_1 \ast A_2 \ast R_2 \ast B_{11} \ast B_{22} \\
-A_1 \ast A_2 \ast R_2 \ast B_{11} \\
-A_1 \ast A_2 \ast R_2 \ast B_{22} \\
A_1 \ast A_2 \ast R_2 \ast (B_{12} \ast B_{21}) / (A_1 \ast A_2 \ast R_2)
\end{align*} \]

\[ K_{12} := (D(A_1, A_{L2}) \ast D(A_2, A_{L1}) \ast U_1 \ast R_1 \ast R_2) \]

\[ + D(A_1, A_{L2}) \ast D(A_2, A_{L1}) \ast U_1 \ast R_1 \ast R_2 \]

\[ + 2 \ast D(A_1, A_{L2}) \ast D(A_2, A_{L1}) \ast U_1 \ast R_1 \ast R_2 \ast B_{22} \ast B_{13} \]

\[ - 2 \ast D(A_1, A_{L2}) \ast D(A_2, A_{L1}) \ast U_1 \ast R_1 \ast R_2 \ast B_{21} \ast B_{23} \]

\[ + 2 \ast D(A_1, A_{L2}) \ast D(A_2, A_{L1}) \ast U_1 \ast R_1 \ast R_2 \ast B_{13} \]

\[ + D(A_1, A_{L2}) \ast D(A_2, A_{L1}) \ast V_1 \ast R_1 \ast R_2 \]

\[ + D(A_1, A_{L2}) \ast D(A_2, A_{L1}) \ast V_1 \ast R_1 \ast R_2 \]

\[ + 2 \ast D(A_1, A_{L2}) \ast D(A_2, A_{L1}) \ast V_1 \ast R_1 \ast R_2 \ast B_{11} \ast B_{23} \]

\[ - 2 \ast D(A_1, A_{L2}) \ast D(A_2, A_{L1}) \ast V_1 \ast R_1 \ast R_2 \ast B_{12} \ast B_{13} \]

\[ + 2 \ast D(A_1, A_{L2}) \ast D(A_2, A_{L1}) \ast V_1 \ast R_1 \ast R_2 \ast B_{23} \]

\[ - D(A_1, A_{L2}) \ast D(U_1, A_{L2}) \ast A_1 \ast V_1 \ast R_1 \ast R_2 \]

\[ - D(A_1, A_{L2}) \ast D(U_1, A_{L2}) \ast A_1 \ast V_1 \ast R_1 \ast R_2 \]

\[ - D(A_1, A_{L2}) \ast D(U_1, A_{L2}) \ast A_1 \ast R_1 \ast R_2 \ast B_{11} \ast B_{23} \]

\[ + D(A_1, A_{L2}) \ast D(U_1, A_{L2}) \ast A_1 \ast R_1 \ast R_2 \ast B_{12} \ast B_{13} \]

\[ - D(A_1, A_{L2}) \ast D(U_1, A_{L2}) \ast A_1 \ast R_1 \ast R_2 \ast B_{23} \]
\[ +DF(A_1, A_2) \times DF(V_1, A_2) \times A_1 \times U_1 \times R_1 \times R_2 \]
\[ +DF(A_1, A_2) \times DF(V_1, A_2) \times A_1 \times U_1 \times R_1 \times R_2 \]
\[ +DF(A_1, A_2) \times DF(V_1, A_2) \times A_1 \times R_1 \times R_2 \times EB_22 \times EB_23 \]
\[ -DF(A_1, A_2) \times DF(V_1, A_2) \times A_1 \times R_1 \times R_2 \times EB_21 \times EB_23 \]
\[ +DF(A_1, A_2) \times DF(V_1, A_2) \times A_1 \times R_1 \times R_2 \times EB_13 \]
\[ +DF(A_1, A_2) \times DF(EB_11, A_2) \times A_1 \times U_1 \times R_1 \times R_2 \times EB_23 \]
\[ -DF(A_1, A_2) \times DF(EB_22, A_2) \times A_1 \times V_1 \times R_1 \times R_2 \times EB_13 \]
\[ -DF(A_1, A_2) \times DF(EB_12, A_2) \times A_1 \times U_1 \times R_1 \times R_2 \times EB_13 \]
\[ +DF(A_1, A_2) \times DF(EB_21, A_2) \times A_1 \times V_1 \times R_1 \times R_2 \times EB_23 \]
\[ -DF(A_1, A_2) \times DF(EB_13, A_2) \times A_1 \times U_1 \times R_1 \times R_2 \times EB_12 \]
\[ -DF(A_1, A_2) \times DF(EB_13, A_2) \times A_1 \times V_1 \times R_1 \times R_2 \times EB_22 \]
\[ -DF(A_1, A_2) \times DF(EB_13, A_2) \times A_1 \times V_1 \times R_1 \times R_2 \]
\[ +DF(A_1, A_2) \times DF(EB_23, A_2) \times A_1 \times U_1 \times R_1 \times R_2 \times EB_11 \]
\[ +DF(A_1, A_2) \times DF(EB_23, A_2) \times A_1 \times U_1 \times R_1 \times R_2 \]
\[ +DF(A_1, A_2) \times DF(EB_23, A_2) \times A_1 \times V_1 \times R_1 \times R_2 \times EB_21 \]
\[ +DF(A_1, A_2) \times A_1 \times A_2 \times U_1 \times W_1 \times R_1 \]
\[ +DF(A_1, A_2) \times A_1 \times A_2 \times U_1 \times W_1 \times R_1 \times R_2 \]
\[ -DF(A_1, A_2) \times A_1 \times A_2 \times U_1 \times R_1 \times R_2 \times EB_11 \times EB_22 \]
\[ -DF(A_1, A_2) \times A_1 \times A_2 \times U_1 \times R_1 \times R_2 \times EB_11 \]
\[ -DF(A_1, A_2) \times A_1 \times A_2 \times U_1 \times R_1 \times R_2 \times EB_22 \]
\[ +DF(A_1, A_2) \times A_1 \times A_2 \times U_1 \times R_1 \times R_2 \times EB_12 \times EB_21 \]
\[
\begin{align*}
\text{+DF}(A_1, AL_2) & \cdot A_1 \cdot A_2 \cdot U_1 \cdot R_1 \cdot R_2 \\
\text{+DF}(A_1, AL_2) & \cdot A_1 \cdot A_2 \cdot W_1 \cdot R_1 \cdot R_2 \cdot E_{B22} \cdot E_{B13} \\
\text{-DF}(A_1, AL_2) & \cdot A_1 \cdot A_2 \cdot W_1 \cdot R_1 \cdot R_2 \cdot E_{B21} \cdot E_{B23} \\
\text{+DF}(A_1, AL_2) & \cdot A_1 \cdot A_2 \cdot W_1 \cdot R_1 \cdot R_2 \cdot E_{B13} \\
\text{+DF}(A_1, AL_2) & \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2 \cdot E_{B22} \cdot E_{B13} \\
\text{-DF}(A_1, AL_2) & \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2 \cdot E_{B21} \cdot E_{B23} \\
\text{+DF}(A_1, AL_2) & \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2 \cdot E_{B13} \\
\text{+DF}(A_2, AL_1) & \cdot \text{DF}(U_1, AL_1) \cdot A_2 \cdot V_1 \cdot R_1 \cdot R_2 \\
\text{+DF}(A_2, AL_1) & \cdot \text{DF}(U_1, AL_1) \cdot A_2 \cdot V_1 \cdot R_1 \cdot R_2 \\
\text{+DF}(A_2, AL_1) & \cdot \text{DF}(U_1, AL_1) \cdot A_2 \cdot R_1 \cdot R_2 \cdot E_{B11} \cdot E_{B23} \\
\text{-DF}(A_2, AL_1) & \cdot \text{DF}(U_1, AL_1) \cdot A_2 \cdot R_1 \cdot R_2 \cdot E_{B12} \cdot E_{B13} \\
\text{+DF}(A_2, AL_1) & \cdot \text{DF}(U_1, AL_1) \cdot A_2 \cdot R_1 \cdot R_2 \cdot E_{B23} \\
\text{-DF}(A_2, AL_1) & \cdot \text{DF}(V_1, AL_1) \cdot A_2 \cdot U_1 \cdot R_1 \cdot R_2 \\
\text{-DF}(A_2, AL_1) & \cdot \text{DF}(V_1, AL_1) \cdot A_2 \cdot U_1 \cdot R_1 \cdot R_2 \\
\text{-DF}(A_2, AL_1) & \cdot \text{DF}(V_1, AL_1) \cdot A_2 \cdot R_1 \cdot R_2 \cdot E_{B22} \cdot E_{B13} \\
\text{+DF}(A_2, AL_1) & \cdot \text{DF}(V_1, AL_1) \cdot A_2 \cdot R_1 \cdot R_2 \cdot E_{B21} \cdot E_{B23} \\
\text{-DF}(A_2, AL_1) & \cdot \text{DF}(V_1, AL_1) \cdot A_2 \cdot R_1 \cdot R_2 \cdot E_{B13} \\
\text{-DF}(A_2, AL_1) & \cdot \text{DF}(E_{B11}, AL_1) \cdot A_2 \cdot U_1 \cdot R_1 \cdot R_2 \cdot E_{B23} \\
\text{+DF}(A_2, AL_1) & \cdot \text{DF}(E_{B22}, AL_1) \cdot A_2 \cdot V_1 \cdot R_1 \cdot R_2 \cdot E_{B13} \\
\text{+DF}(A_2, AL_1) & \cdot \text{DF}(E_{B12}, AL_1) \cdot A_2 \cdot U_1 \cdot R_1 \cdot R_2 \cdot E_{B13}
\end{align*}
\]
\( -D(A_2, A_1) \times D(E_{B21}, A_1) \times A_2 \times V_1 \times R_1 \times R_2 \times E_{B23} \\
+D(A_2, A_1) \times D(E_{B13}, A_1) \times A_2 \times U_1 \times R_1 \times R_2 \times E_{B12} \\
+D(A_2, A_1) \times D(E_{B13}, A_1) \times A_2 \times V_1 \times R_1 \times R_2 \times E_{B22} \\
+D(A_2, A_1) \times D(E_{B13}, A_1) \times A_2 \times V_1 \times R_1 \times R_2 \\
-D(A_2, A_1) \times D(E_{B23}, A_1) \times A_2 \times U_1 \times R_1 \times R_2 \times E_{B11} \\
-D(A_2, A_1) \times D(E_{B23}, A_1) \times A_2 \times U_1 \times R_1 \times R_2 \\
-D(A_2, A_1) \times D(E_{B23}, A_1) \times A_2 \times V_1 \times R_1 \times R_2 \times E_{B21} \\
+D(A_2, A_1) \times A_1 \times A_2 \times V_1 \times W_1 \times R_1 \times R_2 \\
+D(A_2, A_1) \times A_1 \times A_2 \times V_1 \times W_1 \times R_2 \\
+D(A_2, A_1) \times A_1 \times A_2 \times V_1 \times R_1 \times R_2 \\
-D(A_2, A_1) \times A_1 \times A_2 \times V_1 \times R_1 \times R_2 \times E_{B11} \times E_{B22} \\
-D(A_2, A_1) \times A_1 \times A_2 \times V_1 \times R_1 \times R_2 \times E_{B11} \\
-D(A_2, A_1) \times A_1 \times A_2 \times V_1 \times R_1 \times R_2 \times E_{B22} \\
+D(A_2, A_1) \times A_1 \times A_2 \times V_1 \times R_1 \times R_2 \times E_{B12} \times E_{B21} \\
+D(A_2, A_1) \times A_1 \times A_2 \times W_1 \times R_1 \times R_2 \times E_{B11} \times E_{B23} \\
-D(A_2, A_1) \times A_1 \times A_2 \times W_1 \times R_1 \times R_2 \times E_{B12} \times E_{B13} \\
+D(A_2, A_1) \times A_1 \times A_2 \times W_1 \times R_1 \times R_2 \times E_{B23} \\
+D(A_2, A_1) \times A_1 \times A_2 \times R_1 \times R_2 \times E_{B11} \times E_{B23} \\
-D(A_2, A_1) \times A_1 \times A_2 \times R_1 \times R_2 \times E_{B12} \times E_{B13} \\
+D(A_2, A_1) \times A_1 \times A_2 \times R_1 \times R_2 \times E_{B23} \)
\[\begin{align*}
-\text{DF}(U1, AL1) \times \text{DF}(U1, AL2) & \times A1 \times A2 \times R1 \times R2 \\
-\text{DF}(U1, AL1) \times \text{DF}(U1, AL2) & \times A1 \times A2 \times R1 \times R2 \\
-\text{DF}(U1, AL1) \times \text{DF}(EB22, AL2) & \times A1 \times A2 \times R1 \times R2 \times EB13 \\
+\text{DF}(U1, AL1) \times \text{DF}(EB21, AL2) & \times A1 \times A2 \times R1 \times R2 \times EB23 \\
-\text{DF}(U1, AL1) \times \text{DF}(EB13, AL2) & \times A1 \times A2 \times R1 \times R2 \times EB22 \\
-\text{DF}(U1, AL1) \times \text{DF}(EB13, AL2) & \times A1 \times A2 \times R1 \times R2 \\
+\text{DF}(U1, AL1) \times \text{DF}(EB23, AL2) & \times A1 \times A2 \times R1 \times R2 \times EB21 \\
-\text{DF}(U1, AL2) \times \text{DF}(EB22, AL1) & \times A1 \times A2 \times R1 \times R2 \times EB13 \\
+\text{DF}(U1, AL2) \times \text{DF}(EB21, AL1) & \times A1 \times A2 \times R1 \times R2 \times EB23 \\
-\text{DF}(U1, AL2) \times \text{DF}(EB13, AL1) & \times A1 \times A2 \times R1 \times R2 \times EB22 \\
-\text{DF}(U1, AL2) \times \text{DF}(EB13, AL1) & \times A1 \times A2 \times R1 \times R2 \\
+\text{DF}(U1, AL2) \times \text{DF}(EB23, AL1) & \times A1 \times A2 \times R1 \times R2 \times EB21 \\
-\text{DF}(U1, AL2) \times A1 \times A2 \times W1 \times R1 \times R2 \\
-\text{DF}(U1, AL2) \times A1 \times A2 \times W1 \times R2 \\
-\text{DF}(U1, AL2) \times A1 \times A2 \times R1 \times R2 \\
+\text{DF}(U1, AL2) \times A1 \times A2 \times R1 \times R2 \times EB11 \times EB22 \\
+\text{DF}(U1, AL2) \times A1 \times A2 \times R1 \times R2 \times EB11 \\
+\text{DF}(U1, AL2) \times A1 \times A2 \times R1 \times R2 \times EB22 \\
-\text{DF}(U1, AL2) \times A1 \times A2 \times R1 \times R2 \times EB12 \times EB21 \\
-\text{DF}(V1, AL1) \times \text{DF}(V1, AL2) & \times A1 \times A2 \times R1 \times R2
\end{align*}\]
\[
\begin{align*}
-\text{DF}(V1, AL1) \times \text{DF}(V1, AL2) \times A1 \times A2 \times R1 \times R2 \\
-\text{DF}(V1, AL1) \times \text{DF}(EB11, AL2) \times A1 \times A2 \times R1 \times R2 \times EB23 \\
+\text{DF}(V1, AL1) \times \text{DF}(EB12, AL2) \times A1 \times A2 \times R1 \times R2 \times EB13 \\
+\text{DF}(V1, AL1) \times \text{DF}(EB13, AL2) \times A1 \times A2 \times R1 \times R2 \times EB12 \\
-\text{DF}(V1, AL1) \times \text{DF}(EB23, AL2) \times A1 \times A2 \times R1 \times R2 \times EB11 \\
-\text{DF}(V1, AL1) \times \text{DF}(EB23, AL2) \times A1 \times A2 \times R1 \times R2 \\
-\text{DF}(V1, AL1) \times A1 \times A2 \times W1 \times R1 \\
-\text{DF}(V1, AL1) \times A1 \times A2 \times W1 \times R1 \times R2 \\
+\text{DF}(V1, AL1) \times A1 \times A2 \times R1 \times R2 \times EB11 \times EB22 \\
+\text{DF}(V1, AL1) \times A1 \times A2 \times R1 \times R2 \times EB11 \\
+\text{DF}(V1, AL1) \times A1 \times A2 \times R1 \times R2 \times EB22 \\
-\text{DF}(V1, AL1) \times A1 \times A2 \times R1 \times R2 \times EB12 \times EB21 \\
-\text{DF}(V1, AL1) \times A1 \times A2 \times R1 \times R2 \\
-\text{DF}(V1, AL2) \times \text{DF}(EB11, AL1) \times A1 \times A2 \times R1 \times R2 \times EB23 \\
+\text{DF}(V1, AL2) \times \text{DF}(EB12, AL1) \times A1 \times A2 \times R1 \times R2 \times EB13 \\
+\text{DF}(V1, AL2) \times \text{DF}(EB13, AL1) \times A1 \times A2 \times R1 \times R2 \times EB12 \\
-\text{DF}(V1, AL2) \times \text{DF}(EB23, AL1) \times A1 \times A2 \times R1 \times R2 \times EB11 \\
-\text{DF}(V1, AL2) \times \text{DF}(EB23, AL1) \times A1 \times A2 \times R1 \times R2 \\
-\text{DF}(W1, AL1) \times \text{DF}(W1, AL2) \times A1 \times A2 \times R1 \times R2 \\
-\text{DF}(W1, AL1) \times \text{DF}(W1, AL2) \times A1 \times A2 \times R1 \times R2 \\
\end{align*}
\]
\[+DF(W_1, AL_1) \cdot DF(EB_{11}, AL_2) \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2 \cdot EB_{22}\]
\[+DF(W_1, AL_1) \cdot DF(EB_{12}, AL_2) \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2 \cdot EB_{21}\]
\[+DF(W_1, AL_1) \cdot DF(EB_{21}, AL_2) \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2 \cdot EB_{12}\]
\[+DF(W_1, AL_1) \cdot A_1 \cdot A_2 \cdot V_1 \cdot R_1\]
\[+DF(W_1, AL_1) \cdot A_1 \cdot A_2 \cdot V_1 \cdot R_1 \cdot R_2\]
\[+DF(W_1, AL_1) \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2 \cdot EB_{11} \cdot EB_{23}\]
\[+DF(W_1, AL_1) \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2 \cdot EB_{12} \cdot EB_{13}\]
\[+DF(W_1, AL_1) \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2 \cdot EB_{23}\]
\[+DF(W_1, AL_2) \cdot DF(EB_{11}, AL_1) \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2 \cdot EB_{22}\]
\[+DF(W_1, AL_2) \cdot DF(EB_{12}, AL_1) \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2 \cdot EB_{21}\]
\[+DF(W_1, AL_2) \cdot DF(EB_{21}, AL_1) \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2 \cdot EB_{12}\]
\[+DF(W_1, AL_2) \cdot A_1 \cdot A_2 \cdot U_1 \cdot R_1 \cdot R_2\]
\[+DF(W_1, AL_2) \cdot A_1 \cdot A_2 \cdot U_1 \cdot R_2\]
\[+DF(W_1, AL_2) \cdot A_1 \cdot A_2 \cdot R_1 \cdot R_2 \cdot EB_{22} \cdot EB_{13}\]
\[-\text{DF}(W_1, AL_2)*A_1 \times A_2\times R_1\times R_2 \times \text{EB}_{21}\times\text{EB}_{23}\]

\[+\text{DF}(W_1, AL_2)*A_1 \times A_2\times R_1\times R_2 \times \text{EB}_{13}\]

\[-\text{DF}(E_{B11}, AL_1)*A_1\times A_2 \times V_1\times R_1 \times R_2 \times \text{EB}_{22}\]

\[-\text{DF}(E_{B11}, AL_1)*A_1\times A_2 \times V_1\times R_1 \times R_2\]

\[-\text{DF}(E_{B11}, AL_1)*A_1\times A_2 \times W_1\times R_1 \times R_2 \times \text{EB}_{23}\]

\[-\text{DF}(E_{B11}, AL_1)*A_1\times A_2 \times R_1 \times R_2 \times \text{EB}_{23}\]

\[-\text{DF}(E_{B11}, AL_2)*A_1 \times A_2\times U_1\times R_1\times R_2 \times \text{EB}_{22}\]

\[-\text{DF}(E_{B11}, AL_2)*A_1 \times A_2\times U_1\times R_1\times R_2\]

\[-\text{DF}(E_{B22}, AL_1)*A_1\times A_2 \times V_1\times R_1 \times R_2 \times \text{EB}_{11}\]

\[-\text{DF}(E_{B22}, AL_1)*A_1\times A_2 \times V_1\times R_1 \times R_2\]

\[-\text{DF}(E_{B22}, AL_2)*A_1 \times A_2\times U_1\times R_1\times R_2 \times \text{EB}_{11}\]

\[-\text{DF}(E_{B22}, AL_2)*A_1 \times A_2\times U_1\times R_1\times R_2\]

\[-\text{DF}(E_{B22}, AL_2)*A_1 \times A_2\times W_1\times R_1\times R_2 \times \text{EB}_{13}\]

\[-\text{DF}(E_{B22}, AL_2)*A_1 \times A_2\times W_1\times R_1\times R_2 \times \text{EB}_{13}\]

\[+\text{DF}(E_{B12}, AL_1)*A_1\times A_2 \times V_1\times R_1 \times R_2 \times \text{EB}_{21}\]

\[+\text{DF}(E_{B12}, AL_1)*A_1\times A_2 \times W_1\times R_1 \times R_2 \times \text{EB}_{13}\]

\[+\text{DF}(E_{B12}, AL_1)*A_1\times A_2 \times R_1 \times R_2 \times \text{EB}_{13}\]

\[+\text{DF}(E_{B12}, AL_2)*A_1 \times A_2\times U_1\times R_1\times R_2 \times \text{EB}_{21}\]

\[+\text{DF}(E_{B21}, AL_1)*A_1\times A_2 \times V_1\times R_1 \times R_2 \times \text{EB}_{12}\]

\[+\text{DF}(E_{B21}, AL_2)*A_1 \times A_2\times U_1\times R_1\times R_2 \times \text{EB}_{12}\]
\[
\begin{align*}
&+DF(EB21, AL2) * A1 * A2 \cdot W1 \cdot R1 \cdot R2 * EB23 \\
&+DF(EB21, AL2) * A1 * A2 \cdot R1 \cdot R2 * EB23 \\
&+DF(EB13, AL1) * A1 \cdot A2 \cdot W1 \cdot R1 \cdot R2 * EB12 \\
&+DF(EB13, AL1) * A1 \cdot A2 \cdot R1 \cdot R2 * EB12 \\
&-DF(EB13, AL2) * A1 \cdot A2 \cdot W1 \cdot R1 \cdot R2 * EB22 \\
&-DF(EB13, AL2) * A1 \cdot A2 \cdot W1 \cdot R1 \cdot R2 \\
&-DF(EB13, AL2) * A1 \cdot A2 \cdot R1 \cdot R2 * EB22 \\
&-DF(EB13, AL2) * A1 \cdot A2 \cdot R1 \cdot R2 \\
&-DF(EB23, AL1) * A1 \cdot A2 \cdot W1 \cdot R1 \cdot R2 * EB11 \\
&-DF(EB23, AL1) * A1 \cdot A2 \cdot W1 \cdot R1 \cdot R2 \\
&-DF(EB23, AL1) * A1 \cdot A2 \cdot R1 \cdot R2 * EB11 \\
&-DF(EB23, AL1) * A1 \cdot A2 \cdot R1 \cdot R2 \\
&+DF(EB23, AL2) * A1 \cdot A2 \cdot W1 \cdot R1 \cdot R2 * EB21 \\
&+DF(EB23, AL2) * A1 \cdot A2 \cdot R1 \cdot R2 * EB21 \\
&+DF(EB23, AL2) * A1 \cdot A2 \cdot R1 \cdot R2 * EB21 \\
&-A1 \cdot A2 \cdot u1 \cdot v1 \cdot R1 \\
&-A1 \cdot A2 \cdot u1 \cdot v1 \cdot R2 \\
&-A1 \cdot A2 \cdot u1 \cdot R1 \cdot R2 \cdot EB11 \cdot EB23 \\
&+A1 \cdot A2 \cdot u1 \cdot R1 \cdot R2 \cdot EB12 \cdot EB13 \\
&-A1 \cdot A2 \cdot u1 \cdot R1 \cdot R2 \cdot EB23 \\
&-A1 \cdot A2 \cdot v1 \cdot R1 \cdot R2 \cdot EB22 \cdot EB13 
\end{align*}
\]
\[ 2 \ 2 \]
\[ +A_1 \* A_2 \* V_1 \* R_1 \* R_2 \* E_B_{21} \* E_B_{23} \]

\[ 2 \ 2 \]
\[ -A_1 \* A_2 \* V_1 \* R_1 \* R_2 \* E_B_{13} \/ (A_1 \* A_2 \* R_1 \* R_2 \) \]