THE INFLUENCES OF HALL EFFECT AND WALL CONDUCTIVITY ON MAGNETOHYDRODYNAMIC COUETTE FLOW

by

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DELFt - THE NETHERLANDS

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Summary

In this report the influences of Hall effect and wall conductivity on MHD couette flow are studied. Under the present assumptions even negative values of the velocity component $U_x$ appear in some cases. It is concluded that it will be of importance to study the influence of Hall effect and wall conductivity also in two-dimensional magnetohydrodynamic channel flows.
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of symbols</td>
<td>p.ii</td>
</tr>
<tr>
<td>1 Introduction</td>
<td>p.1</td>
</tr>
<tr>
<td>2 The equations for an ionized gas in the two-fluid approximation</td>
<td>p.3</td>
</tr>
<tr>
<td>3 Couette flow of an ionized gas in the two-fluid approximation</td>
<td>p.10</td>
</tr>
<tr>
<td>3.1. Formulation of the problem</td>
<td>p.11</td>
</tr>
<tr>
<td>3.2. Formulation of the boundary conditions and solution of the problem</td>
<td>p.16</td>
</tr>
<tr>
<td>4 Estimation of the order of magnitude of the flow-parameters</td>
<td>p.26</td>
</tr>
<tr>
<td>5 Discussion of results and conclusions</td>
<td>p.29</td>
</tr>
<tr>
<td>References</td>
<td>p.43</td>
</tr>
<tr>
<td>Tables I-IV</td>
<td>p.44</td>
</tr>
<tr>
<td>Figures 1-18</td>
<td></td>
</tr>
<tr>
<td>Appendix - The Algol programme used for the numerical calculations.</td>
<td></td>
</tr>
</tbody>
</table>
List of symbols

\( B \)  
magnetic induction

\( B_0 \)  
strength of externally applied magnetic induction

\( d \)  
thickness of lower wall

\( \frac{\partial}{\partial t} \)  
local time derivative

\( \frac{D}{Dt} \)  
material derivative = \( \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla) \)

\( e \)  
proton charge

\( E \)  
electric field strength

\( f_1 - f_2 \)  
constants in expressions for \( u_x, u_z \)  
(equations 3.2.25 to 28)

\( G \)  
"Generalized" electric field strength  
(equation 2.6)

\( h \)  
distance between the walls

\( i \)  
\( i = \sqrt{-1} \)

\( I \)  
\( I = \int_{0}^{1} i \, dy \)  
(paragraph 3)

\( j \)  
electrical current density

\( k \)  
Boltzmann's constant
\( k_1 - k_6 \)  
constants in expressions for \( I_x, I_z \)  
(equations 3.2.29 to 32).

\( m \)  
mass

\( M \)  
Hartmann number = \( B_0 h \sqrt{\frac{\sigma f}{\eta}} \).

\( M_1 - M_2 \)  
constants (equation 3.2.20)

\( n \)  
total number density = \( n_e + n_i + N \)

\( N \)  
number density of neutral molecules

\( p \)  
hydrostatic pressure

\( P_{\alpha\beta} \)  
viscous stress tensor

\( q \)  
heat flux

\( R_m \)  
magnetic Reynolds number = \( c_f \mu U_0 h \).

\( t \)  
time

\( T \)  
temperature

\( u \)  
flow velocity

\( U \)  
gravitational potential

\( U_0 \)  
speed of the upper wall

\( \nu(B_0) \)  
"velocity of the applied magnetic field"  
(equation 5.5).

\( x, y, z \)  
coordinates of Cartesian reference frame

\( Z \)  
number of heavy particles (protons and neutrons) in an ion
\[ \delta \]
degree of ionization \( = \frac{n_e}{n} \)

\[ \delta_{\alpha\beta} \]
Kronecker symbol \( = 1 \) if \( \alpha = \beta \),
\( = 0 \) if \( \alpha \neq \beta \).

\[ \varepsilon \]
dielectric constant

\[ \eta \]
coefficient of viscosity

\[ \lambda \]
coefficient of heat conduction

\[ \mu \]
permeability

\[ \mu_0 \]
permeability of free space

\[ \rho \]
mass density

\[ \rho_0 \]
mass density at Standard temperature and pressure (paragraph 4)

\[ \rho_q \]
charge density

\[ \sigma \]
coefficient of electrical conductivity

\[ \Sigma \]
wall conductance parameter \( = \frac{\sigma_d}{\sigma_f h} \)

\[ \tau \]
mean collision time for electron-ion collisions.

\[ \tau_2 \]
mean collision time for ion-ion collisions.

\[ \omega \]
electron gyration frequency \( = \frac{eB}{m_e} \)
\[ \omega \]  
ion gyration frequency \( = \frac{eB}{m_i} \)

\[ \omega \tau \]  
Hall parameter \( = \frac{\sigma_f B_0}{n_e} \)

\[ \nabla \]  
nabla operator

**Indices**

\[ e \]  
electron

\[ f \]  
fluid

\[ i \]  
ion

\[ w \]  
wall

\[ x, y, z \]  
vector components

\[ \alpha, \beta \]  
vector or tensor components

\[ * \]  
physical quantity (paragraph 3 only)

\[ \sim \]  
complex quantity (equation 3.2.1)

\[ // \]  
component parallel to \( B \)

\[ \perp \]  
component perpendicular to \( B \).
1 Introduction.
In problems concerning the motion of an ionized gas through a magnetic field the electrical and thermal conductivities may be considered as scalar quantities as long as the mean collision frequencies are much larger than the gyration frequencies of charged particles around the lines of magnetic induction. If at least one of the gyration frequencies and one of the collision frequencies are of the same order of magnitude this is no longer true and the transport coefficients will be tensor quantities. The effects which then occur are studied for the simple geometry of the Couette flow between two parallel infinite plane walls of which the lower one is at rest and the other one is moving with a constant speed. A constant magnetic field is applied in the direction normal to the walls. The equations to be used have been derived by Chapman and Cowling \(^1\) by considering the ionized gas as a mixture of two components, one consisting of positively and the other one of negatively charged particles. The generalized Ohm's law obtained by Chapman and Cowling can be shown to be identical with the expression given by Spitzer \(^2\) (for the stationary case). The derivation of the equations according the Chapman and Cowling is used in this report because Spitzer does not give an expression for the heat flux vector or the viscous stress tensor for the case that the spiralling motion of the charged particles has to be taken into account.
The solution of the Couette flow will be given for various conducting properties of the lower wall. So far Couette flow in the two-fluid approximation has been studied by Van Wijngaarden \(^3\) but only for a superconducting and for an nonconducting lower wall. The effect of the wall conductivities on Couette flow has been studied before by Chang and Yen \(^4\) but only in the case that the transport coefficients are scalar quantities. The report can thus be considered as an extension of the
results of both Van Wijngaarden and Chang and Yen. In paragraph 2 the two-fluid equations of Chapman and Cowling will be discussed and they are applied in paragraph 3 to obtain a solution of the Couette flow problem. In paragraph 4 the order of magnitude of the significant parameters is estimated and the solution obtained in paragraph 3 can then be calculated numerically. The program used for these calculations has been included in the appendix. Some of the results, which appear to be of major significance, have been given in the figures which are discussed in paragraph 5.
The equations for an ionized gas in the two-fluid approximation according to Chapman and Cowling.

To derive the equations of continuity, motion and energy for an ionized gas Chapman and Cowling \(^1\) consider the gas to be a mixture of two components. The first component consists of positively the second component of negatively charged particles and both components are assumed to have equal temperatures.

From the Boltzmann equation a separate set of equations for the ion gas and for the electron gas can be derived. By suitably defining average flow quantities the equations for the two components can be combined into one set of equations describing the behaviour of the mixture as a whole.

In this section the additional assumptions are made that only singly charged ions have to be considered while the mixture as a whole will be very close to electric neutrality. Thus \(n_e - n_i\), although on a small length scale small charge densities \(\rho_q = e(n_i - n_e)\) may occur. This implies that the electrostatic force \(\rho_q E\) can be neglected with respect to the Lorentz force \(j \times B\) and that the convection current \(\rho_q u\) can be neglected with respect to the conduction current \(j\).

It is further assumed that the only other external force acting on the particles is the gravity force. With these assumptions the equations derived by Chapman and Cowling can be written in the form

\[
\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1}
\]

\[
\rho \frac{D\mathbf{u}}{Dt} = -\frac{\partial}{\partial x_\beta} p_{\alpha\beta} + (j \times B)_\alpha - \rho \frac{\partial}{\partial x_\alpha} \mathbf{u} \tag{2}
\]

\[
\frac{D}{Dt} \left( \frac{3}{2} nkT \right) = -\frac{3}{2} nkT (\nabla \cdot \mathbf{u}) - p_{\alpha\beta} \frac{\partial u_\alpha}{\partial x_\beta} - (\nabla \cdot \mathbf{u}) + j \cdot (E + \mathbf{u} \times B) \tag{3}
\]
where M.K.S. units are used throughout. In the equations
greek subscripts denote vector or tensor components and the
summation convention applies. These equations have to be
supplemented by Maxwell's equations which are:

\[ \nabla \cdot \mathbf{B} = 0 \]

\[ \varepsilon \nabla \cdot \mathbf{E} = \rho \]

\[ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \]  \hspace{1cm} (4)

\[ \nabla \times \mathbf{B} = \mu \mathbf{j} + \varepsilon \mu \frac{\partial \mathbf{E}}{\partial t} \]

Finally expressions for \( \mathbf{j} \), \( \rho \) and \( \rho_{\alpha \beta} \) are needed which can also
be derived from kinetic theory.

When superscripts "\( \parallel \)" and "\( \perp \)" are used to denote components
parallel and perpendicular to the magnetic field the results
of Chapman and Cowling for the electric current density can be
written in the following form:

\[ \mathbf{j}^\parallel = \sigma \mathbf{G}^\parallel \]  \hspace{1cm} (5)

\[ \mathbf{j}^\perp = \frac{\sigma}{1 + (\omega t)^2} \left[ \mathbf{G}^\parallel + \frac{\omega t}{B} \mathbf{B} \times \mathbf{G}^\perp \right] \]

Here \( \mathbf{G} \) is a "generalized" electric field given by

\[ \mathbf{G} = \mathbf{E} + \mathbf{u} \times \mathbf{B} + \frac{\nabla \rho}{2 n_e e} \]  \hspace{1cm} (6)

and

\[ \sigma = \frac{n_e e^2 \tau}{m_e} \]  \hspace{1cm} (7)
The parameter $\omega \tau$ is the ratio of the electron gyration frequency $\omega$ and the mean electron-ion collision frequency $\frac{1}{\tau}$ (thus $\tau$ is the corresponding mean collision time). It should be mentioned that (5) and (6) have been obtained in this form from the results of Chapman and Cowling by using the facts that the electron mass can be neglected with respect to the ion mass and that $n_e \sim n_i$.

The equations (5) can be combined into one expression by using the relations

$$G^\theta = \frac{1}{B^2} (B \cdot G) B$$

$$G^\perp = G - G^\theta = \frac{1}{B^2} B \times (G \times B)$$

The ultimate form of the "generalized" Ohm's law is then

$$\mathbf{j} = \frac{1}{1 + \omega^2 \tau^2} \left\{ \frac{1}{B^2} B \times (G \times B) + \frac{\omega \tau}{B} (B \times G) + \frac{1 + \omega^2 \tau^2}{B^2} (B \cdot G) B \right\}$$

(8)

This relation can also be written in the equivalent form

$$\mathbf{j} = \sigma G - \frac{\omega \tau}{B} (\mathbf{j} \times B)$$

(9)

In this form it is clear that the "generalized" Ohm's law according to Chapman and Cowling is identical with the expression derived by Spitzer \(^2\) for the stationary case. It has been mentioned already that Chapman and Cowling are followed in this report since Spitzer does not give an expression for the heat flux vector or the viscous stress tensor for the case that the influence of the Larmor gyration of the charged particles has to be taken into account. It should be pointed out that thermal diffusion has been neglected in the above
forms of the "generalized" Ohm's law.
The results of Chapman and Cowling for the heat flux vector $\mathbf{q}$
consistent with the expression (5) for $\mathbf{j}$ may be given in the form

$$
\mathbf{q} = -\lambda (\nabla T)^{\parallel}
$$

$$
\mathbf{q}^{\perp} = -\frac{\lambda}{(1 + \omega^2 \tau^2)} \left\{ (\nabla T)^{\parallel} + \frac{\omega t}{B} \mathbf{B} \times (\nabla T)^{\parallel} \right\}
$$

(10)

It should be noted that Chapman and Cowling consider the
electron and ion gas separately in deriving expressions for
the heat flux vector and the viscous stress tensor. The
expression (10) has been obtained from these results by using
the fact that the electrons mainly provide for the transport
of heat (which Chapman and Cowling derive from a simple free-
path theory). The expressions (10) can be combined in one
equation of the form

$$
\mathbf{q} = -\frac{\lambda}{(1 + \omega^2 \tau^2)} \left\{ \frac{1}{B^2} \mathbf{B} \times \left[ (\nabla T) \times \mathbf{B} \right] + \frac{\omega t}{B} \mathbf{B} \times (\nabla T) \right\}
$$

$$
+ \left[ \frac{1 + \omega^2 \tau^2}{B^2} \right] \left[ \mathbf{B} \cdot (\nabla T) \right] \frac{\mathbf{B}}{B}
$$

(11)

It should be pointed out that in this expression the
diffusion-thermo effect has been neglected.
In the results of Chapman and Cowling for the stress tensor
$\rho_{\alpha\beta}$ a new parameter appears, which is the product of the ion
gyration frequency $\omega_i$ and the mean ion-ion collision time $\tau_2$.
This is the only new parameter since it can be shown that the
non-hydrostatic part of the electron stress tensor can be
neglected with respect to that of the ion stress tensor as
pointed out by Van Wijngaarden. This parameter $\omega_i \tau_2$ can be
expressed in terms of the parameter $\omega \tau$ appearing in the expressions for $i$ and $q$ namely (see for instance Van Wijngaarden p. 29)

$$\omega i \tau_2 = \left( \frac{m_e}{m_i} \right)^{1/2} \omega \tau.$$ 

When an ion consists of $Z$ heavy particles (protons and neutrons)

$$\frac{m_i}{m_e} \approx Z \times 1840$$

and

$$\omega i \tau_2 = \frac{1}{43} \frac{\omega \tau}{\sqrt{Z}}.$$

In paragraph 4 the order of magnitude of the parameters is estimated for high temperature air for which $Z = 16$ and the maximum value of $\omega \tau$ is $\omega \tau = 6$. Then $\omega i \tau_2 \approx 0.04$ and may reasonably be neglected compared to one. In that case there is no influence of the magnetic field on the stress tensor and the well-known expression of the non-conducting, viscous medium can be used i.e.

$$p_{\alpha \beta} = p_{\beta \alpha} - \eta \left( \frac{\partial u_{\alpha}}{\partial x_{\beta}} + \frac{\partial u_{\beta}}{\partial x_{\alpha}} \right) + \frac{2}{3} \eta b_{\alpha \beta} (\nabla \cdot u) \tag{12}$$

In this way the set of equations needed to describe the behaviour of the ionized gas has been completed and in the next paragraph it will be applied to the problem of the Couette flow.

It has to be pointed out, however, that two questions remain in connection with these equations. The first question is whether the equations can only be applied to a fully ionized gas. The second is for what values of $\omega \tau$ the equations will be valid. Although it is clear that the two-fluid model strictly applies only to flows of fully ionized gases the influence of the neutral molecules on the behaviour of the flow may quite often be neglected in flows of partially ionized gases.
Delcroix 5) defines the limit of the domain of weakly ionized gases (in which the influence of the neutral molecules is quite essential) by the point where the electron-ion collision frequency and the electron-neutral collision frequency are equal. For hydrogen gas at a temperature of 10000°K this limit occurs at an average degree of ionization of about one half per cent. The definition of this limit is rather arbitrarily.

There will be a transition region where the influence of the neutral molecules will be small but not negligible. In paragraph 4 it is assumed (also rather arbitrarily) that with air at temperatures of around 10000°K the two-fluid model may be applied above a degree of ionization of 25 per cent.

The other problem of the range of \( \omega_r \)-values for which the equations apply is also largely unsolved. Van Wijngaarden 3) states that because the equations are obtained by means of a Chapman-Enskog expansion assuming a small departure from the Maxwell-Boltzmann equilibrium distribution the validity of the equations is restricted to \( \omega_r < 1 \). This may be strictly true but is probably too restrictive and in fact Van Wijngaarden's reasoning indicates only that the limitation to the first term of the Chapman-Enskog expansion, which is generally done in deriving the equations, is not correct if \( \omega_r > 1 \). Marshall 6) also determined the equations for an ionized gas according to the two-fluid model by means of the first term of the Chapman-Enskog expansion and implicitly suggests that the highest value of \( \omega_r \) for which his equations are valid is of the order of 5. This limit will be approximately pursued in this report where \( \omega_r = 6 \) is the highest value considered.

Finally it should be mentioned that the coefficients in the expressions for the heat flux vector and the current density as derived by Chapman and Cowling differ significantly from those obtained by Marshall although the general form of the
expressions are very similar. This difference occurs since
the first term of the Chapman-Enskog expansion is expressed as
a series of orthogonal functions. Chapman and Cowling only
determine the first term in this series while Marshall considers
the entire series. Thus Marshall's results are more exact
but due to the simplicity of the results obtained by Chapman
and Cowling these are more convenient when analytical solutions
are desired. For this reason the equations of Chapman and
Cowling are used in this report. An additional advantage is
that similar equations are frequently used in the recent
literature so that more comparisons can be made with previous
results.
Couette flow of an ionized gas in the two-fluid approximation.
The equations discussed in the preceding paragraph will now
be applied to the study of the Couette flow. Since it is
convenient to include dimensionless variables the physical
quantities will be denoted in this paragraph with asterisks.
The flow configuration is as follows. The viscous, electrically
conducting gas flows between two parallel walls of infinite
length and width. One of the walls coincides with the $x^*z^*$
plane of a right-handed Cartesian reference frame, it is at
rest and it may be a conductor. The other wall is an insulator,
lies in the plane $y^* = h$ and moves in this plane with a constant
velocity $U_o^*$ in the direction of the positive $x^*$ axis. It is
assumed that $U_o^*$ is so small that compressibility effects
will be negligible.

A constant homogeneous magnetic field $B_o^*$ is externally applied
and is directed along the positive $y^*$ axis. The solution to
be constructed will depend on the electrical conductivity of
the lower wall assuming that the current through the wall
cancels the total current through the fluid. The electrical
conductivity of the fluid will be denoted by $\sigma_f^*$ to distinguish
it from the wall conductivity denoted by $\sigma_w^*$.

So far magnetohydrodynamic Couette flow has been studied by
Van Wijngaarden 3) according to the two fluid approximation
but only for two values of the conductivity of the lower wall
namely $\sigma_w^* = 0$ and $\sigma_w^* = \infty$. The effect of the conducting proper-
ties of both the upper and the lower wall on magnetohydro-
dynamic Couette flow has been studied by Chang and Yen 4) but
only for the case that the Hall effect can be neglected
($\omega \tau \ll 1$) so that the electrical and thermal conductivity
of the medium are scalar quantities. It is felt, however,
that varying the conductivity of the upper wall will not
cause any new phenomena to occur and therefore this will not
be considered.
3.1. Formulation of the problem.

It has been mentioned that the physical quantities are denoted with asterisks. Dimensionless quantities are then defined by

\[
x^* = xh; 
\hat{y}^* = yh; 
\hat{z}^* = zh; 
\hat{U}^* = \frac{U}{U_0}; 
\hat{B}^* = \frac{B}{B_0}; 
\hat{E}^* = \frac{E}{U_0 B_0}; 
\hat{\alpha}^* = \frac{1}{\sigma_f U_0 B_0}; 
\hat{p}^* = \frac{p}{h}; 
\hat{T}^* = \frac{T}{\lambda}. 
\]

(2)

It is clear from the description of the flow configuration that the flow is steady and that the variables may be assumed to be independent of \(x\) and \(z\). The only exception to this rule may be the pressure. Sherman and Sutton \(^7\) namely showed that the pressure can be in general a linear function in \(x\) and \(z\).

It is, however, assumed that no pressure differences will be applied in the \(x\) or \(z\) directions so that the pressure will be a function of \(y\) only. Moreover the transport coefficients are assumed to be constant and gravity effects will be neglected.

With these conditions the equations to be solved can be simplified and the consequences will now be considered.

Since the flow is incompressible the continuity equation (2.1) gives

\[
\frac{d\hat{U}y}{dy} = 0 
\]

(2)

With the boundary condition that \(u_y\) must vanish at the walls this shows that

\[
\hat{u}_y = 0 
\]

(3)

Because all quantities depend on \(y\) only several conclusions can be drawn from Maxwell's equations. Equation (2.4\(^1\)) shows that

\[
\hat{B}_y = \frac{\hat{B}_y}{B_0} = 1 
\]

(4)
The $x$ and $z$ components of equation (2.4$^3$) show that

$$E_x = \text{constant}, \quad E_z = \text{constant}$$  \hspace{1cm} (5)

Finally the $y$ component of equation (2.4$^4$) shows that

$$j_y = 0$$  \hspace{1cm} (6)

The equation of motion together with equations (1) and (3) gives

$$M^2 \mathbf{j} \times \mathbf{B} - \nabla p + \nabla^2 \mathbf{u} = 0$$  \hspace{1cm} (7)

Here $M$ is the Hartmann number which represents the ratio between electromagnetic and viscous dissipation. It is defined by

$$M = B_o h \sqrt{\frac{\sigma r}{\eta}}$$  \hspace{1cm} (8)

The "generalized" Ohm's law in the form of equation (2.9) is on account of (2.6) and (1)

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} + \frac{\omega r}{2M^2} \nabla p - \omega r \mathbf{j} \times \mathbf{B} - \mathbf{j} = 0$$  \hspace{1cm} (9)

In this expression the relation

$$\omega r = \frac{\sigma r B_o}{n_e e}$$  \hspace{1cm} (10)

has been used which is obtained from the definition of $\omega$ and equation (2.7). Moreover it has been used in (9) that the induced magnetic field will be negligibly small compared with the externally applied field (in the range of parameters considered, as may be verified from the results).

The $x$ and $z$ components of the equations (7) and (9) together with the boundary conditions completely specify the electrical
current density and the flow velocity. These four equations have to be solved simultaneously and are

\[-M^2 j_z + \frac{d^2 u_z}{dy^2} = 0\]  \hspace{1cm} (11)

\[M^2 j_x + \frac{d^2 u_x}{dy^2} = 0\]  \hspace{1cm} (12)

\[E_x - u_z + \omega \tau j_z - j_x = 0\]  \hspace{1cm} (13)

\[E_z + u_x - j_z - \omega \tau j_x = 0\]  \hspace{1cm} (14)

This set of simultaneous equations has also been solved by Van Wijngaarden \(^3\) but with the assumption \(\omega \tau \ll 1\) while in the following a solution will be given for arbitrary values of \(\omega \tau\). Van Wijngaarden also considered only two cases of boundary conditions namely \(\sigma_w = 0\) and \(\sigma_w = \infty\). Here the solution will be obtained for any value of \(\sigma_w\).

The induced magnetic fields can be determined from the \(x\) and \(z\) components of Maxwell's equation (2.4 \(^4\)) which are on account of (1)

\[\frac{dB_z}{dy} = R_m j_x\]  \hspace{1cm} (15)

\[\frac{dB_x}{dy} = -R_m j_z\]  \hspace{1cm} (16)

In these equations the magnetic Reynolds number \(R_m\) appears, representing the ratio between the work done by the Lorentz force and the Joule dissipation. It is defined by

\[R_m = \sigma_f \mu U_0 h\]  \hspace{1cm} (17)
The temperature distribution depends on the form of the heat flux vector $q^*$. Assuming that the temperature depends on $y^*$ only, the expression for $q^*$ is from equation (2.11)

$$q^* = -\frac{\lambda}{1 + \omega^2 \tau^2} \left[ x \left\{ \omega^2 \tau^2 \frac{B^*}{(B^*)^2} - \omega \frac{B_z^*}{(B^*)^2} \right\} + \gamma \left\{ 1 + \omega^2 \tau^2 \frac{B_o^*}{(B^*)^2} \right\} \right] \frac{dT^*}{dy^*}$$

(18)

In the energy equation $q^*$ appears in the form $\nabla \cdot q^*$ which can be written as

$$\nabla \cdot q^* = -\frac{\lambda}{1 + \omega^2 \tau^2} \left\{ 1 + \omega^2 \tau^2 \frac{B_o^*}{(B^*)^2} \right\} \frac{d^2 T^*}{d(y^*)^2}$$

(19)

It has been mentioned before that the induced magnetic fields will be negligibly small compared with the externally applied field in the range of parameters considered so that the expression (19) can be further reduced to

$$\nabla \cdot q^* = -\lambda \frac{d^2 T^*}{d(y^*)^2}$$

(20)

Using the equations (20), (1), (3) and (2.12) the energy equation (2.3) then takes the form

$$\left( \frac{d^2 u_x}{dy^2} + \frac{d^2 u_z}{dy^2} + \frac{d^2 T}{dy^2} + M^2 (j_x E_x + j_z E_z + j_z u_x - j_x u_z) \right) = 0$$

(21)
From this equation the temperature distribution can be determined. Finally when the solution of equations (11) to (16) is known the \( y \) components of equations (7) and (9) determine \( \frac{dp}{dy} \) and \( E_y \) respectively and from Maxwell's equation \((2.4^2)\rho_q\) can be obtained. These quantities, however, are not of very great interest and will not be considered further.

The system of equations (11) to (14) has to be solved simultaneously after completion by boundary conditions from which the constant values of \( E_x \) and \( E_z \) can be determined. The boundary conditions will be formulated in the next paragraph and the solution will then be constructed.
3.2. **Formulation of the boundary conditions and solution of the problem.**

It is convenient to introduce the complex quantities

\[
\begin{align*}
\tilde{u} &= u_x + iu_z \\
\tilde{j} &= j_z - ij_x \\
\tilde{E} &= E_x + iE_z \\
\tilde{B} &= B_x + iB_z
\end{align*}
\]

(1)

Equations (3.1.11) and (3.1.12) can then be taken together in the form

\[
\frac{d^2\tilde{u}}{dy^2} - M^2\tilde{j} = 0
\]

(2)

Equations (3.1.13) and (3.1.14) can also be combined and may be written as

\[(1 + i\omega t)\tilde{j} = \tilde{u} - i\tilde{E}\]

(3)

Finally combination of equations (3.1.15) and (3.1.16) is possible and gives

\[
\frac{d\tilde{E}}{dy} = -R_m\tilde{j}
\]

(4)

Equations (3) and (4) will be used to obtain the electromagnetic boundary conditions. Since it is more convenient to perform this in physical rather than in dimensionless quantities the equations are first rewritten in physical form. Equation (3) gives

\[
(1 + i\omega t)\tilde{k}\frac{1}{c} = \tilde{u}*B_0 - i\tilde{E}*
\]

(3*)

and equation (4) is
\[
\frac{dB^*}{dy^*} = -\mu^* y^* \tag{4}
\]

The well-known electromagnetic boundary conditions (see for instance Jackson \(^8\)) state that the tangential components of the electric field and the normal component of the magnetic field must be continuous across the interfaces between the fluid and the wall. When only finite conductivities are involved as supposed in this report the tangential components of the magnetic field must also be continuous across the interfaces. It should be mentioned that the "generalized" Ohm's law \((3^*)\) can also be applied in the wall. For solid conductors, however, the "Hall parameter" \(\omega r\) is very small. For instance for copper using equation \((3.1.10)\) and the values \(n_e = 10^{28}/m^3\), \(\sigma = 5 \times 10^7\text{mho/m}\) (see Jackson) and \(B_0 = 0.15\text{weber/m}^2\) (the greatest strength of the applied magnetic field to be considered in paragraph 4) the "Hall parameter" has the value \(5 \times 10^{-3}\). It is therefore reasonable to neglect the "Hall effect" in the wall.

Indicating fields, conductivity etc. in the fluid with subscript \(f\) and in the wall with subscript \(w\) and putting \((\omega r)^*_w = 0\), \((\omega r)^*_f = \omega r\) the boundary condition \(E^*_f = E^*_w\) gives at the lower wall \((y^*_w = 0)\) using \((3^*)\) and \((4^*)\)

\[
\tilde{u}^*_f B^*_o + \frac{1 + i\omega r}{(\sigma \mu)^*_f} \left(\frac{dB^*_w}{dy^*_w}\right)_f = \tilde{u}^*_w B^*_o + \frac{1}{(\rho \mu)^*_w} \left(\frac{dB^*_w}{dy^*_w}\right)_w \tag{5}
\]

At \(y^*_w = 0\) \(\tilde{u}^*_f = \tilde{u}^*_w = 0\) and excluding the possibility of ferromagnetic walls \(\mu^*_f = \mu^*_w = \mu_o\). Thus equation \((5)\) can be simplified to the form

\[
\frac{1 + i\omega r}{\sigma_f} \left(\frac{dB^*_w}{dy^*_w}\right)_f = \frac{1}{\sigma_w} \left(\frac{dB^*_w}{dy^*_w}\right)_w \tag{6}
\]
Now equation (3*) shows that \( j^* \) is constant so that it follows from equation (4*) that \( B^*_w \) is linear in \( y^* \). Since \( B^* \) is assumed to be zero outside the channel and because at the interface between fluid and wall \( B^*_w = B^*_f \) it follows that

\[
\left( \frac{d\tilde{B}^*_w}{dy^*_w} \right) = \frac{\tilde{B}^*_f}{d}
\]

(7)

if \( d \) denotes the thickness of the lower wall. Using equation (7) the boundary conditions on the magnetic field in the fluid at \( y^* = 0 \) can be written as

\[
\left( \frac{d\tilde{B}^*_f}{dy^*_f} \right) - \frac{1}{1 + i\omega} \frac{\sigma_f}{\sigma_w} \tilde{E}^*_f = 0
\]

(8)

Equation (8) corresponds to the boundary condition used by Chang and Yen 4) but for the present purpose it is more convenient to express the boundary conditions in terms of the electric field \( E^* \).

First it is observed that at the interface fluid-upper wall \( y^* = h \) \( B^*_f = 0 \) due to the continuity of \( B^* \) at the interfaces and since \( B^* \) doesn't change in the nonconducting upper wall. Integrating equation (4*) from 0 to \( h \) (i.e. over the fluid height) one obtains

\[
\mu_0 \int_0^h \gamma^* dy^* = \tilde{B}^*_f (0)
\]

(9)

Furthermore the equations (3*) and (4*) show that at \( y^* = 0 \)

\[
\left( \frac{d\tilde{B}^*_f}{dy^*_f} \right) = \frac{\sigma_f \mu_0}{1 + i\omega} i \tilde{E}^*_f
\]

(10)

Substituting equations (9) and (10) into equation (8) the electromagnetic boundary condition at the lower wall requires
\[ i\xi^* - \frac{1}{\sigma_w^d} \int_0^h \gamma^* \, dy^* = 0 \quad (11) \]

Now equation (11) can be easily separated into real and imaginary parts and converting physical into dimensionless quantities the following equations result

\[ E_x + \frac{\sigma_f h}{\sigma_w^d} \int_0^1 j_x \, dy = 0 \quad (12) \]

\[ E_z + \frac{\sigma_f h}{\sigma_w^d} \int_0^1 j_z \, dy = 0 \quad (13) \]

With equations (12) and (13) the boundary conditions are not yet complete. It has been observed already that the electromagnetic boundary conditions require at the upper wall \( y^* = h \)

\[ B_x^* = 0 \quad \text{or} \quad B_y^* = 0 \quad \text{at} \quad y = 1 \quad (14) \]

Since the fluid is viscous the boundary conditions on the fluid velocity expressed in dimensionless quantities are

at \( y = 0 \) \quad \( u_x = u_z = 0 \quad \text{or} \quad \tilde{u} = 0 \quad (15) \]

at \( y = 1 \) \quad \( u_x = 1, u_z = 0 \quad \text{or} \quad \tilde{u} = 1 \quad (16) \]

Finally for the boundary conditions on the temperature the choice is made

\[ T^* = T_0 \quad \text{or} \quad T = \frac{\lambda T_0}{\eta U_0^2} \quad \text{at} \quad y = 0,1 \quad (17) \]
At this point it is worthwhile to state once again that the electromagnetic boundary conditions are derived under the assumption that $\mathcal{B}$ is zero outside the channel. This amounts to the assumption that the total current through the fluid is exactly cancelled by the total current through the lower wall. Finally it should be pointed out that the lower wall of the channel can be considered as an external load resistance.

In fact the situation could also be viewed as follows. It can be assumed that a third wall of thickness $d$ and with conductivity $\sigma_w$ acting as a separate load resistance for the return of the currents generated in the channel is placed far below the lower wall. If at the same time the lower wall of the channel is replaced by a non-conducting wall all phenomena inside the channel will be the same as before. This can be seen from the following argument. If all quantities depend only on $y$ the continuity of the tangential components of the electric field at all interfaces requires $E_1$ and $E_3$ to be the same in the external load as in the channel. The conductivity of the load resistance then determines the constant values of $E_1$ and $E_3$ in the same way as before and the only difference with the configuration considered there is that due to the continuity of the tangential components of the magnetic field at all interfaces $B_1$ and $B_3$ will have constant non-zero values in the space between the lower wall of the channel and the load resistance.

Now the solution of the problem can be considered employing the complex quantities introduced in equation (1). With (1) the equations (3.1.11) to (3.1.14) can be combined into equations (2) and (3) which can be solved directly in complex quantities. Substitution of (3) into (2) gives

$$
\frac{d^2\tilde{u}}{dy^2} - \frac{M^2}{1 + i\omega} \tilde{u} = - \frac{iM^2}{1 + i\omega} \tilde{E}
$$

(18)
Since $\widetilde{E}$ is constant the solution of (18) is

$$\widetilde{u} = a \sinh(M_1 - iM_2)y + b \cosh(M_1 - iM_2)y + i\widetilde{E}$$  \hspace{1cm} (19)

with

$$M_1 - iM_2 = \frac{M}{(1 + i\omega t)^{1/2}}$$  \hspace{1cm} (20)

The constants $a$ and $b$ in (19) can be calculated from the boundary conditions (15) and (16) and one finds

$$\widetilde{u} = (1 - i\widetilde{E}) \frac{\sinh(M_1 - iM_2)y}{\sinh(M_1 - iM_2)} + i\widetilde{E} \frac{\cosh(M_1 - iM_2)\sinh(M_1 - iM_2)y}{\sinh(M_1 - iM_2)}$$

$$+ i\widetilde{E} \left\{ 1 - \cosh(M_1 - iM_2)y \right\}$$  \hspace{1cm} (21)

In (21) the constant $\widetilde{E}$ is still unknown but $\widetilde{E}$ can be determined according to equations (12) and (13) from an integral of $\tilde{j}$. It is therefore appropriate to calculate

$$\widetilde{I} = \int_0^1 \tilde{j} \, dy$$  \hspace{1cm} (22)

Using (3) and (21) one obtains

$$\widetilde{I} = \frac{1}{(1 + i\omega t)(M_1 - iM_2)} \left\{ \frac{\cosh(M_1 - iM_2) - 1}{\sinh(M_1 - iM_2)} + i\widetilde{E} \left[ \frac{1 - 2 \cosh(M_1 - iM_2)}{\sinh(M_1 - iM_2)} \right. \right.$$  

$$\left. + \frac{\cosh^2(M_1 - iM_2)}{\sinh(M_1 - iM_2)} - \sinh(M_1 - iM_2) \right] \right\}$$  \hspace{1cm} (23)

To determine the expressions for $u_x$ and $u_z$ separately the equations (21) and (23) have to be separated into real and imaginary parts. We have the relations

$$\sinh(M_1 - iM_2) = \sinh M_1 \cos M_2 - i \cosh M_1 \sin M_2$$
\[
cosh(M_1 - iM_2) = \cosh M_1 \cos M_2 - i \sinh M_1 \sin M_2 \tag{24}
\]

and from straightforward but tedious calculations the following relations for \(u_x\) and \(u_z\) are obtained from (21)

\[
u_x = E_z (\cosh M_1 y \cos M_2 y - 1) - E_x \sinh M_1 y \sin M_2 y
\]
\[
+ f_1 \sinh M_1 y \cos M_2 y + f_2 \cosh M_1 y \sin M_2 y \tag{25}
\]
\[
u_z = E_x (1 - \cosh M_1 y \cos M_2 y) - E_z \sinh M_1 y \sin M_2 y
\]
\[
+ f_2 \sinh M_1 y \cos M_2 y - f_1 \cosh M_1 y \sin M_2 y \tag{26}
\]

where

\[
f_1 = \frac{1}{\sinh^2 M_1 + \sin^2 M_2} \left\{ (1 + E_z) \sinh M_1 \cos M_2 + E_x \cosh M_1 \sin M_2
\right.
\]
\[
- E_z \sinh M_1 \cosh M_1 - E_x \sin M_2 \cos M_2 \right\} \tag{27}
\]
\[
f_2 = \frac{1}{\sinh^2 M_1 + \sin^2 M_2} \left\{ (1 + E_z) \cosh M_1 \sin M_2 - E_x \sinh M_1 \cos M_2
\right.
\]
\[
+ E_x \sinh M_1 \cosh M_1 - E_z \sin M_2 \cos M_2 \right\} \tag{28}
\]

Separation into real and imaginary parts of equation (23) gives

\[
I_x = \int_0^1 j_x dy = - k_5 - k_3 E_x + k_4 E_z \tag{29}
\]
\[ I_z = \int_0^1 j_z \, dy = k_6 - k_4 E_x - k_3 E_z \] (30)

with

\[ k_3 = \frac{(M_1 + \omega t M_2)k_2 - (M_2 - \omega t M_1)k_1}{(1 + \omega^2 \tau^2)(M_1^2 + M_2^2)} \]

\[ k_4 = \frac{(M_2 - \omega t M_1)k_2 + (M_1 + \omega t M_2)k_1}{(1 + \omega^2 \tau^2)(M_1^2 + M_2^2)} \] (31)

\[ k_5 = \frac{\{(M_2 - \omega t M_1) \sinh M_1 - (M_1 + \omega t M_2) \sin M_2\}\{(\cosh M_1 - \cos M_2)\}}{(1 + \omega^2 \tau^2)(M_1^2 + M_2^2)(\sinh^2 M_1 + \sin^2 M_2)} \]

\[ k_6 = \frac{\{(M_1 + \omega t M_2) \sinh M_1 + (M_2 - \omega t M_1) \sin M_2\}\{(\cosh M_1 - \cos M_2)\}}{(1 + \omega^2 \tau^2)(M_1^2 + M_2^2)(\sinh^2 M_1 + \sin^2 M_2)} \]

and

\[ k_1 = \frac{\frac{1}{2} \cosh M_1 \sin M_2 (4 + \cos 2M_2 - \cosh 2M_1) - \sin 2M_2}{\sinh^2 M_1 + \sin^2 M_2} \]

\[ + \cosh M_1 \sin M_2 \]

\[ k_2 = \frac{\frac{1}{2} \sinh M_1 \cos M_2 (4 + \cosh 2M_1 - \cos 2M_2) - \sinh 2M_1}{\sinh^2 M_1 + \sin^2 M_2} \] (32)

\[ - \sinh M_1 \cos M_2 \]
Substitution of (29) into (12) and of (30) into (13) gives two equations for the two unknowns $E_x$ and $E_z$. Putting $\Sigma = \frac{\sigma wd}{\sigma RH}$ the solution of these equations is

\[
E_x = \frac{k_6 x k_4 - k_5 x (k_5 - \Sigma)}{(k_5 - \Sigma)^2 + k_4^2} \tag{33}
\]

\[
E_z = \frac{k_5 x k_4 + k_6 x (k_5 - \Sigma)}{(k_5 - \Sigma)^2 + k_4^2} \tag{34}
\]

With (20), (25) to (28) and (31) to (34) the velocities $u_x$ and $u_z$ can now be calculated for all values of the parameters $M$, $\omega r$ and $\Sigma$. Rather than to determine the complete expressions for the other quantities of interest these will be expressed in terms of $u_x$ and $u_z$. Separating equation (3) into real and imaginary parts we find

\[
j_x = \frac{E_x + \omega r E_z - u_z + \omega r u_x}{(1 + \omega^2 \tau^2)} \tag{35}
\]

\[
j_z = \frac{E_z - \omega r E_x + u_x + \omega r u_z}{(1 + \omega^2 \tau^2)} \tag{36}
\]

From (3.1.15) and (3.1.16) using the boundary conditions (14) and the equations (12) and (13) the magnetic fields $B_x$ and $B_z$ can be found in terms of $j_x$ and $j_z$ as follows

\[
B_x = -R_m \left\{ \int_0^y j_z \, dy + \Sigma E_z \right\} \tag{37}
\]

\[
B_z = R_m \left\{ \int_0^y j_x \, dy + \Sigma E_x \right\} \tag{38}
\]
With the aid of equations (35) and (36) $B_x$ and $B_z$ can then be expressed in terms of $u_x$ and $u_z$.

Finally in the energy equation (3.1.21) the known terms can be expressed in terms of $u_x$ and $u_z$ by means of (3.1.11) and (3.1.12). The result is

$$\frac{d^2 T}{dy^2} + \left( \frac{du_x}{dy} \right)^2 + u_x \frac{d^2 u_x}{dy^2} + \left( \frac{du_z}{dy} \right)^2 + u_z \frac{d^2 u_z}{dy^2} + E_z \frac{d^2 u_x}{dy^2} - E_x \frac{d^2 u_z}{dy^2} = 0$$

(39)

Because $E_x$ and $E_z$ are constant this can also be written as

$$\frac{d^2}{dy^2} \left\{ T + \frac{u_x^2 + u_z^2}{2} + E_z u_x - E_x u_z \right\} = 0$$

(40)

Integration of (40) using the boundary conditions (17) then gives

$$\frac{\lambda(T - T_0)}{\eta U_0^2} = (E_z + \frac{1}{2}) y + E_z u_z - E_x u_x - \frac{u_x^2}{2} - \frac{u_z^2}{2}$$

(41)

With these formulae all quantities of interest can be calculated for all parameter values. In the next paragraph the order of magnitude of the parameters $M$, $\omega r$ and $R_m$ will be estimated. It is clear that the parameter $\Sigma = \frac{\sigma u d}{\sigma r h}$ can assume all values in the range $0 \leq \Sigma \leq \infty$ when the possibility of a superconducting wall is excluded.
4. **Estimation of the order of magnitude of the flow-parameters.**

In this paragraph the order of magnitude of the flow-parameters appearing in the study of the Couette flow i.e. M, $\omega t$ and $R_m$ will be estimated. To that end the work of Peng and Pindroh ⁹ will be used in which the composition and the transport properties of high temperature air have been calculated. From the results the tables (I) to (IV) have been calculated showing the electron density, degree of ionization, electrical conductivity and viscosity in the temperature range of 5000 to 15000⁰K and in the density range of 10 to $10^{-5}$ amagats (a density of e.g. 10 amagats meaning $\frac{\rho}{\rho_o} = 10$ where $\rho_o$ is the air density at standard temperature $\rho_o$ and pressure).

The parameters have been defined in equations (3.1.8), (3.1.10) and (3.1.17) as

$$M = B_o h \sqrt{\frac{\sigma f}{\eta}}$$

$$\omega t = \frac{\sigma B_o}{n_e e}$$

$$R_m = \sigma f U_o h$$

These definitions show that with a value of $B_o$ chosen at a certain temperature and density of the medium, M can be varied independent of $\omega t$ by selecting different values of h. $R_m$ can still be varied independent of $\omega t$ and M by choosing different values of $U_o$. For an estimate of the parameters the following values are taken

$$U_o = 50 \text{ m/sec}$$

$$h = 1 \text{ cm} = 10^{-2} \text{m}$$

(2)
\( B_0 = 10^3 \text{ gauss} = 0,1 \text{ weber/m}^2 \)

Then
\[
\frac{m}{\eta} = 10^{-3} \frac{\sigma_f}{\eta} \quad (3)
\]
\[
\omega T = 6,17 \times 10^{17} \times \frac{\sigma_f}{n_e} \quad (4)
\]
\[
R_m = 2\pi \times 10^{-7} \times \Theta_f \quad (5)
\]

where the values \( \mu = \mu_0 = 4\pi \times 10^{-7} \text{ Henry/m} \) and \( e = 1,62 \times 10^{-19} \text{ Coulomb} \) have been used. In figures (1) to (3) the degree of ionization \( \delta \), the Hartmann number \( M \) and the Hall parameter \( \omega T \) have been plotted as functions of the temperature for different values of the density using the tables (I) to (IV) and the equations (3) and (4). It has been mentioned in paragraph 2 that the equations are assumed to be valid for a degree of ionization exceeding the value 0.25. Figure (1) shows that this is reached for \( \frac{\rho}{\rho_0} = 10^{-3} \) if \( T \geq 12000^0 K \) and for \( \frac{\rho}{\rho_0} = 10^{-5} \) if \( T \geq 9500^0 K \).

In connection with this it should be remarked that the definition in this report of the degree of ionization \( T = \frac{n_e}{n_e + n_i + N} \) differs from the definition usually encountered in the literature. The definition used has the disadvantage that for a fully ionized gas where \( N = 0 \) the degree of ionization is \( \delta = 0,5 \) (since \( n_e = n_i \)). It can be seen from the definition that \( \delta = 0,25 \) corresponds to the condition \( N = n_e + n_i \) that is the situation where there are as many charged as neutral particles.

It has also been mentioned in paragraph 2 that the equations are assumed to remain valid up to \( \omega T = 6 \). Now figure 2 shows that for \( \frac{\rho}{\rho_0} = 10^{-3} \) and \( T \geq 12000^0 K \) a value of \( \omega T = 6 \) cannot be reached with reasonable values of the magnetic field \( B_0 \).

The further discussion will therefore be limited to the
conditions \( \frac{\rho}{\rho_0} = 10^{-5} \) and \( T \geq 9500^\circ K \).

Figure (2) shows that under these conditions \( 4.0 \leq \omega \tau \leq 6.3 \)
while figure (3) shows that \( 5.3 \leq M \leq 37 \). Varying the value
of \( B_0 \) at \( \frac{\rho}{\rho_0} = 10^{-5} \) and \( T \geq 9500^\circ K \) to obtain \( \omega \tau = 1 \) the

corresponding range in Hartmann numbers is \( 0.8 \leq M \leq 8 \). When
the same thing is done to obtain \( \omega \tau = 6 \) the range of Hartmann
numbers is \( 5 \leq M \leq 47 \). From this it is concluded that a

range of Hartmann numbers \( 5 \leq M \leq 80 \) can be made to correspond
to all values of the Hall parameter in the range \( 1 \leq \omega \tau \leq 6 \)
by varying \( h \) between \( 10^{-2} \) and \( 10^{-1} m \).

Finally it follows from table (IV) that for \( \frac{\rho}{\rho_0} = 10^{-5} \) and

\( T \geq 9500^\circ K \) the magnetic Reynolds number will be in a range of
about \( 1.3 \times 10^{-3} \leq R_m \leq 2.5 \times 10^{-2} \) at the chosen value of \( U_0 \)
when \( h \) is varied between \( 10^{-2} \) and \( 10^{-1} m \). In paragraph 5 numerical
results which have been obtained at \( M = 5, 10, 15, 20; \)
\( \omega \tau = 0, 2, 4, 6 \) and same values of the parameter \( \Sigma \)
will be discussed. Some of the most interesting results are
given in figures.
5. Discussion of results and conclusions.

The quantities of interest have been calculated numerically by using the equations (3.2.20), (3.2.25) to (3.2.28), (3.2.31) to (3.2.38) and (3.2.41) for the following values of the parameters

\[ \begin{align*}
M &= 5, 10, 15, 20 \\
\omega t &= 0, 2, 4, 6 \\
\Sigma &= 0, 0.5, 1, 10, 100, 10^4
\end{align*} \]  

(1)

These numerical calculations have been performed on the Telefunken TR-4 computer of the Technological University, Delft; the program used was written in Algol-60 and has been included in the appendix. From the numerical results the figures (4) to (18) have been drawn giving \( u_x, u_z, T, j_x \) and \( j_z \) as function of \( y \). A selection had to be made of the parameter combinations in (1) for which figures should be drawn. It was also concluded that no absolute necessity existed to include also figures of \( B_x \) and \( B_z \) as function of \( y \). Thus before starting with a discussion of the figures (4) to (18) the conclusions should be mentioned which can be drawn from those numerical results which are not included in these figures. These conclusions (for the range of parameters considered in (1)) are the following.

1) \( E_z \) is always negative; at \( \Sigma = 0 \), \( E_z = -0.5U_oB_o \) independent of the values of \( M \) and \( \omega t \); when \( \Sigma \) increases the absolute value of \( E_z \) decreases; at \( \Sigma = 10^4 \), \( \left| \frac{E_z}{U_o B_o} \right| \) varies between \( 2 \times 10^{-5} \) and \( 2 \times 10^{-6} \); there is some influence of \( M \) and \( \omega t \) on the value of \( E_z \) for \( \Sigma > 0.5 \) but this appears to be less important than the influence of \( \Sigma \).

2) \( E_x = 0 \) at \( \Sigma = 0 \) or \( \omega t = 0 \) independent of the value of \( M \); for \( \Sigma > 0.5 \) and \( \omega t > 2 \) \( E_x \) is negative and its absolute
value decreases as $\Sigma$ increases; at $\Sigma = 0.5$, $\omega \gg 2$, $|E_x/\nu_0B_0|$ varies between $10^{-1}$ and $2 \times 10^{-2}$; at $\Sigma = 10^4$, $\omega \gg 2$, $|E_x/\nu_0B_0|$ varies between $10^{-5}$ and $10^{-6}$; there is thus some influence of $M$ and $\omega \tau$ on the value of $E_x$ for $\Sigma \gg 0.5$, $\omega \tau \gg 2$ but this appears to be less important than the influence of $\Sigma$.

3) $B_x$ and $B_z$ are always small; for instance the maximum value of $B_x$ occurs at $M = 5$, $\omega \tau = 0$, $\Sigma = 10^4$ and amounts to $10^{-2}B_0$ at the maximum value of $R_m$. It indicates that the induced magnetic fields can almost be neglected with respect to the externally applied magnetic field. This conclusion has been used already in the equations (3.1.9) and (3.1.20).

4) The influence of an increase in $\Sigma$ on the variables shown in the figures (4) to (18) will be slight above $\Sigma = 10$. In that case the curves will be very close together. Therefore $\Sigma = 10$ is the largest value of $\Sigma$ used in the figures (4) to (18).

Now the figures (4) to (18) will be discussed and an attempt will be made to give some explanation of the influence of the parameters $M$, $\omega \tau$ and $\Sigma$. These figures will not be considered in the order in which they appear since the influence of $\Sigma$ can be discussed more easily if we consider first the influences of $M$ and $\omega \tau$ when $\Sigma = 0$.

In this case it appears from the figures that, for all values of the parameters $M$ and $\omega \tau$, $[u_x - (U_0/2)]$, $u_z$, $j_x$ and $j_z$ are odd functions of the coordinate $[(Y/h) - 0.5]$ (positive for $(Y/h) > 0.5$, negative for $(Y/h) < 0.5$). At the same time the figures show that $(T - T_0)$ is an even function of the coordinate $[(Y/h) - 0.5]$ in this case. This can easily be explained by considering first the following configuration.
It should be noted that with \( \Sigma = 0 \) both walls are insulators which makes it possible to apply symmetry-antisymmetry arguments. Indicating the quantities in the configuration (a) by primes these arguments easily show that \( u_x^1, u_z^1, j_x^1 \) and \( j_z^1 \) are odd functions of the coordinate \((y/h) - 0.5\) and that \( E_x^1 \) and \( E_z^1 \) will be zero. The configuration (a), differs from the situation studied in this report, which can be drawn schematically as follows.

It is clear, that this configuration can be easily obtained from configuration (a) by a simple transformation (in which the coordinate axes remain stationary with respect to each other). The relation between the (unprimed) variables of the configuration (b) and the primed variables will be in general
\[ u_x = u^1_x + \frac{U_0}{2}, \quad u_z = u_z \]

\[ j_x = j^1_x + \frac{U_0}{2} \rho q, \quad j_z = j^1_z \]

\[ E_x = E^1_x, \quad E_z = E^1_z - \frac{U_0 B_0}{2} \]

It should be noted that the approximation

\[ \rho q u \ll \lambda \]

has been used throughout in this report and thus the arguments above explain why \( u_x - (U_0/2) \), \( u_z \), \( j_x \) and \( j_z \) are odd functions of the coordinate \( (y/h) - 0.5 \). Moreover these arguments show that for \( \Sigma = 0 \) always \( E_x = 0 \) and \( (E_z/U_0 B_0) = -0.5 \) as already stated before. The fact that \( (T-T_0) \) is an even function of \( (y/h) - 0.5 \) is clear from what has been said above and from the fact that equation (3.2.41) can be written in general in the form (in physical quantities)

\[
\frac{(T-T_0)}{\eta u^2/\lambda} = \left( \frac{E_z}{U_0 B_0} + 0.5 \right) \frac{v}{h} - \frac{1}{2} \left[ \frac{u_z - (E_x/B_0)}{U_0} \right]^2 - \frac{1}{2} \left[ \frac{u_x + (E_z/B_0)}{U_0} \right]^2 + \frac{E^2_x + E^2_z}{2U_0^2 B_0^2}
\]

At the same time this expression suggests that for instance \( E_z/B_0 \) can be interpreted as a velocity; for the explanation of the influence of the magnetic field on the quantities in
the figures (4) to (18) it is indeed convenient to think in terms of the "velocity of the applied magnetic field" defined by

\[ v(\mathbf{B}_0) = \frac{\mathbf{E} \times \mathbf{B}_0}{\mathbf{B}_0^2} \]  

(5)

It should be pointed out that this concept may give some confusion when it is interpreted too literally as for instance the velocity with which the coil generating \( \mathbf{B}_0 \) has to move. This velocity has no relation at all with \( v(\mathbf{B}_0) \) since the value of \( \mathbf{E} \) is exclusively determined by the geometry and the boundary conditions (in this report by the value of \( \Sigma \)). The equation (5) together with the values of \( \mathbf{E} \) and \( \mathbf{E}_1 \) which have been found indicates that "\( \mathbf{B}_0 \) is stationary" in configuration (a) while "it moves with the velocity \( v_x(\mathbf{B}_0) = \frac{\mathbf{E}_2}{\mathbf{B}_0} = \frac{\mathbf{U}_0}{2} \)" in configuration (b) corresponding with the transformation performed in order to go from configuration (a) to (b).

The quantity \( [u_x - (\mathbf{U}_0/2)] \) can thus be interpreted as the velocity of slip relative to the magnetic field \( (u_x - v_x(\mathbf{B}_0)). \)

Now it is well known that the magnetohydrodynamic effects on the velocity can be explained by noting that it is difficult for the fluid elements to cross the magnetic field lines. As the magnetohydrodynamic effects increase corresponding in this report with an increase of \( M \) "the fluid elements become more rigidly attached to the field lines" in other words the velocity of slip \( |u_x - v_x(\mathbf{B}_0)| \) decreases. This is clearly illustrated by the figures (5c) and (5d) and the effect of the Hartmann number \( M \) on \( u_x \) is thus easily explained.

From figure (5) the effect of \( \omega t \) on \( u_x \) appears to be in general opposed to the influence of \( M \). This can be elucidated by considering the differential equation for \( u_x \) which may be written down from the equations (3.1.11), (3.1.13), (3.1.14) and (5)
in the form (in physical quantities)

\[ n^2 \frac{d^2 u_x}{d y^2} = \frac{M^2}{1 + (\omega t)^2} \left\{ (u_x - v_x(B_0)) + \omega t (u_z - v_z(B_0)) \right\} \]  \hspace{1cm} (6)

Thus to a first approximation (neglecting the influence of \( u_z - v_z(B_0) \)) the behaviour of \( u_x \) is governed by the combination of parameters \( \frac{M^2}{1 + (\omega t)^2} \) indicating that an increase of \( \omega t \) has the reverse effect as an increase of \( M \). The influence of \( u_z - v_z(B_0) \) on \( u_x \) doesn't show this behaviour and is apparently of less importance in general.

In order to explain the effects of \( M \) and \( \omega t \) on \( u_z \) it should be noted that \( u_z \) vanishes when there are no magnetohydrodynamic effects that is when \( M = 0 \). It is well known that \( u_z \) vanishes also when \( M \neq 0 \) but when the Hall parameter \( \omega t \) is zero. This can easily be verified from the differential equation for \( u_z \) and indicates that, at least for small values of \( M \) and \( \omega t \), both an increase of \( M \) at a constant value of \( \omega t (>0) \) and an increase of \( \omega t \) at a constant value of \( M (>0) \) will result in an increase of \( |u_z| \). The figure (8) indeed shows this behaviour. It also shows that \( |u_z| \) will only increase up to a certain value of \( M \) or \( \omega t \) and is followed by a decrease if \( M \) or \( \omega t \) is further increased. Since it appears from figure (8) that the influences of \( M \) and \( \omega t \) depend on the region of \( y \) considered it is difficult to draw more precise conclusions from this figure or to give a more detailed explanation of the effects of \( M \) and \( \omega t \).

For a discussion of the effects of \( M \) and \( \omega t \) on the temperature distribution (figure 11) the equation (4) can again be used. It should be noted that the contribution of \( (u_x - v_x(B_0)) \) in equation (4) is much more important than the contribution of
\( u_2 (V_2^{(B_0)} = 0) \) since figure (5) shows that the maximum of \\
\((u_x - V_2^{(B_0)})^2 / U_0^2 \) is equal to 0.25 while the maximum of \( u_2^2 / U_0^2 \) \\
is according to figure (8) of the order of 0.02 or less (the \\
term in equation (4) which is linear in \( y \) vanishes and \\
\( E_2^2 / U_0^2 B_0^2 = 0.25 \)). Thus the influences of \( M \) and \( \omega \tau \) on \((T - T_0)\) \\
can be explained by means of equation (4) when the contribution \\
of \( u_2 \) to \((T - T_0)\) is neglected. It is clear from figure (5) \\
that with increasing \( M \) the regions where a large slip \\
\( |u_x - V_2^{(B_0)}| \) occurs shift ever more to regions close to the \\
walls. Examining equation (4) this explains why the profile \\
of \((T - T_0)\) in figure (11) becomes more rectangular as \( M \) in- \\
creases. A more physical explanation of the influence of \( M \) \\
on \((T - T_0)\) can be given by making a comparison with viscous \\
Couette flow \((M = 0)\) when the profile of \((T - T_0)\) is a parabola. \\
Compared with viscous Couette flow extra Joule dissipation \\
appears in magnetohydrodynamically Couette flow which is largest \\
in the regions close to the walls since there \( |j_x| \) and \( |j_z| \) \\
are largest (see the figures 14 and 17). The viscous dis- \\
sipation in the middle of the channel is reduced since there \\
\( du_x / dy \) is smaller (in viscous Couette flow \( du_x / dy \) is constant) \\
while the viscous dissipation increases close to the walls \\
because \( du_x / dy \) is much larger in those regions (the viscous \\
dissipation due to the motion in the \( z \) direction is also \\
mainly concentrated in the regions close to the walls). The \\
effect of extra dissipation in the regions close to the walls \\
is then to distort the parabolic temperature profile to a more \\
rectangular one. For the explanation of the effect of \( \omega \tau \) on \\
the temperature distribution equation (4) will again be used \\
and the contribution of \( u_2 \) to \((T - T_0)\) will be neglected as \\
before. It has been discussed above that the effect of \( \omega \tau \) on \\
\((u_x - V_x^{(B_0)})\) is opposite to the influence of \( M \). It may thus \\
be expected and it is confirmed by figure (11) that the effects
of $M$ and $\omega t$ on the temperature distribution are also opposite so that as $\omega t$ increases the temperature profiles become less rectangular.

The explanation of the effects of $M$ and $\omega t$ on the electrical current densities, $j_z$, will first be considered. This will be done by using the equation for $j_z$ which can be written from the equations (3.1.11) and (6) in the form (in physical quantities)

$$\frac{j_z}{\sigma t U_o B_o} = \frac{1}{U_o(1 + \omega^2 \tau^2)} \left\{ (u_x - V_x^{(B_0)}) + \omega t (u_z - V_z^{(B_0)}) \right\}$$  \hspace{1cm} (7)

It shows that $j_z$ is generated by the velocities of slip relative to the magnetic field. It has appeared already that $|u_x - V_x^{(B_0)}|$ decreases when $M$ increases which shows that then $|j_z|$ will also decrease at least for small values of $\omega t$. This is confirmed by figure (17c). At large values of $\omega t$ $|j_z|$ may increase when $M$ increases because $|u_z|$ will then in general increase ($V_z^{(B_0)} = 0$). This is shown, although not very convincingly, by figure (17d). To explain the influence of $\omega t$ one may use that in general both $|u_x - V_x^{(B_0)}|$ and $|u_z|$ increase when $\omega t$ increases. Due to the fact that the conductivity in the $z$ direction decreases corresponding to the factor $1/(1 + \omega^2 \tau^2)$ in equation (7) $|j_z|$ will in general decrease as shown by the figures (17a) and (17b).

The effects of $M$ and $\omega t$ on $j_x$ can be derived from an equation corresponding with the equation above namely

$$\frac{j_x}{\sigma t U_o B_o} = \frac{1}{U_o(1 + \omega^2 \tau^2)} \left\{ \omega t (u_x - V_x^{(B_0)}) - (u_z - V_z^{(B_0)}) \right\}$$  \hspace{1cm} (8)

When $\omega t = 0$, $u_z = 0$ and it is clear that $j_x = 0$; when $\omega t$ increases $|j_x|$ will thus increase at least for small values.
of \( \omega t \). Due to the factor \( \frac{1}{1 + \omega^2 \tau^2} \) in equation (8) this will go on until a certain value of \( \omega t \) after which \( |j_x| \) will decrease when \( \omega t \) is further increased. This is confirmed by the figures (14a) and (14b). The feature, shown in the figures (14c) and (14d), that \( |j_x| \) generally decreases when \( M \) increases can be explained from the fact that then \( |u_x - v_x^{(B_0)}| \) decreases and although \( |u_x| \) will increase in general in this case the total effect is that \( |j_x| \) decreases since always \( |u_x| < \omega t |u_x - v_x^{(B_0)}| \).

It should be noted that in the case \( M = 0 \) which arises when \( \sigma_r \) or \( B_0 \) is zero the quantities \( \frac{j_x}{\sigma_r U_0 B_0} \) and \( \frac{j_z}{\sigma_r U_0 B_0} \) appearing in the figures (13) to (18) are meaningless. Therefore it is useless for the explanation of the effect of \( M \) on the curves in these figures to start from the case \( M = 0 \) (although the equations (7) and (8) show that in this case \( j_x \) and \( j_z \) are both zero).

With this the effects of \( M \) and \( \omega t \) on the quantities in the figures (4) to (18) for the case \( \Sigma = 0 \) have been completely discussed. As announced above the remainder will be concerned with the discussion of the influence of \( \Sigma \) at constant values of \( M \) and \( \omega t \).

In this case it is most convenient to start with the influence of \( \Sigma \) on the electrical current densities \( j_x \) and \( j_z \). As \( \Sigma \) increases the currents will flow more easily through the lower wall than through the medium adjacent to it in other words for small values of \( \Sigma \) \( |j_x| \) and \( |j_z| \) will decrease when \( \Sigma \) increases. This is shown by the figures (13) to (18) although not very pronounced in some cases. These figures also show (in some cases not very clearly) that \( j_x \) and \( j_z \) increase in the upper part of the channel when \( \Sigma \) increases. This can be explained to a first approximation by considering the contribution of the slip in the \( x \) direction \( (u_x - v_x^{(B_0)}) \) in the equations (7) and (8). It has been noted above that \( |E_z| \) and thus also \( v_x^{(B_0)} \) decreases when \( \Sigma \) increases; close to the upper
wall \( u_x \) must approach the speed of the upper wall \( U_o \) and thus \((U_x - V_x(B_o))\) has to increase in the upper part of the channel explaining why \( j_x \) and \( j_z \) increase in this region.

The influence of \( \Sigma \) on \( u_z \) can be explained by considering the Lorentz force in the z direction which generates the motion in this direction and which can be written from its definition as \( j_x B_o \). It is clear from the remarks made before about the influence of \( \Sigma \) on \( j_x \) that the Lorentz force in the z direction will increase in the upper part of the channel while it becomes less negative for small values of \( y \) as \( \Sigma \) increases. This implies that \( u_z \), which is always positive in the upper part of the channel will increase in this region while it will become less negative for small values of \( y \) as \( \Sigma \) increases. This is clearly shown in the figures (7) and (9). These figures also show that in some circumstances \( u_z \) can even become positive for all values of \( y \).

Now turning to the influence of \( \Sigma \) on \( u_x \) it will be clear that this can be explained mainly by considering the influence of \( \Sigma \) on \( V_x(B_o) \) at a constant value of \( M \) and \( \omega \) at which "the fluid elements are attached to a certain extent to the magnetic field lines". It has been noted before that \( V_x(B_o) \) decreases when \( \Sigma \) increases and it can thus be expected that, at least in the middle of the channel where the viscous forces are not very important, \( u_x \) will then also decrease. This is illustrated by the figures (4) and (6) and is particularly clear in those cases where "the fluid elements are sufficiently attached to the magnetic field lines" i.e. in the cases \( M = 10, \omega = 0 \) (figure 4\(^c\)) and \( M = 20, \omega = 6 \) (figure 6\(^b\)). It also appears from the figures (4) and (6) that \( u_x \) can even become negative in some cases which seems an interesting result and thus deserves a further discussion. In order to determine the conditions to be fulfilled when \( u_x \) becomes negative the equation (7) will be used and the equation (3.1.14) which can be written in the
form (in physical quantities)

\[ j_z + \omega \tau j_x = \sigma B_0 (u_x - V_x^{(B_0)}) \]  

(9)

It will be clear that a negative value of \( u_x \) can only be generated when the Lorentz force in the \( x \) direction has a negative value. The Lorentz force is equal to \((- j_z B_0)\) which shows that in the region where \( u_x \) is negative \( j_z \) must be positive. Since the right hand side of equation (9) is negative in the region where \( u_x \) is negative \((V_x^{(B_0)} > 0)\) and since \( j_z \) must be positive there it is clear that \( j_x \) must be negative and moreover that the inequality \( \omega \tau/j_x>j_z \) must be satisfied in this region. Summarizing it can be said that \( u_x \) will become negative in the region where the following inequalities are simultaneously satisfied

\[ j_z > 0 \]

\[ j_x < 0 \]

\[ \omega \tau/j_x > j_z \]  

(10)

which is confirmed by the figures (4) to (18).

Examining the condition that \( j_z \) must be positive in the region where \( u_x \) is negative more closely by means of equation (7) it is clear that this can never result from the term \((u_x - V_x^{(B_0)})\) which is always negative in this region. It is therefore clear that a positive value of \( j_z \) can only occur in this region if the contribution of \((u_x - V_x^{(B_0)})\) to \( j_z \) is positive i.e. if

\[ u_x - V_x^{(B_0)} > 0 \]  

(11)

Moreover the equation (7) shows that a positive value of \( j_z \) in this region only results if the contribution of \((u_x - V_x^{(B_0)})\) is more important than the (counteracting) influence of the term \((u_x - V(B_0))\) i.e. if
\[ \omega r (u_z - v_z^{(B_0)}) > |u_x - v_x^{(B_0)}| \]  \hspace{1cm} (12)

With this the conditions for the appearance of negative values of \( u_x \) have been completely established. The physical mechanism which causes this fact will now also be clear. The situation may be roughly summarized by noting that negative values of \( u_x \) only appear if (among other things) in the lower part of the channel, which is the region where these values of \( u_x \) are most apt to occur, the inequality (11) is satisfied (noting that \( v_z^{(B_0)} < 0 \) if \( \Sigma \gg 0.5 \)). At this point it should be mentioned, however, that this condition is possibly only satisfied by an assumption made in solving the problem namely that \( \frac{\partial p}{\partial z} = 0 \). It was mentioned in section 3.1 that \( \frac{\partial p}{\partial z} \) is a constant which may be chosen arbitrarily (depending on the conditions which are assumed to be satisfied at infinity). Instead of assuming \( \frac{\partial p}{\partial z} = 0 \) the value of \( \frac{\partial p}{\partial z} \) is quite often chosen in such a manner that the total mass flow in the \( z \) direction vanishes i.e. that

\[ \int_{0}^{h} u_z \, dy = 0 \]  \hspace{1cm} (13)

Although this assumption does not make the situation more realistic it will completely alter the solution. It is difficult to draw conclusions from the solution obtained in this report, which can be applied to the case where \( \frac{\partial p}{\partial z} \) is chosen in such a way that equation (13) is satisfied. If it is assumed that \( v_z^{(B_0)} \) remains unaltered it will, however, be clear that the condition (11) is no longer satisfied in the lower part of the channel since, due to equation (13). In contrast with some of the curves in the figures (7) and (9) \( u_z \) has to become negative in this region. It will be interesting to study the situation where the value of \( \frac{\partial p}{\partial z} \) is
determined by the condition (13) although, as already mentioned, this situation is not more realistic than the situation studied in this report. It is even more interesting to determine $\frac{\partial p}{\partial z}$ explicitly that is to attempt the solution of the corresponding channel flow problem with the influence of the side walls taken into account. To the author's knowledge a solution for this flow problem with the combined influence of Hall effect and wall conductivity is not yet known.

Finally the influence of $\Sigma$ on the temperature distribution has to be considered. From equation (4) it has been seen above that the first term is zero when $\Sigma = 0$; as $\Sigma$ increases, $|E_z|$ decreases so that the first term will become more important. Moreover it has appeared above that large values of $(u_x - V_x^{(B_0)})$ and $(u_z - V_z^{(B_0)})$ shift to larger values of $y/h$ as $\Sigma$ increases. This implies that the linear term in equation (4) dominates ever more for small values of $y$ as $\Sigma$ increases while the maximum value of $(T-T_0)$ increases ever more in this case. This is clearly shown by the figures (10) and (12) and can also be explained on physical grounds. Obviously the dissipation decreases in the lower part of the channel when $\Sigma$ increases since then the velocity gradients and the electrical currents decrease in this region. On the other hand the figures show that the velocity gradients and the electrical currents increase in the upper part of the channel when $\Sigma$ increases leading to increased dissipation in this region. It also explains the increase of the maximum temperature reached in the channel.

With this the influence of $\Sigma$ on the quantities in the figures (4) to (16) has been completely discussed. It should be noted that the effects of $M$ and $\omega$ are still to be discussed for the case that $\Sigma > 0$. This will not be done since the arguments used for the case $\Sigma = 0$ can easily be generalized to be applicable in this case also.
Summarizing it may be concluded that, at a constant value of the Hartman number $M$, the wall conductance parameter $\Sigma$ and the Hall parameter $\omega t$ greatly influence the velocity profiles, temperature profiles etcetera. In connection with the negative values of $u_x$, appearing in some solutions obtained in this report, it is concluded that it will be interesting to study the effects of $\omega t$ and the conducting properties of the walls also in two dimensional magnetohydrodynamic channel flows.
References.
8. Jackson, J.D. Classical electrodynamics
Table I  Electron density, degree of ionization, electrical conductivity and viscosity coefficient for high temperature air at \( \frac{\rho}{\rho_0} = 10 \) (from Peng and Pindroh\(^9\)).

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<th>( \eta )</th>
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Table II Electron density, degree of ionization, electrical conductivity and viscosity coefficient for high temperature air at $\rho/\rho_o = 10^{-1}$ (from Peng and Pindroh).

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Table III: Electron density, degree of ionization, electrical conductivity and viscosity coefficient for high temperature air at $\frac{p}{p_0} = 10^{-3}$ (from Peng and Pindroh\textsuperscript{9}).

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Table IV  Electron density, degree of ionization, electrical conductivity and viscosity coefficient for high temperature air at \( \frac{p}{p_0} = 10^{-5} \) (from Peng and Pindroh\(^9\)).

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<td>( 1.08 \times 10^{-4} )</td>
</tr>
<tr>
<td>11000</td>
<td>( 4.06 \times 10^{20} )</td>
<td>( 4.16 \times 10^{-1} )</td>
<td>( 2.63 \times 10^3 )</td>
<td>( 2.08 \times 10^{-5} )</td>
</tr>
<tr>
<td>13000</td>
<td>( 5.11 \times 10^{20} )</td>
<td>( 4.81 \times 10^{-1} )</td>
<td>( 3.51 \times 10^3 )</td>
<td>( 4.97 \times 10^{-6} )</td>
</tr>
<tr>
<td>15000</td>
<td>( 5.21 \times 10^{20} )</td>
<td>( 4.88 \times 10^{-1} )</td>
<td>( 3.98 \times 10^3 )</td>
<td>( 2.88 \times 10^{-6} )</td>
</tr>
</tbody>
</table>
Appendix  The Algod programme used for the numerical calculations.

'begin'  'real' q, e1, e3, ul, u3, j1, j3, b1, b3, t, alfa, ml, m2, 
        sinhml, coshml, a, c, g, h, k1, k3, k4, k5, 
        k6, cosm2, sinm2, d1, d2, d3, d4, f1, f2, y, blv, 
        b3v, jlv, j3v, sigma; 
'integer' m, x, s; 
'boolean' p; 'procedure' se4; 'code'; 
p := 'true'; 
'for' m := 5, 10, 15, 20 'do' 
'begin'  'for' q := 0, 2.0, 4.0, 6.0 'do' 
        'begin'  alfa := arctan(q); 
            ml := m * cos(alfa/2) * sqrt(cos(alfa)); 
            m2 := m * sin(alfa/2) * sqrt(cos(alfa)); 
            sinhml := (exp(ml) - exp(-ml))/2; 
            coshml := (exp(ml) + exp(-ml))/2; 
            a := 1/(1 + q 'power' 2); 
            c := 1/(ml 'power' 2 + m2 'power' 2); 
            cosm2 := cos(m2); 
            sinm2 := sin(m2); 
            g := ml + (q * m2); 
            h := m2 - (q * ml); 
            k1 := (coshml * sinm2 * (4 + cos(2 * m2) 
                 - (1 + 2 * (sinhml 'power' 2))/2 - sin(2 * m2))/ 
                  (sinhml 'power' 2 + sinm2 'power' 2) + coshml * sinm2; 
            k2 := (sinhml * cosm2 * (4 + 2 * sinhml 'power' 2 
                 - cos(2 * m2))/2 - 2 * sinhml * coshml)/ 
                  (sinhml 'power' 2 + sinm2 'power' 2) - sinhml * cosm2; 
            k3 := (g * k2 - h * k1) * m * c; 
            k4 := (h * k2 + g * k1) * m * c; 
            k5 := (h * sinhml * (coshml - cosm2) 
                  + g * sinm2 * (cosm2 - coshml)) * m * c)/ 
                  (sinhml 'power' 2 + sinm2 'power' 2); 
            k6 := ((g * sinhml * (coshml - cosm2) 
                  - h * sinm2 * (cosm2 - coshml)) * m * c)/ 

(sinhml\textasciipower2+sinm2\textasciipower2);
'for'\textasciisigmaxeq0,0.5,1,10,100,10\textasciitilde4'do'
'begin' \textasciie1 :=\frac{(k5\times k4-k5\times (-\textasciisigma+k3))}{(k4\textasciipower2+-\textasciisigma+k3)\textasciipower2)};
\textasciie3 :=\frac{(k5\times (-\textasciisigma+k3)+k5\times k4)}{(k4\textasciipower2+-\textasciisigma+k3)\textasciipower2)};
'if'\textasciip'\textasciithen'
p :='false'
'else'
'begin' p :='true'; nlcr(2);
'end';
write('m='); vasko(2,0,m);
write('q='); vasko(1,1,q);
write('\sigma=');
'if'\sigma<2'\textasciithen'
vasko(1,1,\sigma)'else'
'begin' 'if'\sigma<15'\textasciithen'vasko(2,0,\sigma)
'else'
'begin' 'if'\sigma<105'\textasciithen'vasko(3,0,\sigma)
'else'
'begin' 'if'\sigma<10\textasciitilde5'\textasciithen'
vasko(5,0,\sigma);
'end';
'end';
'end';
write('e1='); se4(e1);
write('e3='); se4(e3); nlcr(2);
write('y t u l u3 j l

\begin{align*}
f1 := & (1+e3)\times \text{sinhml}\times \text{cosm2} \\
& + e1 \times \text{coshml}\times \text{sinm2}\ - e3 \times \text{sinhml}\times \text{coshml} \\
& - e1 \times \text{sinm2}\times \text{cosm2}\) / (\text{sinhml}\textasciipower2+\text{sinm2}\textasciipower2); \\
f2 := & (1+e3)\times \text{coshml}\times \text{sinm2} \\
& - e1 \times \text{sinhml}\times \text{cosm2} + e1 \times \text{sinhml}\times \text{coshml} \\
& - e3 \times \text{sinm2}\times \text{cosm2}\) / (\text{sinhml}\textasciipower2+\text{sinm2}\textasciipower2); \\
\end{align*}
blv:=-((e3+w*e3);
b3v:=(e3+w*e1);
jlv:=(e1+w*e3)*a;
j3v:=(e3-w*e1)*a;
'for'x:=0'step'50'until'800,'825'step'25
'until'1000'do'
'begin' y:=x/1000;vasko(1,3,y);
d1:=exp(m1*y);
d2:=exp(-m1*y);d3:=sin(m2*y);
d4:=cos(m2*y);
u1:=(e3*(d1+d2)*d4-2)-e1*(d1-d2)*d3 +f1*(d1-d2)*d4+f2*(d1+d2)*d3)/2;
u3:=(e1*(2-(d1+d2)*d4)-e3*(d1-d2)*d3 +f2*(d1-d2)*d4-f1*(d1+d2)*d3)/2;
t:=(e3+0.5)*y+elwul-e3*wul -ul'power'2+u3'power'2)/2;
vasko(1,6,t,u1,u3);
'if'x'less'45'then'
'begin' vasko(1,6,jlv,j3v,blv,b3v);
nlcz(1);
'end'
'else'
'begin' j1:=ax(e1+w*e3-u3+w*u1);
j3:=ax(e3-w*e1+ul+w*u3);
'if'x'less'805'then's:=40'else'
s:=80;
bl:=(e3+j3v)/s;
b3:=b3v+(j1+jlv)/s;
jlv:=j1; j3v:=j3;
biv:=bl; b3v:=b3;
vasko(1,6,j1,j3,bl,b3);
nlcr(1); 'end'; 'end'; 'end'; 'end'; 'end'; 'end'; 'end';
Figure 1

Degree of ionization $\delta$

$\frac{\rho}{\rho_0} = 10^{-5}$

$10^{-3}$

$10^{-1}$

$10^1$

Temperature $T \text{K}$
FIGURE 2

HALL PARAMETER \( \omega \tau \)

- \( \frac{\rho}{\rho_o} = 10^{-5} \)
- \( \frac{\rho}{\rho_o} = 10^{-3} \)
- \( \frac{\varphi}{\varphi_o} = 10^{-1} \)
- \( \frac{\varphi}{\varphi_o} = 10 \)

Temperature \( T \) in °K:
- 5000
- 7000
- 9000
- 11000
- 13000
- 15000
Fig. 4
The influence of the Hall parameter \( \omega T \)
and the wall conductance parameter \( \Sigma \)
on the profile of the dimensionless velocity
\( u_x/U_0 \) at constant Hartmann number.
\( M=10 \)
Fig 5
The influence of the Hall parameter $\omega \tau$ and the Hartmann number $M$ on the profile of the dimensionless velocity $u_x / u_0$ at constant wall conductance parameter, $\Sigma = 0$
Fig. 6
The influence of the wall conductance parameter \( \Sigma \) and the Hartmann number \( M \) on the profile of the dimensionless velocity \( u_x/U_0 \) at constant Hall parameter, \( \omega \Sigma = 6 \).
Fig. 7
The influence of the Hall parameter \( \omega T \) and the wall conductance parameter \( \Sigma \) on the profile of the dimensionless velocity \( \frac{u_z}{U_0} \) at constant Hartmann number, \( M=10 \).
Fig. 8
The influence of the Hall parameter $\omega Z$ and the Hartmann number $M$ on the profile of the dimensionless velocity $u_z/U_0$ at constant wall conductance parameter, $\Xi = 0$. 
Fig. 9

The influence of the wall conductance parameter \( \Sigma \) and the Hartmann number \( M \) on the profile of the dimensionless velocity \( u_z/U_0 \) at constant Hall parameter, \( \omega\nu = 6 \).
Fig 10
The influence of the Hall parameter $\omega \tau$ and the wall conductance parameter $\Sigma$ on the profile of the dimensionless temperature $T-T_0$ at constant Hartmann number, $M=10$.
Fig. 11
The influence of the Hall parameter $\omega t^*$ and the Hartmann number $M$ on the profile of the dimensionless temperature $\frac{T - T_0}{\Delta T_0} / \frac{a}{\lambda}$ at constant wall conductance parameter $\Sigma = 0$. 
Fig. 12
The influence of the wall conductance parameter $\Sigma$ and the Hartmann number $M$ on the profile of the dimensionless temperature $\frac{T-T_0}{\frac{T-T_0}{2U_0/\lambda}}$ at constant Hall parameter, $\omega T^*=6$. 
Fig. 13
The influence of the Hall parameter $\omega T$ and the wall conductance parameter $\Sigma$ on the profile of the dimensionless electrical current density $jx/\sigma_0 U_0 B_0$ at constant Hartmann number, $M=10$. 
Fig. 14
The influence of the Hall parameter $\omega_T$ and the Hartmann number $M$ on the profile of the dimensionless electrical current density $j_x/(\sigma U_0 B_0)$ at constant wall conductance parameter, $\Xi = 0.$
Fig. 15

The influence of the wall conductance parameter $\Sigma$ and the Hartmann number $M$ on the profile of the dimensionless electrical current density $j_x/G_l U_0 B_0$ at constant Hall parameter, $\omega \tau = 6$. 
Fig. 16
The influence of the Hall parameter $\omega T^*$ and the wall conductance parameter $\Xi$ on the profile of the dimensionless electrical current density $j_x/\sigma x_0 B_0$ at constant Hartmann number, $M=10$. 
Fig. 17
The influence of the Hall parameter \( \omega \xi^* \) and the Hartmann number \( M \) on the profile of the dimensionless electrical current density \( J_{2x}/\nu \sigma \xi U_0 B_0 \) at constant wall conductance parameter, \( \Xi = 0 \)
Fig. 10
The influence of the wall conductance parameter Σ and the Hartmann number $M$ on the profile of the dimensionless electrical current density $j z / \Sigma U_0 B_0$ at constant Hall parameter, $\alpha^* = 6$.